Local Concepts in Synthetic Differential Geometry and Germ Representability

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INTRODUCTION

Synthetic differential geometry (SDG) started by the consideration of an axiom of *jet representability*. This axiom is stated in a ringed elementary topos $(\mathcal{E},R)$. Briefly, it says that there is an object in $\mathcal{E}$, denoted $D_r(n)$, which is the domain of definition for the r-jets of functions in n variables $R^n \to R$. This axiom concerns the infinitesimal aspects of differential geometry. The aim of this paper is to consider a second axiom which establishes the capacity to pass from the infinitesimal to the local. This is an axiom of *germ representability*. Briefly, it says that there is an object in $\mathcal{E}$, denoted $\Delta(n)$, which is the domain of definition for germs of functions in n variables $R^n \to R$.

No specific acquaintance with the theory of SDG is presupposed here and the article is purely synthetic, in the sense that it is written in the internal language of an *elementary* topos. It has two parts. In the first we give an account of the representability of jets in SDG. In the second (main) part we consider local notions in a (ringed) topos and we treat the representability of germs in SDG. In both parts we analyze the basic problem of integration of vector fields. There is also an appendix of semantical content where Grothendieck topoi come into play and where we analyze the validity of the theory in the models. Comments, references and historical remarks are spread throughout the text.

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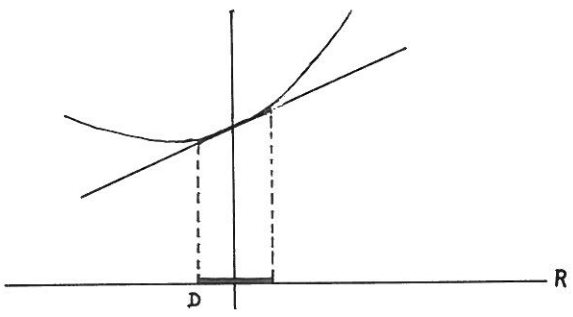
PART I. INFINITESIMAL CONCEPTS IN SDG

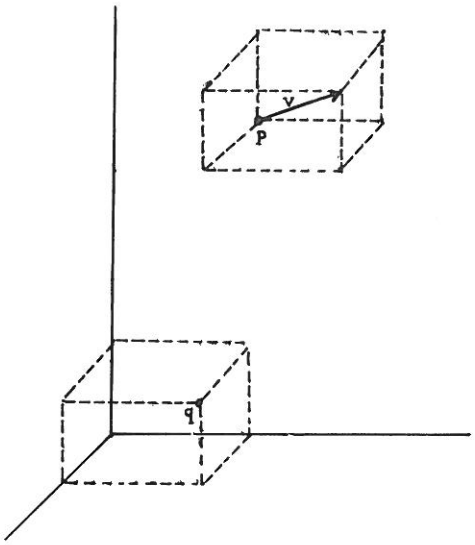
Jets in SDG

Fix a topos $\mathcal{E}$, a ring R in $\mathcal{E}$, called the *line*, and a subobject $D \subset R$, $0 \in D$. The object D is considered to be an *infinitesimal* neighbourhood of zero in the sense that the map

$$R \times R \xrightarrow{\alpha} R^D, \quad (a,b) \mapsto (d \mapsto a+db)$$

is invertible. Thus, if $f: D \to R$, then there is a unique $b \in R$ such that $f(d) = f(0)+bd$. We see then that D is so small that the graph of any function is part of a straight line (of slope b), but large enough so that this line is unique.





Here, f is defined in a part of R that contains D. Of course, b is said to be the value of the *derivative* of f at 0, b = f'(0). Thus, the restriction of f to D is completely characterized by the *1-jet* (at 0) of f, (f(0),f'(0)). Thus, the assumption which says that the map α above is invertible can be interpreted as an *axiom of representability of 1-jets.*

It follows immediately from this assumption that also the map

$$R^n \times R^n \xrightarrow{\alpha} R^{n^D}, \quad (p,q) \mapsto (d \mapsto p+dq)$$

is invertible. Thus, given a map $f:D \to R^n$, there is a unique $q \in R^n$ such that $f(d) = f(0)+dq$. Thus, f is characterized by a *vector* v with principal part q attached to the point p = f(0).

If $f = (f_1, f_2, \ldots, f_n)$, then $q = (f_1'(0), f_2'(0), \ldots, f_n'(0))$. Thus, given a curve in R^n parametrized by a function f defined on the line (or in a part that contains D), the restriction of f to D is completely characterized by the *velocity vector* of f (at time 0). This vector is a *tangent* vector (to the curve) attached to the point p, and lives in the tangent space of R^n at p. Thus, the assumption which says that the map α above is invertible means that the *tangent bundle is representable* by the object D.

All this goes back to a lecture given by F.W. Lawvere at the University of Chicago in 1967 (see [25]). The first successful utilization of such an object D of infinitesimals was the construction of the Lie bracket by G. Wraith (actually of the Lie algebra of a group [36]). It was shown therein how the synthetic arguing of the 19th century geometers can be carried over rigorously (this was later published in [33]). Previously, Wraith had studied the line in algebraic geometry (in the category of schemes), where the object D is determined by the ring structure and by means of the description $D = [\![x \in R \mid x^2 = 0]\!]$ (this is the spec of the ring of dual numbers), and had shown how much differential calculus stems from it, [35]. Influenced by these ideas, A. Kock started the real development of SDG in his two papers [19], [20]. There he states an axiom on a ring R from which all the calculus of derivatives and Taylor series follow. This axiom explicitly states that the isomorphism $R \times R \approx R^D$ is given by the invertibility of the map α considered above, and that $D = [\![x \in R \mid x^2 = 0]\!]$.

1. JET REPRESENTABILITY

Essentially, everything in this section is due to A.Kock and the standard reference is [22]. The Kock-Lawvere axiom, at the base of SDG, is today stated as a comprehensive axiom of representability for Weil bundles, which are a useful generalization of the jet bundles. In order to understand this axiom we shall see first how much we can get from the weaker versions, in particular from the one introduced in the previous considerations. We assume, although it is not always necessary, that the integers are invertible in R. Also, everytime we say *ring* we mean *R-algebra*, and every time we say *ring morphism* we mean *R-algebra morphism*.

Let $D \subset R$ be the object $D = [\![x \in R \mid x^2 = 0]\!]$. Consider the ring of polynomials $R[x]$ (in $\mathcal{E}$) divided by the ideal generated by x^2. We denote this ring by $R[D]$. It is the ring of dual numbers. Let $\epsilon \in R[D]$ correspond to $x \in R[x]$; then every element $\varphi \in R[D]$ is (in a unique way) of the form $\varphi(\epsilon) = a+b\epsilon$ with $a,b \in R$ and $\epsilon^2 = 0$.

1.1 Kock-Lawvere Axiom

The morphism of rings

$$R[D] \xrightarrow{\alpha} R^D, \quad \varphi(\epsilon) \mapsto (d \mapsto \varphi(d))$$

is invertible. (The reader should check how this is a morphism of rings).

Given $f \in R^R$ and $p \in R$, it follows that for all $d \in D$, $f(p+d) = f(p)+db$ for some unique $b \in R$. This defines a function, $f' \in R^R$, the *derivative* of f, which is characterized by the equation

(1) $$f(p+d) = f(p)+d\cdot f'(p), \quad \forall d \in D.$$

Immediate arithmetic calculations show (à la Fermat) that from (1) follows, for $f,g \in R^R$ and $a \in R$ that

(2) $$(f+g)' = f'+g', \ (af)' = af' \text{ and } (fg)' = f'g+fg'.$$

This, in turn, implies that

(3) the derivative of a polynomial coincides with the usual (formal) derivatives.

Suppose now that we know f only on the object D. Then $f'(0)$ is well determined, but not so $f'(t)$ for the others $t \in D$. This requires the knowledge of $f(t+d)$, but $t+d$ may not be in D. Thus, we cannot determine the value of the second derivative $f^{(2)}$ of f at 0. However, $(t+d)^3 = 0$, thus if we know f on the object $D_2 = [\![x \in R \mid x^3 = 0]\!]$, then $f^{(2)}(0)$ is well determined. More generally, let $D_r \subset R$ be the object $D_r = [\![x \in R \mid x^{r+1} = 0]\!]$, where r is any positive integer. Newton's binomial law yields

(4) $$(t \in D_{r-1} \text{ and } d \in D) \Longrightarrow (t+d) \in D_r.$$

This shows that the derivative of a function defined on D_r can be defined on D_{r-1}; that is, we have that

(5) there is a map $R^{D_r} \to R^{D_{r-1}}$, $f \mapsto f'$.

Let $U \subset R$ be such that $D_r \subset U$. From (5) it follows immediately that for all $f,g \in R^U$,

(6) $$f|_{D_r} = g|_{D_r} \Rightarrow f(0) = g(0) \text{ and } f'|_{D_{r-1}} = g'|_{D_{r-1}}.$$

Thus we conclude, for the iterated derivatives

(7) $$f|_{D_r} = g|_{D_r} \Rightarrow f^{(i)}(0) = g^{(i)}(0), \quad \forall i \leq r.$$

Given $f \in R^R$ and $p \in R$, the Taylor polynomial of f at p of degree r, $T_p^r f$, is defined by the formula

(8) $$T_p^r f(x) = \sum_{i=0}^{r} \frac{1}{i!} f^{(i)}(p)(x-p)^i.$$

(Notice that it will be enough to consider f defined on a part $U \subset R$ such that $p \in U$ and $p+D_r \subset U$).

From (3) and (8) we easily derive that

(9) if φ is a polynomial of degree $\leq r$, $T_p^r \varphi = \varphi$.

Our task is now to show Taylor's formula, namely, that the restriction of a function f to the object $p+D_r$ coincides with (and is completely determined by) $T_p^r f$. That is,

TAYLOR'S FORMULA. $f(p+d) = T_p^r f(p+d)$, $\forall d \in D_r$.

To prove this involves being able to *reverse the implication in (7) above.* Notice that the list of derivatives in (7) classically determine the *r-jet* of f at 0. Hence, a stronger axiom is necessary.

Let us be more explicit. We have seen that to determine f'(0), it is necessary to know f on the object D. On the other hand, if we have f(0) and f'(0), Axiom 1.1 says that we can recover f on D by the formula $f(d) = T_0^1 f(d) = f(0)+f'(0)d$. Thus, the information contained in the list f(0), f'(0) is equivalent to the information given by $f|_D$. We have seen also that to determine $f^{(2)}(0)$ it suffices to know the value of f on D_2. But this is not necessary. Actually, it was enough to know f on the smaller object $D+D \subset D_2$. On the other hand, if we have f(0), f'(0) and $f^{(2)}(0)$, it follows from Axiom 1.1 that $f(d) = T_0^2 f(d) = f(0)+df'(0)+d^2\tfrac{1}{2}f^{(2)}(0)$, for all $d \in D+D$. Thus, the list of numbers f(0), f'(0) and $f^{(2)}(0)$ is equivalent to $f|_{D+D}$. It

does not follow that $T_0^2 f = f$ in all of the larger object D_2, neither that $T_0^2 f$ is the only extension to D_2 of a function defined on D+D. In $f|_{D_2}$ there is more information. We next postulate *by the force of an Axiom*, that there is the same information on the list of numbers $f(0), f'(0), \ldots, f^{(r)}(0)$ that in $f|_{D_r}$, for each r. In particular, that given a function defined on D+D+...+D r times, there will be a unique extension to D_r.

Consider the ring of polynomials R[x] divided by the ideal generated by x^{r+1}. We denote this ring $R[D_r]$. Let $\epsilon \in R[D_r]$ correspond to $x \in R[x]$. Then, every element $\varphi \in R[D_r]$ is (in a unique way) of the form $\varphi(\epsilon) = a_0 + a_1\epsilon + \ldots + a_r\epsilon^r$, with $a_i \in R$ and where the symbol ϵ obeys the rule $\epsilon^{r+1} = 0$. In particular, $R[D_r] = R^{r+1}$ as an R-module. The restriction of a polynomial (considered as a function from R to R) to $D_r \subset R$ defines a ring morphism $R[x] \to R^{D_r}$ such that $x^{r+1} \mapsto 0$. This in turn defines the ring morphism in the following axiom.

1.2 Kock-Lawvere Axiom

For each $r \geq 1$, the ring morphism

$$R[D_r] \xrightarrow{\alpha} R^{D_r}, \quad \varphi(\epsilon) \mapsto (d \mapsto \varphi(d))$$

is invertible. This says that for any $f \in R^{D_r}$ there exists a unique polynomial φ of degree $\leq r$ such that $f(d) = \varphi(d)$ for all $d \in D_r$.

1.3 PROPOSITION

Given $U \subset R$ and $f \in R^U$, the Taylor's formula

$$f(p+d) = T_p^r f(p+d) = \sum_{i=0}^{r} d^i \cdot \left[\frac{1}{i!} f^{(i)}(p)\right], \quad \forall d \in D_r$$

holds for all $p \in U$ such that $p+D_r \subset U$.

PROOF. By Axiom 1.2 there is a polynomial φ such that $f(p+d) = \varphi(d)$ for all $d \in D_r$. Then, by (7) we have $f^{(i)}(p) = \varphi^{(i)}(0)$, $\forall i \leq r$. This shows that $T_p^r f(p+x) = T_0^r \varphi(x)$. But by (9), $T_0^r \varphi = \varphi$. This finishes the proof. Notice that only the surjectivity for $r > 1$ in Axiom 1.2 is needed. □

The ring of formal power series $R[\![x]\!]$ and the object D_∞ of all the nilpotent elements of R are associated to the calculus of the Taylor series of a function. We

denote $R[\![x]\!] = R[D_\infty]$. From the equation $\epsilon^{r+1} = 0$ for $\epsilon \in R[D_r]$, it follows that there is an (obvious) ring morphism $R[D_\infty] \to R[D_r]$. Similarly we obtain morphisms $R[D_r] \to R[D_s]$ for $s \leq r$. The ring $R[D_\infty]$ is the inverse limit of the diagram so determined. On the other hand, by definition

$$D \subset D_2 \subset \cdots \subset D_r \subset \cdots \subset D_\infty$$

and D_∞ is the union of this chain. Thus, R^{D_∞} is the inverse limit of the diagram determined by the restriction morphisms. The following diagram commutes by virtue of the definitions involved:

$$\begin{array}{ccccccc} R[D_\infty] & \to & R[D_r] & \to & R[D_s] & \to & R \\ \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \mathrm{id}, \\ R^{D_\infty} & \to & R^{D_r} & \to & R^{D_s} & \to & R \end{array}$$

for all $s < r$.

The first vertical arrow on the left is independently defined since it is always possible to evaluate a formal power series at a nilpotent element. All the other α's are invertible (by Axiom 1.2). Thus

1.4 PROPOSITION

The ring morphism

$$R[\![x]\!] = R[D_\infty] \xrightarrow{\alpha} R^{D_\infty}, \quad \varphi(x) \mapsto (d \mapsto \varphi(d))$$

is invertible. This says that for any $f \in R^{D_\infty}$ there exists a unique formal power series $\varphi(x)$ such that $\varphi(d) = f(d)$ for all $d \in D_\infty$. □

1.5 PROPOSITION

Given $U \subset R$ and $f \in R^U$,

$$f(p+d) = T_p^\infty f(p+d) = \sum_{i=0}^{\infty} d^i \cdot \left[\frac{1}{i!} f^{(i)}(p)\right], \quad \forall d \in D_\infty$$

holds for all $p \in U$ such that $p+D_\infty \subset U$.

PROOF. Since any $d \in D_\infty$ is in D_r for some r, the statement follows directly from Proposition 1.3. □

The partial derivatives for functions of several variables can be defined using only Axiom 1.1. Given a function $f(x,y)$, the partial derivatives are characterized

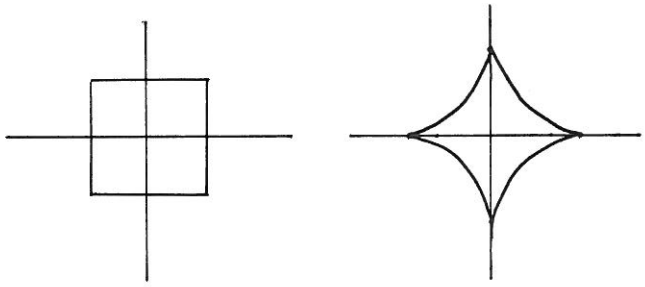
by the formulas

$$f(x+d,y) = f(x,y) + d\cdot\frac{\partial f}{\partial x}(x,y) \quad \forall d \in D$$

$$f(x,y+d) = f(x,y) + d\cdot\frac{\partial f}{\partial y}(x,y) \quad \forall d \in D.$$

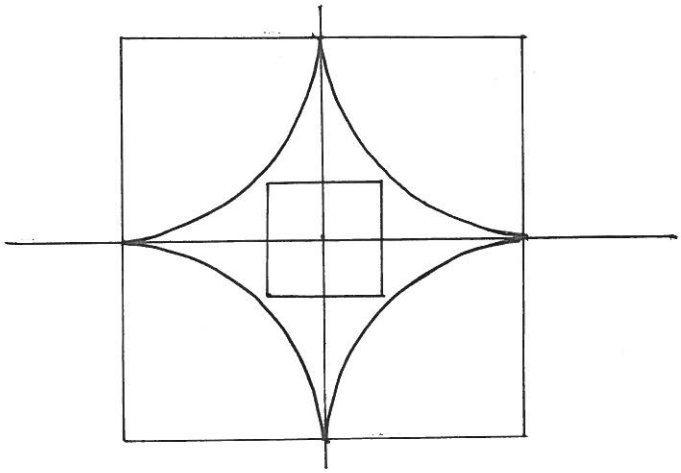
This says that the partial derivative is the derivative of the function of one variable determined by fixing the other.

Suppose that we know the function f on D×D, then we can compute $\frac{\partial f}{\partial x}(0,d)$ for all $d \in D$. Thus we can derive again and compute $\frac{\partial^2 f}{\partial y\,\partial x}(0,0)$. Of course, we can also compute $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$. To compute these last two numbers it is enough to know f in the smaller object $D(2) \subset D\times D$, $D(2) = [\![(x,y) \mid x^2=y^2=xy=0]\!]$, since D(2) contains the D-segment of the axes.



