Categories of Nets

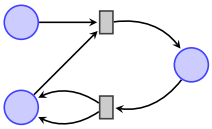
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2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS) | 978-1-6654-4895-6/20/$31.00 ©2021 IEEE | DOI: 10.1109/LICS52264.2021.9470566

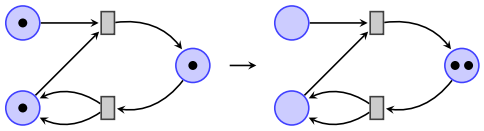
***Abstract*—We present a unified framework for Petri nets and various variants, such as pre-nets and Kock's whole-grain Petri nets. Our framework is based on a less well-studied notion that we call Σ-nets, which allow fine-grained control over whether each transition behaves according to the collective or individual token philosophy. We describe three forms of execution semantics in which pre-nets generate strict monoidal categories, Σ-nets (including whole-grain Petri nets) generate symmetric strict monoidal categories, and Petri nets generate commutative monoidal categories, all by left adjoint functors. We also construct adjunctions relating these categories of nets to each other, in particular showing that all kinds of net can be embedded in the unifying category of Σ-nets, in a way that commutes coherently with their execution semantics.**

I. Introduction

A Petri net is a seemingly simple thing:



It consists of "places" (drawn as circles) and "transitions" (drawn as boxes), with directed edges called "arcs" from places to transitions and from transitions to places. The idea is that when we use a Petri net, we place dots called "tokens" in the places, and then move them around using the transitions:



Thanks in part to their simplicity, Petri nets are widely used in computer science, chemistry, biology and other fields to model systems where entities interact and change state [13], [27].

Ever since the work of Meseguer and Montanari [25], parallels have been drawn between Petri nets and symmetric strict monoidal categories (SSMCs). Intuitively, a Petri net can be interpreted as a *presentation* of such a category, by using its places to generate a commutative monoid of objects, and its transitions to generate the morphisms. An object in the SSMC represents a "marking" of the net—a given placement of tokens in it—while a morphism represents a "firing sequence": a sequence of transitions that carry markings to other markings. One of the advantages of this "execution semantics" for a net is that it can be compositionally interfaced to other structures using monoidal functors.

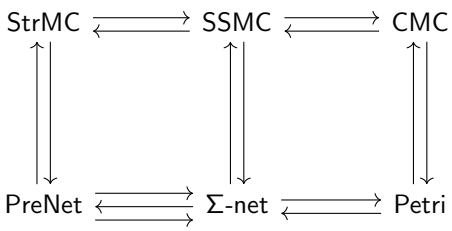
However, the apparent simplicity of Petri nets hides many subtleties. There are various ways to make the definition of Petri net precise. For example: is there a finite *set* of arcs from a given place to a given transition (and the other way around), or merely a natural number? If there is a finite set, is this set equipped with an ordering? Furthermore, what is a morphism between Petri nets? A wide variety of answers to these questions have been explored in the literature.

Different answers are good for different purposes. In the "individual token philosophy", we allow a finite set of tokens in each place, and tokens have their own individual identity. In the "collective token philosophy", we merely allow a natural number of tokens in each place, so it means nothing to switch two tokens in the same place [14].

Moreover, the idea of using SSMCs to represent net semantics, albeit intuitive, presents subtleties of its own. There has been a great deal of work on this subject [1], [5], [7], [11], [12], [24], [28]–[30]. Nevertheless, we still lack a general answer describing the relations between nets and SSMCs.

II. Dramatis personæ

Our goal is to bring some order to this menagerie. Our attitude is that though there may be multiple kinds of Petri net, each should freely generate a monoidal category of an appropriate sort, and these processes should be left adjoint functors. We thus consider three kinds of nets, and three corresponding kinds of monoidal categories:



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On the top row we have:

- StrMC, with strict monoidal categories as objects and strict monoidal functors as morphisms.
- SSMC, with symmetric strict monoidal categories as objects and strict symmetric monoidal functors as their morphisms. A **symmetric strict monoidal category** is a symmetric monoidal category whose monoidal structure is strictly associative and unital; its symmetry may not be the identity.
- CMC, with commutative monoidal categories as objects and strict symmetric monoidal functors as morphisms. A **commutative monoidal category** is a symmetric strict monoidal category where the symmetry is the identity.

Monoidal categories of these three kinds are freely generated by three kinds of nets, on the bottom row of the diagram:

- PreNet, with pre-nets as objects. A **pre-net** consists of a set S of **places**, a set T of **transitions**, and functions $T \xrightarrow{s,t} S^* \times S^*$, where S^* is the underlying set of the free monoid on S. We describe the category PreNet, and its adjunction with StrMC, in Section IV. These ideas are due to Bruni, Meseguer, Montanari and Sassone [5].
- Σ-net, with Σ-nets as objects. A **Σ-net** consists of a set S and a discrete opfibration $T \to PS \times PS^{\mathrm{op}}$, where PS is the free symmetric strict monoidal category generated by a set of objects S and no generating morphisms. We describe the category Σ-net in Section V, and its adjunction with SSMC in Theorem VII.3.
- Petri, with Petri nets as objects. A **Petri net**, as we will use the term, consists of a set S, a set T, and functions $T \xrightarrow{s,t} \mathbb{N}[S] \times \mathbb{N}[S]$, where $\mathbb{N}[S]$ is the free commutative monoid on S. We describe the category Petri, and its adjunction with CMC, in Section III. This material can be found in [1], [24].

These three notions of net obviously have a similar flavor. Their parallel relationships to the three notions of monoidal category is made even clearer when we note that regarded as discrete categories, S^* and $\mathbb{N}[S]$ are respectively the free monoidal category and the free commutative monoidal category on the set S.

Besides the three adjunctions between the categories on the top row and those on the bottom row, in which the left adjoints point upward, there are also adjunctions running horizontally across the diagram: adjoint pairs in the top row and bottom right and an adjoint triple in the bottom left, with left adjoints drawn above their right adjoints. In Section VII, we examine these adjunctions in detail.

Of particular importance are the right adjoint mapping Petri nets to Σ-nets, and the left adjoint mapping pre-nets to Σ-nets. We think of these as "embedding" the collective token world (Petri nets) and the individual token world (pre-nets) into the unifying context of Σ-nets. In the case of Petri nets, the functor is literally an embedding (i.e., fully faithful), and since it is a right adjoint it preserves all limits (though not all colimits). In the case of pre-nets, the functor is faithful but not full, but it is an equivalence onto a slice category of Σ-net, and preserves all colimits and all connected limits (such as pullbacks). These embeddings also respect the most common categorical semantics: in Section VII we will show that the left adjoints PreNet $\to$ SSMC and Petri $\to$ CMC both factor through Σ-net.

The images of pre-nets and Petri nets in Σ-nets have a large intersection, consisting of those nets in which no places are ever duplicated in the inputs or outputs of any transition. These are the nets for which there is no difference between the individual and collective token philosophies. As we shall see, general Σ-nets allow more fine-grained control than either pre-nets or Petri nets: for example, some transitions may obey the individual token philosophy while others obey the collective token philosophy.

Our work is closely related to that of Kock [18]. He refers to Σ-nets as "digraphical species", and sketches a proof, different from ours, that there is an adjunction relating them to SSMC. But his focus is on a fourth notion of net: "whole-grain Petri nets". He sketches a proof that these are the image of pre-nets inside Σ-net, which we detail in Section VIII (so that in particular, whole-grain Petri nets also generate symmetric strict monoidal categories); but says nothing about their relationship to Petri nets as traditionally conceived.

III. Petri Nets

Symmetric monoidal categories are a general algebraic framework to represent processes that can be performed in sequence and in parallel. Because Petri nets represent schematics for such processes, we expect them to freely generate symmetric monoidal categories. In fact they generate a special sort of symmetric monoidal categories: *commutative* ones.

Definition III.1. Let Petri be the category where:

- An object is a **Petri net**: a pair of functions $T \xrightarrow{s,t} \mathbb{N}[S]$, where $\mathbb{N}[S]$ denotes the underlying set of the free commutative monoid on S.
- A morphism from $T_1 \xrightarrow{s_1,t_1} \mathbb{N}[S_1]$ to $T_2 \xrightarrow{s_2,t_2} \mathbb{N}[S_2]$ is a pair of functions $f\colon S_1 \to S_2, g\colon T_1 \to T_2$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{N}[S_1] & \xleftarrow{s_1} & T_1 & \xrightarrow{t_1} & \mathbb{N}[S_1] \\ {\scriptstyle \mathbb{N}[f]}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathbb{N}[f]} \\ \mathbb{N}[S_2] & \xleftarrow[s_2]{} & T_2 & \xrightarrow[t_2]{} & \mathbb{N}[S_2] \end{array}$$

  where $\mathbb{N}[f]$ denotes the unique monoid homomorphism extending f.

Our concept of Petri net morphism is more restrictive than that of Meseguer and Montanari [25]: where we have $\mathbb{N}[f]\colon \mathbb{N}[S_1] \to \mathbb{N}[S_2]$; they allow an arbitrary monoid homomorphism from $\mathbb{N}[S_1]$ to $\mathbb{N}[S_2]$.

Definition III.2. A **commutative monoidal category** is a commutative monoid object in Cat. Equivalently, it is a strict monoidal category $(C, \otimes, I)$ such that for all objects a and b and morphisms f and g in C

$$a \otimes b = b \otimes a \text{ and } f \otimes g = g \otimes f.$$

A morphism of commutative monoidal categories is a strict monoidal functor. We write CMC for the category of commutative monoidal categories and such morphisms between them.

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A commutative monoidal category can be seen as a particularly strict sort of symmetric monoidal category. Ordinarily, symmetric monoidal categories are equipped with "symmetry" isomorphisms

$$\sigma_{x,y} \colon x \otimes y \xrightarrow{\sim} y \otimes x$$

for every pair of objects x and y. In a commutative monoidal category $x \otimes y$ is *equal* to $y \otimes x$, so we can — and henceforth will — make it symmetric by choosing $\sigma_{x,y}$ to be the identity for all x and y. Any morphism of commutative monoidal categories then becomes a strict symmetric monoidal functor.

The following adjunction shows that Petri nets are the right sort of generating data for commutative monoidal categories.

