Category Theoretic Understandings of Universal Algebra and its dual: monads and Lawvere theories, comonads and what?

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Abstract

Universal algebra is often known within computer science in the guise of algebraic specification or equational logic. In 1963, it was given a category theoretic characterisation in terms of what are now called Lawvere theories. Unlike operations and equations, a Lawvere theory is uniquely determined by its category of models. Except for a caveat about nullary operations, the notion of Lawvere theory is equivalent to the universal algebraist's notion of an abstract clone. Lawvere theories were soon followed by a further characterisation of universal algebra in terms of monads, the latter quickly becoming preferred by category theorists but not by universal algebraists. In the 1990's began a systematic attempt to dualise the situation. The notion of monad dualises to that of comonad, providing a framework for studying transition systems in particular. Constructs in universal algebra have begun to be dualised too, with different leading examples. But there is not yet a definitive dual of the concept of Lawvere theory, or that of abstract clone, or even a definitive dual of operations and equations. We explore the situation here.

Keywords: Universal algebra, Lawvere theory, abstract clone, monad, comonad.

1 Introduction

There have been two main category theoretic formulations of universal algebra. The earlier was by Bill Lawvere in 1963 [16]. Nowadays, his central construct is

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usually called a Lawvere theory, more prosaically a single-sorted finite product theory [1,2].

Lawvere made a careful distinction between the notions of Lawvere theory and equational theory. Equational theories are a form of presentation for Lawvere theories: every equational theory determines a Lawvere theory and every Lawvere theory is determined by an infinite class of equational theories. Choosing good presentations for a Lawvere theory and deriving an invariant description of the theory from a presentation are important, but the semantics of a Lawvere theory can be considered independently of that [16]. We give the definitions and outline the situation in Section 2.

Universal algebraists have had the same concerns about the lack of invariance of presentations as category theorists have had. They have long used the notion of clone, of which Lawvere was aware, and they have gradually moved towards that of abstract clone [4]. Subject to a caveat about nullary operations, the notion of abstract clone is equivalent to that of Lawvere theory. So, subject to the caveat, universal algebraists readily recognise the definition of Lawvere theory and accept its significance for universal algebra. We give the definitions and constructions in Section 3.

The second category-theoretic formulation of universal algebra, which was in terms of monads, has a more complicated history and is much less accepted by universal algebraists. Monads typically arise from adjoint pairs of functors. The notion of monad (or triple or standard construction) arose in algebraic topology for reasons distinct from universal algebra, see for instance [6]. In 1965, Eilenberg and Moore noted that in case T is the free group monad, their category of T-algebras is the category of groups [5].

In 1966, Linton made the general connection between monads and Lawvere theories: every Lawvere theory gives rise to a monad on Set whose category of algebras is equivalent to the category of models of the Lawvere theory, and, subject to a generalisation in the definition of Lawvere theory, every monad arises thus, uniquely up to coherent isomorphism [17]. So Linton focused on a generalisation of the notion of a Lawvere theory, one that corresponds exactly to the notion of monad. We give the details in Section 4.

Monads have been the more common category theoretic formulation of universal algebra, see for example [19]. But Lawvere theories relate more closely to universal algebra; they arose directly from universal algebra; and they allow natural constructions that arise in universal algebra, such as the sum or tensor of theories, while monads do not [9]. So it seems little wonder that, although many universal algebraists are aware of monads, they seem generally not to have found them, or an equivalent notion, very helpful. Much of the relevant historical development has been summarised in [9].

Over the past decade or so, category theorists, computer scientists and universal algebraists have all become interested in the dual of this situation,

for a variety of reasons [7,10,14,24,26,29].

A leading example for the interest by computer scientists arises from transition systems, which play a fundamental role in, for example, concurrency [21]. A finitely branching transition system is given by a set S together with a function $t : S \longrightarrow P_f(S)$, where $P_f(S)$ is the set of finite subsets of S. The functor P_f on Set generates a cofree comonad $G(P_f)$, and t corresponds to the $G(P_f)$-coalgebra that sends an element σ of S to the set of all possible streams of transitions generated by t with source σ. We would like to develop a body of theory that is dual to universal algebra and includes this example, but it is not easy. We outline some of the issues in Section 5.

