Toposes in Logic and Logic in Toposes

Marta Bunge

ABSTRACT. The purpose of this paper is to justify the claim that Topos theory and Logic (the latter interpreted in a wide enough sense to include Model theory and Set theory) may interact to the advantage of both fields. Once the necessity of utilizing toposes (other than the topos of Sets) becomes apparent, workers in Topos theory try to make this task as easy as possible by employing a variety of methods which, in the last instance, find their justification in metatheorems from Logic. Some concrete instances of this assertion will be given in the form of simple proofs that certain theorems of Algebra hold in any (Grothendieck) topos, in order to illustrate the various techniques that are used. In the other direction, Topos theory can also be a useful tool in Logic. Examples of this are independence proofs in (classical as well as intuitionistic) Set theory, as well as transfer methods in the presence of a sheaf representation theorem, the latter applied, in particular, to model theoretic properties of certain theories.

0. Introduction

A topos may be viewed as a universe of variable sets, as a generalized topological space, and as a semantical universe for higher order intuitionistic logic. It is, in fact, this many-sided nature of a topos in which its strength lies.

For example, even when using topos methods in Logic (including Set theory and Model theory as well as Proof theory), the geometric aspects can play a key rôle ('Toposes in Logic'). Dually, using Topos theory in branches of Mathematics, which, likw Classical Analysis, have traditionally been impermeable to Logic, it is often the internal logic of a topos that provides far reaching insights ('Logic in Toposes').

To present this double bill I shall discuss, guided by personal preferences, a variety of applications of toposes which may serve to illustrate the two aspects mentioned in the title.

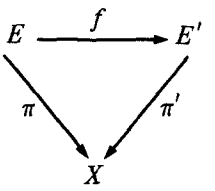
In *Section 2*, the emphasis is on justifying the very use of toposes, while *Section 3* concentrates on various methods employed. *Section 1* introduces the notion of a topos to a reader already familiar with the basic ideas of Category theory. As there are excellent exposition of the subject (c.f. [26]), it seemed preferable to try to gain, instead, quick access to some non-trivial applications.

It is to be hoped, however, that anybody whose curiosity is aroused by reading these pages, will make the effort of studying this beautiful, yet complex, subject, from the very beginning. The bibliography has been compiled, not just in order to be able to refer to the sources when necessary, but also to guide the reader in any further pursuit of the subject. Nevertheless, the already too long list of references is far from complete – a price one has to pay for not wishing to sound too overwhelming.

1. What are toposes?

Before embarking on a precise definition, let us consider a specific, yet illuminating, example of a topos. For X a topological space, a *sheaf* (or *étalé space*) on X is a pair (E, π) where E is a topological space, and $\pi: E \to X$ a *local homeomorphism* (this means that for each $x \in X$, every point in the fiber $\pi^{-1}\{x\}$ possesses an open neighborhood which is mapped, by π, onto an open neighborhood of x).

A *sheaf morphism* from $E \xrightarrow{\pi} X$ to $E' \xrightarrow{\pi'} X$ is given by a continuous map $E \xrightarrow{f} E'$ which preserves the fibers, i.e. such that the diagram



commutes. Denote by $\mathrm{Sh}(X)$ the resulting *category*.

A *section* of $E \xrightarrow{\pi} X$ above an open $U \subseteq X$ is a continuous map $s: U \to E$ such that $\pi.s = \mathrm{id}_U$. Taking sections defines a functor F on the opposite of the category $O(X)$ of open subsets and inclusions, into the category Sets, of sets and functions. The rule which, to an open inclusion $V \subseteq U$ assigns a function $F(U) \to F(V)$ is restriction of a section on U to V; we use the notation $s \longmapsto s|_V$ for this. Let $\{U_\alpha\}$ be any open covering of U, and let $\{s_\alpha\}$be a family of sec-

Topoi **3** (1984), 13–22. 0167-7411/84/0031-0013$01.50.
© 1984 *by D. Reidel Publishing Company.*

tions compatible in the intersections $U_\alpha \cap U_\beta$; there exists a unique section $s \in F(U)$ such that $s|_{U_\alpha} = s_\alpha$ for each α. Abstracting this property gives an alternative definition; any such functor F: $O(X)^{\mathrm{op}} \to$ Sets, satisfying a coherence property given any open covering, is said to be a sheaf on X.

Let F be a sheaf on X, as above. It may be viewed as a "variable set" (c.f. [24]) in the following sense: interpret the elements of $F(U)$, for $U \in O(X)$, as "elements of F defined at stage U". Thus, $\mathrm{Sh}(X)$, in an obvious sense a *generalized topological space*, is a (potential) *universe of variable sets*. What makes the latter into more than just a fancy expression, is the amazing discovery of Lawvere and Tierney (c.f. [13], [31]) that all of higher order logic is interpretable in categories such as $\mathrm{Sh}(X)$, or in their generalizations (Grothendieck toposes). The abstraction of those basic features of such categories arising from Algebraic Geometry and Algebraic Topology, which allow us to view them as surrogates, albeit non-classical, and in a weaker sense, of Set theory, is what led to the notion of a *topos*.

As a generalized topoogical space, the category of *Sets* is trivial: a space with only one point. As a universe of variable sets, the objects in *Sets*, admit no genuine variation, i.e., are constant, or fixed. Yet, it is by lifting set-theoretical constructions to an arbitrary topos that one works – in a first instance, in Topos theory. In so doing, one finds that for the most part, the non-classical aspects of a topos (its logic is intuitionistic), or the weaker Set theory that it embodies (no axiom of replacement, except in the form of bounded comprehension, and no axiom of Choice, in general), are no deterrent for the purposes toposes were conceived of in the first place: to deal more effectively with questions of a *local* nature. From this point of view, the above "defects" are really virtues, as one is then able to give non-trivial definitions which are closer to the mathematical practice and which, from the classical point of view would have no content at all. But, in order to be able to become somewhat more precise, it is now necessary to pause and give a proper definition of what a topos is.