Proposition III.3. *There is an adjunction*

$$\mathsf{Petri} \underset{U_{\mathsf{Petri}}}{\overset{F_{\mathsf{Petri}}}{\rightleftarrows}} \mathsf{CMC}$$

whose left adjoint sends a Petri net P to the commutative monoidal category $F_{\mathsf{Petri}}(P)$ where:

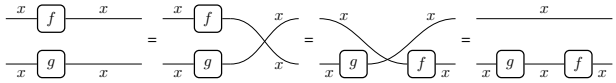
- *Objects are markings of P, i.e., elements of the free commutative monoid on its set of places.*
- *Morphisms are generated inductively by the following rules:*
  - *for each place s there is an identity $1_s \colon s \to s$*
  - *for each transition τ of P, there is a morphism going from its source to its target*
  - *for every pair of morphisms $f \colon x \to y$ and $f' \colon x' \to y'$, there is a morphism $f \otimes f' \colon x \otimes x' \to y \otimes y'$*
  - *for every pair of composable morphisms $f \colon x \to y$ and $g \colon y \to z$, there is a morphism $g \circ f \colon x \to z$*

  and quotiented to satisfy the axioms of a commutative monoidal category.

Proof. This is a special case of [24, Theorem 5.1]. See also [1, Lemma 9]. □

As noted in Section I, we view this construction as associating to each net a monoidal category of its "executions". The objects of this category are markings that accord with the collective token philosophy: for instance, if p and q are places, the object $2p + 3q$ has two tokens on p and three tokens on q, but no way to distinguish between the former two tokens or between the latter three. Similarly, the morphisms in this category are equivalence classes of firing sequences. This interpretation is particularly captivating if we represent morphisms in a monoidal category using string diagrams.

However, the equivalence relation on firing sequences that determines when two define the same morphism is very coarse when we take the commutative monoidal category freely generated by a Petri net. Indeed, if $f, g \colon x \to x$ are morphisms in a commutative monoidal category, the following sequence of equations holds:



These equations imply that given any two firing sequences f and g that start and end at some marking x, the commutative monoidal category cannot distinguish whether they act independently or whether f acts on the tokens already processed by g. When x is the tensor unit, the equations above hold in *any* symmetric monoidal category. But in a commutative monoidal category, the above equations hold for any object x.

IV. Pre-nets

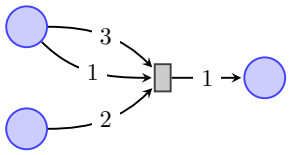
The shortcomings of commutative monoidal categories we presented in the last section are overcome by using symmetric monoidal categories where the symmetries are not necessarily identity morphisms. One approach first builds monoidal categories and then freely adds symmetries. In 1991 Joyal and Street [15] introduced "tensor schemes", which can be used to describe free strict monoidal categories. In 2001, essentially the same idea was introduced by Bruni, Meseguer, Montanari and Sassone [5] under the name "pre-nets". However, for these authors, the use of pre-nets to describe free strict monoidal categories was just the first stage of a procedure to obtain free *symmetric* strict monoidal categories. We recall this procedure now.

Definition IV.1. Let PreNet be the category where:

- An object is a **pre-net**: a pair of functions $T \xrightarrow{s,t} S^*$, where S^* is the underlying set of the free monoid on S.
- A morphism from $T_1 \xrightarrow{s_1,t_1} S_1^*$ to $T_2 \xrightarrow{s_2,t_2} S_2^*$ is a pair of functions $f \colon S_1 \to S_2$, $g \colon T_1 \to T_2$ such that the following diagram commutes, where f^* denotes the unique monoid homomorphism extending f:

$$\begin{array}{ccccc} S_1^* & \xleftarrow{s_1} & T_1 & \xrightarrow{t_1} & S_1^* \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f^*} \\ S_2^* & \xleftarrow[s_2]{} & T_2 & \xrightarrow[t_2]{} & S_2^* \end{array}$$

Graphically, a pre-net looks very similar to a Petri net, and we follow the convention of [2] by decorating arcs with numbers to indicate their input/output order in a transition, as in:



This denotes that the place in the top left is used as the first and third input of the transition, while the place in the bottom left is the second input.

Pre-nets give rise to strict monoidal categories as follows.

Proposition IV.2. *There is an adjunction*

$$\mathsf{PreNet} \underset{U_{\mathsf{PreNet}}}{\overset{F_{\mathsf{PreNet}}}{\rightleftarrows}} \mathsf{StrMC}$$

whose left adjoint sends a pre-net Q to the strict monoidal category where:

- *Objects are elements of the free monoid on the set of places.*

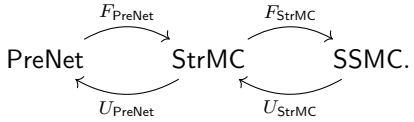
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- *Morphisms are generated inductively by the following rules:*
  - *for each place s there is an identity $1_s \colon s \to s$*
  - *for each transition τ of Q, there is a morphism going from its source to its target*
  - *for every pair of morphisms $f \colon x \to y$ and $f' \colon x' \to y'$, there is a morphism $f \otimes f' \colon x \otimes x' \to y \otimes y'$*
  - *for every pair of composable morphisms $f \colon x \to y$ and $g \colon y \to z$, there is a morphism $g \circ f \colon x \to z$*

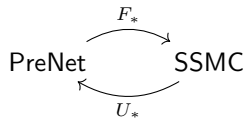
  and quotiented to satisfy the axioms of a strict monoidal category.

Proof. This is [24, Prop. 6.1]. □

The above adjunction can be composed with one defined in Proposition VII.1 to obtain an adjunction between pre-nets and strict symmetric monoidal categories:



The composite adjunction



is used to obtain the categorical operational semantics of pre-nets under the individual token philosophy. This adjunction was first presented by Bruni, Meseguer, Montanari, and Sassone [5] with codomain a full subcategory of SSMC and was later refined to the above form in [24].

This composite adjunction has also been used to give a categorical semantics for Petri nets [4], [5], [28]. For this, given a Petri net P, one first chooses a pre-net Q having P as its underlying Petri net, and then forms the symmetric strict monoidal category $F_*(Q)$. However this semantics is not functorial, due to the arbitrary choice involved.

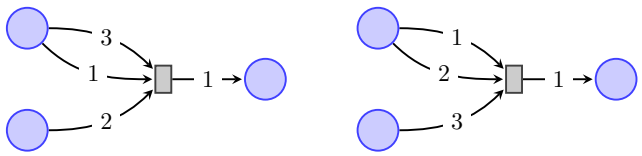
The category PreNet is better behaved than Petri. The latter is not even cartesian closed, for essentially the same reasons described in [6], [23], but the former is cartesian closed, and even a topos:

Proposition IV.3. *The category* PreNet *is equivalent to a presheaf category.*

Proof. It suffices to construct a category C so that functors from C to Set are the same as pre-nets. Let C have an object p, and for every pair of natural numbers (n, m), let C contain an object $t(m, n)$. Here p stands for 'places', while $t(m, n)$ stands for 'transitions with m inputs and n outputs'. Besides identity morphisms, C contains m morphisms $s_i \colon t(m, n) \to p$ representing the source maps and n morphisms $t_j \colon t(m, n) \to p$ representing the target maps. Composition in C is trivial.

A pre-net $T \xrightarrow{(s,t)} S^* \times S^*$ can be identified with the functor $C \to \mathsf{Set}$ that sends p to the set of places S, sends the object $t(m, n)$ to the subset of T consisting of transitions with m inputs and n outputs, and sends the morphisms $s_i, t_j \colon t(m, n) \to p$ to the functions that map each transition to its i-th input and j-th output. A morphism of pre-nets (f, g) can then be identified with a natural transformation between such functors, with the p-component given by g and with the $t(m, n)$-components given by the restrictions of f to the set of transitions with m inputs and n outputs. Naturality follows from the commutative diagrams in Definition IV.1. □

One downside of pre-nets is that ordering the inputs and outputs of transitions seems artificial in many applications where Petri nets are heavily used [11], [31]. This ordering also greatly restricts the available morphisms between pre-nets. For example, there is no morphism between the following pre-nets:



even though there is a morphism between their underlying Petri nets, in which the ordering information has been forgotten.

Furthermore, in the symmetric strict monoidal category $F_*(Q)$ coming from a pre-net Q, *none* of the symmetries $\sigma_{x,y} \colon x \otimes y \to y \otimes x$ are identities, except when x or y is the unit object. Perhaps more importantly, if a transition in the pre-net Q gives a morphism

$$t \colon x_1 \otimes \cdots \otimes x_m \to y_1 \otimes \cdots \otimes y_n,$$

where x_i, y_j are objects coming from places of Q, the composite of t with symmetries that permute the inputs x_i and outputs y_j is only equal to f if both these permutations are the identity.

Thus, the SSMCs obtained from pre-nets exemplify an extreme version of the individual token philosophy. Not only does each token have its own individual identity, switching two tokens before or after executing the morphism corresponding to a transition *always* gives a different morphism.

V. Σ-NETS

We have seen that Petri nets generate symmetric monoidal categories naturally suited to the collective token philosophy, where the tokens have no individual identity, so it makes no sense to speak of switching them. On the other hand, we have just seen that pre-nets generate symmmetric monoidal categories suited to an extreme version of the individual token philosophy, in which switching tokens *always* has an effect.

Now we introduce a new kind of nets, called Σ-nets, which in some sense lie between these two extremes. In a Σ-net, one has control over which permutations of the input or output of a transition alter the morphism it defines and which do not. This finer ability to control the action of permutations allow Σ-nets to behave either like Petri nets or pre-nets—or in a mixed way.

Lemma V.1. *There is a forgetful functor*

$$Q \colon \mathsf{SSMC} \to \mathsf{Set}$$

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that sends a symmetric strict monoidal category to its set of objects and sends a strict symmetric monoidal functor to its underlying function on objects. Q has a left adjoint

$$P\colon \mathsf{Set} \to \mathsf{SSMC}$$

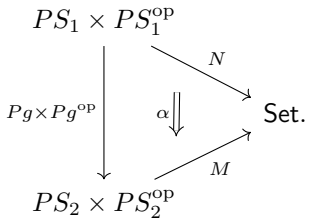
that sends a set S to the symmetric strict monoidal category PS having (possibly empty) words in S as objects, and permutations as morphisms.