One approach to dualising the theory of Lawvere theories is by defining a comodel of a Lawvere theory L in Set to be a model of L in Set^{op}. That line of thought has proved to be valuable for category theorists, for computer scientists, and for universal algebraists [14,24,26]. The category of comodels induces a comonad and is the category of coalgebras for the induced comonad on Set. But this approach does not include transition systems.

Alternatively, one can dualise definitions associated with presentations, carefully dualising the structure of [13,27]. That line of thought has value too [7]. But the fact that Set is a locally finitely presentable category while Set^{op} is not, leads to less elegant results than one would wish [7], and one loses the presentation independence that is central to the notions of Lawvere theory and abstract clone.

Linton's generalised notion of Lawvere theory, corresponding exactly to the notion of monad on Set, suggests a third approach: Linton did not require size conditions [17,18], so dualising his definition is immediate, yielding a definition that is equivalent to that of comonad on Set, thus including transition systems. But sums of monads need not exist, so products of comonads need not exist either; similarly for tensors.

So the question is open, hence the question mark in the title of this paper. We outline the above three proposals in Section 6, and we propose a tentative definition of a dual Lawvere theory in Section 7, leaving its development for further work.

2 Lawvere theories

In his 1963 PhD thesis, Lawvere gave a category theoretic formulation of universal algebra along the following lines.

Definition 2.1 Let $\aleph_0$ denote a skeleton of the category of finite sets and all functions between them, considered as a category with strictly associative coproducts.

Since $\aleph_0$ is equipped with strictly associative finite coproducts given by

the ordinal sum of natural numbers, the opposite category $\aleph_0^{op}$ is equipped with strictly associative finite products. It is equivalent to the free category with finite products on 1 as $\aleph_0$ is equivalent to the free category with finite coproducts on 1.

Definition 2.2 A *Lawvere theory* consists of a small category L with (necessarily strictly associative) finite products and a strict finite-product preserving identity-on-objects functor $I : \aleph_0^{op} \longrightarrow L$. A map of Lawvere theories from L to L' is a (necessarily strict finite-product preserving) functor from L to L' that commutes with the functors I and I'.

Thus the objects of any Lawvere theory L are exactly the objects of $\aleph_0$, and every function between such objects yields a map in L. Note that the functor I need not be an inclusion. One often refers to the maps of a Lawvere theory as *operations*. The notion of map between Lawvere theories encapsulates the idea of an interpretation of one theory in another.

The definitions of Lawvere theory and map between them yield a category Law, with composition given by ordinary composition of functors. The category Law is complete and cocomplete, indeed a locally finitely presentable category.

Given an equational theory, one generates a Lawvere theory by putting $L(n, 1) = Fn$, the free algebra on n generators. This determines $L(n, m)$ for any m as $L(n, m)$ must be the product of m copies of $L(n, 1)$. The composition of L is fully determined by the family of maps

$$(Fp)^n \times Fn \longrightarrow Fp$$

determined by substitution of n terms of p variables into a term of n variables.

For most mathematical purposes, one understands a Lawvere theory by study of its models.

Definition 2.3 A *model* of a Lawvere theory L in a category C with finite products is a finite-product preserving functor $M : L \longrightarrow C$.

Note that one has preservation of finite products here, not strict preservation. Preservation rather than strict preservation of finite products is fundamental: if one demanded strict preservation, the category of models for the Lawvere theory for a monoid would be empty, rather than being the category of monoids as one wants. The reason is that, with the usual set-theoretic definitions, finite products in Set are not strictly associative, whereas they are strictly associative in any Lawvere theory. Preservation rather than strict preservation also allows a smooth account of change of base category along a finite product preserving functor $H : C \longrightarrow C'$.

The requirement that M preserves projections, which is part of what preservation of products means, determines the behaviour of M on all operations of the form If: projections in L amount to coprojections in $\aleph_0$, and

every function f is given by a family of coprojections. So what determines a model is the interpretation of the other operations.

There is a mild difference here between universal algebra and category theory traditions. Some universal algebraists only admit non-empty models [3]. So, for example, a category theorist would regard the empty set as a carrier for the structure of a semigroup, whereas some universal algebraists would not. For a category theorist, the empty semigroup is important as it is the initial object in the category of semigroups.

