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Connected limits, familial representability and Artin glueing

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We consider the following two properties of a functor F from a presheaf topos to the category of sets: (*a*) F preserves connected limits, and (*b*) the Artin glueing of F is again a presheaf topos. We show that these two properties are in fact equivalent. In the process, we develop a general technique for associating categorical properties of a category obtained by Artin glueing with preservation properties of the functor along which the glueing takes place. We also give a syntactic characterization of those monads on **Set** whose functor parts have the above properties, and whose units and multiplications are cartesian natural transformations.

1. Introduction

It has been observed by several authors in different ways (Walters 1989; 1991; Kasangian and Vigna 1991; Johnson and Walters 1992; Kasangian and Labella 1992; 1992; Johnstone 1992) that many set-valued functors of importance in computer science (such as the free-monoid and free-semigroup functors, and the functor that assigns to a set X the set of X-labelled trees) are not representable but are 'very nearly so', in that 'the only reason' why they are not representable is their failure to preserve the terminal object – more formally, they become representable when extended to the free product-completion of their domains. It has also been observed, notably by R. F. C. Walters (Walters 1989a; Walters 1989b; Walters 1992) and by S. H. Schanuel (*cf.* Example 4.6 below), that many of the categories that occur naturally in computer science as fundamental 'building blocks' for categories of datatypes are in fact *presheaf toposes* – that is, categories of set-valued functors on small categories. However, it does not seem to have been previously observed that there is a connection between these two phenomena.

In this paper we show that the 'unexpected' presheaf toposes occur as categories obtained by *Artin glueing* – that is, they are comma categories of the form $(\mathscr{F} \downarrow T)$, where $T : \mathscr{E} \to \mathscr{F}$ is a functor between (presheaf) toposes (and $\mathscr{F}$ is generally the topos of sets.) It has been known for twenty years (Wraith 1974) that the Artin glueing construction yields a topos when ($\mathscr{E}$ and $\mathscr{F}$ are toposes and) T preserves finite limits, and for almost as long that the latter hypothesis can be weakened to 'T preserves pullbacks'. Until recently, however, nobody seems to have asked whether this sufficient condition is also necessary. In this paper we show that it is and, in addition, that glueing along a functor T between presheaf toposes yields a presheaf topos if and only if T preserves arbitrary connected

limits, if and only if (in the case when the codomain is the topos of sets) T is 'very nearly representable' in the sense described in the previous paragraph. This is the previously unobserved connection between the two phenomena mentioned earlier.

The above results are proved in Section 4 of this paper, where we will, in fact, develop a general technique for associating categorical properties of a category obtained by glueing with preservation properties of the functor along which the glueing takes place. For example, we will show that local cartesian closedness of $(\mathscr{F} \downarrow T)$ is already sufficient to imply that the functor T preserves pullbacks (and 'mere' cartesian closedness of $(\mathscr{F} \downarrow T)$ implies that T transforms binary products in its domain to pullbacks over $T1$ in its codomain).

Before doing this, however, we study 'very nearly representable' functors in more generality. The first detailed study of such functors was made by Diers (Diers 1977; Diers 1979) about 15 years ago. He called them 'locally representable' (and later 'multirepresentable'), but we now prefer the term 'familially representable', emphasizing the fact that the 'representor' is not a single object of the domain category but an indexed family of objects. We develop the general theory of such functors in Section 2, including both 'general' and 'special' versions of a familial representability theorem – the former can already be found in Diers' work, but the latter is of more use to us in what follows. We also discuss the relationship between familially representable functors and the partial product functors introduced (in a special case) by B. A. Pasynkov (1965), and studied in more generality by R. Dyckhoff and W. Tholen (1987).

As has been mentioned, familially representable functors arise naturally in computer science, and it will have been observed that the examples we mentioned in the first paragraph all occur as the free functors for finitary algebraic theories – equivalently, they are the functor parts of finitary monads on the category of sets. Since this state of affairs seems to be typical, we devote Section 3 of the paper to investigating those monads whose functor parts are familially representable. We show that, provided we make the further (but not unnatural, in this context) assumption that the unit and multiplication are cartesian natural transformations, such monads can be characterized by a syntactic condition, which we call strong regularity. (This class of monads also appeared in a recent paper of G. M. Kelly (1992), where they were characterized as the clubs (over the free-monoid monad) in the category of sets.) We also show by an example that the additional hypothesis on the unit and multiplication cannot be omitted.

2. Familially representable functors

We begin by reviewing some terminology. The notions of *pullback* and *equalizer* are well known; we will use the term *wide pullback* for what other authors (*e.g.* Paré (1990)) call a fibre product, that is, a limit indexed by a category that is a poset with a terminal object, and in which all maximal chains have length 2. It is well known that all (small) connected limits (that is, limits indexed by small connected categories) can be constructed from wide pullbacks and equalizers (and all finite connected limits can be constructed from ordinary pullbacks and equalizers). R. Paré (1990) investigated the limits that can be constructed from wide pullbacks alone, and showed that they are exactly the limits indexed by small

simply connected categories, that is, those $\mathscr{C}$ with the property that every functor from $\mathscr{C}$ to a groupoid is naturally isomorphic to a constant functor.

When it comes to the preservation of limits by functors, however, there is a strong tendency for those that preserve pullbacks to preserve equalizers as well. The reason is the following well-known folk-theorem.

Lemma 2.1. Let $\mathscr{C}$ be a finitely complete (respectively, complete) category and $T : \mathscr{C} \to \mathscr{D}$ a functor. If T preserves pullbacks (respectively, wide pullbacks), then it preserves all finite (respectively, all small) connected limits.

Proof. We can factor T as

$$\mathscr{C} \xrightarrow{\widehat{T}} \mathscr{D}/T1 \xrightarrow{\Sigma_{T1}} \mathscr{D}$$

where 1 is the terminal object of $\mathscr{C}$, $\widehat{T}(A)$ is T applied to the unique morphism $A \to 1$, and Σ_{T1} is the forgetful functor. Since Σ_{T1} creates (wide) pullbacks, $\widehat{T}$ preserves them, but it also preserves the terminal object by construction, and so preserves all finite (respectively, all small) limits. But Σ_{T1} preserves all connected limits that exist in $\mathscr{D}/T1$. Hence T preserves all finite (respectively, all small) connected limits. □

The presence of a terminal object in $\mathscr{C}$ is essential to this argument. If we take $\mathscr{D}$ to be the category of topological spaces and continuous maps, $\mathscr{C}$ the category of topological spaces and local homeomorphisms, and T the inclusion functor, then T preserves pullbacks but not equalizers, although both types of limit exist in $\mathscr{C}$. (The equalizer in $\mathscr{C}$ of a parallel pair of arrows is the interior of the subspace that is their equalizer in $\mathscr{D}$.)

Next, we need to recall a few facts about the free product completion $\Pi(\mathscr{C})$ of a category $\mathscr{C}$. It is well known that this category can be described as follows: its objects are small families $(X_i)_{i\in I}$ of objects of $\mathscr{C}$, and its arrows $(X_i)_{i\in I} \to (Y_j)_{j\in J}$ are pairs $(\phi, (f_j)_{j\in J})$, where $\phi : J \to I$ is a function and $(f_j)_{j\in J}$ is a family of arrows $f_j : X_{\phi(j)} \to Y_j$ of $\mathscr{C}$. One easily verifies that $\Pi(\mathscr{C})$ has small products and that the obvious functor $(\ )_* : \mathscr{C} \to \Pi(\mathscr{C})$ satisfies the appropriate universal property. Clearly, if $\mathscr{C}$ is locally small, then $\Pi(\mathscr{C})$ is also locally small. As for more general limits in $\Pi(\mathscr{C})$, Diers has already proved (Diers 1977, 4.0.2) the following useful fact.