D×D D(2)

Suppose now that we want to compute the whole 2-jet of f, that is, the list of six numbers $f(0)$, $\frac{\partial f}{\partial x}(0)$, $\frac{\partial f}{\partial y}(0)$, $\frac{\partial^2 f}{\partial x\,\partial y}(0)$, $\frac{\partial^2 f}{\partial x^2}(0)$ and $\frac{\partial^2 f}{\partial y^2}(0)$ (where we wrote 0 in place of (0,0)) determined by all the derivatives of order ≤ 2. Clearly, $D_2\times D_2$ will be enough, but it will also contain the necessary information to compute many more higher derivatives, for example $\frac{\partial^4 f}{\partial x^2\,\partial y^2}(0)$. The reader can check that to compute the 2-jet it is enough to know the value of f on the smaller object $D_2(2) \subset D_2\times D_2$, $D_2(2) = [\![(x,y) \mid x^3=y^3=x^2y=y^2x=0]\!]$, as it can be seen in the picture, where we have displayed $D\times D \subset D_2(2) \subset D_2\times D_2$.



The object $D_2 \times D_2$ also contains $D_2 \times D$. This implies that we can compute the number $\frac{\partial^3 f}{\partial x^2 \partial y}(0)$, etc. It will follow that the lists of the values at 0 of the nine derivatives (including f) with indices ≤ 2 completely determines f in $D_2 \times D_2$, and no more. However, a new axiom (Axiom 1.1.1) is necessary in order to have the analogous situation for the 2-jet (i.e., the list of six derivatives of order ≤ 2) of f at 0 and the object $D_2(2)$. This is not surprising, since to determine the 2-jet of f at 0 it is enough to know f in the object $\{0\} \times D_2 \cup D_2 \times \{0\} \cup D \times D$, as the reader can easily verify. This object, the cross of length D_2 plus the square $D \times D$, is *strictly* contained in $D_2(2)$. We will show briefly now how all this is done in general.

Consider two symbols X,Y and the R-algebra of polynomials R[X,Y] divided by the ideal generated by X^{k+1}, Y^{s+1} (for k,s any integer ≥ 0). We denote this algebra by $R[D_k \times D_s]$. Then, just as in the case of one variable, an element $\varphi \in R[D_k \times D_s]$ is a polynomial $\varphi(\epsilon,\delta)$ on the symbols ϵ,δ which obey the rules $\epsilon^{k+1} = 0$, $\delta^{s+1} = 0$. Thus, $R[D_k \times D_s] \approx R^{(k+1)(s+1)}$ as an R-module. Clearly, there is a morphism of R-algebras:

$$R[D_k \times D_s] \xrightarrow{\beta} R^{D_k \times D_s}, \quad \varphi(\epsilon,\delta) \mapsto ((d,t) \mapsto \varphi(d,t)).$$

That β is invertible follows by the commutativity of the following diagram

$$\begin{array}{ccc} R[D_k \times D_s] \approx R[D_k][D_s] & \overset{\alpha[D_s]}{\rightarrow} & R^{D_k}[D_s] \\ \downarrow \beta & & \downarrow \approx \\ R^{D_k \times D_s} \approx (R^{D_s})^{D_k} & \underset{\alpha}{\leftarrow} & R[D_s]^{D_k} \end{array}$$

The vertical arrow on the right is the identity map: elements in both objects are expressions $f_0+f_1\delta+\cdots+f_s\delta^s$ with $f_i \in R^{D_k}$.

We have then

1.6 PROPOSITION

The map β above is invertible. That is, for any $f \in R^{D_k \times D_s}$ there exists a unique polynomial φ of the form $\varphi(x,y) = \sum_{i \leq k, j \leq s} a_{ij} x^i y^i$ such that $\varphi(d,t) = f(d,t)$ $\forall d \in D_k$, $t \in D_s$. □

As before, we have to show now that this polynomial is the Taylor polynomial defined in terms of the *iterated partial derivatives.* Fix an arbitrary order of iteration. Utilizing this order, the Taylor polynomial is defined in the usual way. Then, as before, it follows that

(a) to determine all the numbers $\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0)$, $\forall i \leq k, j \leq s$, it suffices that the function f be defined on $D_k \times D_s$. Moreover, given any two functions f,g,

$$f|_{D_k \times D_s} = g|_{D_k \times D_s} \Rightarrow \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0) = \frac{\partial^{i+j} g}{\partial x^i \partial y^j}(0), \ \forall i \leq k, j \leq s$$

for any given order of iteration and that

(b) if φ is a polynomial, then it coincides with a sufficiently long Taylor development.

Then, the same argument as the one in Proposition 1.3 shows

1.7 PROPOSITION

Given $U \subset R \times R$ and $f \in R^U$, the formula

$$f(p+d,q+t) = f(p,q) + \sum_{i+j \neq 0}^{k\ s} d^i \cdot t^j \cdot \frac{1}{i!j!} \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(p,q), \quad (d,t) \in D_k \times D_s$$

holds for all $(p,q) \in U$ such that $(p,q)+D_k \times D_s \subset U$. □

In particular, the implication in (a) can be reversed.

1.8 COROLLARY

Iterated derivatives are independent of the order of iteration.

PROOF. The formula in 1.6 holds for any given order of iteration. Thus, the assertion follows from 1.6. □

It is clear from the proofs presented here that all the respective statements in the case of any number of variables hold. Also, the reader can easily check that the proofs of 1.6 and 1.7 can be carried over in the case $r+s = \infty$. Thus, let $R[\![x_1,\ldots,x_n]\!] = R[D_\infty^n]$. Then

1.9 PROPOSITION

The ring morphism

$$R[D_\infty^n] \to R^{D_\infty^n}, \quad \varphi(\bar{x}) \mapsto (\bar{d} \mapsto \varphi(\bar{d}))$$

where $\bar{x} = (x_1,\ldots,x_n)$ and $\bar{d} = (d_1,\ldots,d_n)$, is invertible. This says that for any $f \in R^{D_\infty^n}$, there exists a unique formal power series $\varphi(x)$ such that $\varphi(\bar{d}) = f(\bar{d})$ $\forall \bar{d} \in D_\infty^n$. □

Let us pass now to the problem of finite jets. The r-jet of f involves the list of derivatives up to order r. To this list is associated the object $D_r(n) \subset R^n$, where

$$D_r(n) = [\![\bar{x} = (x_1,\ldots,x_n) \in R^n \mid \text{the product of any } r+1 \text{ of the } x_1,\ldots,x_n \text{ is } 0]\!].$$

For example, $D_1(2) = D(2) = [\![(x,y) \in R^2 \mid x^2=y^2=xy=0]\!]$. Clearly, $D_r(n) \subset D_r^n$ and $D_k(n)\times D_s(m) \subset D_{k+s}(n+m)$. To simplify, we write the above in the case of two variables. Since $D_k \times D_s \subset D_r(2)$ for all k,s such that $k+s \leq r$, it follows from (2) above (here D_0 means $\{0\}$), that

(c) to determine the list of derivatives of f at 0 up to order r, it suffices that f be defined on $D_r(2)$. Moreover, given any two functions f,g,

$$f|_{D_r(2)} = g|_{D_r(2)} \Rightarrow \frac{\partial^{i+j} f}{\partial x^i\, \partial y^j}(0) = \frac{\partial^{i+j} g}{\partial x^i\, \partial y^j}(0), \ \forall i+j \leq r.$$

A new axiom will be necessary to reverse this implication.

Consider the ring of polynomials $R[x_1,\ldots,x_n]$ divided by the ideal generated by all products of r+1 of the symbols $x_1,\ldots,x_n$. We denote this ring $R[D_r(n)]$. Let ϵ_i correspond to x_i. Then every element $\varphi \in R[D_r(n)]$ is (in a unique way) a polynomial expression $\varphi(\epsilon_1,\ldots,\epsilon_n)$ on the symbols ϵ_i. These symbols obey the rule that says that the product of r+1 of them is 0. The restriction of a polynomial (considered as a function from R^n to R) to $D_r(n)$ defines a ring morphism $R[x_1,\ldots,x_n] \to R^{D_r(n)}$ such that the product of any r+1 of the x_i's goes to 0. This in turn defines a ring morphism

$$R[D_r(n)] \xrightarrow{\alpha} R^{D_r(n)}, \quad \varphi(\bar{\epsilon}) \mapsto (\bar{d} \mapsto \varphi(\bar{d})),$$

where $\bar{\epsilon} = (\epsilon_1,\ldots,\epsilon_n)$ and $\bar{d} = (d_1,\ldots,d_n)$. The representability of r-jets concerns the invertibility of this map. That this map is invertible cannot be demonstrated from the Axiom 1.2. Thus, we strengthen the axiom as follows:

1.10 Kock-Lawvere Axiom

For each $r \geq 0$, $n \geq 0$, the ring morphism $R[D_r(n)] \xrightarrow{\alpha} R^{D_r(n)}$, $\varphi(\bar{\epsilon}) \mapsto (\bar{d} \mapsto \varphi(\bar{d}))$ is invertible. This says that for any $f \in R^{D_r(n)}$ there is a unique polynomial φ of (total) degree $\leq r$ such that $f(\bar{d}) = \varphi(\bar{d})$ for all $\bar{d} \in D_r(n)$.

If f is a function in n variables and $p \in R^n$, let $T^r_p f$ be the Taylor polynomial at p of f of order r. Then, from proposition 1.7 and Axiom 1.10 follows (recall also (c) above) the Proposition below.

1.11 PROPOSITION

Given $U \subset R^n$ and $f \in R^U$, then

$$f(p+\bar{d}) = T^r_p f(p+\bar{d}), \ \forall d = (d_1,\ldots,d_n) \in D_r(n)$$

holds for all $p \in U$ such that $p+D_r(n) \subset U$. □

In particular, the implication in (c) above can be reversed. We can see 1.2, 1.3 and 1.10, 1.11 in the following way.

To simplify the notation we write only the case of 2 variables. The general case is a straightforward generalization. (Recall $D_0 = \{0\}$ and $D_2 = D$.)

1.12 PROPOSITION

(1) $D+D+\cdots+D$ (r times) $\subset D_r$ and every function $(D+D+\ldots+D) \to R$ has a unique extension to D_r, for all $r \geq 0$. The values of f on $D+D+\ldots+D$ determine the list of derivatives up to order r of f at 0. The extension is then given by Taylor's formula.

(2) $D_k \times D_s \subset D_r(2)$, for all k,s such that $k+s \leq r$, and every function $D_k \times D_s \to R$ defined in all of the subobjects $D_k \times D_s$ (this means a family of functions which coincide in the intersections) has a unique extension to a function $D_r(2) \to R$. The values of f on the $D_k \times D_s$ determine the list of derivatives up to order r at 0. Then, the extension is given by Taylor's formula. □

This establishes the representability of r-jets in SDG. Classically, to determine the r-jet of a function f at a point p it is necessary to know the function on an *open* neighbourhood around p, in order to compute the derivatives. Here, it is enough to know the function on the infinitesimal neighbourhood $p+D_r(n)$. Moreover, this list of derivatives up to order r of f at p, *the r-jet of p*, completely determines the values of f on this infinitesimal neighbourhood by Taylor's formula.

2. WEIL ALGEBRAS

We have considered so far the family of infinitesimal objects $D_r(n)$. These objects could be called Ehresmann infinitesimal objects, since it was he who introduced the jet notion in differential geometry [15]. The product $D_r(n) \times D_s(m)$ is an infinitesimal object which is not anymore in the family. This means that an iterated jet bundle is not a jet bundle in the sense of Ehresmann. To remedy this, A. Weil in [34] introduced a class of algebras and developed a corresponding theory of jets that generalizes the work of Ehresmann for the algebras $R[D_r(n)]$. Weil algebras appear in SDG in [33], and their systematic use started in [7], where Axiom I in 2.3 is first found expressed in terms of the clasical theory of Weil bundles. In its present form, it appears in [21]. A Weil algebra is essentially a multiplication table on a finite dimensional module. This is coded by a matrix with

coefficients in the ground ring. The entries of this matrix are always *rational* in the applications. To consider Weil algebras over $\mathbb{Q}$ has many advantages. $\mathbb{Q}$ is a field, is a constant sheaf and it is intrinsically defined in any topos.

2.1 DEFINITION [34]

A Weil algebra is an algebra W over $\mathbb{Q}$ equipped with a morphism $W \xrightarrow{\pi} \mathbb{Q}$ such that

1) it is local with maximal ideal $I = \pi^{-1}(0)$;

2) it is a finite dimensional $\mathbb{Q}$-vector space;

3) I is a nilpotent ideal.

There is a unique integer ℓ such that $W \approx \mathbb{Q}^{\ell+1}$. If $e_0, e_1, \ldots, e_\ell$ is a linear base, $e_i \cdot e_j = \sum_{k=0}^{\ell} \gamma_{ij}^k e_k$, in a unique way. This matrix with rational coefficients γ_{ij}^k completely determines the algebra W. It is clear that given any $\mathbb{Q}$-algebra A in any topos $\mathcal{E}$, (actually it would suffice to assume $\mathcal{E}$ to be a category with finite limits), the matrix γ_{ij}^k also determines a A-algebra structure in $A^{\ell+1}$. We denote this algebra $A[W] \approx A^{\ell+1}$. The equations above are a particular presentation for W. More generally, given any presentation $h_i(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ of W, then A[W] is the A-algebra with presentation $h_i(x_1, \ldots, x_n) \in A[x_1, \ldots, x_n]$. A[W] is constructed from A following the same instructions utilized to construct W from $\mathbb{Q}$. It is well established that different presentations of the same W yield the same A[W], and that this process is functorial.

We will prove next a very useful lemma. First, we consider an example: look at the diagram

$$\mathbb{Q}[\theta] \xrightarrow{\varphi} \mathbb{Q}[\epsilon,\delta] \underset{\sigma}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{Q}[\epsilon,\delta], \text{ where}$$

$$\theta^3 = 0, \quad \epsilon^2 = \delta^2 = 0$$

$$\varphi(\theta) = \epsilon+\delta, \quad \sigma(\epsilon) = \delta, \quad \sigma(\delta) = \epsilon.$$

An element of $\mathbb{Q}[\epsilon,\delta]$ is of the form $\xi = a+b\epsilon+c\delta+d\epsilon\delta$. Then, $\mathrm{id}(\xi) = \sigma(\xi)$ if and

only if $b = c$. In this case, the element $\gamma = a+b\theta+\frac{1}{2}d\theta^2$ is such that $\varphi(\gamma) = \xi$, and is the unique such. This shows that the diagram above is an equalizer. We see that also the diagram

$$A[\theta] \xrightarrow{\varphi} A[\epsilon,\delta] \underset{\sigma}{\overset{id}{\rightrightarrows}} A[\epsilon,\delta]$$

will be an equalizer for any $\mathbb{Q}$-algebra A. This is so because there is a construction to obtain the unique element γ such that $\varphi(\gamma) = \xi$, and this construction can be performed in any $\mathbb{Q}$-algebra. We have

2.2 LEMMA (Kock-Lavendhomme, [23])

If $W \to W_i$ is any finite inverse limit of Weil algebras, then so is $A[W] \to A[W_i]$.

PROOF. We should prove two cases. Equalizers and finite products. Consider first an equalizer

$$\begin{array}{ccccc} W & \xrightarrow{h} & H & \underset{g}{\overset{f}{\rightrightarrows}} & S \\ \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\ \mathbb{Q}^{r+1} & \underset{}{\overset{\ell}{\leftrightarrows}} & \mathbb{Q}^{k+1} & \rightrightarrows & \mathbb{Q}^{s+1} \end{array} .$$

The morphisms h,f,g are in particular linear maps between the *vector spaces* in the bottom row. Thus, $\mathbb{Q}^{r+1} \approx \text{kernel}(f-g)$. It follows that there is a linear map ℓ as shown in the diagram, such that $\ell h = id$ and $h\ell = id|_{\text{kernel}(f-g)}$. This map ℓ is a matrix with rational coefficients encoding the information to construct an element $\gamma \in W$ from an element $\xi \in H$, and such that if $f(\xi) = g(\xi)$, then $h(\gamma) = \xi$. This construction is a (matrix)×(vector) multiplication and will yield an element $\gamma \in A[W]$ from any $\xi \in A[H]$ for any $\mathbb{Q}$-algebra A. In the example considered above, it is the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$. A product of Weil algebras is a fiber product over $\mathbb{Q}$; this case is very easy and we leave it to the reader. □

We return now to the ring R. Recall that we had assumed R to be a $\mathbb{Q}$-algebra in order to develop the usual formulas in the calculus of Taylor polynomials. Given

any Weil algebra W with presentation $h_i(x_1,\ldots,x_n)$, we denote by $\bar{W} \subset R^n$ the zero set of the h_i's in R^n, i.e.

$$\bar{W} = [\![(x_1,\ldots,x_n) \in R_n \mid h_i(x_1,\ldots,x_n) = 0 \quad \forall i]\!]$$

Remark that this is actually a particular embedding of $\bar{W}$ in R^n corresponding to the presentation of W. The object $\bar{W}$ is sometimes denoted $\mathrm{Spec}_R(W)$, and it is well established that it defines a (contravariant) functor on W.

The restriction of a polynomial $\varphi \in R[x_1,\ldots,x_n]$ (considered as a function from R^n to A) to $\bar{W}$ defines a ring morphism $R[x_1,\ldots,x_n] \to R^{\bar{W}}$ such that $h_i \mapsto 0$. Thus, it factors through the quotient of $R[x_1,\ldots,x_n]$ by the ideal generated by the h_i's, which is by definition the ring $R[W] \approx R^{D_{q+1}}$. This defined the morphism α in the general form of the first basic axiom in SDG.

2.3 AXIOM I (Kock-Lawvere Axiom)

Given any Weil algebra W, the ring morphism

$$R[W] \xrightarrow{\alpha} R^{\bar{W}}, \quad \xi \mapsto (p \mapsto \varphi(p))$$

is invertible, where $p \in \bar{W}$ and φ is any polynomial that represents $\xi \in R[W]$.

To understand the geometric object $\bar{W}$ as well as the fact that this axiom is an axiom of (generalized) jet representability, the following considerations are useful.

