HIGHER DIMENSIONAL WORD PROBLEM

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The word problem on a monoid admits two natural generalizations :
- the first one is the extension from monoids to categories. In this case the words become the "paths" and the equality problems take the form of diagram commutation problems.
- the second is the extension from monoids to universal algebras. In this case, the words become the "terms", and the rewriting systems set the rules for their equality.

Is it possible to unify these two extensions ? In this paper we answer as follows : the rewriting problem for terms is nothing but a 2-dimensional path problem in a 2-category. This observation leads to the general problem for n-paths in an n-category, or even in an ω-category. A lot of computations made by categoricians are 1-, 2- or 3-dimensional computations, and in fact n-dimensional computations take place in an (n+1)-category (see §1.2). Furthermore, beyond the unity so given to various word problems, the link with combinatorial topology appears, word problems being in this setting refinement of homotopy theory (see §1.3).

1. General setting

We are going to define the n-dimensional word problem ($n\in\mathbb{N}$), and more generally, a word problem of variable dimension which means computing in an ω-category (such a computation being often a reduction to a canonical form, but more generally being the construction of a "homotopy" between two expressions).

1.1. ω-Categories.

One will find in [Bu2] more synthetical definitions of ω-categories, but the definition presented below fits pretty well our purpose. Finally the whole construction amounts to the juxtaposition of an infinity of 2-categories.

An *ω-graph* **G** is the datum of a diagram of sets

$$(*)\qquad G_0 \underset{b_0}{\overset{a_0}{\leftarrow\!\!\leftarrow}} G_1 \underset{b_1}{\overset{a_1}{\leftarrow\!\!\leftarrow}} G_2 \underset{b_2}{\overset{a_2}{\leftarrow\!\!\leftarrow}} G_3 \ \dots\ G_n \underset{b_n}{\overset{a_n}{\leftarrow\!\!\leftarrow}} G_{n+1} \underset{b_{n+1}}{\overset{a_{n+1}}{\leftarrow\!\!\leftarrow}} G_{n+2} \dots$$

such that, for every $n\in\mathbb{N}$, the following equations hold :

$$a_n a_{n+1} = a_n b_{n+1},\quad b_n a_{n+1} = b_n b_{n+1}.$$

The elements of G_n are named n-*cells*, and the following representations of 0-, 1- and 2-cells are well-known :

$$X \qquad X \xrightarrow{f} Y \qquad X \underset{g}{\overset{f}{\Downarrow \lambda}} Y$$

(n=0) (n=1) (n=2)

where $X,Y \in G_0$, $f,g \in G_1$, $\lambda \in G_2$.

Symbols are sometimes omitted, as for instance here : . $\overset{\longrightarrow}{\Downarrow \lambda}$.

We also need higher dimensional cells, and representing them is possible (although difficult), as for instance, for a 3-cell : . $\overset{\Downarrow \to \Downarrow}{\longrightarrow}$.

In fact, as we shall see, , it seems that the main part of the calculus always works in 2-graphs.

A n-*graph* is an ∞-graph G such that $G_p = \varnothing$ for every p>n, and such an n-graph will be identified with the diagram (*) truncated at the level n. In particular a 0-graph is just a set, and a 1-graph is an (oriented) graph in the usual sense.

Starting from an ∞-graph G, we define, for every $0 \le i < j$, a graph G_{ij} :

$$G_i \underset{b_{ij}}{\overset{a_{ij}}{\leftleftarrows}} G_j$$

with : $a_{ij} = a_i\, a_{i+1} \dots a_j$, $b_{ij} = b_i\, b_{i+1} \dots b_j$,

and we define, for every $0 \le i < j < k$, a 2-graph G_{ijk} :

$$G_i \underset{b_{ij}}{\overset{a_{ij}}{\leftleftarrows}} G_j \underset{b_{jk}}{\overset{a_{jk}}{\leftleftarrows}} G_k .$$

In order to obtain an ∞-category on G, we need only a category structure on each G_{ij} in such a way that the 2-graphs G_{ijk} become 2-categories.

