**BRICS LS-01-catnotes: Lecture Notes in Category Theory [Draft]**



**Basic Research in Computer Science**

# Lecture Notes in Category Theory [Draft]

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# 1 Categories, Functors and Natural Transformations

## 1.1 Categories

**Definition 1.1 (Category)** A *category* $\mathcal{C}$ consists of

- a collection of *objects* $\mathbf{ob}(\mathcal{C})$,
- for each pair of objects $X, Y$ a collection $\mathcal{C}(X,Y)$ of *arrows/maps/morphisms*,
- for each triple of objects $X, Y, Z$ a *composition* mapping $\circ_{XYZ} : \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ and
- for each object $X$ an *identity* $\mathsf{id}_X$ in $\mathcal{C}(X,X)$;

satisfying the axioms

$$h \circ_{XZW} (g \circ_{XYZ} f) = (h \circ_{YZW} g) \circ_{XYW} f \qquad f \circ_{XXY} \mathsf{id}_X = f = \mathsf{id}_Y \circ_{XYY} f$$

for any $f$ in $\mathcal{C}(X,Y)$, $g$ in $\mathcal{C}(Y,Z)$ and $h$ in $\mathcal{C}(Z,W)$ where $X, Y, Z, W$ are in $\mathbf{ob}(\mathcal{C})$.

**Notation 1.2** Given a category $\mathcal{C}$, we write $X \in \mathcal{C}$ to mean that $X$ is an object in $\mathbf{ob}(\mathcal{C})$. In general we use capital letters from the beggining or end of the alphabet for objects: $A, B, \ldots, X, Y, Z$. Given an arrow $f$ in $\mathcal{C}(X,Y)$ we say that $X$ is the *domain* of $f$ and $Y$ is the *codomain* of $f$. This is emphasised in the diagrammatic notation for categories where $f$ in $\mathcal{C}(X,Y)$ is pictured as $X \xrightarrow{f} Y$; in the same spirit we sometimes write $f : X \to Y$. In general we use small letters for arrows: $f, g, h, \ldots, u, v, \ldots$. For $f$ in $\mathcal{C}(X,Y)$ and $g$ in $\mathcal{C}(Y,Z)$ we write $f \circ g$ for the composition omitting the subscripts; in some cases when the composition is clear we just write $g\,f$ instead of $g \circ f$:

$$X \underset{g\,f}{\overset{f}{\longrightarrow}} Y \xrightarrow{g} Z \,.$$

Since composition is associative we sometimes omit parenthesis, for example we write $h \circ g \circ f$ to mean either $h \circ (g \circ f)$ or $(h \circ g) \circ f$. We use $\mathbf{arr}(\mathcal{C})$ for $\biguplus_{X,Y \in \mathcal{C}} \mathcal{C}(X,Y)$, the collection of all arrows in $\mathcal{C}$.

**Definition 1.3 (Locally Small Category)** A category $\mathcal{C}$ is *locally small* iff for each pair $X, Y \in \mathcal{C}$ the collection $\mathcal{C}(X,Y)$ is a set.

The category **Set** of (small) sets as objects and total functions between sets as arrows is a locally small category where composition is defined as composition of functions and identities correspond to the the identity functions.

**Definition 1.4 (Small Category)** A locally small category is *small* iff the collection of objects is a set.

Simple examples of small categories are **0** and **1**, the empty category and the singleton category respectively. Another example of small category is given by any partial-order set where the objects are the elements and the arrows are given by the partial-order relation; the identities correspond to reflexivity and composition to transitivity.

**Exercise 1.5** Show that any set can be interpreted as a small category. In general categories of this kind are called *discrete* categories, but not all discrete categories are small.

**Notation 1.6** Henceforth, we use $\mathcal{A}, \mathcal{B}, \mathcal{C} \ldots$ to denote locally small categories and $\mathbb{I}, \mathbb{J}, \mathbb{C} \ldots$ to small categories. Given a locally small category $\mathcal{C}$ we call the collection $\mathcal{C}(X, Y)$ the *hom-set* for $X, Y \in \mathcal{C}$.

**Definition 1.7 (Isomorphism)** An *isomorphism* (or just *iso*) is a pair of arrows

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$.

**Proposition 1.8** Given one arrow of an iso pair, the other arrow is uniquely determined.

**Proof**: Assume $(f, g)$ and $(f, g')$ are iso pairs, where $f : X \to Y$, then $(g' \circ f) \circ g = \mathsf{id}_X \circ g = g$ and $g' \circ (f \circ g) = g' \circ \mathsf{id}_Y = g'$. From associativity of the composition $(g' \circ f) \circ g = g' \circ (f \circ g)$ and then $g = g'$.

□

**Notation 1.9** This proposition allows us to denote an iso pair by just one of the arrows. If $f : X \to Y$ is an arrow of an iso pair, we say that $f$ is an *isomorphism* (or $f$ is an *iso*), and we write $X \overset{f}{\cong} Y$ or $f : X \cong Y$.

**Definition 1.10 (Mono)** The arrow $f : X \to Y$ in the category $\mathcal{C}$ is *mono* (or *monic*) iff for any pair of arrows $g, h$

$$Z \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

we have that $f \circ g = f \circ h$ implies $g = h$.

**Exercise 1.11** Show that the composition of monos gives a mono. Show that if the composition $g \circ f$ is mono then $f$ is mono. What are the monos in **Set**?

**Definition 1.12 (Epi)** The arrow $f : X \to Y$ in the category $\mathcal{C}$ is *epi* iff for any pair of arrows $g, h$ $Z$ and for all $g, h \in \mathcal{C}(Y, Z)$

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} Z$$

we have that $g \circ f = h \circ f$ implies $g = h$.

**Exercise 1.13** Show that the composition of epis gives an epi. Show that if the composition $g \circ f$ is epi then $g$ is epi. What are the epis in **Set**?

## 1.2 Functors

**Definition 1.14 (Functor)** Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A *functor* $F : \mathcal{A} \to \mathcal{B}$ consists of

- a mapping $F_0 : \mathbf{ob}(\mathcal{A}) \to \mathbf{ob}(\mathcal{B})$ and
- for each pair $X, Y \in \mathcal{A}$ a mapping $F_{XY} : \mathcal{A}(X, Y) \to \mathcal{B}\big(F_0(X), F_0(Y)\big)$;

satisfying the axioms

$$F_{XZ}(g \circ f) = F_{YZ}(g) \circ F_{XY}(f) \qquad F_{XX}(\mathsf{id}_X) = \mathsf{id}_{F_0(X)}$$

for any $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$.

**Notation 1.15** For a functor $F : \mathcal{A} \to \mathcal{B}$ we usually write just $F$ for $F_0$ and $F_{XY}$. Sometimes we write $F\,X$ instead of $F(X)$ for $X \in \mathcal{A}$ and similarly with arrows in $\mathcal{A}$. In general we use capital letters from the middle of the alphabet for functors, *e.g.*, $F, G, H, K$.

**Definition 1.16 (Full Functor)** A functor $F : \mathcal{A} \to \mathcal{B}$ is *full* iff for each pair $X, Y \in \mathcal{A}$ the mapping $F : \mathcal{A}(X, Y) \to \mathcal{B}(F\,X, F\,Y)$ is surjective.

**Definition 1.17 (Faithful Functor)** A functor $F : \mathcal{A} \to \mathcal{B}$ is *faithful* iff for each pair $X, Y \in \mathcal{A}$ the mapping $F : \mathcal{A}(X, Y) \to \mathcal{B}(F\,X, F\,Y)$ is injective.

For a functor $F$ which is both full and faithful (*fully faithful*) the mappings $F : \mathcal{A}(X, Y) \to \mathcal{B}(F\,X, F\,Y)$ are all bijections.

**Proposition 1.18** Functors preserve iso: if $(f, g)$ is an iso pair in the category $\mathcal{A}$, and $F : \mathcal{A} \to \mathcal{B}$ is a functor, then $(F\,f, F\,g)$ is an iso pair in $\mathcal{B}$.

**Proof**: Let $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ be iso in $\mathcal{A}$. We are to show that $F\,X \underset{F\,g}{\overset{F\,f}{\rightleftarrows}} F\,Y$ is iso in $\mathcal{B}$. This follows from

$$\begin{aligned}(F\,f) \circ (F\,g) &= F(f \circ g) \\ &= F\,\mathsf{id}_Y \qquad \text{as } (f, g) \text{ is an iso pair,} \\ &= \mathsf{id}_{F\,Y}\ .\end{aligned}$$

Analogously, we can conclude $(F\,g) \circ (F\,f) = \mathsf{id}_{F\,X}$.

□

**Proposition 1.19** If $F : \mathcal{A} \to \mathcal{B}$ is a full and faithful functor, then $F$ reflects isomorphisms: if $F\,X \cong F\,Y$ then $X \cong Y$.

**Proof**: Assume $F\,X \cong F\,Y$. Then there exists a pair of arrows $f, g$

$$F\,X \underset{g}{\overset{f}{\rightleftarrows}} F\,Y$$

in $\mathcal{B}$ such that $f \circ g = \mathsf{id}_{F\,Y}$ and $g \circ f = \mathsf{id}_{F\,X}$. By fullness there exists $f' : X \to Y$ and $g' : Y \to X$ in $\mathcal{A}$ such that $F\,f' = f$ and $F\,g' = g$. We show that $f'$ and $g'$ are mutual inverses and so an iso pair:

$$\begin{aligned}F(f' \circ g') &= F\,f' \circ F\,g' \\ &= f \circ g \\ &= \mathsf{id}_{F\,Y} \\ &= F(\mathsf{id}_Y).\end{aligned}$$

By faithfulness we have $f' \circ g' = \mathsf{id}_Y$. Similarly, we can conclude $g' \circ f' = \mathsf{id}_X$.

□

**Definition 1.20 (Composition of Functors)** Given the functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ their composition $G \circ F : \mathcal{A} \to \mathcal{C}$ is defined as

- $(G \circ F)_0 = G_0 \circ F_0$ and
- $(G \circ F)_{XY} = G_{F_0(X)F_0(Y)} \circ F_{XY}$ for $X, Y \in \mathcal{A}$.

**Exercise 1.21** Check that this definition gives rise to a functor.

It is easy to see that the composition of functors is associative. Moreover, for any category $\mathcal{A}$ we can define the *identity functor* $\mathsf{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ to be the functor which it is the identity on both object and arrow parts. These observations allow us to define categories of categories where arrows are functors. In particular, we define the category **Cat** of small categories and **CAT** of locally small categories.

From the observation that functors are arrows in a category of categories we can specialised the general definitions of iso, epi and mono to functors. For example, if a functor $F$ is defined as both a bijection on objects and arrows we can define a functor $G$ to be the inverse of $F$, and thus $F$ is an isomorphism.

**Exercise 1.22** Why a full and faithful functor is not necessary an isomorphism?

**Definition 1.23 (Subcategory)** A *subcategory* $\mathcal{A}$ of a category $\mathcal{B}$, written as $\mathcal{A} \subseteq \mathcal{B}$, is a collection of objects and arrows in $\mathcal{B}$ such that:

- for each arrow $f$ in $\mathcal{A}$, we have that the domain and codomain of $f$ are in $\mathcal{A}$,
- for each object $A$ in $\mathcal{A}$, we have the arrow $\mathsf{id}_A$ in $\mathcal{A}$, and
- for each pair of composable arrows $f : A \to B$ and $g : B \to C$ in $\mathcal{A}$, we have the composition $g \circ f : A \to C$ in $\mathcal{A}$.

Clearly, the definition above ensures that a subcategory is itself category. The mapping which sends objects and arrows in $\mathcal{A}$ to themselves in $\mathcal{B}$ defines the *inclusion* functor. This functor is faithful and if it is full we say that $\mathcal{A}$ is a *full subcategory* of $\mathcal{B}$.

## 1.3 Natural Transformations

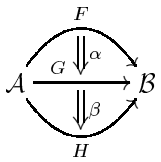
**Definition 1.24 (Natural Transformation)** Given the functors $F, G : \mathcal{A} \to \mathcal{B}$, a *natural transformation* $\alpha : F \Rightarrow G$ consists of a collection of arrows $\langle \alpha_C : F\,C \to G\,C \rangle_{C \in \mathcal{A}}$ in $\mathcal{B}$, such that for any arrow $h : X \to Y$ in $\mathbf{arr}(\mathcal{A})$ the diagram

$$\begin{array}{ccc} F\,X & \xrightarrow{\alpha_X} & G\,X \\ \downarrow{\scriptstyle F\,h} & & \downarrow{\scriptstyle G\,h} \\ F\,Y & \xrightarrow[\alpha_Y]{} & G\,Y \end{array}$$

commutes, *i.e.*, $\alpha_Y \circ F\,h = G\,h \circ \alpha_X$. The diagram above is the *naturality square* associated with $h$.

**Notation 1.25** A natural transformation $\alpha : F \Rightarrow G$ for functors $F, G : \mathcal{A} \to \mathcal{B}$ is a family of arrows in $\mathcal{B}$ indexed by objects in $\mathcal{A}$. Each arrow of such a family is a *component* of the natural transformation, and we use subscripts to denote them, *e.g.*, $\alpha_C : F\,C \to G\,C$. We use angle brackets to denote the family of arrows as in the definition $\langle \alpha_C : F\,C \to G\,C \rangle_{C \in \mathcal{A}}$. Naturality squares are often depicted with the arrow which they are associated with on one side, recall that the arrow and the naturality square may be from different categories.

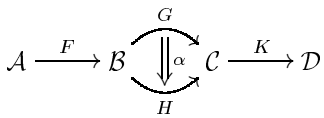
**Definition 1.26 (Composition of Natural Transformations)** The composition of natural transformations is defined componentwise: given two natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$




their composition $\beta \circ \alpha$ at component $A \in \mathcal{A}$ is $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$.

**Exercise 1.27** Check that the componentwise composition of two natural transformations is a natural transformation.

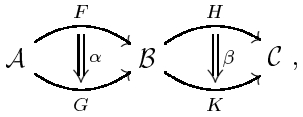
This composition is often called *vertical* composition to distinguish it from the *horizontal* composition of natural transformations – see exercise 1.29 below. Given the functors $F : \mathcal{A} \to \mathcal{B}$ and $K : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha : G \Rightarrow H$




we define the composition $\alpha\,F : G \circ F \Rightarrow H \circ F$ componentwise as $(\alpha\,F)_A = \alpha_{F(A)}$ for $A \in \mathcal{A}$. Similarly the composition $K\,\alpha : K \circ G \Rightarrow K \circ H$ at component $B \in \mathcal{B}$ is defined by $(K\,\alpha)_B = K(\alpha_B)$.

**Exercise 1.28** Check that the composition of natural transformations with functors defined above are natural transformations.

**Exercise 1.29** Given the natural transformations $\alpha$ and $\beta$




we can define the *horizontal* composition $\beta \star \alpha : H \circ F \Rightarrow K \circ G$ – how? **Hint:** The composition of natural transformations with functors is a degenerated case of the composition $\star$ where the functor stands for the identity natural transformation, *i.e.*, a natural transformation which has identities at each component.

## 1.4 Functor Categories

Natural transformations between functors can be (vertically) composed and it is easy to check that this composition is associative. Moreover, given a functor $F : \mathcal{A} \to \mathcal{B}$ we can define the identity $\mathsf{id}_F : F \Rightarrow F$ to be the natural transformation with identity at all components: $(\mathsf{id}_F)_C = \mathsf{id}_{F(C)}$. That actually defines a category of functors and natural transformations.

**Definition 1.30 (Functor Category)** Given the categories $\mathcal{A}$ and $\mathcal{B}$, the *functor category* $[\mathcal{A}, \mathcal{B}]$ is the category where the objects are the functors from $\mathcal{A}$ to $\mathcal{B}$ and the arrows are the natural transformations between them.

As natural transformations are arrows in a functor category we can specialised the notions of iso, mono and epi to them. For example, a natural transformation $\alpha : F \Rightarrow G$ is an iso, called *natural isomorphism*, if there is a natural transformation $\beta : G \Rightarrow F$ such that $\beta \circ \alpha = \mathsf{id}_F$ and $\alpha \circ \beta = \mathsf{id}_G$.

**Proposition 1.31** A natural transformation is an isomorphism of functors iff it is an iso at each component.

**Exercise 1.32** Prove the proposition above.

**Notation 1.33** The category of functors with domain $\mathcal{A}$ and codomain $\mathcal{B}$ is noted by $[\mathcal{A}, \mathcal{B}]$; some authors use $\mathcal{B}^{\mathcal{A}}$. In general we use Greek letters to denote natural transformations. We depict natural transformations as double arrows only when we want to stress we are dealing with natural transformations.

**Exercise 1.34** The category $[\mathcal{A}, \mathcal{B}]$ need not be locally small even though $\mathcal{A}$ and $\mathcal{B}$ are locally small. However, when $\mathcal{A}$ is small $[\mathcal{A}, \mathcal{B}]$ is locally small – why?

**Exercise 1.35** Let $\mathcal{C}$ be a category, a functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor $F^* : [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$ where the action over objects is defined by precomposition with $F$. What is the action on arrows of $F^*$?

# 2 Constructions on Categories

## 2.1 Opposite Category and Contravariance

**Definition 2.1 (Opposite Category)** Let $\mathcal{C}$ be a category, we define the opposite category $\mathcal{C}^{\mathsf{op}}$ as:

- $\mathbf{ob}(\mathcal{C}^{\mathsf{op}}) = \mathbf{ob}(\mathcal{C})$, and
- $\mathcal{C}^{\mathsf{op}}(C, D) = \mathcal{C}(D, C)$ for each pair $C, D \in \mathcal{C}^{\mathsf{op}}$.

In words, $\mathcal{C}^{\mathsf{op}}$ is the category with the same collection of objects as $\mathcal{C}$ where the arrows are reversed. Clearly, the identities in $\mathcal{C}^{\mathsf{op}}$ correspond to the identities in $\mathcal{C}$. Some authors prefer to write $f^{\mathsf{op}}$ for the arrow $f$ in the opposite category, however, in general there is no reason to make such a distinction. The composition of arrows in $\mathcal{C}^{\mathsf{op}}$ can be expressed in terms of the arrows in $\mathcal{C}$ by reading the composition in the reverse order

$$\mathcal{C}^{\mathsf{op}} : X \underset{g\circ f}{\overset{f}{\longrightarrow}} Y \xrightarrow{g} Z \qquad \mathcal{C} : X \underset{f\circ g}{\overset{f}{\longleftarrow}} Y \xleftarrow{g} Z \ .$$

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is essentially a functor which reverses the arrows

$$\begin{array}{ccc} A & & F(A) \\ \downarrow{\scriptstyle f} & \longmapsto & \uparrow{\scriptstyle F(f)} \\ B & & F(B) \ . \end{array}$$

In contrast, the functors as defined in section 1.2 are called *covariant*. It is more convenient to identify a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ with a covariant functor $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$.

**Remark 2.2** There is no difference between $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}^{\mathsf{op}}$ – they are the same collection of functions. Likewise, there is no difference between $F : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$.

**Exercise 2.3** Show the opposite construction gives rise to a functor $(-)^{\mathsf{op}} : \mathbf{CAT} \to \mathbf{CAT}$ acting on objects and arrows

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{C}^{\mathsf{op}} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F^{\mathsf{op}}} \\ \mathcal{D} & & \mathcal{D}^{\mathsf{op}} \ . \end{array}$$

## 2.2 Product of Categories

**Definition 2.4 (Product of Categories)** Given the categories $\mathcal{C}$ and $\mathcal{D}$, we define the product category $\mathcal{C} \times \mathcal{D}$ as:

- $\mathbf{ob}(\mathcal{C} \times \mathcal{D}) = \mathbf{ob}(\mathcal{C}) \times \mathbf{ob}(\mathcal{D})$,
- $\mathcal{C} \times \mathcal{D}\big((C, D), (C', D')\big) = \mathcal{C}(C, C') \times \mathcal{D}(D, D')$ for each $\big((C, D), (C', D')\big) \in \mathcal{C} \times \mathcal{D}$, and
- composition and identities are defined componentwise.

A functor whose domain is a product of categories is called a *bifunctor*..