Given a Weil algebra W ($W \approx D^{q+1}$) with maximal ideal I, the least integer r such that $I^{r+1} = 0$ is called the *order* of W, and the dimension of the vector space I/I^2 is called the (geometric) dimension of $\bar{W}$. Any set of elements $\epsilon_1,\ldots,\epsilon_n$ in I such that their images form a *base* of I/I^2 generate W, and any other presentation needs at least n elements (the reader can verify this by hand, or remark that it follows from Nakayama's lemma). Thus, R^n is the space of smallest dimension in which $\bar{W}$ can be embedded (this is the Krull dimension of W). Since every $\epsilon_i^{r+1} = 0$, there is a quotient map from the ring of formal power series

$$D[[x_1,\ldots,x_n]] \xrightarrow{q} W, \quad x_i \mapsto \epsilon_i$$

and from the equation $I^{r+1} = 0$ follows immediately that $\mathrm{Ker}(q) \subset J^{r+1}$, where J is

the maximal ideal in $\mathbb{Q}[[x_1,\ldots,x_n]]$ generated by the x_i's. That is, J^{r+1} is the ideal generated by all products of r+1 of the symbols $x_1,\ldots,x_n$. This is the ideal presentation of $R[D_r(n)]$. That is, $\bar{W} \subset D_r(n)$. Given a function in n variables, we have seen that $D_r(n)$ is associated to the list of all the derivatives of f up to order r. The object $\bar{W}$ will be associated to some subset of this list. Indeed, let α_i stand for a non-negative integer. Then the set of products $\{\epsilon_1^{\alpha_1}\ \epsilon_2^{\alpha_2}\ \ldots\ \epsilon_n^{\alpha_n} |$ $\alpha_1+\alpha_2+\ldots+\alpha_n \leq r\}$ (where $\epsilon_i^0 = 1$) is a set of linear generators for W (since the ϵ_i's generate, and, of course, a polynomial is nothing but a linear combination of monomials). We select a subset Γ_W of n-tuples $(\alpha_1,\ldots,\alpha_n)$ such that the set $\{\epsilon_1^{\alpha_1}\ \epsilon_2^{\alpha_2}\ \ldots\ \epsilon_n^{\alpha_n} |\ (\alpha_1,\ldots,\alpha_n) \in \Gamma_W\}$ is a linear base of W (this base contains 1 and all the ϵ_i's). Given any $\xi \in W$, there is a *unique* polynomial $f(x_1,\ldots,x_n)$ of the form prescribed by Γ_W that represents x. Consider the list of $\mathbb{Q}+1$ derivatives $\partial/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$ with $(\alpha_1,\ldots,\alpha_n) \in \Gamma_W$. Axiom I says (via Taylor's formula) that the restriction of a function f in n variables to $\bar{W}$, $f|_{\bar{W}}$, is equivalent to the list of values at 0 of these $\mathbb{Q}+1$ derivatives. The object $\bar{W} \subset R^n$ is an infinitesimal neighbourhood of 0, where there is (exactly) enough room to compute this $\mathbb{Q}+1$ derivatives at 0. The vanishing of these derivatives at 0 is equivalent to the vanishing of the function in $\bar{W}$. All the first derivatives $\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}$ are in the list (because the ϵ_i's belong to the base), so $\bar{W}$ contains $D_1(n)$. Thus, we always have $D_1(n) \subset \bar{W} \subset D_r(n)$. We remark that this part of the r-jet is not associated to W, but to a presentation of W with special characteristics; that is, to a special embedding of $\bar{W}$ in R^n. An arbitrary presentation of a Weil algebra W of (geometric) dimension n will in general correspond to an embedding of $\bar{W}$ into a space of higher dimension. For example, let W be

$$W = Q[\epsilon,\delta] \quad \text{with} \quad \epsilon^2 = 0,\ \epsilon-\delta = 0.$$

Then $\bar{W} \rightarrowtail R^2$ is the diagonal of D×D. The vanishing of a function f(x,y) on $\bar{W}$ is equivalent to $f(0) = 0$ and $\frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial y}(0) = 0$. That is, in this case, W is

associated to the (linear with constant coefficients) differential operator $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

3. INFINITESIMAL LINEARITY AND INTEGRATION OF VECTOR FIELDS

Having established (actually by the force of an axiom) that the objects $D_r(n)$ represent the r-jets of functions from R^n to R, it is justified then to *define* the r-jets to be maps from $D_r(n)$ to R, and to say that the r-jet $J^r f$ at $p \in R^n$ of a function $R^n \xrightarrow{f} R$ is its restriction to $p+D_r(n)$. But this definition is clearly available for any object $M \in \mathcal{E}$. In this way, we define the *jet bundles* of M to be

$$M^{D_r(n)} \to M, \quad \xi \mapsto \xi(0),$$

and in particular, the *tangent bundle* to be

$$M^D \to M, \quad \xi \mapsto \xi(0).$$

These bundles will not in general have the properties (that hold when $M = R^m$) required to develop differential geometry in the usual manner. However, a large part of the theory can be developed for objects much more general than R^m; these are the *infinitesimally linear* objects. Before considering these general objects, let us study the vector fields in R^n. We shall write M for R^m. A vector field is a section of the tangent bundle

$$M \xrightarrow{\xi} M^D, \quad \xi(p)(0) = p, \ \forall p \in M.$$

This is equivalent to a map (that we write also ξ)

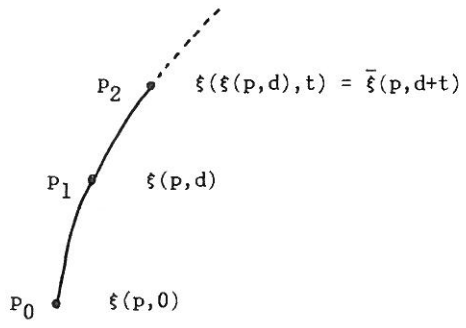
$$M \times D \xrightarrow{\xi} M, \quad \xi(p,0) = p, \ \forall p \in M. \tag{1}$$

It is easy to check that this map is a D-flow. By this we mean that the following equation is satisfied:

$$\xi(p,d+t) = \xi(\xi(p,d),t), \quad \forall d,t \in D \mid d+t \in D. \tag{2}$$

Indeed, $d+t \in D$ iff $(d,t) \in D(2)$. So, for each $p \in M$ both sides in (2) are maps $D(2) \to M$. That is, they are 1-jets in two variables. They will be equal provided they have the same partial derivatives at 0. But this is clear since from (1) it immediately follows that they coincide on the axes. (Actually, we have m-tuples and the argument is done in each coordinate.) The problem of *integration* of vector

fields is the problem of *extension* of the corresponding infinitesimal flow on D. Suppose we are at a point $p = p_0$. The field tells us how to move during an infinitesimal time $d \in D$ to reach the point $p_1 = \xi(p_0,d)$. But there, the field also tells us how to move again for an infinitesimal time $t \in D$. So we reach the point $p_2 = \xi(p_1,t) = \xi(\xi(p_0,d),t)$ in the time $d+t \in D_2$. And so on...



If there is an extension of $\xi: M\times D \to M$ into a flow $\bar{\xi}$ defined on $M\times D_2 \to M$, then the flow equation tells us that $p_2 = \bar{\xi}(p,d+t)$. Actually, it is immediate to see that such an extension exists and that it is unique. For each $p \in R^n$ there is a map:

(3) $$d+t \mapsto \xi(\xi(p,d),t), \quad \text{defined } \forall d,t \in D.$$

But this is the data of a 2-jet in one variable (we can actually compute the first and second derivative at 0). Therefore it uniquely determines a map $\bar{\xi}(p,-): D_2 \to M$ such that

(4) $$\bar{\xi}(p,d+t) = \xi(\xi(p,d),t), \quad \forall d,t \in D.$$

We have to check the flow equation for $\bar{\xi}$. That is,

(5) $$\bar{\xi}(p,d+t) = \bar{\xi}(\bar{\xi}(p,d),t), \quad \forall d,t \in D_2 \mid \ (d,t) \in D_2(2)$$

(if $(d,t) \in D_2(2)$, $(d+t)^3 = d^3+3d^2t+3dt^2+t^3 = 0$, thus $t+d \in D_2$). As before, both sides in (5) are maps $D_2(2) \to M$; that is, they are 2-jets in two variables. They will be equal provided they have the same derivatives of order 2 at 0. From (4) we see that $\bar{\xi}(p,0) = p$. It follows that they coincide on the axes. Thus $\dfrac{\partial^2}{\partial x^2}$ and $\dfrac{\partial^2}{\partial y^2}$ are equal at 0. But (4) also implies that they coincide on $D\times D$. So $\dfrac{\partial}{\partial x}$, $\dfrac{\partial}{\partial y}$ and

$\frac{\partial^2}{\partial x \partial y}$ are in turn equal at 0. And so we can go on. An induction can actually be performed with all rigor. To do this the reader should utilize Proposition 1.12. In this way a D-flow can be extended into a unique D_r-flow for each positive integer r (actually, into a map on D_r that satisfies the flow equation on $D_r(2)$). Thus, it follows that there is a unique extension into a flow on D_∞. Recall that $D_\infty(2) = D_\infty \times D_\infty$. We have

3.1 PROPOSITION

Given any vector field $M \xrightarrow{\xi} M^D$, the associated D-flow $M\times D \xrightarrow{\xi} M$ has a unique extension into a D_∞-flow $M\times D \xrightarrow{\varphi} M$. That is,

i) $\varphi(p,d) = \xi(p,d)$, $\forall d \in D$

ii) $\varphi(p,t+d) = \varphi(\varphi(p,t),d)$, $\forall t,d \in D_\infty$. □

3.2 DEFINITION

A flow in an object M is a family of curves, one for each point $p \in M$. The curve $f(p,-)$ is defined in some part of R that contains D and passes by p at time 0. This means a map

$$U \xrightarrow{f} M, \quad M\times D \subset U \subset M\times R, \quad f(p,0) = p.$$

In addition, the following equation should be satisfied:

(a) $f(p,t+s) = f(f(p,t),s)$, for all p,t,s such that both sides in the equality are defined.

It is an integral flow of a field $M \xrightarrow{\xi} M^D$ if the *velocity vectors* are the field vectors. That is, if

(b) $f(p,t+d) = \xi(f(p,t),d)$, $\forall (p,t) \in M\times R$ such that $(p,t+d) \in U$, $\forall d \in D$.

In the case $M = R^m$, it follows from Axiom 1.1 that the field ξ is of the form

(c) $\xi(p,d) = p+d\cdot g(p)$ with $M \xrightarrow{g} R^m$.

It follows then that to have an integral flow for ξ means to have a solution (on U) for the differential equation

(d) $\frac{df}{dt}(p,t) = g(f(p,t))$, $\forall (p,t) \in M\times R \mid (p,t+d) \in U \ \ \forall d \in D$

with initial value $f(p,0) = p$. This is so since by Axiom 1.1, in equation (b) we have $f(p,t+d) = f(p,t) + d\cdot\frac{df}{dt}(p,t)$, and $\xi(f(p,t),d) = f(p,t)+d\cdot g(f(p,t))$. □

Thus, (almost) by definition, a vector field is already integrated into D, and Proposition 3.1 says that it can always be integrated into D_∞ (when $M = R^m$). It follows that in this case there is a solution for the differential equation (d) on $U = M \times D_\infty$. Since a map $M \times D_\infty \to M$ is equivalent to a map $M \to M^{D_\infty}$, and for $M = R^m$ we have $M^{D_\infty} = (R[[x]])^m$ (Proposition 1.4), we see that for each p there is a (m-tuple of) formal power series $\varphi_p(x)$ that satisfies the differential equation

$$\varphi_p' = g(\varphi_p) \quad \text{with initial value} \quad \varphi_p(0) = p.$$

Thus, it is always possible to integrate a differential equation (on $M = R^n$) by formal power series. This is the meaning of proposition 3.1 expressed in classical terms.

We pass now to the notion of infinitesimal linearity. Given an object M, the r-jet bundle of M is *defined* to be $M^{D_r(n)} \to M$. We have seen that in the case of $M = R^m$, this definition is *justified* since a map $D_r(n) \xrightarrow{f} M$ is completely determined by the list of derivatives up to order r of f at 0. Of course, for a general object M there is no notion of derivative. But imagine for a moment that we have a notion of derivatives for functions with values in M, and that partial derivatives are defined by the method of fixing one variable and then taking derivative with respect to the other. In this case, for example, the 1-jet of a function of two variables is the list of three values, $f(0)$, $\frac{\partial f}{\partial x}(0)$ and $\frac{\partial f}{\partial y}(0)$. To determine these values it is enough to know the function f on the infinitesimal axes, that is, in the cross $D \times D_0 \cup D_0 \times D$. But the 1-jet should be a map $D(2) \to M$. This was decided to be the case for $M = R$ by the force of an axiom (Axiom 1.10). If this is the case for an object M, it establishes a property of M. This is the property of *infinitesimal linearity*. Now, how can we formulate this property *without* the derivatives? The answer is clear; we simply say that every map from the cross $D \times D_0 \cup D_0 \times D \subset D(2)$ has a unique extension into D(2). Consider the diagrams

$$\begin{array}{ccc} 0 & \to & D\times D_0 \\ \downarrow & & \downarrow \\ D_0\times D & \hookrightarrow & D(2) \end{array} \qquad , \qquad \begin{array}{ccc} M^{D(2)} & \to & M^{D\times D_0} \\ \downarrow & & \downarrow \\ M^{D_0\times D} & \to & M \end{array}$$

where in the right hand side, the maps are the restriction maps. Then, obviously, infinitesimal linearity means that this diagram is a pull-back. The diagram on the left hand side corresponds via the functor Spec_R (that is, $W \to \overline{W}$) to the following diagram of Weil algebras

$$\begin{array}{ccc} \mathbb{Q}[\epsilon,\delta] & \to & \mathbb{Q}[\alpha,\beta] \\ \downarrow & & \downarrow \\ \mathbb{Q}[\eta,\mu] & \to & \mathbb{Q} \end{array} \qquad \begin{array}{l} \epsilon^2 = 0, \quad \delta^2 = 0, \quad \epsilon\delta = 0 \\ \alpha = 0, \quad \beta^2 = 0 \\ \eta^2 = 0, \quad \mu = 0 \end{array}$$

which is easily seen to be a pull-back diagram. In this way, the property of infinitesimal linearity is conveniently stated in categorical terms by saying that the combined action of the functors $\mathrm{Spec}_R = \overline{(-)}$, and exponentiation with M, $M^{(-)}$, carries this last pull-back diagram of Weil algebras into a pull-back diagram in $\mathcal{E}$. Similarly, for the general case of $D_r(2)$, M will be *infinitesimally linear* if r-jets of functions of two variables with values in M behave as if they were *the list of derivatives up to order r at 0.* From Proposition 1.13(2) the reader can easily see how to formulate this. Furthermore, it is straightforward to figure out then the corresponding inverse limit diagram of Weil algebras. One more case has to be considered. This corresponds to the fact that higher derivatives are defined by iteration. For example, the first case in 1.12(1) says that if M is infinitesimally linear then every map $D+D \to M$ should have a unique extension into D_2. This can also be stated in terms of finite inverse limits of Weil algebras. Indeed, the equalizer considered in the remarks previous to lemma 2.2 furnishes the solution. To this equalizer it corresponds, via the functor $\mathrm{Spec}_R = \overline{(-)}$, the diagram in $\mathcal{E}$

$$D\times D \underset{g}{\overset{id}{\rightrightarrows}} D\times D \xrightarrow{+} D_2, \quad \sigma(d,t) = (t,d).$$

In turn, taking the exponential with M yields

$$M^{D_2} \to M^{D\times D} \rightrightarrows M^{D\times D}$$

where the arrows are the restriction maps. To say that this is an equalizer diagram in $\mathcal{E}$ means:

(a) for every map $D\times D \xrightarrow{\tau} M$ such that $\tau(t,d) = \tau(d,t)$ for all $(t,d) \in D\times D$, there exists a unique map $D_2 \xrightarrow{\xi} M$ such that $\xi(t+d) = \tau(t,d)$.

From (a) it is immediate to see that every map $D+D \to M$ has a unique extension into D_2. The general case $D+D+\cdots+D \to D_r$ is treated in a similar way. The diagram

$$Q[\theta] \xrightarrow{\varphi} Q[\epsilon_1,\ldots,\epsilon_r] \;\begin{array}{c}\to\\ \vdots \\ \to\end{array}\; Q[\epsilon_1,\ldots,\epsilon_r], \quad \text{with}$$

$$\theta^{r+1} = 0, \quad \epsilon_i^2 = 0, \quad \forall i\ ,$$

where $\varphi(\theta) = \epsilon_1+\epsilon_2+\cdots+\epsilon_r$ and the arows in the right correspond to the $r!$ permutations between the ϵ's is a collective equalizer. This is easily seen. Clearly the elements on this equalizer are the symmetric (polynomial) expressions in the ϵ_i's. Since all $\epsilon_i^2 = 0$, it is immediately seen that such an expression is a linear combination of the elementary symmetric polynomials $\sigma_1,\sigma_2,\ldots,\sigma_r$, $\sigma_1(\epsilon_1,\ldots,\epsilon_r) = \epsilon_1+\epsilon_2+\ldots+\epsilon_r$. The reader can check that in this case one also has $\sigma_1^k = k!\sigma_k$, $k \le r$ and $\sigma_1^{r+1} = 0$. Thus, $\varphi(\theta) = \sigma_1(\epsilon_1,\ldots,\epsilon_r)$, and the claim follows exactly in the same way than as for the case $r = 2$. Then, any object M such that the

functor $M^{\overline{(-)}}$ sends this equalizer into an equalizer in $\mathcal{E}$ will have the property that functions $D+D+\cdots+D \to M$ (r-times D) have a unique extension into D_r. Thus we define

3.3 DEFINITION

An object M is infinitesimally linear if for every finite inverse limit of Weil algebras $W \to W_i$, the diagram $M^{\overline{W}} \to M^{\overline{W_i}}$ is a finite inverse limit in $\mathcal{E}$.

The previous considerations imply that by virtue of the same proof as the one we indicated in the case $M = R^m$, we have

3.4 PROPOSITION

Vector fields on an infinitesimally linear object can be integrated into D_∞, in the sense of proposition 3.1. □

We also have

3.5 PROPOSITION

The line R is infinitesimally linear.

PROOF. By lemma 2.2 and axiom I (2.3). □

The notion in definition 3.3 was first proposed in [1]. (See also [23].) It provides a unified way of expressing the various notions of infinitesimal linearity type considered in the literature (for example in [33], [24], [22]). The algebraic geometers of the Grothendieck school were already aware of the need for properties of this type. See for example [29], page 26, where the property needed to show that addition of tangent vectors is possible is explicitly stated, and addition is obtained as in SDG (compare [24]). G. Wraith in [36] discovered a second property of this type, that was essential, and sufficient, to construct the Lie bracket of vector fields ([33] or [22]). The general notion in definition 3.3 was used in [23],[28] (where infinitesimal linearity is called microlinearity) to show that the synthetic proof of the Ambrose-Palais-Singer theorem on connections and sprays given in [6] was actually valid for any infinitesimally linear object. In questions where only nilpotent infinitesimals (and not local notions) enter into the concepts manipulated, the application of the synthetic method waives the necessity to utilize local results (for example the integration of a vector field), although that is the classical way of proceeding. This is amply illustrated in the references given above. The right assumption seems to be that of infinitesimal linearity. This, in the models, will render a theorem (proved synthetically) true for larger class of objects. We have seen that the class of objects in $\mathcal{E}$ which are infinitesimally linear contains the line R, and it is immediate to prove that it is closed under finite limits and exponentiation by arbitrary objects (all limits in the case of

Grothendieck topoi). Finally, this class will also be closed under descent along etale maps, provided that the notion of etale utilized is the one determined by the objects $\bar{W}$ (for Weil algebras W).