It remains to define a 2-category on a 2-graph $G_0 \underset{b_0}{\overset{a_0}{\leftleftarrows}} G_1 \underset{b_1}{\overset{a_1}{\leftleftarrows}} G_2$;

essentially such a category needs the additional data of two "unity" maps :

$$X \longmapsto X \xrightarrow{id(X)} X, \text{ and } X \xrightarrow{f} Y \longmapsto X \underset{f}{\overset{f}{\Downarrow id(f)}} Y,$$

and of three "composition laws" :

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longmapsto X \xrightarrow{g \circ f} Z,$$

$$X \underset{g}{\overset{f}{\Downarrow \lambda}} Y \underset{g'}{\overset{f'}{\Downarrow \lambda'}} Y \longmapsto X \underset{g' \circ g}{\overset{f' \circ f}{\Downarrow \lambda \circ \lambda'}} Z,$$

$$X \underset{\Downarrow \mu}{\overset{\Downarrow \lambda}{\rightrightarrows}} Y \longmapsto X \underset{}{\overset{\Downarrow \mu \dagger \lambda}{\longrightarrow}} Y.$$

These data must satisfy the neutrality and associativity axioms, and the "Godement rule".

We have the usual notions of homomorphisms between ω-graphs or ω-categories which we define as natural transformations through diagrams and projective limits. So we get two categories, namely ω-**Cat** and ω-**graph**, and a forgetfull functor, U_ω : ω-**Cat** ⟶ ω-**Graph**, and from general category theory, this functor U_ω admits a left adjoint [ ]. The ω-category [**G**] is named the free ω-category generated by the ω-graph **G**.

We know also through general arguments that ω-**Graph** and ω-**Cat** are complete and co-complete.

In the same way we get a functor U_n : n-**Cat** ⟶ n-**Graph**, with a left adjoint [ ]. Let I_n : n-**Cat** ⟶ ω-**Cat** be the functor which just adds units in higher dimensions. Then we have a double commutative square

$$\begin{array}{ccc} \text{n-Cat} & \xrightarrow{I_n} & \omega\text{-Cat} \\ {\scriptstyle U_n}\downarrow\uparrow{\scriptstyle [\,]} & & {\scriptstyle U_\omega}\downarrow\uparrow{\scriptstyle [\,]} \\ \text{n-Graph} & \xrightarrow[\subseteq]{} & \omega\text{-Graph} \end{array}$$

1.2. CW-presentations.

The ω-categories can be constructed by adjoining successive cells as for the CW-complexes, these adjonctions being completed by symetric constructions of collapses. This pair adjonction/collapse is the analogous of the pair generator/relation in the description of algebraic structures.

The *formal* n-*cell*, for every n∈ℕ, is the free ω-category $[e_n]$ generated by the ω-graph e_n and determined through the conditions

$$(e_n)_p = \begin{cases} 2 & \text{if } 0\le p\le n-1 \\ 1 & \text{if } p=n \\ 0 & \text{if } n+1\le p \end{cases},$$

with a_p, b_p : 2⟶2 for 0≤p≤n-1, and a_n, b_n : 1⟶2, given as the constant map on 0 and 1 respectively (we adopt the Von Neuman convention, namely that k={0,1,...,k-1} for every k∈ℕ).

We define the ω-graph ∂e_n, edge of e_n, as being e_n truncated at dimension n-1, and two evident ω-functors

$$[e_{n-1}] \xleftarrow{col_n} [\partial e_n] \xrightarrow{adj_n} [e_n].$$

The ω-functor col_n is the "elementary collapse", and the ω-functor adj_n is the "elementary adjonction".