Definition 2.4 For any Lawvere theory L and any category C with finite products, the category $Mod(L, C)$ is defined to have objects given by all models of L in C, with maps given by all natural transformations between them.

The definition of map in $Mod(L, C)$ is subtle. One can readily prove that any natural transformation between models respects finite products: for any natural transformation α between models M and N, and for any n in $\aleph_0$, the map $\alpha_n : Mn \longrightarrow Nn$ is given by the product of n copies of the map $\alpha_1 : M1 \longrightarrow N1$. Thus the maps in $Mod(L, C)$ could equally be defined to be those natural transformations that respect the product structure of L.

The semantic category C of primary interest is Set. Consider a model M of a Lawvere theory L in Set. The set $M1$ determines Mn up to coherent isomorphism for every n in L: for M preserves finite products of L, equivalently of $\aleph_0^{op}$; these are finite coproducts of $\aleph_0$, which are given by finite sums; and so Mn must be a product of n copies of $M1$. Thus to give a model M is equivalent to giving a set $X = M1$ together with, a function from X^m to X for each map of the form $f : m \longrightarrow 1$ in the category L, subject to the equations given by the composition and product structure of L; and $Mod(L, Set)$ is equivalent to the evident category of such structures. This analysis routinely extends to any category C with finite products.

The category $Mod(L, Set)$ is always complete and cocomplete, with the initial object given by the empty set if L has no nullary operations.

Unlike equational theories, Lawvere theories are semantically invariant. The precise sense in which that is so is as follows. With each Lawvere theory L, we associate the underlying set functor

$$ev_1 : Mod(L, Set) \longrightarrow Set$$

given by evaluation at 1. This is the *semantics* functor of Lawvere [16]. We say that the categories $Mod(L, Set)$ and $Mod(L', Set)$ of models are coherently equivalent if there is an equivalence of categories between them that respects the underlying set functor.

Proposition 2.5 *[16] Given Lawvere theories L and L', if the categories $Mod(L, Set)$ and $Mod(L', Set)$ are coherently equivalent, then the Lawvere theories L and L' are isomorphic in the category Law.*

3 Abstract clones

Let C be an arbitrary category and let X be an object of C for which all finite powers of X also exist in C.

Definition 3.1 Setting $O_X = \coprod_{n>0} C(X^n, X)$, a subset $Cl \subseteq O_X$ is called a *clone* of operations over the object X if it contains all projections $\pi_i : X^n \longrightarrow X$ and is closed under composition, i.e., writing Cl_n for those elements of Cl that lie in $C(X^n, X)$, given $f \,\epsilon\, Cl_n$ and $f_1, \ldots, f_n \,\epsilon\, Cl_k$, the composite $f(f_1, \ldots, f_n)$ is in Cl_k.

This seems the most straightforward possible generalisation of a clone to abstract categories [14]. It is almost verbatim the definition in universal algebra [4] except that the composition under which the clones must be closed is written with the help of tuplings. In particular, putting $C = Set$, this is exactly the notion of a clone as studied in universal algebra [25,28,15].

Note that nullary operations are excluded from the definition, i.e., $C(X^0, X) \not\subseteq O_X$. This follows a convention in universal algebra, which has its advantages but also disadvantages. The notion is naturally connected with that of models of Lawvere theories, although it would be even more natural if one would allow nullary operations, which are less often encountered in universal algebraic literature, e.g., in [20].

Proposition 3.2 *[14] A subset $Cl \subseteq O_X$ is a clone of operations over X if and only if there exists a model $M : L \longrightarrow C$ of a Lawvere theory L in C such that $M(1) = X$ and $Cl = \coprod_{n>0}\{M(f) \mid f \epsilon L(n, 1)\}$.*

The notion of clone is standard within universal algebra and has been so for many years. What is less standard is the abstraction from a base set X to a notion corresponding exactly to that of a Lawvere theory, modulo the above caveat about nullary operations. In order to state a precise equivalence result, in the following, we shall allow abstract clones to have nullary operations.

