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Category Theory as a Conceptual Tool in the Study of Cognition

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1. Introduction

The aim of this paper is to introduce category theory as a conceptual tool in the study of cognition. We believe that category theory has several advantages over set theory as a guide in analyzing and conceptualizing basic intentional abilities of the human mind such as the abilities to refer, to count and to learn.

These advantages result from the following considerations that we hope to clarify and justify as we go along:

1. Category theory is a generalization of set theory in the sense that the latter appears as a limiting case which may be accommodated within the former. As a consequence, category theory is able to provide means to avoid over-determinations resulting from peculiarities of set theory.
2. Category theory provides means to circumscribe and study what is universal in mathematics and other scientific disciplines. By identifying logic with the study of what is universal category theory supplies the means to describe such a logic, the objective logic of the discipline in question.

We illustrate consideration (1) with an example. We all know from high school that some quantities are scalars (temperature, pressure), whereas others are vectors (velocity, force).The latter have not only magnitude but direction. As physics developed, the need for a generalized notion of vector was felt and the question arose how one should define such a notion. The answer turned out to be both deep and simple: a generalized vector is an element of a vector space, i.e., of an algebraic structure satisfying some simple laws. Thus, we do not over-determine a generalized vector by defining it as a kind of "super-arrow" with more directions and further angles (including possibly solid ones)

and satisfying complicated conditions which generalize those of high school vectors. Using an Aristotelian term, we could say that the essence of a vector is its belonging to a vector space.

A similar development has taken place in logic. Modern logic, in the form that originated from Frege, Peano and Russell arose from the study of arithmetic and set theory. As it developed, the need was felt to generalize the notion of set. This need was felt particularly when people tried seriously to apply logic to natural languages. The objects referred to in such languages are ephemeral and changing, unlike numbers and sets, which are timeless and constant. Category theory gives us the means to define a generalized (or variable/cohesive) set as an object of a category satisfying some properties (a topos), without the temptation to go into over-determinations such as: "it is a family of sets varying continuously over a Hausdorff topological space and such that..." Of course, for some toposes such a description of a generalized set is possible. Notice that this is analogous to our vector-space example: generalized vectors may be described in terms of angles, etc. for vector spaces of dimensions up to three.

As an example of this way of thinking, the reader may consult Reyes (1991a), Reyes and Zolfaghari (1991b) for an approach to modal logic that avoids such over-determinations as possible worlds, possible situations, alternative relations and so on. Once again, the resulting modal operators may be described in terms of some of these notions in particular cases.

With respect to consideration (2), some historical notes may help the reader to place the theory of categories in the correct perspective.

Category theory was created in the earlier 1940s by S. Eilenberg and S. MacLane to guide some complicated computations involving passage to the limit in algebraic topology. During the 1950s it was the study of concrete problems in algebraic topology, complex analysis, functional analysis and algebraic geometry that led people like Kan and Grothendieck to further advances such as the discovery of adjoint functors, abelian categories and homological algebra. Since then, and especially during the last 20 years, category theory has increasingly found its vocation as a foundational discipline capable of clarifying (and sometimes even expanding) our understanding of mathematical knowledge and its applications. This change is mainly due to the discovery of F. W. Lawvere that some categories may be viewed as universes of variable/cohesive sets, capable of modelling theories that lack models in the more rigid universe of constant sets. Already, category theory has been applied to a variety of subjects ranging from physics to linguistics.

It was a fundamental discovery that most of the basic notions and constructions in all of these fields may be characterized by so-called

universal properties, properties which may be formulated in terms of a very few primitives: object, morphism, domain (of a morphism), codomain (of a morphism) and composition. These primitives generalize those of set, function, domain and codomain, respectively, of set theory. We shall see several examples of universal properties in this paper. Universality may also be described in terms of adjoint functors (as we shall see in section 3), which Lawvere sees as expressing the objective dialectical relations at the heart of a given field. Considering logic as the study of what is universal, then, we arrive at what Lawvere calls the objective logic of the field, a logic to be discovered rather than postulated at the outset. In this way, classical logic appears as a particular presentation of the objective logic of the category of constant sets. On the other hand, the logic of the universe of graphs is richer and subtler and it exhibits phenomena not to be found in the classical logic (see Reyes and Zolfaghari [1991b]). To see how the search for a logic can be carried out, see section 2, where we argue that counting implies that the logic of finite sets is classical.

Besides these considerations of a rather general nature, there is one which is specific to cognition: the human mind has the ability to carry out a large number of constructions (probably following "blueprints" of some kind) that seem so natural and so universal that they must be severely constrained. (See section 2 for some examples). It follows that we need an adequate theory to describe constructions that are naturally constrained. But as we saw, category theory is precisely such a theory. We may even suggest that universals of the mind may be expressed by means of universal properties in the theory of categories and much of the work done up to now in this area seems to bear out this suggestion.

The paper is divided into two parts. In the first (section 2) we introduce informally some of the main categorical themes by means of an analysis of the process of counting. By discussing the process of counting in some detail, we give evidence that this universal ability of the human mind may be conveniently conceptualized in terms of this theory of universals which is category theory. (For more on category theory as a theory of universals, see Ellerman [1988]).

In the second part (section 3) we give rigorous definitions of categories, functors, natural transformations and adjoint functors and illustrate these notions, whenever possible, with examples from logic and linguistics rather than from mathematics.

2. Counting

Counting is certainly one of the mind's fundamental abilities. We do not want to develop a theory of the mental processes involved in counting

but to express a few basic facts that underly this ability. The first thing required for counting is that the objects to be counted be clearly delimited in such a way that they become definite and distinguishable. For instance, it seems doubtful that we can count all animals. On the other hand, we can count the dogs in a particular room. This "collection into a whole of definite and separate objects" is precisely what Cantor calls a set (*Menge*) and thus we may say that counting applies to sets rather than mereological wholes such as heaps. As Frege pointed out, the same mereological whole consisting of marching soldiers may be counted as one army, ten regiments or 30,000 men. The result of counting the elements of a set is a natural number, the number of elements in the set, provided that the set is finite. We shall write $\#A$ for the number of elements of the (finite) set A. Before going into the process of counting itself, let us remark that counting is simplified by using arithmetical operations such as addition, multiplication and exponentiation between natural numbers. Thus, to count the apples contained in a bag we may put them first into two smaller bags, count the apples in each bag and add the numbers thus obtained. On the other hand, if the objects to be counted are nicely ordered in an array such as a chess board, we may count the columns and the rows separately and then multiply the resulting numbers. The arithmetical operations on natural numbers are the reflection of more basic operations between (finite) sets. The first example exhibits the set-theoretical operation of taking the union of two disjoint sets as the operation underlying addition. In the second example, it is rather the (cartesian) product of the two sets that underlies multiplication. Furthermore, ordinary laws that relate these arithmetical operations turn out also to be reflections of laws that relate the underlying set-theoretical operations.

Before we go into details, let us recall the definitions of the set-theoretical operations mentioned above. Given any sets A and B, we define their *cartesian product* to be the set $A \times B = \{(a, b) : a \in A, b \in B\}$. In other words, the cartesian product of A and B is the set consisting of ordered pairs of elements, the first from A and the second from B. Similarly, given any sets A and B (which may partly overlap or even be equal) we define their *disjoint union* $A + B$ to be the ordinary union of the (disjoint) sets $A \times \{0\}$ and $B \times \{1\}$. This means that an element of the disjoint union is either of the form $(a, 0)$ or of the form $(b, 1)$. Notice that the disjoint union of $\{0, 1\}$ and $\{0, 1, 2\}$ has 5 elements, whereas their ordinary union has only 3. The last operation we consider is exponentiation: given A and B as before, we define B^A to be the set of functions from A to B. In other words, an element of B^A is a function whose domain is A and whose codomain is B.

We may now formulate the relation between ordinary arithmetical

operations and their underlying set-theoretical operations as follows (where A and B are finite sets):

Proposition 2.1.

1. $\#(A + B) = \#A + \#B$
2. $\#(A \times B) = \#A \times \#B$
3. $\#(B^A) = (\#B)^{(\#A)}$

Similarly, we could formulate corresponding relations between arithmetical and set-theoretical operations at the level of laws. As an example, lets us notice that the distributive law $n \times (p + q) = n \times p + n \times q$ follows from the following proposition (where A, B and C are finite sets):

Proposition 2.2. There is a one-to-one correspondance between $A \times (B + C)$ and $A \times B + A \times C$.

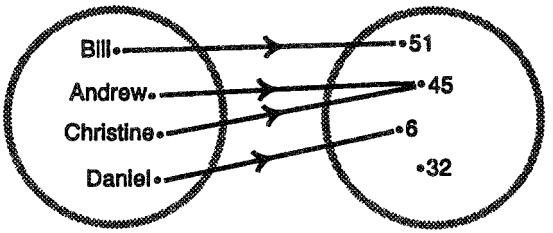
The set-theoretical operations underlying the arithmetical ones are so natural that they are taken for granted when counting, which creates the impression that they are not involved in this process. We believe that these operations constitute universal "blueprints" of the mind and we may ask what distinguishes them from others. Set theory does not give us too much help with this problem. Indeed, from a purely set-theoretical point of view, nothing seems to distinguish these operations from countless others that may be defined starting from two sets. To make things worse, the definition of disjoint union is rather artificial. The main point is to transform the original sets so as to make them disjoint and this could be achieved in a variety of ways. How can one decide which, if any, is the right operation?

