What Is Constructive Mathematics?

Douglas Bridges and Ray Mines

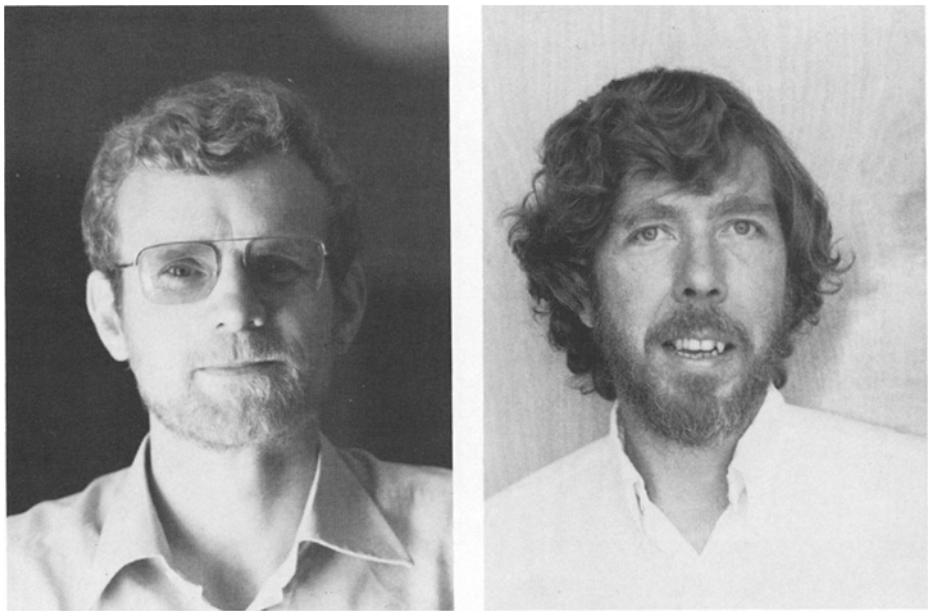
To the memory of our mentor, Errett Bishop (1928–1983)

This article is written in response to recent papers discussing the three most famous schools of mathematical foundations: logicism, formalism, and intuitionism (constructivism) [11,12,20]. In those papers, the first two schools were discussed in detail while the third received only passing mention, perhaps because the authors were unaware of the recent renewed interest in constructive mathematics. We will demonstrate that constructive mathematics is alive and flourishing; that it deals with problems of interest to all mathematicians, not only those with a constructivist philosophy; and that new facets of old problems are revealed by looking at them from the constructive point of view. Our aim is not to convert the mathematical community to constructivism, but to explain the difference between constructive and traditional mathematics. In order to do this, we shall draw on our experience in the constructive seminar at New Mexico State University.

1. Background

Modern constructive mathematics originated in the work of Brouwer, who devoted the best part of his mathematical life to a penetrating criticism of the constructive inadequacy of the mathematics of his day. However, the constructive mathematics that we shall discuss is not the "intuitionistic mathematics" developed by Brouwer and his followers [9], but that of Errett Bishop, who was the first person to show that deep constructive results can be derived with concepts and methods which closely resemble those of everyday mathematics [2].

Brouwer developed his intuitionistic mathematics in an attempt to put mathematics on a firm foundation. His idea was to begin with the intuitive notion of nat-



Douglas Bridges (left), **Ray Mines** (right)

ural number and then reconstruct mathematics using only finite methods. To do this, he abandoned much of the usual mathematical vocabulary and introduced various metamathematical principles which many mathematicians, even among those sympathetic to constructivism, find no more reasonable than the traditional foundations which he rejected. Since his unusual development is difficult to read and understand without a significant investment of time and effort, "for mathematicians, Brouwer's theory has remained a curiosity; it has largely been of interest to logicians" [10].