Proposition 2.2. The product completion $\Pi(\mathscr{C})$ is complete if and only if $\mathscr{C}$ has all small connected limits. Moreover, the universal functor $(\ )_* : \mathscr{C} \to \Pi(\mathscr{C})$ preserves connected limits.

Proof. (Sketch) First suppose $\mathscr{C}$ has connected limits. Since $\Pi(\mathscr{C})$ has products, it suffices to show that it also has equalizers. Given a parallel pair of arrows

$$(X_i)_{i\in I} \underset{(\psi, (g_j)_{j\in J})}{\overset{(\phi, (f_j)_{j\in J})}{\rightrightarrows}} (Y_j)_{j\in J}$$

in $\Pi(\mathscr{C})$, we first form the coequalizer $\chi : I \to K$ of ϕ and ψ in **Set**. For each $k \in K$, we then construct a small category B_k as follows: its objects are those $i \in I$ satisfying $\chi(i) = k$ and those $j \in J$ (presumed disjoint from I) satisfying $\chi\phi(j) = k$; in addition to identities, B_k has one arrow $\phi(j) \to j$ and one arrow $\psi(j) \to j$ for each $j \in J \cap \text{ob } B_k$. From the definition of coequalizers in **Set**, B_k is easily seen to be connected. The objects X_i and Y_j,

with the morphisms f_j and g_j, form a diagram of shape B_k in $\mathscr{C}$, whose limit we denote by L_k. Then it is straightforward to verify that $(L_k)_{k\in K}$ is (the domain of) an equalizer for the given pair of morphisms in $\Pi(\mathscr{C})$.
Conversely, suppose $\Pi(\mathscr{C})$ is complete. Let $(X_i)_{i\in I}$ be the vertices of a connected diagram in $\mathscr{C}$, and let $(L_j)_{j\in J}$ be an object of $\Pi(\mathscr{C})$ providing a limit for (the image under $(\ )_\bullet$ of) this diagram. For each $i \in I$, the ith leg of the limit cone picks out a particular $j(i) \in J$ and a morphism $f_i : L_{j(i)} \to X_i$ in $\mathscr{C}$. However, the connectedness of the diagram forces $j(i)$ to be the same for all $i \in I$, and it is then easy to see that this particular $L_{j(i)}$, together with the f_i, forms a limit for the original diagram in $\mathscr{C}$. In fact it follows that J must be a singleton $\{j(i)\}$, which shows that the canonical functor $(\ )_\bullet$ preserves connected limits. □

The above argument shows that $(\ )_\bullet$ preserves connected limits only under the assumption that $\mathscr{C}$ has all small connected limits. This is all we will need in what follows, though in fact it can be shown that $(\ )_\bullet$ preserves any connected limits that happen to exist in $\mathscr{C}$. For the next result, we recall that a *continuous* functor is one that preserves all small limits.

Corollary 2.3. Let $\mathscr{C}$ be a category with small connected limits, and $\mathscr{D}$ a complete category. Then composition with $(\ )_\bullet : \mathscr{C} \to \Pi(\mathscr{C})$ induces an equivalence between the category of continuous functors $\Pi(\mathscr{C}) \to \mathscr{D}$ and the category of connected-limit-preserving functors $\mathscr{C} \to \mathscr{D}$.

Proof. Clearly, any continuous functor $\Pi(\mathscr{C}) \to \mathscr{D}$ restricts to a connected-limit-preserving functor $\mathscr{C} \to \mathscr{D}$. Conversely, if $T : \mathscr{C} \to \mathscr{D}$ preserves connected limits, it follows easily from the construction of equalizers in the proof of Proposition 2.2 that the unique product-preserving extension of T to $\Pi(\mathscr{C})$ preserves them too. □

We now state the central definition of this paper.

Definition 2.4. Let $\mathscr{C}$ be a locally small category. A functor $T : \mathscr{C} \to \mathbf{Set}$ is called *familially representable* if its extension $\overline{T}$ to a product-preserving functor on $\Pi(\mathscr{C})$ is representable.

So, a functor T is familially representable when there exists a family $(A_i)_{i\in I}$ of objects of $\mathscr{C}$ such that

$$T(X) \cong \Pi(\mathscr{C})((A_i)_{i\in I}, (X)_\bullet) \cong \coprod_{i\in I} \mathscr{C}(A_i, X) .$$

Notice that the representing family is unique up to isomorphism in $\Pi(\mathscr{C})$. Familially representable functors were originally called 'locally representable' by Diers (Diers 1977; Diers 1979), and later 'multirepresentable' (a term invented by the second author of this paper, *cf.* (Johnstone 1979a)); the name 'familially representable' is due to the first author (*cf.* (Johnson and Walters 1992)).

The following characterization of familially representable functors again appears in Diers (1977) as Théorème 3.0.4.

Theorem 2.5. Let $\mathscr{C}$ be a locally small category with all small connected limits, and $T : \mathscr{C} \to \mathbf{Set}$ a functor. The following conditions are equivalent:

(i) T is familially representable.
(ii) T preserves small connected limits and satisfies the solution set condition (as in (Mac Lane 1971, p. 118)).

Proof. By Corollary 2.3, T preserves small connected limits iff $\overline{T}$ preserves all small limits. So we only have to verify that the solution-set condition for T is equivalent to that for $\overline{T}$. But if $(X_i)_{i\in I}$ is a family of objects of $\mathscr{C}$ forming a solution-set for T, let S denote the disjoint union of the sets $T(X_i)$. Then the S-indexed family whose sth member is the particular X_i such that $s \in T(X_i)$ (which we will denote by $(X_{i(s)})_{s\in S}$) forms a singleton solution-set for $\overline{T}$. For if $t \in \overline{T}((Y_j)_{j\in J}) \cong \Pi_{j\in J} T(Y_j)$, then for each $j \in J$ we can choose $i = \phi(j) \in I$ and $s = \psi(j) \in T(X_{\phi(j)})$, together with a morphism $f_j : X_{\phi(j)} \to Y_j$ such that $T(f_j)(\psi(j)) = t_j$, the jth component of t. Then we see that t is in the image of $\overline{T}(\psi, (f_j)_{j\in J}) : (X_{i(s)})_{s\in S} \to (Y_j)_{j\in J}$. Conversely, if $\overline{T}$ is representable by an object $(X_i)_{i\in I}$ of $\Pi(\mathscr{C})$, the family of objects $(X_i)_{i\in I}$ of $\mathscr{C}$ provides a solution-set for T. □

Theorem 2.5 is the 'general' version of a familial representability theorem, whose 'special' version imposes additional conditions on the category $\mathscr{C}$ to ensure that the solution-set condition is automatically satisfied. We give the latter only for the case of a complete category $\mathscr{C}$ – it can be formulated for $\mathscr{C}$ with connected limits, but the hypotheses are more awkward to state.

Theorem 2.6. Let $\mathscr{C}$ be a complete, locally small, well-powered category with a small cogenerating set. Then the following conditions on a functor $T : \mathscr{C} \to \mathbf{Set}$ are equivalent:

(i) T is familially representable.
(ii) T preserves small connected limits.
(iii) T preserves wide pullbacks.