An (n-1)-dimensional *attaching map* towards an ω-category **C** is an ω-functor of the form φ : $[\partial e_n]$⟶**C**. To such a data we associate two ω-categories **C**/φ and **C**[φ], through the two following pushouts :

$$\begin{array}{ccccc} [e_{n-1}] & \xleftarrow{col_n} & [\partial e_n] & \xrightarrow{adj_n} & [e_n] \\ \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow \\ \mathbf{C}/\varphi & \longleftarrow & \mathbf{C} & \longrightarrow & \mathbf{C}[\varphi] \end{array}$$

$\mathbf{C}/\varphi$ is named the *collapse* (of dimension n-1) of $\mathbf{C}$ by φ, and $\mathbf{C}[\varphi]$ is named the *adjonction* (of dimension n) of φ to $\mathbf{C}$.

Every "finitely presented"∞-category $\mathbf{D}$ can be described starting from $\mathbf{C} = \varnothing$ through a finite sequence of operations of these types, namely a sequenceof p adjonctions of $\varphi_1,\ldots,\varphi_p$, followed by q collapses by $\psi_1,\ldots,\psi_q$. Such an analysis of $\mathbf{D}$ will be denoted through :

$$\mathbf{D} = \varnothing[\varphi_1,\ldots,\varphi_p]/(\psi_1,\ldots,\psi_q),$$

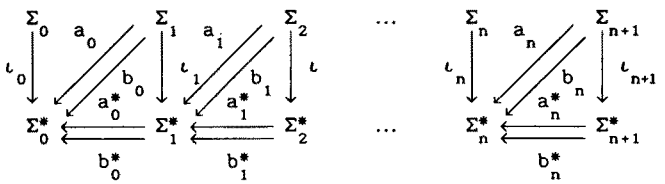
and we name it a CW-*presentation* of $\mathbf{D}$.

In fact, technically, the collapse and the adjonction are essentially of the same nature (think of these operations in the case of CW-complexes, where they yield to the same homotopy type), and we have a canonical epimorphism

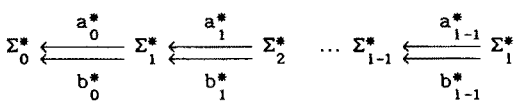
$$\varnothing[\varphi_1,\ldots,\varphi_p,\ \psi_1,\ldots,\psi_q] \longrightarrow\!\!\!\!\!\rightarrow \varnothing[\varphi_1,\ldots,\varphi_p]/(\psi_1,\ldots,\psi_q).$$

But, *in practise*, we proceed in a slightly different manner, adjoining by "blocks", "dimensionwise", and collapsing at the end, all the collapses being of dimension n+1, with $n = \dim(\mathbf{D}) = \sup_{1\le i\le p}\dim(\varphi_i)$. In this way, $\mathbf{D}$ appears directly as an n-category. Our process is summarized as follows :

(**)



where, for each $0\le i\le n$, the partial diagram



represents the underlying i-graph Σ^*_i to the i-category previously obtained through the introduction of j-cells for $0\le j\le i$ (the recurrence begins with $\Sigma_0 = \Sigma^*_0$ and $\iota_0 = \mathrm{id}(\Sigma_0)$) via the graphs $\Sigma^*_j \underset{b^*_j}{\overset{a^*_j}{\leftleftarrows}} \Sigma^*_{j+1}$ which represent the "blocks" made by the union of the j-dimensional attaching maps.

The examples given in 1.3 shall illustrate this construction.

Before concluding, let us informally explain the reasons why the collapses are n-dimensional. First, it is clear that they cannot be of higher dimension ; next, in the case n = 1 so familiar to categoricians, adding invertible morphisms in commutative diagrams is prefered to equalizing objects in a category. Of course, this is not indispensable.

1.3. Examples.

The complexity of the last construction increases with the dimension n.

Case n = 0.