Another constructive approach is due to Markov and uses recursive function theory as its basis [19]. Thus it has a very strict definition of its concepts: each function is thought of in terms of its Gödel number, each real number is a specific recursive function, and so on. The methods used are standard but, because of the reliance on the terminology of recursive function theory, the mathematics is difficult for the outsider to read. Presently, Markov's followers seem to be more interested in complexity theory and its applications to computer science than in the practice of constructive mathematics. (However, in the United States Nerode and others are actively engaged in using recursive function-theoretic methods to obtain interesting constructive counterexamples. As their results are not really within the scope of this article, we will not discuss them further.)[1]

When Bishop's book was published in 1967, constructive mathematics had been in a state of dormancy because it lacked a suitable framework for mathematical practice. By reconstructing a significant part of modern analysis, Bishop was able to revitalize the subject. Among the topics he developed were: measure theory, including the convergence theorems of Lebesgue and a version of Cartan's approach to Haar measure on locally compact groups; the Hahn-Banach theorem and duality in normed linear spaces; and the functional calculus for self-adjoint operators on a separable Hilbert space. Subsequently, Bishop and Cheng produced a new constructive theory of measure based on the Daniell integral [3]. This new theory is extremely elegant and readily disposes of any fear that it would be impossible to construct a countably additive measure on the real line. It also shows that the constructive continuum is uncountable in a strong sense. (An expanded and up-to-date account of the Bishop-Cheng integral will appear in the revised edition of Bishop's book, to be published by Springer-Verlag in 1984.)

While Bishop's work has its roots in that of Brouwer, he was able to free himself from the concepts which appear to have constrained the intuitionists. By avoiding the metamathematical principles of intuitionism, the unnecessary reliance on recursive function-theoretic methods, and any commitment to a formal system, he allowed for future innovations; in this respect, his mathematics contrasts sharply with that of the other schools of constructivism. Moreover, Bishop's use of familiar terminology and notation makes his book easily read by the general mathematical public.[2]

What is novel about Bishop's work is its spirit and execution, which is much more like everyday modern mathematics than anything previously done in a systematic constructive way. Indeed, a (philosophically) unprepared mathematician could pick up Bishop's book and read it as a straight piece of classical Cantorian mathematics. What would be puzzling to him is the more involved choice of certain notions and proofs, unless he also saw in what sense these were dictated by constructive requirements [10].

2. The Nature of Constructive Mathematics

From now on, we shall adopt the stance of a constructive mathematician. In practice, this involves our paying much closer attention to the meaning of mathematical statements, especially those dealing with existence.

What does it mean to say that there exists a well-ordering of the real numbers; or a Hamel basis for the vector space of real numbers over the field of rationals; or an automorphism of the complex number field which maps π to e? In order to justify a claim that there exists an object x with the property P, we must produce a finite routine for the construction of x, together with a finite routine which verifies that x has the property P. (We don't insist that these routines be efficient, but they must be performable by a finite number of beings—or computers—in a finite time. The search for efficient routines is a separate activity, a part of numerical analysis and computer science.)

For example, to construct a real number we must give a finite routine which converts each positive integer n into a rational number x_n, in such a way that $x_1, x_2, \ldots$ is a Cauchy sequence converging to the real number under construction; we must also give an explicit estimate of the rate of convergence of this sequence.

Because of the importance attached to the construction of an object x that is claimed to exist, the proof of the existence of x is an integral part of any proposition that asserts that x exists; the proof explains how the construction of x is to be carried out. This contrasts with the common practice of "pure existence proofs",

[1] Recursive function-theoretic methods are also employed in certain areas of classical algebra: see, for example, *Combinatorial Group Theory*, by R. C. Lyndon and P. E. Schupp (Ergebnisse der Mathematik und ihrer Grenzgebiete 89, Springer-Verlag, 1977).

[2] Not all mathematicians would agree with us on this point: see the review by D. H. Fremlin in Proc. Edinburgh Math. Soc. (1981) 24, pp. 147–148.

in which the "existence" of an object x is established by showing that the assumption "x does not exist" leads to a contradiction. Constructively, the latter proof demonstrates that it is impossible for x not to exist, but it does not provide a method for finding x. (Can you show how to find a well-ordering of the real numbers?) Moreover, even if we have such a method, we may have to perform several additional constructions to show that x has the properties we want; it is not sufficient for us to prove that if x failed to have these properties then we would have a contradiction.