This brings into consideration the second axiom of SDG. The infinitesimal objects $\bar{W}$ should be small or "tiny" in the following sense.

3.6 AXIOM II

For every Weil algebra W, the functor $(-)^{\bar{W}}: \mathcal{E} \to \mathcal{E}$, $M \mapsto M^{\bar{W}}$, has a right adjoint $(-)_{\bar{W}}: \mathcal{E} \to \mathcal{E}$, $M \mapsto M_{\bar{W}}$. (This is equivalent to the fact that exponentiation with $\bar{W}$ preserves epimorphic families (or colimits) in the case of Grothendieck topoi.)

We shall not elaborate on this axiom. Its importance is related to the notion of etale map determined by the objects $\bar{W}$. This axiom was introduced in [24] for the objects D(n). For all Weil algebras it appears in [7]. F.W. Lawvere stressed its importance and advocated that, for example, differential forms $M^D \to R$ should be studied in terms of the corresponding quantities $M \to R_D$.

PART II. LOCAL CONCEPTS IN SDG

Germs in SDG

In order to incorporate into the synthetic method the capacity to treat problems that involve local notions, it is necessary to consider *topological structures* on the line R, and, more generally, on the objects of a topos. Once a topological structure is incorporated, there is no problem to define the notion of *germ of a function* and to construct objects of germs. The synthetic method then postulates that for maps $R^n \to R$, germs are *representable*, in a similar way that, as we have seen it in the first part of this paper, it postulates that for maps $R^n \to R$ jets are representable.

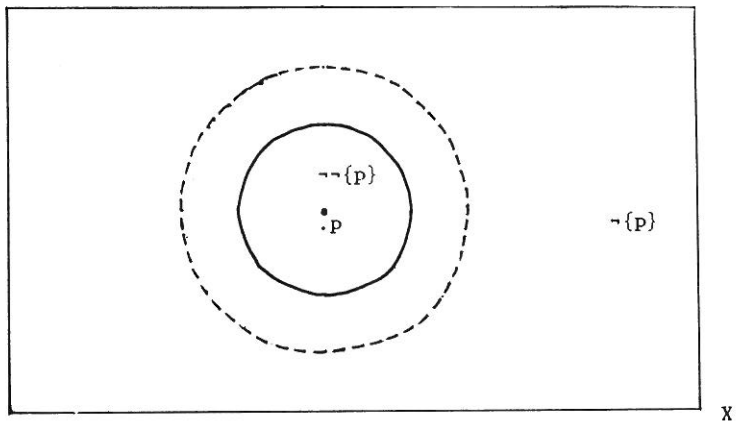
The work in algebraic geometry (models of SDG constructed using the theory of rings) and in the models of SDG constructed using the theory of C^∞-rings, shows that the appropriate notion of *open* subobject is the one determined by the ring structure of R. Namely, there is a basic open that is taken to be the object R^* of invertibles of R, $R^* \hookrightarrow R$, and all other opens are constructed out R^* by

pulling-back and forming unions. This is the Zariski process of building up a topology. In the models constructed using the theory of C^∞-rings, the line R, in addition, has a (strict) order relation.. This means that another topology is available. Namely, the interval or Euclidean topology. Here the basic opens are the intervals $(a-\epsilon,a+\epsilon) \subset R$, for $a \in R$, $\epsilon \in R$, $\epsilon > 0$, and in R^n, just the products of these intervals. Then, all other opens are constructed forming unions. Still another way of considering local notions is available. In all topoi there is an *intrinsic* topological structure on each object. This structure is attached to the logic of the topos [16]. This was (independently) discovered by J.Penon [31] precisely while he was investigating local notions in the models of SDG. He then proved that (in all the models from algebraic geometry and all the well adapted models in the sense of [7],[21]) all three topological structures agree on a large class of objects [31],[32]. This class always includes the line R and all its (finite) products R^n. This result implies that there is a clear choice (all possible alternatives coincide) for the notion of open neighbourhood to be utilized in the axiom of germ representability for maps $R^n \to R$. Let $\mathfrak{P}_0(R^n) \subset R^n$, $0 \in \mathfrak{P}_0(R^n)$ be the intersection of all open neighbourhoods of 0 in R^n. It turns out that $D_\infty(n) \subset \mathfrak{P}_0(R^n)$ (so $\mathfrak{P}_0(R^n)$ is larger than every infinitesimal neighbourhood determined by a Weil algebra W) and that $\mathfrak{P}_0(R^n)$ is equal to the double negation of 0 in the (intuitionistic) logic of the topos, i.e., to $\Delta(n) = [\![x \in R^n \mid \neg\neg(x=0)]\!] = \neg\neg\{0\}$. By its very definition, $\Delta(n)$ is an infinitesimal object that is smaller than all open neighbourhoods and it is the largest such. It was known [22] that in the models of algebraic geometry $\Delta(n) = D_\infty(n)$, which means that there, all infinitesimals are nilpotent. With the introduction of models constructed with C^∞-rings it was realized that there should be larger infinitesimals determined by flat functions [7]. This was actually the case in the model $\mathcal{G}$ (for germ-determined) constructed in [9], cf. [8]. (Cf. also [10],[22],[26] and [32], where $\mathcal{G} = \mathbb{D}$ = Dubuc topos). It was soon discovered [30] that in this model an (external) arrow $\Delta(n) \to R$ in the topos was exactly the same thing that the germ at 0 of a smooth function $\mathbb{R}^n \to \mathbb{R}$. There, [30], the (infinitesimal) inverse function theorem was formulated and proven

to hold in the model. Based on these results, maps from $\Delta(n)$ to R were considered to be germs (at 0) of maps $R^n \to R^m$ in [5], where the pre-image theorem for regular values was derived synthetically. The (internal) axiom of germ representability, which furnishes the key of the passage from the infinitesimal to the local (axiom III in 3.1), was first stated (and shown to hold in the model $\mathcal{G}$) by the second author in a lecture given at McGill University in 1982, in a form which was directly relevant to the problem of local integration of vector fields in SDG. The axiom was expressed in terms of the existence of sufficiently many infinitesimals in $\Delta(n)$. Explicitly, it was stated that if two functions f,g agree on $\Delta(n)$, then the object where they agree, $[\![x \in R^n \mid f(x) = g(x)]\!]$ is a neighbourhood (in the local sense) of 0, and that any function defined on $\Delta(n)$ can be extended to R^n (see [12]). The property of the representability of germs which this axiom actually expresses, was emphasized by the first author in a lecture given at Aarhus University in 1983, along with its analogy with the basic axiom of SDG - namely, with the Kock-Lawvere axiom, if the latter is viewed as an axiom of representability of jets. Since then, the axiom has been used in [32] in connexion with the (local) inverse function theorem.

1. TOPOLOGICAL STRUCTURES IN A TOPOS

Essentially, everything (except definition 1.12 which is new) in this section is due to J. Penon and the standard reference is [32]. Given a topos $\mathcal{E}$ and an object X in $\mathcal{E}$, the basic understanding when considering the topological structure determined on X by the logic of $\mathcal{E}$ is that given any two points x,y in X, to assert the negation of the equality x=y, $\neg(x=y)$, *means* that x and y are *well separated*. Thus, given a point $p \in X$, the points well separated from p are the points $x \in \neg\{p\}$, and the points which are not well separated (i.e., which are very, or infinitesimally, close to p) from p are the points $x \in \neg\neg\{p\}$. Since $\neg\neg\{p\} \cup \neg\{p\} = X$ is false in general, there is still some space left in X. This space is a "no man's land" since



it has no explicit description in the topos (remark that $\neg\neg\neg\{p\} = \neg\{p\}$). A neighbourhood of p is a part $H \subset X$ such that it contains $\neg\neg\{p\}$ and also this "no man's land". This is expressed by the formula $\neg\{p\} \cup H = X$, that is, every point x in X is either well separated from p, or it is in H. We have

1.1 DEFINITION

Given an object X in $\mathcal{E}$, a part $U \subset X$ is an *intrinsic or Penon open* if it is a neighbourhood (in the above sense) of each of its points. That is, if it satisfies

(1) $$\forall x \in U \ (\neg\{x\} \cup U = X)$$

From (1) follows immediately the (strictly weaker) condition:

(2) $$\forall x \in U \ (\neg\neg\{x\} \subset U) \ .$$

We remind the reader that we work in a completely internal sense. Thus (1) above is an abbreviation for the formula

$$\forall x \in X \ [x \in U \Longrightarrow \forall y \in X \ (\neg(y=x) \vee y \in U)].$$

When we say a part $U \subset X$, this actually stands for $U \in \Omega^X$, where Ω is the subobject classifier of the topos.

Recall that a *locale* is a partially ordered object L admitting arbitrary suprema, hence finite infima, and for which the distributive law

$$H \cap \bigcup_{i \in I} G_i = \bigcup_{i \in I} H \cap G_i$$

holds (cf. [17]). The subobject classifier Ω is a locale; thus, for each X in $\mathcal{E}$, the object of parts of X, Ω^X, is a locale. The suprema and infima in Ω^X are the (internal) union and intersection of parts of X. A *sublocale* S of a locale L is a part $S \subset L$ such that:

i) the maximal element of L is in S;

ii) S is closed under finite infima;

iii) S is closed under arbitrary suprema.

It follows that there is an *interior operator* $L \xrightarrow{\mathfrak{Q}} L$ (or coclosure), where for $H \in L$, $\mathfrak{Q}H \in S$, $\mathfrak{Q}H \subset H$, is the largest element of S smaller than H. Notice that iii) implies that $\phi \in S$. The subobject classifier Ω is the *initial* locale and thus it has no proper sublocales.

A *subinf-lattice* B of a locale L is a part $B \subset L$ such that:

i) The maximal element of L is in B;

ii) B is closed under finite infima.

Clearly the part of L consisting of all suprema of parts of B is a locale; we say that it is generated by B.

It is very easy to prove the following.

1.2 PROPOSITION

For every object X in $\mathcal{E}$, the subobject $P(X) \subset \Omega^X$ of Penon opens, $P(X) = [\![U \in \Omega^X \mid \forall p \in U \ \neg\{p\} \cup U = X]\!]$, is a sub-locale of Ω^X. Furthermore, given any $X \xrightarrow{f} Y$, and $U \subset \Omega^Y$, if $U \in P(Y)$, then $f^{-1} \in P(X)$. □

It follows that $P1 = \Omega$, so every part of 1 is Penon open.

1.3 DEFINITION

A *topological structure* on an object X is an object S(X) in $\mathcal{E}$ which is a sublocale of Ω^X, $S(X) \subset \Omega^X$. A *base* is an object B(X) in $\mathcal{E}$ which is a subinf-lattice of Ω^X, $B(X) \subset \Omega^X$. Clearly any base B(X) in X generates a topological structure S(X) such that

$$U \in S(X) \iff \forall x \in U \quad \exists V \in B(X) \ (x \in V \subset U).$$

In this case, we say that B(X) is a base of S(X). As usual, the parts $U \subset X$, such that $U \in S(X)$, are called *open* parts of X, or S-opens when necessary.

An object X in $\mathcal{E}$ together with a topological structure S(X) is exactly the same thing as a *space* in the sense of [17] such that it has sufficiently many points.

We shall now compare a topological structure S(X) with the intrinsic topological structure P(X) on X. In practice topoi very often have a genesis; by this we mean that they have been constructed as the topos of sheaves on a particular given site. For objects X in the sites, if the coverings of X were in S(X), then this will be remembered by the logic of the topos, and the intrinsic opens will all be in S(X). Concretely, for any topos $\mathcal{E}$ we have

1.4 DEFINITION-PROPOSITION

A topological structure S(X) in an object X is said to satisfy the *covering principle* if the following condition holds

$$\forall H,G \in \Omega^X \quad (H \cup G = X \rightarrow \mathbb{Q}H \cup \mathbb{Q}G = X),$$

where $\mathbb{Q}$ is the interior operator corresponding to S(X).

If this is the case, then $P(X) \subset S(X)$. That is, every Penon open is an S-open.

PROOF. Let U be Penon open and let $x \in U$. Then $\neg\{x\} \cup U = X$, thus $\neg\{x\} \cup \mathbb{Q}U = X$. This implies $x \in \mathbb{Q}U$. Thus $U = \mathbb{Q}U$, which shows that U is S-open. □

Given an object X in $\mathcal{E}$, the intrinsic topological structure in X will not in general satisfy the covering principle. When it does, this has remarkable consequences which imply existence conditions always hard to get in intuitionistic mathematics.

1.5 DEFINITION

Let S(X) be a topological structure in an object X, and let H,G be two parts of X, $H \subset G$. We say that G is a *S-neighbourhood* of H if any of the two following equivalent conditions is satisfied.

(1) i) $\forall x \in H \ \exists U \in S(X) \ (x \in U \subset G)$;

ii) $\exists U \in S(X) \ (H \subset U \subset G)$.

We say that H is *well contained* in G if $\neg H \cup G = X$, that is, if the following condition is satisfied:

(2) $\forall x \in X \ (\neg(x \in H) \vee x \in G)$

1.6 PROPOSITION

With the notation as in definition 1.5, we have

(a) If $S(X) \subset P(X)$, and $H = \{p\}$, then (1) $\Rightarrow$ (2).

b) If S(X) satisfies the covering principle, then (2) $\Rightarrow$ (1). That is, if H is well contained in G, then G is a S-neighborhood of H.

(c) In particular, it follows that if the intrinsic topology P(X) satisfies the covering principle and if $p \in X$, then the following two conditions are equivalent:

i) $\neg\{p\} \cup G = X$

ii) $\exists U \in P(X) \quad (p \in U \subset G)$.

PROOF. (a) is very easy, and for (b) do as in 1.4. □

Consider now any two objects X,Y in $\mathcal{E}$. Let, for each $x \in X$, $H_x \subset Y$, $G_x \subset Y$ be parts of Y such that $H_x \subset G_x$. Let $H \subset X \times Y$, $G \subset X \times Y$ be $H = [\![(x,y) \mid y \in H_x]\!]$ and $G = [\![(x,y) \mid y \in G_x]\!]$. Clearly, $H \subset G$. It is immediate to see that $\neg H = [\![(x,y) \mid y \in \neg H_x]\!]$. In turn, from this it follows (also immediately) the following equivalence:

$$\frac{\forall x \in X \ (\neg H_x \cup G_x = Y)}{\neg H \cup G = X \times Y} .$$

This simple observation implies that proposition 1.6 has the following remarkable consequence.

1.7 LEMMA

Let X,Y in $\mathcal{E}$ be such that the intrinsic topology P(X×Y) satisfies the covering principle, $p \in Y$, $H = X \times \{p\}$. Then, if $X \times \{p\} \subset G \subset X \times Y$, G is a P-neighbourhood of $X \times \{p\}$ in X×Y if and only if for each $x \in X$, G_x is a P-neighbourhood of p in Y. □

1.8 THE MONAD OF A POINT

Given a topological structure S(X) on X and a point $x \in X$, we denote by $S_x(X)$ the intersection of all open neighbourhoods of x in X

$$S_x(X) = [\![y \in X \mid \forall U \in \Omega^X\ (U \in SX \wedge x \in U) \Rightarrow y \in U)]\!].$$

In a classical topological space (in Sets) this monad will reduce to the point $\{x\}$ as soon as some minimal separation conditions hold for $S(X)$. Here, such a condition only implies that $S_x(X)$ is infinitesimal. That is, that every point in $S_x(X)$ is not well separated from x.

1.9 DEFINITION-PROPOSITION

A topological structure $S(X)$ in an object X satisfies the separation condition T_1 if for every point x in X, $\neg\{x\}$ is S-open. If this is the case, then for all x in X, $S_x(X) \subset \neg\neg\{x\}$.

PROOF. Let $y \in S_x(X)$ and suppose that $y \in \neg\{x\}$. Then $x \in \neg\{y\}$, and by assumption $\neg\{y\} \in S(X)$. Thus $y \in \neg\{y\}$. This contradiction shows that $y \in \neg\neg\{x\}$. □

We say that a topological structure $S(X)$ is *subintrinsic* if every S-open is Penon open. that is, if $S(X) \subset P(X)$.

1.10 PROPOSITION

Given any subintrinsic T_1 topological structure $S(X)$ on an object X, then for all x in X,

$$P_x(X) = S_x(X) = \neg\neg\{x\}.$$

Notice that this applies to $P(X)$ when $P(X)$ is T_1.

PROOF. $$S_x(X) \subset \neg\neg\{x\} \subset P_x(X) \subset S_x(X).$$

The first inclusion holds by 1.9 since $S(X)$ is T_1, the second holds by 1.1(2), and the third since $S(X) \subset P(X)$. □

1.11 GERMS AND GERM REPRESENTABILITY

Given any two objects X,Y in $\mathcal{E}$ equipped with topological structures $S(X)$, $S(Y)$, an *infinitesimal germ* (for lack of a better name) at $x \in X$ is a map $S_x(X) \xrightarrow{\xi} Y$ which factors $S_x(X) \xrightarrow{\xi} S_{\xi(x)}(Y)$. Let $J^g(X,Y)$ be the object whose elements are pairs (x,ξ) where $x \in X$ and ξ is an infinitesimal germ at x. Clearly there are *source* and *target* maps

$$J^g(X,Y) \xrightarrow{\sigma} X, \quad (x,\xi) \mapsto x$$

$$J^g(X,Y) \xrightarrow{\tau} Y, \quad (x,\xi) \mapsto \xi(x).$$

A continuous function from X to Y is defined as usual. Thus, we have the object of continuous maps $C(X,Y) \subset X^Y$, i.e.

$$f \in C(X,Y) \iff \forall V \in S(Y)\ (f^{-1}V \in S(X)) .$$

Given $U \in S(X)$ and $U \xrightarrow{f} Y$, we say that f is continuous if $\forall V \in S(Y)$, $(f^{-1}V \in S(X))$ (notice that $f^{-1}(V) \subset U$).