A category $\mathbf{E}$ is said to have *finite limits* (*cf.* [26]) if there is a terminal object 1, and for each pair of morphisms $A \xrightarrow{f} C \xleftarrow{g} B$, a pulback $A \xleftarrow{p} P \xrightarrow{q} B$ exists. (If $\mathbf{E}$ is *Sets*, for example, 1 is any singleton, and $P = \{(a, b) \in A \times B | f(a) = g(b)\}$.) As usual, it follows that products $A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$ exist, as well as equalizers $E \rightarrowtail A$ of pairs of maps $A \underset{g}{\overset{f}{\rightrightarrows}} C$. A category $\mathbf{E}$ with finite limits is said to be an (*elementary*) *topos* if power objects exist. To say that $\mathbf{E}$ has *power objects* means that, for each object A there is an object $P(A)$ and a relation $\in_A$ from A to $P(A)$ (i.e. a monomorphism $\in_A \rightarrowtail A \times P(A)$) which is generic in the sense that, given any relation $R \rightarrowtail A \times B$, there is a unique $B \xrightarrow{r} P(A)$ by means of which $R \rightarrowtail A \times B$ is obtained by pullback of $\in_A \rightarrowtail A \times P(A)$ along $\mathrm{id}_A \times r$: $A \times B \to A \times P(A)$. (If $\mathbf{E}$ is Sets, $P(A)$ is the power set of A and $\in_A$ the relation of membership; the connection between R and r is simply that, for $a \in A$, $b \in B$, $(a, b) \in R$ if and only if $a \in r(b)$.)

Thinking of the objects of an arbitrary topos $\mathbf{E}$ as variable sets, for an object A of $\mathbf{E}$, a morphism $B \xrightarrow{f} A$ of $\mathbf{E}$ is to be viewed as a *generalized element of A defined at stage B*, denoted $f \in_B A$. Elements $1 \xrightarrow{a} A$ are said to be *global elements* (or global "sections", a terminology which is drawn from the example of sheaves on a topological space); but objects in an arbitrary topos are not normally determined by their global elements. Thus, any construction done in $\mathbf{E}$ must take all elements of the objects involved into account, and this is an essential aspect of toposes.

The object $\Omega = P(1)$ is equipped with a morphism t: $1 \to \Omega$ (for "true") together with which it becomes a *subobject classifier* (i.e., every subobject $A' \rightarrowtail A$ has a unique "characteristic function" $A \xrightarrow{\varphi} \Omega$, in the sense that u is the pullback of t along φ). Another consequence of the power object axiom is the existence of *finite colimits*; this is best shown by observing that the power object functor P: $\mathbf{E}^{\mathrm{op}} \to \mathbf{E}$ is tripleable (or monadic, c.f. [26], [17]), hence the conclusion follows simply from the assumption of finite limits in $\mathbf{E}$. Using finite limits and colimits, and the universal property of Ω with $1 \xrightarrow{t} \Omega$, other structure maps can be defined, corresponding to logical connectives, i.e., f: $1 \to \Omega$ (for "false"), $\wedge$, $\vee$, $\Rightarrow$: $\Omega \times \Omega \to \Omega$, and $\neg$: $\Omega \to \Omega$, with these, Ω becomes a *Heyting algebra*. In Sets, $\Omega = 2 = \{t, f\}$, while in $\mathrm{Sh}(X)$, the global elements of Ω correspond precisely to the open subsets of X, with this interpretation, conjunction corresponds to intersection, disjunction to union, negation to largest open set disjoint from a given open set, t is X itself, while f is the empty subset.

It also follows from the axioms for a topos that *exponentiation* exists. This property which, together with finite limits and a subobject classifier, provides for an alternative characterization of the notion of a topos, says the following. Given any two objects A, B, there is an object B^A, and a morphism $\mathrm{ev}_{A,B}$: $A \times B^A \to B$, with a universal property leading to the fact that, for each A, the functor $(\ ) \times A$: $\mathbf{E} \to \mathbf{E}$ has a right adjoint, $(\ )^A$: $\mathbf{E} \to \mathbf{E}$. Equivalently, for each object C, there is a bijection (natural in all

three variables) between morphisms $C \to B^A$ and morphisms $C \times A \to B$. The way to construct exponentials in a topos **E** is to mimic the set theoretical construction: a function from A to B is a relation from A to B which is everywhere defined and single-valued. In fact, it can be shown that in a topos **E**, any first order formula $\varphi(x_1, \ldots, x_n)$, with free variables of *sorts* $A_1, \ldots, A_n$ (objects of **E**) admits an *extension* in the form of a subobject $[\![(x_1, \ldots, x_n) | \varphi(x_1, \ldots, x_n)]\!] \rightarrowtail A_1 \times A_2 \times \ldots \times A_n$. *Sheaf semantics* is the laying down of those rules governing these interpretations, and it extends the semantics of Kripke for intuitionistic logic. *Kripke models* are, in fact, toposes. These logical considerations are well explained in various sources, such as [29], [17], [27], [8], [11], [12], among others.

An important class of toposes which encompasses both the spatial toposes $\mathrm{Sh}(X)$ as well as Kripke models, are the *Grothendieck toposes* (c.f. [1], [15]), and were the starting point of the notion of a topos. They can be singled out by saying that they possess arbitrary sex-indexed colimits, as well as a set of generators. These extra features make them particularly useful to work with, e.g., they may be employed as semantic universes from infinitary logic (c.f. [27]). But Grothendieck toposes do not really suffice for all of the topos theoretic applications envisaged, particularly for those of a logical nature (applications in Set theory, Model theory, or Proof theory). In order to obtain significant new examples of non-Grothendieck toposes, categorical methods of varying sophistication are employed (cotriples, fraction categories, glueing) and in these, some kind of consideration of *morphisms* of toposes is unavoidable. It seems to be the case that no genuine application of Toposes in Logic (or even elsewhere) can be made by ignoring that essential aspect of Topos theory which is the study of topos morphisms. But, what are these morphisms?

Thinking of toposes as generalized spaces (even though they sometimes have no points!) the idea is to abstract properties of continuous mappings. A continuous $g: X \to Y$ lifts to the generalized spaces $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ in the form of a functor $g^*: \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$: pulling back along g the étalé spaces. This functor is *left exact* (i.e., preserves finite limits) and has a right adjoint, g_*. Let us say then, that a *geometric morphism* $\mathbf{E} \to \mathbf{F}$ of arbitrary toposes is given by a functor $g_*: \mathbf{E} \to \mathbf{F}$ which has a left exact left adjoint, g^*. (The direction is intended to respect that of the original map g in the motivating example giving rise to this definition.)