To appreciate the sometimes dramatic difference between constructive and classical mathematics, consider the following statement, which Bishop dubbed the *limited principle of omniscience* (LPO for short):

If (a_n) *is a sequence in* $\{0, 1\}$, *then either* $a_n = 0$ *for all* n *or there exists* N *such that* $a_N = 1$.

The constructive interpretation of this proposition is that we have a finite routine which, applied to an arbitrary sequence (a_n) in $\{0, 1\}$, either verifies that each a_n is 0 or constructs an N with $a_N = 1$. If such a routine were available then we would have a uniform method for solving a vast array of open problems, such as Fermat's Last Theorem, the Riemann Hypothesis, and the Goldbach Conjecture. For example, to settle once and for all the Goldbach Conjecture, we could apply the routine to the sequence (a_n) where

$a_n = 0$ if $2k$ is a sum of two primes for each integer k with $1 \leq k \leq n$,
$= 1$ if $2k$ is not a sum of two primes for some integer k with $1 \leq k \leq n$.

Since it seems extremely unlikely that we will ever find a uniform decision method of such wide applicability, we are forced to reject LPO as a working principle of constructive mathematics.[3] Moreover, we regard as essentially nonconstructive any classical theorem which, if proved constructively, would entail LPO. Thus, for example, the trivial classical proposition that there is a mapping f: $\mathbf{R} \to \{0, 1\}$ such that $f(x) = 1$ if and only if x is rational, is essentially nonconstructive. Other classical propositions are essentially nonconstructive because they entail, not LPO, but some other principle of similar constructive status; among propositions in this category is the one which asserts that a continuous, real-valued function on a compact interval attains its supremum.

Incidentally, the most extreme example of a classically valid, but constructively unacceptable, principle is the *law of excluded middle:* for any proposition P, either P or the negation of P is true.

The above discussion of LPO clearly illustrates the main point of our paper:

The proper evaluation of constructive mathematics depends on the correct interpretation of its language.

In constructive mathematics, familiar words are used with new meanings and must be treated with considerable caution, at least until their new usage becomes second nature. The outstanding example of this is the expression "there exists"; a special case of this is the phrase 'P or Q', which means "we have either a proof of P or a proof of Q". Any failure to grasp the significance of its language will almost inevitably lead to a total rejection of constructive mathematics as bizarre, obscurantist, and useless.

Let's illustrate our point with some more examples, beginning with the Fundamental Theorem of Algebra:

Any nonconstant polynomial f *with complex coefficients has at least one complex root.* **(1)**

Perhaps the best known proof of (1) is the one in which we assume that f does not vanish, apply Liouville's Theorem to the reciprocal of f, and conclude that $1/f$, and hence f, is constant. This contradiction completes the proof.

We would argue that this proves not the Fundamental Theorem, but the weaker statement:

A nonvanishing polynomial over the complex numbers is constant. **(2)**

Suppose you had to calculate the roots of a polynomial of degree 100; would the above proof tell you how to begin your calculations? Of course not—it gives no hint of a routine for finding the roots of a polynomial.

In contrast, such a routine is provided by Brouwer's proof of the Fundamental Theorem of Algebra for a monic polynomial [8]. Brouwer began by showing that he could assume that f was a polynomial of positive degree over the Gaussian number field $Q[i]$. Choosing a radius R so large that the leading term dominated $f(x)$, and then using the fact that the winding number of f around the circle with center 0 and radius R was the degree of f, he constructed a Gaussian number z with $f(z)$ small and $f'(z)$ relatively large. Application of Newton-Raphson iteration then enabled him to construct a complex root of f.

Brouwer's routine will produce a root of any given monic polynomial of positive degree over the complex numbers; in particular, armed with his proof, you would be able to construct a root of your polynomial of degree 100. Of course, being of such universal applicability, this routine, while effective, is not efficient. Nevertheless, the availability of such a routine is an encouragement to mathematicians to search for efficient ones in special cases.

Since every constructive proof of a proposition is a classical proof of the same proposition, Brouwer's proof is a classical proof of the Fundamental Theorem

[3] It is worth noting that if you adopt the hypothesis that all sequences of natural numbers are recursive (Church's Thesis), then you can prove that LPO is contradictory.

of Algebra. It is true that his proof is longer than that using Liouville's Theorem; this is only to be expected, as it furnishes us with more information. Similar situations occur in number theory, where a theorem may have a short proof using deep analytic methods, and a longer, more informative, elementary proof.