The 0-dimensional word problem can be identified with the so called "maze problem". Let $\Sigma_0 \underset{b_0}{\overset{a_0}{\leftleftarrows}} \Sigma_1$ be a graph. Does it exist a path from X to Y for $X,Y \in \Sigma_0$? To this "semi-Thue"-problem is associated a Thue-problem (which is the actual maze problem) : $\bar{X} = \bar{Y}$? (with $X \longmapsto \bar{X}$ the canonical surjection $\Sigma_0 \longrightarrow \Sigma_0/(\Sigma_1)$ from Σ_0 to the quotient of Σ_0 by the equivalence relation generated by the relation defining the semi-Thue problem).

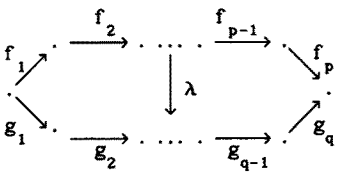
Then, in the diagram (**) for n = 0, Σ_0 represents a 0-graph and Σ_0^* the 0-category generated by Σ_0. Of course, as there are zero operations (in an n-category there are n operations on the n-cells), these two structures can be identified to the sets : $\Sigma_0 = \Sigma_0^*$ and $\iota_0 = \mathrm{id}(\Sigma_0)$. One can also consider Σ_0 as a set of adjonctions of 0-cells to ø. The graph $\Sigma_0 = \Sigma_0^* \underset{b_0}{\overset{a_0}{\leftleftarrows}} \Sigma_1$ represents the adjonction of 1-cells which are used to describe the collapse $\Sigma_0/(\Sigma_1)$.

Case n = 1.

We obtain the classical word problem in monoids (if $\Sigma_0 = 1$) and more generally in the categories.

Σ_1 is what is usually named the alphabet, and $a_0, b_0 : \Sigma_1 \rightrightarrows 1$ are too trivial to appear in the case of a monoid. The datum $\Sigma_0 = \Sigma_0^* \underset{b_0}{\overset{a_0}{\leftleftarrows}} \Sigma_1^*$ represents the underlying graph to the free category generated by the graph $\Sigma_0^* \underset{b_0}{\overset{a_0}{\leftleftarrows}} \Sigma_1$, and $\iota_1 : \Sigma_1 \longrightarrow \Sigma_1^*$ isthe level-1 canonical embedding of a graph Σ into the graph which is underlying the free category Σ^* that it generates.

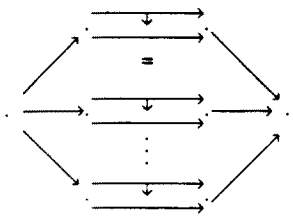
$\Sigma_1^* \underset{b_1}{\overset{a_1}{\leftleftarrows}} \Sigma_2$ represents a set of diagrams of the form



i.e. the adjonction of a set of 2-cells $\lambda \in \Sigma_2$.

As before, this problem entails a semi-Thue problem, and the associated Thue problem is the classical word problem.

The elements of Σ_2^* represent 2-cells admitting a unique decomposition, which is schematically given through



In this figure, every 1-cell represents in fact a path (i.e. an element of Σ_1^*), every 2-cell an element of Σ_2, and the equal sign ("=") within empty spaces means commutativity in the free category generated by the graph $\Sigma_0^* \longleftarrow \Sigma_1$.

Case n = 2.

The construction goes on with the adjonction of 3-cells $\Sigma_2^* \longleftarrow \Sigma_3$. It's easy to imagine, but tedious to describe, how the elements of Σ_3^* are built. Let us merely say that the corresponding Thue problem is nothing but a computation in 2-categories, or the 2-calculus. It has been recently an object of research, especially by P.L. Curien [Cur] and Y. Lafont [Laf], after the works of E.G. Rodeja and his school in Santiago de Compostella. Finally it's also the rewriting problem for terms as we shall show in part 2 below.

