"Critical philosophy begins at the very point where logistic leaves off": Cassirer's Response to Frege and Russell

Jeremy Heis
University of California, Irvine

This paper concerns the reception of the new logic and logicism of Frege and Russell among the German Neo-Kantian philosophers of the Marburg school. Paul Natorp and Ernst Cassirer both object to Frege's and Russell's conception of the ground, nature, and demarcation of logic. This objection depends on the distinctive Marburg school philosophy of logic, first articulated by Hermann Cohen: that "formal" logic (in Kant's sense) depends on a logical investigation of the principles that make mathematical natural science possible. Nevertheless, Cassirer thought that the new logic had profound philosophical implications. First, the new logic provides a technical vindication of some Neo-Kantian theses concerning space and time and mathematical proofs. Second, Cassirer thinks that the new logic allows for a conception of mathematics as the study of relational structures. This conception, in concert with a Kantian notion of objectivity and objecthood, makes possible a justification of modern mathematics.

According to Michael Friedman, Ernst Cassirer's "outstanding contribution [to Neo-Kantianism] was to articulate, for the first time, a clear and coherent conception of formal logic within the context of the Marburg School" (Friedman 2000, p. 30). In his paper "Kant und die moderne Mathematik" (1907), Cassirer argued not only that the new relational logic of Frege[1] and Russell was a major breakthrough with profound

I would like to thank André Carus, Lisa Downing, Scott Edgar, David Hyder, Penelope Maddy, and especially Erich Reck for comments and conversations about the content of this paper. Earlier versions were presented at the Seventh International Conference on the History of Philosophy of Science (HOPOS 2008) and at the University of Nevada, Las Vegas, where I benefited greatly from audience questions and comments.

1. In the paper, Cassirer only discusses Russell and Couturat. He seems not to have read

Perspectives on Science 2010, vol. 18, no. 4
©2010 by The Massachusetts Institute of Technology

philosophical implications, but also that the logicist thesis itself was a "fact" of modern mathematics. Cassirer summarizes his evaluation of Russell's work:

> Here logic and mathematics have been fused into a true, henceforth indissoluble unity; and from this inner connection there arises for each a new concept of its task and of its object. Mathematics is no longer—what it had been considered for centuries—the science of magnitude and number, henceforth it applies to all contents in which complete lawlike certainty and strict deductive combination is within reach. Analogously logic, which up to now hardly observed any other relation than the subsumption of a determinate subject under a more extensive predicate concept, is experiencing a renewal and expansion of its content: it is being expanded to a general logic of relation, which analyzes the various possible types of relation and resolves them to their formal moments. (1907, p. 4)

This paper was significant for Cassirer: it was only his second non-historical publication, and it was the first announcement and working out of the project that became, three years later, his book *Substanzbegriff und Funktionsbegriff*—a systematic work in epistemology and the philosophy of the exact sciences that is arguably the high point of Marburg school Neo-Kantianism.

This paper was also significant in the history of logicism. "Kant und die moderne Mathematik" is a fifty page review of Russell's *Principles of Mathematics* and of Couturat's logicist writings for the readers of *Kant-Studien.* It was among the first discussions of Russell's logicism and logic of relations among German philosophers (Grattan-Guinness 2000, pp. 368, 463–4). Cassirer's interest also inspired his teacher Paul Natorp, whose critical discussion of Russell and Frege largely overlaps with Cassirer's own (Natorp 1910). A. W. Carus (2008, p. 97) has discovered that Carnap's earliest extant philosophical writing endorses a criticism of Frege and Russell given by Cassirer and Natorp. But the reception among contemporary historians has not been so positive. Carus sees significant misunderstandings in Natorp's criticisms of Frege and Russell. And in a recent discussion of Cassirer's reception of logicism (Pulkkinen 2001; cp. Thiel 1997), we read that the debate between Cassirer and Russell is merely verbal, and of no philosophical interest.

On the contrary, as I hope to show, Cassirer and Natorp direct their attacks at one of the most significant (and potentially most vulnerable) fea-

---

Frege until writing *Substanzbegriff und Funktionsbegriff* (translated somewhat misleadingly as "Substance and Function") in 1910.

tures of Frege's and Russell's work: their conception of the ground, nature, and demarcation of logic.[2] Cassirer and Natorp think that Frege and Russell have no explanation of why the principles in Russell's *Principles* or Frege's *Grundgesetze* are true, why these principles get to count as "logical" while others do not, and, even more basically, how the propositions of their logics get to have content at all.

Cassirer's discussions of Frege and Russell take place within the context of the Marburg school of Neo-Kantianism, under whose leading representatives—Hermann Cohen and Paul Natorp[3]—Cassirer had studied. For this reason, this paper begins (in section I) with a fundamental tenet shared by the Marburg Neo-Kantians: that what Kant calls "formal logic" depends on, and is ultimately not separable from, "transcendental logic"—the investigation of the principles that make mathematical natural science possible. In section II, we see that this fundamental tenet in the philosophy of logic is motivated by another fundamental idea: that there cannot be an absolute distinction between "sensibility" (and its forms, space and time) and the "understanding" (and its forms, the forms of judgment and the categories). Given this philosophical conception of logic, it is not surprising that Natorp criticized Russell and Frege for failing to properly ground the new logic in an analysis of the exact sciences (section III).

What is surprising, though, is that Cassirer was so strongly enthusiastic about the philosophical implications of the new logic and the new logicism. Cassirer appreciated that many distinctive theses in the Marburg Neo-Kantian philosophy of mathematics—that arithmetic is independent of properties of time, and that geometrical proofs do not depend on intuition—are given technical proofs in the new logic (section IV). More-

2. Natorp and Cassirer (particularly in his earlier writings) do not always appreciate the differences between Russell's philosophy of logic and mathematics and Frege's. Furthermore, as André Carus has pointed out to me, Natorp in particular often treats Frege separately and more charitably than he does Russell. In this paper, I do not always note when Natorp or Cassirer give a criticism or reading of either Russell or Frege that they do not intend to apply to both. (A fuller discussion would require making the distinctions that I elide here.) Additionally, though I try to interpret Natorp and Cassirer as charitably as possible, I do not consider explicitly whether their readings of Frege and Russell are correct, nor whether their criticisms are just.

3. Natorp and Cohen both taught at Marburg. Cohen habilitated there in 1871 and Natorp a few years later in 1878. Cohen was Cassirer's *Doktorvater.*

Cassirer came to reject some key elements of Marburg Neo-Kantianism in the late teens and early 1920s, culminating in his three volume *Philosophy of Symbolic Forms.* Some of Cassirer's criticisms of Russell's and Frege's logicism in this mature period are significant advances over the earlier criticisms I'm canvassing in this paper. On Cassirer's mature philosophy of mathematics, see Heis forthcoming.

over, as I argue in section V, Cassirer thinks that the new logic and the new logical conception of mathematics defended by Frege and Russell allow a Kantian philosopher to give a philosophical justification for the new "freedom" enjoyed by nineteenth-century mathematicians. This philosophical justification of modern mathematics depends on Cassirer's unique marriage of a Kantian conception of objectivity and objecthood with a generally Russellian conception of mathematics as the study, within the logic of relations, of various relational structures. Nevertheless, though Cassirer recognizes the revolutionary implications of the new logic and logicism, he thinks that the philosophy of logic and mathematics that Russell and Frege attach to their technical projects is in some respects incomplete and in some respects confused (section VI). Having no story of the necessary applicability of mathematics and the new logic in natural science, Cassirer thinks, Frege and Russell cannot make use of Kant's good idea that mathematical judgments get their truth and contentfulness from their use in our experience of empirical objects. This gap in Frege's and Russell's philosophies of logic is exacerbated by a platonizing dualism, and, in Russell's case, by an epistemology of logic based on acquaintance.[4]

I. Cohen and Natorp on Transcendental and Formal Logic

The philosophy of logic defended by Cohen and Natorp is best understood as a reaction to Kant's own. For Kant, pure, general or *formal* logic "abstracts [. . .] from all content of cognition, i.e., from any relation of it to the object, and considers only the logical form in the relation of cognitions to one another, i.e., the form of thinking in general" (A55/B79). Transcendental logic, on the other hand, "expounds the elements of the pure cognition of the understanding and the principles without which no object can be thought at all" (A62/B87). Cohen famously argued that formal logic, if it exists at all, cannot be an independent science, but must ultimately depend on transcendental logic. General logic distinguishes between different kinds of judgments and gives rules for the correct use of these different kinds of judgments in syllogisms. But, Cohen asks, "What insures the correctness of the kinds of judgments that formal logic is concerned to distinguish?" (1885, pp. 241–2). Cohen counsels that we read the "Transcendental Analytic" backwards—for Kant "the way did not lead from the categories to the principles, but from the principles out to the categories" (1885, p. 408).[5] Kant's "principles of the pure understanding" (like the

4. Cassirer and Natorp also give detailed criticisms of Frege's and Russell's definitions of number in terms of concept extensions (or classes). See Heis 2007, ch. 3.