It is at this point that category theory comes to our rescue with a new idea: instead of seeing a cartesian product as a set, let us view it as a structure consisting of a set *together* with the relations between this set and the original ones. This structure satisfies a universal property which essentially characterizes it. We need, of course, to specify the relations between the cartesian product of two sets (as defined above) and the original sets. These relations are given by functions. Since the notion of function is so fundamental, we pause to describe it in some detail.

2.1. *Sets and Functions*

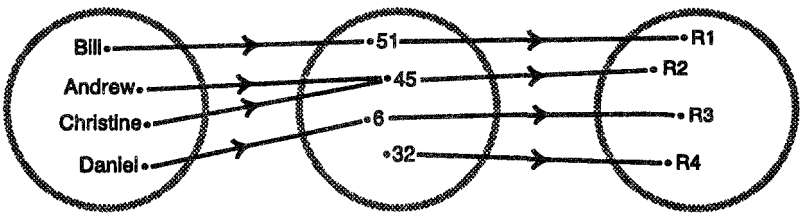
A *function* consists of three things: a starting set called *domain* or *source*, an ending set called *codomain* or *target* and a rule that assigns to each member of the domain, one and only one member of the codomain. As

an example, consider the rule that assigns to each busdriver in Montreal the bus line to which he or she is attached. Defining the set of bus drivers of Montreal as domain and the set of bus lines as codomain, we have defined a function. We can show all these constituents in a diagram like the following:

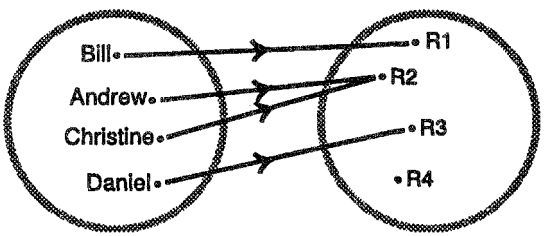


It is easy to see that there is always a very trivial function 1_A between any set A and itself, namely, the one that assigns to each element a of A the same member a. This function is called the *identity function* on A.

Functions are closed under several operations. One of the most basic is *composition*. This concept comes very naturally when we start dealing with functions. In our previous example, we could have introduced another function that would have the set of bus lines as domain, the set of possible bus routes as codomain and that would assign to each bus line the route it regularly follows. We can then represent this situation with the following diagram in which R1 stands for route 1, etc.:

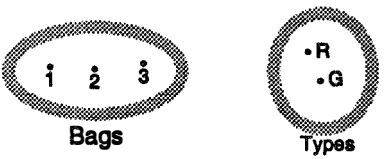


If we follow the arrows from bus drivers to bus lines and then to routes we can now assign a route to each driver. This gives us a function having the set of bus drivers as domain and the set of routes as codomain:

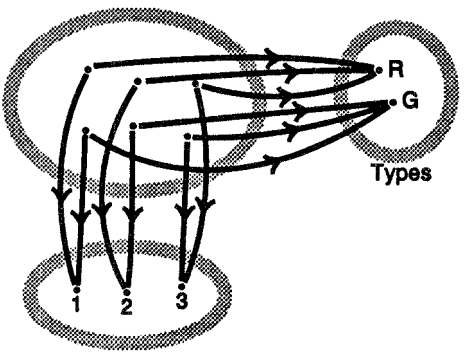


This new function is called *composition* of the original functions. It is clear that we can compose any two functions such that the codomain of one is the same set as the domain of the other. The notation we will use for the composition of $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ will be $A \xrightarrow{g \circ f} C$. It can be seen fairly easily that composition is associative, that is, if we have three functions $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$, then the compositions $h \circ (g \circ f)$ (h composed with $g \circ f$) and $(h \circ g) \circ f$ ($h \circ g$ composed with f) are the same function. The composition with identity functions does nothing, that is, if we have a function $A \xrightarrow{f} B$, then $f \circ 1_A = f = 1_B \circ f$.

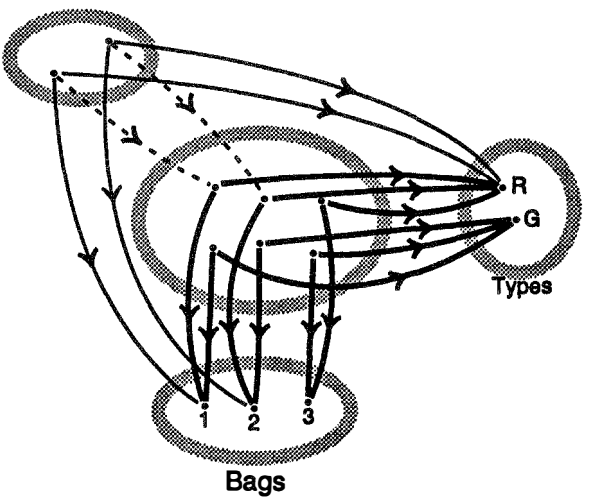
We are now ready to describe the categorical cartesian product of two sets. Suppose that we have three bags of apples denoted by 1, 2, 3, each containing two apples, one green, G and another red, R. Diagrammatically:



We can now make a list of all the elements of the (ordinary) cartesian product as follows: $(1, G)$, $(2, G)$, $(3, G)$, $(1, R)$, $(2, R)$, $(3, R)$. This set is clearly in one-to-one correspondence with the set of all the apples and may be identified with it. For instance, $(2, G)$ corresponds to the green apple in the second bag. The relations between the ordinary cartesian product and the original sets may be described by two functions. The first has the ordinary or set-theoretical cartesian product as domain, the set $\{1, 2, 3\}$ of bags as codomain and assigns to each apple the bag that contains it. The other has also the cartesian product as domain but has the set $\{G, R\}$ as codomain and assigns to each apple its type (green or red).

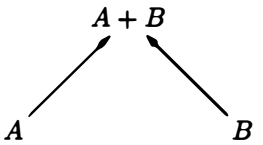


As we said before, the basic idea is now to characterize the structure consisting of the (ordinary) cartesian product *together* with these two special functions (rather than of the cartesian product alone) among sets plus a couple of functions having the same codomains as our two special functions. Let us call such a structure a *fork*. Thus a fork consists of a set X (the "handle") together with a couple of functions having X as domain: the $\{1,2,3\}$-"prong" whose codomain is the set of bags and the $\{G,R\}$-"prong" whose codomain is the set of types. We now consider the fork having the cartesian product as handle, the function which assigns to each apple the bag where it came from as $\{1,2,3\}$-prong and the function which assigns to each apple its type as the $\{G,R\}$-prong. Such a fork has the following universal property: given any other fork, say fork′, there is a unique function f from the handle of fork′ into the product set (handle of our original fork) that respects the prongs in the following sense. If a member x of the handle of fork′ is assigned to bag 2 (by the $\{1,2,3\}$-prong of fork′) and assigned to type R (by the $\{G,R\}$-prong of fork′) then $f(x)$ must be $(2,R)$. The fact that f respects the prongs can be expressed more generally in terms of composition of functions as follows: if we compose a prong of our original fork with f we get the corresponding prong in fork′. A fork having this universal property will be called a *terminal* fork. The following diagram illustrates the situation:

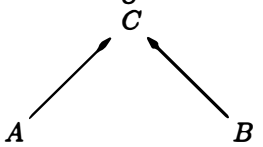


The *categorical cartesian product* is then defined to be any terminal fork. It would be more correct to say *a* product rather that *the* product in this definition, but two terminal forks are "essentially" the same in a sense to be made precise later on.

We can describe in a similar way the sum (or disjoint union) of two sets. Let A and B be two finite sets. We saw before how to form the disjoint union $A + B$ of those sets. The sum of A and B will be defined as the diagram:



instead of just the set $A + B$. A diagram of the form:



will be called a *cofork* for A and B. The sum is clearly a cofork and its specificity is guaranteed by a universal property. The sum is an initial cofork. This means that if we take any other cofork (like the one displayed above) there will be a unique function $A + B \longrightarrow C$ that respects the prongs of the cofork (in a sense analogous to the one we have given in the definition of product).

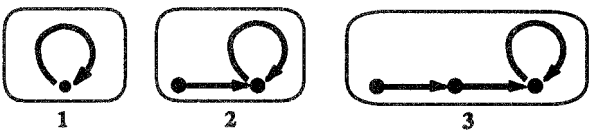
We will see later how the definitions of product and disjoint union (or coproduct) may be formalized in the general context of category theory.

2.2. *Counting as a Process*

Although we have set the stage for counting, counting itself involves other structures that reflect the mind's ability to iterate. To conceptualize this notion, we notice that we iterate or repeat processes and a process is usually conceptualized as a function. To iterate the function is to compose it with itself. This does not always make sense but only at times when, for instance, the domain coincides with the codomain. This leads to the notion of an evolutive set or discrete dynamical system. An *evolutive set* is a function whose domain coincides with its codomain. Thus, an evolutive set is a set with an endofunction.