Definition 3.3 cite [4,14] An *abstract clone* consists of

- for each $n \geq 0$, a set Cl_n, the elements of which are called n-ary operations
- for $1 \leq i \leq n$, an n-ary operation π_i (allowing overloading of notation as, strictly speaking, we have a π_i for each n)
- for each n-ary operation g and m-ary operations $f_1, \ldots, f_n$, an m-ary operation $g(f_1, \ldots, f_n)$

such that, subject to the composites being defined,

- $(h(g_1, \ldots, g_n))(f_1, \ldots, f_m) = h(g_1(f_1, \ldots, f_m), \ldots, g_n(f_1, \ldots, f_m))$
- $\pi_i(f_1, \ldots, f_n) = f_i$ for all $\leq i \leq n$
- $f(\pi_1, \ldots, \pi_n) = f$.

Proposition 3.4 *To give an abstract clone is equivalent to giving a Lawvere theory.*

Proof. Given an abstract clone Cl, put $L_{Cl}(n,m) = Cl_n^m$, with composition determined by the composition of operations in Cl, and with the identity on n given by $(\pi_1, \ldots, \pi_n)$. Observe that L_{Cl} forms a category with strictly associative finite products given by ordinal sum of natural numbers together with tuples of the π_i's, and use the fact that $\aleph_0^{op}$ is the free category with finite products on 1 to generate the functor $I : \aleph_0^{op} \longrightarrow L_{Cl}$. Thus L_{Cl} is a Lawvere theory.

For the converse, given a Lawvere theory L, put $(Cl_L)_n = L(n,1)$, define the π_i's using projections of L, and define composition of operations by the composition of L together with the universal property of finite products. This data readily satisfies the axioms for an abstract clone.

The two constructions are routinely checked to be mutually inverse. □

4 Monads

Soon after Lawvere theories were defined, Linton showed that every Lawvere theory yields a monad on Set [17]. The construction extends to a fully faithful functor from Law to the category Mnd of monads on Set. The functor is not an equivalence of categories. So in this precise sense, a monad on Set is a more general notion than that of Lawvere theory.

Linton also gave a partial converse. One can generalise the definition of Lawvere theory to allow for arities of arbitrary size, with a generalised theory no longer a small category or fully determined by one. The construction of a monad from a Lawvere theory then generalises to an equivalence of categories between the category of generalised Lawvere theories and Mnd. In [18], Linton accordingly generalised Lawvere's treatment of semantics and algebraic structure.

In more detail, for any Lawvere theory L, let $U_L : Mod(L,C) \longrightarrow C$ denote evaluation at 1, cf. Proposition 2.5. If U_L has a left adjoint F_L, as it does whenever C is locally presentable, it exhibits $Mod(L,C)$ as equivalent to the category T_L-Alg for the induced monad T_L on C [1]. Since Set is locally finitely presentable, every Lawvere theory L induces a monad T_L on Set.

Proposition 4.1 *The monad T_L may be described by the following colimit:*

$$T_L X = \int^{n \epsilon \aleph_0} L(n,1) \times X^n$$

This colimit can be constructed by taking the coproduct

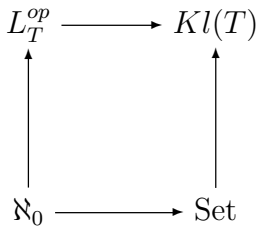
$$\coprod_{n \epsilon \aleph_0} L(n,1) \times X^n$$

then factoring by identifying elements determined by taking projections and diagonal maps of $\aleph_0^{op}$. So it is the set of all equivalence classes of terms generated by the operations of L, with variables among the elements of X, subject to the equalities determined by L.

Proposition 4.2 *The construction sending a Lawvere theory L to the monad T_L extends to a fully faithful functor from Law to Mnd. Moreover, the comparison functor exhibits an equivalence between $Mod(L, Set)$ and T_L-Alg.*

One can readily check that T_L is always finitary. When the base category is *Set*, finitariness characterises the image of the construction, but that was an observation of a later time [12].

For a converse, observe that for any monad T on *Set*, the Kleisli category $Kl(T)$ has all small coproducts and the canonical functor $I : Set \longrightarrow Kl(T)$ preserves them: the canonical functor I has a right adjoint and is identity-on-objects. Restricting I to the full subcategory $\aleph_0$, we obtain (the opposite of) a Lawvere theory as in the diagram.



It is straightforward to show the following.

Proposition 4.3 *The construction sending a monad T on Set to the category $Kl(T)^{op}_{\aleph_0}$ determined by restricting $Kl(T)$ to the objects of $\aleph_0$ extends to a functor $L_- : Mnd \longrightarrow Law$.*

Given a Lawvere theory L, one can readily prove that $L(T_L)$ is isomorphic to L, but the converse is false: the only monads of the form T_L are the finitary ones. Thus we have the following.