Proof. See Sassone [28, Sec. 3] or Gambino and Joyal [9, Sec. 3.1]. □

Definition V.2. A Σ-net is a set S together with a functor

$$N\colon PS \times PS^{\mathrm{op}} \to \mathsf{Set}.$$

A morphism between Σ-nets $PS_1 \times PS_1^{\mathrm{op}} \xrightarrow{N} \mathsf{Set}$ and $PS_2 \times PS_2^{\mathrm{op}} \xrightarrow{M} \mathsf{Set}$ is a pair (g, α) where $g\colon S_1 \to S_2$ is a function and α is a natural transformation filling the following diagram:



This defines the category Σ-net.

The definition of Σ-net may seem unintuitive, but it is easily explained. Suppose $N\colon PS \times PS^{\mathrm{op}} \to \mathsf{Set}$ is a Σ-net. We call S its set of **places**. Objects of PS are words of places. Given $m, m' \in PS$, we call an element of $N(m, m')$ a **transition** with **source** m and **target** m'. In Theorem VII.7 we describe how to freely generate a symmetric strict monoidal category C from the Σ-net N. In this construction, the transitions of N give morphisms that generate all the morphisms in C.

More precisely: the objects of C are words of places. The tensor product of objects is given by concatenation of words, and the symmetry in C acts by permuting places in a word. Each transition $t \in N(m, m')$ gives a morphism $t\colon m \to m'$ in C, and all other morphisms are generated by composition, tensor product and symmetries.

Given a transition $t \in N(m, m')$, the action of the functor $N\colon PS \times PS^{\mathrm{op}} \to \mathsf{Set}$ on morphisms describes what happens to the corresponding morphism $t\colon m \to m'$ when we permute the places in its source and target: it gets sent to some other transition (possibly the same one). We say two transitions $t \in N(m, m')$ and $u \in N(n, n')$ are in the same **transition class** if and only if there exists a morphism $\sigma\colon (m, m') \to (n, n')$ in $PS \times PS^{\mathrm{op}}$ such that

$$N(\sigma)(t) = u.$$

Example V.3. There is a Σ-net N with just two places, say p and q, and just two transitions, $t_1 \in N(pq, \epsilon)$ and $t_2 \in N(qp, \epsilon)$, where ϵ stands for the empty word. There are two morphisms in $PS \times PS^{\mathrm{op}}$ with domain (pq, ϵ), namely the identity and the swap $(pq, \epsilon) \to (qp, \epsilon)$. Since there is a unique function between any two singleton sets, N of this swap must map t_1 to t_2. Thus, both t_1 and t_2 lie in the same transition class, and this Σ-net has just one transition class.

Example V.4. Now consider a Σ-net M with just two places p and q and exactly four transitions, with $M(pq, \epsilon) = \{t_1, u_1\}$ and $M(qp, \epsilon) = \{t_2, u_2\}$. We set M of the swap $(pq, \epsilon) \to (qp, \epsilon)$ to act as the function $\{t_1, u_1\} \to \{t_2, u_2\}$ sending t_1 to t_2 and u_1 to u_2. This Σ-net has exactly two transition classes: t_1 and t_2 represent one transition class, and u_1 and u_2 represent the other. Note that if we instead define the action of M on the swap to send t_1 to u_2 and t_2 to u_1, then we would still have two transition classes; in fact this would be an isomorphic Σ-net.

Examples V.3 and V.4 illustrate the situation when no place occurs more than once in the source or target of any transition. In this case, any two values of the functor N will be related by *at most one* morphism in $PS \times PS^{\mathrm{op}}$, and the functorial action of N on such a morphism provides a way to canonically identify their values.

Example V.5. Next consider a Σ-net O with one place p and one transition, namely $t \in O(pp, \epsilon)$. There are still two morphisms in $PS \times PS^{\mathrm{op}}$ with domain (pp, ϵ), the identity and the swap, but now both have (pp, ϵ) as codomain as well. There is still only one transition class, but now t is mapped to itself by *both* morphisms $(pp, \epsilon) \to (pp, \epsilon)$, the identity and the swap.

Given a group G acting on a set X, the **isotropy group** of $x \in X$ is the subgroup of G consisting of elements that map x to itself. Thus, in Example V.5, unlike Example V.4, we are seeing a transition with a nontrivial isotropy group. In fact, because permutations act *trivially* on all transitions, the Σ-net of Example V.5 belongs to the image of Petri under the functor $G_{\mathrm{pet}}\colon \mathsf{Petri} \to \Sigma\text{-net}$ described in Proposition VII.4.

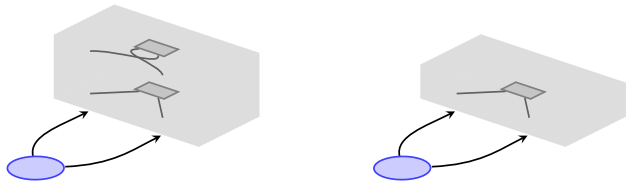
Example V.6. Next consider a Σ-net Q with one place p and precisely two transitions with $Q(pp, \epsilon) = \{t_1, t_2\}$. Suppose that Q of the identity $(pp, \epsilon) \to (pp, \epsilon)$ acts as the identity function (as it must), while Q of the swap acts by $t_1 \mapsto t_2$ and $t_2 \mapsto t_1$. Then t_1 and t_2 represent the same transition class, so there is once again only one transition class. The isotropy groups of t_1 and t_2 are trivial. In fact, because permutations act *freely* on the transitions in every transition class, this Σ-net belongs to the image of PreNet under the functor $F_{\mathrm{pre}}\colon \mathsf{PreNet} \to \Sigma\text{-net}$ described in Theorem VII.3.

Example V.7. Now let us give an example blending features from Examples V.5 and V.6. For this, we create a Σ-net R that has one place p and three transitions $t_1, t_2, u \in R(pp, \epsilon)$, such that t_1 and t_2 are order-sensitive while u is not. This Σ-net maps (pp, ϵ) to $\{t_1, t_2, u\}$ and everything else to the empty set. The action of the swap automorphism of (pp, ϵ) switches t_1, t_2 and fixes u. As a result, this Σ-net has two transition classes: t_1, t_2 are both representatives of one transition class, while u represents the other. This Σ-net is not in the image of $G_{\mathrm{pet}}\colon \mathsf{Petri} \to \Sigma\text{-net}$ or $F_{\mathrm{pre}}\colon \mathsf{PreNet} \to \Sigma\text{-net}$; it mixes the two worlds.

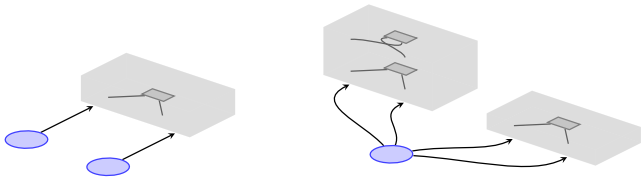
We now introduce some graphical representations for Σ-nets. The first depicts a transition class as a three-dimensional tank containing pictures of the permutations that act trivially

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on an arbitrarily chosen transition in this class; for instance:



On the left is Example V.5, where the identity and the swap both act trivially. On the right is Example V.6, where only the identity acts trivially. Examples V.3 and V.7 are instead:

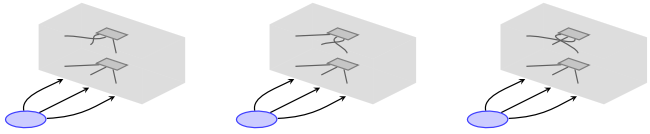


Although each transition class is a set of transitions, and each tank represents a single transition class, the pictures inside that tank do *not* represent the transitions in that class, but rather the *isotropy group* of a *single* transition in the class. For a given transition class, with m inputs and n outputs, say, the number of transitions and the size of the isotropy group are inversely related: their product is the cardinality $m!n!$ of the total symmetry group $S_m \times S_n$.

Of course, this formalism quickly becomes disadvantageous for large nets. Another approach is to draw a Σ-net using the usual two-dimensional representation of a Petri net, with one node for each transition class, but decorated by the relevant isotropy group. When this group is trivial, we can omit it. In this style of drawing, the nets of Examples V.5 and V.6 look as follows:



In the rest of the paper, we will use the three-dimensional representation. However, both representations are potentially misleading, in different ways. The two-dimensional representation does not show how the isotropy group sits inside the ambient group $S_m \times S_n$. The three-dimensional representation indicates this, but it is sensitive to the arbitrary choice of one transition in each class. For instance, the following three pictures all represent the same Σ-net, with one transition class containing three transitions, each with isotropy group isomorphic to S_2. But these three copies of S_2 sit inside S_3 differently, yielding different three-dimensional pictures.



We do not know whether there is a graphical representation of Σ-nets that avoids both of these problems.

VI. Perspectives on the category Σ-net

The definition of Σ-nets in Section V gives what may be called a "profunctorial" perspective: a Σ-net is a functor from $PS \times PS^{\mathrm{op}}$ to Set, which is the same as a profunctor from PS to itself. This perspective will be useful in constructing the adjunction between Σ-net and SSMC in Theorem VII.7. However, there are other perspectives on Σ-nets, leading to two alternative descriptions of Σ-net, useful for other purposes.

A. The presheaf perspective

Theorem VI.1. Σ-net *is equivalent to a presheaf category.*

Proof. We construct a category D so that functors from D to Set can be identified with Σ-nets. To construct D, we take the category C from Proposition IV.3 and throw in extra automorphisms of each object $t(m,n)$, making its automorphism group $S_m \times S_n$. For a source map $s_i \colon t(m,n) \to p$ and an automorphism $(\sigma,\tau) \in S_m \times S_n$, we set the composite $s_i \circ (\sigma,\tau)$ equal to $s_{\sigma(i)}$. Similarly, for a target map $t_j \colon t(m,n) \to p$, we set the composite $t_j \circ (\sigma,\tau)$ equal to $t_{\tau(j)}$. Then, for each Σ-net $N \colon PS \times PS^{\mathrm{op}} \to \mathsf{Set}$, there is a corresponding functor $\nu \colon D \to \mathsf{Set}$ defined as follows. It sends the object $p \in D$ to the set of places of N. It sends each object $t(m,n) \in D$ to the disjoint union of the sets $N(a,b)$ over all $a \in PS$ with length m and $b \in PS$ with length n. It sends the morphisms $s_i, t_j \colon t(m,n) \to p$ to the functions that map any transition to its ith input and jth output. Finally, this functor ν sends the permutations (σ,τ) to the natural actions of the symmetric group on the transitions of N. For a morphism of Σ-nets $(g,\alpha) \colon N \to N'$, there is a natural transformation between their functors whose p-component is given by g and whose $t(m,n)$-components are given by disjoint unions of the components of α. One can check that the resulting functor from Σ-net to Set^D is an equivalence. □

Theorem VI.1 has a lot of nice consequences, such as:

- Σ-net is complete and cocomplete. This is particularly important since many compositional approaches to Petri nets rely on colimits; for example, composition of open Petri nets is done using pushouts, while tensoring them is done using coproducts [1].
- Σ-net is a topos, and thus an adhesive category [21], so it admits a theory of double pushout rewriting [20]. This is relevant as double pushout rewriting is a widely used technique to transform graph-like structures in the literature [8]. The internal logic of toposes is very rich, and understanding its implications for Σ-nets is an interesting direction for future work.