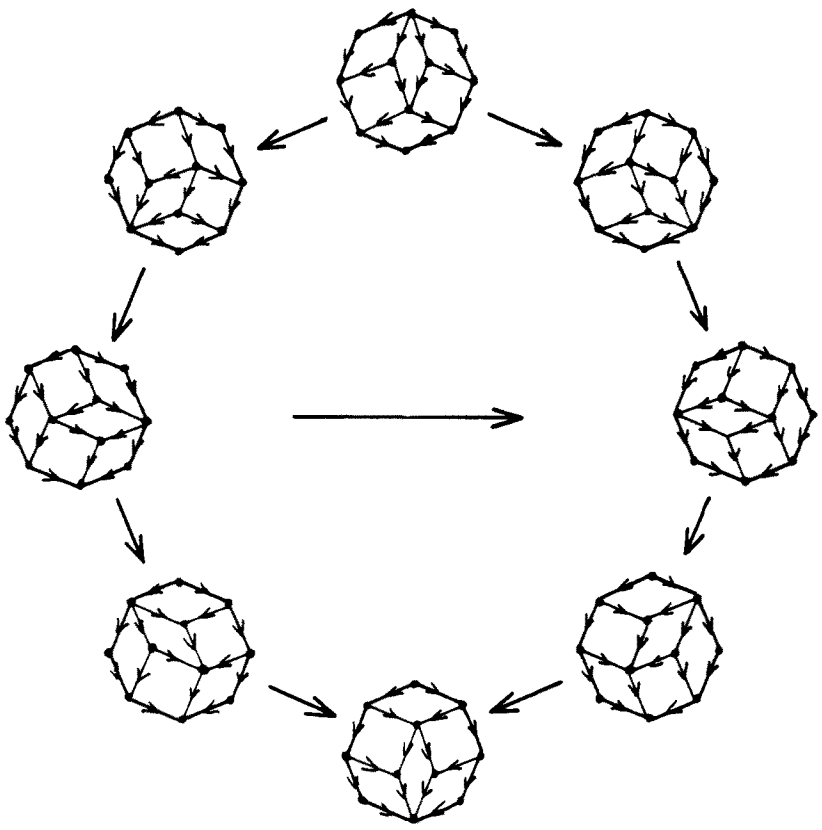
1.4. Relationship with combinatorial topology.

The ∞-graphs - as geometrical objects of combinatorial topology - are too poor to give interesting models. On the other hand the simplicial sets, the cubic sets, and eventually other objects of this nature provide the classical space models. But all (∞-graphs, simplicial sets, etc) can be interpreted as CW-presentations of ∞-categories. For the ∞-graphs it's evident, but, even if the intuition broadly suggests that this is true for the simplicial and cubic sets, the concrete realization of this fact seems to be a very complex problem. This is achieved for simplicial sets in the work of Street [Str].

To illustrate this fact, we shall restrain ourselves to extracting the hereafter picture from [Bu2] ; this picture shows how "reading" a 4-dimensional cubic cell may be interpreted in a 4-category. For the sake of simplicity we omit the 2-cells. In this figure they are all from left to right as shown hereafter and all fill the hexagons.



But there is an essential difference between the word problems and the combinatorial topological problems : here all the cells are oriented and these orientations are essential, except in the collapses where they have a purely technical function (Church-Rosser property).



2. Rewriting systems.

The notion of algebraic theory was defined by Lawvere [Law] as equivalent to the datum of a (strict) cartesian category **T** in which every object is a cartesian power $S^n = S \times S \times \ldots \times S$ ($n \in \mathbb{N}$) of a unique object S of **T**. In this second part we show how such a structure is equivalent to the datum of a special 2-category (more precisely a special 2-monoid) and how the rewriting systems of terms in a universal algebra can be interpreted as CW-presentations of a 2-category. So we are in the context of 2-dimensional word problems.

2.1. 2-monoids.

Let n be an element of $\mathbb{N}$. Let us call n-*monoid* an n-category **C** such that $\mathbf{C}_0 = \langle * \rangle$ (i.e. a single 0-cell). A 1-monoid is of course a monoid.