5. This explains the characteristic hostility within Neo-Kantianism (and post-Kantian philosophy generally) to the Metaphysical Deduction of the categories from the forms of judgment. Cassirer in his 1918 work *Kant's Life and Thought,* writes:

persistence of substance and the principle of causality) are in fact the most basic laws of experience, those laws without which experience is not possible. In the *Critique,* Cohen is suggesting, we begin by taking the existence of mathematics and mathematical natural science as a given, and we then reflect on its presuppositions in order to isolate these fundamental principles. These principles give us the most basic concepts, the categories, which we then use as a guide in distinguishing the kinds of judgments treated of by formal logic. The ultimate justification of the laws of *general logic* (and the principle for isolating and distinguishing the forms treated of by general logic) is found in *transcendental logic's* analysis of the exact sciences themselves. The consequence is the "dependence of logic on the fundamental forms of mathematical natural science" (Cassirer 1912, p. 261).

This dependence of logic is an application of what the Marburg Neo-Kantians called the "transcendental method" or the method of "transcendental logic."[6] According to this approach, philosophy begins not with metaphysical speculation, nor with observation of the psychology of individual human subjects, but with our best current mathematical sciences of nature. These sciences are our paradigms of knowledge and constitute the "fact" whose preconditions it is the task of philosophy to study.[7] No attempt is made to understand the nature of reality, or of thought, in isolation from reflecting on the nature of things as revealed to us by exact science and on the nature of thought as it is expressed in exact science. Now logic, according to Kant, is the science of the rules of *thinking.* But how could a philosopher know what thinking is really like or how it ought to be executed (Cohen 1902b, p. 17)? Certainly not by introspection, nor by a physiological investigation of human psychology.[8] Without an investiga-

---

> While general logic can similarly be employed as the "clue to the discovery of all the pure concepts of the understanding," this is not done with the aim of basing the transcendental concepts on the formal ones, but conversely with the aim of basing the latter on the former, and in that way yielding a more profound understanding of the ultimate ground of their validity. ([1918] 1981, p. 173; cp. [1907] 1922, p. 625)

For a recent critical discussion of this reading of Kant, see Longuenesse 1998, pp. 3, 5.

6. See Cohen 1885, p. 66ff; Cassirer [1906] 1922, pp. 14–18; Cassirer 1912; Natorp 1912; Kinkel [1904] 1920, p. xvi. A recent discussion is Richardson 2006.

7. On the Marburg reading of Kant, Kant first isolated the transcendental method and applied it to Newtonian science; in fact, he mistakenly thought that the transcendental preconditions of Newtonian science were the fixed preconditions for all scientific cognition in all times (Cassirer [1906] 1922, p. 18).

8. Cohen ("Vorwort" in Cohen 1902a) claims that Helmholtz and Friedrich Lange mistakenly try to carry out the project of transcendental logic using psychological facts about the psychophysical constitution of the human mind. See also Cassirer 1912 and Anderson 2005.

tion of thinking in its paradigmatic shape in our mathematical sciences, we can make no progress in determining what thinking is like or what rule-governed method thinking is to follow.

However, once we see that the thinking whose rules logic investigates just is the kind of thinking done in mathematical natural science, the distinction between Kant's pure general logic and the logic of the natural sciences dissolves (Natorp [1911] 1918, p. 45). There is no such distinction between formal and material logic; there is only one logic, properly so called: transcendental logic. Cohen very forcefully draws this conclusion in his late systematic work *Die Logik der reinen Erkenntnis,* a work which

> takes care of that tiresome [leidigen] distinction between a formal and a material [sachliche] logic; let the latter be now *metaphysic, critique of cognition,* or even the *methodology* of the sciences that is incorporated into them. *What is not material is also not formal.* . . . The logic of judgment generates the categories, as pure cognitions, formally from judgment. But these are the materials [Sachen], which in particular constitute the content [Inhalt und Gehalt] of mathematical natural science. The formal judgment generates the material foundation, as the presupposition of science. (1902b, p. 500–1)

II. The Marburg rejection of the sensibility/understanding distinction, and a consequence

The Marburg school's rejection of Kant's distinction between formal and transcendental logic follows from their rejection of another fundamental Kantian dualism. Cassirer, Natorp, and Cohen were famously hostile toward the idea that sensibility—the capacity for objects to be given to us through their affecting us—is separable from or makes a contribution to our knowledge that is independent of the understanding, our capacity to think.[9] This hostility expresses itself in their forceful denials that *sensations* can be both non-conceptual and guide our thinking—that *causal* relations among *subjective* states could of their own make *normative* or *objective* claims on our thinking (Cassirer [1910] 1923, ch. 1). There is not and cannot be something "given" in an experience (Cohen 1902b, pp. 24–5, 48–51). So Cohen argues that, if thinking is synthesis—producing a unity from an initial multiplicity—the multiplicity itself, which provides the content for our thinking, has to be *generated* by thought itself (1902b, pp. 9, 19–25, 48–52). And Natorp writes that if in synthesizing we are relating different relata, we need to be sure that the *termini* of the relata are themselves generated by the relation itself (1910, pp. 99, 103).

9. See here the very helpful discussions in Friedman 2000, ch. 3 and Kim 2003.

Our present discussion does not require that we take on the difficult task of making these ideas precise. Nevertheless, this rejection of the distinction between sensibility and understanding has a consequence of fundamental importance for Cassirer's peculiar response to Frege and Russell. Logic, for Kant, is the science of the forms of the understanding; it is distinguished from Transcendental Aesthetic, an independent investigation of the forms of sensibility, namely space and time (A52/B76). Over against the view of pure intuition as an independent source of cognitions, Walter Kinkel—a student of Cohen's whose introduction to his edition of Kant's *Logik* Cassirer read and cited—follows Cohen in seeing the understanding, as the transcendental unity of apperception, as the source not only of the categories, but also of space and time.[10] Thus, though Kant failed to draw this final conclusion, his own arguments should have driven him to conclude that *both* the categories *and* space and time are "purely conceptual" in nature. But if even space and time, Kant's "pure intuitions," are finally to be reckoned as conceptual representations, it is easy to see that we could never succeed in marking off a distinct discipline that concerns the faculty of concepts in isolation from a distinct faculty of intuitions. "This special position, which [Kant] claimed for space and time, as forms of intuition, prompted him to make that mistaken separation of general from transcendental logic; there is only one, comprehensive logic, which has to do with the possibility of experience: transcendental logic" (Kinkel [1904] 1920, pp. xvi–xvii).[11] Kant claims (A55/B79–80) that a transcendental logic distinct from formal logic is possible only because there are pure intuitions of space and time. But if space and time themselves depend on the understanding, we cannot really distinguish the forms of "sensibility" from the forms of the understanding, and the distinction between transcendental and formal logic collapses.