We now define germs as expected. A germ from X to Y is an equivalence class $[x,f,U]$ of triples (x,f,U) where $x \in X$, $U \in S(X)$, $x \in U$ and $U \xrightarrow{f} Y$ continuous. The equivalence relation is, $(x,f,U) \sim (x',f',U')$ if $x = x'$ and $\exists W \in S(X)$, $x \in W \subset U \cap U'$ such that $f|_W = f'|_W$. We use the notation $f|_x$ to denote the germ of f at x. As before, we call x the *source* and f(x) the target (or value) of $f|_x$. We denote $C^g(X,Y)$ the object of germs from X to Y. There are well defined source and target maps

$$C^g(X,Y) \xrightarrow{\sigma} X, \qquad f|_x \mapsto x$$

$$C^g(X,Y) \xrightarrow{\tau} Y, \qquad f|_x \mapsto f(x).$$

We put $\sigma^{-1}(x) = C^g_x(X,Y)$, $\tau^{-1}(y) = C^g(X,Y)_y$ and $C^g_{x,y}(X,Y) = C^g_x(X,Y) \cap C^g(X,Y)_y$.

It is immediate to show that the restriction to $S_x(X)$ of a continuous map $U \xrightarrow{f} Y$, $x \in U$, factors $U \xrightarrow{f} S_{f(x)}(Y)$. This shows that there is a restriction map that we denote $j: f|_x \mapsto j^g_x f$,

$$C^g(X,Y) \xrightarrow{j} J^g(X,Y) \qquad\qquad C^g_{x,y}(X,Y) \xrightarrow{j} S_y(Y)^{S_x(X)}$$

$$\searrow \quad \swarrow$$

$$X \times Y \qquad\qquad \text{(on the fibers)}$$

Germ representability concerns the invertibility of this map.

We remark that the above definitions of $J^g(X,Y)$ and $C^g(X,Y)$ are (internal) descriptions for a construction in the topos $\mathcal{E}$. Given $x \in X$, $U \in S(X)$, $x \in U$, an arrow $S_x(X) \xrightarrow{f} Y$ or $U \xrightarrow{f} Y$ is a variable $f \in \text{Partial}(X,Y)$, where $\text{Partial}(X,Y) \in \mathcal{E}$ is the object of partial maps from X to Y. In this way, all three variables, x, U and f have a well defined type. Thus, the above descriptions yield objects $J^g(X,Y)$ and $C^g(X,Y)$ in $\mathcal{E}$.

Given a topological structure $S(X)$ in X, the covering principle (1.4) is a useful criterion utilized to show that the intrinsic opens are in $S(X)$. On the other hand, we do not know yet any meaningful general criteria that could be used to show that $S(X)$ is subintrinsic. However, in practice, $S(X)$ is often generated by a base $B(X) \subset P(X)$, therefore it is automatically subintrinsic.

We now pass to consider the concept of a topological structure in the *whole* topos $\mathcal{E}$.

1.12 DEFINITION

A topological structure S in a topos $\mathcal{E}$ is the data consisting of a topological structure $S(X)$ for every object X in $\mathcal{E}$, such that for every X,Y in $\mathcal{E}$, $f \in Y^X$ and $U \in \Omega^Y$,

$$\text{if } U \in S(Y) \text{ then } f^{-1}U \in S(X);$$

that is, such that every map is continuous (in an internal sense).

A base B is defined similarly, except that the condition reads

$$\text{if } U \in B(Y) \text{ then } f^{-1}U \in S(X),$$

where $S(X)$ is the topological structure generated in X by $B(X)$.

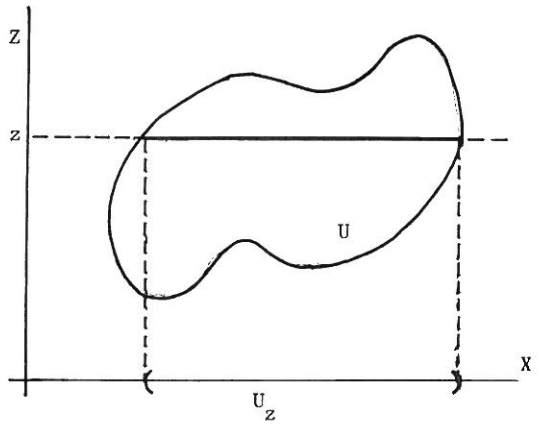
Clearly, the topological structure $S(X)$ generated by $B(X)$ for each X in $\mathcal{E}$ defines a topological structure S in the topos $\mathcal{E}$. This just means that, as usual, it is enough to test continuity on a base.

A base B is said to be *classifiable* if it is of the form $B(X) = B^X$ for some $B \subset \Omega$, necessarily $B = B(1)$. That is, there is a subinf-lattice B of Ω that classifies (internally) the parts of X which are in $B(X)$.

1.13 REMARK

Let X,Z be any two objects in $\mathcal{E}$, and U an open part of $X\times Z$, $U \in S(X\times Z) \subset \Omega^{X\times Z}$. Consider for each $z \in Z$ the map $X \to X\times Z$, $x \mapsto (x,z)$. It follows that $U_z = [\![x \in X \mid (x,z) \in U]\!]$ is an open part of X. That is, $U_z \in S(X)$ for each $z \in Z$. This shows that under the identification $\Omega^{X\times Z} = (\Omega^X)^Z$, there is an inclusion of subobjects $S(X\times Z) \subset S(X)^Z$. This inclusion is always strict except in the case $S(X) = \Omega^X$. Indeed, if $S(X\times Z) = S(X)^Z$, then for $X = 1$ we have $S(Z) = S(1)^Z$. But $S(1) = \Omega$ since it is a sub-locale of Ω. (For a base, however, the equality

$B(X\times Z) = B(X)^Z$ is possible, and means that the base B is classifiable.) In terms of families this means that an S-open part $U \subset X\times Z$ determines always a Z-indexed family of S-open parts in X, $U_z \in S(X)$; but that the contrary is false unless $S(X) = \Omega^X$.



This is clearly understood since it is what the topological intuition acquired in the classical school tells us about space. The intrinsic topology has, however, *good properties* which are remarkable and surprising to the classical intuition. One of them has already been established in 1.7, and it says that if $P(X\times Z)$ satisfies the covering principle, then a family (indexed by Z) of P-neighbourhoods in X is a P-neighbourhood in $X\times Z$. Another is the following (with no assumptions).

1.14 PROPOSITION

Under the identification $\Omega^{X\times Y} = (\Omega^X)^Y = (\Omega^Y)^X$, there is an equality of subobjects $P(X\times Y) = P(X)^Y \cap P(Y)^X$. That is, if a part $U \subset X\times Y$ is such that for all $y \in Y$ and for all $x \in X$ the families $U_y = [\![x \mid (x,y) \in U]\!] \subset X$ and $U_x = [\![y \mid (x,y) \in U]\!] \subset Y$ are Penon opens of X, respectively of Y, then U is a Penon open of $X\times Y$ (the other implication has been proved in Remark 1.13).

PROOF. By assumption we have, for any $p \in X$ (fixed), $q \in Y$ (fixed)

(a) $(p,y) \in U \Longrightarrow \forall z \in Y \quad (z \neq y \vee (p,z) \in U)$

(b) $(x,q) \in U \Longrightarrow \forall w \in X \quad (w \neq x \vee (w,q) \in U)$.

We have to prove:

(c) $(x,y) \in U \to \forall(w,x) \in X\times Y \ ((w,z) \neq (x,y) \vee (w,z) \in U)$.

Let $(x,y) \in U$ Given (w,z), fix $y = q$ in (b). Then $w \neq x \vee (w,y) \in U$. If $w \neq x$ then $(w,z) \neq (x,y)$ and we are done. If $(w,y) \in U$, fix $w = p$ in (a). Then $z \neq y \vee (w,z) \in U$. If $z \neq y$, as before, we are done. In the other case, $(w,z) \in U$. This finishes the proof of (c). □

We finish this section on general topology with the consideration of separation conditions in a topos equipped with a topological structure S. Given X in $\mathcal{E}$, the Haussdorf condition involves pairs of points; that is, the object X×X. In practice,it has been well established that the correct separation condition is formulated utilizing the topological structure S(X×X) in X×X rather than the (coarser) product structure (of obvious definition) of S(X) with S(X). Thus

1.15 DEFINITION

Given a topological structure S in $\mathcal{E}$, an object X is S-separated (satisfies the condition T_2) if the negation of the diagonal, $\neg\Delta_X \subset X\times X$, is an open part of S(X×X).

For the intrinsic topology we have the following characterization,

1.16 PROPOSITION

Let X,Y be any two objects in $\mathcal{E}$ such that P(X) and P(Y) are T_1 (cf. 1.9). Then the following conditions are equivalent:

(a) P(X×Y) is separated (i.e. is T_2)

(b) $(x,y) \neq (p,q) \Rightarrow (x \neq p \vee y \neq q)$

for all x,p in X and y,q in Y.

PROOF. (a) ⇒ (b). It is clear that $T_2 \Rightarrow T_1$, so P(X×Y) is T_1. Thus:

$$(x,y) \neq (p,q) \Rightarrow \forall(w,z) \in X\times Y, \ (w,z) \neq (x,y) \vee (w,z) \neq (p,q).$$

It suffices then to take $(w,z) = (x,q)$ to deduce (b).

(b) ⇒ (a). By Proposition 1.14 applied to (X×Y)×(X×Y) it suffices to show that for all $(p,q) \in X\times Y$, $\neg\{(p,q)\} \in P(X\times Y)$. It is clear that (b) is actually an equivalence. Thus we have,

$$\neg\{(p,q)\} = [\![(x,y) \mid (x,y) \neq (p,q)]\!] = [\![(x,y) \mid x \neq p]\!] \cup [\![(x,y) \mid y \neq q]\!].$$

But these two points are P-open since P(X) and P(Y) are T_1 (and the projections are continuous). □

From the case Y = 1, or directly from 1.14 it follows

1.17 COROLLARY

Given any X in $\mathcal{E}$, for the intrinsic topological structure, X is separated (T_2) if and only if it is T_1; that is, if and only if the following holds

$$x \neq y \Rightarrow \forall z\ (z \neq x \vee z \neq y). \quad \square$$

1.18 SOME FINAL REMARKS

When working with objects in a topos naturally equipped with a topological structure (for example, schemes, manifolds, analytic spaces, algebraic spaces, etc.), this structure will not have, a priori, desirable properties such as the ones described in 1.7 and 1.14. If only because of this fact, it is important to show that on these objects the natural topological structure agrees with the intrinsic one.

We have used the word "topological structure" instead of the much convenient "topology" since the latter is already in use for a different concept. A *topological structure* S in $\mathcal{E}$ is not a *topology* J in $\mathcal{E}$ in the sense currently understood in Topos theory. The first, S, is a sublocale of Ω^X for each X in $\mathcal{E}$. That is, it corresponds to a coclosure or interior operator in Ω^X (for each X). The second, J, is a quotient locale of Ω^X for each X in $\mathcal{E}$. That is, it corresponds to a closure operator in Ω^X (for each X). A topology J is *always* of the form $J(X) = J^X$ (necessarily $J = J(1) \subset \Omega$). A topological structure ($\neq \Omega$) can *never* be of the form $S(X) = S^X$. In this case, $S = S(1) \subset \Omega$ is a sublocale of Ω, thus $S = \Omega$. Therefore, while closed parts with respect to J are classified by $J \subset \Omega$, it can never exist an open part classifier for S in this internal sense.

An external characterization of the concept of topological structure S in a topos $\mathcal{E}$ may be useful. We state it as an exercise since it is not necessary. However, to work out this exercise should be instructive to any reader who wants to develop a feeling for the logic of topoi (cf. [32], Chapter 0, or [2]).

1.19 EXERCISE

A topological structure S in a topos $\mathcal{E}$ is a functor $\mathcal{E}^{op} \xrightarrow{S} \mathcal{E}$ together with a (monic) natural transformation $S \rightarrowtail \Omega^{(-)}$ such that $S(X) \rightarrowtail \Omega^X$ is a sublocale for each X in $\mathcal{E}$, and such that under the identification $\Omega^{X\times Z} = (\Omega^X)^Z$, there is an inclusion of subobjects $S(X\times Z) \subset S(X)^Z$. The exercise consists in showing that this is equivalent to definition 1.12. □

2. TOPOLOGICAL STRUCTURES IN A RINGED TOPOS

Let $\mathcal{E}$ be a topos equipped with a ring object R such that $1 \neq 0$. We do not assume here that R satisfies Axiom I (2.3 part I). Let $R^* \subset R$ be the object of invertible elements of R.

2.1 THE ZARISKI TOPOLOGICAL STRUCTURE

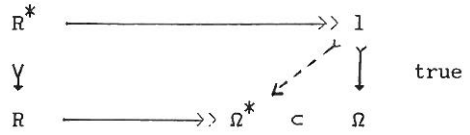
First we define the affine opens. Given any X in $\mathcal{E}$, $Aff(X) \subset \Omega^X$ is defined by the formula,

$$\text{for } U \in \Omega^X, \quad U \in Aff(X) \iff \exists f \in R^X \ (U = f^{-1}R^*) \ .$$

From the fact that the product of any two elements in a ring is invertible if and only if each factor is, it follows that Aff(X) is closed under finite intersections. On the other hand, it is immediate to check that given any $f \in Y^X$ and $U \in Aff(Y)$, $f^{-1}U \in Aff(X)$. Thus Aff is a base in $\mathcal{E}$. The Zariski topological structure in $\mathcal{E}$, denoted Zar, is the structure generated by this base (cf. 1.3, 1.12). Thus, by definition, given $U \in \Omega^X$, we have

$$U \in Zar(X) \iff \forall x \in U \ \exists f \in R^X \ (x \in f^{-1}R^* \subset U) \ .$$

There is another way in which the object of invertibles, $R^* \subset R$ determines a topological structure in $\mathcal{E}$. Consider the image in Ω of the characteristic function of R^*, denoted Ω^*, i.e.



$R^* \to 1$ is epic since $1 \in R^*$; this guarantees that $\text{true} \in \Omega^*$. From the formula $xy \in R^* \Leftrightarrow x \in R^* \wedge y \in R^*$, true in R, it follows that Ω^* is a subinf-lattice of Ω. Then define a classifiable base B (cf. 1.12) by the equation $B(X) = \Omega^{*X}$. We shall denote Zar_2 the topological structure generated by this base. It is clear that $\text{Aff} \subset B$, so $\text{Zar} \subset \text{Zar}_2$. This inclusion is strict. For example, the subobject $1 \overset{\text{true}}{\rightarrowtail} \Omega^*$ obviously belongs to $B(\Omega^*)$ and thus to $\text{Zar}_2(\Omega^*)$. However, there is no reason why it should be in $\text{Aff}(\Omega^*)$ or $\text{Zar}(\Omega^*)$. By definition Aff is generated from the subobject $R^* \subset R$ by (internal) pullbacks, while B is generated from $R^* \subset R$ by *descent* (which generates the subobject $1 \overset{\text{true}}{\rightarrowtail} \Omega^*$) and (internal) pullbacks. However, we shall see (1.5, 1.10 Appendix) that for the object R^n in the models, it is the case that $\text{Zar}_2(R^n) = \text{Zar}(R^n)$.

There is a very natural algebraic property on the ring R that is essentially equivalent to the separation (T_2) of the Zariski topological structure on R. Since $1 \neq 0$, it follows that $R^* \subset \neg\{0\}$. A ring R is said to be a *field of fractions* (cf. [22]) if every nonzero element is invertible (and $1 \neq 0$). Thus,

$$R \text{ is a field of fractions} \Leftrightarrow R^* = \neg\{0\}.$$

But this means that R is T_1-Zariski, (by translation) and also T_2 since $\neg\Delta_R = (x-y)^{-1}\neg\{0\}$.

We shall denote $\Delta(n) \subset R^n$, $\Delta(n) = \neg\neg\{0\}$, where 0 indicates the origin of R^n. It is an easy exercise in intuitionistic logic to show that $\Delta(n) = \Delta^n$, where $\Delta \subset R$ is $\Delta = \Delta(1)$ (use that $\neg(p \vee q) = \neg p \wedge \neg q$). Since $1 \neq 0$, clearly any nilpotent element in R is in $\neg R^*$, that is, $D_\infty \subset \Delta$. Thus, $D_\infty^n \subset \Delta^n$.

2.2 PROPOSITION

If R is a field of fractions, then for all positive integers r,n, $D_r(n) \subset D_\infty(n) \subset \Delta(n)$. □

The T_2-condition for the intrinsic topology, if R is already a field of fractions, turns out to be equivalent to the algebraic property of being a *local ring*. Recall that R is a local ring if $1 \neq 0$ and

(a) $x-y \in R^* \Longrightarrow x \in R^* \vee y \in R^*$.

On the other hand, by 1.17, P(R) is separated (T_2) if

(b) $x \neq y \Rightarrow \forall z\ (z \neq x \vee x \neq y)$.

Then, since we assume $R^* = \neg\{0\}$, the equivalence between (a) and (b) is immediate. In this case we have:

2.3 PROPOSITION

If R is a local ring and a field of fractions, then the Zariski topology is subintrinsic, i.e. $Zar(X) \subset P(X)$ for all objects X in $\mathcal{E}$.

PROOF. Use (b) with $y \neq 0$ and the fact that $\neg\{0\} = R^*$. □

There is a stronger algebraic property, introduced in [30], which turns out to be equivalent to the intrinsic separation of the spaces R^n. This property is essential in order to have a meaningful calculus of determinants and Jacobian matrices (cf. [18] and [32]), and it is a simple natural strengthening of the notion of field of fractions ($x \neq 0 \Rightarrow x \in R^*$). A ring R is a *field in the sense of Kock* if for each positive integer n, the following holds for $(x_1,\ldots,x_n) \in R^n$,

(a) $\qquad (x_1,\ldots,x_n) \neq 0 \Rightarrow x_i \in R^*$ for some i.

Now, by 1.16 (which obviously holds for any finite number of factors) and translation, we see that $P(R^n)$ is separated (T_2) if and only if

(b) $\qquad (x_1,\ldots,x_n) \neq 0 \Rightarrow x_i \neq 0$ for some i.