At the risk of being repetitive, we emphasize the importance of appreciating that, although the statement of the Fundamental Theorem of Algebra is the same in both classical and constructive mathematics, its constructive interpretation is quite different from the classical one. When correctly interpreted as an assertion about the computability of zeros of polynomials, the constructive version is seen to be considerably more informative than the classical. We bring out this distinction by rewriting Brouwer's version in the form:

There is a finite routine which, applied to any nonconstant polynomial f *with complex coefficients, enables us to compute a root of* f. **(3)**

Another example from algebra will help here:

Let f *be a nonconstant polynomial over a field* k. *Then either* f *is irreducible or* f *has a nonconstant factor of lower degree over* k. **(4)**

Classically, (4) is too trivial for words. On the other hand, there are several articles dealing with constructive aspects of (4). In the first of these, Kronecker proved that (4) holds when k is an algebraic number field [15]. In the earlier editions of his well-known text, van der Waerden included a proof that (4) is preserved under separable extensions; later, he pointed out that (4) is not preserved under arbitrary field extensions [21]. Today, Seidenberg and others continue to study conditions which ensure the effective factorization of polynomials [13,17,18].

Why should anyone be interested in a statement which is *prima facie* obvious? To see why, let's interpret (4) constructively:

There is a finite routine which, applied to a nonconstant polynomial f *over the field* k, *either decides that* f *is irreducible or constructs a nonconstant polynomial factor of* f *with lower degree over* k. **(5)**

That there is such a procedure for polynomials over the rational numbers is the surprising content of Kronecker's result. We recommend that the reader look at Kronecker's ingenious proof in [22].

3. What parts of classical mathematics are constructive?

Bishop has written that every classical theorem presents a challenge: to find a constructive version and to give it a constructive proof. The Fundamental Theorem of Algebra is an example where the classical statement remains valid when interpreted constructively, but the usual classical proofs are not constructive. In most cases it is not at all easy to formulate a conjecture about the constructive content, if any, of a classical theorem. Indeed, many classical theorems are unlikely to have any constructive version: an example is the Bolzano-Weierstrass Theorem; a more extreme example is Zorn's Lemma. On the other hand, it frequently happens that a classical theorem has several quite different constructive reinterpretations, arising from statements which are equivalent under the assumption of the law of excluded middle. This "forking of interpretations", which is one of the most interesting and fruitful aspects of constructive mathematics, is well illustrated by the reinterpretations of the Big Picard Theorem of complex analysis:

Let O *be an essential singularity of a complex-valued analytic function* f *defined on the punctured unit disk. Then* f *attains every finite value, with at most one exception, infinitely often in any given neighborhood of* O. **(6)**

A constructive proof of this theorem would involve the computation of the one exceptional value (if there is one), together with a demonstration that every other value is attained infinitely often in the given neighborhood of O. In particular, given a nonexceptional value a, we would have to compute an infinite sequence $z_1, z_2, \ldots$ in the neighborhood such that $f(z_n) = a$ for each n. There is strong evidence that such constructions are impossible [4].

Picard's Theorem has several classically equivalent versions, some of which can be proved constructively. Here are two:

Let f *be a complex-valued analytic function defined on the punctured unit disk, and suppose that* f *omits the values 0 and* 1. *Then we can compute the finite order of* f *at 0.* **(7)**

Let 0 be an essential singularity of a complex-valued analytic function f *defined on the punctured unit disk, and let* a *and* b *be two distinct complex numbers. Then in any neighborhood of 0, we can compute* z *with either* f(z) = a *or* f(z) = b. **(8)**

We have written (7) and (8) so that the constructive interpretation is apparent; but it is easy to see that if the phrase "we can compute" is replaced by the classical "there exists", then (7) and (8) are classically equivalent to (6). Constructively, no two of the three are equivalent.