There are cases where processes are not readily presented as functions. How can we consider the turning of the earth around the sun as an endofunction? What would its domain be? This example and several others seem to require a more abstract conceptualization: a set (whose elements are thought of as processes) together with a binary operation (thought of as composition) satisfying some conditions. These considerations suggest the use of the more abstract notion of *monoid* to conceptualize iteration. However, it is a basic result of Cayley that any abstract monoid may be represented as an evolutive set.

With these structures in place, let us see what is involved when we count a crowd of people in a room. We first choose one person and give her the number 1, proceed to choose a second person and give her the number 2 and so on until everybody has been chosen once and given a number. These successive choices amount to associating the structure of an evolutive set with the set of persons (the endofunction assigns to each person the one that is chosen next and with the last that person herself). The evolutive set is endowed with a starting point (the first person chosen) together with a function from the set of people into the set of natural numbers $1, 2, 3, \ldots$ The number assigned to the last person chosen is precisely the number of people sought. We can picture such a structure by drawing a dot for each member of the set we want to count and an arrow from each dot to the dot counted next. In this way we get things like:



Once again we see a blueprint of the mind in action whose prototype is the evolutive set I of natural numbers $1, 2, 3, \ldots$ with successor as the endofunction and 1 as the starting point. What characterizes this system as distinct from other evolutive sets? The answer (due to Lawvere) is given by the following universal property: it is the initial evolutive set with starting element. More precisely, we have the following:

Proposition 2.3. Given any evolutive set A with endofunction f and starting element a, there is exactly one function from I into A which preserves the structure in the following sense: it assigns 1 to a and whenever n is assigned to x, then $n+1$ is assigned to $f(x)$, the successor of x.

In particular, there is a unique structure preserving function from I into the evolutive system of the people in the room induced by the successive choices. Notice that if there are, say, 15 people, the last person will be assigned to all numbers from 15 on. This gives an equivalent way of counting: the first natural number to which the last person is assigned is the number of people sought.

Counting a set A appears, then, as associating the generic structure of an evolutive set having a starting element with A and mapping the blueprint I into A in the only way that preserves the structure.

Notice that the number of elements of a set is defined in terms of the set in question *enriched* with a structure of evolutive set with starting

element. Since there may be several ways to introduce such a structure on a set (just start with a different person and make different choices for second, third, etc.) it is not clear (from this definition) that the number in question is independent of the choice of such a structure. We have, however, the following fundamental result:

Proposition 2.4. Two evolutive sets with starting elements whose underlying sets are in bijective correspondance have the same number of elements.

These considerations of the actual process of counting have an interesting corollary about the "logic of the stage", that is, about the logic of those "collections into a whole of definite and separate objects": it is Boolean. Indeed, let S be such a collection that may be counted by a natural number and let T be a subcollection, namely those already counted. To keep on going, we need to be able to decide, given an arbitrary element of S whether it has been counted already (i.e., whether it belongs to T) or not. In other words, the subcollections of S constitute a Boolean algebra. We have not started with a ready-made logic that we impose on these collections. Rather, we discover their logic by assuming that the operation of counting is applicable to them. This is one example of a main theme in this paper: objective versus subjective logic.

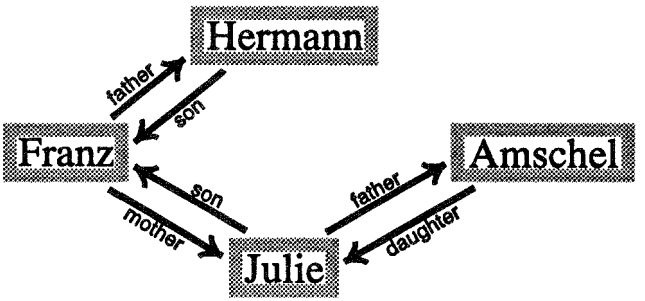
Before we leave this subject, we would like to point out that we have only scratched the surface of a large topic. Fascinating mathematical problems have appeared when dealing with the psychology of counting from a perspective similar to ours (cf. Bénabou [1991]). On the other hand, the psychology of counting (of which this section gives a glimpse) is a largely unexplored area which certainly deserves to be studied.

3. Basic Notions in Category Theory

Notice that in the previous section we did not define products (and disjoint unions) as sets of ordered pairs of elements (as is usually done in set theory), but in terms of universal properties described using only the basic notions of set, function, domain (of a function), codomain (of a function), identity (function) and composition (of functions). Category theory starts with the realization that most constructions in mathematics may be defined by universal properties, provided that we generalize these basic set-theoretical notions. Thus, the basic notions of category theory are: object, morphism, domain (of a morphism), codomain (of a morphism), identity (morphism) and composition (of morphisms).

3.1. Towards a Definition of a Category

Before going into the formal definition, we give an example of a category that is very far from the finite sets that we have (implicitely) described. Its objects will be people (rather than sets) and its morphisms, family relations (rather than functions). Let us take a "universe" consisting of four persons (the objects of the category): Franz, Hermann, Julie and Amschel. Morphisms will be family relations. More precisely, a morphism from (a person) A to (a person) B will be the description of B relative to A within the family. The domain of a morphism from A to B is A and its codomain is B. In the following diagram, morphisms are represented by arrows:



For instance, the arrow going from *Franz* to *Hermann* can be read as: "The father of Franz is Hermann." Now that we have objects and morphisms we naturally want to know how to compose the morphisms and what are the identity morphisms.

The identity morphims on a object A is a distinguished arrow starting from A and ending at A. In our example, the identity arrow on *Franz* will be interpreted as: "Franz is Franz." The notion of composition is also very straightfoward in this case. If we compose the two following arrows:

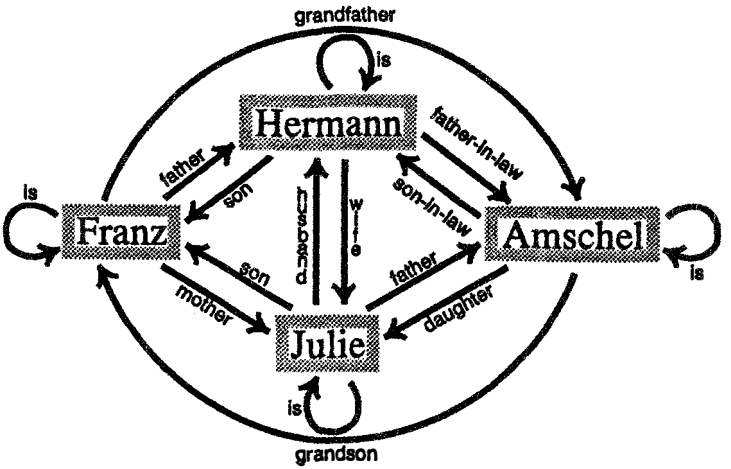


we get



Thus we identify 'The father of the mother of Franz is Amschel' with 'The grandfather of Franz is Amschel'. Of course, if we wanted to apply this to a concrete family, things could become more complicated because we would have to take into account the fact that everybody has two grandfathers and that a father may have more that one son, etc. Clearly, we could solve these difficulties but to do so would make the language heavier and would in no way help the reader to grasp the basic notions of category theory or prepare him to understand the general notion of a category.

If we draw our diagram again and add the identity arrows and all the arrows needed to compose any pair of arrows we get the following (by making the required identifications):



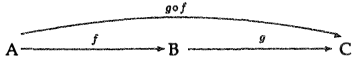
We can ask whether the properties of composition of functions remain true in this new context of family relations (rather than functions). In particular, is composition associative in our example? The reader can convince himself of the associativity of the composition by checking that 'The father of the wife of the man who is the father of Franz' and 'The father of the woman that is the wife of the father of Franz' are equivalent descriptions for 'The grandfather of Franz' (follow the diagram!). The other property of composition is that the composition of

a morphism f with any identity morphism (of the source or the target of f) gives f itself. This is also true in our example, since 'The father of the person that is Franz is Hermann' is the same as 'The father of Franz is Hermann'.

After these examples we are ready to give the definition of a category:

Definition 3.1 A category $\mathfrak{C}$ is a collection of objects $Ob(\mathfrak{C})$ and a collection of morphisms $Ar(\mathfrak{C})$ (also called arrows or maps) such that for each morphism of $Ar(\mathfrak{C})$ we can associate a domain object in $Ob(\mathfrak{C})$ (from which the morphism starts) and a codomain object in $Ob(\mathfrak{C})$ (at which the morphism ends) in a way that the following properties hold:

1. For each object $A \in Ob(\mathfrak{C})$ we have a special morphism $1_A \in Ar(\mathfrak{C})$ that has A both for domain and codomain and that is called identity on A.
2. If we have two morphisms $f, g \in Ar(\mathfrak{C})$ such that the codomain of f is the same object as the domain of g then there exists a morphism $g \circ f$ in $Ar(\mathfrak{C})$ starting from the domain of f and ending at the codomain of g. This morphism is called the composition of f and g.