Proof. The equivalence of (ii) and (iii) is Lemma 2.1, and (i)⇒(ii) again follows from Corollary 2.3. For (ii)⇒(i), we have to show that $\Pi(\mathscr{C})$ inherits well-poweredness and a cogenerating set from $\mathscr{C}$. It is easy to see that a morphism $(\phi, (f_j)_{j\in J}) : (X_i)_{i\in I} \to (Y_j)_{j\in J}$ is a monomorphism in $\Pi(\mathscr{C})$ iff ϕ is surjective and, for each $i \in I$, the family of morphisms $(f_j)_{\phi(j)=i}$ is jointly monic in $\mathscr{C}$. It is thus easy to see that the subobjects of a given object $(Y_j)_{j\in J}$ of $\Pi(\mathscr{C})$ are parametrized by pairs (R, S), where R is an equivalence relation on J and S is a function assigning to each R-equivalence class a subobject of the product in $\mathscr{C}$ of the corresponding Y_j. Now let $\mathscr{G}$ be a cogenerating set for $\mathscr{C}$; we claim that

$$\{(G)_\bullet \mid G \in \mathscr{G}\} \cup \{(1)_\bullet\}$$

is a cogenerating set for $\Pi(\mathscr{C})$. If

$$(X_i)_{i\in I} \underset{(\psi, (g_j)_{j\in J})}{\overset{(\phi, (f_j)_{j\in J})}{\rightrightarrows}} (Y_j)_{j\in J}$$

is a parallel pair of morphisms in $\Pi(\mathscr{C})$ that are not equal, then either $\phi \neq \psi$, in which case we choose $j \in J$ with $\phi(j) \neq \psi(j)$ and consider the unique morphism $(\chi, h) : (Y_j)_{j\in J} \to (1)_\bullet$ where $\chi(*) = j$, or there exists $j \in J$ with $\phi(j) = \psi(j)$ but $f_j \neq g_j$, in which case we find $h : Y_j \to G$ in $\mathscr{C}$ with $G \in \mathscr{G}$ and $hf_j \neq hg_j$, and consider the morphism $(\chi, h) : (Y_j)_{j\in J} \to (G)_\bullet$ with $\chi(*) = j$. In either case we have found a morphism to a member of the given set of objects having unequal composites with the given morphisms. □

Note that the hypotheses of Theorem 2.6 are satisfied whenever $\mathscr{C}$ is a Grothendieck topos; in particular when $\mathscr{C} = \mathbf{Set}$.

Scholium 2.7. In the case $\mathscr{C} = \mathbf{Set}$, the following equivalent condition may be added to those of Theorem 2.6:

(iv) T is isomorphic to a partial product functor $P(p, -)$ in the sense of (Dyckhoff and Tholen 1987).

Proof. An object $(X_i)_{i\in I}$ of $\Pi(\mathbf{Set})$ may be identified with a morphism $p: X \to I$ of **Set**, where X is the disjoint union of the X_i (equivalently, $X_i = p^{-1}(i)$). In terms of this representation, the familially representable functor corresponding to $(X_i)_{i\in I}$ sends a set A to $\coprod_{i\in I} A^{X_i}$, that is, it is the composite

$$\mathbf{Set} \xrightarrow{I^*} \mathbf{Set}/I \xrightarrow{(\ )^p} \mathbf{Set}/I \xrightarrow{\Sigma_I} \mathbf{Set}$$

(where the middle factor is exponentiation in $\mathbf{Set}/I$). But this is exactly the description of the partial product functor $P(p, -)$ given, for example, in Johnstone (1992, 3.1). □

Let $\mathscr{C}$ be a locally small category. By the Yoneda lemma, the category of familially representable functors $\mathscr{C} \to \mathbf{Set}$ (and all natural transformations between them) is equivalent to $\Pi(\mathscr{C})^{\mathrm{op}}$ (or, if you prefer, to the coproduct completion $\coprod(\mathscr{C}^{\mathrm{op}})$ of $\mathscr{C}^{\mathrm{op}}$). In particular, we note that this category has all small coproducts, and if $\mathscr{C}$ has coproducts, then $\Pi(\mathscr{C})^{\mathrm{op}}$ has products. For example, if $M: \mathbf{Set} \to \mathbf{Set}$ is the free-monoid functor, then $M \times M$ is familially representable, the representing family being the coproduct in $\Pi(\mathbf{Set})$ of $([n])_{n\in\mathbb{N}}$ with itself, that is $([n] + [m])_{(n,m)\in\mathbb{N}\times\mathbb{N}}$. (We will return to this observation in Example 4.6 below.)

However, when dealing with pullback-preserving functors, it often makes sense not to consider all natural transformations but only the *cartesian* ones, that is, those for which the naturality squares are pullbacks. In this context, we have the following lemma.

Lemma 2.8. Let $\mathscr{C}$ be a locally small category, and let $S, T: \mathscr{C} \rightrightarrows \mathbf{Set}$ be familially representable functors, with representing families $(X_i)_{i\in I}$ and $(Y_j)_{j\in J}$, respectively. Then a natural transformation $\alpha: S \to T$ is cartesian if and only if the corresponding morphism $(\phi, (f_i)_{i\in I}): (Y_j)_{j\in J} \to (X_i)_{i\in I}$ has the property that each $f_i: Y_{\phi(i)} \to X_i$ is an isomorphism.

Proof. Let $g: A \to B$ be a morphism of $\mathscr{C}$. The assertion that the naturality square for g is a pullback reduces to saying that each f_i is orthogonal to g, that is, that each commutative square

$$\begin{array}{ccc} Y_{\phi(i)} & \xrightarrow{f_i} & X_i \\ \downarrow & & \downarrow \\ A & \xrightarrow{g} & B \end{array}$$

has a unique diagonal fill-in $X_i \to A$. This clearly holds if the f_i are all isomorphisms. For the converse, take g to be f_i itself and the two vertical edges of the square to be identity maps. □

For the particular case $\mathscr{C} = \mathbf{Set}$, we may once again represent set-indexed families of objects $(X_i)_{i\in I}$ by morphisms $p: X \to I$ in **Set**. In this context, a morphism

$(\phi,(f_i)_{i\in I}):(X_i)_{i\in I}\to(Y_j)_{j\in J}$ of $\Pi(\mathbf{Set})^{\mathrm{op}}$ becomes a pair (ϕ,f), where $\phi:I\to J$ as before and $f:\phi^*(Y)\to X$ over I. In particular, such a morphism satisfies the condition of Lemma 2.8 iff f is an isomorphism. Thus we deduce the following corollary.

Corollary 2.9. The category of familially representable functors **Set** $\to$ **Set** and cartesian natural transformations between them is equivalent to the category $\mathscr{A}$ whose objects are morphisms $p:X\to I$ in **Set** and whose morphisms are pullback squares

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ I & \longrightarrow & J \end{array}$$

Since a composite of familially respresentable functors **Set** $\to$ **Set** is familially representable (and composition with a pullback-preserving functor preserves cartesianness of natural transformations), the category $\mathscr{A}$ has a monoidal structure. For future reference, the following lemma gives an explicit description of this structure, at least on objects.