On the other hand, the logical properties making an elementary topos **E** into a universe of sets dictates yet another choice for the notion of morphism. A *logical morphism* $\mathbf{E} \to \mathbf{F}$ of toposes is any functor which preserves the topos structure (e.g., finite limits and power objects, and therefore also Ω, exponentiation, finite colimits; in fact, all interpretation of first order formulas with free variables whose sorts are interpreted as objects of the topos).

Among examples of constructions involving logical morphisms are glueing (c.f. [32]), the ultrapower construction (c.f. [31]), or the free topos (c.f. [22]). Geometric morphisms are central for the construction of the classifying topos of a (finitary) geometric theory (c.f. [17]), or in generalization of the notion of a Grothendieck topology made internal and relative to an arbitrary topos – the latter is related to a factorization of geometric morphisms arising from left exact cotriples (c.f. [32], [17]). These "topologies" j on a topos **E**, giving rise to a new topos $\mathrm{Sh}_j(\mathbf{E})$ over **E**, were originally conceived by Lawvere (c.f. [12] for an explicit demonstration) as modal operators of the kind "it is locally the case that...". In any topos **E**, $\neg\neg: \Omega \to \Omega$ is one such topology, and $\mathrm{Sh}_{\neg\neg}(\mathbf{E})$ is a *Boolean topos* (i.e., a topos in which Ω, rather than being just a Heyting algebra, is a Boolean algebra, hence logic becomes classical therein).

Is it really necessary to work with toposes? Why cannot we be satisfied with the foundations supplied by Set theory? An attempt to answer this question is made in the next section by means of specific examples, also intended to illustrate the two aspects mentioned in the title.

2. Why work with toposes?

Everybody knows that the main bulk of Mathematics (à la Bourbaki) can be based on a set-theoretical foundation as that provided by Zermelo–Fraenkel–Gödel–Bernays Set theory. Why, then try to alter this ideal situation? There are several reasons, both from the conceptual as well as from the practical point of view, which, in my view, justify the claim that, at least sometimes, it is desirable to found a given portion of Mathematics on Topos theory. I shall proceed to mention specific examples from some recent developments which involve utilizing toposes in an essential way.

2.1. *Synthetic foundations of Differential Geometry*

On account of the rigors of the arithmetization to which analysis was subjected in the 19th century, much of the original intuitive aspects of the subject were lost in the

process. On the other hand, in the theory of smooth manifolds (and differential calculus therein), the lack of good categorical properties (such as all finite limits, or exponentials) forced the artificial division between the finte and the infinite-dimensional cases. In order to solve both these problems at once, Lawvere [23] proposed to work in a sufficiently rich category **E** (e.g. a topos) in which the real line would be just a non-trivial commutative ring and in which geometric constructions such as the tangent bundle of R (or of any other object M of **E**, thought of to be "a smooth manifold") became representable. More explicitly, using equalizers (or the interpretation of formulas), one can consider an object $D = [\![x \in R \mid x^2 = 0]\!] \rightarrowtail R$ and require that "every map from D to R be linear" – an assumption which is categorically expressible (cf. [21]) by saying that the morphism

$$R \times R \xrightarrow{\alpha} R^D$$

defined by exponential adjointness to correspond to the morphism

$$R \times R \times D \xrightarrow{\bar{\alpha}} R$$

which in terms of generalized elements $a, b \in R$, $d \in D$ (defined at the same stage) is given by the rule $(a, b, d) \longmapsto a + d.b$, is an isomorphism. It follows (c.f. [21]) that there are no rings 'of line type' in *Sets*; define a function $f\colon D \to R$ by using excluded middle, saying that $g(0) = 0$ and $g(d) = 1$ for $d \neq 0$. Since $D = \{0\}$ is impossible as this would contradict the axiom, we may assume (again, a classical argument) that $\exists\, d_0 \in D\colon \neg(d_0 = 0)$. By the axiom, it is the case that $\forall d \in D\colon g(d) = g(0) + d.b$, for a unique $b \in R$ (interpreted as the derivative $g'(0)$). Applying this to d_0, one easily gets $1 = g(d_0) = 0 + d_0.b$ which, when squared, yields $1 = 0$, a contradiction.

Now, in **E**, every object M has a "tangent space", M^D equipped with a 'base point' morphism $\pi\colon M^D \to M$ which, under suitable assumptions on M including a large class of objects, is fiberwise an R-module. For any morphism $f\colon M \to R$, the *formal derivative* of f at a (generalized) element $a \in M$ is definable as the composite $(\mathrm{df})_a = T_aM \rightarrowtail M^D \xrightarrow{f^D} R^D \xrightarrow{\gamma} R$, where $T_aM = \pi^{-1}\{a\}$, and where γ is the composite $R^D \xrightarrow{\alpha^{-1}} R \times R \xrightarrow{p_2} R$ $(g \longmapsto g'(0))$. On these grounds, a great deal of differential calculus and fifferential geometry (vector fields, differential forms, connections) can be developed (c.f. [21]) and is largely the work of Lawvere, Wraith, Kock and Reyes. Some synthetic aspects of the theory of smooth mappings, such as those which, like stability, are relevant in the classification of singularities, are studied in [5]. Models for the axiom of line type, and of several other axioms introduced along the way to advance the theory, exist (c.f. [21]) and some even contain the category **M** of smooth manifolds in a way that preserves the structure already available therein ("well adapted models for Synthetic Differential Geometries") and where the real numbers become the ring of line type.

2.2. *Constructivist mathematics and Intuitionistic Logic and Set theory*

There is, nowadays, an increasing tendency among mathematicians to work 'constructively'. This does not necessarily imply fully adopting Brower's principles, but it does mean that, whenever possible, one should not use arguments by contradiction (alternatively, be aware that, by so doing, it is not a certain statement φ one proves, but, instead only $\neg\neg\varphi$), or use the axiom of choice. Intuitionism, and its codification by Heyting, partly responded to the constructivist program. Since the internal logic of a topos is intuitionistic, and not all toposes satisfy the axiom of choice (those which do are actually Boolean!), such principles will not enter inadvertently into any proof by topos theoretic methods. The other side of the coin, is that, in order to guarantee that constructions corresponding to manipulating certain formulas, when applied to objects in a topos yield again objects in the topos, all one has to do is use only intuitionistically valid deductions (metatheorem).