Note that Petri is not a presheaf category, whereas the category of directed graphs is. Indeed, as noted in [17], graphs are functors $C_{1,1} \to \mathsf{Set}$, where $C_{1,1}$ is the full subcategory of the above C (or D) on the objects p and $t(1,1)$.

The category D in Theorem VI.1 is equivalent to the opposite of Kock's category of "elementary graphs" [17, 1.5.4]. Thus, Σ-net is equivalent to his category of "digraphical species" [17, 2.1]. Similarly, C is the opposite of Kock's "elementary planar graphs".

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B. The groupoidal perspective

The profunctorial and presheaf perspectives highlight the transitions of a Σ-net over its transition classes. Sometimes, however, we want to work directly with the transition classes; we now describe a third perspective that permits this.

Firstly, it is well-known [22, Theorem 2.1.2] that a functor $N\colon PS \times PS^{\mathrm{op}} \to \mathsf{Set}$ is equivalent to a discrete opfibration $T \to PS \times PS^{\mathrm{op}}$ for some category T. Since $PS \times PS^{\mathrm{op}}$ is a groupoid, T is as well. In addition to this, a morphism of Σ-nets is equivalently a commutative square

$$\begin{array}{ccc} T_1 & \xrightarrow{\quad g \quad} & T_2 \\ \downarrow & & \downarrow \\ PS_1 \times PS_1^{\mathrm{op}} & \xrightarrow[Pf \times Pf^{\mathrm{op}}]{} & PS_2 \times PS_2^{\mathrm{op}} \end{array}$$

Note that this looks much more similar to the definitions of the categories PreNet and Petri. The set of objects of T here is the disjoint union of the sets $N(p, p')$, i.e., the transitions rather than the transition classes. The transition classes are the *isomorphism classes* of the groupoid T. To contract these down to single objects, we can replace T by an equivalent groupoid that is **skeletal**, i.e., there are no morphisms $x \to y$ for objects $x \neq y$, or equivalently each isomorphism class contains exactly one object. After such a replacement the functor $T \to PS \times PS^{\mathrm{op}}$ is no longer a discrete opfibration, but it is still faithful. To compensate for this replacement of discrete opfibrations by faithful functors with skeletal domain, when defining the morphisms of Σ-nets we have to allow the squares to commute up to isomorphism rather than strictly.

Theorem VI.2. *The category of Σ-nets is equivalent to the following category:*

- *Its objects are faithful functors $T \to PS \times PS^{\mathrm{op}}$, where S is a set and T is a skeletal groupoid.*
- *Its morphisms are squares that commute up to specified isomorphism*

$$\begin{array}{ccc} T_1 & \xrightarrow{\quad g \quad} & T_2 \\ \downarrow & \cong_\theta & \downarrow \\ PS_1 \times PS_1^{\mathrm{op}} & \xrightarrow[Pf \times Pf^{\mathrm{op}}]{} & PS_2 \times PS_2^{\mathrm{op}} \end{array}$$

  modulo the equivalence relation that two such morphisms (f, g, θ) and (f', g', θ') are considered equal if $f = f'$ and there is a natural isomorphism $\phi\colon g \Rightarrow g'$ such that

$$\begin{array}{ccc} T_1 & \overset{g}{\underset{g'}{\Downarrow \phi}} & T_2 \\ \downarrow & \cong_{\theta'} & \downarrow \\ PS_1 \times PS_1^{\mathrm{op}} & \xrightarrow[Pf \times Pf^{\mathrm{op}}]{} & PS_2 \times PS_2^{\mathrm{op}} \end{array} \quad = \quad \begin{array}{ccc} T_1 & \xrightarrow{\quad g \quad} & T_2 \\ \downarrow & \cong_{\theta} & \downarrow \\ PS_1 \times PS_1^{\mathrm{op}} & \xrightarrow[Pf \times Pf^{\mathrm{op}}]{} & PS_2 \times PS_2^{\mathrm{op}} \end{array}$$

Note that since $T_2 \to PS_2 \times PS_2^{\mathrm{op}}$ is faithful, such a ϕ is unique if it exists. Thus, the category described in the theorem is in fact *equivalent* to the evident 2-category having as 2-morphisms natural isomorphisms ϕ as above.

Proof. Define a category $\mathcal{P}$ as in the theorem, but where the objects allow T to be any groupoid. Then there is a functor $\Sigma\text{-}\mathsf{net} \to \mathcal{P}$, since discrete opfibrations are faithful and strictly commutative squares also commute up to isomorphism. This functor is faithful, since if θ and θ' are identities so is ϕ, by the faithfulness of $T_2 \to PS_2 \times PS_2^{\mathrm{op}}$. Moreover, any morphism in $\mathcal{P}$ whose target $T_2 \to PS_2 \times PS_2^{\mathrm{op}}$ is a discrete opfibration has a representative that commutes strictly, since we can lift the isomorphism θ to an isomorphism ϕ with θ' an identity. Thus, the functor $\Sigma\text{-}\mathsf{net} \to \mathcal{P}$ is also full.

Let the **pseudo slice 2-category** over a groupoid B be the 2-category with groupoids over B as objects, triangles commuting up to natural isomorphism

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & A' \\ & {\scriptstyle \cong_\theta} & \\ & B & \end{array}$$

as morphisms, and the evident 2-morphisms [26, Definition 3.2]. Any groupoid over B is equivalent, in the pseudo slice 2-category of B, to a fibration [32, Theorem 6.7], which in the groupoid case is the same as an opfibration. If $f\colon A \to B$ is faithful then this opfibration will be as well, so f is equivalent to a discrete opfibration. Since equivalences in the pseudo slice 2-category yield isomorphisms in $\mathcal{P}$, the functor $\Sigma\text{-}\mathsf{net} \to \mathcal{P}$ is also essentially surjective, and hence an equivalence.

The category described in the theorem is a full subcategory of $\mathcal{P}$, so it suffices to show that every object of $\mathcal{P}$ is isomorphic to one where T is skeletal. But any groupoid is equivalent to a skeletal one, and such an equivalence preserves faithfulness and yields an isomorphism in $\mathcal{P}$. □

Note that the construction in the final paragraph taking a groupoid to a skeletal one preserves connected components, while in the output each connected component has exactly one object. Thus, in the representation described in Theorem VI.2 the objects of the groupoid T really are precisely the transition classes. Since the transition classes of a Σ-net correspond to the transitions of a Petri net, we can think of a Σ-net as a Petri net together with, for each transition, (1) a lifting of its source and target multisets to words, and (2) an *isotropy group* that acts faithfully on those words, i.e., maps injectively to the subgroup of $S_m \times S_n$ that fixes both of those words.

Example VI.3. If we start from a transition in a Petri net $t\colon 3a + 2b \to 4c$, then we could lift it to a transition in a Σ-net by defining $t\colon aaabb \to cccc$ and equipping it with any subgroup of $S_3 \times S_2 \times S_4$, which describes the "degree of collectivization" of t. If the isotropy group is trivial, then this particular transition behaves like one in a pre-net—tokens are "fully individualized"—whereas if it is as large as possible then it behaves as a Petri net—tokens are "fully collectivized". This idea is heavily used in the next section to describe the adjunctions between Petri, PreNet and $\Sigma\text{-}\mathsf{net}$.

VII. Description of the Adjunctions

Now we describe in detail all the adjunctions between the categories in play. We again include the diagram of Section II,

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but now with most of the functors labeled.

$$\begin{array}{ccccc} \mathsf{StrMC} & \underset{U_{\mathsf{StrMC}}}{\overset{F_{\mathsf{StrMC}}}{\rightleftarrows}} & \mathsf{SSMC} & \underset{U_{\mathsf{SSMC}}}{\overset{F_{\mathsf{SSMC}}}{\rightleftarrows}} & \mathsf{CMC} \\ \uparrow{\scriptstyle F_{\mathsf{PreNet}}} \downarrow{\scriptstyle U_{\mathsf{PreNet}}} & & \uparrow{\scriptstyle F_{\Sigma\text{-net}}} \downarrow{\scriptstyle U_{\Sigma\text{-net}}} & & \uparrow{\scriptstyle F_{\mathsf{Petri}}} \downarrow{\scriptstyle U_{\mathsf{Petri}}} \\ \mathsf{PreNet} & \underset{H_{\mathrm{pre}}}{\overset{F_{\mathrm{pre}}}{\rightleftarrows}} & \Sigma\text{-net} & \underset{G_{\mathrm{pet}}}{\overset{F_{\mathrm{pet}}}{\rightleftarrows}} & \mathsf{Petri} \end{array} \tag{1}$$

The adjunctions in the top row can be constructed using standard tools, such as the adjoint functor theorem or the adjoint lifting theorem.

Proposition VII.1. *There is an adjunction*

$$\mathsf{StrMC} \underset{U_{\mathsf{StrMC}}}{\overset{F_{\mathsf{StrMC}}}{\rightleftarrows}} \mathsf{SSMC}.$$

Here, U_{StrMC} freely adds symmetries to a strict monoidal category, while U_{StrMC} sends any symmetric strict monoidal category to its underlying strict monoidal category.