3. If $A \xrightarrow{f} B$ in a morphism in $Ar(\mathfrak{C})$ then

$$f \circ 1_A = f = 1_B \circ f$$

4. If $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$ are three morphisms in $Ar(\mathfrak{C})$ then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

3.2. *Examples of Categories*

We now look at some simple examples of categories. The reader is invited to verify that all the properties needed are satisfied in those examples.

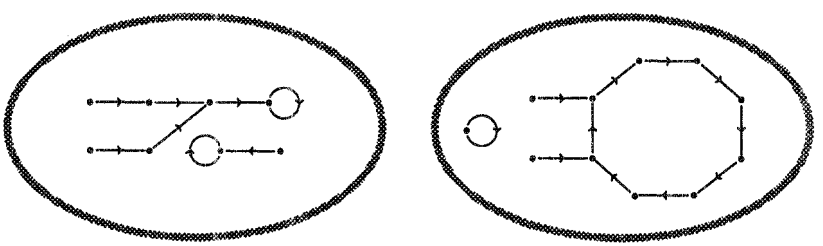
Example 1. Finite sets with functions clearly form a category, as we have seen in the first section.

Example 2. The family consisting of Hermann, Franz, Julie and Amschel together with their (family) relations also form a category.

Example 3. To each formal system we can associate a category whose objects are the well-formed formulas of the formal system. We also choose to put one and only one map going from formula A to formula B if we can prove B under the assumption of A i.e., $A \rightarrow B \iff A \vdash B$. Clearly, from each formula A we can trivially prove A ($A \vdash A$) so that for every object A we have an identity morphism, namely, the only morphism from A to A. Composition also comes very naturally. If we can prove B from A and C from B, then (by modus ponens) we can prove C from A ($A \vdash B$, $B \vdash C \Rightarrow A \vdash C$).

Example 4. We now define what is called "the nominal category of a subject" (introduced by M. LaPalme, J. Macnamara and G. E. Reyes in this volume). Objects are taken to be the count nouns of a subject. Take, for instance, zoology as the subject. The objects will then be the count nouns 'dog', 'mammal', 'whale', etc. We recognize a morphism from A to B if the sentence 'an A is a B' is correct in the subject of interest (correct, in the sense that you will find it in dictionaries). For example, there is a map from 'whale' to 'mammal' because it is the case that 'a whale is a mammal' is a correct sentence. There is no map from 'dog' to 'cat' because a dog is not a cat. The domain of a map is a count noun in subject position. The codomain is a count noun in predicate position. Identity maps are given by valid sentences of the form 'a whale is a whale'. Finally, composition is given by modus ponens: 'a whale is a mammal', 'a mammal is an animal' therefore 'a whale is an animal'.

Example 5. The category of evolutive sets (or discrete dynamical systems) whose objects are of the type (A, f) where A is a set and $A \xrightarrow{f} A$ is a function. We can give a very suggestive graphic representation of objects of this kind. First, we draw a dot for each member of the set A. Next, for each member $x \in A$, we draw an arrow that starts at the dot representing x that points at the dot representing $f(x)$ (the image of x by f). We get diagrams like these:



The justification of the name "dynamical system" appears through this representation. If we select a dot in the diagram – as initial state of the system – , then each time we apply f we move (in a completely

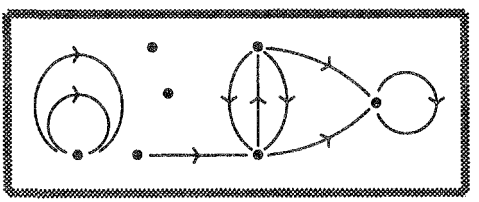
determinate manner) to the next state of the system and so on. We usually call f the "evolution" of the dynamical system.

Clearly, they are a generalisation of sets because we can consider a plain set A as the dynamical system $(A, 1_A)$ (where 1_A is the identity function on A) where no movement exists. Dynamical systems are sets with a further structure on them. In fact, we can consider dynamical systems as "sets that internally develop" (as Lawvere as pointed out). One of our main points can be formulated here. Category theory gives us much richer structures than plain set theory. This gives us one reason to believe that category theory is a better source than set theory for models of cognitive processes.

Remember that morphisms in a category are very important constituents. So, we must give a precise definition of morphisms between two evolutive sets (A, f) and (B, g) to put a category structure on this universe. Perhaps we could define such a morphism as a simple function $\Phi : A \longrightarrow B$. Clearly, all the conditions imposed in the definition of a category would be satisfied by this choice. But, it would not be a very clever choice because morphisms of this form do not take into account the evolutions f and g of the discrete dynamical systems. With these morphisms we would get a category that is essentially the same as the category of sets. The evolution functions f and g should, then, play a role in the definition of morphisms.

We give them a role by defining a morphism $(A, f) \xrightarrow{\Phi} (B, g)$ as a function $\Phi : A \longrightarrow B$ with further restrictions. We want Φ to preserve the evolutions in the following sense: If Φ sends $a \in A$ to $b = \Phi(a) \in B$, then the evolution of a (namely $f(a)$) must be sent by Φ to the evolution of b (namely $g(b)$). Using a mathematical notation, this fact is expressed by $\Phi(f(a)) = g(\Phi(a))$ for all $a \in A$. These morphisms are structure-preserving. They are clearly what we were looking for.

Example 6. Our last example will be the category of (multiple oriented irreflexive) graphs. Graphs are objects composed of dots and arrows. Each arrow has a source dot and a target dot. When the source and the target are the same for a particular arrow we call it a loop. The following picture shows an example of a graph:



Mathematically we describe a graph G as a set of dots G_0, a set of arrows

G_1 and two functions $s, t : G_1 \longrightarrow G_0$ that assign to each arrow its source and its target respectively. We use the notation (G_0, G_1, s, t) to refer to such a structure.

To get a category, we also need to define a notion of morphism between graphs. We will say that a dot and an arrow are incident if the dot is the source or the target of the arrow (we also say in this case that they are in a relation of incidence). If $G = (G_0, G_1, s_G, t_G)$ and $H = (H_0, H_1, s_H, t_H)$ are two graphs, we define a morphism of graphs $\Phi : G \longrightarrow H$ as a pair of functions (Φ_0, Φ_1) (where $\Phi_0 : G_0 \longrightarrow H_0$ and $\Phi_1 : G_1 \longrightarrow H_1$) that respect the incidence relations, i.e., $\Phi_0(s(f)) = s(\Phi_1(f))$ and $\Phi_0(t(f)) = t(\Phi_1(f))$ for all arrows $f \in G_1$. These mathematical relations only mean that the source of an arrow f in G must be sent by Φ_0 to the source of the arrow to which Φ_1 sends f in H (the same restrictions apply for the targets). These retrictions on Φ rule out morphisms that would separate an arrow from its incident dots by sending the dots to a place different from where the arrow is sent. Graphs can also be viewed as a generalisation of sets if we consider graphs with no arrows.

3.3. *Generalized Elements in a Category*

The last five examples given above are far removed in nature from simple sets. In fact, neither a formula nor a count noun is a set; evolutive sets and graphs are not just plain sets but sets with further structures on them. A fortiori, morphisms in these categories are not plain functions. Nevertheless, there is a sense in which objects of a category may be represented as sets of generalized elements and morphisms, as functions between such sets. To see this, let $1 = \{0\}$ be the singleton whose only element is 0. We claim that there is a one-to-one correspondence between elements of A and functions from 1 into A: with $a \in A$ we associate the function $a : 1 \longrightarrow A$ which sends the unique element of its domain, 0, into a. Conversely, with $f : 1 \longrightarrow A$ we associate the element $f(0) \in A$. These associations establish the required one-to-one correspondence. We may thus identify elements of A with functions $1 \longrightarrow A$ and this allows us to view elements as particular functions, namely, those whose domains are singletons. Let us define a *generalized element* of an object A of a category to be a morphism $X \longrightarrow A$, where X is again an object of the category. If we want to make explicit the dependency on X, we shall talk of an *X-element* of A or an element of A parametrized by X. A morphism $f : A \longrightarrow B$ may be represented as an actual function from the generalized elements of A into the generalized elements of B as follows: given a generalized X-element of A, $a : X \longrightarrow A$, the composition $f \circ a$ is a generalized X-element of B.

We now give an example of generalized elements in the category of graphs. Let X be the graph consisting of only one dot a and one loop l. We consider all the morphisms from X to an arbitrary graph G. This collection is the collection of X-elements of G. What does an element of this collection look like? To construct a morphism $X \xrightarrow{\Phi} G$, we first need to give an image to a, that is, to select a dot $\Phi_0(a)$ in G. We also need to give an image to the loop l and so we select an arrow $\Phi_1(l)$ in G. According to the rules for the morphisms in graphs, the source of $\Phi_1(l)$ and its target must both be $\Phi_0(a)$ (because a and l have this incidence relation in X by the very definition of loops). We have found that an X-element of G is simply a loop of G so that X is a special graph that has the ability to extract the loops from any graph. The reader is encouraged to see how this representation works in our four other examples of categories.