Lemma 2.10. Let $p:X\to I$ and $q:Y\to J$ be functions, corresponding to familially representable functors $S,T:\mathbf{Set}\rightrightarrows\mathbf{Set}$, respectively. Then the composite functor ST corresponds to the vertical composite in the diagram

$$\begin{array}{ccccc} & & Z & \longrightarrow & Y \\ & & \downarrow & & \downarrow{\scriptstyle q} \\ X & \longleftarrow & W & \xrightarrow{\rho} & J \\ {\scriptstyle p}\downarrow & & \downarrow & & \\ I & \xleftarrow{\lambda} & S(J) & & \end{array}$$

in which the squares are pullbacks, and λ and ρ are the universal maps associated with the partial product $S(J)=P(p,J)$.

Proof. By definition, an element of $ST(A)$ corresponds to a pair (i,f) where $i\in I$ and $f:X_i\to T(A)$. But since T is familially representable by $(Y_j)_{j\in J}$, f in turn corresponds to a pair (g,h) where $g:X_i\to J$ and $h:\coprod_{x\in X_i}Y_{g(x)}\to A$. Now the pair (i,g) defines an element of $S(J)$, so ST is representable by the $S(J)$-indexed family whose (i,g)th member is $\coprod_{x\in X_i}Y_{g(i)}$. But this corresponds precisely to the composite $Z\to S(J)$ in the above diagram, since elements of W correspond to triples (i,g,x) with $(i,g)\in S(J)$ and $x\in X_i$, and $\rho(i,g,x)=g(x)$. □

It is now straightforward to describe the effect of the monoidal structure on morphisms of $\mathscr{A}$, and we leave this as an exercise for the reader.

3. Strongly regular algebraic theories

In this section we investigate which monads on **Set** correspond to monoids in the monoidal category $\mathscr{A}$ defined in Corollary 2.9. We recall that monads on **Set** may be presented in terms of (possibly infinitary) operations and equations between compound operations. In practice, we are mainly interested in finitary monads, so we will write all our proofs in terms of finitary operations and equations; but the diligent reader may check that the results remain true if the natural numbers that appear as the arities of operations are replaced by arbitrary (linearly orderable) sets. Another possible generalization is to many-sorted algebraic theories, that is, to monads on $\mathbf{Set}^n$ for some n: the notion of strong regularity, to be defined below, makes sense for such monads, and the link with familial representability remains valid. However, for simplicity we will deal only with the single-sorted case.

Definition 3.1. An equation between compound operations will be called *strongly regular* if the same variables occur on each side of the equation, in the same order and without repetition. An algebraic theory (or the monad on **Set** that corresponds to it) will be called *strongly regular* if it has a presentation by operations and strongly regular equations.

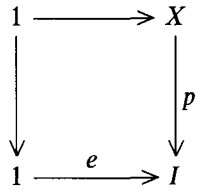
For example, both the associative law $m(x, m(y, z)) = m(m(x, y), z)$ and the unit law $m(x, e) = x$ (where m is a binary operation and e is nullary) are strongly regular, but the commutative law $m(x, y) = m(y, x)$ and the idempotent law $m(x, x) = x$ are not. (Universal algebraists use the term *regular equation* (*cf.* (Jónsson and Nelson 1974)) for an equation in which the same variables appear on each side. However, permutation and repetition of variables are allowed, so the two latter equations are regular but not strongly regular.)

In 1992, Kelly observed that the strongly regular monads on **Set** are precisely those given by clubs (over the free-monoid monad) in his sense. We present another characterization in the following proposition.

Proposition 3.2. A monad on **Set** is strongly regular iff its functor part is familially representable and its unit and multiplication are cartesian natural transformations.

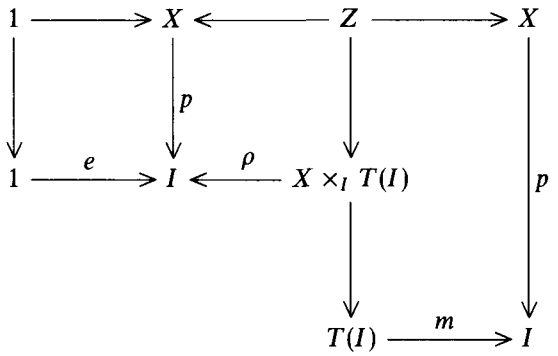
Proof. First suppose $\mathbb{T}$ is a strongly regular theory. We define a representing family $(X_i)_{i \in I}$ for the free $\mathbb{T}$-algebra functor as follows: I is the free $\mathbb{T}$-algebra on one generator, that is, the set of all unary derived operations of $\mathbb{T}$ (which we may think of as arbitrary compounds of the primitive operations of $\mathbb{T}$ with all variables set equal to some particular variable x, modulo the identifications generated by the primitive equations). For each $i \in I$, X_i is the set of all occurrences of the variable x in i (the strong regularity assumption ensures that this is a well-defined concept). Let F be the familially representable functor corresponding to this family. Then, for any set A, an element of $F(A)$ consists of an element i of I together with a function $X_i \to A$; thus we can regard it as an operation of the theory evaluated at a tuple of elements of A, that is, an element of the free $\mathbb{T}$-algebra on A. Thus we have identified the functor part of $\mathbb{T}$ with the functor F. Moreover, if we let $e: 1 \to I$ be the function picking out the identity unary operation, we have a pullback

square



(where, as usual, X is the disjoint union of the X_i), and it is easy to see that the cartesian natural transformation from the identity functor to F induced by this square is the unit of the monad. Similarly, the multiplication is induced by a morphism $m: F(I) \to I$ for which we have an appropriate pullback square: $F(I)$ is the set of operations applied to elements of I, and we obtain a single element of I from such an entity by evaluating it (that is, by composing operations).

Conversely, suppose we are given that the functor part T of $\mathbb{T}$ is represented by the family $(X_i)_{i \in I}$, and that its unit and multiplication are cartesian. By Corollary 2.9 and Lemma 2.10, the latter must be induced by maps $e: 1 \to I$ and $m: T(I) \to I$ for which we have a diagram



in which all the squares are pullbacks. Then we may write down a presentation of the corresponding theory as follows: take one operation ω_i, of arity card X_i, for each element i of I, together with the equations

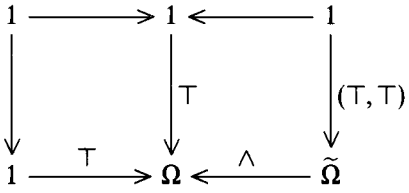
$$\omega_{i_0}(x) = x$$

(where i_0 is the unique element in the image of e) and

$$\omega_{m(i,(j_1,\ldots,j_n))}(x_1,\ldots,x_p) = \omega_i(\omega_{j_1}(x_1,\ldots,x_{m_1}),\ldots,\omega_{j_n}(x_{p-m_n+1},\ldots,x_p)),$$

where ω_i has arity n, ω_{j_r} has arity m_r $(1 \leq r \leq n)$ and $p = \sum_{r=1}^{n} m_r$; the condition that the right-hand cell in the diagram above is a pullback then ensures that $\omega_{m(i,(j_1,\ldots,j_n))}$ has arity p. (This notation must be interpreted *cum grano salis* in the case when the operations involved are not all finitary; but the above equations do still make sense.) By inspection, these equations are strongly regular; and it is straightforward to verify that they give a presentation of the given theory, since any compound operation applied to a sequence of elements of A may be reduced to a single operation ω_i applied to the same sequence, that is, to an element of $T(A)$. □