According to statement (7), if we know that f is analytic in the punctured disk and omits the values 0 and 1, then we can either show that f extends to a function analytic at 0 or compute the order of pole of f at 0. None of the classical proofs of (7) enables us to do this. Statement (8) has a closer resemblance to (6), yet it also says more: there is a routine which, applied to the distinct complex numbers a and b and the function f,

computes z in the given neighborhood such that $f(z) = a$ or $f(z) = b$. (Recall the interpretation of the phrase "either $f(z) = a$ or $f(z) = b$": we can determine which of the equations is satisfied.)

Classically, (8) implies that one of the values a and b is assumed infinitely often. Constructively, it does not. Given a positive integer n, we can find z_n such that $|z_n| < 1/n$ and either $f(z_n) = a$ or $f(z_n) = b$; but there is no routine which looks at the infinite sequence z_1, z_2, . . . and decides which of the two statements $f(z) = a$ $f(z_n) = b$ holds for infinitely many n.

4. The impact of constructive mathematics

Since much of the constructive development of mathematics to date has followed closely standard classical ideas (such as those involved in the Big Picard Theorem), one might wonder if constructive mathematics is forever doomed to feed at the expense of its classical host. In this section, we indicate that symbiosis, rather than parasitism, better describes the relation between the two types of mathematics.

It will surprise many readers to learn that one benefit of a constructive approach is the provision of better proofs of certain theorems. For example, the constructive proofs of (7) and (8) in [4] are more natural than the standard classical ones. They are also simpler, avoiding all use of normal families or modular functions and requiring only basic properties of analytic functions.

Topology, in particular dimension theory, offers a further example of the insight gained from our approach. While looking for a constructive proof of the fact that the complement of an $(n - 2)$-dimensional subset of Euclidean n-space is arc-connected, Julian et al. were led to a new concept of dimension and thence to a new classical theorem [14]. To discuss this, we need some definitions.

Let X be a metric space. Given positive ϵ, we say that a finite cover F of X is an *ϵ-cover* if each member of F has diameter less than ϵ. A definition of Lebesgue says that X has *covering dimension at most n* if, for each positive ϵ, there is an ϵ-cover F of X such that each point of X is bounded away from all but $n + 1$ members of F. What happens if, for a certain positive ϵ, we can construct an ϵ-cover F of X such that each point of X is bounded away from all but $n + 1$ members of F? In that case we say that X has *ϵ-dimension at most n*. Thus X has covering dimension at most n if and only if it has ϵ-dimension at most n for each positive ϵ.

The skin of a coral snake is an example of a space which has ϵ-dimension at most 1: The red, black, and yellow bands on the skin are the elements of the cover, and ϵ is approximately the sum of the diameter of the snake's cross-section and the width of a band. However, if the coral snake swallows a large mouse, then the snake loses this at most one-dimensionality. That is the zoological content of the main theorem in [14]:

> *If X is a compact subset of Euclidean* n*-space with ϵ-dimension at most* n*, and if* p *and* q *are two points whose distance from* X *is at least $\epsilon/\sqrt{2}$, then there is a path which joins* p *to* q *and is bounded away from* X. **(9)**

Another branch of mathematics in which the application of constructive methods can produce considerable dividends is numerical analysis. Consider, for example, the theorem which lies at the heart of numerical approximation theory:

> *Let* X *be a finite-dimensional linear subspace of a real normed space* E*. Then to each* a *in* E *there corresponds an element* b *in* X *such that the distance from* a *to* b *equals the distance from* a *to* X. **(10)**

The constructive substance of all the classical proofs of (10) is the computability of the distance from a to X, rather than the attainment of this distance at some point b of X. In fact, it is not hard to show that (10) as it stands is essentially nonconstructive.

In order to produce a good constructive substitute for (10), we introduce the following definitions. Let E be a metric space, X a nonvoid subset of E, and a an element of E. We say that a has *at most one best approximant* in X if for each pair of distinct elements x, y of X, there exists z in X such that

$$\max\{d(a,x),\ d(a,y)\} > d(a,z)$$

On the other hand, an element b of X is a *best approximant* of a in X if $d(a,x) \geqslant d(a,b)$ for all x in X.