We should have expected this anyway, even without the detour through reflecting on space and time. For Kant, the possibility of marking out formal logic as a distinct discipline from transcendental logic requires that we consider the understanding (or "thinking") and its rules in complete abstraction from sensibility—and this in the strong sense that we consider thinking in complete isolation from the fact that it relates to objects. However, from the Marburg point of view, there can be no strict distinction between receptivity, "as a particular, distinct way in which objects

10. The Marburg Neo-Kantians saw this as Kant's considered view at the end of the discussion of space and time in the B Transcendental Deduction. See Kinkel [1904] 1920, pp. xiv–xvi on *Critique of Pure Reason* sections 24 and 26, esp., B160–1n. See also Cassirer 1907, p. 35, and the recent discussion of Cassirer's reading in Longuenesse 2005, p. 66ff.

11. This reading of Kant is severely criticized by Klaus Reich ([1932] 1992, p. 116, note).

can be given" (Natorp 1910, p. 201) and spontaneity or "thinking." And if the distinction between sensibility and the understanding cannot be maintained, neither can the distinction between transcendental and formal logic.

III. Natorp's Criticism of Russell's "Formal" Logic

These two distinctive views, first of the relation between formal and transcendental logic, and, second, of the intellectual status of space, make all the difference in Natorp's and Cassirer's responses to Frege and Russell. Natorp thinks that these two views make trouble not only for traditional Aristotelian formal logic, but even for the new logic of Russell, Frege, and Couturat.[12] For Natorp, the significant question for the logicist is this: "Has therefore mathematics not become one, rather *the* logic, a logic entirely of the kind and in the general sense of the old formal logic, only with a much wider scope?" (Natorp 1910, pp. 4–5) Natorp's statement concludes that, although the new logic has made great technical progress, in its fundamental orientation it is does not differ from formal logic "in the old, Aristotelian and perhaps Wolffian sense" (1912, p. 196). The *quid juris?* of the fundamental concepts and principles of logistic must, on Natorp's view, reside in a transcendental analysis of the exact sciences. Such an analysis would show that the principles in Russell's *Principles* are conditions of the possibility of (natural scientific) experience in the same sense that Kant's Principles—the principle of causality, the permanence of substance—would be. Similarly, such an analysis would show that the concepts of the new logic—like the concept of relation or function—are just as necessary for mathematical, natural science as Kantian categories would be. Without appealing to the exact sciences themselves, Frege and Russell have no justification for the truth or correctness of their logic—indeed, they have no justification for claiming that the sentences of logistic have content at all. Having forsaken that, Natorp argues, the only alternatives for the logicists are to ground their logic in brute facts about human psychology—a position Natorp realizes is the exact opposite of Frege and Russell's intentions—or to refuse to ground their concepts and principles at all, and thus let logic slide into empty formalism (Natorp 1910, p. 8). Thus, it is not enough for logicists to claim that they are not interested in being "a logic of 'experience' in the Kantian sense, a logic of natural science" (Natorp [1911] 1918, p. 46), since transcendental logic is the only game in town.[13]

12. Natorp cites these three in his criticism of logicism in Natorp 1910, p. 3 and singles out Russell 1903 as the most significant expression of "modern logistic" in Natorp [1911] 1918, p. 46.

13. Natorp sometimes, rephrasing his criticism, argues that the only grounding that

IV. The significance of the new logic for Cassirer: the independence of mathematics from space and time

As we'll see in section VI, Cassirer sympathizes with Natorp's fundamental criticism of Russell and Frege. Nevertheless, Cassirer sees the real philosophical and mathematical import of the new logic and the new logicism. Here is how Cassirer characterizes the importance of the new logic:

> However you may think about its significance for the entirety of philosophical problems, it cannot be denied that ["logistic"] has renewed 'formal logic' and has filled it in turn with the lifeblood of science. Logistic . . . can never displace or set aside 'transcendental' logic; but it is not to be doubted that it serves as a rich stimulus for the individual critical problems in their modern form and contains a more secure 'leading thread' than Kant possessed in the traditional logic of his time. (1907, p. 8)

I see Cassirer arguing in two ways that the new relational logic serves as a "more secure leading thread" for philosophy. First, the new logic allows us to give actual mathematical proofs that mathematics is independent of the pure sensible representations of space and time. Second, it gives us a mathematical way of extending Kant's idea that "concepts rest on functions" (A68/B93). I'll begin with the first way in this section, and discuss the second way in the next section (section V).

As we've seen, Cohen and Natorp's rejection of the given in experience—that our sensibility makes an independent contribution to our knowledge—led them to argue that space and time, the forms of sensibility, are just as much intellectual representations as the categories. For this reason, Cohen and Natorp argue against the inferential use of properties of space and time in mathematical proofs and against an essential ap-

---

Frege and Russell could have for the contentfulness and truth of the new logic would be a *circular* justification. See 1910, pp. 5, 8. The circularity charge also appears in Natorp's criticism of Frege's extension-theoretic definition of number (1910, p. 114, 115, 118, 123). (Similarly, when Cassirer criticizes Frege's definition of number—from his earliest discussions (Cassirer [1910] 1923, pp. 46, 50, 51) to his latest (Cassirer [1929] 1957, p. 378; Cassirer 1950, pp. 61–6)—he accuses it of circularity.) Thiel (1997) rightly points out that Natorp sometimes charges Frege of committing a technical circularity, and that in those places Natorp has simply misunderstood the technical details of Frege's definitions. Thiel, however, does not note that a significant part of Natorp's circularity charge is directed against Frege's and Russell's philosophy of logic. (Carus [2008, pp. 99–101] helpfully sees Natorp's objections as fundamentally directed against Frege's "conceptual realism." I think, however, that Natorp's criticisms can be read still more charitably than Carus does.)

peal to time in arithmetic. Now the independence of arithmetic from temporal notions and of mathematical proofs from anything besides logical laws were, of course, among the most important claims Russell thought himself to have proved in *Principles of Mathematics.*

> There was, until very lately, a special difficulty in the principles of mathematics. It seemed plain that mathematics consists of deductions, and yet the orthodox accounts of deduction were largely or wholly inapplicable to existing mathematics. [. . .] In this fact lay the strength of the Kantian view, which asserted that mathematical reasoning is not strictly formal, but always uses intuitions, i.e., the *a priori* knowledge of space and time. Thanks to the progress of Symbolic Logic, especially as treated by Professor Peano, this part of the Kantian philosophy is now capable of a final and irrevocable refutation. (Russell 1903, §4)

Needless to say, though, if your Kantian philosophy is that of Cohen, Natorp, and Cassirer, this progress in Symbolic Logic will not subject Kantian philosophy to final refutation. On the contrary, Cassirer writes:

> And it is important and significant, that the immanent development [Fortbildung] of the Kantian doctrine has led of itself to the same result that is more and more clearly demanded by the progress [Fortgang] of science. Just as 'logistic,' so has modern critical logic walked away from Kant's doctrine of pure sensibility. And in fact sensibility signifies an epistemological [erkenntniskritisches] problem, but no more a self-standing and peculiar source of certainty. [Here Cassirer footnotes Cohen's *Logik der reinen Erkenntnis,* pp. 11ff., 128.] Thus it agrees in its fundamental thoughts with the tendency of which the work of Russell and Whitehead is the fulfillment: in the demand of a pure logical derivation of mathematical fundamental principles, through which we first fully understand and learn to master conceptually 'intuition' itself, as well as space and time. (1907, pp. 31–2)

What a Marburg Neo-Kantian should find of use in Russell's work, though, is not just a serendipitous confluence of ideas: he finds actual proofs. Cohen, for example, had argued that geometrical constructions play no inferential role in mathematical proofs, which are purely deductive.