Proposition VII.2. *There is an adjunction*

$$\mathsf{SSMC} \underset{U_{\mathsf{SSMC}}}{\overset{F_{\mathsf{SSMC}}}{\rightleftarrows}} \mathsf{CMC}.$$

F_{SSMC} takes a symmetric strict monoidal category and imposes a law saying that all symmetries are identity morphism, while U_{SSMC} sends any commutative monoidal category to its underlying symmetric strict monoidal category.

The adjunction between Petri and CMC was recalled in Proposition III.3, while that between PreNet and StrMC was recalled in Proposition IV.2. We now cover the middle column and bottom row of the diagram, which are new.

Theorem VII.3. *There is a triple of adjoint functors*

$$\mathsf{PreNet} \underset{H_{\mathrm{pre}}}{\overset{F_{\mathrm{pre}}}{\overset{\longleftarrow G_{\mathrm{pre}} \longrightarrow}{\rightrightarrows}}} \Sigma\text{-net}.$$

Proof. For this proof it is most convenient to work with the presheaf perspective. In the proof of Proposition IV.3 we described a category C such that $\mathsf{PreNet} \cong [\mathsf{C}, \mathsf{Set}]$ and in the proof of Theorem VI.1 we described a category D such that $\Sigma\text{-net} \cong [\mathsf{D}, \mathsf{Set}]$. Recall that D is built by starting with the objects and morphisms of C and adding new morphisms and equations. The inclusion gives a functor $i\colon \mathsf{C} \to \mathsf{D}$ which induces a functor

$$\Sigma\text{-net} \cong [\mathsf{D}, \mathsf{Set}] \xrightarrow{(-)\circ i} [\mathsf{C}, \mathsf{Set}] \cong \mathsf{PreNet}$$

given by precomposition with i. The composite functor above is the forgetful functor G_{pre}. Therefore G_{pre} has a left adjoint $F_{\mathrm{pre}}\colon \mathsf{PreNet} \to \Sigma\text{-net}$ given by left Kan extension along i and a right adjoint $H_{\mathrm{pre}}\colon \mathsf{PreNet} \to \Sigma\text{-net}$ given by right Kan extension along i. □

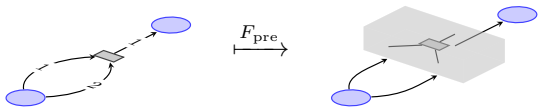
Let us spell out what the functors F_{pre}, G_{pre}, H_{pre} do in detail, using our three-dimensional graphical representation.

F_{pre}) For this functor we work in the groupoid representation of Σ-nets. A pre-net $T \xrightarrow{(s,t)} S^* \times S^*$ is sent to the Σ-net $T \xrightarrow{F_{\mathrm{pre}}(s,t)} PS \times PS^{\mathrm{op}}$, where T denotes the *discrete* groupoid having T as underlying set of objects. Since T is discrete, the functor $F_{\mathrm{pre}}(s,t)$ only needs to be defined on objects, which we do by taking the composite

$$T \xrightarrow{(s,t)} S^* \times S^* \to PS \times PS^{\mathrm{op}}$$

using the fact that S^* is the set of objects of PS. A morphism of pre-nets $(f,g)\colon (s_1,t_1) \to (s_2,t_2)$ induces a morphism of Σ-nets: $g\colon T_1 \to T_2$ lifts to a morphism between discrete groupoids and $f\colon S_1 \to S_2$ lifts to a functor $PS_1 \times PS_1^{\mathrm{op}} \to PS_2 \times PS_2^{\mathrm{op}}$. The relevant square as in Theorem VI.2 commutes strictly.

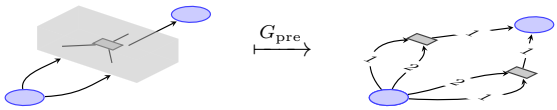
F_{pre} takes a pre-net and builds from it a Σ-net with trivial isotropy groups. Graphically, this amounts to enclosing every transition of the given pre-net in a tank:



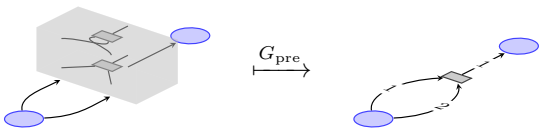
In particular, the *transition classes* of $F_{\mathrm{pre}}(N)$ are the transitions of N.

G_{pre}) For this functor we work in the profunctor representation. A Σ-net $PS \times PS^{\mathrm{op}} \xrightarrow{N} \mathsf{Set}$ is sent to the pre-net having S as its set of places and the disjoint union of all sets $N(a,b)$, for any a, b objects of PS, as its set of transitions. For each transition, input and output places are defined using the inverse image of N. That is, the transitions of $G_{\mathrm{pre}}N$ are the *transitions* of N, with their grouping into classes and their isotropy groups forgotten.

We can give a different interpretation of this using the groupoid perspective. Suppose $T \xrightarrow{N} PS \times PS^{\mathrm{op}}$ is a Σ-net. Then for each object t of T such that $N(t)$ is a pair of strings of length m and n there will be $S_m \times S_n / \hom_T(t,t)$ transitions in $G_{\mathrm{pre}}N$, where S_n denotes the group of permutations over a string of n elements. Graphically, this is represented by "exploding" a tank with m inputs and n outputs and introducing $m!n!/k$ pre-net transitions, where k is the number of elements in the tank.

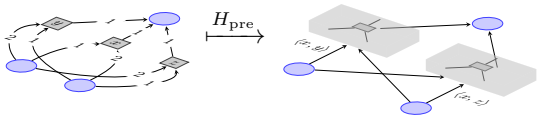


In the image above, we see the behavior of G_{pre} on a Σ-net having a transition with trivial isotropy group, while in the image below G_{pre} is used on a Σ-net having a transition with 2-element isotropy group.

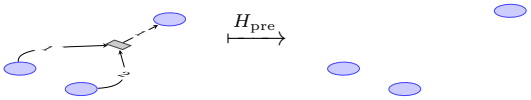


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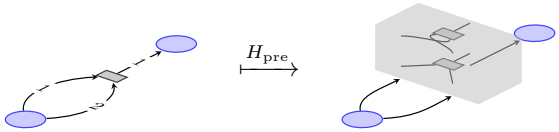
H_{pre}) While F_{pre} builds as many tanks as we can get from a pre-net's transitions, H_{pre} bundles pre-net transitions sharing the same inputs/outputs modulo permutations, whenever they complete their corresponding symmetry groups. For instance, in the figure below transitions x and y complete the permutation group $S_2 \times S_1$, and hence they give rise to the tank denoted with $\langle x, y\rangle$. The same happens for transitions x and z, giving rise to tank $\langle x, z\rangle$.



The following pre-net does not have enough transitions to complete the symmetry group of its inputs/outputs. As such, H_{pre} cannot match this transition with anything, and does not produce any tank.



In the following case, the pre-net has a repeated input. H_{pre} is then able to match the transtion with itself, producing a maximally commutative tank.



Looking at these examples, we see that in general the transitions of N do not correspond directly to either the transitions of $H_{\mathrm{pre}}(N)$ or the transition classes of $H_{\mathrm{pre}}(N)$.

Proposition VII.4. *There is an adjunction*

$$\Sigma\text{-net} \underset{G_{\mathrm{pet}}}{\overset{F_{\mathrm{pet}}}{\rightleftarrows}} \mathsf{Petri}.$$

Proof. Note first that Petri is by definition precisely the comma category $(\mathsf{Set} \downarrow (\mathbb{N}[-] \times \mathbb{N}[-]))$. Similarly, if we identify a Σ-net with a functor $N\colon PS \times PS^{\mathrm{op}} \to \mathsf{Set}$ and thereby with a discrete opfibration $N \to PS \times PS^{\mathrm{op}}$, then Σ-net becomes identified with the full subcategory of the comma category $(\mathsf{Cat} \downarrow (P(-) \times P(-)^{\mathrm{op}}))$ consisting of the discrete opfibrations.

Now note that Set is a reflective full subcategory of Cat, with reflector π_0 that takes the set of connected components of a category. Moreover, we have $\pi_0(PS \times PS^{\mathrm{op}}) \cong \mathbb{N}[S] \times \mathbb{N}[S]$. Thus, Lemma VII.5, proven below (and applied with $\mathsf{D} = \mathsf{Cat}$, $\mathsf{C} = \mathsf{E} = \mathsf{Set}$, and $K = P(-) \times P(-)^{\mathrm{op}}$), shows that we have an adjunction

$$(\mathsf{Cat} \downarrow (P(-) \times P(-)^{\mathrm{op}})) \underset{G}{\overset{F}{\rightleftarrows}} (\mathsf{Set} \downarrow (\mathbb{N}[-] \times \mathbb{N}[-])) = \mathsf{Petri}$$

in which the left adjoint F applies π_0 to both domain and codomain, and the right adjoint G pulls back along the unit $PS \times PS^{\mathrm{op}} \to \mathbb{N}[S] \times \mathbb{N}[S]$. Therefore, it suffices to observe that this right adjoint takes values in discrete opfibrations, hence in Σ-net. □

Lemma VII.5. *Let* E *be a reflective subcategory of* D*, with reflector* $\pi\colon \mathsf{D} \to \mathsf{E}$*, and let* $K\colon \mathsf{C} \to \mathsf{D}$ *be a functor where* D *has pullbacks. Then there is an adjunction*

$$(\mathsf{D} \downarrow K) \underset{G}{\overset{F}{\rightleftarrows}} (\mathsf{E} \downarrow (\pi \circ K)).$$

Proof. Let $\eta_X\colon X \to \pi X$ denote the unit of the reflection. Then for any $f\colon S_1 \to S_2$ in C, we have $\eta_{KS_2} \circ Kf = \pi Kf \circ \eta_{KS_1}$ by naturality; we denote this common map by η_f. Now there is a profunctor between $(\mathsf{D} \downarrow K)$ and $(\mathsf{E} \downarrow (\pi \circ K))$ defined to take $T_1 \to KS_1$ and $T_2 \to \pi KS_2$ (where $T_1 \in \mathsf{D}$ and $T_2 \in \mathsf{E}$) to the set of pairs (f, g) where $f : S_1 \to S_2$ in C and $g : T_1 \to T_2$ in D make the following square commute:

$$\begin{array}{ccc} T_1 & \xrightarrow{\ g\ } & T_2 \\ \downarrow & & \downarrow \\ KS_1 & \xrightarrow[\eta_f]{} & \pi KS_2. \end{array}$$