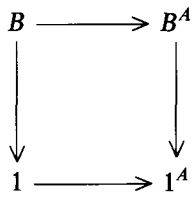
Examples 3.3. (*a*) The best-known example of a strongly regular algebraic theory is of course the theory of monoids, as mentioned in the Introduction. As pointed out by Bénabou (1991), the free monoid functor remains 'familially representable' (that is, a partial product functor) over any topos $\mathscr{E}$ with a natural number object. Over any such topos, the result of Proposition 3.2 remains valid if we interpret all notions relating to operations and equations in a suitable 'internal' sense.
(*b*) Of course, any free algebraic theory (that is, a theory generated by operations satisfying no equations) is strongly regular. As observed by Kelly (1992), the free theory on a single binary operation yields the free-algebra functor that to a set A assigns the set $B(A)$ of binary trees whose leaves are labelled with elements of A. If instead we take the free theory with one generating operation of each (finite) positive arity, we obtain the functor that sends A to $T(A)$, the set of all (finitely-branching) trees whose leaves are labelled with elements of A. These functors, and the categories obtained by glueing along them, play an important role in the work of Kasangian and Vigna (Kasangian and Vigna 1991; Kasangian and Vigna 1991a; Kasangian and Labella 1992).
(*c*) Any unary theory (that is, a theory of M-sets, for some monoid M) is strongly regular. The free functor is simply $(-) \times M$, which is representable by an M-indexed family of singletons (that is, by $1 : M \to M$), and the unit and multiplication are induced by the unit and multiplication of M, which give rise to pullback diagrams in an obvious way.
(*d*) Another (and rather trivial) example is provided by the theory of pointed sets, for which the free functor is simply $(-) + 1$. This example becomes more interesting when we consider its topos-theoretic generalization, which was studied by Kock (1991). The free functor is then the partial-map classifier operation $X \mapsto \widetilde{X}$; the fact that this is a partial product functor does not seem to have been noticed before (in fact it is $P(\top, -)$, where $\top : 1 \to \Omega$ is the generic subobject), but it is implicit in (Johnstone 1977, Example 1.45). The unit and multiplication of the monad are, respectively, induced by the pullback squares



where $\widetilde{\Omega}$ is identified with the order-relation $\Omega_1 \rightarrowtail \Omega \times \Omega$, and $\wedge$ is the binary meet operation on subobjects.

Remark 3.4. The restriction to cartesian natural transformations in the statement of Proposition 3.2 seems not unnatural, given that we are dealing with functors that preserve pullbacks. It cannot, however, be omitted, as the following example shows. Let A be a fixed set. Then the functor $(-)^A : \mathbf{Set} \to \mathbf{Set}$ is representable, and it has a monad structure whose unit and multiplication are induced by composition with the unique map $A \to 1$ and the diagonal $A \to A \times A$. However, these transformations are not cartesian: indeed,

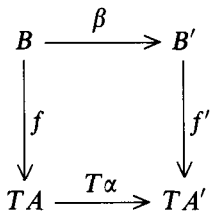
the naturality square



is never a pullback unless either A or B is a singleton, or B is empty and A is not. The algebraic theory corresponding to this monad is the theory $\mathbb{H}_n$ generated by a single n-ary hyperaffine operation (where n is the cardinality of A), described in Section 4 of (Johnstone 1990). (For $n = 2$, this theory is better known to semigroup-theorists as the theory of *rectangular bands*.) A presentation of $\mathbb{H}_n$ by operations and equations is given in (Johnstone 1990); it may be seen that it is not even regular. (By the result of Jónsson and Nelson (1974), this follows from the fact that the two-element semilattice is not a rectangular band.)

4. Artin glueing

Let $\mathscr{E}$ and $\mathscr{F}$ be categories, and $T : \mathscr{E} \to \mathscr{F}$ a functor. By the *category obtained by (Artin) glueing along* T, we mean the comma category $(\mathscr{F} \downarrow T)$, whose objects are triples (A, B, f) with $A \in \mathrm{ob}\mathscr{E}$, $B \in \mathrm{ob}\mathscr{F}$ and $f : B \to TA$ in $\mathscr{F}$, and whose morphisms $(A, B, f) \to (A', B', f')$ are pairs $(\alpha : A \to A', \beta : B \to B')$ such that



commutes. The name 'Artin glueing' is in honour of M. Artin, who established sufficient conditions for $(\mathscr{F} \downarrow T)$ to be a Grothendieck topos (Artin *et al.* 1972, IV 9.5); the corresponding result for elementary toposes was first proved by G. C. Wraith (1974). It was observed by several people that the sufficient condition given by Wraith (*viz.* that $\mathscr{E}$ and $\mathscr{F}$ should be toposes and T should preserve finite limits) was not necessary, in that the condition on T could be weakened to preservation of pullbacks; the first appearance in print of this result seems to have been in (Rosebrugh and Wood 1991), but it was surely known much earlier.

However, until recently no-one seems to have considered whether preservation of pullbacks by T is necessary for $(\mathscr{F} \downarrow T)$ to be a topos. In this section, we will show that it is; more generally, we will establish how various categorical properties of $(\mathscr{F} \downarrow T)$ correspond to categorical properties of $\mathscr{E}$ and $\mathscr{F}$, and preservation properties of T.

Our main theorem is simplest to state in the special case when T does in fact preserve the terminal object.

Theorem 4.1. Suppose $\mathscr{E}$ has a terminal object and $T : \mathscr{E} \to \mathscr{F}$ preserves it. Then, for each of the following categorical properties:

(i) being cartesian closed with an initial object;
(ii) being locally cartesian closed with an initial object;
(iii) being a quasitopos;
(iv) being an (elementary) topos;
(v) being a presheaf topos (that is, equivalent to a functor category $[C^{\mathrm{op}}, \mathbf{Set}]$ where C is small),

the comma category $(\mathscr{F} \downarrow T)$ has the given property if and only if both $\mathscr{E}$ and $\mathscr{F}$ have the property, and T preserves finite products in case (i), finite limits in cases (ii)–(iv) and all (small) limits in case (v).

(In fact case (i) of Theorem 4.1 is not quite an 'if and only if', unless we adopt the (nonstandard) convention that cartesian closedness entails the existence of all finite limits, and not just finite products. The construction of exponentials in $(\mathscr{F} \downarrow T)$ requires the existence of pullbacks in $\mathscr{F}$; however, the proof of 'only if' does not make any use of pullbacks.)

The general version of the theorem (that is, when T is not assumed to preserve the terminal object) may be reduced to the particular case considered above by the following argument (which we will refer to as the 'standard reduction'):

Lemma 4.2. Suppose $\mathscr{E}$ has a terminal object 1. Then, if we factor T as

$$\mathscr{E} \xrightarrow{\widehat{T}} \mathscr{F}/T1 \xrightarrow{\Sigma_{T1}} \mathscr{F},$$

as in Lemma 2.1, the comma categories $(\mathscr{F} \downarrow T)$ and $((\mathscr{F}/T1) \downarrow \widehat{T})$ are isomorphic.