> [W]e cannot view this method [of proof in mathematics] as different [from the deductive one] on account of the fact that the construction makes use of intuition—in the way that Kant, because of

> his esteem for pure intuition, was enabled to take on this view. Already Plato had warned against seeing the proof of the proposition in the drawn figure. And Nicholas of Cusa and Descartes have expressed the same thoughts forcefully. (Cohen 1902b, pp. 477–8)

Cassirer agrees, but his reasons are importantly different. He first traces out the historical evolution of geometrical methods up through the deductive, axiomatic method presented for the first time in Pasch's work, he points out the role played in this development by considerations arising from geometrical duality, and he then finally argues that the demonstration of the purely deductive character of mathematical proofs is complete only now with a richer theory of deduction from Frege and Russell (Cassirer 1907).

Similarly, Natorp had argued that the science of arithmetic is grounded on a series of relations among elements (namely the natural numbers ordered by the successor relation) and that the sequence so defined is not identical to or dependent on temporal sequence (1910, p. 107; cf. Cassirer 1907, p. 34). Natorp understandably takes as an ally the mathematician Richard Dedekind, who defines the numbers as members of the (smallest) chain of elements from some element under a specified relation (Dedekind [1888] 1963). But, from Cassirer's point of view, this argument is incomplete. Dedekind takes as a primitive notion the idea of a "mapping" or "relation" or "coordination," and the door is still open for an objector, in a Kantian spirit, to argue that this coordination is a temporal notion. Russell's calculus of relations blocks this move. Similarly, with Dedekind's purely arithmetical concept of continuity (Dedekind [1872] 1963), a property thought to be essentially spatio-temporal—the continuity of the real number system—is shown to depend entirely on intellectual or logical laws, and an essential property of space—its continuity—turns out not to be required for geometrical constructions after all. Now, this is just as the Marburg idea that space is generated by the act of thinking would suggest. But again, Dedekind's construction provides a mathematical proof for this idea originally pushed for other reasons; the new logic of relations then completes the proof by showing that the ordering relation on the rationals is independent of spatio-temporal notions.

V. The significance of the new logic for Cassirer: concepts as functions and the objectivity of modern mathematics

There is also a more surprising use that Cassirer makes of the new logic of relations. Cassirer sees the richer expressive power of Russell's relational logic as intimately tied to the Kantian idea that concepts "rest on functions."

> Whereas all intuitions, as sensible, rest on affections, concepts rest on *functions.* By 'function' I mean the unity of the *act* of bringing various representations under one common representation. Concepts are based on the *spontaneity* of thought, sensible intuitions on the receptivity of impressions. (A68/B93, emphasis added)

When Kant says that concepts rest on functions, he means that all conceptual representations—indeed, all of our knowledge—ultimately consist in *rule-governed acts* of combining (or "synthesizing") representations into unified wholes.[14] These rule-governed activities of the mind ("functions") include bringing subject concepts under predicate concepts in judgments (as in this passage), but also producing intuitive representations of particular objects out of sensory manifolds (A79/B104–5), forming concepts from intuitions (A78/B103), and even combining all of our knowledge together into a unity of consciousness (A108, B133).

Since in Russell's relational logic functions (in the mathematical sense[15]) are just one-one or many-one relations, Cassirer sometimes will call the new logic of relations—playing up the Kantian overtones—the logic of *functions.* There is thus a double allusion in the title of Cassirer's *Substanzbegriff und Funktionsbegriff* ["Substance-concept and Function-concept"]: Kant's theory of concepts as functions and Frege's and Russell's new logic of relations. Indeed, we've understood the fundamental argument of this book when we see why Cassirer saw such a tight connection between, on the one hand, Kant's theory of concepts and, on the other hand, the nineteenth century development of a new, more abstract mathematics that made possible the foundation of mathematics in the enriched new logic.

Cassirer announces his all-important distinction between substance and function in his two writings from 1907—three years before *Substance and Function.* First, in "Kant und die moderne Mathematik," he introduces the distinction as following from the work of Russell:

> [In the pre-Russellian logic] logic remains in the viewpoint of substance and thereby bound to the fundamental form of the judgment

14. "Hence we say that we cognize an object if we have effected synthetic unity in the manifold of intuition. But this is impossible if the intuition could not have been produced through a function of synthesis in accordance with a rule that makes the reproduction of the manifold necessary *a priori* and a concept in which this manifold is united possible" (A105).

15. Note that in the quotation from A68/B93 Kant's use of "function" is entirely non-mathematical. Cassirer of course recognized that Kant's use of "function" to denote a rule-governed activity is different from Russell's use of "function" in the mathematical sense.

> of predication, while living scientific thought [that is, mathematics itself] more clearly aims at the concept of *function* as its own systematic middle point. One recognizes in this connection the value and necessity of the new foundation on which Russell has sought to found logic. Mathematics in his presentation is nothing other than a special application of the general logic of relations: but the concept of relation goes back to the more fundamental thought of functionality.[16]

Here Cassirer is explaining the greater expressive power of the new logic. The old logic was primarily restricted to categorical judgments (like "Socrates is mortal"), which express the predication of an attribute of a substance. The new logic, on the other hand, can express compound relations and functional dependencies rich enough to capture all of modern mathematics.

The second appearance of the distinction between substance and function is in the long discussion of Kant's *Critique of Pure Reason* that concludes Cassirer's two-volume historical work *Das Erkenntnisproblem.* Characteristically, Cassirer introduces a fundamental insight of Kant's as following from a recognition that Kant had about the development of mathematical natural science:

> For Kant, the critical meaning of reality finds a new confirmation in the progress that exact science itself has made gradually and ever more clearly in the formulation of its task . . . Mathematical physics has progressed from its first beginnings to a new, fundamental point of view. From the essence of things, physics turned to its numerous orders and connections, from its *substantial* inwardness [substantiellen Inneren] to its *functional,* mathematical structure. ([1907] 1922, pp. 670–1)

To see what Cassirer thinks is so important in Kant's philosophy, and what this has to do with the new logic of relations, we'll need to say a bit more about what Cassirer thought to be perhaps the single most important innovation of the *Critique:* a new, non-metaphysical notion of objectivity.

> 'Objective' ['Gegenständlich'], in the critical sense of the word, means that which is 'constant' ['beständig'] in our cognition: but

16. 1907, p. 7. Cassirer here cites Russell's *Principles of Mathematics* (1903, §254), where Russell says that "functionality does not differ from relation"—that is, functions are many-one or one-one relations.

> what is constant is only what is prescribed through the laws of knowledge for all time[17] and therefore does not need to be borrowed from the objects. The traditional concept of an object, as something that is foreign and external to thinking, destroys the objectivity of knowing. . . . For all metaphysical theories presuppose the very thing which is here in question: they start from a world existing [bestehenden] for itself, which is supposed to be brought in some way to consciousness for the I, which for its part is also thought of as a self-subsisting substantial essence. (Cassirer [1907] 1922, p. 639)

This is fundamentally a story about the objectivity of science, which Cassirer finds principally in the B edition Transcendental Deduction. For Cassirer, as for Kant, to say that our cognition is objective or has 'objective validity' is another way of saying that our knowledge relates to an object.[18] But which notion explains which? For Cassirer's Kant, the order of explanation is clear.[19] When we give an account of science's objectivity, we should not do it by saying that there is a world and there are human subjects, and the world causes us to have certain empirical cognitions that are objective, or relate to and are true of objects. It's not that this way of speaking is false: it is just unexplanatory.[20] What makes science objective

17. In this passage, Cassirer is endorsing the basic Kantian idea that the objectivity of our knowledge consists in there being laws of knowledge. But, as I noted above, Cassirer and the other Marburg Neo-Kantians do not think that the laws of Newtonian science (or even our current best science) are "prescribed . . . for all time." Indeed, Cassirer denies that we could ever know with certainty whether the presuppositions of our current science will be presuppositions of all future science.