A 2-monoid can also be interpreted as a *strict monoidal category*, that is a 3-uple $(\mathbf{C}, I, \otimes)$, where **C** is a category, I an object of **C**, and $\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$

a functor verifying the neutrality and associativity relations :

$$id(I)\otimes f = f = f\otimes id(I),\ (f\otimes g)\otimes h = f\otimes(g\otimes h)$$

for every morphisms f,g,h of **C** (the non necessarily strict monoidal categories are those where equalities are replaced with coherent natural equivalences).
The transformation which exhibits $(\mathbf{C},I,\otimes)$ as a 2-monoid is similar to a "suspension" in topology : one "augments" the dimension of the elements of **C**. First one adds a 0-cell *, then each object X of **C** becomes a 1-cell $X : * \longrightarrow *$ and, finely, every morphism $f : X \longrightarrow Y$ of **C** becomes a 2-cell of the 2-monoid

$$\begin{array}{ccc} & \xrightarrow{\quad X \quad} & \\ * & \downarrow f & * \\ & \xrightarrow[\quad Y \quad]{} & \end{array}$$

An homomorphism $F : (\mathbf{C},I,\otimes)\longrightarrow(\mathbf{C}',I',\otimes')$ between two strict monoidal categories (identified to 2-categories) is a functor $F : \mathbf{C}\longrightarrow\mathbf{C}'$ which commutes with the operations : $F(I) = I'$ and $F(f\otimes g) = F(f)\otimes'F(g)$ (so it is a strict monoidal functor). We write **2-Mon** the category of 2-monoids and homomorphisms between them. It is of course a full sub-category of **2-Cat**.

An example of monoidal category is given with the cartesian categories $(\mathbf{C},1,\times)$, where 1 is a final object of **C** and $\times : \mathbf{C}\times\mathbf{C}\longrightarrow\mathbf{C}$ a cartesian product functor. Here we shall behave with these categories as if they were strict : in fact, we do nothing but follow the customs of algebrists and logicians (to make no difference between $S\times(S\times S)$, $(S\times S)\times S$ and $S\times S\times S$). The only cartesian category which will really interest us (see $\mathbb{F}^{op}$ in 2.2) is cartesian strict.

A first step towards the "elimination" of the universal property of the catesian product - for the benefit of an equational system - is given with the following result :

Proposition : *The strict monoidal category* $(\mathbf{C},1,\times)$ *is cartesian if and only if there exists two natural transformations* ε, δ, *which, for every* $X\in\mathbf{C}_0$, *define two morphisms*

$$1\xleftarrow{\varepsilon(X)}X\xrightarrow{\delta(X)}X\times X$$

satisfying the relations :

$$(id(X)\times\varepsilon(X))\circ\delta(X) = id(X) = (\varepsilon(X)\times id(X))\circ\delta(X)$$

$$(id(X)\times\varepsilon(Y))\times(\varepsilon(X)\times id(Y))\circ\delta(X\times Y) = id(X\times Y)$$

$$\varepsilon(1) = id(1)$$

where X,Y are arbitrary objects in **C**.
When they exist, these data are unique up to isomorphism.

2.2. A CW-presentation of $\mathbb{F}$.

Let **Ens** be the category of sets and $\mathbb{F}$ the full subcategory of **Ens** whose objects are the natural integers ($\mathbb{F}_0 = \mathbb{N}$), where each integer n is the set

$\{0,1,\ldots,n-1\}$.

$\mathbb{F}$ is a cartesian category, but it's also a cocartesian one, so $\mathbb{F}^{op}$ is cartesian and it is this last structure which interests us : the products in $\mathbb{F}^{op}$ are given by the coproducts in $\mathbb{F}$, and coincides with the usual addition in $\mathbb{N}$. For $p,p',q,q' \in \mathbb{N}$, the canonical injections of the coproduct $p \xleftarrow{\lambda} p+p' \xleftarrow{\lambda'} p'$ are defined through $\lambda(i) = i$ for $0 \leq i \leq p-1$, and through $\lambda'(i) = p+1$ for $0 \leq i \leq p'-1$. For $u : p \longrightarrow q$ and $u' : p' \longrightarrow q'$ in $\mathbb{F}$, the coproduct $u+u'$ is given with :

$$(u+u')(j) = \begin{cases} u(j) & \text{if } 0 \leq j \leq p-1 \\ u'(j-1) & \text{if } p \leq j \leq p+p'-1 \end{cases}$$

Dualizing, this structure is then transformed in a cartesian category $(\mathbb{F}^{op},0,+)$ and one easily proves that it is strict. So we get an example of a 2-monoid which will play an essential part in 2.3.