Proof. We identify an object (A, B, f) of $(\mathscr{F} \downarrow T)$ with the object (A, b, f) of $((\mathscr{F}/T1) \downarrow \widehat{T})$, where b is the composite $\widehat{T}(A)f : B \to TA \to T1$. The remaining details are straightforward. □

We may now state the general form of our theorem. The necessary and sufficient condition for (mere) cartesian closedness of $(\mathscr{F} \downarrow T)$ is not very transparent, but the other four properties translate as follows:

Theorem 4.3. Let $T : \mathscr{E} \to \mathscr{F}$ be an arbitrary functor. Then, for each of the properties (ii)–(v) of Theorem 4.1, $(\mathscr{F} \downarrow T)$ has the property iff $\mathscr{E}$ and $\mathscr{F}/T1$ have the property and T preserves pullbacks (cases (ii)–(iv)) or wide pullbacks (case (v)).

Note that we cannot hope, in the case when T does not preserve 1, to deduce anything about the structure of $\mathscr{F}$ from that of $(\mathscr{F} \downarrow T)$, since objects of $\mathscr{F}$ that admit no morphisms to $T1$ will never occur as the second projections of objects of $(\mathscr{F} \downarrow T)$. Thus the form of Theorem 4.3 is 'best possible' if we do not assume anything *a priori* about $\mathscr{E}$ and $\mathscr{F}$. However, if we know that they are categories of the appropriate kind, we can rephrase Theorem 4.3 as follows:

Corollary 4.4. Suppose $\mathscr{E}$ and $\mathscr{F}$ are categories having one of the properties (ii)–(v) of Theorem 4.1, and let $T : \mathscr{E} \to \mathscr{F}$ be a functor. Then $(\mathscr{F} \downarrow T)$ has the property in question iff T preserves pullbacks (cases (ii)–(iv)) or wide pullbacks (case (v)).

Corollary 4.4 follows at once from Theorem 4.3 because each of the properties (ii)–(v) is stable under slicing (that is, if $\mathscr{F}$ has the property, so does $\mathscr{F}/B$ for any B) – for the proof of this in case (v), see Johnstone (1977, 2.18).

Combining 4.3(v) with Theorem 2.6, we note the following corollary, which fulfils a promise made in the Introduction.

Corollary 4.5. Let $\mathscr{E}$ be a topos and $T : \mathscr{E} \to \mathbf{Set}$ a functor. Then $(\mathbf{Set} \downarrow T)$ is a presheaf topos iff $\mathscr{E}$ is a presheaf topos and T is familially representable.

Example 4.6. As an application of Corollary 4.5, we recover the observation of S. H. Schanuel (unpublished) that the category of computads in the sense of Street (1976) is a presheaf topos. A *computad* (to be thought of as a 'presentation' of a 2-category) consists of three types of entities: objects, arrows and 2-cells; each arrow has a source and target that are objects, and each 2-cell has a source and target that are formal composites of arrows, that is, a parallel pair of morphisms in the free category on the directed graph formed by the objects and arrows. Let $\mathscr{E}$ be the topos of directed graphs (that is, the functor category $[C^{\mathrm{op}}, \mathbf{Set}]$, where C is the category $(\bullet \rightrightarrows \bullet)$). The functor $T : \mathscr{E} \to \mathbf{Set}$, which assigns to a directed graph G the set of parallel pairs of morphisms in the free category FG, is familially representable: the representing family consists of all graphs $G(n, m)$ having two strings of arrows (one of length n, the other of length m) joining the same pair of vertices. And the category of computads is simply $(\mathbf{Set} \downarrow T)$. (If we restrict our attention to one-object computads, that is, to presentations of strict monoidal categories, we obtain the comma category $(\mathbf{Set} \downarrow M \times M)$, where M is the free-monoid functor; we have already observed in Section 2 that $M \times M$ was familially representable. It should also be clear that we can 'iterate' the construction in this example: by glueing along the appropriate (familially representable) functor from computads to **Set**, we obtain a category whose objects are presentations of 3-categories, and so on.)

We now embark on the proofs of Theorems 4.1 and 4.3. Note that each of the categorical properties (i)–(v) entails the existence of a terminal object; thus the 'if' parts of Theorem 4.3 follow immediately from the corresponding parts of Theorem 4.1 by the standard reduction. For the 'only if' parts, we need to know that the existence of a terminal object in $(\mathscr{F} \downarrow T)$ implies that $\mathscr{E}$ has one too; but this follows at once from the following lemma.

Lemma 4.7. The projection functor $P_1 : (\mathscr{F} \downarrow T) \to \mathscr{E}$ has a full and faithful right adjoint R_1.

Proof. We define $R_1(A) = (A, TA, 1_{TA})$. Then it is clear that a morphism $(\alpha, \beta) : (A', B', f') \to R_1(A)$ is entirely determined by $\alpha : A' = P_1(A', B', f') \to A$, since we must have $\beta = T(\alpha)f'$. This establishes the adjunction, and the counit map $P_1R_1(A) \to A$ is clearly the identity, so R_1 is full and faithful. □

Thus $\mathscr{E}$ may be identified with a reflective subcategory of $(\mathscr{F} \downarrow T)$; in particular, it has any limits or colimits that exist in $(\mathscr{F} \downarrow T)$.

It thus suffices to prove Theorem 4.1, so from now on we assume that the standard reduction has been made. We begin by disposing of the 'if' parts of the Theorem. Case (iv) is well known, being a corollary of the main result of (Wraith 1974) (see also Johnstone (1977, 4.27)). The fact that Artin glueing works for quasitoposes (case (iii)) is less well known, but it follows easily from (Wyler 1991, 50.5). Inspection of the proof (*cf.* (Johnstone 1977, 2.32) or (Wyler 1991, 50.2)) will show that the cartesian closedness of $(\mathscr{F} \downarrow T)$ follows from that of $\mathscr{E}$ and $\mathscr{F}$ plus the preservation of finite products by T (and the existence of pullbacks in $\mathscr{F}$, as mentioned earlier). And it is easily seen that $(\mathscr{F} \downarrow T)$ inherits any

types of colimits that exist in both $\mathscr{E}$ and $\mathscr{F}$ (in fact the two projection functors P_1 and P_2 jointly create them), so this disposes of case (i). For case (ii), let (A, B, f) be an object of $(\mathscr{F} \downarrow T)$, and let T' be the composite functor

$$\mathscr{E}/A \xrightarrow{T/A} \mathscr{F}/TA \xrightarrow{f^*} \mathscr{F}/B,$$

where T/A denotes T applied to objects and morphisms over A, and f^* denotes pullback along f. Then T' preserves finite products, and it is easy to see (much as in Lemma 4.2) that $(\mathscr{F} \downarrow T)/(A, B, f)$ is isomorphic to $((\mathscr{F}/B) \downarrow T')$. So (ii) follows from (i).

Thus it only remains to establish (v). For this we use the concept of profunctor collage, which we briefly recall.