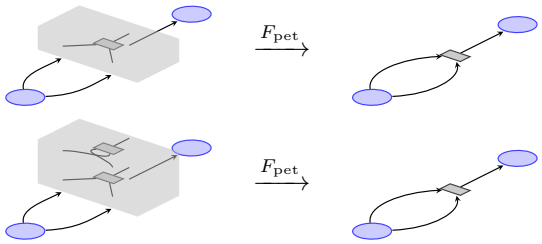
This profunctor is representable on both sides, because any such square factors uniquely in both of the following ways:

$$\begin{array}{ccccc} T_1 & \xrightarrow{\eta_{T_1}} & \pi T_1 & \dashrightarrow & T_2 \\ \downarrow & & \downarrow & & \downarrow \\ KS_1 & \xrightarrow[\eta_{KS_1}]{} & \pi KS_1 & \xrightarrow[\pi Kf]{} & \pi KS_2 \end{array} \qquad \begin{array}{ccccc} T_1 & \dashrightarrow & \bullet & \longrightarrow & T_2 \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ KS_1 & \xrightarrow[Kf]{} & KS_2 & \xrightarrow[\eta_{KS_2}]{} & \pi KS_2. \end{array}$$

On the left, the factorization is by the universal property of η_{T_1}, while on the right it is by the universal property of the pullback. Therefore, there is an adjunction $F \dashv G$ as desired, where F takes $T_1 \to KS_1$ to $\pi T_1 \to \pi KS_1$, and G takes $T_2 \to \pi KS_2$ to the pullback of T_2 to KS_2. □

Note that by construction, this adjunction is a reflection, i.e., the right adjoint G_{pet} is fully faithful. We can illustrate the action of F_{pet} and G_{pet} with examples.

F_{pet}) In the groupoid perspective, this functor takes a Σ-net $T \xrightarrow{N} PS \times PS^{\mathrm{op}}$ and maps it to the Petri net having the underlying set of objects of T as transitions, S as places, and input/output functions induced by the mapping on objects of N. The action on a morphism (g, f) is obtained by restricting the functor g to its mapping on objects. Graphically, F_{pet} deflates tanks, replacing each tank by a single transition:



In particular, the transitions of $F_{\mathrm{pet}}(N)$ are the *transition classes* of N.

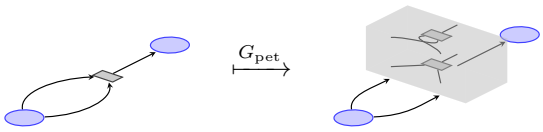
G_{pet}) Petri nets are mapped under G_{pet} to corresponding Σ-nets that have the largest isotropy groups possible. Consider a transition t in a Petri net N. Its inputs and outputs will be a couple of unordered strings of length n, m, respectively. Pick any ordering for these strings, and call them a, b, respectively. Finally, let

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G_t be the subgroup of $S_n \times S_m$ that fixes the pair of strings (a, b).

N is mapped to a Σ-net $T \xrightarrow{G_{\mathrm{pet}} N} PS \times PS^{\mathrm{op}}$ whose groupoid T has transitions of N as objects and, for each t in T, G_t as its group of automorphisms. N maps t to the ordering (a, b) we have chosen before. It can be seen that picking different orderings of the input/output of each transition gives isomorphic results.

Graphically, out of each Petri net we build a corresponding Σ-net that has its tanks as full as possible:



Thus, the *transition classes* of $G_{\mathrm{pet}}(N)$ are the transitions of N.

Remark VII.6. Note that F_{pre} and G_{pet} both build a Σ-net whose transition classes are the transitions of a pre-net or Petri net. On the other hand, G_{pre} and F_{pet} are "dual", in that they build a pre-net or Petri net whose transitions are, respectively, the transitions or the transition classes of a Σ-net. In particular, the composite $F_{\mathrm{pet}} \circ F_{\mathrm{pre}}$ preserves transitions: it is the functor $\mathsf{PreNet} \to \mathsf{Petri}$ that simply forgets the ordering of inputs and outputs. Its right adjoint $G_{\mathrm{pre}} \circ G_{\mathrm{pet}}$ explodes each transition of a Petri net into as many transitions of a pre-net as possible, giving its inputs and outputs all possible orderings.

The last adjunction to construct is the one in the middle column:

Theorem VII.7. *There is an adjunction*

$$\Sigma\text{-net} \underset{U_{\Sigma\text{-net}}}{\overset{F_{\Sigma\text{-net}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{SSMC}.$$

Two proofs of Theorem VII.7 were sketched by Kock [18, §§6–7]. Our proof is more similar to the proof of [24, Theorem 5.1], which is a generalization of Propositions IV.2 and III.3 involving a Lawvere theory Q: these two propositions follow by taking Q to be the theory of commutative monoids and the theory of monoids, respectively. In that proof, an adjunction between Q-nets and Q-categories (i.e., Q-algebras in Cat) was obtained as the composite of two adjunctions where the intermediate category consists of Q-graphs: graphs internal to the category of Q-algebras. Note that these Q-graphs have operations coming from the Lawvere theory Q, which act both on vertices and edges, but they lack the ability to compose edges (i.e., morphisms) that one has in a Q-category.

Our desired adjunction here is not a special case of [24, Theorem 5.1], since the symmetries in a symmetric monoidal category cannot be represented by a structure on the object set alone. However, we can perform a similar factorization through a category containing only the monoidal operations. We begin by reducing the problem from strict symmetric monoidal categories to (colored) props.

Definition VII.8. A **(colored) prop** consists of a set S, a strict symmetric monoidal category B, and a strict symmetric monoidal functor $i\colon PS \to B$ that is bijective on objects. A morphism of props consists of a function $S \to S'$ and a strict symmetric monoidal functor $B \to B'$ making the evident square commute. We denote the category of props by PROP.

Lemma VII.9. *There is an adjunction*

$$\mathsf{PROP} \underset{U_2}{\overset{F_2}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{SSMC}.$$

Proof. For a prop (S, B, i) we define $F_2(S, B, i) = B$. And for a strict symmetric monoidal category B, we let S be the set of objects of B, so that we have a strict symmetric monoidal functor $PS \to B$. Now we factor this functor as a bijective-on-objects functor $i\colon PS \to B'$ followed by a fully faithful one $p\colon B' \to B$. Then B' can be given a symmetric strict monoidal structure making both i and p strict symmetric monoidal functors, and we define $U_2(B) = (S, B', i)$. □

Therefore, it will suffice to construct an adjunction between Σ-net and PROP. We work with the profunctor representation of Σ-nets. Let $U_1\colon \mathsf{PROP} \to \Sigma\text{-net}$ be the functor sending (S, B, i) to (S, N) where $N(a_1, a_2) = \hom_B(i(a_2), i(a_1))$. This is the functor we aim to construct a left adjoint of. As in [24], we do this "fiberwise" for a fixed A, then piece the fiberwise adjunctions together.

Lemma VII.10. *We have a commutative triangle*

$$\begin{array}{ccc} \Sigma\text{-net} & \xleftarrow{U_1} & \mathsf{PROP} \\ & \searrow \quad \swarrow & \\ & \mathsf{Set} & \end{array}$$

in which the two diagonal functors are split fibrations and U_1 is cartesian.

Proof. The two diagonal functors send (S, N) to S and (S, B, i) to S, respectively. To show the left-hand diagonal functor is a split fibration, let $(S, H) \in \Sigma\text{-net}$ and $g : S' \to S$; then $(S', N \circ (Pg \times Pg^{\mathrm{op}}))$ is the domain of a cartesian lifting. For the right-hand functor, given $(S, B, i) \in \mathsf{PROP}$ and $g\colon S' \to S$, the composite $i \circ g\colon PS' \to B$ may no longer be bijective on objects, but we can factor it as a bijective-on-objects functor $i'\colon PS' \to B'$ followed by a fully faithful one $g'\colon B' \to B$. These are both again strict symmetric monoidal functors, and the induced map $(S', B', i') \to (S, B, i)$ is cartesian. Finally, U_1 is cartesian by construction, since g' is fully faithful. □

Let $U_{1,S}\colon \Sigma\text{-net}_S \to \mathsf{PROP}_S$ denote the restriction of U_1 to the fibers over a particular set S. We will construct a left adjoint $F_{1,S}$ of this functor, then piece these together fiberwise.

Following the proof of [24], we need to decompose the structure of a prop with object set S into the "monoidal piece" and the "composition piece". This can be accomplished as follows. Batanin and Markl [3] define a **duoidal category** to be a category C with two monoidal structures $(\star, J)$ and $(\diamond, I)$ and additional natural morphisms

$$I \to J \qquad I \to I \star I \qquad J \diamond J \to J$$
$$(A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

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satisfying axioms that say $(\star, J)$ is a pseudomonoid structure on $(\mathsf{C}, \diamond, I)$ in the 2-category of lax monoidal functors. It is **$\diamond$-symmetric** if $\diamond$ is a symmetric monoidal structure and the above maps commute with the symmetry in an evident way.

In a duoidal category, the monoidal structure $\star$ lifts to a monoidal structure on the category of $\diamond$-monoids. A $\star$-monoid in this monoidal category of $\diamond$-monoids is called a **duoid**. Similarly, if the duoidal category is $\diamond$-symmetric, then $\star$ lifts to the category of commutative $\diamond$-monoids, and a monoid therein is called a **$\diamond$-commutative duoid**.

Lemma VII.11. *There is a $\diamond$-symmetric duoidal structure on $\Sigma\text{-net}_S$ whose category of $\diamond$-commutative duoids is equivalent to PROP_S.*

Proof. Note that $\Sigma\text{-net} = \mathbf{Prof}(PS, PS)$ is the hom-category of PS in the bicategory $\mathbf{Prof}$ of categories and profunctors. Since this is an endo-hom-category in a bicategory, it has a monoidal structure given by composition in $\mathbf{Prof}$, which we call $\star$ (thus J is the hom-functor of A). The monoidal structure $\diamond$ is given by convolution:

$$(H \diamond K)(x, z) = \int^{a,b,c,d} PS(x, ab) \times H(a, c) \times K(b, d) \times PS(cd, z).$$

with $I(x, y) = PS(x, \epsilon) \times PS(\epsilon, y)$.