In general a mapping $M$ on objetcs and arrows in $\mathcal{C} \times \mathcal{D}$ to objects and arrows in $\mathcal{E}$ determines mappings $M(C, -)$ from $\mathcal{D}$ to $\mathcal{E}$ for each $C \in \mathcal{C}$ defined as follows

$$\begin{array}{ccc} D & & M(C, D) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle M(C,g)=M(\mathsf{id}_C,g)} \\ D' & & M(C, D') \ . \end{array}$$

Similarly $M$ defines mappings $M(-, D)$ for every $D \in \mathcal{D}$. Notice that $M$ is not necessarily a bifunctor, and from the following proposition we cannot in general conclude that $M$ is a bifunctor even when $M(C, -)$ and $M(-, D)$ are.

**Proposition 2.5** Suppose $F$ is a mapping on objects and arrows in $\mathcal{C} \times \mathcal{D}$ such that

- $F(C, D) \in \mathcal{E}$ for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$, and
- $F(f, g) \in \mathcal{E}\big(F(C, D), F(C', D')\big)$ if $f \in \mathcal{C}(C, C')$ and $g \in \mathcal{D}(D, D')$.

$F$ is a bifunctor from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ iff

1. $F(C, -)$ is a functor from $\mathcal{D}$ to $\mathcal{E}$ for all $C \in \mathcal{C}$,

2. $F(-, D)$ is a functor from $\mathcal{C}$ to $\mathcal{E}$ for all $D \in \mathcal{D}$, and

3. $F(f, g) = F(C', g) \circ F(f, D) = F(f, D') \circ F(C, g)$, *i.e.*, the diagram

$$\begin{array}{ccc} F(C,D) & \xrightarrow{F(f,D)} & F(C',D) \\ {\scriptstyle F(C,g)}\downarrow & \searrow^{F(f,g)} & \downarrow{\scriptstyle F(C',g)} \\ F(C,D') & \xrightarrow[F(f,D')]{} & F(C',D') \end{array}$$

commmutes for all $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$.

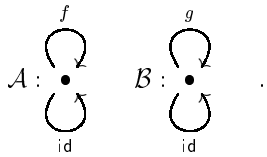
**Proof**:
*"Only-if"*: if $F$ is a functor then $F(C, -)$ and $F(-, D)$ are functors, and $F$ must satisfy 3 since $(f, g) = (f, \mathsf{id}_{D'}) \circ (\mathsf{id}_C, g) = (\mathsf{id}_{C'}, g) \circ (f, \mathsf{id}_D)$.
*"If"*: we need to prove $F$ satisfies the functor laws. It is easy to show $F(\mathsf{id}_{(C,D)}) = \mathsf{id}_{F(C,D)}$ by using the fact that $F(C, -)$ is a functor (or equivalently that $F(-, D)$ is a functor). It remains to prove $F$ preserves composition; assume $C \xrightarrow{f} C' \xrightarrow{f'} C''$ and $D \xrightarrow{g} D' \xrightarrow{g'} D''$, then

$$\begin{aligned}
F\big((f', g') \circ (f, g)\big) &= F(f' \circ f, g' \circ g) && \text{by comp. in } \mathcal{C} \times \mathcal{D}, \\
&= F(C'', g' \circ g) \circ F(f' \circ f, D) && \text{by 3}, \\
&= \Big(F(C'', g') \circ F(C'', g)\Big) \circ \Big(F(f', D) \circ F(f, D)\Big) && \text{by 1 and 2}, \\
&= F(C'', g') \circ \Big(F(C'', g) \circ F(f', D)\Big) \circ F(f, D) && \text{by associativity}, \\
&= F(C'', g') \circ \Big(F(f', D') \circ F(C', g)\Big) \circ F(f, D) && \text{by 3}, \\
&= F(f', g') \circ F(f, g) && \text{by associativity and twice 3}.
\end{aligned}$$

□

The next example considers a mapping satisfying conditions 1 and 2 but not 3.

**Example 2.6** Let $\mathcal{A}$ and $\mathcal{B}$ be one-object categories




.

Let $F$ be a mapping from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{B}$ which sends $(\bullet, \bullet)$ to the only object in $\mathcal{B}$ and acts on arrows as follows

$$\begin{aligned}
F(f, f) &= g \\
F(\mathsf{id}, f) &= \mathsf{id} = F(\bullet, f) \\
F(\mathsf{id}, \mathsf{id}) &= \mathsf{id} = F(\bullet, \mathsf{id}) = F(\mathsf{id}, \bullet) \\
F(f, \mathsf{id}) &= \mathsf{id} = F(f, \bullet).
\end{aligned}$$

One can check $F(\bullet, -)$ and $F(-, \bullet)$ are functors from $\mathcal{A}$ to $\mathcal{B}$, but $F$ does not satisfy condition 3 of Proposition 2.5

$$\begin{gathered}
F(f, f) = g \\
F(\mathsf{id}, f) \circ F(f, \mathsf{id}) = \mathsf{id} \circ \mathsf{id} = \mathsf{id} \ .
\end{gathered}$$

Hence,

$$F\big((\mathsf{id}, f) \circ (f, \mathsf{id})\big) = F(f, f) = g \neq F(\mathsf{id}, f) \circ F(f, \mathsf{id}) \ .$$

Therefore, $F$ is not a functor.

## 2.3 Natural Transformations between Bifunctors

Given the bifunctors $F, G : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, consider the natural transformation $\alpha$

$$\mathcal{C} \times \mathcal{D} \;\underset{G}{\overset{F}{\Downarrow \alpha}}\; \mathcal{E} \ .$$

From the definition of natural transformation $\alpha$ is a family of arrows in $\mathcal{E}$ indexed by objects in $\mathcal{C} \times \mathcal{D}$ such that given an arrow $(f, g)$ in $\mathcal{C} \times \mathcal{D}$ the square

$$\begin{array}{ccc} (C,D) & & F(C,D) \xrightarrow{\alpha_{C,D}} G(C,D) \\ {\scriptstyle (f,g)}\downarrow & & {\scriptstyle F(f,g)}\downarrow \qquad\qquad \downarrow{\scriptstyle G(f,g)} \\ (C',D') & & F(C',D') \xrightarrow[\alpha_{C',D'}]{} G(C',D') \end{array}$$

commutes. This definition induces two natural transformations by fixing one of the arguments:

- $\alpha_{C,-} : F(C,-) \Rightarrow G(C,-)$ where $(\alpha_{C,-})_D = \alpha_{C,D}$, and
- $\alpha_{-,D} : F(-,D) \Rightarrow G(-,D)$ where $(\alpha_{-,D})_C = \alpha_{C,D}$.

**Proposition 2.7** Given the bifunctors $F, G : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, a family of maps $\langle \alpha_{C,D} : F(C,D) \to G(C,D) \rangle_{(C,D) \in \mathcal{C} \times \mathcal{D}}$ in $\mathcal{E}$ is a natural transformation iff

1. $\alpha_{C,-} = \langle \alpha_{C,D} \rangle_{D \in \mathcal{D}}$ is a natural transformation from $F(C,-)$ to $G(C,-)$ for any $C \in \mathcal{C}$, and
2. $\alpha_{-,D} = \langle \alpha_{C,D} \rangle_{C \in \mathcal{C}}$ is a natural transformation from $F(-,D)$ to $G(-,D)$ for any $D \in \mathcal{D}$.

**Proof**:
*"Only-if"*: left as exercise.
*"If"*: It follows from exhibing the arrow

$$(C,D) \xrightarrow{(f,g)} (C',D')$$

as the composition

$$(C,D) \xrightarrow{(f,\mathsf{id}_D)} (C',D) \xrightarrow{(\mathsf{id}_{C'},g)} (C',D') \ .$$

Consider the associated squares:

$$\begin{array}{ccc} (C,D) & & F(C,D) \xrightarrow{(\alpha_{-,D})_C = \alpha_{C,D}} G(C,D) \\ {\scriptstyle (f,\mathsf{id}_D)}\downarrow & & {\scriptstyle F(f,\mathsf{id}_D)=F(f,D)}\downarrow \qquad\qquad \downarrow{\scriptstyle G(f,\mathsf{id}_D)=G(f,D)} \\ (C',D) & & F(C',D) \xrightarrow{(\alpha_{-,D})_{C'} = \alpha_{C',D} = (\alpha_{C',-})_D} G(C',D) \\ {\scriptstyle (\mathsf{id}_{C'},g)}\downarrow & & {\scriptstyle F(\mathsf{id}_{C'},g)=F(C',g)}\downarrow \qquad\qquad \downarrow{\scriptstyle G(\mathsf{id}_{C'},g)=G(C',g)} \\ (C',D') & & F(C',D') \xrightarrow{(\alpha_{C',-})_{D'} = \alpha_{C',D'}} G(C',D') \end{array}$$

From 2 we have the commutativity of the upper square and from 1 of the lower square. Hence the outermost square commutes

$$\begin{aligned} \alpha_{C',D'} \circ F(f,g) &= (\alpha_{C',-})_{D'} \circ F\big((\mathsf{id}_{C'},g) \circ (f,\mathsf{id}_D)\big) \\ &= (\alpha_{C',-})_D \circ F(\mathsf{id}_{C'},g) \circ F(f,\mathsf{id}_D) && \text{as } F \text{ is functor,} \\ &= G(\mathsf{id}_{C'},g) \circ G(f,\mathsf{id}_D) \circ (\alpha_{-,D})_C && \text{by commutativity of the squares,} \\ &= G\big((\mathsf{id}_{C'},g) \circ (f,\mathsf{id}_D)\big) \circ (\alpha_{-,D})_C && \text{as } G \text{ is functor,} \\ &= G(f,g) \circ \alpha_{C,D} \ . \end{aligned}$$

Therefore we can conclude $\alpha$ is a natural transformation from $F$ to $G$.

□

Thus, naturality may be checked in each argument separately when dealing with bifunctors.

**Exercise 2.8** Prove the *"Only-if"* part in the proof of Proposition 2.7.

**Remark 2.9 (Functoriality and Naturality)** Suppose $E(X)$ is an expression with a an argument $X$ such that given an object $A \in \mathcal{A}$, $E(A)$ stands for an object in the category $\mathcal{C}$; and given an arrow $f \in \mathbf{arr}(\mathcal{A})$, $E(f)$ stands for an arrow in $\mathcal{C}$. We say that $E(X)$ is *functorial* in $X$ if this assignment is a functor from $\mathcal{A}$ to $\mathcal{C}$. One example of functorial expressions are hom-sets.

Traditionally, arguments in expressions are denoted by means of place-holders like for example $\mathcal{C}(-,+)$. This notation might lead to some confusion specially in distinguishing the scope where each argument is defined. To save this potential source of confution we sometimes use $\lambda$-notation to indicate arguments in expressions. Thus, assuming $E(X)$ is functorial in $X$ of type $\mathcal{A}$ then the expression $\lambda X.\, E(X)$ denotes a functor from $\mathcal{A}$ to $\mathcal{C}$.

We say that the expressions $E_1(X)$ and $E_2(X)$ both functorial in $X$ are *naturally* isomorphic in $X$ or equivantly that there is an isomorphism *natural* in $X$, iff there is a natural isomorphism between $\lambda X.\, E_1(X)$ and $\lambda X.\, E_2(X)$.

**Exercise 2.10** Prove that a bifunctor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ defines a unique *curryied* functor $\lambda X.\, \lambda Y.\, F(X,Y) : \mathcal{A} \to [\mathcal{B}, \mathcal{C}]$. Similarly, show how a natural transformation $\alpha : F \Rightarrow G$ for functors $F, G : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ uniquely defines a natural transformation from $\lambda X.\, \lambda Y.\, F(X,Y)$ to $\lambda X.\, \lambda Y.\, G(X,Y)$.

## 2.4 Examples

**Evaluation Functor:** Define the *evaluation* functor $eval : [\mathcal{C}, \mathcal{D}] \times \mathcal{C} \to \mathcal{D}$ for categories $\mathcal{C}$ and $\mathcal{D}$ as follows

$$\begin{array}{ccc} (F, C) & \longmapsto & F(C) \\ \downarrow{\scriptstyle (\alpha, g)} & \longmapsto & \downarrow{\scriptstyle F'(g) \circ \alpha_C = \alpha_{C'} \circ F(g)} \\ (F', C') & \longmapsto & F'(C') \end{array}$$

where $\alpha : F \Rightarrow F'$ and $g : C \to C'$. The arrow $g$ induces the natural square which justifies the action of *eval* on the arrow $(\alpha, g)$

$$\begin{array}{ccc} C & FC & \xrightarrow{\alpha_C} & F'C \\ \downarrow{\scriptstyle g} & \downarrow{\scriptstyle Fg} & & \downarrow{\scriptstyle F'g} \\ C' & FC' & \xrightarrow{\alpha_{C'}} & F'C'. \end{array}$$

We use Proposition 2.7 to show *eval* is a bifunctor:

1. $eval(F, -) : \mathcal{C} \to \mathcal{D}$ which sends $g : C \to C'$ to $F\,g : F\,C \to F\,C'$ is a functor since $F$ is a functor;
2. $eval(-, C) : [\mathcal{C}, \mathcal{D}] \to \mathcal{D}$ which sends $\alpha : F \Rightarrow F'$ to $\alpha_C : F\,C \to F'\,C$ is a functor since the identity natural transformation has identities in each component and the composition of natural transformations is defined componentwise; and
3. finally we want

$$eval(\alpha, g) = eval(F', g) \circ eval(\alpha, C) = eval(\alpha, C') \circ eval(F, g)$$

from the definition of *eval* this is

$$eval(\alpha, g) = F'\,g \circ \alpha_C = \alpha_{C'} \circ F\,g$$

and this follows from the naturality square above.

**Hom-functor:** Recall that the hom-set $\mathcal{C}(C,D)$ is defined to be the set of arrows with domain $C$ and codomian $D$ in the locally small category $\mathcal{C}$. Here we show how hom-sets extend to bifunctors.

A *hom-functor* for a category $\mathcal{C}$ is a functor $\mathcal{C}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$ which acts on objects and arrows by sending $(f,g) : (C,D) \to (C',D')$ to

$$\mathcal{C}(C,D) \xrightarrow{\mathcal{C}(f,g)=g\circ f} \mathcal{C}(C',D')$$

where $\mathcal{C}(f,g)$ is a function mapping any arrow $C \xrightarrow{h} D$ to the arrow $C' \xrightarrow{f} C \xrightarrow{h} D \xrightarrow{g} D'$. Informally, to obtain an arrow from $C'$ to $D'$ out of an arrow from $C$ to $D$, we need both a way to go from $D$ to $D'$ and a way to go from $C'$ to $C$.

**Exercise 2.11** Check that the definition of the hom-functor above is actually a bifunctor.

From Proposition 2.5 we have a functor $\mathcal{C}(-,D) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ which acts on objects and arrows

$$\begin{array}{ccc} C & & \mathcal{C}(C,D) \\ \downarrow{\scriptstyle f} & & \uparrow{\scriptstyle \mathcal{C}(-,D)=-\circ f} \\ C' & & \mathcal{C}(C',D)\ . \end{array}$$

Notice that the arrow $f$ drawn above is in the category $\mathcal{C}$ not $\mathcal{C}^{\mathsf{op}}$; that explains why the direction of the arrow is reversed after applying $\mathcal{C}(-,D)$. In the same manner we can define a functor $\mathcal{C}(C,-) : \mathcal{C} \to \mathbf{Set}$.

**Yoneda Functor:** Given the category of functors and natural transformations $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$, we define the *Yoneda* functor $\mathcal{Y} : \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ which takes an object $C \in \mathcal{C}$ to the functor $\mathcal{C}(-,C)$ and acts on arrows as follows

$$\begin{array}{ccc} C & & \mathcal{Y}\,C = \mathcal{C}(-,C) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathcal{Y}\,g=\langle g\circ - \rangle_{E\in\mathcal{C}}} \\ D & & \mathcal{Y}\,D = \mathcal{C}(-,D). \end{array}$$

**Exercise 2.12** Check that $\mathcal{Y}$ defines a functor from $\mathcal{C}$ to $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$.

Dually, we define the contravariant version of the Yoneda functor $\mathcal{Y}^o : \mathcal{C}^{\mathsf{op}} \to [\mathcal{C}, \mathbf{Set}]$ which acts on objects and arrows

$$\begin{array}{ccc} C & & \mathcal{Y}^o\,C = \mathcal{C}(C,-) \\ \downarrow{\scriptstyle g} & & \uparrow{\scriptstyle \mathcal{Y}^o\,g=\langle -\circ g \rangle_{E\in\mathcal{C}}} \\ D & & \mathcal{Y}^o\,D = \mathcal{C}(D,-)\ . \end{array}$$

**Constant Functor:** Given $Y \in \mathcal{D}$, we write $\Delta\,Y : \mathcal{C} \to \mathcal{D}$ for the constantly $Y$ functor which acts over objects and arrows as follows:

$$\begin{array}{ccc} C & & \Delta\,Y(C) = Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Delta\,Y(f)=\mathsf{id}_Y} \\ C' & & \Delta\,Y(C') = Y \end{array}$$

This may be extended to the functor $\Delta- : \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$ which sends $g : D \to D'$ to $\Delta g : \Delta D \Rightarrow \Delta D'$ where $\Delta g$ is just the arrow $g$ at all components.

**Exercise 2.13** Check that $\Delta-$ is a functor from $\mathcal{D}$ to $[\mathcal{C}, \mathcal{D}]$.

# 3 Yoneda Lemma and Universal Properties

## 3.1 Yoneda Lemma

**Proposition 3.1** Let$F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ be a functor, we have that each element $x \in F(C)$ for some $C \in \mathcal{C}$ induces a natural transformation $\widetilde{x} : \mathcal{Y}\,C = \mathcal{C}(-, C) \Rightarrow F$ whose component at an arbitrary $D \in \mathcal{C}$ is

$$\widetilde{x}_D = F(-)(x)\ . \tag{1}$$

**Proof**: We need to prove that for an arbitrary arrow $g : E \to D$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} \mathcal{C}(D,C) & \xrightarrow{\widetilde{x}_D} & F(D) \\ \Big\downarrow{\scriptstyle -\circ g} & & \Big\downarrow{\scriptstyle F(g)} \\ \mathcal{C}(E,C) & \xrightarrow{\widetilde{x}_E} & F(E) \end{array}$$

commutes. For any $f \in \mathcal{C}(D, C)$ we have

$$F(f \circ g)(x) = F(g)(F(f)(x))$$

since $F$ is a functor. This gives the commutativity required. Recall $F$ is a contravariant functor and so reverses the order of composition in $\mathcal{C}$.

□

**Proposition 3.2** Let $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ be a functor; given a natural transformation $\alpha : \mathcal{C}(-, C) \Rightarrow F$ we have $\alpha = \widetilde{\alpha_C(\mathsf{id}_C)}$.

**Proof**: Given an arbitrary $f \in \mathcal{C}(D, C)$, we need to check that $\alpha_D(f) = F(f)(\alpha_C(\mathsf{id}_C))$. This equality is obtained by chasing the identity $\mathsf{id}_C$ around the naturality square determined by $f$.

□

**Theorem 3.3 (Yoneda Lemma)** The mappping $\theta_{C,F}$ which takes $\alpha$ in $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F)$ to $\alpha_C(\mathsf{id}_C) \in F(C)$ is an isomorphism

$$[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F) \overset{\theta_{C,F}}{\cong} F(C)$$

natural in $C \in \mathcal{C}$ and $F \in [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$.

**Remark 3.4** Notice that the naturality condition in the statement of the Yoneda lemma requires functorial expressions. The right hand side corresponds to the application of the evaluation functor $eval : \mathcal{C}^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}] \to \mathbf{Set}$. The right hand side is a more complex expression that involves composition of functors:

$$\mathcal{C}^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}] \xrightarrow{\mathcal{Y}\times\mathsf{id}} [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}] \xrightarrow{[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](-,+)} \mathbf{Set}.$$

**Proof**: We show that $\widetilde{(\ )}$ defined by (1) is the inverse of $\theta_{C,F}$. From Proposition 3.2 we already have that $\widetilde{\theta_{C,F}(\alpha)} = \widetilde{\alpha_C(\mathsf{id}_C)} = \alpha$. Let $x$ be an element in $F(C)$

$$\begin{aligned} \theta_{C,F}(\widetilde{x}) &= \widetilde{x}_C(\mathsf{id}_C) && \text{by definition of } \theta_{C,F}, \\ &= F(\mathsf{id}_C)(x) && \text{by definition of } \widetilde{(\ )}, \\ &= \mathsf{id}_{F(C)}(x) && \text{as } F \text{ is a functor}, \\ &= x\ . \end{aligned}$$

Thus we complete the prove that $\theta_{C,F}$ is an isomorphism of sets.