18. See, for instance, B137. Kant characterizes 'objective validity' in other ways, too.

19. "The 'revolution in method' which Kant brought to theoretical philosophy rests on the fundamental idea that the relation between cognition and its object, generally accepted until then, must be radically modified . . . [T]he crucial question always remains whether we seek to understand the function by the structure or the structure by the function, which one we choose to 'base' upon the other . . . [T]he fundamental principle of critical thinking [is] the principle of the 'primacy' of the function over the object" ([1923] 1955, pp. 78–9). In a late essay (Cassirer [1939] 1993, p. 202), Cassirer says that he is a "Neo-Kantian" fundamentally because he accepts a Kantian conception of objectivity and objecthood.

20. See Cassirer [1910] 1923, pp. 291–2. In his early essay "On the Objective and Subjective Grounding of Knowledge" (1887), Paul Natorp considers the view that "objects in themselves are there, outside and independent of all subjectivity, without any original relation to it." He replies:

> Perhaps this actually means something correct, but at least as it is presented, it is not really an answer to the question [What is objective validity?] The being-in-itself of the object is itself an enigma [Rätsel] and thus cannot serve as a solution to our present enigma. If we understood what it meant to say the object in itself is there independent of all subjectivity and then is appropriated in our subjectivity

is that science has a shared method that it follows, a method that expresses itself in fundamental laws, like the law of causality, that allow our different experimental results and our different scientists to come to agreement with one another. To use Kant's language (B137), if in the concept of an atom or number our representations are united interpersonally and transtemporally, then atoms or numbers are objects and our thoughts about them are objectively valid.[21] The concept of an object thus has no content beyond that of necessary agreement in science.

Cassirer puts this point by claiming, in opposition to Russell's perceptual model for the epistemology of mathematics, that "the 'spontaneity' of thought is thus not the opposite but the necessary correlate of 'objectivity,' which can only be reached by means of it" ([1910] 1923, p. 317). When Kant claims in A68/B93 that concepts rest on functions, he attributes these functions (these rule-governed activities) to the spontaneity of the mind. Thus, on Kant's view, *objectivity* is explained in terms of *functions:* for us to have knowledge of an object is for our various representations to have been combined together into a unity by means of a rule.[22] Stripping this Kantian idea of the psychologistic language of individual minds and their particular acts and representations, we arrive at Cassirer's "Kantian" notion of objectivity: that our thinking (or concept-deployment) is knowledge and represents objects, *because* in our science we are governed by a shared method, or obey laws.[23] In this way Cassirer, in

---

by knowledge, there would be no problem in the knowing of objects or in the objectivity of knowledge. ([1887] 1981, p. 253)

Scott Edgar has pointed out to me that Natorp's notion of objectivity in [1887] 1981 differs subtly from Cassirer's notion, and in ways that would make it harder for Natorp to acknowledge the objectivity of modern mathematics. On Natorp's notion of objectivity, see Edgar 2008.

21. Thus, Cassirer characterizes the objectivity of physical science in this way: "In so far as our intellectual construction is extended and takes up new elements into itself, it appears that it does not proceed according to caprice, but follows a certain law of progress. This law is the ultimate criterion of 'objectivity'; for it shows us that the world-system of physics more and more excludes all the accidents of judgment, such as seem unavoidable from the standpoint of the individual observer, and discovers in their place that necessity that is universally the kernel of the concept of the object" ([1910] 1923, p. 187). Compare the similar comments in Natorp [1887] 1981, p. 261.

22. "Now this concept [of the transcendental object] cannot contain any determinate intuition at all, and therefore concerns nothing but that unity which must be encountered in a manifold of cognition insofar as it stands in relation to an object. This relation, however, is nothing other than the necessary unity of consciousness, thus also of the synthesis of the manifold through a common function of the mind for combining it in one representation" (A109).

23. I speak here of scientists *obeying laws* in their scientific activities; earlier I claimed

appropriating Kant's association of functions and concepts, is expressing this new non-psychologistic and non-metaphysical notion of objectivity.

This notion of objectivity relates back to the opposition to the given in Natorp and Cohen. If there were something independent of our thinking given to us, the objectivity of our cognition would depend on receptivity, on that independent object (a "substance") that acts on us autonomously in sensation, and thinking, if it is to remain objective, would copy or mirror the properties of the object given to us (Cohen 1902b, p. 55). But if there is nothing given to us independently of our thinking, our account of objectivity has to depend on spontaneity (on "functions"): on our science having a shared method that it follows, on there being functions that are actualized in the historical activity of scientists.

Pretty clearly, Russell had none of this in mind. For Cassirer, though, the new logic of relations fits in with the analysis of the transcendental method, objectivity, and concepts as functions that Cassirer takes over from the Marburg school.[24] The transcendental method requires that philosophy take modern mathematics as a fact; it has no privileged position from which it could judge an established result of mathematics as false or meaningless.

> The modern development of mathematics thus created a new 'fact' that the critical philosophy, which does not seek to direct the sciences but to understand them, can no longer overlook. (Cassirer 1907, p. 31)

Now, modern mathematics includes the more exotic varieties of nineteenth century mathematics, like ideal numbers, imaginary points, non-

---

that for the Marburg Neo-Kantians, the shared methods of scientists express themselves in *fundamental laws,* like the law of causality. Cassirer has a story about how the fact that natural scientists have a shared method that allows for inter-personal and trans-temporal agreement depends on the fact that scientists discover *laws of nature.* (Readers of Kant will recognize an echo of the argument of the Second Analogy.) Unfortunately, I won't be able to elaborate on this point here. Space considerations further prevent me from discussing how the way that laws of nature provide for objectivity in natural science differs from the way that laws of mathematics—like the Peano Axioms—provide for objectivity in pure mathematics.

24. Here is another connection between the new logic and the Kantian notion of objectivity. Cassirer is a mathematical "structuralist": he sides with Dedekind's "logicism" and departs from Frege's and Russell's. (For a fuller discussion, see Heis forthcoming and Heis 2007, ch. 3.) But Cassirer thinks this position draws on their work simply because without the new logic it would not be possible to express—with the required abstractness and precision—the various relational structures that mathematicians study. And Cassirer thinks that a Dedekindian "structuralist" philosophy of mathematics would be impossible without the new conception of objectivity and objecthood that Kant introduced.

Euclidean geometries, and transfinite cardinals. This "freedom" of mathematics, to repeat the phrase that Cassirer takes from Cantor (1907, p. 48), would be endangered by the traditional (and orthodox Kantian) conception of mathematics as the science of number and magnitude.[25] And these advances in geometry and number theory would make no sense if we thought of them as tied to the properties of space and time.

Now, Cohen and Natorp were already arguing, on other grounds, that since there is no independent sensibility, space and time are intellectual representations, and that arithmetic is an intellectual science, not dependent on space and time. Russell's logic of relations actually gives a mathematical demonstration that mathematical concepts and inferences can be free from the strictures of space and time. But this demonstration requires the greater expressive power of the new logic of relations—the "logic of functions" not of "substances" in Cassirer's evocative terminology (Cassirer 1907. p. 7). And once we have this new logic of relations in hand, we can develop all of mathematics from a single point of view, as a science of pure relational structures. (See Cassirer 1907, p. 4, quoted at the opening of this paper.) In this science of pure relational structures, all the new, abstract elements that have become essential to modern mathematics will no longer seem to be outliers. We can understand how even transfinite cardinals or n-dimensional complex spaces have a place in the modern mathematical method. On this Kantian notion of objectivity, this is sufficient to secure the objectivity of modern mathematics; moreover, on this Kantian notion of objecthood, this objectivity is precisely what we need to prove that there are such objects.