A by-product of the uses of intuitionistic (rather than classical) logic is that it forces mathematicians to be more discriminating in their choice of notions. For example, to say that a non-trivial commutative ring A is a *field*, classically we may choose from any of the following statements (c.f. [18]):

$$F_1\colon \forall a \in A\,[a = 0 \vee a \in U(A)]$$
$$F_2\colon \forall a \in A\,[\neg(a = 0) \Rightarrow a \in U(A)]$$
$$F_3\colon \forall a \in A\,[\neg(a \in U(A)) \Rightarrow a = 0],$$

where "$a \in U(A)$" means "$\exists x \in A\ (a.x = 1)$". But there are interesting "field-like' objects in a topos which (such as the Dedekind reals) satisfy F_2, or F_3, but not F_1. In effect, F_2 corresponds to fields which arise as fields of fractions of integral domains, while F_3 to those which are residue fields of local rings. For a prime ideal P, the field of fractions of the integral domain A/P equals the residue field of the local ring A_p, and either yields a field as in F_1 (called 'geometric') on account of the formula expressing this property being

preserved by the inverse image part, g^*, of any geometric morphism (g^*, g_*) of toposes). By working intuitionistically, these notations are kept separate and it is easier to point to what is one really using, which is not always the case when working classically.

Another example, this time involving the axiom of choice rather than excluded middle, is the "construction" of the *algebraic closure* of a field. Even for geometric fields, algebraic closures need not exist in a topos (c.f. [19]) although for a Grothendieck topos **E**, a theorem of Barr (c.f. [17]) says that a topos **F** satisfying the axiom of choice, and a *surjection* $\mathbf{F} \xrightarrow{f} \mathbf{E}$ (i.e., a geometric morphism in which f^* is faithful) always exist so that, by "moving" to **F**, we can find all the algebraic closures we wish – the problem being that, in general, in view of the poor preservation properties of f_*, it is not possible to come back without losing much of what was gained. (Even so, this is a powerful method.) By contrast, the theory of ordered fields has a more constructive flavor, and much work on it is still undergoing, particularly in connection with Real Algebraic Geometry. One aspect of this is reflected in the fact (proven in [3]) that in a Grothendieck topos **E**, the real closure of an ordered field always exists.

Although we have concentrated on Algebra above, other branches of Mathematics, such as Topology (c.f. [14]), are, at present, the object of much interest for constructivist mathematics, and the connections with toposes are becoming stronger everyday. For example, a specific topos has been constructed by Joyal (c.f. [9], [30]) in which the unit interval (of the Dedekind reals), is non-compact. Hence, such a property of the real numbers cannot be established within intuitionistic set theory without Choice.

2.3. *Sheaf representations and Model theory*

To take an example of a particular interest to logicians, let us mention that the study of the model theoretic properties of a theory T_0 in a first order language can be largely simplified if one knows that the models of T_0 are precisely the global sections of sheaves which are models of a theory T in the topos of sheaves, and if the model theoretic properties of T are known. This ideas was first put to use by Macintyre [25], who deduced the model completeness of the theory T_0^* of monically closed (von Neumann) regular rings with no minimal idempotents, from the (positive) model completeness of the theory T^* of algebraically closed fields. An entirely topos theoretic proof of Macintyre's theorem and related results has since been given by Reyes and myself [6]. In it, a regular ring R is explicitly viewed as having two components (X, K); a Boolean space X, the Stone space of the Boolean algebra B of idempotents of R, and a (geometric) field K in the topos $\mathrm{Sh}(X)$, having the property that R is isomorphic to the global sections of K. This same approach, based on a Boolean representation theory involving a theory T_0 (regular rings) and a theory T (fields), where models of T_0 are representable by models of T in a topos $\mathrm{Sh}(X)$ where X is a Boolean space, can be applied in order to obtain other model theoretic results, e.g. amalgamation, cofinality, or the existence of prime model extensions (c.f. [6], [3], [4]). Examples of such pairs (T_0, T) other than the pair (regular rings, fields) are given by (regular f-rings, ordered fields), (differential rings, differential fields), etc.

The use of sheaf representations is, of course, not limited to model-theoretic proeprties. In the case of regular rings it is of interest to use this approach in order to develop Algebraic and Real Algebraic Geometry over regular and regular f-rings, respectively, and work in this direction will be contained in the doctoral dissertation of my student W. MacCaull. The interest in regular rings stems from Functional Analysis, were the main examples of such rings (of continuous functions) arise. But in order to apply methods of sheaf representations, or "transfer" methods, it becomes necessary to first acquire knowledge of the subjects to be transferred in toposes of the sort involved in the representation theory one uses, e.g., toposes of sheaves on a Boolean space.

2.4. *Independence proofs in Set theory*

One of the earliest applications of toposes (c.f. [13], [31]) was to give a new proof of the independence of the continuum hypothesis from the axioms of Set theory, proved originally by Cohen by the method of forcing. The Set theory on which the new proof is based is an $\in$-free formulation based on Topos theory (hence, on Category theory). By a *model of Set theory* one understands, in this context, a Boolean topos **S**, with a natural numbers object ω, which satisfies the axiom of Choice (as pointed out earlier, this already implies Booleaness), and which is two-valued, meaning that $\Omega = 2 = \{t, f\}$. This system of axioms has been shown (c.f. [17]) to be basically equivalent to a weak form (no Replacement, only bounded comprehension) of Zermelo–Fraenkel Set theory with Choice.

The question of the continuum can be posed, categorically, as follows: is there a set X such that there are mono-

morphisms $\omega \rightarrowtail X \rightarrowtail 2^{\omega}$, with $\omega \not\cong X \not\cong 2^{\omega}$? The continuum hypothesis ($2^{\aleph_0} = \aleph_1$) says that there is no such X, and is consistent with the axioms of Set theory. Cohen showed that it is also consistent to assume its negation. The steps in the topos-theoretic proof of this fact are the following: (i) one begins by formulating the statement internally in the topos, in such a way that taking global sections the statement becomes precisely what one wants to prove (above, this means constructing internally the objects $\mathrm{Epi}(\omega, X)$ and $\mathrm{Epi}(X, 2^{\omega})$ and asserting that they are the initial object, 0); (ii) construct a partially ordered set $\mathbf{P}$ of forcing conditions (in this case its elements are all partial approximations to an uncountable object X with $\omega \rightarrowtail X \rightarrowtail 2^{\omega}$) and collect this information together into a single variable set X in the topos $\mathrm{Sets}^{\mathbf{P}}$ of set-valued functors on $\mathbf{P}$; (iii) prove that when passing to $\mathrm{Sh}_{\neg\neg}(\mathrm{Sets}^{\mathbf{P}})$ the statement one wishes to establish (negation of the continuum hypothesis), true almost by construction for X in $\mathrm{Sets}^{\mathbf{P}}$, remains true; (iv) pass from the Boolean topos $\mathrm{Sh}_{\neg\neg}(\mathrm{Sets}^{\mathbf{P}})$, which is actually a Boolean-valued-model of Set theory in that it satisfies the axiom of choice, to a two-valued model $\mathbf{S}$ by applying an ultrafilter construction in the form of a logical functor which thus preserves the validity of the statement.