Theorem : *The 2-monoid* $(\mathbb{F}^{op},0,+)$ *is finitely presented (in the sense of CW-presentations).*

A CW-presentation of this 2-monoid can be explicited. But, as $\mathbb{F}$ is more familiar than $\mathbb{F}^{op}$, we will give a presentation for the "dual" 2-monoid $(\mathbb{F},0,+)$, where only composition ("$\circ$") is dualized.

- adjonctions :

-one 0-cell written : $*$

-one 1-cell written : $1 : * \longrightarrow *$

-three 2-cells written : $* \underset{1}{\overset{0}{\Downarrow \eta}} *$, $* \underset{1}{\overset{2}{\Downarrow \mu}} *$, $* \underset{2}{\overset{2}{\Downarrow \tau}} *$

where the integer n represents the "path" :1+1+...+1 (n times).

- collapses :

these ones are written hereafter by means of equations, instead of arrows, which lets the choice for their orientations (for the possibility of a choice giving a canonical system, see [La]). We did not avoid redundance (and we wrote n instead of id(n) for $n \in \mathbb{N}$) :

(1) $\mu \circ (\eta+1) = 1$

(2) $\mu \circ (1+\eta) = 1$

(3) $\mu \circ (\mu+1) = \mu \circ (1+\mu)$

(4) $\tau \circ (1+\eta) = \eta+1$

(5) $\tau \circ (\eta+1) = 1+\eta$

(6) $\tau \circ (\mu+1) = (1+\mu) \circ (\tau+1) \circ (1+\tau)$

(7) $\tau \circ (1+\mu) = (\mu+1) \circ (1+\tau) \circ (\tau+1)$

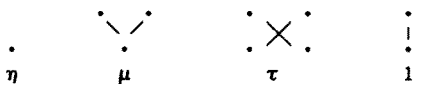
(8) $\tau \circ \tau = 2$

(9) $(1+\tau) \circ (\tau+1) \circ (1+\tau) = (\tau+1) \circ (1+\tau) \circ (\tau+1)$

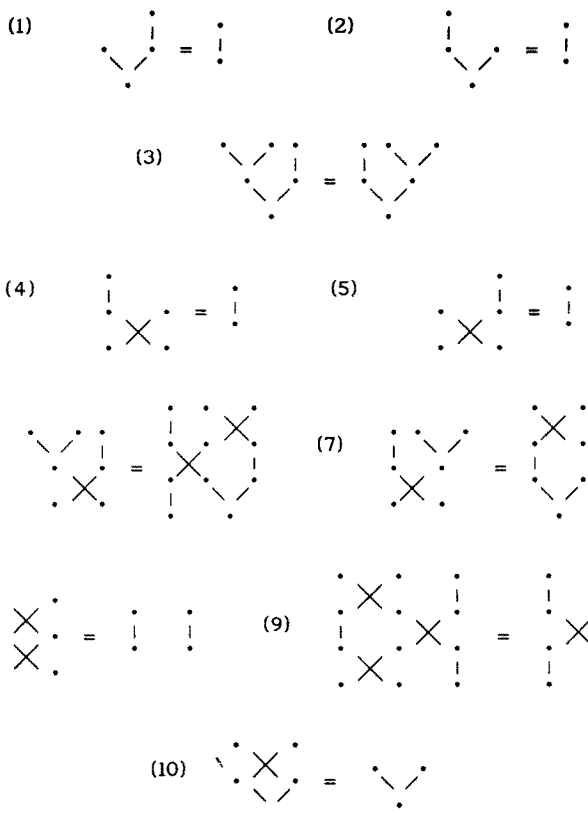
(10) $\mu \circ \tau = \mu$

The formulation given in [Bu 1] was incomplete : the relation (10), which plays a subtile role in this system, was missing. We are thankful to Y. Laffont for pointing this out to us.