Let PROF be the bicategory of small categories and profunctors between them; so, an arrow $\phi : C \to D$ in PROF is an object ϕ of $[C^{\mathrm{op}} \times D, \mathbf{Set}]$ whose value on a pair (c, d) we will denote, following Lawvere (1973), as $\phi(d, c)$. Composition in PROF is profunctor composition: if $\psi : D \to E$ is another composable profunctor, then

$$(\psi \otimes_D \phi)(e, c) = \int^d \psi(e, d) \times \phi(d, c) \, .$$

Profunctor composition with a given ϕ induces a cocontinuous functor

$$\phi \otimes_D (-) : [D^{\mathrm{op}}, \mathbf{Set}] \to [C^{\mathrm{op}}, \mathbf{Set}] \, ,$$

and this has a right adjoint $\hom^{\phi}$, which can easily be calculated as

$$\hom^{\phi}(X)(d) = [C^{\mathrm{op}}, \mathbf{Set}]\,(\phi(d, -), X) \, .$$

Hence $\phi \mapsto \hom^{\phi}$ defines a homomorphism of bicategories $\hom : \mathsf{PROF}^{\mathrm{co}} \to \mathsf{CAT}$. In the converse direction, if $T : [C^{\mathrm{op}}, \mathbf{Set}] \to [D^{\mathrm{op}}, \mathbf{Set}]$ is any continuous functor, the special adjoint functor theorem ensures that T has a left adjoint L; and restricting L along the Yoneda embedding yields a functor $D \to [C^{\mathrm{op}}, \mathbf{Set}]$ – equivalently, a profunctor $\phi : C \to D$ – such that L is isomorphic to the left Kan extension of this functor, that is, $L \cong \phi \otimes_D (-)$. Hence $T \cong \hom^{\phi}$. Thus we have shown the following lemma.

Lemma 4.8. The assignment $\phi \mapsto \hom^{\phi}$ defines an equivalence of categories from $\mathsf{PROF}(C, D)^{\mathrm{op}}$ to the category of continuous functors $[C^{\mathrm{op}}, \mathbf{Set}] \to [D^{\mathrm{op}}, \mathbf{Set}]$ (and arbitrary natural transformations between them).

Recall now that the *collage* (see Carboni *et al.* (1991)) of a profunctor $\phi : C \to D$ is the category $\mathrm{Coll}(\phi)$ whose set of objects is the disjoint union of the objects of C and of D, and whose hom is defined by cases as follows: $\mathrm{Coll}(\phi)(c, c') = C(c, c')$, $\mathrm{Coll}(\phi)(c, d) = \phi(d, c)$, $\mathrm{Coll}(\phi)(d, c) = \emptyset$ and $\mathrm{Coll}(\phi)(d, d') = D(d, d')$. The collage of a profunctor has in PROF a universal property similar to the one that the glueing of a functor has in CAT: that is, it is a lax limit (*cf.* (Wraith 1974)) for a diagram of shape $(\bullet \to \bullet)$.

Lemma 4.9. Let $\phi : C \to D$ be a profunctor. Then the canonical functor

$$[\mathrm{Coll}(\phi)^{\mathrm{op}}, \mathbf{Set}] \to ([D^{\mathrm{op}}, \mathbf{Set}] \downarrow \hom^{\phi})$$

is an equivalence.

Proof. A presheaf on $\mathrm{Coll}(\phi)$ consists of a presheaf X on C, a presheaf Y on D and a family of functions $\alpha(d,c): \phi(d,c) \to X(c)^{Y(d)}$ such that for all $f: c \to c'$ and all $g: d' \to d$, the squares

$$\begin{array}{ccc} \phi(d,c') & \xrightarrow{\alpha(d,c')} & X(c')^{Y(d)} \\ {\scriptstyle \phi(d,f)}\downarrow & & \downarrow{\scriptstyle X(f)^{Y(d)}} \\ \phi(d,c) & \xrightarrow{\alpha(d,c)} & X(c)^{Y(d)} \end{array} \quad \text{and} \quad \begin{array}{ccc} \phi(d',c) & \xrightarrow{\alpha(d',c)} & X(c)^{Y(d')} \\ {\scriptstyle \phi(g,c)}\downarrow & & \downarrow{\scriptstyle X(c)^{Y(g)}} \\ \phi(d,c) & \xrightarrow{\alpha(d,c)} & X(c)^{Y(d)} \end{array}$$

commute. So, transposing, for all $d \in D_0$ we get a function

$$\alpha_d: Y(d) \to [C^{\mathrm{op}}, \mathbf{Set}](\phi(d,-), X),$$

which is natural in d, that is, a natural transformation

$$\alpha: Y \to [C^{\mathrm{op}}, \mathbf{Set}](\phi(-,-), X) = \hom^{\phi}(X),$$

and one can easily see that every natural transformation in $[\mathrm{Coll}(\phi)^{\mathrm{op}}, \mathbf{Set}]$ gives rise in a full and faithful way to a morphism in the comma category. Clearly every natural transformation $Y \to \hom^{\phi}(X)$ can be described as above, hence the equivalence is established. □

Combining the last two results, we have completed the proof of the 'if' part of Theorem 4.1(v).

We now embark on proving the 'only if' parts of Theorem 4.1. First we need a companion for Lemma 4.7.

Lemma 4.10. If $\mathscr{E}$ has a terminal object and $T: \mathscr{E} \to \mathscr{F}$ preserves it, the projection $P_2: (\mathscr{F} \downarrow T) \to \mathscr{F}$ has a full and faithful right adjoint R_2.

Proof. We define $R_2(B) = (1, B, B \to 1)$. Then it is clear that a morphism $(\alpha, \beta): (A', B', f') \to R_2(B)$ is entirely determined by $\beta: B' = P_2(A', B', f') \to B$, since α is the unique morphism $A \to 1$. Thus R_2 is right adjoint to P_2. And the counit $P_2R_2(B) \to B$ is clearly the identity, so R_2 is full and faithful. □

Thus we may identify $\mathscr{F}$, like $\mathscr{E}$, with a reflective subcategory of $(\mathscr{F} \downarrow T)$, and so deduce that it too has any limits or colimits that exist in $(\mathscr{F} \downarrow T)$. In particular, each of conditions (i)–(v) implies the existence of an initial object, so we now know that $\mathscr{F}$ has an initial object. But we have the following lemma.

Lemma 4.11. Suppose $\mathscr{F}$ has an initial object. Then $P_1: (\mathscr{F} \downarrow T) \to \mathscr{E}$ has a left adjoint L_1, as well as a right adjoint.

Proof. We define $L_1(A) = (A, 0, 0 \to TA)$, where 0 is the initial object of $\mathscr{F}$. As before, the adjunction is easy to verify. □

For the next result, we need to know that the initial object of $\mathscr{F}$ is *strict*, that is, that any morphism $B \to 0$ in $\mathscr{F}$ is an isomorphism. It is well known that the initial object, if it exists, in a cartesian closed category is strict; however, strictness of initial objects is not in general inherited by reflective subcategories (for a simple counterexample, take any

category with a non-strict initial object and freely adjoin a new (strict) initial object to it – the original category appears as a reflective subcategory of the new one). But in this particular case, we have the following lemma.

Lemma 4.12. If $(\mathscr{F} \downarrow T)$ has a strict initial object, then so has $\mathscr{F}$.

Proof. Let (U, V, f) be the initial object of $(\mathscr{F} \downarrow T)$. If $g : B \to V$ is any morphism of $\mathscr{F}$ with codomain V, then $(1_U, g)$ is a morphism $(U, B, fg) \to (U, V, f)$ in $(\mathscr{F} \downarrow T)$. Hence g is an isomorphism. □

Lemma 4.13. Suppose that $\mathscr{F}$ has a strict initial object, and that $(\mathscr{F} \downarrow T)$ has finite products. Then the 'Frobenius reciprocity' condition holds for the adjunction $(L_1 \dashv P_1)$; that is, the canonical morphism

$$L_1(A \times P_1(A', B', f')) \longrightarrow L_1(A) \times (A', B', f')$$

is an isomorphism for all A and all (A', B', f').