Other independence proofs (c.f. [2], [30]) can be done topos theoretically by similar procedures. It is important to observe here that what Topos theory does for these proofs is to make explicit the topos needed in forcing proofs, rendering, at the same time, perhaps less mysterious the choice of forcing conditions in each case.

In the next section we shall deal, in more detail, with some of the methods which, in addition to those already mentioned above, are employed by toposophers. The emphasis will be, as already in this section, either in the rôle of Logic in toposes, or in the applications of toposes to Logic.

3. Topos theoretic methods in applications

Let us now give a few sample applications of Topos theory to the Calculus of Variations, Proof Theory, Model theory, and Algebra, with emphasis on the methods employed, while continuing to try to illustrate the interaction between the geometric and the logical aspects of toposes.

3.1. *Embedding a category into a topos: Calculus of Variations*

It was mentioned in Section 2.1 that well adapted models for Differential Geometry exist. Let $\mathbf{M} \xrightarrow{i} \mathbf{E}$ be a full embedding of the category of smooth manifolds, with $i(\mathbb{R}) = R$, of line type. Preservation properties of i (transversal pullbacks and open coverings are preserved by i) may be used to deduce several algebraic properties of R (c.f. [21]); e.g., R admits an order which is total when restricted to the invertible elements, but is such that $D \subseteq [0,0]$; also, R is a local ring. For the applications to classical problems (i.e., problems involving smooth manifolds), it is useful to consider a well adapted model $(\mathbf{E}, R)$ for which the global sections functor, $\Gamma\colon \mathbf{E} \to \mathrm{Sets}$, takes the image $\bar{M}$ of a smooth manifold M under i, into (the underlying set of) M again. Since Γ preserves finite limits, at least all algebraic constructions are preserved. But more can be inferred in concrete instances.

Consider now an example drawn from the *classical calculus of variations* (c.f. [16], [7]) which is conceptually advantageous to deal with by embedding into a well adapted model.

For a smooth path $c\colon [0,1] \to \mathbb{R}^n$, with $c(0) = p$, $c(1) = q$, the energy (or action) integral is defined as

$$E(c) = \int_0^1 \sum_{i=1}^{n} (c_i'(t))^2 \, \mathrm{d}t,$$

where $c = (c_1, \ldots, c_n)$, and $c_i'(t)$ indicates the first derivative. Classically, it is shown that the "critical" paths (*ad hoc* definition, by analogy with the finite dimensional case) for the energy integral are precisely the geodesics, or paths of minimal length. Embedding $\mathbf{M}$ into $\mathbf{E}$, with $\mathbb{R}$ becoming R, the object of paths is given by the extension of the formula

$$\Lambda_{p,q} = [\![c \in R^{n[0,1]} \mid c(0) = p \wedge c(1) = q]\!] .$$

If $\mathbf{E}$ satisfies, furthermore, the axiom of integration (c.f. [21], it says that primitives exist and it allows for a development of integration on "manifolds", i.e., objects of $\mathbf{E}$, including Stoke's theorem), it is possible to mimic the definition of the energy integral which then becomes a morphism

$$E\colon \Lambda_{p,q} \to R$$

and therefore (c.f. Section 2.1), it has formally a derivative at c,

$$(dE)_c\colon T_c(\Lambda_{p,q}) \to R .$$

Synthetically, and by contrast to the classical counterpart,

the treatment of morphisms between manifolds (whether "finite or infinite dimensional") is thus unified (c.f. [23]). In particular, the notion of "critical" path if just the usual notion in this context, i.e. c is critical for E provided $(dE)_c = 0$.

By means of further axioms it is then possible to argue as in the classical treatment (i.e., by contradiction), and conclude that, for any $c \in \Lambda_{p,q}$, if c is critical for E then c is almost a geodesic, i.e. $\forall t \in [0,1]: \neg\neg(c''(t) = 0)$. Having verified that one of the standard well adapted models satisfies the additional axioms, it remains to argue that if c is a global element of $\Lambda_{p,q}$ (equivalently, since the embedding is fully faithful), if c is the image of an actual path $[0,1] \to \mathbb{R}^n$, then c is almost a geodesic if and only if c is a geodesic (i.e., $\forall t \in [0,1]: c''(t) = 0$). The latter is based on the remark that the object

$$\Delta = [\![x \in R \mid \neg\neg(x = 0)]\!]$$

has no global elements other than $0 \in R$.

The object Δ, classically trivial, is to be thought of, synthetically, as the "monad of 0" (although it is quite different from the corresponding notion in Non-standard Analysis); it was considered by Penon [28] and can be seen to represent the important notion of germ of a smooth mapping and, as such, it is exploited in [5]. Closely related to this is a notion (also due to Penon) of "open" which again, although geometrically close to our intuitions, is classically trivial and can only be considered on account of the (intuitionistic nature of the) logic of a topos. A subobject $U \rightarrowtail M$ of an arbitrary object of $\mathbf{E}$ is said to be an open neighborhood of any one of its elements, say $a \in U$, if the following holds:

$$\forall x \in M[\neg(x = a) \vee x \in U]$$

i.e., either x is "far away from a" (in which case we are not interested in it), or else it belongs to U. Equivalently, M is the union, in the topos, of the topos complement of the subobject $\{a\} \rightarrowtail M$ and of $U \rightarrowtail M$.