Theorem 4.4 *The constructions sending L to T_L and that sending T to L_T exhibit Law as a full coreflective subcategory of Mnd, the category of monads on Set.*

Because this is not an equivalence of categories, Linton generalised the definition of Lawvere theory to consider a locally small category L with all small products, together with a strict product preserving identity-on-objects functor from the opposite of a skeleton of *Set* to L. With this generalised notion of Lawvere theory, Linton showed that the construction of Proposition 4.2 extends, and in the corresponding version of Theorem 4.4, one has an equivalence of categories [17].

The different range of generality of the ideas of monads and Lawvere theories extends to the connection between (generalised) Lawvere theories and monads. One can consider monads on any category, while Lawvere theories correspond to (finitary) monads on Set. On the other hand, a monad on a category C has algebras, i.e., models, only in C, while a Lawvere theory naturally has models in any category with products. So while monad maps between monads on Set (see [1] for this notion of monad map) correspond directly to maps of Lawvere theories, there is nothing in the world of monads (at least nothing to which one has immediate access) corresponding to the functoriality of $Mod(L, C)$ in C.

5 Comonads and transition systems

The notions of monad and algebra dualise to those of comonad and coalgebra: that is easy.

Definition 5.1 A *comonad* on a category C is a monad on C^{op}. Given a comonad G on C, a *G-coalgebra* is a G-algebra for G *qua* monad on C^{op}.

So the body of abstract theory initiated by Eilenberg and Moore for algebra immediately yields a body of abstract theory for coalgebra [5], and that has proved to be of considerable importance for computer science over the past twenty years or so [10]. A leading example is as follows.

Definition 5.2 Given a set A, a *finitely branching A-labelled transition system* is a pair (S, t) consisting of a set S and a function $t : S \longrightarrow P_f(A \times S)$, where $P_f(X)$ is the set of finite subsets of a set X.

The transition function t tells you, given a machine in state σ, to what states it might pass in one A-labelled move. The notion is fundamental to the theory of concurrency, for instance, playing a central role in CCS [21].

The axiomatic situation is that in a category C such as Set, one considers a pair (X, x) consisting of an object X of C and a map $x : X \longrightarrow H(X)$, where H is an endofunctor on C. Such a pair is called an H-coalgebra [10]. In such axiomatic terms, a finitely branching A-labelled transition system is precisely a $P_f(A \times -)$-coalgebra.

The structure of the category of $P_f(A \times -)$-coalgebras can be used to characterise the critical notion of bisimulation in concurrency [21]. In fact, the body of theory of bisimulation can be defined and developed axiomatically in terms of H-coalgebras for an arbitrary endofunctor H satisfying axiomatically defined conditions [10].

The most fundamental construct one makes in coalgebra is the construction of the cofree comonad $G(H)$ on H if it exists. To give an H-coalgebra is

equivalent to giving a $G(H)$-coalgebra, where the term coalgebra is overloaded, as H is treated as an endofunctor while $G(H)$ is treated as a comonad.

Theorem 5.3 *[7] For any finitary endofunctor H on any locally finitely presentable category C, a cofree comonad $G(H)$ on H exists.*

Although the statement of this theorem is dual to the statement of a theorem about algebras [11], the proof is different, not dual to the proof of the corresponding theorem for algebras. The reason for the difference is that Set is a locally finitely presentable category, while Set^{op} is not. Much of the category-theoretic effort involved with coalgebra revolves around handling that fact.

Although a cofree comonad on a finitary endofunctor on a locally finitely presentable category necessarily exists, it typically is not finitary.

Example 5.4 [7] Given a set A, consider the endofunctor $H = A \times -$ on Set. The cofree comonad $G(H)$ sends a set X to the set of infinite streams of elements of $A \times X$. So, for any countably infinite set X, the set $G(H)(X)$ contains a stream involving infinitely many different elements of X. Such a stream cannot be given by a finite subset of X, and so although $G(H)$ has a rank, that rank is necessarily greater than $\aleph_0$.