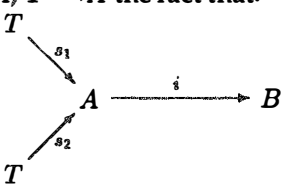
Using these generalized elements, we will now discuss the generalization of the notions of a monic, epic and bijective function from the theory of sets to the context of category theory.

3.4. *Monomorphism, Epimorphism and Isomorphism*

Let $f : A \longrightarrow B$ be an ordinary function between sets. Recall that this function is monic (or one-to-one) if, whenever a and a' are two elements of A which are sent by f into the same element of B, i.e., $f(a) = f(a')$, then $a = a'$. In other words, different elements of the domain have different values in the codomain. As an example consider a classroom during a lecture in which nobody is standing. The function which assigns to each person the chair he is sitting in is monic precisely when no two people sit in the same chair.

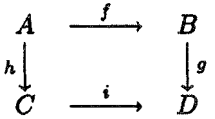
The natural generalization of this notion to the context of category theory is by defining a morphism $f : A \longrightarrow B$ to be monic or a monomorphism if the actual function from generalized elements of A into generalized elements of B which represents f is monic. We thus arrive at the following:

Definition 3.2. A morphism $A \xrightarrow{i} B$ is a monomorphism if for any object T and morphisms $T \xrightarrow{s_1} A$, $T \xrightarrow{s_2} A$ the fact that:



commutes (meaning that $i \circ s_1 = i \circ s_2$) must imply that $s_1 = s_2$. In this case we write $A \overset{i}{\hookrightarrow} B$ instead of just $A \xrightarrow{i} B$.

We say that a diagram commutes if, for any pair of objects in the diagram, all paths between them give rise to the same morphism if we compose the morphisms in each path. For example, to say that the following diagram commutes:

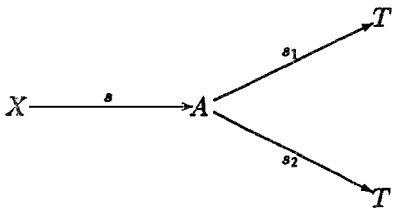


means that $i \circ h = g \circ f$ because there are two paths going from A to D (first f followed by g and also h followed by i).

Recall that a function $f : A \longrightarrow B$ between sets is epic or surjective if every element in the codomain comes from an element of the domain; if, in other words, for each $b \in B$ there is an $a \in A$ such that $f(a) = b$. If we go on to define a morphism $f : A \longrightarrow B$ in a category to be epic or to be an epimorphism by requiring that the representing function (between generalized elements) should be epic or surjective, we obtain a very strong notion. Indeed, assuming that for every generalized element $b : X \longrightarrow B$, there is a generalized element $a : X \longrightarrow A$ such that $f \circ a = b$. In particular, this must be true for $1_B : B \longrightarrow B$ and hence we obtain a generalized element $s : B \longrightarrow A$ such that $f \circ s = 1_B$. In other words, f must have a right inverse (s), generally referred to as a section (of f). Conversely, if f has a section, one may conclude that the representing function of f is surjective. This notion is important in several categorical contexts and is called a *split epi*. For some purposes, however, it is too strong and we would like to define a weaker notion of epimorphism.

To do so, let us return to our definition of products and coproducts. The reader may have observed that these are very symmetric. In fact, one may reverse all morphisms in the definition of either to obtain the definition of the other. We say that they are duals. This gives a hint to the solution of our problem: define epimorphism to be the dual of monomorphism. Thus we arrive at the following:

Definition 3.3. A morphism $X \xrightarrow{s} A$ is called epimorphism if, for any object T and each pair of morphisms $A \xrightarrow{s_1} T$, $A \xrightarrow{s_2} T$, we have that $s_1 = s_2$ from the fact that following diagram commutes:



Finally, let us see how to generalize the notion of bijection to the categorical context. Following the straightforward approach of representing a morphism $f : A \longrightarrow B$ as a function between generalized elements, it is natural to define f to be an isomorphism if its representing function is bijective, i.e., monic and epic. A moment's reflection allows us to reformulate this definition as follows:

Definition 3.4. A morphism $A \xrightarrow{f} B$ is called isomorphism if there is a morphism $B \xrightarrow{g} A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

3.5. *Universal Constructions in Categories*

We are now in a position to define products and coproducts as well as some other operations defined by universal properties in an arbitrary category.

3.5.1. *Initial and Terminal Objects*

In section 3.3 we tried to capture the notion of an element of a set purely in terms of the basic notions of set, function, domain, codomain, identity and composition. We would have succeeded but for the fact that we had to introduce a special set 1 having exactly one element (that we called "0"), i.e., a singleton. The question naturally arises whether we can define singletons in terms of our basic notions. The answer is yes: a singleton S is characterized by the universal property that there is exactly one function from any set X into S (indeed, the function which assigns the only element of S to each element of the set X). This allows us to define the corresponding categorical notion:

Definition 3.5. A terminal object of a category $\mathfrak{C}$ is an object $1 \in Ob(\mathfrak{C})$ such that, for every object $X \in Ob(\mathfrak{C})$, there is one and only one morphism:

$$X \xrightarrow{!} 1$$

By the process of reversing all the arrows we may define the dual notion:

Definition 3.6. An initial object of a category $\mathfrak{C}$ is an object $0 \in Ob(\mathfrak{C})$ such that, for every object $X \in Ob(\mathfrak{C})$, there is one and only one morphism:

$$0 \xrightarrow{!} X$$

An important fact is that any two terminal objects (as well as any two initial objects) in a category are uniquely isomorphic. In other words, if T and T' are two terminal objects, then there is a unique isomorphism

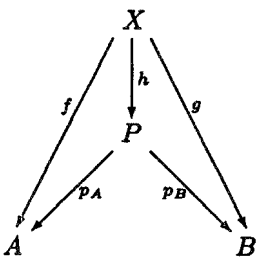
between the two. Because of this, it is customary to collapse all terminal objects into a representative and talk about *the* terminal object.

Looking back at our examples, we find that in the category of finite sets the initial object is the empty set $\emptyset$. This may seem rather twisted at first because a function from the empty set to another set seems to mean nothing. However, the function that does nothing satisfies our definition of function and it is clearly unique because we cannot construct two different functions $\emptyset \longrightarrow A$ (To do so we would have to give a different image to an element of $\emptyset$ in each function. But by definition $\emptyset$ does not have elements). The notions of terminal and initial objects do not exist in the nominal category nor in the category of family relations. In the category of well-formed formulas of a formal system, initial objects are contradictions (statements that are always false like $P \wedge \neg P$) for the simple reason that from them we can prove anything and that morphisms between two objects in this category are always unique when they exist. Still, in this category the terminal objects are clearly tautologies (statements that are always true, like $P \vee \neg P$) for the reason that we can prove a tautology from any statement. In the category of evolutive sets the terminal objects are of the form $(\{*\}, Id)$ where $\{*\}$ is a singleton and Id is the identity function on this singleton. The initial object (unique in this case) is the empty evolutive set having as evolution the only possible endofunction $\emptyset \xrightarrow{f} \emptyset$. We invite the reader to find initial and terminal objects in the category of graphs.

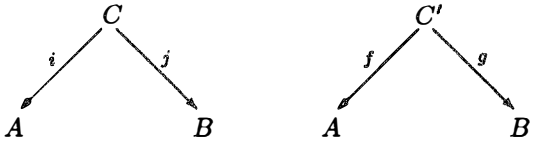
3.5.2. *Products and Coproducts*

We have seen in section 2.1 how to define the notion of product in terms of sets and functions alone. We may now give the definition of a (categorical) product of two objects in a category:

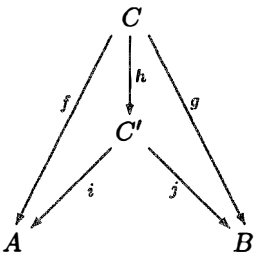
Definition 3.7. Let $\mathfrak{C}$ be a category. A product of two objects $A, B \in Ob(\mathfrak{C})$ is an object $P \in Ob(\mathfrak{C})$ together with two morphisms $P \xrightarrow{p_A} A$ and $P \xrightarrow{p_B} B$ such that, for any object $X \in Ob(\mathfrak{C})$ equipped with a pair of morphisms $X \xrightarrow{f} A$, $X \xrightarrow{g} B$, there is one and only one morphism $X \xrightarrow{h} P$ (usually we use the notation $\langle f, g \rangle$ for h) such that the following diagram commutes:



We can also see the product as a terminal fork. To do this, we first need to define the category of forks on two objects A and B. The objects of this category are forks on A and B. If we have two forks:



then a morphism from the C fork to the C' fork is defined as a morphism $C \xrightarrow{h} C'$ (this last morphism is actually in the category $\mathfrak{C}$) such that the following diagram commutes:



It is clear that a terminal object in this category of forks is a product of A and B.