We now have:

> *Let* X *be a finite-dimensional linear subspace of a real normed space* E*, and* a *an element of* E *with at most one best approximant in* X*. Then we can compute a best approximant of* a *in* X. **(11)**

Classically, (11) is equivalent to (10); but the constructive proof of (11) embodies an algorithm of very wide applicability. In particular, (11) suffices to cover the case of minimax approximation by polynomials of degree n, in which E is the space of continuous, real-valued functions on the interval [0, 1], and the monomials $1, x, x^2, \ldots, x^n$ form a basis of X. In this connection, the reader should consult [5].

There is a well known algorithm, due to Remes, for the computation of minimax polynomial approximations of degree n [16, pp.105–106]. A few simple modifications suffice to make this algorithm fully constructive; in particular, we must avoid any appeal to the essentially nonconstructive proposition which asserts the attainment of the supremum of a continuous, real-valued function on [0, 1]. However, in all the classical

references the proof of convergence of the Remes algorithm is not constructive: what is proved is that nonconvergence of the algorithm is impossible; what is not given is information about the rate of convergence of the algorithm in general. (Note that there are special cases where rates of convergence have been given [16, p. 111]).

It seems to us that an algorithm is incompletely described unless it is equipped with some estimate of its rate of convergence. With the help of constructive methods, it is possible to give such an estimate for the Remes algorithm [6]. Of course, this estimate is crude as it covers even the most pathological examples of minimax polynomial apoproximation. But the inefficiency of such an estimate doesn't mean that we cannot learn from it something about the process under study. In this context we are reminded of the Chebyshev inequality in probability theory: In practice, it is not hard to obtain better estimates than those given by Chebyshev's inequality, but the latter still gives valuable insight into the meaning of the concept of standard deviation. Moreover, there remains the possibility that efficient estimates for the convergence of the Remes algorithm can be obtained from the general ones in special cases of practical importance.

Recently, one of us used constructive methods to obtain Lipschitz constants and continuity estimates for the best Chebyshev approximation operator [7]. Although these quantities were known to exist in the classical sense, the constructive approach had the merit of simplicity and directness, and yielded just the sort of explicit estimates that classical approximation theorists want but often cannot obtain with their indirect methods.

5. Postscript: on logic and mathematics

The examples we have given illustrate only a few of the areas which the constructive mathematician has explored since Bishop's pioneering work. But while much has been done, much more is needed before we can expect to see "the . . . day when constructive mathematics will be the accepted norm" [2, preface].

At no stage in our presentation so far have we suggested that constructive methods are the best, let alone the only proper, ones for mathematics. At the same time, it is hard to imagine serious involvement in the practice of constructive mathematics without some commitment to, or sympathy with, an underlying constructivist philosophy. Certainly, research in constructive mathematics requires enough commitment to enable one to resist any temptation to introduce classical reasoning where it might "simplify" matters.

This raises interesting questions about the status of recent results in mathematical logic [1]. Logicians often urge us not to bother trying to prove certain theorems constructively because they fall into a category of propositions whose proof in a formal system with classical logic can be translated into a proof in a formal system using intuitionistic (constructive) logic. Now as techniques of classical mathematical logic, these translations are often very interesting and important. But seen from a constructive viewpoint, they do not offer a satisfactory means of proof, for at least two reasons.

First, any attempt to use formal systems to establish results within, rather than about, constructive mathematics is contrary to the spirit of constructivism, in which numbers, for example, have a meaning quite apart from any formal representation of them by logical symbols. (In other words, formalism and constructivism, as philosophies, are not on speaking terms!) This point is reinforced by the observation that a strict constructivist would see no meaning whatsoever in the result of translating a meaningless (classical) theory into another formal theory which happened to use a formalization of constructive logic.

For example, what serious constructive analyst would be happy with the suggestion in a recent review that, in order to establish the existence of approximate eigenvectors for a self-adjoint operator on a separable Hilbert space, he invoke the trivial classical existence theorem and apply a logical metatheorem which says that the constructive result then follows?

A second, and practically more important, argument against the free use of logical metatheorems in constructive mathematics clearly applies to this last example: The generality of the logical translations under discussion means that, in any specific case, they are unlikely to yield the neat, intuitive proofs that an analyst might want. In each case, there will usually be direct, constructive, analytic proofs which are far more elegant and incisive than the "proofs" obtained by logical manipulation.