3.2. *Glueing: realizability*

A formulation and proof of *realizability* using toposes has been given by Freyd (unpublished, c.f. also [22]). The method employed is known under the name of Artin-Wraith glueing (c.f. [32]). In this formulation, realizability says that, in the free topos, 1 is indecomposable and projective. Take the rule (where $\vdash_T$ is intuitionistic provability) which says

$$\frac{\vdash_T \varphi \vee \psi}{\vdash_T \varphi \text{ or } \vdash_T \psi}$$

where φ, ψ are sentences in a first order language. Translating the meaning of this in terms of their extensions in a topos, the above says that 1 cannot be expressed as the union of two proper subobjects, i.e., 1 is *indecomposable*. Take now the rule:

$$\frac{\vdash_T \ x \in A \ \varphi(x)}{\text{there exists } a \in A \text{ such that } \vdash_T \varphi(a)}$$

where $\varphi(x)$ is a formula and x a free variable of sort A, for some A interpreted as an objects in a topos. The hypothesis says that the canonical map into 1

$$[\![x \in A \mid \varphi(x)]\!] \twoheadrightarrow 1$$

is epi, and the conclusion, that it splits (i.e., has a section, thus, 1 is *projective* (in effect, "support splits").

For an arbitrary category $\mathbf{C}$ with a terminal object 1, and where $\Gamma = \mathrm{Hom}(1, -): \mathbf{C} \to \mathit{Sets}$ is the global sections functor, the "comma category" $\hat{\mathbf{C}} = (\mathit{Sets}, \Gamma)$ is the category whose objects are functions of the form

$$S \xrightarrow{f} \Gamma(C)$$

with $S \in \mathit{Sets}$, $C \in \mathbf{C}$, and whose morphisms are commutative diagrams (a morphism from $S \longrightarrow \Gamma(C)$ to $S' \xrightarrow{f'} \Gamma(C')$, that is)

$$\begin{array}{ccc} S & \xrightarrow{s} & S' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \Gamma(C) & \xrightarrow{\Gamma(c)} & \Gamma(C'). \end{array}$$

Such a construction produces a topos $\hat{\mathbf{C}}$ out of any such $\mathbf{C}$; moreover, the projection

$$\hat{\mathbf{C}} \xrightarrow{p_2} \mathbf{C}$$

given by $S \xrightarrow{f} \Gamma(C) \longmapsto C$; $(s, c) \longmapsto c$, is a *logical* functor.

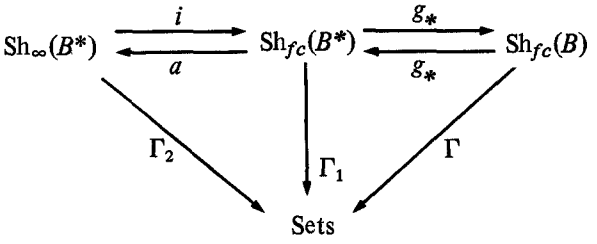
Take $\mathbf{C}$ to be the free topos. This means that $\mathbf{C}$ is a topos (which can be shown to exist) having the property that for any topos $\mathbf{E}$, there is a unique logical morphism $\mathbf{C} \to \mathbf{E}$. In particular, the unique logical morphism $\mathbf{C} \to \hat{\mathbf{C}}$, when composed with $\mathbf{C} \xrightarrow{p_2} \hat{\mathbf{C}}$, must be the identity. It is easy to prove now that $\hat{1} = 1 \to \Gamma(1)$ is (externally) indecomposable and projective in $\hat{\mathbf{C}}$. Given U, V subobjects of 1 in $\mathbf{C}$, with $U \cup V = 1$, it follows that $\hat{U}$, $\hat{V}$ are subobjects of $\hat{1}$ in $\hat{\mathbf{C}}$, and that $\hat{U} \cup \hat{V} = \hat{1}$ in $\hat{\mathbf{C}}$. Therefore, either $\hat{U} = \hat{1}$ or $\hat{V} = \hat{1}$.

Applying p_2, this yields that either $U = 1$ or $V = 1$. Hence, 1 is indecomposable in **C**. A similar argument shows that 1 is projective in **C**.

3.3. *Transfer methods: cofinality of models*

In Section 2.3 we mentioned transfer methods in order to establish certain model theoretic results for theories of global sections of structures which are models of another theory in a topos of sheaves on a Boolean space. In order to do this in practice, it is often required to pass to a Boolean valued model of Set theory, where the usual theorems of Zermelo–Fraenkel Set theory with Choice (not involving the fact that there are exactly two truth values) are true. As mentioned in Section 2.2, one such method is to use a theorem of Barr; however, sometimes one needs the passage to be done via a morphism $\mathbf{F} \xrightarrow{f} \mathbf{E}$ which has further properties than just being a surjection. Let us deal with a concrete property: the transfer of *cofinality* (c.f. [6]) in the context of a Boolean representation theory, as presented in Section 2.3. Assume that (T_0, T) is a pair in a Boolean representation theory, and that T^* is a theory such that Mod $T^* \hookrightarrow$ Mod T (assume T^* is a quotient of T, i.e., has more axioms) has the property (cofinality) that every model of T admits an extension to a model of T^* (this true for models in *Sets*). Define T_0^* as the theory of global sections of sheaves over Boolean spaces, which are models of T^* in the corresponding topos where they live. Prove that Mod $T_0^{\mathrm{op}} \hookrightarrow$ Mod T_0 is cofinal. (Taking the theories T_0 = regular rings, T = fields; T^* = algebraically closed fields, for example, T_0^* = monically closed regular rings, and the theorem would say that every regular ring can be embedded into one which is monically closed, i.e., for which *monic* polynomials with coefficient in the ring, have roots in the ring.) Now, as mentioned in Section 2.2, there is a little problem here concerning the possible transfer of the corresponding theorem which would say: "every field can be embedded into an algebraically closed field" since it is not true in an arbitrary topos – in fact, it is not true in general, even if the topos is of the form $\mathrm{Sh}(X)$, X a Boolean space, basically because the axiom of Choice need not hold there. Still, the result about regular rings is known to be true. The key lies in an appropriate change of universe in order to establish this fact, as follows.