It remains to prove naturality in $C \in \mathcal{C}$ and $F \in [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$. By Proposition 2.7, this is equivalent to proving naturality in $C$ and $F$ separately.

**Naturality in $C$:** Given an arrow $f : C \to D$ in $\mathcal{C}$, we need to prove

$$\begin{array}{ccc} [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F) & \xrightarrow{\theta_{C,F}} & F(C) \\ \uparrow{\scriptstyle -\circ(\mathcal{Y}\,f)} & & \uparrow{\scriptstyle F(f)} \\ [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,D, F) & \xrightarrow{\theta_{D,F}} & F(D) \end{array}$$

commutes. Given $\beta \in [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,D, F)$, we have

$$\begin{aligned} \theta_{C,F}(\beta \circ (\mathcal{Y}\,f)) &= (\beta \circ (\mathcal{Y}f))_C(\mathsf{id}_C) && \text{by definition of } \theta_{C,F}, \\ &= (\beta_C \circ (\mathcal{Y}f)_C)(\mathsf{id}_C) && \text{as composition is componentwise,} \\ &= (\beta_C \circ (f \circ -))(\mathsf{id}_C) && \text{by definition of } \mathcal{Y}, \\ &= \beta_C(f) \\ &= F(f)(\beta_D(\mathsf{id}_D)) \end{aligned}$$

which gives the commutativity of the diagram above.

**Naturality in $F$:** Given $\alpha \in [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F)$ and $\nu : F \Rightarrow G$. Since the composition of natural transformations is defined componentwise we have

$$(\nu \circ \alpha)_C(\mathsf{id}_C) = \nu_C(\alpha_C(\mathsf{id}_C)) \ ,$$

which proves the commutativity of the diagram

$$\begin{array}{ccc} F & [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, F) & \xrightarrow{\theta_{C,F}} F(C) \\ \Downarrow{\scriptstyle \nu} & \downarrow{\scriptstyle \nu\circ -} & \downarrow{\scriptstyle \nu_C} \\ G & [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}\,C, G) & \xrightarrow{\theta_{C,G}} G(C) \ . \end{array}$$

□

The Yoneda lemma is stated for an arbitrary set-valued functor $F$. One special case arises when $F$ is itself a hom-functor, say $\mathcal{C}(-, D)$. In this setting the Yoneda lemma tells us that there exists an isomorphism between the morphisms in $\mathcal{C}(C, D)$ and the natural transformations in $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}C, \mathcal{Y}D)$. As one might expect this isomorphism corresponds to the Yoneda functor acting on arrows.

**Corollary 3.5** The Yoneda functor $\mathcal{Y} : \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ is full and faithful, i.e., given $C, D \in \mathcal{C}$ the Yoneda functor defines a bijection between $(\mathcal{Y}\,D)\,C = \mathcal{C}(C, D)$ and $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}C, \mathcal{Y}D)$.

**Proof**: Given $f \in \mathcal{C}(C, D)$ we show $\widetilde{f} = \mathcal{Y}\,f$. Given an object $E$ in $\mathcal{C}$, from the definition of $\widetilde{(\ )}$ in (1) we have

$$\widetilde{f}_E = \big(\mathcal{C}(-, D)\big)(f) = f \circ - = (\mathcal{Y}f)_E \ .$$

□

**Proposition 3.6** Given $C, D \in \mathcal{C}$, we have that $C \cong D$ iff $\mathcal{Y}\,C \cong \mathcal{Y}\,D$ in $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$.

**Proof**: Left as exercise.

□

**Exercise 3.7** Prove Proposition 3.6 and state its dual.

**Exercise 3.8** State the Yoneda Lemma for the Yoneda contravariant functor case.

## 3.2 Representability

**Definition 3.9 (Representation)** A *representation* for a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ consists of an object $C \in \mathcal{C}$ and a natural isomorphism

$$\mathcal{Y}\, C = \mathcal{C}(-, C) \overset{\alpha}{\cong} F \ .$$

We also say that $F$ is *representable*.

**Remark 3.10** We have a dual definition; a functor $G : \mathcal{C} \to \mathbf{Set}$ is representable iff there exist an object $D \in \mathcal{C}$ and a natural isomorphism $\mathcal{Y}^o\, D = \mathcal{C}(D, -) \overset{\beta}{\cong} G$

**Definition 3.11 (Universal Element)** Given a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$, a pair $(C, e)$ where $C \in \mathcal{C}$ and $e \in F(C)$ is a *universal element* of $F$ iff for any $C' \in \mathcal{C}$ and $e' \in F(C')$ there exists a **unique** $f : C' \to C$ in $\mathcal{C}$ such that $e' = F(f)(e)$.

**Theorem 3.12** There exists a universal element for a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ iff $F$ is representable.

**Proof**:
*"Only-if"*: Let $(C, e)$ be a universal element for $F$. From Proposition 3.1 we have that $e$ induces a natural transformation $\widetilde{e} : \mathcal{Y}\, C \Rightarrow F$ whose component at $C' \in \mathcal{C}$ is $\widetilde{e}_{C'} = F(-)(e)$. As $e$ is a universal element we have that $\widetilde{e}_{C'}$ is a bijection and then by Proposition 1.31 we conclude that $\widetilde{e}$ is a natural isomorphism.
*"if"*: Suppose $\mathcal{Y}\, C \overset{\alpha}{\cong} F$, then from for any $e' \in F(C')$ for some $C' \in \mathcal{C}$ there exists a unique arrow $f \in \mathcal{C}(C', C)$ such that $\alpha_{C'}(f) = e'$. From Proposition 3.2 we have that $e' = \widetilde{\alpha_C(\mathsf{id}_C)}_{C'}(f) = F(f)(\alpha_C(\mathsf{id}_C))$ and then $(\alpha_C(\mathsf{id}_C), C)$ is a universal element.

□

**Definition 3.13 (Universal Properties)** An object $C \in \mathcal{C}$ satisfies a *universal property* when $C$ and some specified natural isomorphism form a pair which represents a (contravariant) set-valued functor, say $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$.

In this setting we may understand an element in $F(D)$, for any $D \in \mathcal{C}$, as uniquely determined by the universal element and an arrow $f \in \mathcal{C}(C, D)$. The arrow $f$ sometimes is called *mediating arrow* and it is drawn in diagrams with a dotted line.

**Proposition 3.14** Representations are unique up to isomorphisms: if $(A, \alpha)$ and $(B, \beta)$ are both representations of a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ then $A \cong B$ in $\mathcal{C}$.

**Proof**: A direct consequence of Proposition 3.6.

□

Therefore two objects satisfying the same universal properties are isomorphic.

**Theorem 3.15 (Parametrised Representability)** Let $F : \mathcal{A} \times \mathcal{B}^{\mathsf{op}} \to \mathbf{Set}$ be a bifunctor such that for every $A \in \mathcal{A}$ there exists a representation $(\mathcal{B}(A), \theta^A)$ for the functor $F(A, -) : \mathcal{B}^{\mathsf{op}} \to \mathbf{Set}$. Then there is a **unique** extension of the mapping $A \mapsto \mathcal{B}(A)$ to a functor $\mathcal{B}(-) : \mathcal{A} \to \mathcal{B}$ such that

$$\mathcal{B}(-, \mathcal{B}(A)) \overset{\theta^A}{\cong} F(A, -)$$

is natural in $A \in \mathcal{A}$.

**Proof**: By *currying* $F$ we have the functor $\lambda X.\, \lambda Y.\, F(X, Y) : \mathcal{A} \to [\mathcal{B}^{\mathsf{op}}, \mathbf{Set}]$ – see Exercise 2.10. Then, given $f : A \to A'$ we can construct the commuting square

$$\begin{array}{ccc} \mathcal{Y}(\mathcal{B}(A)) & \xrightarrow{\cong \theta^A} & F(A, -) \\ {\scriptstyle (\theta^{A'})^{-1} \circ F(f,-) \circ \theta^A}\downarrow & & \downarrow{\scriptstyle F(f,-) = \langle F(f,B) \rangle_{B \in \mathcal{B}}} \\ \mathcal{Y}(\mathcal{B}(A')) & \xrightarrow{\cong \theta^{A'}} & F(A', -) \ . \end{array}$$

Since $\mathcal{Y}$ is full and faithful, there must exist a **unique** arrow, say $\mathcal{B}(f)$, from $\mathcal{B}(A)$ to $\mathcal{B}(A')$ in $\mathcal{B}$ such that $\mathcal{Y}(\mathcal{B}(f)) = (\theta^{A'})^{-1} \circ F(f,-) \circ \theta^A$. We need to check that this extension to $\mathcal{B}(-)$ indeed satisfies the definition of functor:

- Given the arrows $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathcal{A}$, we have

$$\begin{aligned}
\mathcal{Y}\big(\mathcal{B}(f' \circ f)\big) &= (\theta^{A''})^{-1} \circ F(f' \circ f, -) \circ \theta^A \\
&= (\theta^{A''})^{-1} \circ F(f', -) \circ F(f, -) \circ \theta^A && \text{as } \lambda X.\, \lambda Y.\, F(X,Y) \text{ is a functor,} \\
&= (\theta^{A''})^{-1} \circ F(f', -) \circ \theta^{A'} \circ (\theta^{A'})^{-1} \circ F(f, -) \circ \theta^A \\
&= \mathcal{Y}(\mathcal{B}(f')) \circ \mathcal{Y}(B(f)) \\
&= \mathcal{Y}\big(\mathcal{B}(f') \circ \mathcal{B}(f)\big) && \text{as } \mathcal{Y} \text{ is a functor;}
\end{aligned}$$

  therefore $\mathcal{Y}(\mathcal{B}(f' \circ f)) = \mathcal{Y}(\mathcal{B}(f) \circ \mathcal{B}(f'))$, and as $\mathcal{Y}$ is faithful we have $\mathcal{B}(f' \circ f) = \mathcal{B}(f') \circ \mathcal{B}(f)$ as required.

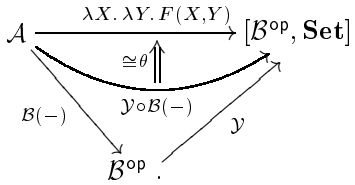
- Given the identity $\mathsf{id}_A$ in for $A \in \mathcal{A}$ we have

$$\begin{aligned}
\mathcal{Y}(\mathcal{B}(\mathsf{id}_A)) &= (\theta^A)^{-1} \circ F(\mathsf{id}_A, -) \circ \theta^A \\
&= (\theta^A)^{-1} \circ \mathsf{id}_{F(A,-)} \circ \theta^A && \text{as } \lambda X.\, \lambda Y.\, F(X,Y) \text{ is a functor,} \\
&= \mathsf{id}_{\mathcal{Y}(\mathcal{B}(A))} \\
&= \mathcal{Y}(\mathsf{id}_{\mathcal{B}(A)}) && \text{as } \mathcal{Y} \text{ is a functor;}
\end{aligned}$$

  therefore $\mathcal{Y}(\mathcal{B}(\mathsf{id}_A)) = \mathcal{Y}(\mathsf{id}_{\mathcal{B}(A)})$, and as $\mathcal{Y}$ is faithful we have $\mathcal{B}(\mathsf{id}_A) = \mathsf{id}_{\mathcal{B}(A)}$ as required.

□

The composition of the functor $\mathcal{B}(-)$ and $\mathcal{Y}$ gives rise to a functor which is isomorphic to the *curryied* version of $F$ where the natural isomorphism is given by $\theta = \langle \theta^A \rangle_{A \in \mathcal{A}}$




**Exercise 3.16** State the dual version of parametrised representability theorem, *i.e.*, where representables are of the form $\mathcal{Y}^o(\mathcal{B}(A))$.

**Exercise 3.17** Let $n \in \omega$ and let $Prod : \mathbf{Set} \to \mathbf{Set}$ be the functor which sends the set $X$ to $X^n$ and acts on arrows accordingly, i.e., it sends $f : X \to Y$ to $f^n : X^n \to Y^n$. In this setting, $f^n(x_1, \ldots, x_n) = (fx_1, \ldots, fx_n)$. Suppose $\theta : Prod \Rightarrow Prod$ is a natural isomorphism. Show that there is a permutation $\pi$ on the sequence $1, \ldots, n$ such that

$$\theta_X(x_1, \ldots, x_n) = (x_{\pi(1)}, \ldots, x_{\pi(n)}) \ .$$

**Hint:** Use the contravariant Yoneda embedding $\mathcal{Y}^o : \mathbf{Set}^{\mathsf{op}} \to [\mathbf{Set}, \mathbf{Set}]$ noting that $X^n \cong \mathbf{Set}(\vec{n}, X)$, where $\vec{n} = \{1, \ldots, n\}$.

# 4 Limits and Colimits

## 4.1 Definition of Limit

We take limits over functors whose domains are small categories ($\mathbb{I}, \mathbb{J}, \mathbb{K} \ldots$ ) and codomains are locally small categories ($\mathcal{C}, \mathcal{D} \ldots$ ). In this setting these functors are sometimes called (small) *diagrams*, *e.g.*, for $F : \mathbb{I} \to \mathcal{C}$ we say $F$ is an $\mathbb{I}$-indexed diagram in $\mathcal{C}$.

**Definition 4.1 (Limit Abstractly)** A *limit* for a diagram $F : \mathbb{I} \to \mathcal{C}$ is a representation for the functor $[\mathbb{I}, \mathcal{C}](\Delta -, F) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$:

$$\mathcal{C}(-, L) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{C}](\Delta -, F) \, .$$

Notice that the functor $[\mathbb{I}, \mathcal{C}](\Delta -, F) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ in the definition above is obtained as the composition

$$\mathcal{C}^{\mathsf{op}} \xrightarrow{\Delta -} [\mathbb{I}, \mathcal{C}]^{\mathsf{op}} \xrightarrow{[\mathbb{I}, \mathcal{C}](-, F)} \mathbf{Set}.$$

Take $X \in \mathcal{C}$ in the definition above, then we have $\mathcal{C}(X, \varprojlim_{\mathbb{I}} F) \overset{\theta_X}{\cong} [\mathbb{I}, \mathcal{C}](\Delta X, F)$, *i.e.*, an arrow $f : X \to \varprojlim_{\mathbb{I}} F$ is mapped to a natural transformation $\theta_X(f) = \gamma : \Delta X \Rightarrow F$. Given an arrow $u : I \to J$ in $\mathbb{I}$ we have the associated naturality square in $\mathcal{C}$

$$\begin{array}{c} I \\ \downarrow u \\ J \end{array} \qquad \begin{array}{ccc} \Delta X(I) = X & \xrightarrow{\gamma_I} & F(I) \\ \downarrow{\scriptstyle \Delta X(u) = \mathsf{id}_X} & & \downarrow{\scriptstyle F(u)} \\ \Delta X(J) = X & \xrightarrow{\gamma_J} & F(J) \, . \end{array} \tag{2}$$

This diagram can be drawn as

$$\begin{array}{c} I \\ \downarrow u \\ J \end{array} \qquad \begin{array}{ccc} & \overset{\gamma_I}{\nearrow} & F(I) \\ X & & \downarrow{\scriptstyle F(u)} \\ & \underset{\gamma_J}{\searrow} & F(J); \end{array}$$

from the commutativity of (2) we have that this triangle commutes, *i.e.*, $\gamma_J = F(u) \circ \gamma_I$. A natural transformation of this kind is called a *cone*.

**Definition 4.2 (Cone)** Let $F : \mathbb{I} \to \mathcal{C}$ be a diagram. A *cone* $\alpha$ from $X \in \mathcal{C}$ to $F$ (or a *cone* for the $\mathbb{I}$-indexed diagram $F$ in $\mathcal{C}$ with vertex $X$) is a collection of arrows $\langle \alpha_I : X \to F(I) \rangle_{I \in \mathbb{I}}$ in $\mathcal{C}$ such that for any arrow $u : I \to J$ in $\mathbb{I}$ the diagram

$$\begin{array}{ccc} & \overset{\gamma_I}{\nearrow} & F(I) \\ X & & \downarrow{\scriptstyle F(u)} \\ & \underset{\gamma_J}{\searrow} & F(J) \end{array}$$

commutes.

Therefore, the functor $[\mathbb{I}, \mathcal{C}](\Delta -, F)$ takes an object $X \in \mathcal{C}^{\mathsf{op}}$[1] to the set of cones from $X$ to $F$. Given an arrow $f : X' \to X$ in $\mathcal{C}$ and a cone $\gamma = \langle \gamma_I : X \to F(I) \rangle_{I \in \mathbb{I}}$, we can extend $\gamma$ to a cone $\gamma' = \langle f \circ \gamma_I : X' \to F(I) \rangle_{I \in \mathbb{I}} = \gamma \circ \Delta f$. This explains how $[\mathbb{I}, \mathcal{C}](\Delta -, F)$ acts on arrows.

From the definition above a limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ is given by a representation $\mathcal{C}(-, L) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{C}](\Delta -, F)$. From the proof of Theorem 3.12, we have that $(\theta_L(\mathsf{id}_L), L)$ is a universal element of $[\mathbb{I}, \mathcal{C}](\Delta -, F)$. Then given a cone $\gamma \in [\mathbb{I}, \mathcal{C}](\Delta X, F)$ there exists a **unique** $f : X \to L$ such that

$$\begin{aligned} \gamma &= \big([\mathbb{I}, \mathcal{C}](\Delta f, F)\big)\big(\theta_L(\mathsf{id}_L)\big) \\ &= \big(\theta_L(\mathsf{id}_L)\big) \circ \Delta f \\ &= \langle (\theta_L(\mathsf{id}_L))_I \circ f \rangle_{I \in \mathbb{I}} \end{aligned}$$

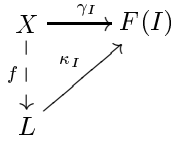
Then each component $\gamma_I : X \to F(I)$ is the composition of arrows

$$X \xrightarrow{f} L \xrightarrow{(\theta_L(\mathsf{id}_L))_I} F(I) \, .$$

This straightforwardly gives a more concrete definition of limit in terms of cones where $\theta_L(\mathsf{id}_L)$ is a limiting cone.

[1] Recall $\mathbf{ob}(\mathcal{C}^{\mathsf{op}}) = \mathbf{ob}(\mathcal{C})$.

**Proposition 4.3 (Limit Concretely)** Given a diagram $F : \mathbb{I} \to \mathcal{C}$, a *limit* for $F$ determines and is determined by a cone $\kappa = \langle \kappa_I : L \to F(I) \rangle_{I \in \mathbb{I}}$ from some $L \in \mathcal{C}$ to $F$ such that for any other cone $\gamma = \langle \gamma_I : X \to F(I) \rangle_{I \in \mathbb{I}}$ there exists a **unique** arrow $f : X \to L$ such that for every $I \in \mathbb{I}$ the diagram




commutes. The cone $\kappa$ is called *limiting cone*.

**Proof**: Given a representation $\mathcal{C}(-, L) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{C}](\Delta -, F)$ from Theorem 3.12 we have that $(\theta_L(\mathsf{id}_L), L)$ is a universal element. To prove the converse we just need to show $(\kappa, L)$ is a universal element of $[\mathbb{I}, \mathcal{C}](\Delta -, F)$, and then by Theorem 3.12 we have the representation $\mathcal{C}(-, L) \overset{\tilde{\kappa}}{\cong} [\mathbb{I}, \mathcal{C}](\Delta -, F)$.

□

**Remark 4.4** In general, we use the word "limit" to denote the object part of the representation in the definition. Thus, the object $L$ in the definition of limit is often written as $\varprojlim_{\mathbb{I}} F$ or $\lim_{\mathbb{I}} F$. For a given diagram, however, there might exist more than one limit. As a corollary of Proposition 3.14 we have that limits are unique up to isomorphism. In other words, a specific limit for a given diagram $F : \mathbb{I} \to \mathcal{C}$ involves a choice among all possible limits of $F$. In the abstract definition this choice is made explicit through the representation for the functor $[\mathbb{I}, \mathcal{C}](\Delta -, F) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$.

**Definition 4.5 (Complete Category)** A category $\mathcal{C}$ is *complete* iff the limits for all diagrams exist in $\mathcal{C}$.