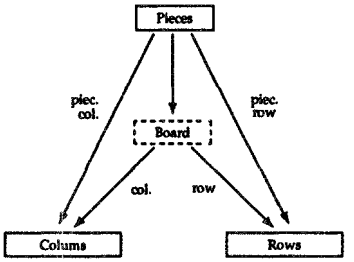
We use a Gentzen notation to express universal mapping properties. For the product, we get:

$$\frac{C \to A \times B}{C \to A \ , \ C \to B}$$

This means that with each morphism from an object C to the product object $A \times B$ we can associate a morphism from C to A and another from C to B and conversely. The reader is invited to go back to the example of bags of apples to verify his understanding of the above definition.

Another example of product is the following: a chess board is a square divided into 8 rows and 8 columns. Each player has a set of pieces that are placed on the board. A possible way to describe all the positions of the pieces on the board at any time in the game is to systematically draw the board with each of the pieces in its position.

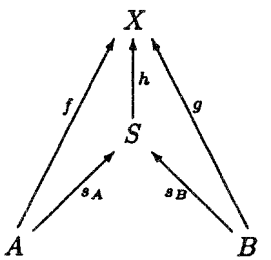
Another way would be to assign to each column, a letter (from A to H) and to each row, a number (from 1 to 8). We could then say where a particular piece is placed by giving the letter of its column and the number of its row (like C3). The reason why we can do this is that the board is the product of the set of possible rows and the set of possible columns. The following diagram describes the integration of the notion of product in this example:



Our last example of product will be the product in the category of a formal system. What could satisfy the definition of product in this category? The product of two objects A and B must be an object P with two morphisms $P \dashrightarrow A$ and $P \dashrightarrow B$ such that, for every other object C and morphism $C \dashrightarrow A$, $C \dashrightarrow B$, there is one and only one morphism $C \dashrightarrow P$ such that a given diagram commutes (see the definition of product). In our particular category the product of formulas A and B must be a formula P from which we can prove both A and B and such that it is provable from any other formula C that could also prove both A and B. There is clearly one way of constructing the product: by letting $P = A \wedge B$ (the conjuction of the two formulas). We could have let $P = \neg(\neg A \vee \neg B)$ but this formula is clearly equivalent to the first. In fact, we can prove easily that all the products of two given objects in a category are isomorphic (this follows from the fact that any two terminal objects are isomorphic).

The dual notion of product is called "coproduct" and is defined as follows:

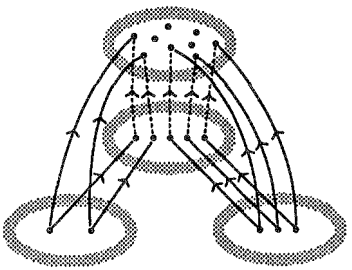
Definition 3.8. Let $\mathfrak{C}$ be a category. A co-product of two objects $A, B \in Ob(\mathfrak{C})$ is an object $S \in Ob(\mathfrak{C})$ with two morphisms $A \xrightarrow{s_A} S$ and $B \xrightarrow{s_B} S$ such that, for all objects $X \in Ob(\mathfrak{C})$ and all pairs of morphisms $A \xrightarrow{f} X$, $B \xrightarrow{g} X$, there exists one and only one morphism $S \xrightarrow{h} X$ such that the following diagram commutes:



Again we can synthetise this definition in the following notation:

$$\frac{A + B \rightarrow C}{A \rightarrow C \ , \ B \rightarrow C}$$

What does the coproduct look like in our category of finite sets? A picture is enough to give the idea:



The coproduct in the category of finite sets is *sum*. We see in the above picture that $2 + 3 = 5$. We begin to see the advantages of expressing concepts in the categorial language. We have found a very intimate relation between product and sum. Naturally, the coproduct in the category of a formal system will be disjunction of formulas. We leave the verification of these facts to the reader.

3.5.3. *Equalizers and Coequalizers*

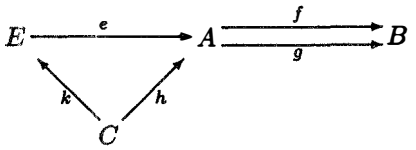
To introduce the notion of an equalizer we imagine a schoolteacher that in the middle of a term decides to assign new seats to his students. Suppose he wants to know which students remained in their old places. To make the change amounts to defining two functions o and n from the set S of students to the set C of chairs. These functions assign to each student his old and new chair respectively. The question is then to find the students who did not change places. From a set-theoretical point of view, the answer is obvious: it is the set $E = \{x \in S : o(x) = n(x)\}$. This set E is called the (set-theoretical) equalizer of o and n and we may try to characterize this set purely in terms of sets and functions. Just as in the case of products, the solution is to characterize the structure consisting of E *together* with the inclusion map e of E into S, rather than the set E alone. We can picture this structure as the diagram:

$$E \xrightarrow{\ e\ } S \underset{n}{\overset{o}{\rightrightarrows}} C$$

By going from sets to an arbitrary category and representing E as a set of generalized elements and e as a function between sets of generalized elements, we arrive naturally at the following definition:

Definition 3.9. A pair consisting of an object E and an arrow $E \xrightarrow{e} A$ in a category $\mathfrak{C}$ is called an equalizer of a pair of arrows $f, g : A \to B$ if

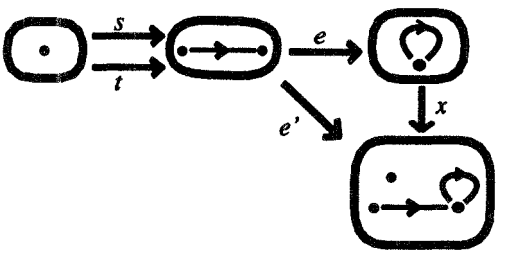
$f \circ e = g \circ e$ and, for every arrow $C \xrightarrow{h} A$ such that $f \circ h = g \circ h$, there is a unique arrow $C \xrightarrow{k} E$ such that the following diagram commutes:



As in previous cases, two equalizers are uniquely isomorphic.

The dual notion of an equalizer is called a coequalizer. We do not give the precise definition since it is the symmetric counterpart of the above definition: just reverse all the arrows! A simple example of a coequalizer can be given in the category of graphs.

Suppose we have two graphs G_1 and G_2 such that G_1 consists of only one dot (no arrows) and G_2 consists of two dots with an arrow linking them. We define two morphisms $s, t : G_1 \longrightarrow G_2$. The morphism s sends the dot of G_1 to the source of the arrow in G_2 and the morphism t sends the dot of G_1 to the target of the arrow in G_2. A coequalizer for this pair of morphisms must be a graph X and a morphism $e : G_2 \longrightarrow X$ such that $e \circ s = e \circ t$ and that is universal among similar objects. It is easy to see that the two dots in G_2 must be sent to a unique dot in X to satisfy the required condition. Consequently, there must be a loop in X for a morphism like e to exist. We must also verify that X together with e has the universal property of coequalizer; that is, if we have another morphism $e' : G_2 \longrightarrow X'$ such that $e' \circ s = e' \circ t$, then there must exist a unique morphism $X \xrightarrow{x} X'$ such that $x \circ e = e'$. Since X' has the same property as X, it must also contain a loop. The arrow in G_2 must be sent to a loop l in X' by e' so that there is a unique way of sending the loop X to l.



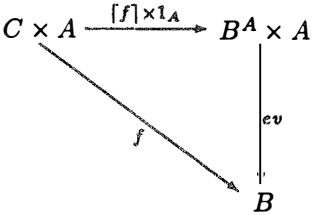
3.5.4. *Exponential Objects*

In the first section we pointed out that the set-theoretical operation of exponentiation B^A (for the set of all functions from A to B) generalizes the corresponding operation between natural numbers. This notation is very suggestive if we take into account our previous formula telling

that $\#(B^A) = \#A^{\#B}$. How do we characterize this operation by means of a universal property?

Let us imagine a computer that is programmed to compute the values of functions from A to B when presented with elements of A. You could give the computer a function $A \xrightarrow{f} B$ (i.e., an element of B^A) and an element $a \in A$, and it would give as answer the element of $b \in B$ where f sends a. The program in the computer would be a function $B^A \times A \xrightarrow{ev} B$ that does the following: $ev(\langle f, a\rangle) = f(a)$ (given a pair consisting of an function f and an element $a \in A$, it computes $f(a)$). This function is called the *evaluation function*. This function is, in a way, the best possible choice among all other functions of the form $C \times A \xrightarrow{g} B$. This is expressed precisely in the following definition:

Definition 3.10. Let $\mathfrak{C}$ be a category in which there are products, and let $A, B \in Ob(\mathfrak{C})$. A morphism $B^A \times A \xrightarrow{ev} B$ is called an evaluation morphism if, for all morphisms $C \times A \xrightarrow{f} B$, there exists a unique morphism $\lceil f \rceil : C \longrightarrow B^A$ for which the following diagram commutes:



The universal property of the exponential object can be expressed by a very simple correspondence:

$$\frac{C \to B^A}{C \times A \to B}$$

An interesting example of exponential objects can be found in our category of a formal system. We are looking for an object B^A (a formula) that has the property that, if we can deduce it from C, then we can deduce B from $C \times A$ (that is $C \wedge A$); conversely, if we can deduce B from $C \wedge A$, then we should be able to deduce B^A from C alone. Symbolically we have:

$$\frac{C \vdash B^A}{C \wedge A \vdash B}$$

The obvious candidate for B^A is $A \Rightarrow B$. Of course, if $C \vdash A \Rightarrow B$, then $C \wedge A \vdash (A \Rightarrow B) \wedge A$ so that $C \wedge A \vdash B$ by modus ponens. Conversely, if $C \wedge A \vdash B$, then by Herbrand's deduction theorem we

have that $C \vdash A \Rightarrow B$. Categories for which cartesian products as well as exponentials of two objects exist are called *cartesian closed categories.*

3.6. *Functors*

We have so far introduced as basic concepts the notions of object and morphisms. Morphisms, as we have seen, can be viewed as links between objects. In the present section, we investigate links between categories, the functors.