VI. Cassirer's Criticism of Russell and Frege

For Cassirer, then, the most significant service of the new logic is to present a coherent and systematic conception of mathematical concepts and inferences that is demonstrably sufficient to encompass *all* of modern mathematics without relying on facts about space and time, and without

25. Similarly, on Cassirer's view, the great accomplishment of Kant's critical turn was to secure—through the new notion of objectivity—the right for Newtonian scientists to speak about action at a distance, the continuity of space, and the applicability of the calculus in physics, without having to bring in metaphysical theories about God, freedom, and immortality. On Kant's view, "there cannot be two absolutely divided, heterogeneous kinds of substances [i.e., the "inner" and the "outer", "mind" and "nature"], because the category of substance [Substanz], because the function [Funktion] that this concept serves, is only one; because there thus exists from the very beginning a common forum of cognition, before which both kinds of objectivity belong. The metaphysical theories of physical influx, of pre-established harmony, and the supernatural assistance all together suffer from the same affliction: that they bypass and eliminate this forum" (Cassirer [1907] 1922, p. 725).

restricting its new-found freedom. The new logic provides a new and comprehensive method for introducing mathematical propositions, concepts, and objects, and—given the Kantian conception of objectivity—this goes almost all the way towards quelling the legitimate worries that one might have about mathematical truth or existence. Of course, Frege and Russell wanted more from the laws and objects of mathematics—they wanted them to be genuinely *logical* laws and *logical* objects. Thus, Russell at various times tried to show that his fundamental concepts and principles are logical because they are universal,[26] or formal.[27] Cassirer—true to his Neo-Kantianism and sensitive to the kernel of truth in Natorp's criticism of Frege and Russell—thinks that this project is both unnecessary and confused. It is unnecessary because the philosophical justification of the objectivity and object-directedness of modern mathematics makes no use of the purported fact that the new logic really is "logic." And the project is confused because, as Cohen and Natorp recognized, there is no way to distinguish among the necessary principles of the exact sciences which are "formal" and which are "material": there is "no way . . . back to the 'classical' formal logic."[28]

Although Cassirer thinks that it is idle to worry about whether the new logic is really "logic,"[29] he maintains that there is nevertheless an addi-

26. See Russell [1901] 1957, pp. 70–1; Russell 1903, §7; Frege [1884] 1953, §3.

27. See Russell [1913] 1992, pp. 97–8; Russell 1920, ch. 18. In some of his writings, Russell seems unconcerned about showing that his principles are logical: see Russell 1973. It is fortunately not necessary for us to engage with the large literature on whether Russell or Frege had an overarching conception of logic, and, if so, what this conception was.

28. See Cassirer [1928] 1993, p. 132:

> Nevertheless, for me no way leads from this recognition back to the 'classical' formal logic. For, if this kind of logic, according to the Kantian definition, consists in this, that it "abstracts from all objects of cognition and their differences" [(Bix)]—so the philosophical theory of the concept, which I sought to construct, presents rather the diametrically opposed task. [. . .] A true 'general' logic can therefore only draw on a transcendental logic, that is, draw on a logic of thinking an object. [. . .] [My] intention was [not] to replace the kinds of activity handed down in 'formal logic' by some other new ones: to overturn the old form of this 'logic' in order to put a new one in its place. [. . .] It matters little to me, whether one next to it wants to secure a special task or in some measure a place of honor and security in the system of philosophical cognition for the 'formal' logic, in the old sense of the word.

29. For Michael Friedman, "Cassirer's outstanding contribution was to articulate, for the first time, a clear and coherent conception of formal logic within the context of the Marburg School. Cassirer in *Substance and Function* identifies formal logic with the new theory of relations developed especially by Bertrand Russell in *The Principles of Mathematics*" (2000, p. 30). But Cassirer did not think of himself as identifying the new logic with *formal* logic: indeed, he rejected the very possibility of formal logic.

tional philosophical task that Frege and Russell overlook. From Cohen, Natorp, and Cassirer's Kantian point of view, the only objects in the strict sense are the empirical objects of natural science, and the only kind of objectivity is the objectivity secured by reference to these objects (Cassirer 1907, pp. 42–3). The meaningfulness of mathematics (and of the new logic) therefore depends on the fact that propositions about mathematical objects play an essential role in the sciences of physical objects—that in mathematics, "we are dealing in no sense with some transcendent object, but only with the objective certainty of our empirical knowledge itself" (1907, p. 48). Demonstrating this fact is a task specifically for transcendental philosophy and falls outside of the purview of the technical projects of Frege and Russell.

> Thus a new task begins at the very point where logistic leaves off. What the critical philosophy seeks and what it must demand is a logic of objective [gegenständliche] knowledge. [. . .] The critique of knowledge begins with the question that the mathematician does not know and does not need to know. Its particular problem is not so much the content of mathematical principles as it is the role which they play in the construction of our concept of an 'objective' reality [Wirklichkeit]. (1907, pp. 44, 48)

It would take another paper to discuss how Cassirer thinks he can show the necessary applicability of mathematics in the exact sciences. Cassirer hopes to show that—just as modern, pure mathematics can no longer be understood as the science of number and magnitude—modern natural science requires a more expansive and abstract mathematics of pure relational structures.[30] This would explain how the new "logic" gets its content and avoids sliding into the empty formalism that Natorp warned of.

Yet Frege and Russell not only overlooked the necessity of this task, some of their philosophical commitments preclude its completion. When Frege and Russell insist that the propositions of the new logic are formal or general and so are isolated and grounded in a different way from those principles that give the conditions of the possibility of the mathematical sciences of nature (namely, the principles of transcendental logic), they introduce an unbridgeable gap between the laws of logic and experience. Kant himself needed an elaborate and controversial argument (the Meta-

30. In chapter 4 of [1910] 1923, Cassirer argues that abstract, modern mathematics is necessary for empirical science because modern scientific theories are themselves more abstract; theory formation does not consist in simple inductive generalizations of observed facts, but in positing abstract relational structures that are related to experience only mediately and holistically. For a fuller but still incomplete discussion, see Heis forthcoming.

physical Deduction) to show that the logical forms—mediated through the application of the categories—apply to objects of experience. In his writings on Kant, Cohen cuts through the Gordian knot of the Metaphysical Deduction: since the laws of "logic" are themselves derived from an analysis of the mathematical sciences of nature, it becomes superfluous to show that they apply to objects of experience—"and so there is no miracle" to be explained (Cohen 1885, p. 78). Frege and Russell, though, are obligated to bridge the gap between logic and experience. And it is not obvious to Cassirer and Natorp how this could work:

> Couturat tries to explain the generality and a priori necessity of mathematical truths with their purely "formal" character. [Cassirer cites (Couturat 1905, p. 207).] But this solution might be permissible and comprehensible for the critical theory of knowledge and its determination of the concept of "possible experience": for Logistic it is not. How would it justify that we impose logical laws, which we have arrived at completely independently of the consideration of things, additionally on things; how is it able to prove that future experience will accord with the consequences that we have extracted from purely logical premises and without any consideration of intuition and observation? (1907, p. 44; cp. Natorp [1911] 1918, p. 43)

Demarcating formal logic as an independent discipline already threatens to create a problem of how logic is applied in empirical science; Frege and Russell's platonistic rhetoric about abstract, logical objects and Russell's epistemology of acquaintance threaten to make the gap between logic and experience even more severe. Russell claims that the mind "passively receives" knowledge of logical objects through immediate acquaintance with abstract logical entities (Cassirer [1910] 1923, p. 316).[31] It is clear, Cassirer thinks, what is driving Russell to think that our knowledge of logical truths is to be understood in a way exactly parallel to our immediate observational knowledge of physical objects: he thinks that the *objectivity* of logic and mathematics can only be explained by positing an immediate relation between the subject and some *object.*[32] Now, the problem

31. Cassirer here cites and paraphrases Russell (1903, p. 33): "The mind . . . is as purely receptive in inference as common sense supposes it to be in perception of sensible objects." He might also have cited Russell 1903, p. xv, where Russell says that a goal of his discussion is to "make others see clearly the entities concerned [i.e. the fundamental terms of logic and mathematics], in order that the mind may have that kind of acquaintance with them which it has with redness or the taste of pineapple."