Then, since $R^* = \neg\{0\}$ (case n=1), the equivalence between (a) and (b) is immediate. It also follows that R^n is Zariski separated (T_2). Indeed, since (a) is clearly an equivalence, we have for $0 \in R^n$,

$$\neg\{0\} = \bigcup_{i=1}^{n} [\![(x_1,\ldots,x_n) \mid x_i \in R^*]\!],$$

thus, a Zariski open since a union of Zariski opens. This shows, by translation, that R^n is Zariski T_1, and also T_2 since $\neg\Delta_{R^n} = (\pi_1-\pi_2)^{-1}\neg\{0\}$, where π_1,π_2 are the two projections $R^n \times R^n \to R^n$. We have then:

2.4 PROPOSITION

If R is a field in the sense of Kock, or, equivalently, if $P(R^n)$ is separated (T_2) for each positive integer n, then so is the case for $Zar(R^n)$. Furthermore, $Zar(X) \subset P(X)$ for all X in $\mathcal{E}$, and

$$D_\infty(n) \subset P_0(R^n) = Zar_0(R^n) = \Delta(n).$$

PROOF. By 2.2, 2.3, the previous remarks, and 1.10. □

It remains now to analyze whether every Penon open is a Zariski open. That is, whether $P \subset Zar$ in the topos, or, at least, $P(R^n) \subset Zar(R^n)$. There is no synthetic reason why this should be the case. It is actually related to the role the Zariski opens played in the genesis of the topos $\mathcal{E}$. Thus, in the abstract theory we can only postulate:

2.5 POSTULATE A

For each positive integer n,

(a) the intrinsic topological structure $P(R^n)$ is separated (T_2)

(b) the Zariski topological structure $Zar(R^n)$ satisfies the covering principle.

[Recall that (a) is equivalent to the algebraic fact that the line R is a field in the sense of Kock.]

It follows that $P(R^n) = Zar(R^n)$, thus both topologies are separated (T_2) and satisfy the covering principle. This makes results such as the one in 1.17 available, either for P or Zar in R^n.

This part of the theory suffices to cover the algebraic models. In the models constructed using C^∞-rings, the line R has another topological structure, the classical euclidean topology determined by the open intervals. The local notion associated to open intervals has to be considered if we want the theory to be well adapted to the classical work in differential geometry.

The Euclidean Topological Structure

We assume that the ring R has a relation "$<$" such that

R1) $x > 0 \wedge y > 0 \Rightarrow xy > 0 \wedge x+y > 0$ (compatible with the ring structure)
(assume also $1 > 0$)

R2) $\neg(x > x)$ (strict)

R3) $y < x \Rightarrow \forall z\ (z < x \vee z > y)$ (local)

It follows easily that "$<$" is a transitive relation, thus a strict order in R. If a,b are in R, the *open interval* (a,b) is $(a,b) = [\![x \in R \mid a<x<b]\!]$. Given ϵ, δ in R,

the following holds,

(1) $$\epsilon > 0 \wedge \delta > 0 \rightarrow \exists\gamma > 0\ (\gamma < \epsilon \wedge \gamma < \delta).$$

PROOF. Since $\epsilon < 2\epsilon$, it follows that $\delta < 2\epsilon$ or $\epsilon < \delta$. In the first case set $\gamma = \delta/2$ and in the second set $\gamma = \epsilon/2$. (We took this argument from [32].) □

Given $x = (x_1,\ldots,x_n) \in R^n$ and $\epsilon > 0$, let $B(x,\epsilon)$ be the product of the intervals $(x_i-\epsilon, x_i+\epsilon)$, $x \in B(x,\epsilon) \subset R^n$,

$$B(x,\epsilon) = \bigcap_{i=1}^{n} [\![y \in R^n \mid y_i - x_i \in (-\epsilon,\epsilon)]\!].$$

Then, the Euclidean topological structure on R^n is defined by

$$U \in E(R^n) \iff \forall x \in U\ \exists\epsilon > 0\ (B(x,\epsilon) \subset U).$$

It is clear that to show that $E(R^n) \subset \Omega^{R^n}$ is a sublocale it only remains to see that it is closed under finite infima. But this follows immediately from (1) above.

EXERCISE

Show that any interval (a,b) belongs to E(R).

2.6 DEFINITION

Given any object X in $\mathcal{E}$ which is a subobject $X \subset R^n$, the Euclidean topology E(X) is defined to be the "subspace" topology. Namely, for $U \in \Omega^X$,

$$U \in E(X) \iff \forall x \in U\ \exists\epsilon > 0\ \ (X \cap B(x,\epsilon) \subset U).$$

It is clear how to show that this topological structure in E(X) has (internally) the universal property of the subspace topology. (The subspace topology can be defined in any part of R^n; however, in this paper we consider only topological structures in objects of the topos, cf. definition 1.3.)

2.7 PROPOSITION

The topological structure $E(R^n)$ is the n-times product of the topological structure E(R), meaning that it has internally the corresponding universal property. It follows that given $X \subset R^n$, $Y \subset R^s$, then $E(X\times Y)$ is the product of E(X) with E(Y).

PROOF. First, it is clear by the definition that the projections are continuous. Second, given any Z in $\mathcal{E}$ together with a topological structure S(Z), the projections induce a map in $\mathcal{E}$

$$\text{Continuous}(Z,R^n) \rightarrow \text{Continuous}(Z,R)^n.$$

It is straightforward to see that this map is invertible, as well as to check the statement about X and Y. □

2.8 COROLLARY

Given any objects X,Y, $X \subset R^n$, $Y \subset R^m$, then under the identifivation $\Omega^{X\times Y} = (\Omega^X)^Y$, there is an inclusion of subobjects $E(X\times Y) \subset E(X)^Y$.

PROOF. Do as in remark 1.13. □

2.9 PROPOSITION

The euclidean topology is subintrinsic. That is, given any X in $\mathcal{E}$, $X \subset R^n$, $E(X) \subset P(X)$.

PROOF. It is enough to show that given any $x \in R$ and $\epsilon \in R$, $\epsilon > 0$, the interval $(x-\epsilon,x+\epsilon)$ is Penon open. In fact, any interval (a,b), $a,b \in R$ is Penon open. Let $x \in (a,b)$ and $z \in R$. From R3) we have $(z > a \vee z < x)$ and $(z > x \vee b > z)$. Then using R2) it follows immediately that $z \neq x \vee z \in (a,b)$. □

It is now the turn to impose the separation condition in the euclidean topology. Because of the compatibility of the order with the ring structure, it suffices to establish that it is T_1 at the origin

(1) $$x \neq 0 \Rightarrow x > 0 \vee x < 0.$$

This says that $\neg\{0\} = (-\infty,0)\cup(0,+\infty)$. Thus $P(R)$ is T_1 (by translation) and also T_2. This is a classical argument. Let $(x,y) \in \neg\Delta_R$. Then from (1) it follows that $x > y$ or $x < y$. Suppose $x > y$. Then $z = \frac{x+y}{2}$ is such that $x < z < y$. Thus $(x,y) \in (-\infty,z)\times(z,+\infty)$ and also $(-\infty,z)\times(z,+\infty) \subset \neg\Delta_R$. To have the separation of the space $E(R^n)$ it is necessary to strengthen (1) above into the corresponding statement for each positive integer n.

R4) $$(x_1,\ldots,x_n) \neq 0 \Rightarrow (x_i > 0 \vee x_i < 0) \text{ for some } i.$$

The reader can check, arguing as above, that R4) implies that $E(R^n)$ is separated (T_2). In turn, this implies that the euclidean topology is also separated in any $X \in \mathcal{E}$, $X \subset R^n$. We have:

2.10 PROPOSITION

If R has a relation "<" such that R1, R2, R3 and R4 above hold, then $E(R^n)$ is separated (T_2) for each positive integer n, $E(R^n) \subset P(R^n)$ (thus $P(R^n)$ is also

separated) and

$$P_0(R^n) = E_0(R^n) = \Delta(n).$$

PROOF. By 2.9, the previous remarks, and 1.10. □

It remains now to analyze whether every Penon open of R^n is an euclidean open. As for the Zariski topology there is no synthetic reason why this should be the case. Thus, we can only postulate

2.11 POSTULATE WA1

(a) The ring R has a relation "<" which satisfies R1), R2), R3) amd R4).

(b) The euclidean topological structure $E(R^n)$ satisfies the covering principle. [Remark that R1, R2, R3 and R4 mean that "<" is compatible with the ring structure, strict, local and separated, respectively. We may include also (b) for all subobjects of R^n.]

It follows that $P(R^n) = E(R^n)$, thus both topologies are separated (T_2) and satisfy the covering principle. Notice that this implies that every function $R^m \to R^n$ is (internally) continuous in the euclidean topological structure (since this is the case for the intrinsic structure).

Finally, if R is a field of fractions (i.e. $R^* = \neg\{0\}$), it follows that R4) is equivalent to the fact that R is a field in the sense of Kock. Also, in this case there is a complete identification $Zar(R^n) = E(R^n) = P(R^n)$.

From the fact that $x \neq 0$ means invertibility (that is $R^* = \neg\{0\}$) it follows easily that "<" is total on R^*, that is, $x \in R^* \Leftrightarrow x > 0 \vee x < 0$. Also,

2.12 PROPOSITION

The conditions R1, R2, R3 and R4 are equivalent to the following conditions

R1) Compatible with the ring structure.

R2) $\neg(0 < 0)$.

R3) $\forall z\ (z < 1 \vee z > 0)$.

R4) $(x_1,\ldots,x_n) \neq 0 \Rightarrow (x_i > 0 \vee x_i < 0)$ for some i. □

2.13 FINAL REMARK

It is straightforward to check that Propositions 2.4 and 2.10 hold for any X in $\mathcal{E}$, $X \subset R^n$ and $x \in X$. In this case, replace $\Delta(n)$ by $\neg\neg\{x\}$, and remark also that $\neg\neg\{x\} = (x+\Delta(n))\cap X$.

3. GERM REPRESENTABILITY AND INTEGRATION OF VECTOR FIELDS

Let $\mathcal{E}$ be a topos and R a ring object ($1 \neq 0$) in $\mathcal{E}$. Assume that the intrinsic topology $P(R^n)$ is separated (T_2). In this case (see 1.10), $P_0(R^n) = \neg\neg\{0\} = \Delta(n)$. Recall (1.8) that $P_0(R^n)$ is the intrinsic monad of the origin in R^n, and that $\Delta(n)$ is just a notation for $\neg\neg\{0\}$. Recall also (1.11) that $C^g_0(R^n,R)$ denotes the object (in this case a ring) of germs at 0 of real valued functions in n-variables, and that $J^g_0(R^n,R) = R^{\Delta(n)}$ is the respective object of infinitesimal germs - both now considered with respect to the intrinsic topological structure.

3.1 AXIOM III

For each positive integer n, the restriction map

(a) $$C^g_0(R^n,R) \xrightarrow{j} R^{\Delta(n)}$$

is invertible. This axiom is divided in two parts, the injectivity and the surjectivity of j. Given $f,g \in \mathrm{Partial}(R^n,R)$, $U = \mathrm{Domain}(f)\cap\mathrm{Domain}(g)$, $0 \in U \in P(R^n)$,

III_1: $f|_{\Delta(n)} = g|_{\Delta(n)} \Rightarrow \exists V(0\in V\in P(R^n)$ and $f|_V = g|_V)$ or, equivalently,

$\forall x[\neg(x\neq 0) \Rightarrow f(x) = g(x)] \Rightarrow \forall x[(x\neq 0) \vee f(x) = g(x)]$

and

III_2: $\forall f \in R^{\Delta(n)}\ \exists g \in \mathrm{Partial}(R^n,R)[0\in\mathrm{Domain}(g)\in P(R^n)$ and $(g|_{\Delta(n)} = f)]$ which could, by abuse of notation, be written:

$\forall f \in R^{\Delta(n)}[\exists U(0\in U\in P(R^n)$ and $\exists g \in R^U(g|_{\Delta(n)} = f)]$.

In the presence of Postulate A (2.5), III_1 and III_2 also hold for the Zariski topological structure, and under Postulate WA1(2.11), they also hold for the euclidean topology. Thus, the existential quantifiers actually mean $\exists\epsilon \in R, \epsilon > 0$ etc.

Actually, a stronger form of Axiom III will sometimes be needed. This concerns germs defined around some "closed submanifold" rather than around a point of R^n.

Given X,Y equipped with topological structures S(X), S(Y), germs from X to Y around parts F of X, $F \in \Omega^X$, are defined exactly in the same way as in 1.11, where $F = \{x\}$, $x \in X$. For each $F \in \Omega^X$, we have an object $C^g_F(X,Y)$ whose elements are equivalence classes of pairs (f,U), where $F \subset U \in S(X)$, and $U \xrightarrow{f} Y$ continuous. The equivalence relation is $(f,U) \sim (f',U')$ if $\exists W \in S(X)$, $F \subset W \subset U \cap U'$ such that $f|_W = f'|_W$. More precisely, and without abuse of notation, a representative for a germ around F is an element $f \in \text{Partial}(X,Y)$ such that $F \subset \text{Domain}(f) \in S(X)$, and f continuous.

The stronger form of AXIOM III is the following:

For any two positive integers k,n, the restriction map

$$(b) \qquad C^g_{R^k x\{0\}}(R^k x R^n, R) \xrightarrow{j} R^{R^k x \Delta(n)}$$

is invertible. We have: under the assumption that $P(R^n)$ satisfies the covering principle for each n, the injectivity of the map j in (a) implies the injectivity of the map j in the stronger form (b).

In fact, let $f,g \in \text{Partial}(R^k x R^n, R)$, $U = \text{Domain}(f) \cap \text{Domain}(g)$, $R^k x\{0\} \subset U \in P(R^k x R^n)$, be such that $f|_{R^k x \Delta(n)} = g|_{R^k x \Delta(n)}$. Let G be the object $G := \{(p,x) \in U \mid f(p,x) = g(p,x)\}$. By III_1, for each $p \in R^k$ there exists $H_p \in P(R^n)$, $0 \in H_p$, such that $H_p \subset G_p = \{x \mid (p,x) \in G\}$. Then, by lemma 1.7 there exists $W \in P(R^k x R^n)$, $R^k x\{0\} \subset W \subset G$. this proves the claim.

However, for the surjectivity we cannot argue as above. We thank M. Makkai who pointed out to us that such an argument, used in an earlier version of this paper in Proposition 3.5, required an unlawful use of choice.

In practice, the validity of III_2, either in (a) or (b) is often guaranteed by the fact that the restriction map from global functions, $R^{R^k x R^n} \to R^{R^k x \Delta(n)}$, is an epimorphism. This is notably the case in all well adapted models in the sense of [7].

Generally speaking, in the algebraic models, III_1 holds and III_2 fails (cf. [32]). We see this synthetically as follows. In the algebraic models, maps $R^n \to R$ are polynomials in n variables and maps $U \to R$, where U is an affine Zariski open,

are rational functions (with denominator any polynomial h such that $U = h^{-1}(R^*)$). On the other hand, $\Delta(n) = D_\infty(n)$, so by Proposition 1.9. (Part I) maps $\Delta(n) \to R$ are formal power series in n variables. Remark now that affine Zariski opens are a base for the Zariski topology, so that in Axiom III we can consider all the Zariski open neighbourhoods of 0 to be affine. Thus, III_1 says that any two rational functions with same Taylor series are equal, which holds. But III_2 says that any formal power series is the Taylor development of a rational function, which fails.

In the models constructed with C^∞-rings we can survey the situation (informally) as follows.

In these models, maps $R^n \to R$ are essentially "smooth maps" in n variables, and the open neighbourhoods in Axiom III are euclidean opens. In the smaller models, maps from $\Delta(n)$ to R actually extend to the whole space; so III_2 holds; but $\Delta(n)$ is too small to render the validity of III_1. For example, in the Cahier Topos [7], and in the topos $\mathcal{F}$ defined by universally closed ideals (for the Frechet topology) of $C^\infty(\mathbb{R}^n)$, $\Delta(n) = D_\infty(n)$ again. So III_1 says that any two smooth functions which have the same derivatives at 0 agree in an euclidean neighbourhood of 0 - which fails due to the existence of flat functions. In the model $\mathcal{G}$ [9] defined by germ determined ideals of $C^\infty(\mathbb{R}^n)$, III_2 holds as before, and this time there are enough infinitesimals in $\Delta(n)$ to make III_1 also true. In larger models, for example in the topos $\mathcal{Z}$ [27] defined by all ideals of $C^\infty(\mathbb{R}^n)$, the object $\Delta(n)$ is not representable, and the situation of axiom III is not clear to us at the moment.

We shall now illustrate the use of axiom III in the problem of local integration of vector fields. We have seen (3.1, part I) that given a vector field on R^n, there exists a unique integral flow defined in $R^n \times D_\infty$. However, there is no synthetic reason why we should be able to go beyond. Thus, in the abstract theory, we have to postulate

3.2 POSTULATE WA2

The restriction map

$$\mathrm{Flow}(R^n \times \Delta, R^n) \to \mathrm{Flow}(R^n \times D, R^n)$$

is invertible (where Flow indicates the object of flows).

This means (see 3.1, 3.2 part I) that given a vector field $R^n \overset{\xi}{\to} R^{n^D}$ there exists a unique flow $R^n \times \Delta \to R$ such that $f(x,t+d) = \xi(f(x,t),d)$ for all $t \in \Delta$, $d \in D$.

On account of 3.2 (d)(part I) we can write this postulate in the following equivalent form.

3.3 POSTULATE WA2 (ALTERNATIVE FORMULATION)

The map

$$\mathrm{Flow}(R^n \times \Delta, R^n) \to R^{n^{R^n}}, \quad f \mapsto \frac{df}{dt}(x,t)|_{t=0}$$

is invertible.

This means that given a function $R^n \overset{g}{\to} R^n$, the differential equation $\frac{df}{dt}(x,t) = g(f(x,t))$ with initial values $f(x,0) = x$, has a unique solution $f(x,t)$ defined for all $x \in R^n$ and $t \in \Delta$.

We now make precise what we mean by a local integral flow of a vector field in R^n.

3.4 DEFINITION

Given a vector field $R^n \overset{\xi}{\to} R^{n^D}$, a local integral flow of ξ is an integral flow $G \overset{f}{\to} R^n$ (cf. 3.2 part I) such that $G \subset R^n \times R$ is a P-neighbourhood (cf. 1.5) of $R^n \times \{0\}$. Assume now the validity of Postulate WA1 (2.11). Then $P = E$ in $R^n \times R$. Thus G is an euclidean neighbourhood. This means the following (use 2.7): for all $p \in R^n$, there exists $U \in E(R^n)$ with $p \in U$, and there exists $\epsilon \in R$, $\epsilon > 0$, such that $U \times (-\epsilon,\epsilon) \subset G$ and $f(x,s+t) = f(f(x,s),t)$ for all $x \in U$, $s \in (-\epsilon,\epsilon)$, $t \in (-\epsilon,\epsilon)$ (take ϵ sufficiently small so that the points $(x,s+t)$ and $(f(x,s),t)$ belong to G). Thus, $f(x,t)$ is a solution of the differential equation associated to ξ (3.2 part I) defined on $U \times (-\epsilon,\epsilon)$ (for each $p \in R^n$).