Let M be a model of T_0, identified with the pair (X, K) where $R = \Gamma(K)$, and K is a model of T in $\mathrm{Sh}(X)$. It is well known (c.f. [10]) that $\mathrm{Sh}(X)$ may also be viewed as $\mathrm{Sh}_{fc}(B)$, where now B is regarded as a category using the preorder



relation with a Grothendieck topology whose basic coverings of $b \in B$ are those families $\{b_i\}_{i=1,\ldots,n}$ for which $b = \bigvee_{i=1}^n b_i$. If B^* is the Dedekind–McNeille completion, of B, we may still consider $\mathrm{Sh}_{fc}(B^*)$, but in this case it makes sense to consider also arbitrary coverings, hence the topos $\mathrm{Sh}_\infty(B^*)$. These three toposes are related by geometric morphisms with special properties, namely with $a \dashv i$, $g^* \dashv g_*$, i flat (i.e., preserves first-order logic) and g^* faithful. Moreover, $\mathrm{Sh}_\infty(B^*) = \mathrm{Sh}_{\neg\neg}(\mathrm{Sh}_{fc}(B^*))$, hence is a Boolean topos; in fact (c.f. [10]), it is a Boolean-valued model of Set theory, and the direct image parts of the geometric morphisms above, commute with the global sections functors. Assume now that all theories involved are *geometric*, in the sense that their models be preserved by inverse image parts of arbitrary geometric morphisms (this notion of a geometric theory, as well as that of a coherent, or finitely geometric theory is studied in [27] – we shall come back to it later on). Of T_0 furthermore, it is needed that its models be preserved by g_*. Since the latter arises from the inclusion $B \subseteq B^*$, this translates into the synthactical condition that T_0 be axiomatizable in a language with very poor structure – this will be satisfied if. e.g. T_0 is algebraic. Now, since $K \in \mathrm{Mod}_{\mathrm{Sh}_{fc}(B)}(T)$, it follows by our assumptions (true in the examples, by the way) that $ag^*K \in \mathrm{Mod}_{\mathrm{Sh}_\infty(B^*)}(T)$ and since the latter is a Boolean-valued model, there is an extension $ag^*K \rightarrowtail K'$ to some $K' \in \mathrm{Mod}_{\mathrm{Sh}_\infty(B^*)}(T^*)$. By exponential adjointness (c.f. Section 1), this corresponds to a morphism $K \rightarrowtail g_* i K'$, which is an extension by yet a further assumption made on morphisms whose domains are models of T. Since i is flat, $iK' \in \mathrm{Mod}_{\mathrm{Sh}_{fc}(B^*)}(T^*)$; since g_* preserves models of T_0, $g_*(iK') \in \mathrm{Mod}_{\mathrm{Sh}_{fc}(B)}(T_0)$. To finish, we apply global sections, which gives a monomorphism

$$R = \Gamma(K) \rightarrowtail \Gamma(g_* i K') = \Gamma_1(iK') = R';$$

noticing that, by definition of T_0^*, R' is a model of T_0^*. The rôle played by geometric morphisms is here essential, and this limits, to a certain extent, the type of theories this argument applies to. But geometric theories have a particular attractiveness, in view of the completeness theorem (of

Gödel–Henkin–Tarski) applied to geometric fragments (c.f. [27], [20]) an application of which is given in the next section.

3.4. *Geometric logic: Hilbert's Nullstellensatz*

In [4] it is established by transfer, a *nullstellensatz* (or thoerem of zeroes) for various theories: regular rings, regular f-rings, and differential rings in zero characteristic. This is done by methods similar to those employed in Section 3.3. above. In particular, it is not necessary that the theorem in question be valid in the topos $\mathrm{Sh}(X)$ itself, which gives us more freedom with respect to its formulation. But another formulation of Hilbert's nullstellensatz, equally well suitable for transfer, is valid in any Grothendieck topos. An argument, due to W. MacCaull, is given as an application of a metatheorem which is the logical counterpart of a theorem of Barr and Deligne (c.f. [17]) and which, in turn, is a consequence, for geometric infinitary logic, of the completeness theorem of Gödel (c.f. [27], [20]). Let us outline this result and the above mentioned application of it in what follows.

A *geometric sequent* σ is the universal closure of a formula $\varphi(\vec{x}) \Rightarrow \psi(\vec{x})$, with free variables $\vec{x} = (x_1, \ldots, x_n)$, where $\varphi(\vec{x})$ and $\psi(\vec{x})$ are either t, f or are built up using only $\wedge$, $\exists$ and (possibly infinitary) $\vee$. A *geometric theory,* is a theory axiomatizable by means of geometric sequents. The metatheorem in question states that if T is a geometric theory in a countable fragment of $\mathscr{L}_{\infty\omega}$, and σ is a geometric sequent such that $T \vdash_C \sigma$ (i.e., σ is *classically* deducible from T), then for every Grothendieck topos $\mathbf{E}$, any model of T in $\mathbf{E}$ satisfies σ.

In view of this metatheorem, it will be enough to produce a geometric formulation of Hilbert's nullstellensatz, which can be done as follows. Let $\mathscr{L}$ be a language in a countable fragment of $\mathscr{L}_{\infty\omega}$, to which sorts K, K' are added. A theory T_1 in this language is considered whose models are given by: a (geometric) field K, an algebraically closed (geometric) field K', and an extension $K \hookrightarrow K'$. Let f, $f_1, \ldots, f_m \in K[\vec{y}]$ be polynomials over K.

Consider the (geometric) sequents

$$\sigma_1(\vec{a})\colon\ (\forall \vec{y} \in (K')^n\ [\bigwedge_{j=1}^{m} f_j(\vec{y}) = 0 \Rightarrow f(y) = 0])$$

and

$$\sigma_2(\vec{a})\colon\ (t \Rightarrow \bigvee_{r=1}^{\infty} \exists x_1 \ldots x_m \in K[\vec{y}]\,(f^r = \sum_{j=1}^{m} x_j f_j))$$

where $\vec{a}$ are the parameters (i.e. the coefficients of the f, $f_1, \ldots, f_m$).

Hilbert's nullstellensatz is true classically, i.e.,

$$T_1 \vdash_C \sigma_1(\vec{a}) \Rightarrow \sigma_2(\vec{a});$$

By the Deduction theorem,

$$T_1 \cup \{\sigma_1(\vec{a})\} \vdash_C \sigma_2(\vec{a})$$

and by the metatheorem (since $T = T_1 \cup \{\sigma_1(\vec{a})\}$ is a geometric theory)

$$T_1 \cup \{\sigma_1(\vec{a})\} \models_{G.T} \sigma_2(\vec{a})$$

meaning that every model of $T_1 \cup \{\sigma_1(\vec{a})\}$ in a Grothendieck topos is also a model of $\sigma_2(\vec{a})$. Using now the Deduction theorem again (as it is intuitionistically valid), we obtain

$$T_1 \models_{G.T.} \sigma_1(\vec{a}) \Rightarrow \sigma_2(\vec{a})$$

and by the rule of universal generalization,

$$T_1 \models_{G.T} \forall \vec{a}\,(\sigma_1(\vec{a}) \Rightarrow \sigma_2(\vec{a})).$$

This completes the proof.