**Proposition 4.6** Two results on limiting cones:

*(i)* Suppose $\kappa$ is a limiting cone from $X$ to $F$, and $Y \overset{f}{\cong} X$. Then $\kappa \circ \Delta f$ is a limiting cone from $Y$ to $F$.

*(ii)* Suppose $\kappa^X$ is a limiting cone from $X$ to $F$ and that $\kappa^Y$ is a limiting cone from $Y$ to $F$. Then there is an isomorphism $Y \overset{f}{\cong} X$ such that $\kappa^Y = \kappa^X \circ \Delta f$.

**Proof**: Left as an exercise.

□

**Exercise 4.7** Prove the proposition above (use the concrete understanding of limiting cone).

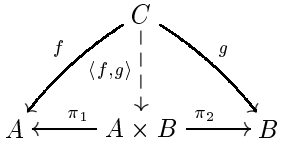
## 4.2 Examples of Limits

**Definition 4.8 (Terminal Object)** An object $\mathbf{1}_{\mathcal{C}}$ in a category $\mathcal{C}$ is called a *terminal object* (*final, one*) iff there is exactly one arrow from every $A \in \mathcal{C}$ to $\mathbf{1}_{\mathcal{C}}$.

A limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ where $\mathbb{I}$ is the empty category is a terminal object in $\mathcal{C}$. Notice that this limit might not exist. Any $A \in \mathcal{C}$ is the vertex for an empty cone to $F$, then there must be a **unique** arrow from $A$ to $\varprojlim_{\mathbb{I}} F = \mathbf{1}_{\mathcal{C}}$.

**Exercise 4.9** What is the terminal object in **Set**?

**Definition 4.10 (Product)** A *product* of two objects $A$ and $B$ in the category $\mathcal{C}$ is an object $A \times B$ in $\mathcal{C}$ together with two *projections* $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, such that for any pair of arrows $f : C \to A$ and $g : C \to B$ there is a **unique** (*mediating*) arrow $\langle f, g \rangle : C \to A \times B$ making the diagram

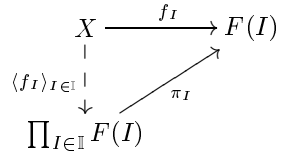


commute, *i.e.*, $f = \pi_1 \circ \langle f, g\rangle$ and $g = \pi_2 \circ \langle f, g\rangle$.

Let $\mathbb{I} = \{1, 2\}$ be the discrete category with two objects. A limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ is the product $F(1) \times F(2)$ in $\mathcal{C}$ where the limiting cone gives the projections $\pi_1$ and $\pi_2$.

One can extend the definition of binary product to general products:

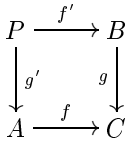
**Definition 4.11** Assume $\mathbb{I}$ is a discrete category. Given a diagram $F : \mathbb{I} \to \mathcal{C}$ a *product of* $F$ consists of an object $\prod_{I\in\mathbb{I}} F(I)$ in $\mathcal{C}$, and a collection of *projections* $\pi_I : \prod_{I\in\mathbb{I}} F(I) \to F(I)$ for each $I \in \mathbb{I}$, such that given any $X \in \mathcal{C}$ together with a collection of arrows $\langle f_I : X \to F(I)\rangle_{I\in\mathbb{I}}$ there is a **unique** arrow $\langle f_I\rangle_{I\in\mathbb{I}} : X \to \prod_{I\in\mathbb{I}} F(I)$ such that for each $I \in \mathbb{I}$ the diagram




commutes, *i.e.*, $f_I = \pi_I \circ \langle f_I\rangle_{I\in\mathbb{I}}$.

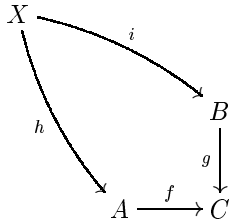
**Definition 4.12 (Pullback)** A *pullback* of a pair of arrows $f : A \to C$ and $g : B \to C$ in a category $\mathcal{C}$ is an object $P \in \mathcal{C}$, sometimes written $A \times_C B$, together with a pair of arrows $g' : P \to A$ and $f' : P \to B$ such that

1. $f \circ g' = g \circ f'$, *i.e.*, the diagram

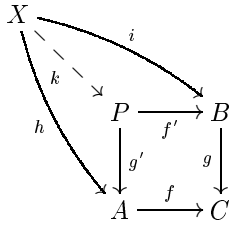



   commutes, and

2. for any $X \in \mathcal{C}$ together with arrows $h : X \to A$ and $i : X \to B$ for which $f \circ h = g \circ i$, *i.e.*, the diagram

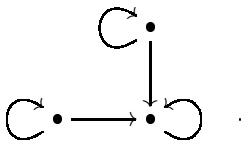



   commutes, then there exists a **unique** arrow $k : X \to P$ such that $h = g' \circ k$ and $i = f' \circ k$, *i.e.*, the two triangles in



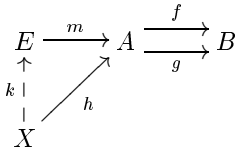

   commute.

**Exercise 4.13** Let $\mathbb{I}$ be the category represented by the following graph



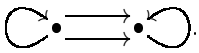
Show that a limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ is a pullback in $\mathcal{C}$.

**Definition 4.14 (Equaliser)** An *equaliser* of a pair of parallel arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in a category $\mathcal{C}$ is an arrow $m : E \to A$ such that $f \circ m = g \circ m$ and with the universal property that for any arrow $h : X \to A$ such that $f \circ h = g \circ h$ there is a **unique** arrow $k : X \to E$ making $m \circ k = h$, *i.e.*, the triangle in the diagram




commutes.

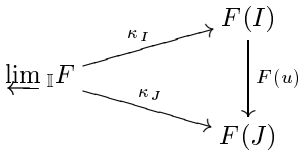
**Exercise 4.15** Let $\mathbb{I}$ be the category represented by the following graph



Show that a limit for a diagram $F : \mathbb{I} \to \mathcal{C}$ is an equaliser in $\mathcal{C}$. **Hint:** The equaliser corresponds to one of the arrows in the limiting cone.

## 4.3 Limits in Set

Consider a diagram $F : \mathbb{I} \to \mathbf{Set}$; from the definition of limit for any $u : I \to J$ the induced triangle




commutes, where $\kappa = \langle \kappa_I \rangle_{I \in \mathbb{I}}$ is a limiting cone. Then for any $x \in \varprojlim_{\mathbb{I}} F$ we have that $\big(F(u) \circ \kappa_I\big)(x) = \kappa_J(x)$.

We define our choice of limitis in **Set** to be

$$\varprojlim_{\mathbb{I}} F = \big\{ \langle x_I \rangle_{I \in \mathbb{I}} \mid x_I \in F(I) \;\&\; \forall\, u : I \to J \;\; x_J = (F(u))\, x_I \big\}$$

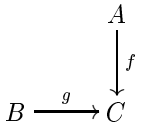
and the limiting cone $\kappa = \langle \kappa_I \rangle_{I \in \mathbb{I}}$ by letting $\kappa_I$ be the projection on the $I$'th component. Clearly, any other cone $\gamma = \langle \gamma_I : X \to F(I) \rangle_{I \in \mathbb{I}}$ factors uniquely through our choice of limit in **Set** by the mapping $z \in X \mapsto \langle \gamma_I(z) \rangle_{I \in \mathbb{I}}$.

**Products in Set:** Consider the diagram $F$ whose domain is a discrete category with two objects $\{1, 2\}$ and codomain is **Set**. As we have seen before the limit of $F$ is the product $F(1) \times F(2)$ in **Set**. Using the definition of limits in **Set** above we obtain

$$\varprojlim_{\{1,2\}} F = \{ \langle x_1, x_2 \rangle \mid x_1 \in F(1) \wedge x_2 \in F(2) \}$$

which is the usual definition of product of sets.

**Pullbacks in Set:** Let us assume $F$ is a diagram in **Set** whose limit gives a pullback for the arrows




(the identities arrows are not drawn). From the definition of limits in **Set**, the pullback $P$ is

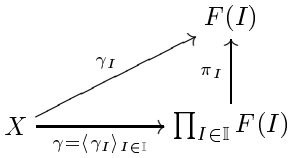
$$\begin{aligned} P &= \{ \langle a, c, b \rangle \mid a \in A \wedge b \in B \wedge c \in C \wedge f(a) = c \wedge g(b) = c \} \\ &= \{ \langle a, b \rangle \mid a \in A \wedge b \in B \wedge f(a) = g(b) \}. \end{aligned}$$

**Exercise 4.16** What are equalisers in **Set**?

**Exercise 4.17** The set $[\mathbb{I}, \mathcal{C}](\Delta\, X, F)$ is itself (isomorphic to) a limit, of which diagram?

## 4.4 Limits as Products and Equalisers

Assume a category $\mathcal{C}$ with all products. Given a small diagram $F : \mathbb{I} \to \mathcal{C}$, then a cone $\gamma$ uniquely defines a mediating arrow $\langle \gamma_I \rangle_{I \in \mathbb{I}} : X \to \prod_{\mathbb{I}} F(I)$ such that for every $I \in \mathbb{I}$ the diagram

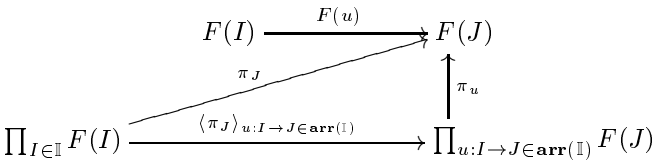



commutes. This is just a consequence of the definition of product.

We can also define in $\mathcal{C}$ the product $\prod_{u:I \to J \in \mathbf{arr}(\mathbb{I})} F(J)$: the product of $F(J)$'s such that there is an arrow $u : I \to J \in \mathbf{arr}(\mathbb{I})$. The projections of this product are indexed by the arrows in $\mathbb{I}$.

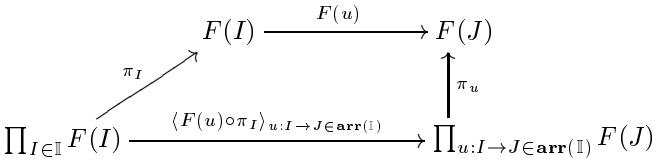
We have two mediating arrows from $\prod_{I \in \mathbb{I}} F(I)$ to $\prod_{u:I \to J \in \mathbf{arr}(\mathbb{I})} F(J)$:

1. $\langle \pi_J \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$ such that for an arbitrary $u : I \to J$ the diagram




   commutes; and

2. $\langle F(u) \circ \pi_I \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$ such that for an arbitrary $u : I \to J$ the diagram




   commutes.

**Theorem 4.18** A category $\mathcal{C}$ has all small limits iff $\mathcal{C}$ has all products and equalisers.

**Proof**:
*"Only-if"*: Obvious since products and equalisers are limits.
*"If"*: Given a diagram $F : \mathbb{I} \to \mathcal{C}$, a cone $\gamma$ from $X$ to $F$ equalises the mediating arrows $\langle \pi_J \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$ and $\langle F(u) \circ \pi_I \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$. On the other hand, any arrow equalising these arrows determines a cone on $F$. So a limit of $F$ is an equaliser for $\langle \pi_J \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$ and $\langle F(u) \circ \pi_I \rangle_{u:I \to J \in \mathbf{arr}(\mathbb{I})}$ .

□

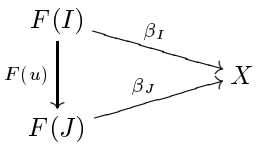
## 4.5 Definition of Colimit

**Definition 4.19 (Colimit Abstractly)** A *colimit* for a diagram $F : \mathbb{I} \to \mathcal{D}$ is a representation for $[\mathbb{I}, \mathcal{D}](F, \Delta -) : \mathcal{D} \to \mathbf{Set}$:

$$\mathcal{D}(C, -) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{D}](F, \Delta -) .$$

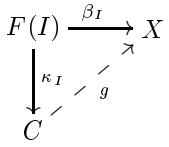
The object $C$ is often written as $\varinjlim_{\mathbb{I}} F$ or $\mathrm{colim}_{\mathbb{I}} F$.

The component at $X$ of the natural isomorphism $\theta$ in the definition above takes arrows in $\mathcal{D}(C, X)$ to cones from $F$ to $X$ (or a *cocone* for the $\mathbb{I}$-indexed diagram $F$ in $\mathcal{D}$ with vertex $X$). A cone $\beta$ from $F$ to $X$ is a collection of arrows $\langle \beta_I : F(I) \to X \rangle_{I \in \mathbb{I}}$ such that for any $u : I \to J$ in $\mathbb{I}$ the diagram




commutes.

**Proposition 4.20 (Colimit Concretely)** Given a diagram $F : \mathbb{I} \to \mathcal{D}$, a *colimit* for $F$ determines and is determined by a cone $\kappa = \langle \kappa_I : F(I) \to C \rangle_{I \in \mathbb{I}}$ from $F$ to some $C \in \mathcal{D}$ such that given any other cone $\beta = \langle \beta_I : F(I) \to C \rangle_{I \in \mathbb{I}}$ there exists a **unique** arrow $g : C \to X$ such that for every $I \in \mathbb{I}$ the diagram




commutes.

**Proof**: Left as exercise.

□

**Exercise 4.21** Prove the proposition above by using the dual of Theorem 3.12.

All the remarks we made for limits are applied to colimits in the dual version.

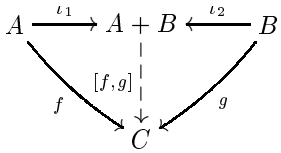
**Definition 4.22 (Cocomplete Category)** A category $\mathcal{D}$ is cocomplete iff the colimits for all diagrams exist in $\mathcal{D}$.

## 4.6 Examples of Colimits

**Definition 4.23 (Initial Object)** An object $\mathbf{0}_{\mathcal{D}}$ in a category $\mathcal{D}$ is called an *initial object* (*zero*) iff there is exactly one arrow from $\mathbf{0}_{\mathcal{D}}$ to every $A \in \mathcal{D}$.

A colimit for a diagram $F : \mathbb{I} \to \mathcal{D}$ where $\mathbb{I}$ is the empty category is an initial object in $\mathcal{D}$. Notice that this colimit might not exist. Any $A \in \mathcal{D}$ is the vertex for an empty cone from $F$, then there must be a **unique** arrow from $\varinjlim_{\mathbb{I}} F = \mathbf{0}_{\mathcal{D}}$ to $A$.

**Definition 4.24 (Coproduct)** A *coproduct* of two objects $A$ and $B$ in the category $\mathcal{D}$ is an object $A + B$ in $\mathcal{D}$ together with two *injections* $\iota_1 : A \to A + B$ and $\iota_2 : B \to A + B$, such that for any pair of arrows $f : A \to C$ and $g : B \to C$ there is exactly one arrow $[f, g] : A + B \to C$ such that the diagram

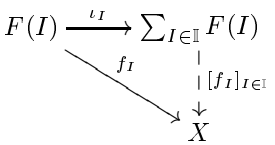



commutes, *i.e.*, $f = [f, g] \circ \iota_1$ and $g = [f, g] \circ \iota_2$.

Let $\mathbb{I} = \{1, 2\}$ be the discrete category with two objects. A colimit for a diagram $F : \mathbb{I} \to \mathcal{D}$ is a coproduct $F(1) + F(2)$ in $\mathcal{D}$ where the limiting cone defines the injections $\iota_1$ and $\iota_2$.

We can extend this to general coproducts:

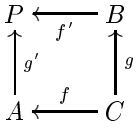
**Definition 4.25** Assume $\mathbb{I}$ is a discrete category. Given a diagram $F : \mathbb{I} \to \mathcal{D}$ a colimit for $F$ is called a coproduct and consists of an object $\sum_{I \in \mathbb{I}} F(I)$ and a cone $\langle \iota_I : F(I) \to \sum_{I \in \mathbb{I}} F(I) \rangle_{I \in \mathbb{I}}$ such that for any other cone $\langle f_I : F(I) \to X \rangle_{I \in \mathbb{I}}$ there exists a **unique** arrow $[f_I]_{I \in \mathbb{I}}$ such that for each $I \in \mathbb{I}$ the diagram




commutes.

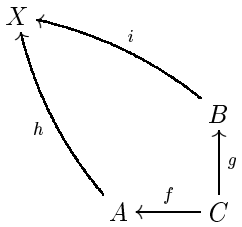
**Definition 4.26 (Pushout)** A *pushout* of a pair of arrows $f : C \to A$ and $g : C \to B$ is an object $P$ together with a pair of arrows $f' : B \to P$ and $g' : A \to P$ such that

1. $f' \circ g = g' \circ f$, *i.e.*, the diagram

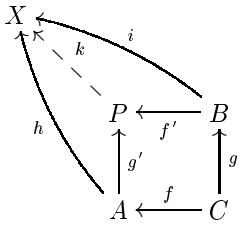



commutes, and

2. for any object $X$ together with arrows $h : A \to X$ and $i : B \to X$ for which $h \circ f = i \circ g$, *i.e.*, the diagram

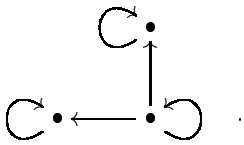



commutes, then there exists a **unique** arrow $k : P \to X$ such that $h = k \circ g'$ and $i = k \circ f'$, *i.e.*, the two triangles in



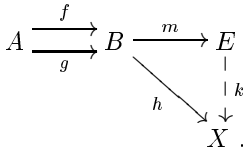

commute.

**Exercise 4.27** Let $\mathbb{I}$ be a category defined by the graph



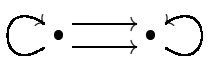
Show that colimit for a diagram $F : \mathbb{I} \to \mathcal{D}$ is a pushout in $\mathcal{D}$.

**Definition 4.28 (Coequaliser)** A *coequaliser* of a pair of parallel arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in a category $\mathcal{D}$ is an arrow $m : B \to E$ such that $m \circ f = m \circ g$ with the universal property that for any arrow $h : B \to X$ such that $h \circ f = h \circ g$ there is a **unique** arrow $k : E \to X$ making $k \circ m = h$, *i.e.*, the triangle in the diagram




commutes.

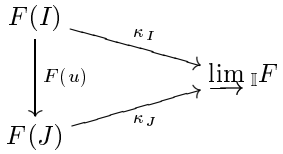
**Exercise 4.29** Let $\mathbb{I}$ be the category defined by the graph



Show that a colimit for a diagram $F : \mathbb{I} \to \mathcal{D}$ is a coequaliser in $\mathcal{D}$.

## 4.7 Colimits in Set

Consider a diagram $F : \mathbb{I} \to \mathbf{Set}$, from the definition of colimits we have that for any $u : I \to J$ the triangle




commutes, where $\langle \kappa_I \rangle_{I \in \mathbb{I}}$ is a colimiting cone. Then for any $x \in F(I)$ we have $\kappa_I(x) = (\kappa_j \circ F(u))(x)$.

We define our choice of colimits in **Set** to be $\varinjlim_{\mathbb{I}} F = \biguplus_{I \in \mathbb{I}} F(I) / \sim$, where the relation $\sim$ represents the constraints imposed by the arrows in $\mathbb{I}$. We define the relation $\sim$ as the least equivalence relation on $\biguplus_{I \in \mathbb{I}} F(I) \times \biguplus_{I \in \mathbb{I}} F(I)$ such that

$$(I, x) \sim (J, y) \Leftrightarrow \exists\ u : I \to J \in \mathbf{arr}(\mathbb{I})\ (F(u))x = y\ . \tag{3}$$

For $(I, x) \in \biguplus_{I \in \mathbb{I}} F(I)$ we let $[(I, x)]_\sim$ denote the equivalence class of $\sim$ containing $(I, x)$. The colimiting cone is given by the collection of injection arrows $\kappa_I : x \mapsto [(I, x)]_\sim$.