By way of introducing this notion, consider the canonical interpretation of the nominal category $\mathfrak{N}$ into the category of sets that assigns to the count noun 'dog', say, the set $\|DOG\|$ of all dogs. Recalling that morphisms in the nominal category were identifications of the type 'a dog is a mammal', it seems natural to interpret such a morphism as the function from the set $\|DOG\|$ of all dogs to the set $\|MAMMAL\|$ of all mammals that assigns to each dog his counterpart in the set of mammals. Identity morphisms of the type 'a dog is a dog' may be interpreted as the identity function 1_{DOG}. Finally, composition of morphisms of the nominal category may be interpreted naturally as composition of the corresponding functions. In other words, the canonical interpretation is just a function from objects and morphisms of $\mathfrak{N}$ into objects and morphisms of SET which preserve, in an obvious sense, the structure of the category. We thus arrive at the following:

Definition 3.11. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories. A functor $F : \mathfrak{C} \to \mathfrak{D}$ is a mapping that assigns to each object $A \in Ob(\mathfrak{C})$, an object $F(A) \in Ob(\mathfrak{D})$ and to each morphism $A \xrightarrow{f} B$ in $\mathfrak{C}$, a morphism $F(f) : F(A) \longrightarrow F(B)$ in $\mathfrak{D}$ such that $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

We can give another simple example of functors. To do so, let us define a very simple category $\mathfrak{L}$. It has only one object (which will be called "$*$"), the identity morphism 1 and an infinite class of morphisms $\alpha, \alpha^2, \alpha^3 \ldots$. In pictures:



Clearly, all these morphisms start and end at the only object available.

The composition of morphisms in this category is naturally defined by the following rule:

$$\alpha^m \circ \alpha^n \equiv \alpha^{m+n}$$

For example, $\alpha^3 \circ \alpha = \alpha^4$. The composition with the identity morphism is forced by the definition of a category, i.e., $1 \circ \alpha^m = \alpha^m$. This may seem very abstract since no meaning was actually given to the object or the morphisms. The flexibility of the definition of a category allows us to do that.

In itself, $\mathfrak{L}$ does not appear to be very interesting but it becomes so when we look at functors $F : \mathfrak{L} \to \mathfrak{Set}$ (where $\mathfrak{Set}$ is the category of set and functions). In fact, such a functor is simply a set $F(*)$ (image of the object of $\mathfrak{L}$ by F) and a class of functions $F(\alpha^m)$ (images of the morphisms in $\mathfrak{L}$). By the definition of functor we have that:

$$F(\alpha^m) = F(\alpha \circ \alpha^{m-1}) = F(\alpha) \circ F(\alpha^{m-1}) = \ldots = \underbrace{F(\alpha) \circ \ldots \circ F(\alpha)}_{m \text{ times}}$$

This last equation tells us that the image of α^m for any given m is strictly determined by composing $F(\alpha)$ with itself m times. We can conclude from this that a functor $F : \mathfrak{L} \to \mathfrak{Set}$ is simply a set A (corresponding to $F(*)$) and a function $A \xrightarrow{f} A$ (corresponding to $F(\alpha)$), i.e., an evolutive set or discrete dynamical system.

3.7. Natural Transformations

Natural transformations are, in spite of their names, usually not very "natural" for a beginner in category theory. In fact, it is a complicated notion which, however, is fundamental in category theory. So fundamental is it that Mac Lane has said that categories and functors were precisely invented to give a precise meaning to this particular notion. Natural transformations are links between functors, and so, links between links (see Mac Lane 1988).

As we did for functors, we introduce this notion through an example, namely, the category of evolutive sets or discrete dynamical systems. We leave justifications for the notion of natural transformation in the context of linguistics to M. La Palme, J. Macnamara and G. Reyes (this volume). The main observation is that evolutive systems may be represented as functors. Natural transformations are what correspond to morphisms between such systems.

Once we have represented evolutive sets as functors, the question is: how to represent morphisms of evolutive sets? Recall that a morphism $(A, f) \xrightarrow{\Phi} (B, g)$ in the category of evolutive sets is a function $A \xrightarrow{\Phi} B$ that preserves the evolution in the following sense: for each $a \in A$ we

require that $\Phi(f(a)) = g(\Phi(a))$. In words, the evolution of a point $a \in A$ must be sent by Φ to the evolution of the point $\Phi(a)$ in B. This can also be expressed by the commutativity of the following diagram:

$$\begin{array}{ccc} F(*) & \xrightarrow{F(\alpha)} & F(*) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ G(*) & \xrightarrow{G(\alpha)} & G(*) \end{array}$$

If we remember that dynamical systems are functors of the type $F : \mathfrak{L} \to \mathfrak{Set}$, we see that the morphisms between dynamical systems are in fact morphisms between functors. Following this lead one arrives at the following:

Definition 3.12. Let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories. Let $F, G : \mathfrak{C} \to \mathfrak{D}$ be two functors. A natural transformation $\tau : F \dot{\longrightarrow} G$ is a class of morphisms containing a morphism $F(X) \xrightarrow{\tau_X} G(X)$ in $Ar(\mathfrak{D})$ for each $X \in Ob(\mathfrak{C})$ and such that, for every morphism $X \xrightarrow{f} Y$ in $Ar(\mathfrak{C})$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle\tau_X}\downarrow & & \downarrow{\scriptstyle\tau_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

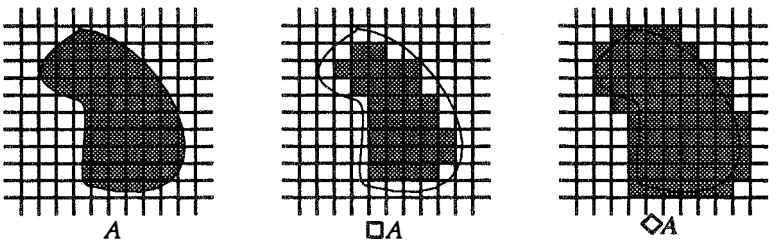
Notice that natural transformations themselves between functors from a category $\mathfrak{C}$ to a category $\mathfrak{D}$ form a category. The reader is invited to define identity morphisms and composition. Now that functors have been integrated in the context of categories, we can apply to them the language developed in the previous sections. For example, two functors $F, G : \mathfrak{C} \to \mathfrak{D}$ are isomorphic if there are natural transformations $\tau : F \dot{\longrightarrow} G$ and $\eta : G \dot{\longrightarrow} F$ such that $\eta \circ \tau = 1_F$ and $\tau \circ \eta = 1_G$.

3.8. Adjoint Functors

In the present section we give some simple examples of adjoint functors. The first example is concerned with modal operators in logic, more particularly with a pictorial way to represent these operators. The second example will show how to view products and coproducts in a category as adjoints to a functor.

Our first example requires the definition of a special category whose objects are figures of the plane. This category will have one and only one morphism between a figure A and a figure B if and only if the figure A is contained in the figure B. This category is usually called $\mathcal{P}(\mathbf{R}^2)$. Now, suppose that there is a square grid covering the whole plane. We now define two functors $\Box, \Diamond : \mathcal{P}(\mathbf{R}^2) \to \mathcal{P}(\mathbf{R}^2)$ relative to this grid.

The $\Box$ functor assigns to a figure A all the squares in the grid that are "necessarily" contained in A; that is the squares that are completely covered by A. The $\Diamond$ functor assigns to a figure A all the squares in the grid that are "possibly" contained in A; that is the squares that are partly or totally covered by A. The next pictures show how the functors operate:



The reader may check that we have the following relation between these functors: if X and Y are figures, then X is contained in $\Box Y$ if and only if $\Diamond X$ is contained in Y or, with our usual notation:

$$\frac{X \to \Box Y}{\Diamond X \to Y}$$

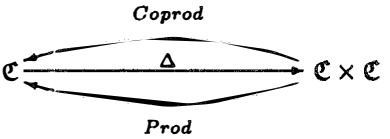
We shall say that $\Box$ and $\Diamond$ are adjoint functors and, more precisely that $\Diamond$ is a left adjoint of $\Box$. In symbols: $\Diamond \dashv \Box$. The reader familiar with modal logic will recognize a particular determination of the operators of necessity and possibility.