Proof. Since P_1 preserves products (by Lemma 4.11), we know that the first component of the product on the right-hand side must be $A \times A'$. The second component of this product admits a morphism to $P_2 L_1(A) = 0$, so it must be isomorphic to 0. Thus both components of the morphism $L_1(A \times A') \to L_1(A) \times (A', B', f')$ are isomorphisms. □

For later reference, we note the following corollary.

Corollary 4.14. Suppose, in addition to the hypotheses of Lemma 4.13, that $\mathscr{E}$ and $(\mathscr{F} \downarrow T)$ are cartesian closed. Then the projection P_1 preserves exponentials.

Proof. For a functor between cartesian closed categories having a left adjoint, preservation of exponentials is well known to be equivalent to the Frobenius reciprocity condition. □

More immediately, however, by specializing Lemma 4.13 to the case $A = 1$, we obtain the following corollary.

Corollary 4.15. Under the hypotheses of Lemma 4.13, the functor L_1 induces an equivalence of categories between $\mathscr{E}$ and $(\mathscr{F} \downarrow T)/U$, where $U = L_1(1)$.

Proof. Putting $A = 1$ in the Frobenius reciprocity condition, we obtain a natural isomorphism $L_1 P_1(X) \cong X \times U$ for $X \in \mathrm{ob}(\mathscr{F} \downarrow T)$; moreover, the isomorphism identifies the counit of the adjunction $(L_1 \dashv P_1)$ with the projection $X \times U \to X$. The comultiplication of the comonad structure on $L_1 P_1$ is similarly identified with the natural transformation induced by the diagonal $U \to U \times U$, and so $\mathscr{E}$, being a coreflective subcategory of $(\mathscr{F} \downarrow T)$, is equivalent to the category of coalgebras for this comonad, which is simply $(\mathscr{F} \downarrow T)/U$. □

Note that, since the comonad structure on $(-) \times U$ is idempotent, U must be a subobject of 1 in $(\mathscr{F} \downarrow T)$.

Corollary 4.15 immediately implies, in each of cases (ii)–(v) of Theorem 4.1, that $\mathscr{E}$ must inherit the appropriate property from $(\mathscr{F} \downarrow T)$, since these properties are stable under slicing. For case (i), we use a different argument involving the right adjoint R_1 of P_1. We know that R_1 is full and faithful and its left adjoint P_1 preserves binary products, and a well-known theorem of Day ((Day 1972); see (Wyler 1991, 9.4) or (Freyd and Scedrov 1990, 1.857)) asserts that a reflective subcategory of a cartesian closed category

is an *exponential ideal* (that is, if X is isomorphic to an object in the subcategory, so is X^Y for any Y) iff the reflector preserves binary products. Hence, in particular, the image of R_1 is closed under exponentials in $(\mathscr{F} \downarrow T)$. Since it is also full and closed under binary products, it is cartesian closed.

Next, we turn our attention to the image of R_2.

Lemma 4.16. For an object X of $(\mathscr{F} \downarrow T)$, the unit map $X \to R_2P_2(X)$ is an isomorphism iff $P_1(X) \cong 1$.

Proof. Immediate from the definition of R_2. □

Corollary 4.17. If $(\mathscr{F} \downarrow T)$ is cartesian closed and has an initial object, then

(i) the image of R_2 is an exponential ideal in $(\mathscr{F} \downarrow T)$;
(ii) P_2 preserves binary products.

Proof. (i) follows from the fact that P_1 preserves exponentials (Corollary 4.14) and $1^A \cong 1$ for any $A \in \mathrm{ob}\,\mathscr{E}$.
(ii) follows from (i) by the converse of the result used above to prove that the image of R_1 is an exponential ideal. □

Corollary 4.17 completes the proof of the 'only if' part of case (i) of Theorem 4.1, as we now know that both $\mathscr{E}$ and $\mathscr{F}$ are equivalent to exponential ideals in $(\mathscr{F} \downarrow T)$, and $T = P_2R_1$ is a composite of two functors each of which preserves binary products (and we already knew it preserved 1). We can deduce case (ii) from case (i) much as we did in the opposite direction, by expressing $(\mathscr{F} \downarrow T)/(A, B, f)$ as the category obtained by glueing along a functor $\mathscr{E}/A \to \mathscr{F}/B$. Since any object of $\mathscr{E}$ (respectively, $\mathscr{F}$) can occur as the first (respectively, second) component of an object of $(\mathscr{F} \downarrow T)$, this shows that all the slices $\mathscr{E}/A$ and $\mathscr{F}/B$ are cartesian closed. Moreover, by considering objects of the form $R_1(A)$, we see that $T/A : \mathscr{E}/A \to \mathscr{F}/TA$ must preserve binary products for every A – equivalently, T preserves pullbacks over A for every A. Once again, since we know that T preserves 1, this implies that it preserves all finite limits.

To complete the proof of 'only if' in cases (iii) and (iv), we have only to prove that the subcategory of $(\mathscr{F} \downarrow T)$ described in Lemma 4.16 is a topos (respectively, quasitopos) if $(\mathscr{F} \downarrow T)$ is. For toposes, this is well known, since the condition $X \times U \cong U$ characterizes the objects of the closed subtopos complementary to $(\mathscr{F} \downarrow T)/U$ (*cf.* (Johnstone 1977, 3.53)). For quasitoposes, the result is less well known, but may be proved in exactly the same way. (Alternatively, we could argue that $\mathscr{F}$ is a reflective subcategory of $(\mathscr{F} \downarrow T)$ for which the reflector preserves finite limits (*cf.* (Johnstone 1977, 4.15) or (Wyler 1991, 53.1)).

To deduce the 'only if' part of case (v), we need one further result, which is well known but does not seem to be in print anywhere (though part of it appears as Example 13 in (Johnstone 1979)).

Lemma 4.18. Let $\mathscr{P}$ be a presheaf topos. Then any closed subtopos $\mathscr{C}$ of $\mathscr{P}$ is essential (that is, the associated sheaf functor $\mathscr{P} \to \mathscr{C}$ has a left adjoint), and it is also a presheaf topos.

Proof. Let $\mathscr{P} = [E^{\mathrm{op}}, \mathbf{Set}]$ for a small category E. Then subobjects of 1 in $\mathscr{P}$ correspond to *sieves* in E, that is to full subcategories R of E such that if $A \to B$ is a morphism of E with $B \in \mathrm{ob}\,R$, then $A \in \mathrm{ob}\,R$. The open subtopos corresponding to a sieve R is

$[R^{\mathrm{op}}, \mathbf{Set}]$, and an object F of $[E^{\mathrm{op}}, \mathbf{Set}]$ belongs to the complementary closed subtopos iff $F(A) \cong 1$ for all $A \in \mathrm{ob}\, R$. But this condition characterizes the right Kan extensions to E of presheaves on the cosieve complementary to R, that is, the full subcategory S of E on those objects that are not in R. Thus we can identify the closed subtopos complementary to $[R^{\mathrm{op}}, \mathbf{Set}]$ with $[S^{\mathrm{op}}, \mathbf{Set}]$; in particular, the associated sheaf functor $[E^{\mathrm{op}}, \mathbf{Set}] \to [S^{\mathrm{op}}, \mathbf{Set}]$ is identified with restriction along the inclusion functor $S \to E$, so it has adjoints on both sides. □

Applying this result in the case when $(\mathscr{F} \downarrow T)$ is a presheaf topos, we deduce that $\mathscr{F}$ is a presheaf topos and that P_2 preserves all limits, whence $T = P_2 R_1$ does so too.

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