**Exercise 4.30** Show this choice indeed gives a colimit in **Set**.

**Exercise 4.31** What are coproducts, pushouts and coequalisers in **Set**?

## 4.8 Limits with Parameters

A special case of parametrised representability arises when considering limits of diagrams as representations:

**Proposition 4.32 (Parametrised Limits)** Let $F : \mathbb{I} \times \mathcal{A} \to \mathcal{C}$ be a functor such that for every $A \in \mathcal{A}$ the functor (small diagram in $\mathcal{C}$) $F(-, A) : \mathbb{I} \to \mathcal{C}$ has as limit the representation $\mathcal{C}(-, \varprojlim_{\mathbb{I}} F(-, A)) \overset{\theta^A}{\cong} [\mathbb{I}, \mathcal{C}](\Delta -, F(-, A))$. Then the mapping $A \mapsto \varprojlim_{\mathbb{I}} F(-, A)$ extends uniquely to a functor $\lambda W.\ \varprojlim_{\mathbb{I}} F(-, W) : \mathcal{A} \to \mathcal{C}$ such that

$$\mathcal{C}\big(X, \varprojlim_{\mathbb{I}} F(-, A)\big) \overset{(\theta^A)_X}{\cong} [\mathbb{I}, \mathcal{C}]\big(\Delta X, F(-, A)\big) \tag{4}$$

natural in $A \in \mathcal{A}$ and $X \in \mathcal{C}$.

**Proof**: This proposition is just a corollary of Theorem 3.15 (Parametrised Representability) applied to the functor $\lambda A, X.\ [\mathbb{I}, \mathcal{C}](\Delta X, F(-, A)) : \mathcal{A} \times \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$.

□

Given the functor $F : \mathbb{I} \times \mathcal{A} \to \mathcal{C}$ in the proposition, an arrow $f : A \to B$ in $\mathcal{A}$ has associated the naturality square

$$\begin{array}{ccc} A & \mathcal{C}\big(X, \varprojlim_{\mathbb{I}} F(-, A)\big) & \xrightarrow{\cong \theta^A_X} [\mathbb{I}, \mathcal{C}]\big(\Delta\, X, F(-, A)\big) \\ \Big\downarrow f & \Big\downarrow \varprojlim_{\mathbb{I}} F(-, g) \circ - & \Big\downarrow F(-, g) \circ - \\ G & \mathcal{C}\big(X, \varprojlim_{\mathbb{I}} F(-, B)\big) & \xrightarrow{\cong \theta^B_X} [\mathbb{I}, \mathcal{C}]\big(\Delta\, X, F(-, B)\big) \end{array}$$

for an arbitrary $X \in \mathcal{C}$. Taking $X = \varprojlim_{\mathbb{I}} F(-, A)$ and chasing the identity $\mathsf{id}_{\varprojlim_{\mathbb{I}} F(-,A)}$ around the diagram we have

$$\theta^B_{\varprojlim_{\mathbb{I}} F(-,A)}\big(\varprojlim_{\mathbb{I}} F(-, g)\big) = F(-, g) \circ \theta^A_{\varprojlim_{\mathbb{I}} F(-,A)}(\mathsf{id}_{\varprojlim_{\mathbb{I}} F(-,A)}),$$

*i.e.*, $\varprojlim_{\mathbb{I}} F(-, g)$ is the mediating arrow associated with the cone $F(-, g) \circ \theta^A_{\varprojlim_{\mathbb{I}} F(-,A)}(\mathsf{id}_{\varprojlim_{\mathbb{I}} F(-,A)})$.

**Exercise 4.33** Verify that $F(-, g) \circ \theta^A_{\varprojlim_{\mathbb{I}} F(-,A)}(\mathsf{id}_{\varprojlim_{\mathbb{I}} F(-,A)})$ is a cone from $\varprojlim_{\mathbb{I}} F(-, A)$ to $F(-, B)$.

An important special case of Proposition 4.32 arises by taking $\mathcal{A} = [\mathbb{I}, \mathcal{C}]$ and $F$ to be the evaluation functor $eval : \mathbb{I} \times [\mathbb{I}, \mathcal{C}] \to \mathcal{C}$ where $\mathcal{C}$ has all limits for $\mathbb{I}$-indexed diagrams. Hence, from the definition of the evaluation functor, for every $G \in [\mathbb{I}, \mathcal{C}]$ there is a representation

$$\mathcal{C}\big(-, \varprojlim_{\mathbb{I}} eval(-, G)\big) \cong [\mathbb{I}, \mathcal{C}]\big(\Delta -, eval(-, G)\big).$$

Notice that $eval(-, G) = G$. Then from parametrised limits we have a functor $\lambda W.\ \varprojlim_{\mathbb{I}} eval(-, W) = \varprojlim_{\mathbb{I}} - : [\mathbb{I}, \mathcal{C}] \to \mathcal{C}$, the *limit functor*, such that

$$\mathcal{C}(X, \varprojlim_{\mathbb{I}} F) \cong [\mathbb{I}, \mathcal{C}](\Delta\, X, F)$$

natural in $X \in \mathcal{C}$ and $F \in [\mathbb{I}, \mathcal{C}]$. From the observation above, given a natural transformation $\alpha : F \Rightarrow G$ the arrow $\varprojlim_{\mathbb{I}} \alpha$ is just the mediating arrow associated with the cone resulting from the composition of the limiting cone for $\varprojlim_{\mathbb{I}} F$ with $\alpha$.

**Exercise 4.34** State the dual of the Proposition 4.32 for colimits and conclude that if $\mathcal{C}$ has all limits for $\mathbb{I}$-indexed diagrams there is a colimit functor $\varinjlim_{\mathbb{I}} - : [\mathbb{I}, \mathcal{C}] \to \mathcal{C}$.

**Remark 4.35** The definition of the limit functor gives in reality a family of functors. A specific limit functor depends on the choice for the limit of each diagram $F \in [\mathbb{I}, \mathcal{C}]$. The limit functors, however, are isomorphic and then we can talk about "the limit functor" with respect to some particular choice.

**Exercise 4.36** Check that the definition of $\varprojlim_{\mathbb{I}} -$ satisfies the axioms in the definition of functor. In principle that is not necessary since parametrised representability ensures that the extension of the mapping $F \mapsto \varprojlim_{\mathbb{I}} F$ is a functor!

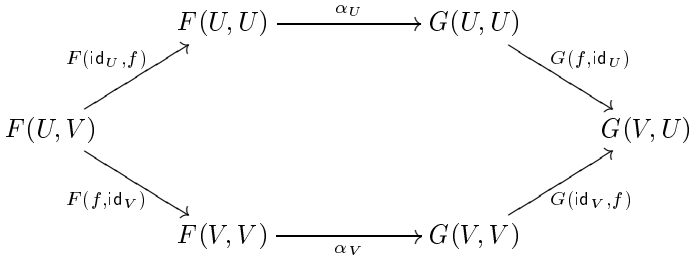
**Remark 4.37** In general there are categories where limits for some diagrams are missing. We can still define the limit functor, however, by restricting to the subcategory of diagrams for which all limits exists Thus, given a category $\mathcal{A}$ for which not necessarily all limits of type $\mathbb{I}$ exist, we define the full subcategory $Diag[\mathbb{I}, \mathcal{A}] \subseteq [\mathbb{I}, \mathcal{A}]$ of diagrams with limits in $\mathcal{A}$. Notice that if $\mathcal{A}$ is not the empty category then $Diag[\mathbb{I}, \mathcal{A}]$ contains at least the constant diagrams since their limits always exist in $\mathcal{A}$.

# 5 Ends

## 5.1 Dinaturality

We have considered bifunctors and natural transformations between them in § 2.2. We now study a special kind of bifunctors with mixed-variant domain, e.g. $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$, and *dinatural transformations* between them. The main motivation for dinatural transformations here is to present *ends* and dually *coends* as representations for suitable functors.

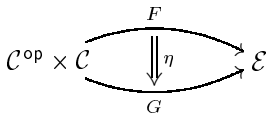
**Definition 5.1 (Dinatural Transformation)** Let $F, G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ be functors. A *dinatural transformation* $\alpha : F \ddot{\to} G$ consists of a family $\langle \alpha_U : F(U, U) \to G(U, U) \rangle_{U \in \mathcal{C}}$ of arrows in $\mathcal{D}$ such that for every arrow $f : V \to U$ in $\mathcal{C}$ the diagram



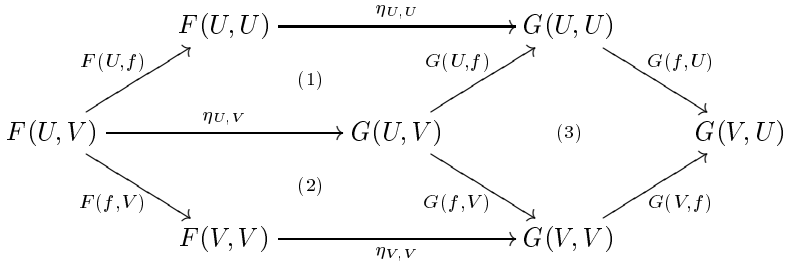

commutes.

Recall that $F(\mathsf{id}_U, f) = F(U, f)$ where $F(U, -) : \mathcal{C} \to \mathbf{Set}$ is the functor obtained by fixing the object $U$ in the first argument of $F$ (see § 2.2). Henceforth we use the more compact $F(U, f)$ instead of $F(\mathsf{id}_U, f)$. Similarly, we use $F(f, V)$ instead of $F(f, \mathsf{id}_V)$.

Every ordinary natural transformation gives rise to a dinatural transformation: Let $\eta$




be a natural transformation then the family of arrows $\langle \eta_{U,U} : F(U,U) \to G(U,U) \rangle_{U \in \mathcal{C}}$ is a dinatural transformation. Given the arrow $f : V \to U$ we construct the diagram




where

(1) is the naturality square associated with $(\mathsf{id}_U, f) : (U, V) \to (U, U)$,

(2) is the naturality square associated with $(f, \mathsf{id}_V) : (U, V) \to (V, V)$, and

(3) commutes since $G$ is a bifunctor and then $G(f, f) = G(V, f) \circ G(f, V) = G(f, U) \circ G(U, f)$ (see § 2.2).

Therefore,

$$\begin{aligned} G(f,U) \circ \eta_{U,U} \circ F(U,f) &= G(f,U) \circ G(U,f) \circ \eta_{U,V} && \text{(1) commutes,} \\ &= G(V,f) \circ G(f,V) \circ \eta_{U,V} && \text{(3) commutes,} \\ &= G(V,f) \circ \eta_{V,V} \circ F(f,V) && \text{(2) commutes.} \end{aligned}$$

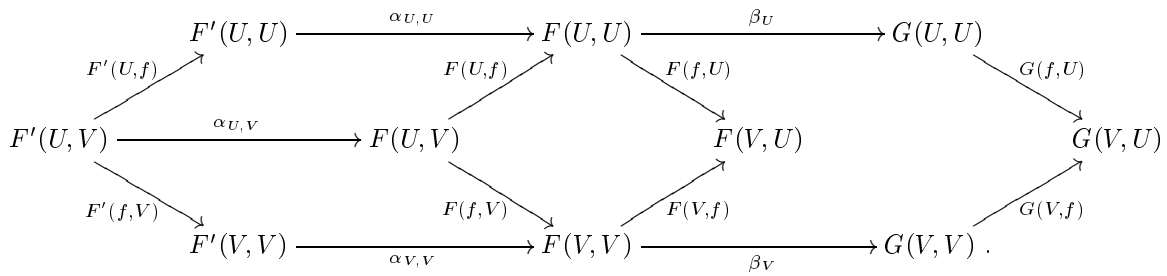
Thus $\langle \eta_{U,U} \rangle_{U \in \mathcal{C}}$ is a dinatural transformation.

In general dinatural transformations do not compose. But dinatural transformations obtained from natural transformations compose with arbitrary dinatural transformations.

**Exercise 5.2** Given $\alpha : F' \Rightarrow F$, $\beta : F \ddot{\to} G$ and $\gamma : G \Rightarrow G'$ (assume functors of the right type), show that

1. $\beta \circ \langle \alpha_{U,U} \rangle_U$;
2. $\langle \gamma_{U,U} \rangle_U \circ \beta$; and
3. $\langle \gamma_{U,U} \rangle_U \circ \beta \circ \langle \alpha_{U,U} \rangle_U$

are dinatural transformations, where the composition is defined componentwise. **Hint:** For 1 consider the diagram



The composition of dinatural transformations with natural transformations is functorial, i.e., we have a functor

$$\mathbf{Dinat}(-,+) : [\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}]^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}] \to \mathbf{Set}$$

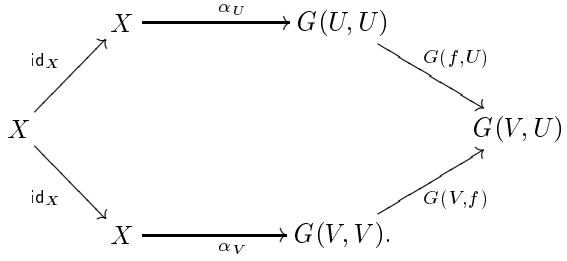
where $-$ and $+$ are place-holders used to stress the mixed-variance of the functor. $\mathbf{Dinat}(-,+)$ acts on functors and natural transformations as follows:

$$\begin{array}{ccc} F & G & \mathbf{Dinat}(F,G) \\ \eta \Uparrow & \Downarrow \varphi & \Big\downarrow \mathbf{Dinat}(\eta,\varphi) = \varphi \circ - \circ \eta \\ H & K & \mathbf{Dinat}(H,K)\ . \end{array}$$

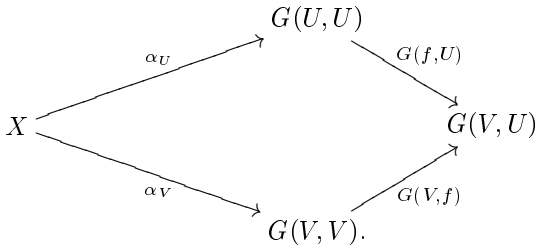
## 5.2 Special Cases of Dinatural Transformations

In the next examples we consider functors $F, G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ and dinatural transformations from $F$ to $G$.

**One-side Constant:** Assume $F = \Delta X$ is a constant functor for some $X \in \mathcal{D}$ and consider a dinatural transformation $\alpha : \Delta X \ddot{\to} G$. Then given $f : V \to U$ in $\mathcal{C}$ the diagram

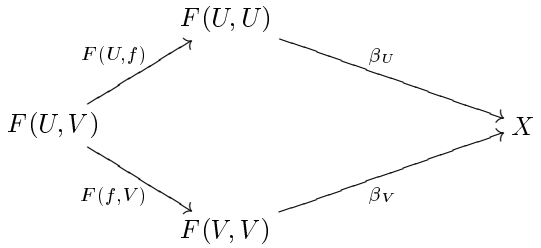



commutes. This diagram can be drawn as




Such an $\alpha$ is called an *extra* natural transformation, a *super* natural transformation, or a *wedge* from $X$ to $G$.

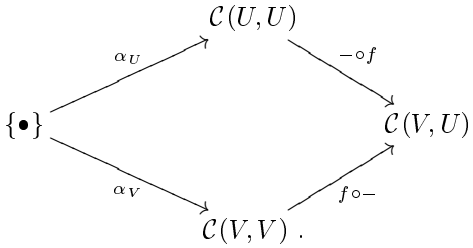
Likewise, if $G$ is a constant functor we can consider the dinatural transformation $\beta : F \ddot{\to} \Delta X$ for some $X \in \mathcal{D}$. Then we have for any $f : V \to U$ that the diagram




commutes. This is a wedge from $F$ to $X$.

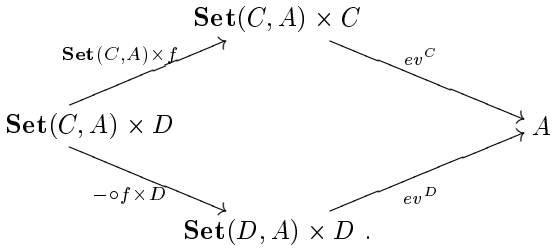
As an example of the first case consider the set-valued functors: $\Delta\{\bullet\}$ and $\mathcal{C}(-,+)$. The collection $\alpha$ whose component at $U$ is the function $\bullet \mapsto \mathsf{id}_U$ is a dinatural transformation from $\Delta\{\bullet\}$ to $\mathcal{C}(-,+)$.

For any $f : V \to U$ in $\mathcal{C}$ we construct the diagram




By chasing $\bullet$ in the diagram it is easy to check that the diagram commutes.

An example of the second case is given by the family of arrows $\langle ev^B : \mathbf{Set}(B,A) \times B \to A \rangle_{B \in \mathbf{Set}}$ for $A \in \mathbf{Set}$ where the component $ev^B$ is defined by $(g,b) \mapsto g(b)$. Given an arrow $f : D \to C$ in $\mathbf{Set}$ we construct the diagram




By chasing $(k,x) \in \mathbf{Set}(C,A) \times D$ in the diagram above

$$(k,x) \xmapsto{ev^C \circ \mathbf{Set}(C,A)\times f} k(f(x)), \text{ and}$$

$$(k,x) \xmapsto{ev^D \circ (-\circ f \times D)} (k \circ f)(x) = k(f(x)).$$

Therefore $\langle ev^B \rangle_{B \in \mathbf{Set}}$ is a dinatural transformation from $\big(\mathbf{Set}(-,A) \times +\big)$ to $\Delta\, A$.

**"Dummy" in One Argument:** A functor $F : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ is *dummy* in its first argument if there exists a functor $F_0 : \mathcal{C} \to \mathcal{D}$ such that $F$ is the composition

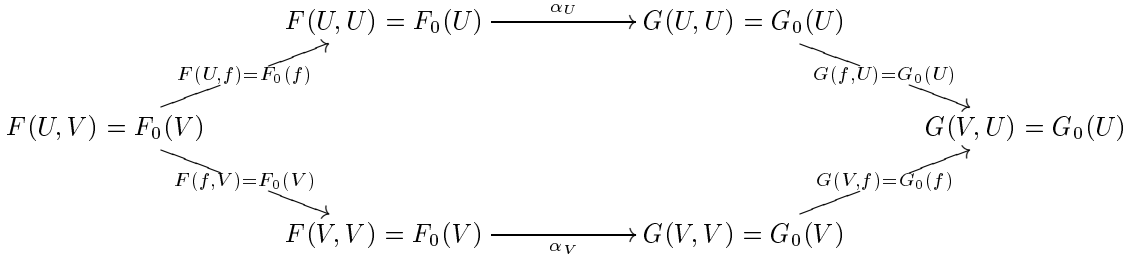
$$\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\pi_2} \mathcal{C} \xrightarrow{F_0} \mathcal{D}$$

where $\pi_2 : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$ is the projection to the second argument. Analogously, a functor $G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ is dummy in its second argument if there exists a functor $G_0 : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ such that $G$ is the composition

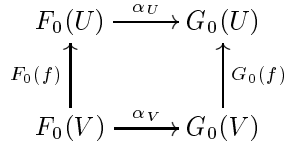
$$\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\pi_1} \mathcal{C}^{\mathsf{op}} \xrightarrow{G_0} \mathcal{D}$$

where $\pi_1 : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}^{\mathsf{op}}$ is the projection to the first argument.

Suppose $F, G : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$ are both dummy in the first argument and $\alpha : F \ddot{\to} G$ is a dinatural transformation. Then for any $f : V \to U$ in $\mathcal{C}$ the diagram




commutes. Hence the diagram



commutes and $\alpha$ defines a natural transformation from $F_0$ to $G_0$.

If $F$ is dummy in its second argument and $G$ is dummy in its first argument, a dinatural transformation $\alpha : F \ddot{\to} G$ is a family $\langle \alpha_U : F_0(U) \to G_0(U) \rangle_{U \in \mathcal{C}}$ of arrows in $\mathcal{D}$ where $F_0 : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ and $G_0 : \mathcal{C} \to \mathcal{D}$ are functors such that given $f : V \to U$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} F_0(U) & \xrightarrow{\alpha_U} & G_0(U) \\ {\scriptstyle F_0(f)}\uparrow & & \downarrow{\scriptstyle G_0(f)} \\ F_0(V) & \xrightarrow[\alpha_V]{} & G_0(V) \end{array}$$

commutes. The case where $F$ is dummy in its first argument and $G$ is dummy in its second argument is similar.

## 5.3 Definition of End

We take ends over functors whose domains are products of the form $\mathbb{I}^{\mathsf{op}} \times \mathbb{I}$, where $\mathbb{I}$ is a small category.