This phenomenon has been studied extensively and generalised by Worrell [29]. The key consequence of these issues for us is that dualising notions such as that of Lawvere theory is subtle, and we cannot expect to obtain as neat a relationship as that between Lawvere theories and monads.

6 Dualising Lawvere theories

One approach to dualising the body of theory of Lawvere theories is as follows.

Definition 6.1 A *comodel* of a Lawvere theory L in a category with finite coproducts C is a model of L in C^{op}.

Comodels in C generate a category $Comod(L, C)$, with maps given by natural transformations, and a forgetful functor $U_L : Comod(L, C) \longrightarrow C$ given by evaluation at 1, just as for models as in Section 2. Moreover, albeit with a different proof to that for models, we have the following, cf. Proposition 4.3.

Theorem 6.2 *[26] For any Lawvere theory L and locally finitely presentable category C, the forgetful functor U_L has a right adjoint, generating a comonad G_L on C, whereupon $Comod(L, C)$ is coherently equivalent to G_L-$Coalg$.*

This dual of the theory of Lawvere theories appeared in [26], where it was used to model arrays. If L_S be the (countable) Lawvere theory for global state, as described in [23,8], the category $Comod(L_S, Set)$ is equivalent to the

category of arrays [26]. Lawvere theories and their comodels have also been used, extending [23], to model operational semantics [24]. And they have been used, in terms of coclones, in [14]. But this dualisation of the theory of Lawvere theories does not include transition systems [10].

A second approach to dualising algebra is to start not with the notion of Lawvere theory but rather with that of equational theory, dualise it to a notion of co-equational theory, then look for an invariant, allowing us to mimic Proposition 2.5. There is an axiomatic account of the notions of operations, equations, algebras and monads in [13,27] that has been dualised in [7]. The idea is as follows.

Example 6.3 Consider three binary operations and no equations. This may be seen as a single binary operation with codomain 3, cf. the way in which one generates a Lawvere theory from an equational theory [13,27]. A model is a set X together with a function of the form $X^2 \longrightarrow X^3$. Dualise this to consider a function of the form $2^X \longrightarrow 3^X$.

Axiomatically, in [7], one retained the natural numbers as arities, defined the notions of co-operation, co-equation and coalgebra in the spirit of Example 6.3, and proved that any family of co-operations and co-equations generates a comonad G on Set such that the category of G-coalgebras is isomorphic to the category of coalgebras for the co-operations and co-equations. Finitely branching A-labelled transition systems provided a leading example.

A third approach to a dual notion of Lawvere theory is generated by Linton's work [17,18], as his generalised notion of Lawvere theory as discussed in Section 4 does not depend upon Set being locally finitely presentable, and so generates a dual just as the notion of monad does. As for algebra, this has the drawback of not allowing duals of basic universal algebraic constructs such as sum.

These three approaches, together with a recent characterisation of Lawvere theories in [22], collectively suggest a tentative definition of dual Lawvere theory, which we give in Section 7.

7 Further Work: a proposal for dualising

There are several different but equivalent formulations of the notions of Lawvere theory and model. Starting with the usual definition of Lawvere theory as in Definition 2.2, to give a model of L in Set (Definition 2.3) is equivalent to giving a functor $M : L \longrightarrow Set$ such that the composite MI is of the form $Set(J-, X)$, where J is the inclusion of $\aleph_0$ in Set: it follows from this definition that M preserves finite products; the converse is given by putting $X = M1$.

With considerably more effort, one can prove that to give a Lawvere theory

is equivalent to giving a small category L together with an identity-on-objects functor $I : \aleph_0^{op} \longrightarrow L$ such that I preserves all finite limits in $\aleph_0^{op}$: this does not imply that L has all finite limits, although it does follow that L has all finite products [22].

So one possible notion of dual to investigate is as follows: a *dual Lawvere theory* is a small category L together with an identity-on-objects functor $I : \aleph \longrightarrow L$ that preserves all finite limits in $\aleph$. Note the dropping of $(-)^{op}$. A *comodel* in Set is a functor $C : L \longrightarrow Set$ for which CI is of the form $Set(X, J-)$.

This seems to bear comparison with the definitions in [7] and seems to restrict Linton's abstract work [17,18], which does not require size. It is not clear yet how it relates to comodels of Lawvere theories qua finite coproduct preserving functors. We propose this as further work.

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