We see how Axiom III furnishes the way to pass from the infinitesimal to the local in the following proposition.

3.5 PROPOSITION

Given a flow $R^n \times \Delta \xrightarrow{f} R^n$, there exists (uniquely) a local flow that extends f. The uniqueness means that any two such extensions agree in a neighbourhood of $R^n \times \{0\}$.

Proof. Axiom III directly says that there are extensions $G \xrightarrow{g} R$, $R^n \times \{0\} \subset G \subset R^n \times R$, and that any two such extensions agree in a neighbourhood of $R^n \times \{0\}$. It remains to check the flow equation. Let $H = [\![(x,s,t) \mid (x,s+t) \in G \text{ and } (g(x,s),t) \in G]\!]$. Clearly $R^n \times \{0\} \subset H \subset R^n \times R^2$ and $H \in P(R^n \times R^2)$. Let g_1 and g_2 defined on H be, $g_1(x,s,t) = g(x,s+t)$, and $g_2(x,s,t) = g(g(x,s),t)$. Since f is a flow and g extends f, g_1 and g_2 agree on $R^n \times \Delta(2)$. Then, again by axiom III, they also agree in an open $W \in P(R^n \times R^2)$, $R^n \times \{0\} \subset W \subset H$. Since addition is an open map for the euclidean topology E (trivial verification) and P = E, it follows that g satisfy the flow equation in a P-neighbourhood (possible smaller than G) of $R^n \times \{0\}$. □

A consequence of this proposition is that maps $R^n \times \Delta \to R^n$, (or their equivalent forms $R^n \to R^{n^{\Delta}}$, $\Delta \to R^{R^n}$) express in a simple way all the data involved in a local flow; i.e., Δ-infinitesimal flows are already local flows. A solution of a differential equation for all $x \in R^n$ and all $t \in \Delta$ is already a local solution (of course, all this holds only in the presence of axiom III). This should lead to considerable simplifications in any concept, construction or calculation that involves local solutions of differential equations.

The passage from the local to the global involves the notion of compactness, and it can be proved by purely synthetic arguments without need of any further postulates, except for the archimedeaness of the relation "<" in R.

We say that "<" is archimedian if it satisfies

R5) $\forall x \in R \quad \exists n \in \mathbb{N}\ (x \in (-n,n))$,

(where $\mathbb{N}$ is the natural number of object in $\mathcal{E}$).

We recall now briefly the notion of compactness in a topos introduced and developed in [14].

3.6 DEFINITION

An object K in $\mathcal{E}$ is *compact* if:

(1) $\forall G \in \Omega^K \quad \forall H \in \Omega \; [G \cup \pi^{-1}H = K \Rightarrow \forall_\pi G \cup H = \text{true}]$,

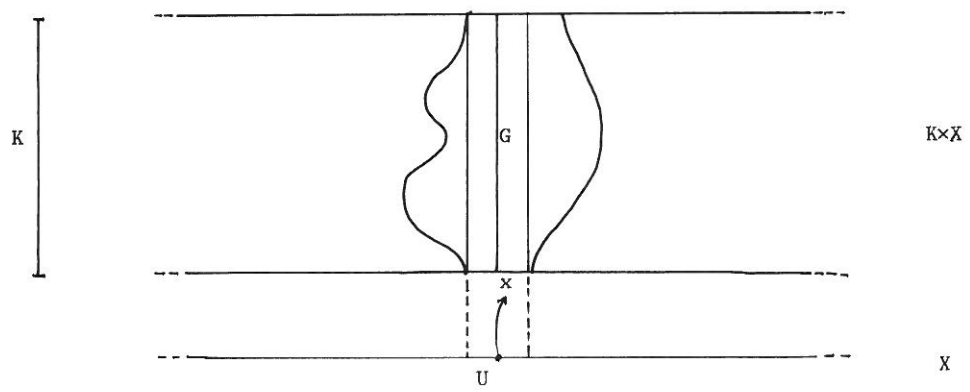
where π is the unique map $K \xrightarrow{\pi} 1$. Notice that $H \subset 1$ and $\pi^{-1}H = K \times H$. From (1) it immediately follows

(2) Given any X in $\mathcal{E}$, $G \subset K \times X$ and $H \subset X$,

$G \cup \neg(K \times H) = K \times X \Rightarrow \forall_\pi G \cup \neg H = X$,

where π is the projection $K \times X \to X$.

For example, if H is a point $x \in X$, this says that if the axis $K \times \{x\}$ is well contained in G (in particular, if G is a P-neighbourhood of $K \times \{x\}$), then $\forall_\pi G$ is a P-neighbourhood of x. This means that there is an open U, $x \in U \in P(X)$ such that $\pi^{-1}U \subset G$: a characteristic property of compactness.



It is shown in [14] that this definition means essentially that the projections $K \times K \to X$ are closed maps. It is also shown that in all the algebraic models, if K is a scheme, then K is compact if and only if it is complete in the classical sense.

In the well adapted models, if K is a manifold, then K is compact if and only if it is compact in the classical sense. More generally, if K is a C^∞-scheme (including all representable objects), then K is compact if and only if the set of global sections $\Gamma(K)$ (furnished with its natural euclidean topology induced by the classical topology on $\mathbb{R}^n$) is a compact space in the usual sense.

3.7 DEFINITION

A vector field $R^n \times D \xrightarrow{\xi} R^n$, respectively a flow $R^n \times \Delta \xrightarrow{f} R^n$, is of *compact support* if there is a compact K and a part $H \subset R^n$ such that $K \subset R^n$, $K \cup H = R^n$ and for every $x \in H$, $\xi(x,d) = x$, $\forall d \in D$, respectively $f(x,t) = x$, $\forall t \in \Delta$. This means that the function $g: R^n \to R^n$, $\xi(x,d) = x+g(x)d$ which defines the differential equation associated to ξ is zero on H. Thus outside K, since clearly $\neg K \subset H$.

3.8 PROPOSITION

If R satisfies R5) (i.e., is Archimedian) then every flow $R^n \times \Delta \xrightarrow{f} R^n$ of compact support has a unique global extension $R^n \times R \xrightarrow{h} R^n$.

PROOF. Take any local extension $G \xrightarrow{g} R^n$, $R^n \times \{0\} \subset G \subset R^n \times R$ given by 3.5. We can assume that $g(x,t) = x$ for all $(x,t) \in G \cap H \times R$. Indeed, by assumption $g(x,t) = x$ for all $x \in H$, $t \in \Delta$. Then arguing as in 3.5 the claim follows (for a possibly smaller G). Now, let F be $F = G \cap (K \times R)$, $K \times \{0\} \subset F \subset K \times R$. F is a P- (or E) neighbourhood of $K \times \{0\}$ (cf. 2.6). Take $\epsilon \in R$, $\epsilon > 0$, $K \times (-\epsilon, \epsilon) \subset F$ (such an ϵ is given by 3.6 since P = E). The maps $K \times (-\epsilon,\epsilon) \xrightarrow{g} R^n$ and $H \times (-\epsilon,\epsilon) \xrightarrow{\ell} R^n$, $\ell(x,t) = x$ are compatible in the intersection. This defines a flow $R^n \times (-\epsilon,\epsilon) \xrightarrow{g} R^n$ that extends $R^n \times \Delta \xrightarrow{f} R^n$. A classical argument now shows that g has a unique global extension. Let $g^t: R^n \to R^n$ denote the map $g(-,t)$. These maps are defined for all $t \in (-\epsilon,\epsilon)$. Let now t be any $t \in R$. Take integers n,k such that $x \in (-n,n)$ and $1/\epsilon \in (-k,k)$. Let h^t be $h^t = g^{t/nk} \circ \ldots \circ g^{t/nk}$; that is, the map $g^{t/nk}$ composed with itself nk times. It readily follows (see exercise below) that this definition is independent of n,k and that it defines the unique global flow extending h. Any two extensions of f coincide on some (maybe smaller) ϵ-interval. It is clear then (by construction) that h is unique with respect to f. □

EXERCISE. Show that any map $(-\epsilon,\epsilon) \overset{\varphi}{\to} M$ into a monoid M such that $\varphi(0) = e$ (neutral element) and, for any $t,s,t+s \in (-\epsilon,\epsilon)$, $\varphi(t+s) = \varphi(t)\varphi(s)$, has a unique extension into a morphism of monoids $R \to M$ (R archimedean).

We finish with this. There is, however, one more axiom that has to be considered (although it is not utilized in this paper). The Axiom I (of jet representability) has a companion axiom which establishes the "tinyness" (or infinitesimal character) of the representing objects $D_r(n)$. This is Axiom II. In the same way we state a companion axiom for the Axiom III (of germ representability).

3.9 AXIOM IV

The functor $(-)^{\Delta(n)}: \mathcal{E} \to \mathcal{E}$, $x \mapsto x^{\Delta(n)}$, has a right adjoint, $(-)_{\Delta(n)}: \mathcal{E} \to \mathcal{E}$, $x \mapsto x_{\Delta(n)}$. (This is equivalent to the fact that exponentiation with $\Delta(n)$ preserves epimorphic families (or colimits) in the case of Grothendieck topoi.)

In conclusion, we can say that SDG has an infinitesimal aspect, and a local aspect.

Infinitesimal Aspect

AXIOM I, AXIOM II. Jet representability and tinyness of the representing objects.

It applies to all models. The notion of *infinitesimal linearity* establishes the right level of generality for the objects to work with.

Local Aspect.

POSTULATE A.

It is divided in two parts,

a) The line R is a field in the sense of Kock. This means that the Zariski topological structure is subintrinsic and that the space R^n is separated.

b) There are enough Zariski opens in R^n (covering principle). In particular, the Zariski and the intrinsic topological structures coincide in R^n.

It is oriented to the algebraic models, but it applies to all the models.

POSTULATE WA1.

It is divided in two parts,

a) The line R has an order relation which is: R1) compatible with the ring structure; R2) strict; R3) local; R4) separated. In particular, this means that the euclidean topological structure is subintrinsic and separated (in R^n and all its subobjects). We may additionally require, R5) archimedian.

b) There are enough euclidean opens in R^n (covering principle). In particular, the euclidean and the intrinsic topological structures coincide in R^n.

It applies to all models constructed using C^∞-rings; when R5) is required, only to the well adapted models ([7],[21]).

POSTULATE WA2

Δ-integration of vector fields in R^n.

It applies to all well adapted models constructed using C^∞-rings. The validity of this postulate in the larger models, [27], is not clear to us at the moment. This postulate is not meaningful unless Axiom III holds (for example, if $\Delta = D_\infty$ it is empty of content).

AXIOM III, AXIOM IV

Germ representability and tinyness of the representing objects.

It applies to the study of classical differential geometry and manifolds. The only known model is the topos $\mathcal{G}$, [9] (the validity of the larger models [27] is unclear to us at the moment). The general objects for which Axiom III holds seem to be very close to the classical notion of manifold. In [32] it is shown that an internally defined (of finite presentation) $\Delta(n)$-manifold is necessarily a manifold in the classical sense (this is not surprising).

Axiom III, Postulate W2 and the results about integration of vector fields (3.5, 3.8) can be generalized by the classical local methods to any $\Delta(n)$-manifold. but probably only to these objects; this remains to be investigated.

APPENDIX. SURVEY OF THE MODELS

All models are constructed following a similar pattern.

1.1

There is a category $\mathcal{C}$ together with a class $\mathcal{L} \subset \mathcal{C}$ of maps such that

i) isomorphisms are in $\mathcal{L}$

ii) closed under pull-back (and composition).

In addition, there is a Grothendieck (pre) topology in $\mathcal{C}$ such that

a) all maps in every covering family are in $\mathcal{L}$.

The model is the topos $\mathcal{E}$ of sheaves over $\mathcal{C}$ (cf. [11], [32]).

Axiom I has been extensively analyzed in the models. Its validity in the algebraic models was recognized by Lawvere before the existence of SDG. We will not consider it here. Axioms III and IV hold essentially because by construction, in all the models, the objects $D_r(n)$, $\bar{W}$ in general and $\Delta(n)$ do not admit any proper covers. We pass now to analyze the content of Part II.

The class $\mathcal{L} \subset \mathcal{C}$ determines a topological structure in the topos $\mathcal{E}$. This structure, inherited from the construction of the topos, plays an important role.

1.2 THE SPECTRAL TOPOLOGICAL STRUCTURE

Consider for every C in $\mathcal{C}$ the class of subobjects, denoted $\mathcal{L}_C$, defined by, $U \rightarrowtail C \in \mathcal{L}_C$ if there exists a family $C_\alpha \xrightarrow{f_\alpha} C$ in $\mathcal{L}$ such that each f_α factors $C_\alpha \to U \rightarrowtail C$ and the family $C_\alpha \to U$ is epimorphic in $\mathcal{E}$.

That is, $U \rightarrowtail C$ is an (external) union of subobjects $U_\alpha \rightarrowtail C$ of the form $U_\alpha = \text{Image}(f_\alpha)$ for $f_\alpha \in \mathcal{L}$.

It is straightforward to check that the following holds,

1) Given any map $D \to C$ in $\mathcal{C}$, and a pull-back diagram

$$\begin{array}{ccc} W & \to & U \\ \downarrow & & \downarrow \\ D & \to & C \end{array}$$

$$U \in \mathcal{L}_C \quad \Rightarrow \quad W \in \mathcal{L}_D.$$

2) Given any covering family $C_\alpha \to C$, and pull back diagrams

$$\begin{array}{ccc} W_\alpha & \to & U \\ \downarrow & & \downarrow \\ C_\alpha & \to & C \end{array}$$

$$\forall\alpha\ W_\alpha \in \mathcal{L}_{C_\alpha} \Rightarrow U \in \mathcal{L}_C.$$

Consider now a subobject $B \subset \Omega$. For every C in $\mathcal{C}$ it determines a class of subobjects $\mathcal{L}_C$ as follows,

(*)
$$\frac{U \rightarrowtail C \in \mathcal{L}_C}{\begin{array}{ccc} C & \to & \Omega \\ & \searrow & \uparrow \\ & & B \end{array}}$$

where the map $C \to \Omega$ is the characteristic function of U. The correspondence (*) establishes an equivalence between classes of subobjects such that 1) and 2) above and subobjects $B \subset \Omega$.

PROOF. Clearly, given $B \subset \Omega$, 1) and 2) above are satisfied. For the other implication, define a function on the objects of $\mathcal{C}$,

$$\mathcal{C}^{op} \to \text{Sets}, \quad C \mapsto \mathcal{L}_C.$$

This defines a pre-sheaf by 1), which is a sheaf by 2). Thus it determines an object B in $\mathcal{E}$, which by construction is a subobject B of Ω and satisfies the correspondence (*). □

The class of subobjects $\mathcal{L}_C$ defined above contains the maximal subobject $C \overset{id}{\rightarrowtail} C \in \mathcal{L}_C$ and is closed under intersections, thus the corresponding object, that we denote B_s in this case, is a subinf-lattice of Ω. The *spectral topological structure* in the topos $\mathcal{E}$ is the topological structure generated by the classifiable base B_s. Thus, by definition we have (cf. [32])

$$U \in \text{Spec}(X) \iff \forall x \in U \ \ \exists V \in B_s^X (x \in V \subset U)$$

1.3

For every C in $\mathcal{C}$, Spec(C) satisfies the covering principle (cf. [32]).

PROOF. Let $H,G \in \Omega^C$ be such that $H \cup G = C$. We have $D \xrightarrow{H} \Omega^C$, $D \xrightarrow{G} \Omega^C$ for some D in $\mathcal{C}$. The corresponding subobjects $H \rightarrowtail C\times D$, $G \rightarrowtail C\times D$ are such that $H \cup G = C\times D$. This means that there is a cover $E_\alpha \xrightarrow{f_\alpha} C\times D$ such that for each α either $E_\alpha \to H$ or $E_\alpha \to G$. The unions (of the respective) Image(f_α) furnish two subobjects U,V in $\mathcal{L}_{C\times D}$ such that $U \subset H$, $V \subset G$ and $UUV = C\times D$. Then $C\times D \xrightarrow{U} B_s$ thus $D \xrightarrow{U} B_s^C$. This means that $U \in B_s^C$. Similarly $V \in B_s^C$. But $B_s^C \subset \mathrm{Spec}(C)$. Thus $U,V \in \mathrm{Spec}(C)$, $U \subset H$, $V \subset G$ and $U \cup V = C$ (clearly). □

Let us denote by $\mathcal{E} \xrightarrow{\Gamma}$ Sets the global sections functor. Then, by definition, $\mathcal{L}_C = \Gamma(B_s^C)$ is a locale in Sets. It follows easily (using the fact that the objects of $\mathcal{C}$ generate $\mathcal{E}$) that for any X in $\mathcal{E}$, $\Gamma(B_s^X)$ is also a locale in Sets; but not a sublocale of $2^{\Gamma(X)}$ unless Γ preserves unions.