Before stating our second example, we must define a particular category that can be built from any category $\mathfrak{C}$. We will denote this category by $\mathfrak{C} \times \mathfrak{C}$. Its objects are pairs of objects (A, B) where $A, B \in Ob(\mathfrak{C})$. A morphism $(A, B) \to (C, D)$ is simply a pair of morphisms (f, g) where $A \xrightarrow{f} C$ and $B \xrightarrow{g} D$.

There is clearly a functor $\Delta : \mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$ that assigns to each object $A \in Ob(\mathfrak{C})$ the object $(A, A) \in Ob(\mathfrak{C} \times \mathfrak{C})$ and, to each morphism $A \xrightarrow{f} B$ in $Ar(\mathfrak{C})$, the morphism $(f, f) \in Ar(\mathfrak{C} \times \mathfrak{C})$. This functor gives rise to two different adjunctions as follows: Assume that $\mathfrak{C}$ has products as well as coproducts. Then we can define two functors $Prod, Coprod : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ as $Prod(A, B) \equiv A \times B$ (the categorical product) and $Coprod(A, B) \equiv A + B$ (the coproduct). The following properties hold for them:

$$\frac{X \to Prod(A, B)}{\dfrac{X \to A \ , \ X \to B}{\dfrac{(X, X) \to (A, B)}{\Delta(X) \to (A, B)}}} \qquad \frac{Coprod(A, B) \to Y}{\dfrac{A \to Y \ , \ B \to Y}{\dfrac{(A, B) \to (Y, Y)}{(A, B) \to \Delta(Y)}}}$$

The situation can be pictured as:

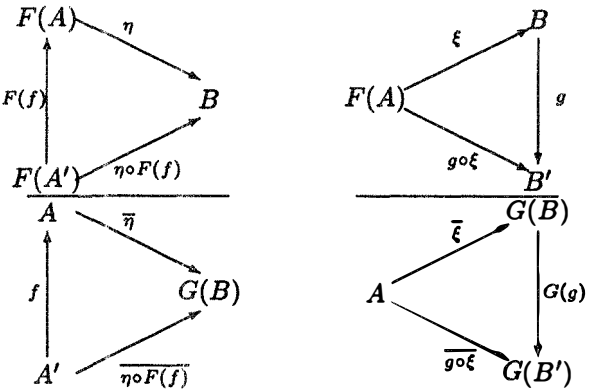


We say in this case that *Prod* is the right adjoint of Δ and that *Coprod* is the left adjoint of Δ (i.e., $Coprod \dashv \Delta \dashv Prod$). We are now ready to give a precise meaning to the notion of adjunction:

Definition 3.13. Let $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{D} \to \mathfrak{C}$ be two functors between categories $\mathfrak{C}$ and $\mathfrak{D}$. We say that F is left adjoint to G (or that G is right adjoint to F) if we have a bijective correspondence:

$$\frac{F(A) \xrightarrow{x} B}{A \xrightarrow[\bar{x}]{} G(B)}$$

for all $A \in Ob(\mathfrak{C})$ and all $B \in Ob(\mathfrak{D})$. Furthermore, this bijection must be natural in the following sense: For each morphism $A' \xrightarrow{f} A$ and $B \xrightarrow{g} B'$ we must have the following correspondences (where all the triangles commute):



which mean in particular that $\overline{g \circ \xi} = G(g) \circ \bar{\xi}$ and $\bar{\eta} \circ f = \overline{\eta \circ F(f)}$. In this case we write $F \dashv G$.

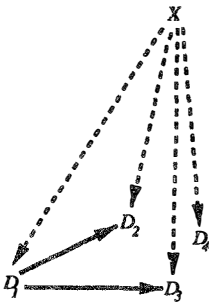
3.9. Limits and Colimits

Our last subject will be limits and colimits. These notions will synthesize almost all universal properties we have defined so far in categories. As their name suggests, they are dual notions. For this reason, we will only give here the definition of limit and leave to the care of the reader the formal description of the notion of colimit.

We will use the formulation 'D is a diagram of $\mathfrak{C}$' to describe a family $\{D_i\}_{i \in I}$ of objects of $\mathfrak{C}$ with some morphisms $D_i \xrightarrow{x} D_j$ (of $\mathfrak{C}$) linking

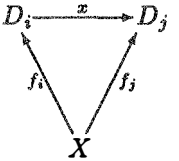
these objects. The word 'diagram' still bears the same meaning we have used informally in the previous sections when we were talking about commutativity of diagrams. A diagram is, in some sense, just a part of a category.

Sometimes, in particular categories, we can build a kind of super-fork for a diagram. Let $\mathfrak{C}$ be a category and D a diagram in $\mathfrak{C}$. A super-fork for D must have an handle $X \in Ob(\mathfrak{C})$ and a prong $X \xrightarrow{f_i} D_i$ for each object D_i in the diagram D. We also require a super-fork to be compatible with the diagram D. To do this, we require that the prongs of the super-fork obey the following: whenever two prongs form a triangle with an arrow in D, this triangle must be commutative. The next picture shows the context of a super-fork:

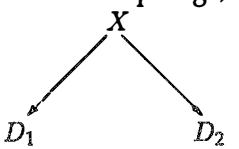


The category-theory representative of super-forks is usually called *cone*. We now give a precise definition for it:

Definition 3.14. Let D be a diagram in a category $\mathfrak{C}$. A cone for D is an object X of $\mathfrak{C}$ together with a family of arrows $\{X \xrightarrow{f_i} D_i\}_{i \in I}$ (where $\{D_i\}_{i \in I}$ is the family of objects of D) such that, for all arrows $D_i \xrightarrow{x} D_j$ of D, the following diagram commutes:

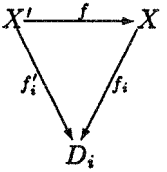


It is easy to see that super-forks are a generalisation of forks. If we take the diagram consisting of only two objects D_1 and D_2 (no arrows) in $\mathfrak{C}$ then we realize that a super-fork for this diagram is simply a fork for D_1 and D_2 (an handle X and two prongs):



As we have noticed before, a product of two objects in a category is a terminal object in the category of forks on those two objects. If we pursue the generalisation, we are led to define a category of super-forks and define limits as terminal super-forks. The objects of this category are super-forks for the diagram D. If X and X' are two super-forks for D, then we define a morphism from X' to X to be a morphism $X' \xrightarrow{f} X$ in $\mathfrak{C}$ such that, for each object D_i in D, the triangle formed by the X'-prong on D_i, the X-prong on D_i and f commutes. We then simply define the notion of limit as a terminal object in this category of super-forks. Another clearly equivalent and perhaps simpler definition of the notion of limit is the following:

Definition 3.15. A limit for a diagram D (of a category $\mathfrak{C}$) is a cone X for D such that, for every other cone X' of D, there is one and only one morphism f such that:



commutes for all the D_i.

As we have seen a categorical product of A and B is clearly a limit for the diagram consisting of only the objects A and B. The colimit of the same diagram would naturally lead us to the coproduct (or sum). The limit of the empty diagram (no objects and therefore no arrows) is simply a terminal object for the category. We can see this by verifying the definitions. What is a cone for the empty diagram? Clearly, every object of the category is a cone for the empty diagram. So that a terminal cone is just a terminal object. We leave to the reader the verification that an equalizer is a limit for:

$$D_1 \rightrightarrows D_2$$

We use the epithet *complete* for categories in which limits of all diagrams exist. We call a category co-complete if colimits of all diagrams exist.

As a last remark we want to emphasize the fact that there is a special relation between limits and left adjoints. If a functor $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ is the left adjoint of some functor and when $\mathfrak{C}$ and $\mathfrak{D}$ are complete categories, then F must preserve the limits. For example, in the case of product this implies that $F(A \times B) = F(A) \times F(B)$. Further, if $\mathfrak{C}$ and $\mathfrak{D}$ are co-complete categories then right adjoints preserve colimits.

Acknowledgments

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References

Bénabou, J. (1991). Seminaire 1990-1991, *Cahiers de poétique comparée, Mezura* 25

Ellerman, D. P. (1988). Category theory and concrete universals. *Erkenntnis* 28: 409–429

Goldblatt, R. (1984). *Topoi: The Categorical Analysis of Logic.* 2th Ed. New York, NY: North Holland

La Palme, Marie, John Macnamara and Gonzalo E. Reyes (this volume). Reference, kinds and predicates

Lawvere, F. W. and S. H. Schanuel (1991). *Conceptual Mathematics: A First Introduction to Categories.* Buffalo, NY: Buffalo Workshop.

Mac Lane, S. (1988). *Categories for the Working Mathematician.* Graduate Texts in Mathematics 5, New York, NY: Springer-Verlag

Pawlak, Z. (1982). Rough sets. *International Journal of Information and Computer Science* 11: 145–172

Reyes, G. E. (1991a). A topos-theoretic approach to reference and modality. *Notre Dame Journal of Formal Logic* 32: 3, 359-391

Reyes, G. E. and Houmann Zolfaghari (1991b). Topos-theoretic approaches to modality. *Proceedings of the 1990 Como Meeting on Category Theory,* forthcoming in Springer-Verlag Lecture Notes in Mathematics