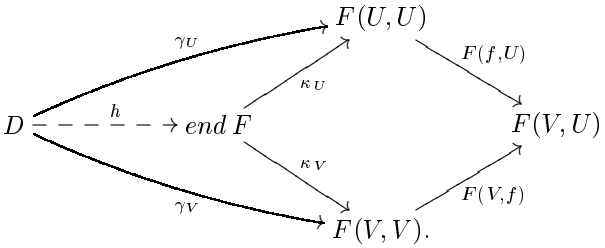
**Definition 5.3 (End Abstractly)** An *end* for a functor $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ is a representation for $\mathbf{Dinat}\big(\Delta -, F\big) : \mathcal{D}^{\mathsf{op}} \to \mathbf{Set}$

$$\mathcal{D}(-, end\, F) \overset{\theta}{\cong} \mathbf{Dinat}\big(\Delta -, F\big).$$

The set $\mathbf{Dinat}\big(\Delta X, F\big)$ for $X \in \mathcal{D}$ is the set of wedges from $X$ to $F$. We can recover a more concrete characterisation of ends as we have done with limits. Given a wedge $\gamma \in \mathbf{Dinat}\big(\Delta X, F\big)$ by Theorem 3.12 there exists a **unique** arrow $h : X \to end\, F$ such that

$$\begin{aligned} \gamma &= \mathbf{Dinat}\big(\Delta h, F\big)\big(\theta_{end\, F}(\mathsf{id}_{end\, F})\big) \\ &= \big(\theta_{end\, F}(\mathsf{id}_{end\, F})\big) \circ \Delta h \\ &= \langle \big(\theta_{end\, F}(\mathsf{id}_{end\, F})\big)_I \circ h \rangle_{I \in \mathbb{I}} \ . \end{aligned}$$

In other words, the *universal* or *ending* wedge $\theta_{end\, F}(\mathsf{id}_{end\, F})$ is uniquely extended by the arrow $h$ resulting in the wedge $\gamma$. Given $f : V \to U$ the situation is illustrated in the following commutative diagram where $\kappa = \theta_{end\, F}(\mathsf{id}_{end\, F})$




**Proposition 5.4 (End Concretely)** An *end* for a functor determines and is determined by a universal wedge.

**Proof**: By using Theorem 3.12 following the same reasoning done with limits.

□

**Notation 5.5** In general we write $\int_I F(I,I)$ for the object $end\, F \in \mathcal{D}$, where $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$; the properties of ends shall justify this notation.

**Limits are Ends:** Let $F_0 : \mathbb{I} \to \mathcal{D}$ be a functor with limit the representation $\mathcal{D}(-, \varprojlim_{\mathbb{I}} F_0) \overset{\theta}{\cong} [\mathbb{I}, \mathcal{D}](\Delta_{\mathbb{I}} -, F_0)$. The functor $F_0$ induces a functor $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ dummy in its first argument. The natural transformations from $\Delta_{\mathbb{I}} X$ to $F_0$ are the dinatural transformations from $\Delta_{\mathbb{I}^{\mathsf{op}} \times \mathbb{I}} X$ to $F$ – see § 5.2. Thus we have

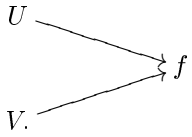
$$\mathcal{D}(X, \varprojlim_{\mathbb{I}} F_0) \overset{\theta_X}{\cong} [\mathbb{I}, \mathcal{D}](\Delta_{\mathbb{I}} X, F_0) = \mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}} \times \mathbb{I}} X, F\big)$$

natural in $X$, *i.e.*, $(\varprojlim_{\mathbb{I}} F_0, \theta)$ is a representation for $\mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}}\times\mathbb{I}} -, F\big)$ and consequently $\varprojlim_{\mathbb{I}} F_0 = \int_I F(I,I)$.

**Notation 5.6** This justifies writing $\int_I F_0(I)$ for $\varprojlim_{\mathbb{I}} F_0$.

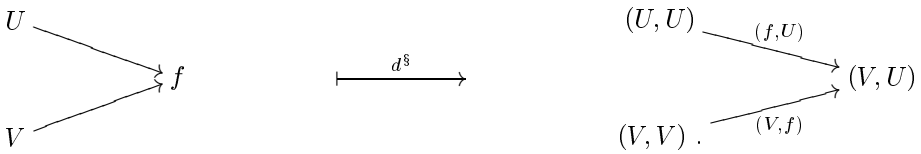
**Ends are Limits:** Let $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ be a functor with end the representation $\mathcal{D}(-, \int_I F(I,I)) \overset{\theta}{\cong} \mathbf{Dinat}\big(\Delta -, F\big)$ . We construct a category $\mathbb{I}^{\S}$ and a functor $d^{\S} : \mathbb{I}^{\S} \to \mathbb{I}^{\mathsf{op}} \times \mathbb{I}$ such that $\int_I F(I,I) = \varprojlim_{\mathbb{I}^{\S}} (F \circ d^{\S})$. The category $\mathbb{I}^{\S}$ is built from the objects and arrows in the category $\mathbb{I}$ as follows:

- $\mathbf{ob}(\mathbb{I}^{\S}) = \mathbf{ob}(\mathbb{I}) \cup \mathbf{arr}(\mathbb{I})$; and
- $\mathbf{arr}(\mathbb{I}^{\S})$ is the collection of identities and for every $f : V \to U$ in $\mathbb{I}$ the arrows




The only possible composition in $\mathbb{I}^{\S}$ is with identities.

Now, we define the functor $d^{\S} : \mathbb{I}^{\S} \to \mathbb{I}^{\mathsf{op}} \times \mathbb{I}$ acting on objects and arrows




Observe (left as an exercise) that the cones in $[\mathbb{I}^{\S}, \mathcal{D}](\Delta_{\mathbb{I}^{\S}} X, F \circ d^{\S})$ are exactly the wedges in $\mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}}\times\mathbb{I}} X, F\big)$:

$$\mathcal{D}(-, \textstyle\int_I F(I,I)) \overset{\theta}{\cong} \mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}}\times\mathbb{I}} -, F\big) = [\mathbb{I}^{\S}, \mathcal{D}](\Delta_{\mathbb{I}^{\S}} -, F \circ d^{\S}) \ .$$

Therefore $\int_I F(I,I)$ is a limit for $F \circ d^{\S}$.

**Exercise 5.7** Show that $[\mathbb{I}^{\S}, \mathcal{D}](\Delta_{\mathbb{I}^{\S}} -, F \circ d^{\S}) = \mathbf{Dinat}\big(\Delta_{\mathbb{I}^{\mathsf{op}}\times\mathbb{I}} -, F\big)$.

As a consequence of these two results we can deduce that if $\mathcal{D}$ has all limits then $\mathcal{D}$ has all ends; we also have a choice for ends in **Set** given by the choice for limits and a parametricity result.

## 5.4 Ends in Set

Let $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathbf{Set}$ be a functor. A choice for the end of $F$ is given by

$$\begin{aligned}
\textstyle\int_I F(I,I) &= \varprojlim_{\mathbb{I}^{\S}} (F \circ d^{\S}) \\
&= \big\{ \langle x_A \rangle_{A \in \mathbb{I}^{\S}} \mid x_A \in (F \circ d^{\S})(A) \ \&\ \forall f : V \to U \in \mathbf{arr}(\mathbb{I}) \big( F(f,U)(x_U) = x_f \ \&\ F(V,f)(x_V) = x_f \big) \big\} \\
&= \big\{ \langle x_U \rangle_{U \in \mathbb{I}} \mid x_U \in F(U,U) \ \&\ \forall f : V \to U \in \mathbf{arr}(\mathbb{I}) \ : \ F(f,U)(x_U) = F(V,f)(x_V) \big\} \qquad (5)
\end{aligned}$$

where the projections form the ending wedge.

## 5.5 Ends with Parameters

**Proposition 5.8 (Parametrise Ends)** Let $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \times \mathcal{B} \to \mathcal{D}$ be a functor such that for every $B \in \mathcal{B}$ the end $\int_I F(I,I,B)$ of the functor $F(-,+,B) : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ exists in $\mathcal{D}$. Then the function $B \mapsto \int_I F(I,I,B)$ extends uniquely to a functor from $\mathcal{B}$ to $\mathcal{D}$ such that

$$\mathcal{D}\big(D, \textstyle\int_I F(I,I,B)\big) \cong \mathbf{Dinat}\big(\Delta\, D, F(-,+,B)\big)$$

natural in $D \in \mathcal{D}$ and $B \in \mathcal{B}$.

**Proof**: The proof follows from parametrised representability (Theorem 3.15) applied to the functor $\lambda B, D.\, \mathbf{Dinat}\big(\Delta D, F(-,+,B)\big)$.

□

**Exercise 5.9** Define the "end functor" $end - : [\mathbb{I}^{\mathsf{op}} \times \mathbb{I}, \mathcal{C}] \to \mathcal{C}$ where $\mathcal{C}$ has all ends of type $\mathbb{I}$.

# 6 Limits and Ends

## 6.1 Some Useful Isomorphisms

**Curry and Uncurry:** Letting $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, the isomorphism $curry : [\mathcal{A} \times \mathcal{B}, \mathcal{C}] \cong [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ with inverse *uncurry* (see exercise 2.10) gives an isomorphism

$$[\mathcal{A} \times \mathcal{B}, \mathcal{C}](F, G) \overset{\textbf{curry}}{\cong} [\mathcal{A}, [\mathcal{B}, \mathcal{C}]](\lambda A.\, \lambda B.\, F(A, B), \lambda A.\, \lambda B.\, G(A, B)) \tag{6}$$

which sends a natural transformation $\alpha : F \Rightarrow G$ to $\langle \langle \alpha_{A,B} \rangle_{B \in \mathcal{B}} \rangle_{A \in \mathcal{A}}$. Its inverse, *uncurry*, acts by sending $\beta$ to $\langle (\beta_A)_B \rangle_{A,B \in \mathcal{A} \times \mathcal{B}}$. The isomorphism is natural in $F$ and $G$.

**Exercise 6.1** Swap?

**Limits and Ends as Representations:** Assume $\mathcal{D}$ has all limits, as a consequence of parametrised representability for limits there is an isomorphism

$$\mathcal{D}(X, \textstyle\int_{\mathbb{I}} F(I)) \overset{\theta_{X,F}}{\cong} [\mathbb{I}, \mathcal{D}](\Delta\, X, F)$$

natural in $X \in \mathcal{D}$ and $F \in [\mathbb{I}, \mathcal{D}]$ – see § 4.8. Moreover, from Theorem 3.12 the isomorphism is defined by

$$f \in \mathcal{D}\big(X, \textstyle\int_I F(I)\big) \longmapsto^{\theta_{X,F}} \kappa \circ \Delta f \in [\mathbb{I}, \mathcal{D}](\Delta\, X, F) \tag{7}$$

where $\kappa = \theta_{\int_I F(I), F}(\mathsf{id}_{\int_I F(I)})$ is the limiting cone of $F$.

The case for ends goes similarly. As $\mathcal{D}$ has all limits then it has all ends, then from parametrised representability for ends there is an isomorphism

$$\mathcal{D}(X, \textstyle\int_I G(I, I)) \overset{\sigma_{X,G}}{\cong} \mathbf{Dinat}\big(\Delta\, X, G\big)$$

natural in $X \in \mathcal{D}$ and $G \in [\mathbb{I}^{\mathsf{op}} \times \mathbb{I}, \mathcal{D}]$, where $\sigma_{X,G}$ is defined by

$$f \in \mathcal{D}\big(X, \textstyle\int_I G(I, I)\big) \longmapsto^{\sigma_{X,G}} \gamma \circ \Delta f \in \mathbf{Dinat}\big(\Delta X, G\big) \tag{8}$$

where $\gamma = \sigma_{\int_I G(I,I), G}(\mathsf{id}_{\int_I G(I,I)})$ is the universal wedge of $G$.

**Dinaturality Formula:** Letting $F, G : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$, then

$$\mathbf{Dinat}\big(F, G\big) = \textstyle\int_I \mathcal{D}\big(F(I, I), G(I, I)\big) \tag{9}$$

provided we make the specific choice for ends of set-valued functors as given by (5). The right hand side of this formula is the end taken over the functor $\lambda I^-, I^+.\, \mathcal{D}\big(F(I^+, I^-), G(I^-, I^+)\big)$. The two sides of (9) are the same (left as an exercise) functorially in $F$ and $G$ (by virtue of the identity isomorphism).

**Exercise 6.2** Prove the dinaturality formula by using our explicit choice for ends in **Set**.

**Naturality Formula:** We can specialise the dinaturality formula to an end characterisation of natural transformations. Letting $F, G : \mathbb{I} \to \mathcal{D}$, then

$$\mathbf{Nat}\big(F, G\big) = [\mathbb{I}, \mathcal{D}](F, G) = \textstyle\int_I \mathcal{D}\big(F(I), G(I)\big) \tag{10}$$

where the end of the right hand side is taken over the functor $\lambda I^-, I^+.\, F(I^-, I^+)$. Again with respect to the specific choice for ends in **Set** the two sides are the same functorially in $F$ and $G$.

## 6.2 Limits in Functor Categories

Let $F : \mathbb{I} \times \mathbb{J} \to \mathcal{D}$ be a functor; if for each $J \in \mathbb{J}$ the functor $F(-, J) : \mathbb{I} \to \mathcal{D}$ has a limit in $\mathcal{D}$ then by parametrised limits (Theorem 4.32) there is a functor $\lambda J.\ \int_I F(I, J) : \mathbb{J} \to \mathcal{D}$. This limit functor is in fact isomorphic to the limit $\int_I \lambda J.\ F(I, J)$. In other words, limits in functor categories are obtained pointwise provided the pointwise limits exist.

**Proposition 6.3** Assume a functor $F : \mathbb{I} \times \mathbb{J} \to \mathcal{D}$ is such that for each object $J \in \mathbb{J}$ there is a limiting cone $\kappa^J$ for $F(-, J)$. Then $\big\langle \langle \kappa_I^J \rangle_{J \in \mathbb{J}} \big\rangle_{I \in \mathbb{I}}$ is a limiting cone for $\lambda I.\, \lambda J.\, F(I, J)$ with vertex $\lambda J.\ \int_I F(I, J)$. Assuming every functor $F$ in $[\mathbb{I} \times \mathbb{J}, \mathcal{D}\,]$ has a limit $\int_I F(I, J)$ for all object $J \in \mathbb{J}$, the isomorphism

$$\int_I \lambda J.\, F(I, J) \cong \lambda J.\ \int_I F(I, J)\ ,$$

determined by universality, is natural in $F$.

**Proof**: We show $[\mathbb{J}, \mathcal{D}]\big(Y, \lambda J.\ \int_I F(I, J)\big) \cong [\mathbb{I}, [\mathbb{J}, \mathcal{D}]]\big(\Delta_{\mathbb{I}} Y, \lambda I.\, \lambda J.\, F(I, J)\big)$ natural in $Y \in [\mathbb{I}, \mathcal{D}]$. This natural isomorphism is obtained as the composition of the following natural isomorphisms:

$$\begin{aligned}
[\mathbb{J}, \mathcal{D}]\big(Y, \lambda J.\ \textstyle\int_I F(I, J)\big) &= \textstyle\int_J \mathcal{D}\big(Y(J), \int_I F(I, J)\big) && \text{by naturality formula (10),}\\
&\cong \textstyle\int_J [\mathbb{I}, \mathcal{D}]\big(\lambda I.\, Y(J), \lambda I.\, F(I, J)\big) && \text{by representation of limits,}\\
&= [\mathbb{J}, [\mathbb{I}, \mathcal{D}]]\big(\lambda J.\, \lambda I.\, \Delta(Y(J)), \lambda J.\, \lambda I.\, F(I, J)\big) && \text{by naturality formula (10),}\\
&\cong [\mathbb{I} \times \mathbb{J}, \mathcal{D}]\big(\lambda I, J.\, Y(J), \lambda I, J.\, F(I, J)\big) && \text{by uncurrying and swapping,}\\
&\cong [\mathbb{I}, [\mathbb{J}, \mathcal{D}]]\big(\lambda I.\, \lambda J.\, (Y\ J), \lambda I.\, \lambda J.\, F(I, J)\big) && \text{by currying.}
\end{aligned}$$

As each isomorphism is natural in $Y \in [\mathbb{J}, \mathcal{D}]$ we obtain the required representation. A natural transformation $\varphi \in [\mathbb{J}, \mathcal{D}]\big(Y, \lambda J.\ \int_I F(I, J)\big)$ is sent to $\big\langle \langle \kappa_I^J \circ \varphi_J \rangle_{J \in \mathbb{J}} \big\rangle_{I \in \mathbb{I}}$ as shown below:

$$\varphi = \langle \varphi_J \rangle_{J \in \mathbb{J}} \longmapsto \langle \kappa^J \circ \Delta \varphi_J \rangle_{J \in \mathbb{J}} \longmapsto \langle \kappa_I^J \circ \varphi_J \rangle_{I, J \in \mathbb{I} \times \mathbb{J}} \longmapsto \big\langle \langle \kappa_I^J \circ \varphi_J \rangle_{J \in \mathbb{J}} \big\rangle_{I \in \mathbb{I}}\ .$$

In particular the identity $\mathsf{id}_{\lambda J.\ \int_I F(I, J)}$ is sent to $\big\langle \langle \kappa_I^J \rangle_{J \in \mathbb{J}} \big\rangle_{I \in \mathbb{I}}$ which by Theorem 3.12 is a limiting cone for $\lambda I.\, \lambda J.\, F(I, J)$. As each isomorphism is natural in $F$ as well, by parametrised representability, we have

$$\int_I \lambda J.\, F(I, J) \cong \lambda J.\ \int_I F(I, J)$$

natural in $F$.

□

## 6.3 Fubini Theorem

**Proposition 6.4** Let $F : \mathbb{I} \times \mathbb{J} \to \mathcal{D}$ be a functor and assume $\int_I F(I, J)$ exists for any object $J$ in $\mathbb{J}$, then

$$\int_J \int_I F(I, J) \cong \int_{(I, J)} F(I, J)$$

meaning if one limit exists so does the other. If $\mathcal{D}$ is complete the isomorphism is natural in $F$.

**Proof**: By parametrised representability and since $\int_I F(I, J)$ exists for any $J \in \mathbb{J}$ we have the functor $\lambda J.\ \int_I F(I, J)$. By representation of limits

$$\begin{aligned}
\mathcal{D}\big(-, \textstyle\int_J \int_I F(I, J)\big) &\cong [\mathbb{J}, \mathcal{D}]\big(\Delta -, \lambda J.\ \textstyle\int_I F(I, J)\big) \text{ and}\\
\mathcal{D}\big(-, \textstyle\int_{(I, J)} F(I, J)\big) &\cong [\mathbb{I} \times \mathbb{J}, \mathcal{D}]\big(\Delta -, \lambda I, J.\, F(I, J)\big)\ .
\end{aligned}$$

We prove that the right hand sides are isomorphic:

$$\begin{aligned}
[\mathbb{J}, \mathcal{D}]\big(\Delta V, \lambda J.\ \textstyle\int_I F(I, J)\big) &\cong \textstyle\int_J \mathcal{D}\big(V, \int_I F(I, J)\big) && \text{by naturality formula (10),}\\
&\cong \textstyle\int_J [\mathbb{I}, \mathcal{D}]\big(\lambda I.\, V, \lambda I.\, F(I, J)\big) && \text{by representation for limits,}\\
&\cong [\mathbb{J}, [\mathbb{I}, \mathcal{D}]]\big(\lambda J.\, \lambda I.\, V, \lambda J.\, \lambda I.\, F(I, J)\big) && \text{by naturality formula (10),}\\
&\cong [\mathbb{I} \times \mathbb{J}, \mathcal{D}]\big(\lambda I, J.\, V, \lambda I, J.\, F(I, J)\big) && \text{by uncurring and swapping.}
\end{aligned}$$

Notice that each isomorphism is natural in $V \in \mathcal{D}$ and $F \in [\mathbb{I} \times \mathbb{J}, \mathcal{D}]$ (provided we assume $\mathcal{D}$ is complete).

□

**Corollary 6.5 (Fubini for Limits)** Let $F : \mathbb{I} \times \mathbb{J} \to \mathcal{D}$ be a functor such that

- $\int_I F(I, J)$ exists for all object $J$ in $\mathbb{J}$; and
- $\int_J F(I, J)$ exists for all object $I$ in $\mathbb{I}$,

then

$$\int_J \int_I F(I, J) \cong \int_I \int_J F(I, J) \ ,$$

meaning if one side exists so does the other. Assuming $\mathcal{D}$ is complete the isomorphism is natural in $F$.