A $\star$-monoid is a monad on PS in the bicategory $\mathbf{Prof}$, which is well-known to be equivalent to a category B with a bijective-on-objects functor $PS \to B$. Applying the Yoneda lemma, we find that a $\diamond$-monoid structure on such a B consists of morphisms

$$B(a, c) \times B(b, d) \to B(ab, cd)$$

that are suitably compatible. This extends the monoidal structure of PS to the arrows of B (it is already defined on the objects of B since they are the same as the objects of A). Compatibility with the duoidal exchange morphism says that this action is functorial, while compatibility with the map $J \diamond J \to J$ says that it extends the functorial action of the monoidal structure on PS. The associativity and unitality of a $\diamond$-monoid says B has a strict monoidal structure and the functor $PS \to B$ is strict monoidal. Finally, the symmetry of $\diamond$ switches H and K and composes with the symmetry isomorphisms in PS on either side; thus $\diamond$-commutativity of a duoid makes B a *symmetric* strict monoidal category and $PS \to B$ a strict symmetric monoidal functor. □

In fact, an analogous result holds with PS replaced by any symmetric monoidal category. A more abstract construction of this duoidal structure was given by Garner and López Franco [10, Proposition 51], while the identification of its duoids follows from their Proposition 49 and the remarks after Proposition 54. Note that the adjective "commutative" in [10] is used with a different meaning than ours; we repeat that for us, "$\diamond$-commutative" simply means that the monoid structure with respect to $\diamond$ is commutative in the ordinary sense for a monoid object in a symmetric monoidal category.

Note that both monoidal structures $\star$ and $\diamond$ of $\Sigma\text{-net}_S$ preserve colimits in each variable. We can now work at a higher level of abstraction.

Lemma VII.12. *For any cocomplete $\diamond$-symmetric duoidal category C such that $\star$ and $\diamond$ preserve colimits in each variable, the forgetful functor*

$$\diamond\colon \mathsf{CommDuoid}(\mathsf{C}) \to \mathsf{C}$$

has a left adjoint.

Proof. Recall that free commutative monoids exist in any cocomplete monoidal category whose tensor product preserves colimits in each variable, given by

$$FX = \coprod_n X^{\diamond n}/\Sigma_n$$

where $X^{\diamond n}/\Sigma_n$ denotes the n^{th} tensor power of X quotiented by the action of the n^{th} symmetric group. Indeed, commutative monoids are monadic over such a category. Thus, the category of commutative $\diamond$-monoids in our C is monadic over C.

Moreover, since $\diamond$ preserves colimits in each variable, by standard arguments it preserves reflexive coequalizers and sequential colimits in both variables together. Thus $X \mapsto X^{\diamond n}$ also preserves reflexive coequalizers and sequential colimits, hence so does the functor F and thus the monad for commutative $\diamond$-monoids. It follows that reflexive coequalizers and sequential colimits in the category of commutative $\diamond$-monoids are computed as in C, and therefore are preserved in each variable by the lifted tensor product $\star$. Therefore, by [19], the free $\star$-monoid on a commutative $\diamond$-monoid exists. Composing these two free constructions, we find that free $\diamond$-commutative duoids exist. □

Lemma VII.13. *There is an adjunction*

$$\Sigma\text{-net} \underset{U_1}{\overset{F_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{PROP}.$$

Proof. By Lemmas VII.11 and VII.12, each fiber functor $U_{1,S}$ has a left adjoint $F_{1,S}$; thus it remains to piece these adjoints together. Suppose $(S', N) \in \Sigma\text{-net}$ and $(S, B, i) \in \mathsf{PROP}$. By Lemma VII.10, a morphism $(S', N) \to U_1(S, B, i)$ is equivalently given by a function $g\colon S' \to S$ and a morphism $(S', N) \to U_{1,S'}(S', B', i')$ in $\Sigma\text{-net}_{S'}$, where $PS' \xrightarrow{i'} B' \xrightarrow{g'} B$ is the factorization of $i \circ Pg$ as a bijective-on-objects functor followed by a fully faithful one. But the latter morphism is equivalently a morphism $F_{1,S'}(S', N) \to (S', B', i')$ in $\mathsf{PROP}_{S'}$, hence a morphism $F_{1,S'}(S', N) \to (S, B, i)$ in PROP. Thus, defining $F_1(S', N) = F_{1,S'}(S', N)$ yields a left adjoint to U_1. (Note that it is unnecessary to ask whether F_1 is cartesian.) □

Proof of Theorem VII.7. Combining Lemma VII.9 and VII.13 we obtain the composite adjunction:

$$\Sigma\text{-net} \underset{U_1}{\overset{F_1}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{PROP} \underset{U_2}{\overset{F_2}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{SSMC}. \qquad \square$$

We end this section by considering the commutativity properties of the squares in eq. (1).

Proposition VII.14. *There is a natural isomorphism $G_{\text{pre}} \circ U_{\Sigma\text{-net}} \cong U_{\mathsf{PreNet}} \circ U_{\mathsf{StrMC}}$. Therefore, there is also a natural isomorphism $F_{\mathsf{StrMC}} \circ F_{\mathsf{PreNet}} \cong F_{\Sigma\text{-net}} \circ F_{\text{pre}}$.*

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Proof. With our precise definitions, the first isomorphism is actually a strict equality: both functors take a symmetric strict monoidal category C to the pre-net whose places are the objects of C and whose transitions from a word p to a word q are the morphisms in C from the tensor product of p to the tensor product of q. The second isomorphism follows by passage to left adjoints. □

Recalling from Section IV that the composite $F_{\mathsf{StrMC}} \circ F_{\mathsf{PreNet}}$ has been used to give a categorical semantics for pre-nets, we see that this semantics factors through Σ-nets.

Proposition VII.15. *There is a natural isomorphism* $G_{\mathrm{pet}} \circ U_{\mathsf{Petri}} \cong U_{\Sigma\text{-net}} \circ U_{\mathsf{SSMC}}$. *Therefore, there is also a natural isomorphism* $F_{\mathsf{SSMC}} \circ F_{\Sigma\text{-net}} \cong F_{\mathsf{Petri}} \circ F_{\mathrm{pet}}$.

Proof. Again, the first isomorphism is a strict equality: both functors take a commutative monoidal category C to the Σ-net whose places are the objects of C and whose transitions from p to q are the morphisms in C from the tensor product of p to the tensor product of q, with symmetries acting trivially. The second isomorphism follows by passage to left adjoints. □

Though analogous to Proposition VII.14, Proposition VII.15 does not imply that the categorical semantics of Petri nets factors through Σ-nets. However, that is also true:

Proposition VII.16. *There is a natural isomorphism* $F_{\mathsf{Petri}} \cong F_{\mathsf{SSMC}} \circ F_{\Sigma\text{-net}} \circ G_{\mathrm{pet}}$.

Proof. Let N be a Petri net and C a commutative monoidal category; since G_{pet} is fully faithful we have natural isomorphisms

$$\begin{aligned}\mathsf{Petri}(N, U_{\mathsf{Petri}}(C)) &\cong \Sigma\text{-}\mathsf{net}(G_{\mathrm{pet}}(N), G_{\mathrm{pet}}(U_{\mathsf{Petri}}(C)))\\ &\cong \Sigma\text{-}\mathsf{net}(G_{\mathrm{pet}}(N), U_{\Sigma\text{-net}}(U_{\mathsf{SSMC}}(C)))\\ &\cong \mathsf{SSMC}(F_{\Sigma\text{-net}}(G_{\mathrm{pet}}(N)), U_{\mathsf{SSMC}}(C))\\ &\cong \mathsf{CMC}(F_{\mathsf{SSMC}}(F_{\Sigma\text{-net}}(G_{\mathrm{pet}}(N))), C).\end{aligned}$$

Thus $F_{\mathsf{SSMC}} \circ F_{\Sigma\text{-net}} \circ G_{\mathrm{pet}}$ is left adjoint to U_{Petri}, hence isomorphic to F_{Petri}. □

VIII. Relation to whole-grain Petri nets

We now clarify the relation of our work to Kock's "whole-grain Petri nets" [18]. We show that a whole-grain Petri net can be thought of as a special sort of Σ-net: one that is free on a pre-net. We first recall Kock's definition:

Definition VIII.1. A **whole-grain Petri net** is a diagram

$$S \longleftarrow I \longrightarrow T \longleftarrow O \longrightarrow S$$

in which the fibers of the functions $I \to T$ and $O \to T$ are finite. A morphism of whole-grain Petri nets, sometimes called an *etale map*, is a diagram

$$\begin{array}{ccccccccc} S & \longleftarrow & I & \longrightarrow & T & \longleftarrow & O & \longrightarrow & S \\ \downarrow & & \downarrow & \lrcorner & \downarrow & \llcorner & \downarrow & & \downarrow \\ S' & \longleftarrow & I' & \longrightarrow & T' & \longleftarrow & O' & \longrightarrow & S' \end{array}$$

This defines the category WGPet.

Theorem VIII.2. *The category* WGPet *is equivalent to the full image of* $F_{\mathrm{pre}} \colon \mathsf{PreNet} \to \Sigma\text{-}\mathsf{net}$. *In other words, there are functors*

$$\mathsf{PreNet} \xrightarrow{Z_1} \mathsf{WGPet} \xrightarrow{Z_2} \Sigma\text{-}\mathsf{net}$$

such that Z_1 *is essentially surjective,* Z_2 *is fully faithful, and the composite* $Z_2 \circ Z_1$ *is isomorphic to* F_{pre}.

Proof. Given a pre-net $s, t : T \to S^* \times S^*$, let I be the set of transitions $u \in T$ equipped with a choice of an element of $s(u)$, and define O similarly using $t(u)$. There are forgetful functions $I \to T$ and $O \to T$, and maps $I \to S$ (resp. $O \to S$) that select the chosen element of $s(u)$ (resp. $t(u)$). This defines a whole-grain Petri net $Z_1(s, t)$. Note that the fibers of $I \to T$ and $O \to T$ are not just finite but equipped with a linear ordering, and the morphisms in the image of Z_1 (which is faithful) are precisely those that preserve these orderings.

To see that Z_1 is essentially surjective, given a whole-grain Petri net N we choose linear orderings on each fiber of the maps $I \to T$ and $O \to T$. These orderings associated each element of T to two elements of S^*, yielding a pre-net whose image under Z_1 is isomorphic to N.