Before we completely alienate our logician friends, we had better make clear that we are not criticising their work on the relation between classical and constructive mathematics. Indeed, apart from its intrinsic interest as logic, their work contains some very important negative results which clearly tell us that certain classical theorems are very unlikely to have direct constructive analogues. What we are saying is that the application of their metatheorems is not the ideal way to tackle the problems of mathematics, as distinct from logic; the most natural, elegant, and numerically fruitful approach to constructive mathematics is the direct one. We have argued above that if we take this approach, and if we interpret constructive statements correctly, then we make available more information than is possible with the classical interpretation of "existence". Thus, dealing with questions of computability, constructive proofs should be of interest to any mathematician, even one who does not subscribe to a constructivist philosophy of mathematics.

References

1. Beeson, M. (1978) Some Relations between Classical and Constructive Mathematics, J. Symbolic Logic, 43, 228–246.
2. Bishop, Errett (1967) Foundations of Constructive Analysis, New York: McGraw-Hill.
3. Bishop, Errett, and Henry Cheng (1972) Constructive Measure Theory, Memoirs of the Amer. Math. Soc., 116.
4. Bridges, D. S., A. Calder, W. Julian, R. Mines, and F. Richman (1982) Picard's Theorem, Trans. Amer. Math. Soc., Vol. 269, No. 2, 513–520.
5. Bridges, D. S. (1982) Recent Progress in Constructive Approximation Theory, North-Holland: The L. E. J. Centenary Symposium (A. S. Troelstra and D. van Dalen, eds.).
6. Bridges, D. S. (1981) A Constructive Analysis of the Remes Algorithm, J. Approximation Theory, Vol. 32, No. 4, 257–270.
7. Bridges, D. S. (1982) Lipschitz Constants and Moduli of Continuity for the Chebyshev Projection, Proc. Amer. Math. Soc., Vol. 85, No. 4, 557–561.
8. Brouwer, L. E. J., and B. de Loor (1975) Intuitionistischer Beweis der Fundamentalsatzes der Algebra, North-Holland: Collected works of L. E. J. Brouwer, Vol. 1 (ed. A. Heyting).
9. Dummett, M. A. E. (1977) Elements of Intuitionism, Oxford: Clarendon Press.
10. Feferman, S. (1979) Constructive Theories of Functions and Classes, North-Holland: Logic Colloquium '78 (M. Buffa, D. van Dalen, and K. McAloon, eds.).
11. Goodman, N. D. (1979) Mathematics as an Objective Science, Amer. Math. Monthly, Vol. 86, 540–551.
12. Hersch, R. (1979) Some Proposals for Reviving the Philosophy of Mathematics, Advances in Math., Vol. 31, 31–50.
13. Julian, W., R. Mines, and F. Richman (1978) Algebraic Numbers, a Constructive Development, Pacific J. Math., Vol. 74, 91–102.
14. Julian, W., R. Mines, F. Richman, and A. Calder (1979) ϵ-covering Dimension, Pacific J. Math, Vol. 95, No. 2, 257–262.
15. Kronecker, L. (1882) Grundzüge einer arithmetischer Theorie der algebraischen Grossen, J. Reine Angew. Math., Vol. 92, 1–222.
16. Meinardus, G. (1967) Approximation of Functions: Theory and Numerical Methods, Tracts in Natural Philosophy, 13, Berlin: Springer Verlag.
17. Mines, R., and F. Richman (1982) Separability and Factoring Polynomials, Rocky Mountain J. Math., Vol. 12, No. 1, 43–54.
18. Seidenberg, A. (1978) Constructions in a Polynomial Ring over the Ring of Integers, Amer. J. Math., Vol. 100, 678–703.
19. Shanin, N. (1963) On the Constructive Interpretation of Mathematical Judgments, Amer. Math. Soc. Translations, (2), 23, 109–189.
20. Snapper, E. (1979) What is Mathematics? Amer. Math. Monthly, Vol. 86, 551–557.
21. van der Waerden, B. L. (1930) Eine Bemerkung über die Unzerlegbarkeit von Polynomen. Math. Ann., Vol 102, 738–739.
22. van der Waerden, B. L. (1953) Modern Algebra (second edition), Ungar.

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