**Proof**: Apply the Proposition 6.4 twice.

□

As ends are limits we also have a Fubini theorem for ends:

**Corollary 6.6 (Fubini for Ends)** Given $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \times \mathbb{J}^{\mathsf{op}} \times \mathbb{J} \to \mathcal{D}$ such that

- $\int_I F(I, I, J^-, J^+)$ exists for all $J^-$,$J^+ \in \mathbb{J}$; and
- $\int_J F(I^-, I^+, J, J)$ exists for all $I^-$,$I^+ \in \mathbb{I}$,

then

$$\int_I \int_J F(I, I, J, J) \cong \int_J \int_I F(I, I, J, J) \ ,$$

meaning if one side exists so does the other. Moreover this isomorphism is natural in $F$ if $\mathcal{D}$ is complete.

## 6.4 Preservation of Limits

**Definition 6.7 (Preservation of Limits)** Let $F : \mathbb{I} \to \mathcal{C}$ be a diagram with limiting cone $\kappa = \langle \kappa_I : \int_I F(I) \to F(I) \rangle_{I \in \mathbb{I}}$, a functor $G : \mathcal{C} \to \mathcal{D}$ preserves the limit of $F$ iff $G\,\kappa = \langle G(\kappa_I) : G(\int_I F(I)) \to G(F(I)) \rangle_{I \in \mathbb{I}}$ is a limiting cone in $\mathcal{D}$.

**Proposition 6.8** The hom-functor $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ preserves all limits.

**Proof**: Suppose $F : \mathbb{I} \to \mathcal{C}$ is a diagram with limiting cone $\kappa = \langle \kappa_I \rangle_{I \in \mathbb{I}}$, as a consequence of our choice of limits in **Set** (a special case of naturality formula (10)):

$$\mathcal{C}(C, \textstyle\int_I F(I)) \cong [\mathbb{I}, \mathcal{C}](\Delta C, F) = \int_I \mathcal{C}(C, F(I)).$$

That does not ensure that $\mathcal{C}(C, -)\,\kappa$ is limiting. Let us examine the diagram for $\kappa_I$

$$\begin{array}{ccc} \mathcal{C}(C, \int_I F(I)) & \xrightarrow{\mathcal{C}(C,\kappa_I) = \kappa_I \circ -} & \mathcal{C}(C, F(I)) \\ \Big\downarrow{\scriptstyle \cong \theta} & \nearrow_{\pi_I} & \\ \int_I \mathcal{C}(C, F(I)) \ , & & \end{array}$$

given $f \in \mathcal{C}(C, \int_I F(I))$, we have from the representation for limits (7)

$$f \xmapsto{\ \pi_I \circ \theta\ } \kappa_I \circ f \ .$$

Thus $\pi_I \circ \theta = \kappa_I \circ - = \mathcal{C}(C, \kappa_I)$, and by Proposition 4.6*(i)* we conclude $\langle \mathcal{C}(C, \kappa_I) \rangle_{I \in \mathbb{I}}$ is a limiting cone in **Set**.

□

Clearly, if $F$ and $G$ are isomorphic functors then $F$ preserves $\mathbb{I}$-indexed limits iff $G$ does. Hence from Proposition 6.8, we conclude that representable functors preserve limits.

**Proposition 6.9** Given a diagram $F \in [\mathbb{I}, \mathcal{C}]$ and a functor $G : \mathcal{C} \to \mathcal{D}$ such that the limits of $F$ and $G \circ F$ exists, then $G$ preserves the limit of $F$ iff the mediating arrow defined by universality of limit is an isomorphism.

**Proof**: Left as exercise.

□

**Exercise 6.10** Prove the result above by using Proposition 4.6

To prove preservation of limits by closely following the definition might be tedious. In general it is easier to prove that a functor preserves limiting objects, however, this in general gives a necessary but not sufficient condition.

**Exercise 6.11** Show an example of a functor $G : \mathcal{C} \to \mathcal{D}$ such that for some diagram $F : \mathbb{I} \to \mathcal{C}$ there is an isomorphism $G(\int_I F(I)) \cong \int_I (G \circ F)(I)$ but $G$ does not preserve the limit of $F$, i.e., $G$ sends a limiting cone in $\mathcal{C}$ to a non-limiting cone in $\mathcal{D}$.

Fortunately, naturality comes to the rescue:

**Theorem 6.12** Assume categories $\mathcal{C}$ and $\mathcal{D}$ with limits for all $\mathbb{I}$-indexed diagrams. The functor $G : \mathcal{C} \to \mathcal{D}$ preserves limits of $\mathbb{I}$-indexed diagrams iff

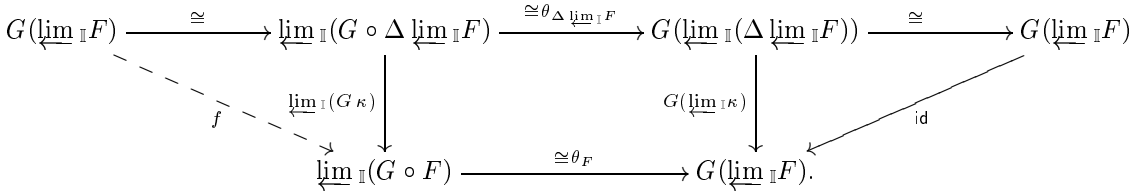
$$\varprojlim_{\mathbb{I}}^{\mathcal{D}} (G \circ F) \cong G(\varprojlim_{\mathbb{I}}^{\mathcal{C}} F)$$

natural in $F$, i.e., the square of categories and functors

$$\begin{array}{ccc} [\mathbb{I}, \mathcal{C}] & \xrightarrow{\varprojlim_{\mathbb{I}} -} & \mathcal{C} \\ {\scriptstyle G\circ -}\downarrow & & \downarrow{\scriptstyle G} \\ [\mathbb{I}, \mathcal{D}] & \xrightarrow{\varprojlim_{\mathbb{I}} -} & \mathcal{D} \end{array}$$

commutes up to isomorphism.

**Proof**:
*"Only-if"*: If $G$ preserves limits then by Proposition 6.9 the mediating arrow from $G(\varprojlim_{\mathbb{I}} F)$ to $\varprojlim_{\mathbb{I}}(G \circ F)$ for a given diagram $F$ is an isomorphism. It is easy to verify that those mediating arrows are the components of a natural transformation.
*"If"*: Assume $\theta_F : \varprojlim_{\mathbb{I}}(G \circ F) \cong G \circ \varprojlim_{\mathbb{I}}$ natural in $F \in [\mathbb{I}, \mathcal{C}]$. Take $\kappa$ to be the limiting cone defined by the limit of an arbitrary diagram $F : \mathbb{I} \to \mathcal{C}$. The cone $\kappa$ is an arrow in the category $[\mathbb{I}, \mathcal{C}]$; this defines the naturality square in the diagram




Regardless the choice for limits we clearly have $G(\varprojlim_{\mathbb{I}}(\Delta \varprojlim_{\mathbb{I}} F)) \cong G(\varprojlim_{\mathbb{I}} F)$ where the isomorphism is given by the mediating arrow. Thus, by uniqueness of the mediating arrows the right hand side triangle in the diagram above commutes and $G(\varprojlim_{\mathbb{I}} \kappa)$ is an isomorphism. By a similar argument the left hand side triangle commutes as well, where $f$ is the mediating arrow defined by the universal property of limit. By commutative of the square the arrow $\varprojlim_{\mathbb{I}}(G\,\kappa)$ is an isomorphism which in turn implies that $f$ is an isomorphism. Therefore, by Proposition 6.9 we conclude $G$ preserves the limit of $F$.

□

We can generalise the result of Theorem 6.12 to consider the case where not all limits exist. We take instead of $[\mathbb{I},\mathcal{C}]$ the category $Diag[\mathbb{I},\mathcal{C}] \subseteq [\mathbb{I},\mathcal{C}]$ of diagrams whose limits exist in $\mathcal{C}$ as defined in Remark 4.37:

**Corollary 6.13** A functor $G : \mathcal{C} \to \mathcal{D}$ preserves those $\mathbb{I}$-indexed limits which exist in $\mathcal{C}$ iff

*(i)* whenever $F : \mathbb{I} \to \mathcal{C}$ has a limit in $\mathcal{C}$ then $G \circ F : \mathbb{I} \to \mathcal{D}$ has a limit in $\mathcal{D}$; and

*(ii)* $\varprojlim_{\mathbb{I}} (G \circ F) \cong G(\varprojlim_{\mathbb{I}} F)$ natural in $F$, where $F \in Diag[\mathbb{I},\mathcal{C}]$.

**Proof**: Clearly the constant diagrams are objects in $Diag[\mathbb{I},\mathcal{D}]$ since their limit exist. Then the proof of Theorem 6.12 works within the subcategory $Diag[\mathbb{I},\mathcal{D}]$.

□

**Proposition 6.14** The Yoneda embedding $\mathcal{Y} : \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ preserves limits.

**Proof**: As a consequence of Propositions 6.8 and 6.3 there is an isomorphism

$$(\mathcal{Y} \int_I F(I))(C) \cong (\int_I \mathcal{Y}\, F(I))(C)$$

natural in $C \in \mathcal{C}$ and $F \in Diag[\mathbb{I},\mathcal{C}]$. As $[\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]$ is complete the functor $\mathcal{Y} \circ F$ has limit for all $F$, then from Corollary 6.13 we can conclude that $\mathcal{Y}$ preserves limits.

□

**Proposition 6.15** The bifunctor $eval : [\mathbb{J},\mathcal{D}] \times \mathbb{J} \to \mathcal{D}$ induces a functor $eval(-,J) : [\mathbb{J},\mathcal{D}] \to \mathcal{D}$ for every $J \in \mathbb{J}$. Suppose $F : \mathbb{I} \to [\mathbb{J},\mathcal{D}]$ is a functor such that $\lambda I.\,(F\,I)J$ has a limit for all $J \in \mathbb{J}$; then $eval(-,J)$ preserves the limit of $F$.

**Proof**: Again we use Proposition 6.3 to conclude that there is an isomorphism

$$\big(\int_I F(I)\big)\,J \cong \int_I \big(F(I)J\big)$$

natural in $F$ and $J$. By definition $eval(\int_I F(I), J) = (\int_I F(I))(J)$ and $\int_I eval(F(I), J) = \int_I \big(F(I)J\big)$. Thus, from Corollary 6.13 we have $eval(-,J)$ preserves the limit of $F$.

□

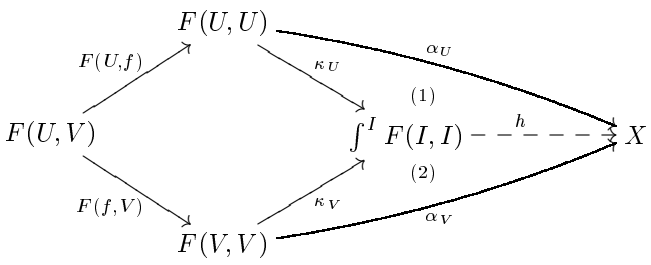
# 7 Coends

## 7.1 Definition of Coend

**Definition 7.1 (Coend Abstractly)** A *coend* for $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$ is a representation for $\mathbf{Dinat}\big(F, \Delta -\big) : \mathcal{D} \to \mathbf{Set}$

$$\mathcal{D}\big(\int^I F(I,I), -\big) \overset{\theta}{\cong} \mathbf{Dinat}\big(F, \Delta -\big)\ .$$

An element of the set $\mathbf{Dinat}\big(F, \Delta X\big)$ is a wedge from $F$ to $X$. By using the dual form of Theorem 3.12 we can recover a concrete definition for coends in terms of couniversal wedges. Thus given a wedge $\alpha \in \mathbf{Dinat}\big(F, \Delta X\big)$ there exists a **unique** arrow $h : \int^I F(I,I) \to X$ such that $\alpha = \Delta\, h \circ \theta_{\int^I F(I,I)} \Big(\mathsf{id}_{\int^I F(I,I)}\Big)$ where $\theta_{\int^I F(I,I)} \Big(\mathsf{id}_{\int^I F(I,I)}\Big)$ is the couniversal wedge. Therefore, given an arrow $f : V \to U$ in $\mathbb{I}$ we have that in the diagram



(1) and (2) commutes, where $\kappa = \theta_{\int^I F(I,I)}\left(\mathsf{id}_{\int^I F(I,I)}\right)$.

By means of the dinaturality 9 formula we can define coends in terms of ends, given a functor $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathcal{D}$

$$\mathcal{D}\big(\int^I F(I,I), V\big) \cong \mathbf{Dinat}\big(F, \Delta V\big) = \int_I \mathcal{D}(F(I,I), V) \tag{11}$$

natural in $V$ (and also natural in $F$ if $\mathcal{D}$ is a complete category).

The relation between colimits and coends allows to extend our choice for colimits of set-valued functors to coends in **Set**. Thus, given a functor $F : \mathbb{I}^{\mathsf{op}} \times \mathbb{I} \to \mathbf{Set}$

$$\int^I F(I,I) = \biguplus_{I \in \mathbb{I}} F(I,I) / \sim \tag{12}$$

where the relation $\sim$ is the least equivalence relation on $\biguplus_{I \in \mathbb{I}} F(I,I) \times \biguplus_{I \in \mathbb{I}} F(I,I)$ such that

$$(U,x) \sim (V,y) \text{ if } \exists f : V \to U \in \mathbf{arr}(\mathbb{I})\, \exists z \in F(U,V)\,.\,(x = F(U,f)z)\,\&\,(y = F(f,V)z)\ .$$

The universal wedge is given by the collection of injection arrows $x \mapsto [(U,x)]_\sim$.

The results for ends have a dual version for coends: parametricity coends, Fubini Theorem and pointwise coends in functor categories.

## 7.2 Density Formula

Given an object $X$ in the functor category $[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$, by Yoneda Lemma (Theorem 3.3) and naturality formula 10

$$X(U) \cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(\mathcal{Y}\,U, X\big) = \int_V \mathbf{Set}\big((\mathcal{Y}\,U)V, X(V)\big). \tag{13}$$

**Notation 7.2** Henceforth we shall denote hom-sets in **Set** with square brackets, for instance $[A,B]$ instead of $\mathbf{Set}(A,B)$. Rewriting (13) to $X(U) \cong \int_V [\mathbb{C}(V,U), X(V)]$.

We now present a coend formula for $X$. The *density formula* expresses $X(U)$ in terms of the coend

$$X(U) \cong \int^W \mathbb{C}(U,W) \times X(W)$$

natural in $U \in$ and $X \in [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$. Notice that the right hand side is a coend for the functor $\lambda W^-, W^+.\, \mathbb{C}(U,W^+) \times X(W^-) : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$ where $U \in \mathbb{C}$.

**Notation 7.3** In a category $\mathcal{C}$ with all small coproducts, for $S \in \mathbf{Set}$ and $A \in \mathcal{C}$, the *copower* $S \cdot A$ is the coproduct $\sum_{s \in S} A$.

Let us rewrite the density formula using copowers:

$$\begin{aligned}
X &\cong \lambda U.\ \textstyle\int^W X(W) \times \mathbb{C}(U,W) \\
&\cong \textstyle\int^W \lambda U.\, X(W) \times \mathbb{C}(U,W) && \text{as coends are computed pointwise,} \\
&\cong \textstyle\int^W X(W) \cdot (\lambda U.\, \mathbb{C}(U,W)) \\
&= \textstyle\int^W X(W) \cdot \mathcal{Y}W
\end{aligned} \tag{14}$$

where (14) is justified by

$$\sum_{s \in X(W)} (\lambda U.\, \mathbb{C}(U,W)) \cong \lambda U. \sum_{s \in X(W)} \mathbb{C}(U,V) = \lambda U.\, X(W) \times \mathbb{C}(U,W)\ .$$

**Proof of the density formula:** Let $Y \in [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$

$$\begin{aligned}
[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]\big(\textstyle\int^W X(W) \cdot \mathcal{Y}W, Y\big) &= \textstyle\int_U [(\int^W X(W) \cdot \mathcal{Y}W)U, Y(U)] && \text{by naturality formula (10),}\\
&\cong \textstyle\int_U [\int^W X(W) \times \mathbb{C}(U,W), Y(U)] && \text{as coends are computed pointwise,}\\
&\cong \textstyle\int_U \int_W [X(W) \times \mathbb{C}(U,W), Y(U)] && \text{by (11),}\\
&\cong \textstyle\int_U \int_W [X(W), [\mathbb{C}(U,W), Y(U)]] && \text{by currying,}\\
&\cong \textstyle\int_W \int_U [X(W), [\mathbb{C}(U,W), Y(U)]] && \text{by Fubini,}\\
&\cong \textstyle\int_W [X(W), \int_U [\mathbb{C}(U,W), Y(U)]] && \text{by dinaturality formula,}\\
&\cong \textstyle\int_W [X(W), Y(W)] && \text{by Yoneda Lemma,}\\
&= [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](X, Y) && \text{by dinaturality formula,}
\end{aligned}$$

natural in $Y$. By Proposition 3.6 we have $X \cong \int^W X(W) \cdot \mathcal{Y}W$.

## 7.3 Recasting the Density Formula

The density formula can be rewritten in two ways. Both make use of a construction called the *category of elements* of a set-valued functor.

**Definition 7.4** Let $G : \mathcal{A} \to \mathbf{Set}$ be a functor. Define its *category of elements* $\mathbf{els}(G)$ (sometimes $\int G$ though this clashes with our use for ends and coends) to be the category consisting of

- $\mathbf{ob}(\mathbf{els}(G)) = \biguplus_{A \in \mathcal{A}} G(A)$; and
- arrows $h : (A, z) \to (A', z')$ where $h : A \to A'$ is an arrow in $\mathcal{A}$ such that $G\,h(z) = z'$.

Often we will take the category of elements defined by a contravariant functor $G : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}$, in this case $h : (C, z) \to (C', z')$ is an arrow of $\mathbf{els}(G)$ if $h : C' \to C$ is an arrow in $\mathcal{C}$ and $(G\,h)(z) = z'$.

Note that a product of sets $X(W) \times \mathbb{C}(U,W)$ can be written as a coproduct in two ways

$$\sum_{x \in X(W)} \mathbb{C}(U,W) = \textstyle\int^{x \in X(W)} \mathbb{C}(U,W) \tag{15}$$

or

$$\sum_{f \in \mathcal{C}(U,W)} X(W) = \textstyle\int^{f \in \mathbb{C}(U,W)} X(W) \tag{16}$$

where we treat the sets $X(W)$ and $\mathbb{C}(W,U)$ as discrete categories (i.e. all maps are identities) when using the "integral" notation (here representing colimits).

From the density formula, we have

$$X(U) \cong \textstyle\int^W X(W) \times \mathbb{C}(U,W) = \int^W \int^{x \in X(W)} \mathbb{C}(U,W)\ .$$

By using $\mathbf{els}(X)$ we can replace the "double integral" by a "single integral"

$$\textstyle\int^W \int^{x \in X(W)} \mathbb{C}(U,W) \cong \int^{(W,x) \in \mathbf{els}(X)} \mathbb{C}(U,W)\ . \tag{17}$$

Notice that while the left hand side is a coend for the functor $\lambda W^-, W^+.\ \int^{x \in X(W^-)} \mathbb{C}(U, W^+)$ the right hand side is a colimit for the functor $(W,x) \mapsto \mathbb{C}(U,W)$ from $\mathbf{els}(X)$ to $\mathbf{Set}$.

**Exercise 7.5** Considering our particular choice for coends and colimits in **Set** prove that the isomorphism (17) holds and conclude that it is natural in $U$.

Hence, $X(U) \cong \int^{(W,x) \in \mathbf{els}(X)} \mathbb{C}(U,W)$, so

$$X \cong \lambda U.\ \textstyle\int^{(W,x) \in \mathbf{els}(X)} \mathbb{C}(U,W) \cong \int^{(W,x) \in \mathbf{els}(X)} \lambda U.\ \mathbb{C}(U,W)\ , \tag{18}$$

exhibiting $X$ as a colimit of representables. Similarly, by considering (16)

$$X(U) \cong \textstyle\int^W \int^{f \in \mathbb{C}(U,W)} X(W) \cong \int^{(W,f) \in \mathbf{els}(\mathbb{C}(U,-))} X(W).$$

**Exercise 7.6** Show this isomorphism by considering the concrete construction of coends and colimits in **Set**.

# 8 Adjunctions

## 8.1 Definition of Adjunctions

**Definition 8.1 (Adjunction)** Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be functors where $\mathcal{C}$ and $\mathcal{D}$ are categories. An *adjunction* in which $F$ is the *left adjoint* and $G$ is the *right adjoint* consists of an isomorphism

$$\mathcal{D}(F(X), Y) \overset{\theta_{X,Y}}{\cong} \mathcal{C}(X, G(Y))$$

natural in $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. When such a natural isomorphism exists we write $F \dashv G$.