Now, given a whole-grain Petri net $S \leftarrow I \to T \leftarrow O \to S$, we define a Σ-net in the presheaf perspective. Its set of places is S, and its (m, n)-transitions are elements $u \in T$ equipped with a linear ordering on the fibers of I and O over u, which we require to have m and n elements respectively. These linear orderings enable us to define the source and target maps picking out places, while the permutations act on the linear orderings. This defines the functor Z_2.

Note that the set T in a whole-grain Petri net N is naturally isomorphic to the set of transition *classes* of $Z_2(N)$. Thus, to show that Z_2 is fully faithful it remains to show that a morphism $\alpha : Z_2(N) \to Z_2(N')$ uniquely determines the maps $I \to I'$ and $O \to O'$. Given $i \in I$ lying over $t \in T$, choose any ordering on the fibers over t, in which i appears as the k^{th} element of its fiber. This choice determines a transition $\hat{t}$ of $Z_2(N)$, and hence a transition $\alpha(\hat{t})$ of $Z_2(N')$, which is an element $\breve{\alpha}(\hat{t})$ of T' with ordered fibers. Then the function $I \to I'$ can and must send i to the k^{th} element of the I-fiber over $\breve{\alpha}(\hat{t})$. This is independent of the choice of ordering because α commutes with the permutation actions, and it is straightforward to check that it indeed defines a morphism $N \to N'$.

Finally, the composite $Z_2 \circ Z_1$ preserves the set of places and replaces each (m, n)-transition by $m!\, n!$ transitions with free permutation action; but this is the same as F_{pre}. □

Another construction of the functor Z_2 appears in [18], as a restricted Yoneda embedding or "nerve". Recall the categories C and D from Proposition IV.3 and Theorem VI.1. In fact D is the full image of the composite of the Yoneda embedding $C \hookrightarrow [C^{\mathrm{op}}, \mathsf{Set}] \simeq \mathsf{PreNet}$ with $Z_1 : \mathsf{PreNet} \to \mathsf{WGPet}$; we can then define Z_2 as the composite $\mathsf{WGPet} \to [\mathsf{WGPet}^{\mathrm{op}}, \mathsf{Set}] \to [D^{\mathrm{op}}, \mathsf{Set}] \simeq \Sigma\text{-}\mathsf{net}$. In addition, Theorem VIII.2 can be viewed as a two-sided version of the relationship between symmetric and nonsymmetric collections, as in [16, 2.4.4].

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IX. Conclusion and future work

In this work we have systematized the theory of Petri nets, their variants, and their categorical semantics. To this end, we have shown that the notion of Σ-net, almost absent from standard Petri net literature, is in fact central. Our framework gives a consistent view of the relations between these interacting notions of net in terms of adjunctions, such that the most important adjunctions present in the literature can be recovered as composites of our fundamental ones.

Our work makes substantial use of tools from homotopy theory and related fields, such as groupoids and fibrations. We believe this will open up exciting new directions of research in the study of distributed systems and network theory in general.

In fact, the relationships between the various notions of net in our work have analogues in topology. On one hand, a manifold can always be given "local coordinates", but it is too restrictive to ask that such coordinates be preserved strictly by maps between manifolds. Such coordinates can be regarded as analogous to the orderings on sources and targets in a pre-net. On the other hand, when a group acts on a manifold, the quotient topological space may no longer be a manifold, but has singularities at points of non-free action. This "coarse moduli space" can be regarded as analogous to a Petri net, where symmetry information has been lost. Kock's whole-grain Petri nets are analogous to abstract manifolds themselves: they are free of undesirable "coordinates", but neither can they have singularities. Finally, our Σ-nets play the role of *orbifolds*, coordinate-free manifold-like structures that retain the information of "isotropy groups" at singular points, yielding a better-behaved notion of quotient.

References

[1] J. C. Baez and J. Master, Open Petri nets, *Math. Str. Comp. Sci.* **30** (2020), 314–341. Available as arXiv:1808.05415. (Referred to on page 1, 2, 3, 6.)

[2] P. Baldan, R. Bruni, U. Montanari, Pre-nets, read arcs and unfolding: a functorial presentation, in *Recent Trends in Algebraic Development Techniques*, Springer, Berlin, 2003, pp. 145–164. (Referred to on page 3.)

[3] M. Batanin and M. Markl, Centers and homotopy centers in enriched monoidal categories, *Adv. Math.* **23** (2012), 1811—1858. Available as arXiv:1109.4084. (Referred to on page 10.)

[4] R. Bruni, J. Meseguer, U. Montanari and V. Sassone, Functorial semantics for Petri nets under the individual token philosophy, *Electron. Notes Theor. Comput. Sci.* **29** (1999). (Referred to on page 4.)

[5] R. Bruni, J. Meseguer, U. Montanari and V. Sassone, Functorial models for Petri nets, *Information and Computation* **170** (2001), 207–236. (Referred to on page 1, 2, 3, 4.)

[6] E. Cheng, A direct proof that the category of 3-computads is not cartesian closed. Available as arXiv:1209.0414. (Referred to on page 4.)

[7] P. Degano, J. Meseguer and U. Montanari, Axiomatizing net computations and processes, in *Logic in Computer Science, 1989*, IEEE, New Jersey, pp. 175–185. Available at https://www.computer.org/csdl/proceedings/lics/. (Referred to on page 1.)

[8] H. Ehrig, M. Pfender and H.J. Schneider, Graph-grammars: An algebraic approach, in *Switching and Automata Theory, 1973. SWAT'08. IEEE Conference Record of 14th Annual Symposium*, IEEE, 1973, pp. 167–180. (Referred to on page 6.)

[9] N. Gambino and A. Joyal, On operads, bimodules and analytic functors, *Memoirs AMS* **249**, 2017. Available as arXiv:1405.7270. (Referred to on page 5.)

[10] R. Garner and I. López Franco, Commutativity, *Jour. Pure Appl. Algebra* **220** (2016), 1707–1751. Available as arXiv:1507.08710. (Referred to on page 11.)

[11] F. Genovese, A. Gryzlov, J. Herold, M. Perone, E. Post and A. Videla, Computational Petri nets: adjunctions considered harmful. Available as arXiv:1904.12974. (Referred to on page 1, 4.)

[12] F. Genovese and J. Herold, Integer Petri nets are compact closed categories, *Electron. Proc. Theor. Comput. Sci.* **287** (2019), 127–144. Available as arXiv:1805.05988. (Referred to on page 1.)

[13] C. Girault and R. Valk, *Petri Nets for Systems Engineering: a Guide to Modeling, Verification, and Applications*, Springer, Berlin, 2013. (Referred to on page 1.)

[14] R. J. van Glabbeek and G. D. Plotkin, Configuration structures, event structures and Petri nets, *Theoretical Computer Science* **410** (2009), 4111–4159. Available as arXiv:0912.4023. (Referred to on page 1.)

[15] A. Joyal and R. Street. The geometry of tensor calculus, I. *Adv. Math.* **88** (1991), 55–112. (Referred to on page 3.)

[16] J. Kock, Polynomial functors and trees, *Int. Math. Res. Not.* **3** (2011), 609–673. Available as arXiv:0807.2874. (Referred to on page 12.)

[17] J. Kock, Graphs, hypergraphs, and properads, *Collectanea Mathematica* **67** (2016), 155–190. Available as arXiv:1407.3744. (Referred to on page 6.)

[18] J. Kock, Elements of Petri nets and processes. Available as arXiv:2005.05108. (Referred to on page 2, 10, 12.)

[19] S. Lack, Note on the construction of free monoids, *Appl. Cat. Str.* **18** (2010), 17–29. Available as arXiv:0802.1946. (Referred to on page 11.)

[20] S. Lack and P. Sobociński, Adhesive categories, in *International Conference on Foundations of Software Science and Computation Structures*, Springer, Berlin, 2004, pp. 273–288. (Referred to on page 6.)

[21] S. Lack and P. Sobociński, Toposes are adhesive, in *International Conference on Graph Transformations*, Lecture Notes in Computer Science **4178**, Springer, Berlin, 2006, pp. 184–198. (Referred to on page 6.)

[22] F. Loregian and E. Riehl, Categorical notions of fibration. Available as arXiv:1806.06129. (Referred to on page 7.)

[23] M. Makkai and M. Zawadowski, The category of 3-computads is not cartesian closed, *J. Pure Appl. Alg.* **212** (2008), 2543–2546. Available as arXiv:0710.5202. (Referred to on page 4.)

[24] J. Master, Petri nets based on Lawvere theories, *Math. Struct. Comp. Sci.* **30** (2020), 833–864. Available as arXiv:1904.09091. (Referred to on page 1, 2, 3, 4, 10.)

[25] J. Meseguer and U. Montanari, Petri nets are monoids, *Information and Computation* **88** (1990), 105–155. (Referred to on page 1, 2.)

[26] E. Palmgren, Groupoids and local cartesian closure, Uppsala University Department of Mathematics Technical Report **21** (2003). Available at http://www2.math.uu.se/∼palmgren/gpdlcc.pdf. (Referred to on page 7.)

[27] J. L. Peterson, *Petri Net Theory and the Modeling of Systems*, Prentice–Hall, New Jersey, 1981. (Referred to on page 1.)

[28] V. Sassone, Strong concatenable processes: an approach to the category of Petri net computations, *BRICS Report Series*, Dept. of Computer Science, U. Aarhus, 1994. Available at https://tidsskrift.dk/brics/article/view/21610/19059. (Referred to on page 1, 4, 5.)

[29] V. Sassone, On the category of Petri net computations, in *CAAP'92: 17th Colloquium on Trees in Algebra and Programming*, Lecture Notes in Computer Science **581**, Springer, Berlin, 1992. Available at https://eprints.soton.ac.uk/261951/1/strong-conf.pdf. (Referred to on page 1.)

[30] V. Sassone, An axiomatization of the algebra of Petri net concatenable processes, *Theor. Comput. Sci.* **170** (1996), 277–296. Available at https://eprints.soton.ac.uk/261820/1/P-of-N-Off.pdf. (Referred to on page 1.)

[31] Statebox Team, The mathematical specification of the Statebox language. Available as arXiv:1906.07629. (Referred to on page 4.)

[32] N. P. Strickland, $K(n)$-local duality for finite groups and groupoids, *Topology* **39** (2000), 733–772. (Referred to on page 7.)

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