32. See Cassirer [1910] 1923, p. 317 (already quoted above). From Cassirer's point of view, Russell's platonic atomism is an over-reaction to the plain facts that the truth of a

is not the talk of "mathematical objects" *per se.* However, if we interpret this talk metaphysically, as if there were a separate kind of object constituting a separate realm from that of empirical objects, we will be left with no way to re-establish the necessary relation of mathematical and empirical objects.[33] Once objects are divided into closed off ontological types, it can only be a lucky accident that they are able to come together again.[34]

The philosophy of logic articulated by Cassirer and the other Marburg Neo-Kantians is a genuine alternative to the better known philosophies of logic defended in the early decades of the twentieth century. Natorp and Cassirer not only reject what they take to be the objectionable platonism in Frege's and Russell's philosophies, they reject as confused the project of demarcating formal logic from "material" or "contentful" science. And so their philosophy of logic differs fundamentally from alternatives that secure a place for logic by making it analytic or empty.[35] Further, though Cassirer and Natorp's criticisms dovetail with the objections to Frege's and Russell's logicist project posed by Wittgenstein,[36] they would have no sympathy with the Tractarian project of determining once for all the preconditions of *all* languages. According to the transcendental method,

---

logico-mathematical proposition is independent of its being thought and that mathematics is not about the mind of the mathematician. Both of these facts can be respected without positing logical objects and immediate acquaintance.

33. In his discussion of Frege's definition of number, Natorp claims that "Frege bases number on something like Platonic ideas, namely objective thought contents. That is, he is looking for thought contents, as Plato does, purely, but in a one-sided ontological way" (1910, p. 115).

34. Cassirer elaborates on this point in Cassirer [1913] 1993. Cassirer thinks this objection applies not only to Frege and Russell, but also to the "Southwest" Neo-Kantians like Windelband and Rickert. As he summarizes his idea: "How these [formal and material] moments come together, and how they have brought about a whole reciprocally adapted to each other, no longer constitutes a significant question in need of solution, because only the factors' being in and with one another, and not their being isolated next to each other, can be a *datum* of cognition" (1913, p. 16).

35. Cassirer argues that the new logic is synthetic in Cassirer 1907, p. 36ff.

36. See *Tractatus* 5.5521. David Pears nicely summarizes Wittgenstein's objection to Russell's philosophy of logic.

> [Russell's theory of logic] is an attempt to exploit the resources of two worlds. [. . .] We need acquaintance with abstract objects in another world, because, without such acquaintance we would be unable to understand the forms of propositions or the logical words that occur in their expression. Any theory of this kind has to face the question, where the boundary between the two worlds should be drawn. [. . .] It is also necessary for any such theory to face the question, why elements taken from the two worlds can be fitted together. But Russell never faces this question, and, for all that he says, the harmony might be pure coincidence. Finally a two-world theory, like any other theory of language and logic, has to explain

knowledge of what can and cannot be expressed in the sciences can only be arrived at through detailed analysis of our best current science. Since the principles of logic are derived from an analysis of the sciences, and these sciences are constantly progressing, even the study of the forms of judgment, Cohen and Cassirer argue, is fallible and open to being revised as the exact sciences progress (Cohen 1902, pp. 499–500; Cassirer [1906] 1922, p. 19).

In arguing that there is no way to distinguish the judgments of formal logic from other kinds of "material" judgments and in leaving open the possibility that the laws of logic might change with the progress of science, this Neo-Kantian philosophy of logic in some ways anticipates later Quinean theses. However, though the Marburg Neo-Kantians share with Quine a fallibilism about logic, they differ from him in a more fundamental way. Indeed, any defense of Cassirer's conception of logic as the study of the preconditions of the exact sciences would need to substantiate—in the face of stock Quinean objections—the particular theory of the "relativized a priori" at the center of Marburg Neo-Kantian philosophy.[37] On Cohen and his followers' reading, Kant had given a defense of the non-empirical truth of both Euclidean geometry and the principle of causality by showing that they are preconditions of the possibility of Newtonian science. In the same way, Cassirer claims that Riemannian differential geometry and Einstein's principle of general covariance are conditions of the possibility of general relativity (Cassirer [1921] 1923, p. 415).[38] Cassirer's philosophical justification of modern mathematics thus depends on showing that the pure relational structures expressible in the new logic are—like the other principles of "transcendental" logic—preconditions of the possibility of empirical science.[39]

---

what logical truth is. It is on this point that Russell's theory is weakest. (Pears 1977, p. 183)

The overlap with Cassirer's and Natorp's criticisms should be obvious.

37. A spirited defense of the relativized a priori against Quinean arguments is presented in Friedman 2001. Friedman there also argues—partially overlapping with Cassirer [1921] 1923—that there are a priori elements in the theory of general relativity.

It should be noted that Cassirer's own theory of the a priori differs in important ways from the theories of the relativized a priori defended by the early logical positivists. See Heis forthcoming; Friedman 2000, p. 115ff.; Richardson 1998, ch. 5; Ryckman 2005, ch. 2.

38. For a defense of the plausibility of Cassirer's argument that the principle of general covariance is a priori, see Ryckman 2005.

39. In particular, such a defense would have to show that Cassirer's arguments that empirical science has non-empirical preconditions are not simply arguments against confirmational atomism. This is particularly pressing for Cassirer, since many of his preferred objections to empiricism, like his claim in chapter 1of Cassirer [1921] 1923 that even the

I noted at the beginning of this paper that Natorp's and Cassirer's criticisms of Frege and Russell—when not simply ignored by historians of philosophy—have often been panned as confused, trivial, and fraught with technical misunderstandings. I have no desire to defend all of the details of the Marburg Neo-Kantians' reading of Frege and Russell, and I have not argued that the criticisms discussed in this paper are in fact fatal. But I think it is by now clear that these criticisms are not trivial or simply confused. Moreover, it is ironic that Cassirer's Kantian response to Frege and Russell resembles the common response of philosophers in the analytic tradition today: that we should recognize the revolutionary philosophical implications of the new logic, even if we need ultimately to discard the philosophical packaging in which it first came wrapped.

References

Anderson, R. Lanier. 2005. "Neo-Kantianism and the Roots of Anti-Psychologism." *British Journal for the History of Philosophy* 13: 287–323.

Carus, A. W. 2008. *Carnap and Twentieth Century Thought: Explication as Enlightenment.* Cambridge: Cambridge University Press.

Cassirer, Ernst. (1906) 1922. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit. Erster Band.* 3rd ed. Berlin: Bruno Cassirer.

Cassirer, Ernst. (1907) 1922. *Das Erkenntnisproblem in der Philosophie und Wissenschaft der neuren Zeit. Zweiter Band.* 3rd ed. Berlin: Bruno Cassirer.

Cassirer, Ernst. 1907. "Kant und die moderne Mathematik." *Kant-Studien* 12: 1–40.

Cassirer, Ernst. (1910) 1923. *Substanzbegriff und Funktionsbegriff. Untersuchungen über die Grundfragen der Erkenntniskritik.* Berlin: Bruno Cassirer. Translated by Swabey and Swabey in *Substance and Function & Einstein's Theory of Relativity.* Chicago: Open Court.

Cassirer, Ernst. 1912. "Hermann Cohen und die Erneuerung der Kantischen Philosophie." *Kant-Studien* 17: 252–273.

Cassirer, Ernst. (1913) 1993. "Erkenntnistheorie nebst den Grenzfragen der Logik." Reprinted in *Erkenntnis, Begriff, Kultur,* edited by. R. Bast. Hamburg: Felix Meiner Verlag.

Cassirer, Ernst. (1918) 1981. *Kants Leben und Lehre.* Berlin: Bruno Cassirer. Translated by Haden as *Kant's Life and Thought.* New Haven: Yale University Press.

---

simplest empirical measurements have presuppositions, could arguably be accommodated in an holistic empiricism like Quine's.

It should also be clear that Cassirer's defense of modern mathematics will face special problems with mathematical theories that are not applied in empirical science. On this issue, see Heis forthcoming.