A graphic schematization of these equalities with the alphabet



is given through



2.3. The main theorem.

An algebraic theory (with one sort) is presented by a system (Ω,E) of operations and equations (or rewriting system). This generates a strict cartesian category $\mathbf{T} = \mathbf{T}(\Omega,E)$ which has the following property : the datum of an (Ω,E)-algebra amounts to the datum of a strict cartesian functor $F : \mathbf{T} \longrightarrow \mathbf{Ens}$. In fact, if we take as canonical projections the products

$$X \xleftarrow{id(X)\times\varepsilon(Y)} X\times Y \xrightarrow{\varepsilon(X)\times id(Y)} Y ,$$

it suffices to consider F as a strict monoidal functor, because of the relations $F(1) = 1$, $F(f\times g) = F(f)\times F(g)$, and because of the unicity of $F(\varepsilon(X)) = \varepsilon(F(X))$.

Theorem : *The 2-monoid* $T = T(\Omega,E)$ *is finitely presented, in the sense* 1.2. *of* CW-*presentations, if and only if the algebraic theory* T *in the sense of Lawvere is finitely presented* (Ω *and* E *finite).*

In order to prove it, one first constructs, with the help of the ε,δ,τ as defined in 2.2, and for every $n\in\mathbb{N}$, two applications in $\mathbb{F}$:

$$0 \xrightarrow{\varepsilon_n} n \xleftarrow{\delta_n} n+n$$

Besides, the universal property of F determines a strict cartesian functor $i_T : \mathbb{F}^{op}\longrightarrow T$ providing the images in T of the former applications :

$$1 \xleftarrow{\varepsilon_n} S^n \xrightarrow{\delta_n} S^{2n}.$$

(In order to simplify, we have written $\varepsilon_n = i_T(\varepsilon_n)$, $\delta_n = i_T(\delta_n)$). These morphisms provide candidates for two natural transformations ε,δ, through setting $\varepsilon(n) = \varepsilon_n$, $\delta(n) = \delta_n$. A priori (coming from $\mathbb{F}^{op}$) the relations in proposition 1.2 are satisfied, so we need only verify that they are actually transformations natural in T (and not only in $\mathbb{F}^{op}$). Precisely, for every $\alpha : S^p\longrightarrow S^q$ of T, the following diagrams must commute :

$$\begin{array}{ccc} & & S^p & \xrightarrow{\delta_p} & S^{2p} \\ & \overset{\varepsilon_p}{\swarrow} & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha\times\alpha} \\ 1=S^0 & & & & \\ & \underset{\varepsilon_q}{\nwarrow} & S^q & \xrightarrow[\delta_q]{} & S^{2q} \end{array}$$

We show it is sufficient that this diagram commutes for the α's in Ω. Thus, if Ω is finite, we then have a finite number of equations to add to E.

Remark 1. In order to do computations in a T-algebra $F : T^{op}\longrightarrow$ **Ens** (and not only in the theory T) it suffices to make computations in the theory T_F which is obtained through the adjonction of the elements of F (i.e. of F(S)) as constants, and throughthe relations they satisfy. At last, if F is finitely presented, it is sufficient to add the finite data of this presentation : T_F will still be finitely presented (whenever T is).

Remark 2. This result extends to the many- sorted algebraic theories. The part played by $\mathbb{F}$ will be played by $\mathbb{F}^I$, for I the set of sorts.

This second part has been exposed in [Bu 1] and the proofs are in [Bu 2].

References.

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