1.4 THE ZARISKI TOPOLOGY

In all the models, $R \in \mathcal{C}$, $1 \neq 0$, and $R^* \subset R \in \mathcal{L}_R$ (it is always trivial to see this). Thus the characteristic function φ of R^* factors through B_s. It follows that $\Omega^* \subset B_s$ (cf. 2.1 part II). Therefore for each X in $\mathcal{E}$, $\mathrm{Aff}(x) \subset \Omega_*^X \subset B_s^X$. Thus,

$$\mathrm{Zar}(X) \subset \mathrm{Zar}_2(X) \subset \mathrm{Spec}(X).$$

The external affine opens of $X \in \mathcal{E}$ are the subobjects $U \rightarrowtail C$ of the form $U = f^{-1}R^*$ for some map $X \to R$ in $\mathcal{E}$. They form a subinflattice of $\Gamma(\Omega^X)$. The set of *external Zariski opens* of X, denoted $\mathrm{ExZar}(X)$ consists of all the (external) unions in $\mathcal{E}$ of affine opens. It is a locale in Sets, but not a sublocale of $2^{\Gamma(X)}$ unless Γ preserves unions. Since $\mathrm{Aff}(X) \subset B_s^X$, $\Gamma(\mathrm{Aff}(X)) \subset \Gamma(B_s^X)$. It follows

$$\mathrm{ExZar}(X) \subset \Gamma(B_s^X).$$

For an arbitrary X in $\mathcal{E}$ this will always be a proper inclusion. But not so for objects in $\mathcal{C}$. In the Zariski topos and in all models constructed using C^∞-rings, the class $\mathcal{L} \subset \mathcal{C}$ is the class of (external) affine opens. Thus $\Gamma(B_s^C) = \mathcal{L}_C \subset \mathrm{ExZar}(C)$. Therefore, we have

$$\mathrm{ExZar}(C) = \Gamma(B_s^C) = \mathcal{L}_C \quad \text{for all } C \in \mathcal{C}.$$

1.5

If $R \in \mathcal{C}$ and the class $\mathcal{L} \subset \mathcal{C}$ is the class of (external) affine opens, then

$$\mathrm{Zar}(C) = \mathrm{Zar}_2(C) = \mathrm{Spec}(C) \quad \text{for all } C \in \mathcal{C}.$$

PROOF. It only remains to see $\mathrm{Spec}(C) \subset \mathrm{Zar}(C)$. It will be enough to show $B_s^C \subset \mathrm{Zar}(C)$. Let $U \in B_s^C$, then $D \xrightarrow{U} B_s^C$ for some D in $\mathcal{C}$. By definition the corresponding subobject $U \rightarrowtail C\times D \in \mathcal{L}_{C\times D} = \mathrm{ExZar}(C\times D)$. But clearly we have $\mathrm{ExZar}(C\times D) \subset \Gamma(\mathrm{Zar}(C\times D))$. Since we also have $\mathrm{Zar}(C\times D) \subset \mathrm{Zar}(C)^D$ (cf. part II) it follows that $1 \xrightarrow{U} \mathrm{Zar}(C)^D$. Thus $D \xrightarrow{U} \mathrm{Zar}(C)$. This shows that $U \in \mathrm{Zar}(C)$. □

1.6

For every C in $\mathcal{C}$, $\mathrm{Zar}(C)$ satisfies the covering principle (by 1.3 and 1.5). □

Now it is necessary to assume further properties in the sets $\mathcal{C}$.

1.7

We assume that the Grothendieck (pre) topology in 1.1 in addition to (a) also satisfies

b) If $C_\alpha \to C$ covers, then $\Gamma C_\alpha \to \Gamma C$ is a surjective family of sets

c) If $C_\alpha \to C$ are in $\mathcal{L}$, and $\Gamma C_\alpha \to \Gamma C$ is a surjective family of sets, then $C_\alpha \to C$ covers.

These assumptions are satisfied by definition in all the algebraic models and in all the well adapted models. 1.5 and 1.6 were proven under the assumption that for all C in $\mathcal{C}$, the maps in $\mathcal{L}$ were Zariski affine open subobjects. With (a) and (b) we can weaken considerably this assumption so as to include the other algebraic models: Topos Etale, Topos Flat. Assume,

d) Every affine open $U \rightarrowtail C$ is in $\mathcal{L}$, and every $D \xrightarrow{f} C$ in $\mathcal{L}$ is such that $\mathrm{Im}(\Gamma(f)) \subset \Gamma(C)$ is a union of sets of the form $\Gamma(U)$ with U affine open.

It is well known in algebraic geometry that flat (in particular etale) morphisms are open maps. In particular $\mathrm{Im}(\Gamma(f))$ is a Zariski open of $\Gamma(C)$. Thus (d) is satisfied.

1.8 (Penon)

If $D \xrightarrow{f} C$ is in $\mathcal{L}$, then $\mathrm{Im}(f) \subset C$ is a union in $\mathcal{E}$ of affine opens.

PROOF. By (d) there exist affine opens $U_\alpha \rightarrowtail C$ such that, $\Gamma(\mathrm{Im}(f))$ is the union of $\Gamma(U_\alpha)$. Consider the diagrams

$$\begin{array}{ccc} W_\alpha & \to & U_\alpha \\ \downarrow & & \downarrow \\ D & \xrightarrow{f} & C \end{array} \quad , \quad \begin{array}{ccccc} \Gamma(W_\alpha) & \to & \Gamma(U_\alpha) & & \\ \downarrow & & \downarrow & \searrow & \\ \Gamma(D) & \twoheadrightarrow & \mathrm{Im}(\Gamma(f)) & \rightarrowtail & C \end{array}$$

where the square in the left is a pull-back by definition. Since $f \in \mathfrak{L}$, $W_\alpha \to U_\alpha \in \mathfrak{L}$, but $\Gamma(W_\alpha) \to \Gamma(U_\alpha)$ is surjective (since the right square is also a pull-back) thus by (c) it follows that $W_\alpha \to U_\alpha$ covers. On the other hand, $\Gamma(W_\alpha) \to \Gamma(D)$ is also surjective. By the same reason then $W_\alpha \to D$ covers

$$\begin{array}{ccccc} W_\alpha & \to & U_\alpha & & \\ \downarrow & & \downarrow & \searrow & \\ D & \twoheadrightarrow & \mathrm{Im}(f) & \to & C \end{array}$$

The arrow $U_\alpha \to \mathrm{Im}(f)$ exists because $W_\alpha \to U_\alpha$ covers, and since $W_\alpha \to D$ covers it follows that $U_\alpha \to \mathrm{Im}(f)$ is an epimorphic family in $\mathcal{C}$. This proves the claim. □

1.9

$$\mathrm{ExZar}(C) = \Gamma(B_S^C) = \mathfrak{L}_C \quad \text{for all} \quad C \in \mathcal{C}.$$

PROOF. Since every affine open is in $\mathfrak{L}_C$, clearly $\mathrm{ExZar}(C) \subset \mathfrak{L}_C$. The other inclusion follows immediately from 1.8. □

1.10

If $R \subset \mathcal{C}$ and the site $\mathcal{C}$ satisfies 1.1(a) and 1.7(b)(c)(d), then

$$\mathrm{Zar}(C) = \mathrm{Zar}_2(C) = \mathrm{Spec}(C) \quad \text{for all} \quad C \in \mathcal{C}.$$

PROOF. Same proof as 1.5. □

1.11

For every C in $\mathcal{C}$, Zar(C) satisfies the covering principle (by 1.3 and 1.10).

□

It is well known (cf [22] for example) that a simple argument in sheaf semantics shows (in all the models) that $\neg\{0\} = R^*$. So, we will assume this.

We say that the site $\mathcal{C}$ has finite affine covers when every affine open $U \to C$ is in $\mathcal{L}$, and finitely many $U_\alpha \to C$ covers if $\Gamma(U_\alpha) \to \Gamma(C)$ is a surjective family of sets.

1.12

If $\neg\{0\} = R^*$ and $\mathcal{C}$ has finite affine covers, then R is a field in the sense of Kock.

PROOF. Clearly, given $0 \in R^n$, $\pi_i^{-1}\Gamma(\neg\{0\}) \to \Gamma(\neg\{0\})$ is a surjective family of subsets of $\Gamma(R^n)$ (π_i = projections). Thus $\Gamma\pi_i^{-1}(R^*) \to \Gamma(\neg\{0\})$ also is surjective. It follows that given any $C \xrightarrow{f} \neg\{0\}$, the family $f^{-1}\pi_i^{-1}(R^*) \to C$ covers. This shows that $\pi_i^{-1}(R^*) \to \neg\{0\}$ is epimorphic in $\mathcal{E}$. □

1.13

Postulate A holds in all the models. Furthermore it holds for all representables C. Also $Zar_2(C) = Zar(C)$.

PROOF. By 1.6, 1.11 and 1.12.

We now pass to consider the Euclidean Topology. This topology concerns primarily the models constructed with C^∞-rings, which were considered originally as a way to relate SDG with classical differential geometry and manifolds. The use of C^∞-rings was already suggested by Lawvere in his 1967 lectures in Categorical Dynamics [25]. Let $\mathcal{R}$ be the category of open sets in the euclidean spaces $\mathbb{R}^n$ (all non negative integers) and smooth maps. There should be an embedding of $\mathcal{R}$ into the model $\mathcal{R} \to \mathcal{E}$ which takes $\mathbb{R}$ into the line R. The essential characteristic of a C^∞-manifold (actually by definition) is that it is a topological space which locally is an open set of euclidean space. Also, the implicit function theorem means that manifolds can also be seen as topological spaces which are locally cut out of euclidean space by finite sets of smooth independent equations. The leading idea in the introduction of the concept of well adapted model in [7] was that this fact should remain true in the model. This actually forces two conditions on the embedding $\mathcal{R} \to \mathcal{E}$:

(1) Preservation of transversal pull-backs

(2) Preservation of all open covers.

The first such model was constructed in [7] by means of somehow primitive techniques. The fundamental theorem for the construction of models using C^∞-rings says that the functor

$$\text{Manifolds} \to (C^\infty\text{-rings})^{op}, \quad M \mapsto C^\infty(M)$$

preserves transversal pull-backs. A basic corollary of this is that given any open set $U \subset \mathbb{R}^n$ and smooth map $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ such that $U = f^{-1}\mathbb{R}^*$, the ring $C^\infty(U)$ is the C^∞-localization $C^\infty(\mathbb{R}^n)\{f^{-1}\}$. This theorem and its corollary were first proved in [9]. They mean that the well developed techniques utilized in algebraic geometry can be used to construct models with C^∞-rings satisfying condition (1) above. In [9] it was also introduced the notion of ideal of local character (that is, the ideals of definition for germ determined C^∞-rings) and shown that in the category (germ determined C^∞-rings)^{op} the open cover topology is subcanonical (cf. [10] also). This provides the (well adapted) models satisfying also condition (2) above. These techniques were subsequently utilized in [26], [27], where it is proved that the finite open cover topology is subcanonical in the category of all C^∞-rings (of finite type). This provides models satisfying condition (2) only for finite covers, which permits the existence of invertible infinitesimals.

1.14 The Euclidean Topology

Given a site $\mathcal{C}$ as in 1.1, we assume that there is an embedding

$$\mathcal{R} \xrightarrow{i} \mathcal{C}, \quad \mathbb{R} \mapsto R$$

which preserves transversal pull-backs, and that $R^* = \neg\{0\}$. We have seen in the comments above that this is satisfied by all the models constructed using C^∞-rings. We recall that finite products and inverse images of open subsets are particular cases of transversal pull-backs. Thus they are preserved by the embedding.

The ring R has a relation "<" which we shall denote $R_< \subset R\times R$. By definition, $R_< = i\mathbb{R}_<$, where $\mathbb{R}_<$ is the open set $\mathbb{R}_< \subset \mathbb{R}\times\mathbb{R}$ defined by the usual order relation on $\mathbb{R}$.

1.15

Given any $a,b \in \mathbb{R}$, $a < b$, $i(a,b) = (a,b)$. It follows that if $x \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$, then $iB(x,\epsilon) = B(x,\epsilon)$.

PROOF. Consider the diagrams

$$\begin{array}{ccc} (-\infty,b) & \to & R_< \\ \downarrow & & \downarrow \\ \mathbb{R} & \xrightarrow{(id,b)} & \mathbb{R}\times\mathbb{R} \end{array} \quad , \qquad \begin{array}{ccc} (-\infty,b) & \to & R_< \\ \downarrow & & \downarrow \\ R & \xrightarrow{(id,b)} & R\times R \end{array} \quad .$$

They are both pull-backs, and the left square is transversal. Thus $i(-\infty,b) = (-\infty,b)$. Similarly $i(a,+\infty) = (a,+\infty)$. By intersection it follows that $i(a,b) = (a,b)$. The rest is straightforward. □

1.16

$i\mathbb{R}^* = R^*$. This follows similarly utilizing the transversal pull-back,

$$\begin{array}{ccc} R^* & \rightarrowtail & \mathbb{R}\times\mathbb{R} \\ \downarrow & & \downarrow \text{ product} \\ 1 & \xrightarrow{1} & \mathbb{R} \end{array} \quad , \ \square$$

We assume now that in $\mathcal{C}$ we have finite open covers. This means that if $U \subset \mathbb{R}^n$ is open, then $iU \subset R^n \in \mathcal{L}$, and that if finitely many U_α cover $U \subset \mathbb{R}^n$, then the iU_α cover iU in $\mathcal{C}$.

1.17

$R_<$ satisfies R1), R2), R3) and R4), and if $\mathcal{C}$ has all open covers, then it also satisfies R5). (Cf. 2.12 Part II)

PROOF. R1) Compatibility with the ring structure. It is clear.

R2) $\neg(0 < 0)$. Suppose $1 \overset{(0,0)}{\rightarrowtail} R_<$. Then $1 \overset{(0,0)}{\rightarrowtail} \mathbb{R}_<$, which is not the case.

R3) $R = (-\infty,1)\cup(0,+\infty)$. Since $\mathbb{R} = (-\infty,1)\cup(0,+\infty)$. Use 1.15.

R4) $\pi_i^{-1}[(-\infty,0)\cup(0,+\infty)] \rightarrowtail \neg\{0\} \subset R^n$ is an epimorphic family in $\mathcal{E}$. We argue as in 1.12 to show that $\pi_i^{-1}(R^*) \rightarrowtail \neg\{0\}$ is an epimorphic family in $\mathcal{E}$. Since $(-\infty,0)\cup(0,+\infty) = \mathbb{R}^*$, we have $(-\infty,0)\cup(0,+\infty) = R^*$ in $\mathcal{E}$ (use 1.15 and 1.16). This clearly finishes the proof.

R5) $(-n,n) \rightarrowtail R$ all $n \in \mathbb{N}$ is an epimorphic family in $\mathcal{E}$. This is clear. Use 1.15. □

In order to prove the covering principle we make the following assumptions on the site.

1.18

(a) Every C in $\mathcal{C}$ is a subobject $C \rightarrowtail R^n$ for some non negative integer n, and every member of the class $\mathcal{L}$ is a subobject $V \rightarrowtail C$ of the form $V = C \cap iU$ for some open subset $U \subset \mathbb{R}^n$.

(b) If $V_\alpha \rightarrowtail C$ covers, $V_\alpha = C \cap iU_\alpha$, then $\Gamma(C) \cap U_\alpha \rightarrowtail \Gamma(C)$ is a union of subsets of $\mathbb{R}^n$.

(c) If $U_\alpha \subset U$ is a union of (open) subsets in $\mathbb{R}^n$, then $iU_\alpha \rightarrowtail iU$ is a covering family in $\mathcal{C}$.

Conditions (a) and (b) hold by definition in all the models, and (c) in all the well adapted models.

1.19

$$\Gamma(B_S^C) = \mathcal{L}_C \subset \Gamma(E(C)) \quad \text{for all} \quad C \in \mathcal{C}.$$

Recall that the equality in the left is true by definition, and that $E(C)$ denotes the euclidean topological structure on C.

PROOF. Let $V \rightarrowtail C$ be in $\mathcal{L}_C$. It follows from (a) that V is a union in $\mathcal{E}$ (cf. 1.2) of subobjects $V_\alpha \rightarrowtail C$ in $\mathcal{L}$, which are of the form $C \cap iU_\alpha$ for some open $U_\alpha \subset \mathbb{R}^n$. From (b) it follows in turn that $\Gamma(V)$ is a union in $\mathbb{R}^n$ or the subsets $\Gamma(C) \cap U_\alpha$. Now, each U_α is a union of basic opens of the form $B(x,\epsilon)$ (in $\mathbb{R}^n$). Then, from (c) it follows that V is the union in $\mathcal{E}$ of the corresponding $C \cap B(x,\epsilon)$. The claim follows then by 2.6 in part II and 1.15. □

1.20

Under the assumptions (a), (b), (c) in 1.18,

$$\mathrm{Spec}(C) \subset E(C) \quad \text{for all} \quad C \text{ in } \mathcal{C}.$$

PROOF. We do as in 1.5. It will be enough to show $B_S^C \subset E(C)$. Let $U \in B_S^C$, then $D \overset{U}{\rightarrow} B_S^C$ for some D in $\mathcal{C}$. By definition the corresponding subobject

$U \rightarrowtail C\times D \in \mathcal{L}_{C\times D} \subset \Gamma(E(C\times D))$. By 2.8 part II we have $E(C\times D) \subset E(C)^D$. It follows that $1 \xrightarrow{U} E(C)^D$. Thus $D \xrightarrow{U} E(C)$. This shows $U \in E(C)$. □

1.21

For every C in $\mathcal{C}$, $E(C)$ satisfies the covering principle (by 1.3 and 1.20). □

From 1.17 and 1.21 we have

1.22

Postulate WA1 holds in all the well adapted models. Furthermore, it holds for all representables C. □

This probably holds also in the larger models. In [27] it is shown that $E(R) = P(R)$ in the smooth Zariski model $\mathcal{Z}$. The proof can probably be adapted to show that for all representable objects $C \in \mathcal{C}$, the equation $E(C) = P(C)$ can be proved utilizing only finite open covers. Then by 1.11 and 1.13 $E(C) = Zar(C)$ and it satisfies the covering principle (we have not checked this).

Summarizing, we see (by 2.12 in Part II and 1.12) that there is a complete identification between all the natural topological structures that can be defined on the representable objects.

$$Zar(C) = Zar_2(C) = E(C) = P(C) = Spec(C).$$

We recall that all these topological structures on the objects of a ringed topos (except Zar_2) were introduced by J. Penon who also proved this result in all the algebraic cases (this is 1.12 here) and for the well adapted model $\mathcal{G}$ (cf. [32], where $\mathcal{G} = \mathbb{D}$).

It remains to analyze Axiom III and Postulate WA2. Their validity narrows the class of models to the single model $\mathcal{G}$ [at present the only model known to satisfy both; the situation in the larger models is unclear to us, and in the smaller models Axiom III does not hold]. The proof of Axiom III and Postulate WA2 for $\mathcal{G}$ depends heavily in the local character of the ideals of definition of the objects in the site $\mathcal{C}$. We send the reader to [13]. The proof of Axiom III can also be found in [32] and [12]. Utilizing the (infinitary) algebraic theory of germ determined C^∞-rings introduced in [10] larger models which satisfy Axiom III and Postulate WA2 can be constructed. We quote from [9] where the model $\mathcal{G}$ was introduced: "The

purpose of this note is twofold: first, to incorporate differential geometry into the field of applicability of the techniques and tools of algebraic geometry; second, to construct a model of SDG sufficiently general and with enough good properties to suggest stronger axioms."

The first axiom concerns R-valued jets (domain of definition $D_r(n)$). The second concerns R-valued germs (domain of definition $\Delta(n)$). A new necessary axiom will (probably) concern R-valued function spaces (compact domain of definition). A deeper study of the models is necessary. It has been initiated by O. Bruno in [3] and [4], where vector fields and intrinsic opens on exponential objects are analyzed.

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