Cassirer, Ernst. (1921) 1923. *Zur Einstein'schen Relativitätstheorie.* Berlin: Bruno Cassirer Verlag. Translated by Swabey and Swabey in *Substance and Function & Einstein's Theory of Relativity.* Chicago: Open Court.

Cassirer, Ernst. (1923) 1955. *Philosophie der symbolischen Formen. Erster Teil: Die Sprache.* Berlin: Bruno Cassirer. Translated by Hendel and Woglom as *The Philosophy of Symbolic Forms. Volume One: Language.* New Haven: Yale University Press.

Cassirer, Ernst. (1928) 1993. "Zur Theorie des Begriffs." Reprinted in *Erkenntnis, Begriff, Kultur.* Edited by. R. Bast. Hamburg: Felix Meiner Verlag.

Cassirer, Ernst. (1929) 1957. *Philosophie der Symbolischen Formen. Dritter Teil: Phänomenologie der Erkenntnis.* Berlin: Bruno Cassirer. Translated by Hendel and Woglom as *Philosophy of Symbolic Forms. Volume Three: The Phenomenology of Knowledge.* New Haven: Yale University Press.

Cassirer, Ernst. (1939) 1993. "Was ist Subjektivismus?" Reprinted in *Erkenntnis, Begriff, Kultur.* Edited by. R. Bast. Hamburg: Felix Meiner Verlag.

Cassirer, Ernst. 1950. *The Problem of Knowledge: Philosophy, Science, and History since Hegel.* Translated by Woglom and Hendel. New Haven: Yale University Press.

Cohen, Hermann. 1885. *Kants Theorie der Erfahrung.* 2nd ed. Berlin: Dümmler.

Cohen, Hermann. 1902a. "Biographisches Vorwort und Einleitung mit kritischem Nachtrag." In Lange, 1902, pp. v–xiii, 437–535.

Cohen, Hermann. 1902b. *Logik der reinen Erkenntnis.* Berlin: Bruno Cassirer.

Couturat, Louis. 1905. *Les principes des mathématiques: avec un appendice sur la philosophie des mathématiques de Kant.* Paris: F. Alcan.

Dedekind, Richard. (1872) 1963. "Continuity and Irrational Numbers." Pp. 1–27 in *Essays on the Theory of Numbers.* Translated by Beman. New York: Dover.

Dedekind, Richard. (1888) 1963. "The Nature and Meaning of Numbers." Pp. 29–115 in *Essays on the Theory of Numbers.* Translated by Beman. New York: Dover.

Edgar, Scott. 2008. "Paul Natorp and the Emergence of Anti-psychologism in the Nineteenth Century." *Stud. Hist. Phil. Sci.* 39: 54–65.

Frege, Gottlob. (1884) 1953. *Die Grundlagen der Arithmetik. Eine logisch-mathematische Untersuchung über den Begriff der Zahl.* Breslau: Koebner. Translated by J. L. Austin as *The Foundations of Arithmetic.* Oxford: Blackwell.

Friedman, Michael. 2000. *A Parting of the Ways: Carnap, Cassirer, and Heidegger.* Chicago: Open Court Press.
Friedman, Michael. 2001. *Dynamics of Reason.* Stanford: CSLI Publications.
Grattan-Guinness, I. 2000. *The Search for Mathematical Roots, 1870–1940.* Princeton: Princeton University Press.
Heis, Jeremy. 2007. "The Fact of Modern Mathematics: Geometry, Logic, And Concept Formation In Kant And Cassirer." PhD diss, University of Pittsburgh.
Heis, Jeremy. Forthcoming. "Ernst Cassirer's Neo-Kantian Philosophy of Geometry."
Kant, Immanuel. (1781/1787) 1998. *Critique of Pure Reason.* Translated by Paul Guyer and Allen Wood. Cambridge: Cambridge University Press.
Kim, Alan. 2003. "Paul Natorp." In *Stanford Encyclopedia of Philosophy.* http://plato.stanford.edu/entries/natorp.
Kinkel, Walter. (1904) 1920. *Immanuel Kants Logik: Ein Handbuch zu Vorlesungen, dritte Auflage, neu herausgegeben, mit einer Einleitung sowie einem Personen- und Sachregister versehen.* 3rd ed. Leipzig: Felix Meiner.
Lange, Friedrich Albert. 1902. *Geschichte des Materialismus und Kritik seiner Bedeutung in der Gegenwart.* Leipzig: J. Baedeker.
Longuenesse, Beatrice. 1998. *Kant and the Capacity to Judge.* Translated by Charles Wolfe. Princeton: Princeton University Press.
Longuenessse, Beatrice. 2005. *Kant on the Human Standpoint.* Cambridge: Cambridge University Press.
Natorp, Paul. (1887) 1981. "Ueber objektive und subjektive Begründung der Erkenntniss (Erster Aufsatz)." *Philosophische Monatshefte* 23: 257–286. Translated as "On the Objective and Subjective Grounding of Knowledge," by L. Phillips and D. Kolb. *Journal of the British Society for Phenomenology* 12: 245–266.
Natorp, Paul. 1910. *Die logischen Grundlagen der exakten Wissenschaft.* Leipzig and Berlin.
Natorp, Paul. (1911) 1918. *Philosophie: Ihr Problem und ihre Probleme: Einführung in den kritischen Idealismus.* 2nd ed. Göttingen: Vandenhoeck & Ruprecht.
Natorp, Paul. 1912. "Kant und die Marburger Schule." *Kant-Studien* 17: 193–221.
Pears, David. 1977. "The Relation between Wittgenstein's Picture Theory of Propositions and Russell's Theories of Judgment." *The Philosophical Review.* 86: 177–196.
Pulkkinen, Jarmo. 2001. "Cassirer and Couturat's Critique of Kant's Philosophy of Mathematics." In *Kant und die Berliner Aufkärung: Akten des*

IX. Internationalen Kant-Kongresses, Band V, 315–322. NY: Walter de Gruyter.

Reich, Klaus. (1932) 1992. *Die Vollständigkeit der kantischen Urteilskraft.* Berlin. Translated as *The Completeness of Kant's Table of Judgments,* by Jane Kneller and Michael Losonsky. Stanford: Stanford University Press.

Richardson, Alan. 1998. *Carnap's Construction of the World: The Aufbau and the Emergence of Logical Empiricism.* Cambridge: Cambridge University Press.

Richardson, Alan. 2006. "'The Fact of Science' and Critique of Knowledge: Exact Science as Problem and Resource in Marburg Neo-Kantianism." Pp. 211–226 in *The Kantian Legacy in Nineteenth-century Science.* Edited by Friedman and Nordmann. Cambridge, MA: MIT Press.

Russell, Bertrand. (1901) 1957. "Mathematics and the Metaphysicians." In *Mysticism and Logic.* New York: Doubleday Anchor Books.

Russell, Bertrand. 1903. *The Principles of Mathematics.* New York: Norton.

Russell, Bertrand. (1913) 1992. *Theory of Knowledge: The 1913 Manuscript.* Edited by Elizabeth Ramsden Eames and Kenneth Blackwell. NY: Routledge.

Russell, Bertrand. 1920. *Introduction to Mathematical Philosophy.* 2nd ed. New York: Macmillan.

Russell, Bertrand. 1973. "The Regressive Method of Discovering the Premises of Mathematics." In *Essays in Analysis.* Edited by D. Lackey. London: George Allen and Unwin Ltd.

Ryckman, Thomas. 2005. *The Reign of Relativity: Philosophy in Physics, 1915–1925.* New York: Oxford University Press.

Thiel, Christian. 1997. "Natorps Kritik an Freges Zahlbegriff." Pp. 123–128 in *Frege in Jena.* Würzburg.

Wittgenstein, Ludwig. 2001. *Tractatus Logico-Philosophicus.* Translated by David Francis Pears and Brian McGuinness. NY: Routledge.