Our list of examples concludes here. It is to be hoped that, having exposed the readers to a variety of applications, he or she will find at least one of these worth pursuing. One conclusion that may, perhaps, be drawn from this display, is the remark that toposophers are just amateur logicians who are interested in too many different subjects to be able to become experts in any one of them. But, who knows, this description (which certainly applies to the present writer), might gain us some sympathizers among the philosophers....

References

[1] Artin, M.: 1962, 'Grothendieck Topologies', Lecture Notes, Harvard University.

[2] Bunge, M.: 1974, 'Topos theory and Souslin's hypothesis', *J. Pure and Applied Algebra* **4**, 159–187.

[3] Bunge, M.: 1981, 'Sheaves and Prime model extensions', *J. of Algebra* **68**, 79–96.

[4] Bunge, M.: 1982, 'On the transfer of an abstract nullstellensatz', *Comm. in Algebra* **10** (17), 1891–1906.

[5] Bunge, M.: 1983, 'Synthetic aspects of C^∞-mappings', *J. Pure and Applied Algebra* **28**.

[6] Bunge, M. and G. E. Reyes: 1981, 'Boolean spectra and model completions', *Fundam. Mathem.* **CXIII**, 165–173.

[7] Bunge, M. and M. Heggie: forthcoming, 'Synthetic Calculus of Variations' (to appear in the *Proceedings of the Special Session on Applied Category Theory of the 89th Annual meeting of the Amer. Math. Soc., Denver, Jan. 1983*).

[8] Boileau, A. and A. Joyal: 1981, 'La logique des topos', *J. of Symb. Logic* **46**, 6–16.

[9] Fourman, M. and M. Hyland: 1979, 'Sheaf models for analysis', in *Applications of Sheaves, Springer Lecture Notes in Math.* 753, 280–301.

[10] Fourman, M. and D. Scott: 1979, 'Sheaves and logic', in *Applications of Sheaves, Springer Lecture Notes in Math.* 753, 302–402.

[11] Freyd, P.: 1972, 'Aspects of Topoi', *Bull. of the Australian Math. Soc.* 7, 1–76.

[12] Goldblatt, R.: 1979, *Topoi, the Categorical Analysis of Logic*, North-Holland, Amsterdam.

[13] Gray, J.: 1971, 'The meeting of the Midwest Category Seminar in Zürich', in *Reports of the Midwest Category Seminar, V. Springer Lecture Notes in Mathem.* **195**, 248–255.

[14] Grayson, R.: 1981, 'Concepts of general topology in constructive mathematics and in sheaves', *Annals of Math. Logic* **20**, 1–41.

[15] Grothendieck, A. and J. L. Verdier: 1972, 'Théorie des Topos' (SGA 4, exposés I–VI) *Springer Lecture Notes in Math.* **269–270**.

[16] Heggie, M.: 1982, *An Approach to Synthetic Variational Theory*, M.Sc. thesis, McGill University.

[17] Johnstone, P. T.: 1977, *Topos Theory*, Academic Press, New York.

[18] Johnstone, P. T.: 1977, Rings, fields, and spectra', *J. of Algebra* **49**, 238–260.

[19] Joyal, A.: 1975, 'Les théorèmes de Chevalley-Tarski et remarques sur l'Algèbre constructive', *Cahiers de Top. et Géom. Diff.* **XVI**, 256–258.

[20] Kock, A. and G. E. Reyes: 1977, 'Doctrines in categorical logic', in J. Barwise (ed.), *Handbook of Mathematical Logic*, North-Holland, Amsterdam.

[21] Kock, A.: 1981, *Synthetic Differential Geometry*, London Mathem. Soc. Lecture Notes Series **51**.

[22] Lambek, J. and P. Scott: 1981, 'Independence of premisses and the free topos', *Proceedings of Symposium in Constructive Mathematics, Springer Lecture Notes in Math.* 873, 191–207.

[23] Lawvere, F. W.: 1979, 'Categorical dynamics', in *Topos Theoretic Methods in Geometry, Aarhus Math. Institute Var. Pub. Series No.* **30**.

[24] Lawvere, F. W.: 1976, 'Variable quantities and variable structures in topoi', in A. Heller and M. Tierney (eds.), *Algebra, Topology and Category Theory: a collection of papers in honor of Samuel Eilenberg*, Academic Press, New York, 101–131.

[25] Macintyre, A.: 1973, 'Model completeness and sheaves of structures', *Fund. Math.* **81**, 73–89.

[26] Maclane, S.: 1971, *Categories for the Working Mathematician*, Graduate Texts in Mathematics no. 5, Springer-Verlag.

[27] Makkai, M. and G. E. Reyes: 1977, *First Order Categorical Logic, Springer Lecture Notes in Math.* **611**.

[28] Penon, J.: 1981, 'Infinitésimeaux et intuitionisme', *Cahiers de Top. et Géom. Diff.* **22**, 67–72.

[29] Reyes, G. E.: 1974, 'From Sheaves to Logic', in A. Daigneault (ed.), *Studies in Algebraic Logic, Studies in Math.* 9, The Mathem. Assoc. of America.

[30] Scedrov, A.: 1981, 'Sheaves and forcing and their mathematical applications', Ph.D. Thesis, SUNY at Buffalo, to appear in *Memoirs Amer. Math. Soc.*

[31] Tierney, M.: 1973, 'Axiomatic sheaf theory: Some constructions and applications', *Proc. CIME Conference on Categories and Commutative Algebra, Varenna, 1971*, Edizioni Cremonese, 249–326.

[32] Wraith, G.: 1975, 'Lectures on Elementary Topoi', in *Model Theory and Topoi, Springer Lecture Notes in Math* **445**, 114–206.

Dept. of Mathematics and Statistics
McGill University
Montreal, Quebec H3A1GS
Canada