By dual form of parametrised representability (Theorem 3.15), it is sufficient to specify an adjunction via a representation for the functor $\mathcal{C}(X, G(-))$

$$\mathcal{D}(F[X], -) \overset{\theta^X}{\cong} \mathcal{C}(X, G(-))$$

where $F[X] \in \mathcal{D}$ for each $X \in \mathcal{C}$. Therefore, the mapping which sends $X$ to $F[X]$ extends to a functor $F : \mathcal{C} \to \mathcal{D}$ such that

$$\mathcal{D}(F(X), Y) \overset{\theta^X_Y}{\cong} \mathcal{C}(X, G(Y))$$

natural in $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, *i.e.*, $F$ is the left adjoint of $G$.

From the dual form of Theorem 3.12, the representation for $\mathcal{C}(X, G(-))$ defines a universal element $\eta_X = \theta^X_{F[X]}(\mathsf{id}_{F[X]})$, *i.e.*, for any $f \in \mathcal{C}(X, G(Y))$ where $Y \in \mathcal{D}$ there exists a **unique** $\overline{f} \in \mathcal{D}(F[X], Y)$ such that

$$\begin{aligned} f &= \big(\mathcal{C}(X, G(-))(\overline{f})\big)(\eta_X) \\ &= \big(G(\overline{f}) \circ -\big)(\eta_X) \\ &= G(\overline{f}) \circ \eta_X. \end{aligned}$$

Then the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G(F[X]) \\ & \searrow^{f} & \downarrow{\scriptstyle G(\overline{f})} \\ & & G(Y) \end{array} \qquad \begin{array}{c} F[X] \\ \downarrow{\scriptstyle \overline{f}} \\ Y \end{array}$$

commutes. The collection of universal elements $\langle \eta_X \rangle_{X \in \mathcal{C}}$ is called the *unit* of the adjunction. Thus we obtain the *freeness* condition sufficient to establish an adjunction.

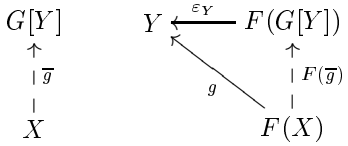
**Definition 8.2 (Freeness Condition)** Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Say a pair $(F[X], \eta_X)$, where $F[X]$ is an object of $\mathcal{D}$ and $\eta_X : X \to G(F[X])$ is a map in $\mathcal{C}$, is *free* over $X$ with respect to $G$ iff for any map $f : X \to G(Y)$ in $\mathcal{C}$ with $Y \in \mathcal{D}$, there is a **unique** map $\overline{f} : F[X] \to Y$ in $\mathcal{D}$ such that $f = G(\overline{f}) \circ \eta_x$.

Suppose $G : \mathcal{D} \to \mathcal{C}$ is a functor, a necessary and sufficient condition for there to be a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F \dashv G$ is that for each object $X \in \mathcal{C}$ there is an object $F[X] \in \mathcal{D}$ and an arrow $\eta_X : X \to G(F[X])$ in $\mathcal{C}$ such that $(F[X], \eta_X)$ is free over $X$. The family $\langle \eta_X \rangle_{X \in \mathcal{C}}$ is called the *unit* of the adjunction.

**Exercise 8.3** Show that the action of the induced functor $F$ over an arrow $f : C \to C'$ in $\mathcal{C}$ is the unique arrow determined by the freeness condition of $\eta_C$ for the arrow $\eta_{C'} \circ f$. Then conclude that the collection $\langle \eta_X \rangle_{X \in \mathcal{C}}$ is a natural transformation from $\mathsf{id}_{\mathcal{C}}$ to $G \circ F$.

Dually, suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor, a necessary and sufficient condition for there to be a functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \dashv G$ is that for each object $Y \in \mathcal{D}$ there is an object $G[Y] \in \mathcal{C}$ and

an arrow $\varepsilon_Y : F(G[Y]) \to Y$ in $\mathcal{D}$ such that $(G[Y], \varepsilon_Y)$ is *co-free* over $Y$ with respect to $F$, i.e. the diagram




commutes. The family of arrows $\langle \varepsilon_Y \rangle_{Y \in \mathcal{D}}$ is called the *co-unit* of the adjunction.

## 8.2 The Naturality Laws for Adjunctions

Given an adjunction $F \dashv G$ for functors $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ we denote the natural isomorphism

$$\mathcal{D}(F(X), Y) \cong \mathcal{C}(X, G(Y))$$

by $\overline{(-)}$ in both directions. Traditionally this correspondence is represented by

$$\frac{F(X) \xrightarrow{g} Y}{X \xrightarrow{\overline{g}} G(Y)} \qquad \text{and} \qquad \frac{X \xrightarrow{f} G(Y)}{F(X) \xrightarrow{\overline{f}} Y} \ .$$

Given that both the isomorphism and its inverse are written as $\overline{(-)}$ we must have $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$. Given an arrow $h$ we called the arrow $\overline{h}$ the *transpose* of $h$. Using this notation, we investigate the consequences of naturality.

**Naturality in $X$:** For $h : X' \to X$ in $\mathcal{C}$ the square

$$\begin{array}{ccc} X & \mathcal{D}(F(X), Y) & \cong \quad \mathcal{C}(X, G(Y)) \\ {\scriptstyle h}\uparrow & \downarrow{\scriptstyle -\circ F\,h} & \qquad \downarrow{\scriptstyle -\circ h} \\ X' & \mathcal{D}(F(X'), Y) & \cong \quad \mathcal{D}(X', G(Y)) \end{array}$$

commutes, *i.e.*, given $f \in \mathcal{D}(F(X), Y)$

$$\frac{F(X') \xrightarrow{F\,h} F(X) \xrightarrow{f} Y}{X' \xrightarrow{h} X \xrightarrow{\overline{f}} G(Y)} \ .$$

Reading from top to bottom ($\downarrow$) $\overline{f \circ F\,h} = \overline{f} \circ h$, and reading from bottom to top ($\uparrow$) $\overline{g \circ h} = \overline{g} \circ F\,h$.

**Naturality in $Y$:** For $k : Y \to Y'$ in $\mathcal{D}$ the square

$$\begin{array}{ccc} Y & \mathcal{D}(F(X), Y) & \cong \quad \mathcal{C}(X, G(Y)) \\ {\scriptstyle k}\downarrow & \downarrow{\scriptstyle k\circ -} & \qquad \downarrow{\scriptstyle G\,k\circ -} \\ Y' & \mathcal{D}(F(X), Y') & \cong \quad \mathcal{D}(X, G(Y')) \end{array}$$

commutes, *i.e.*, given $f \in \mathcal{D}(F(X), Y)$

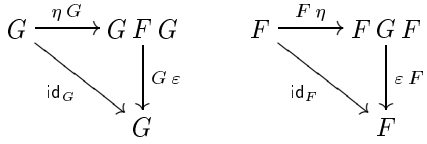
$$\frac{F(X) \xrightarrow{f} Y \xrightarrow{k} Y'}{X \xrightarrow{\overline{f}} G(Y) \xrightarrow{G\,k} G(Y')} \ .$$

Reading from top to bottom ($\downarrow$) $\overline{k \circ f} = G\,k \circ \overline{f}$, and reading from bottom to top ($\uparrow$) $\overline{G\,k \circ g} = k \circ \overline{g}$.

## 8.3 The Triangle Identities

Assume the functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ .

**Proposition 8.4** Given an adjunction $F \dashv G$ with unit $\eta$ and counit $\varepsilon$ the diagrams of functors and natural transformations




commute. They are called the *triangle identities*.

**Proof**: Recall from exercise 8.3 (and its dual) that the unit and counit of an adjunction are natural transformations. We prove only the left hand identity above, the other follows analogously. Given $Y \in \mathcal{D}$, we have

$$\frac{G(Y) \xrightarrow{\eta_{G(Y)}} G(F(G(Y))) \xrightarrow{G(\varepsilon_Y)} G(Y)}{F(G(Y)) \xrightarrow{\overline{\eta_{G(Y)}} = \mathsf{id}_{G(Y)}} F(G(Y)) \xrightarrow{\varepsilon_Y} Y,}$$

and by definition $\overline{\varepsilon_Y} = \mathsf{id}_{G(Y)}$. Then $\overline{\overline{G(\varepsilon_Y) \circ \eta_{G(Y)}}} = \mathsf{id}_{G(Y)}$, *i.e.*, $G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathsf{id}_{G(Y)}$.

□

**Theorem 8.5 (Adjunction)** A pair of natural transformations $(\eta : \mathsf{id}_{\mathcal{C}} \Rightarrow G \circ F, \varepsilon : F \circ G \Rightarrow \mathsf{id}_{\mathcal{D}})$ satisfying the triangular identities define an adjunction as follows:

- Given an arrow $f : X \to G(Y)$ in $\mathcal{C}$ the transpose $\overline{f}$ is $\varepsilon_Y \circ F(f)$, and
- given an arrow $g : F(X) \to Y$ in $\mathcal{D}$ the transpose $\overline{g}$ is $\eta_X \circ G(f)$.

**Proof**: We need to check these functions give a natural bijection. Naturality is inherited from naturality of $\eta$ and $\varepsilon$, *e.g.*, given an arrow $h : X' \to X$ in $\mathcal{C}$ and $g \in \mathcal{D}(F(X), Y)$

$$\begin{aligned}
\overline{g} \circ h &= G(g) \circ \eta_X \circ h && \text{by def. of } (\overline{-}),\\
&= G(g) \circ G(F(h)) \circ \eta_{X'} && \text{by naturality of } \eta,\\
&= G(g \circ F(h)) \circ \eta_{X'}\\
&= \overline{g \circ F(h)}.
\end{aligned}$$

Now, we check the functions are mutually inverse. Given $f : X \to G(Y)$ in $\mathcal{C}$

$$\begin{aligned}
\overline{\overline{f}} &= G(\varepsilon_Y \circ F(f)) \circ \eta_X && \text{by def. of } (\overline{-}),\\
&= G(\varepsilon_Y) \circ G(F(f)) \circ \eta_X\\
&= G(\varepsilon_Y) \circ \eta_{G(Y)} \circ f && \text{by naturality of } \eta,\\
&= f && \text{by triangle identity.}
\end{aligned}$$

Similarly given an arrow $g : F(X) \to Y$ we can prove $\overline{\overline{g}} = g$.

□

**Corollary 8.6** An adjunction $F \dashv G$ is completely determined by its unit and counit.

**Proof**: Since the unit and counit are natural transformations satisfying the triangle identities and the definition of the transpose in the theorem above corresponds to the freeness conditions.

□

## 8.4 Limits and Adjunctions

**Theorem 8.7** Suppose the functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ form an adjunction $F \dashv G$. Then $G$ preserves all limits which exist in $\mathcal{D}$, and $F$ preserves all colimits which exist in $\mathcal{C}$.

**Proof**: Assume $D : \mathbb{I} \to \mathcal{D}$ is a diagram whose limit exists in $\mathcal{D}$. Composing isomorphisms

$$\begin{aligned}
\mathcal{C}\big(U, G(\textstyle\int_I D(I))\big) &\cong \mathcal{D}\big(F(U), \textstyle\int_I D(I)\big) && \text{by the adjunction,}\\
&\cong \textstyle\int_I \mathcal{D}\big(F(U), D(I)\big) && \text{by Proposition 6.8,}\\
&\cong \textstyle\int_I \mathcal{C}\big(U, G(D(I))\big) && \text{by the adjunction,}\\
&\cong \mathcal{C}\big(U, \textstyle\int_I G(D(I))\big) && \text{by representation of limits,}
\end{aligned}$$

by Proposition 3.6 we have

$$G(\textstyle\int_I D(I)) \cong \textstyle\int_I G(D(I))$$

natural in $U \in \mathcal{C}$ and $D \in Diag[\mathbb{I}, \mathcal{D}]$ (see Remark 4.37). It follows that the limit for $G \circ D$ exists for any $D \in Diag[\mathbb{I}, \mathcal{D}]$ and thus from Corollory 6.13 we can conclude that $G$ preserves limits. Dually $F$ preserves colimits.

□

**Exercise 8.8** Complete the proof by showing that left adjoints preserve colimits.

**Exercise 8.9** Show $m : A \to B$ is monic in a category $\mathcal{D}$ iff

$$\begin{array}{ccc} A & \xrightarrow{\mathsf{id}_A} & A \\ {\scriptstyle \mathsf{id}_A}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[m]{} & B \end{array}$$

is a pullback. Deduce right adjoints preserve monics. Why do left adjoints preserve epis?

## 8.5 Representation Functors

Assume a functor $F : \mathbb{C} \to \mathcal{D}$ where $\mathcal{D}$ has all small colimits (i.e. $\mathcal{D}$ is cocomplete). We can obtain an adjunction

$$[\mathbb{C}^{\mathsf{op}}, \mathbf{Set}] \underset{F_!}{\overset{F^*}{\rightleftarrows}} \mathcal{D} \quad (\top)$$

The functor $F_!$ is got by extending $F$ along the Yoneda embedding $\mathcal{Y}$ (an example of a left Kan extension):

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\mathcal{Y}} & [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}] \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle F_!} \\ & & \mathcal{D}. \end{array}$$

Define $F_! = \lambda X.\ \int^C X(C) \cdot F(C)$ and $F^* = \lambda Y.\, \lambda C.\, \mathcal{D}(F(C), Y)$. To show $F_! \dashv F^*$:

$$\begin{aligned}
\mathcal{D}(F_!(X), Y) &= \mathcal{D}\big(\textstyle\int^C X(C) \cdot F(C), Y\big) && \text{by definition,}\\
&= \textstyle\int_C \mathcal{D}(X(C) \cdot F(C), Y) && \text{by representation for coends,}\\
&= \textstyle\int_C \mathcal{D}(\sum_{x \in X(C)} F(C), Y) && \text{by definition of copower,}\\
&\cong \textstyle\int_C \prod_{x \in X(C)} \mathcal{D}(F(C), Y) && (19)\\
&\cong \textstyle\int_C [X(C), \mathcal{D}(F(C), Y)]\\
&\cong \textstyle\int_C [X(C), (F^*(Y))(C)] && \text{by definition of } F^*,\\
&\cong [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}](X, F^*(Y)) && \text{by naturality formula,}
\end{aligned}$$

where all the isomorphisms are natural in $X \in [\mathbb{C}^{\mathsf{op}}, \mathbf{Set}]$ and $Y \in \mathcal{D}$.

Note that treating the set $X(C)$ as a discrete category we have

$$\sum_{x \in X(C)} F(C) = \int^{x \in X(C)} F(C) \quad \text{and} \quad \prod_{x \in X(C)} F(C) = \int_{x \in X(C)} F(C),$$

so step (19) is a special case of the representation for colimits.

In particular, presheaf categories $\widehat{\mathbb{D}} = [\mathbb{D}^{\mathsf{op}}, \mathbf{Set}]$ have all colimits (they are obtained pointwise from colimits in **Set**). So given $F : \mathbb{C} \to \widehat{\mathbb{D}}$ we obtain

$$\widehat{\mathbb{C}} \underset{F_!}{\overset{F^*}{\leftrightarrows}} \widehat{\mathbb{D}}. \quad (\top)$$

Note that $F$ might arise from a functor $F_0 : \mathbb{C} \to \mathbb{D}$ by composition with Yoneda:

$$F : \mathbb{C} \xrightarrow{F_0} \mathbb{D} \xrightarrow{\mathcal{Y}} \widehat{\mathbb{D}}.$$

## 8.6 Adjoints and Initial Objects

## 8.7 The Initial Object Lemma

## 8.8 The General Adjoint Functor Theorem

## 8.9 Well-powered categories

## 8.10 Generators and Cogenerators

## 8.11 The Special Adjoint Functor Theorem

## 8.12 Cartesian-closed Categories

**Definition 8.10 (Cartesian Closed Categories)** A *cartesian-closed category* (ccc) is a category $\mathcal{C}$ with

- all finite products specifically given (so there is a terminal object $\mathbf{1}_{\mathcal{C}}$ and binary products $A \times B$ for all $A, B \in \mathcal{C}$); and
- a representation for the functor $\mathcal{C}(- \times B, C)$:

  $$\mathcal{C}(-, [B, C]) \overset{\theta}{\cong} \mathcal{C}(- \times B, C)$$

  for all $B, C \in \mathcal{C}$.

**Notation 8.11** The object $[B, C] \in \mathcal{C}$ is called the *exponential* of $B$ and $C$ and sometimes is written as $C^B$.

By parameterised representability (Theorem 3.15) for a ccc $\mathcal{C}$ there is a unique way to extend the mapping $(B, C) \mapsto [B, C]$ to an *exponentiation* functor $[-, +]$ from $\mathcal{C}^{\mathsf{op}} \times \mathcal{C}$ to $\mathcal{C}$ such that

$$\mathcal{C}(A, [B, C]) \cong \mathcal{C}(A \times B, C)$$

natural in $A, B, C$. Notice that from this isomorphism we have $- \times B \dashv [B, -]$. In fact, provided $\mathcal{C}$ has specified finite products an adjunction with $- \times B \dashv [B, -]$ for all $B \in \mathcal{C}$, conversely determines a cartesian closed structure on $\mathcal{C}$.

**Exercise 8.12** What are the unit and counit of $- \times B \dashv [B, -]$?

**Proposition 8.13** Presheaf categories are cartesian-closed.

**Proof**: The category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ has products, computed pointwise, like all limits:

$$(X \times Y)(U) \cong X(U) \times Y(U)$$

natural in $U \in \mathcal{C}$. We can make an educated guess at the form of the exponentiation $[Y, Z]$, for $Y, Z \in \widehat{\mathbb{C}}$. It should satisfy:

$$\begin{aligned} [Y,Z](U) &\cong \widehat{\mathbb{C}}(\mathcal{Y}\,U, [Y,Z]) && \text{by the Yoneda Lemma (Theorem 3.3)} \\ &\cong \widehat{\mathbb{C}}(\mathcal{Y}\,U \times Y, Z) && \text{by the adjunction property.} \end{aligned}$$

So we define $[Y, Z] = \lambda U.\, \widehat{\mathbb{C}}(\mathcal{Y}\,U \times Y, Z)$ and check that $\widehat{\mathbb{C}}(X, [Y, Z]) \cong \widehat{\mathbb{C}}(X \times Y, Z)$ natural in $X$:

$$\begin{aligned} \widehat{\mathbb{C}}(X,[Y,Z]) &\cong \textstyle\int_U [X(U), [Y,Z](U)] && \text{by naturality formula,} \\ &\cong \textstyle\int_U [X(U), \widehat{\mathbb{C}}(\mathcal{Y}\,U \times Y, Z)] && \text{by definition of } [Y,Z], \\ &\cong \textstyle\int_U \left[X(U), \int_V [\mathbb{C}(V,U) \times Y(V), Z(V)]\right] && \text{by naturality formula,} \\ &\cong \textstyle\int_U \int_V \left[X(U), [\mathbb{C}(V,U) \times Y(V), Z(V)]\right] && \text{by representation for ends,} \\ &\cong \textstyle\int_U \int_V \left[X(U) \times \mathbb{C}(V,U), [Y(V), Z(V)]\right] && \text{by currying and uncurrying,} \\ &\cong \textstyle\int_V \int_U \left[X(U) \times \mathbb{C}(V,U), [Y(V), Z(V)]\right] && \text{by Fubini (Corollary 6.6),} \\ &\cong \textstyle\int_V \left[\int^U X(U) \times \mathbb{C}(V,U), [Y(V), Z(V)]\right] && \text{by representation of coends,} \\ &\cong \textstyle\int_V \left[X(V), [Y(V), Z(V)]\right] && \text{by density formula,} \\ &\cong \textstyle\int_V [X(V) \times Y(V), Z(V)] && \text{by uncurring,} \\ &\cong \textstyle\int_V [(X \times Y)(V), Z(V)] && \text{product is got pointwise (Proposition 6.3,} \\ &\cong \widehat{\mathbb{C}}(X \times Y, Z) && \text{by naturality formula,} \end{aligned}$$

all natural in $X \in \widehat{\mathcal{C}}$.

□

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