Journal of Mathematical Sociology
1972, Vol. 2, pp 37–61

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Printed in Birkenhead, England

KINSHIP SYSTEMS AND INVERSE SEMIGROUPS

JOHN PAUL BOYD, JOHN H. HAEHL and LEE D. SAILER

University of California at Irvine

Inverse semigroups are introduced as a model for kinship structures. It is shown that semigroups are implied by the traditional anthropological law of uniform descent, and that inverses are suggested by the principle of reciprocity. A separation into a 'surface' and a 'deep' level of analysis was induced by viewing a kinship system as a partition on an underlying inverse semigroup. Some Omaha-type kinship systems are decomposed into direct and semidirect products. Additional assumptions are made to derive a theoretical similarity measure, which is then compared with a judged similarity task on Tenejapan Indians. The results of this preliminary experiment seem to be encouraging.

1. INTRODUCTION

A kinship system is the way in which the members of a particular society name their relatives. What is interesting from a cross-cultural point of view is not that different peoples use different words, but that the patterns of application of their words are different. Thus Americans use the same word 'uncle' for both mother's brother and father's brother and use a different word 'father' for father. Other societies having the so-called bifurcate-merging system have the same word for father as for father's brother and a different word for mother's brother. Thus it is particularly obvious in the domain of kinship that word-for-word translations are impossible. In the case of the bifurcate-merging terminology, for example, how would a missionary translate the phrase, 'God, the Father,' without bringing in unknown associations? Malinowski (1955) suggested that for some purpose 'God, the Mother's Brother' would be a better rendering for the Trobriand Islanders.

It was this same Malinowski (1955) who dismissed kinship studies as 'merely algebra' at a time when no real algebraic studies had been made. Since then true algebraic studies have been made by Lévi-Strauss (1949), Harrison White (1963), Lounsbury (1964), Boyd (1969) and others. This article will extend the group theoretic methods employed in White (1963) and Boyd (1969) to apply to all types of kinship systems. The main theoretical tool that has made this possible is the concept of an inverse semigroup, which is a generalization of a group.

2. KINSHIP GRAMMARS

Lounsbury's (1964) article on the Crow and Omaha type kinship terminologies represents a major breakthrough in the formal analysis of kinship systems. We shall

briefly present his system using Romney and D'Andrade's (1964) notation in order to prepare the way for the algebraic discussion in the succeeding sections.

The primitive symbols of the notation used in the present paper are ♂ for 'male', ♀ for 'female', + for 'parent of', − for 'child of' and * for 'sibling of.' Other relatives are obtained by forming 'strings' of these primitive symbols. For example, the string ♂+♀ is interpreted as 'a man's mother.' ♀+♂+♂ is read 'a woman's father's father' and so on. The sibling relation * can also be defined in these terms by either +♂−, 'a father's child' or as +♀− 'a mother's child.' Sociologically speaking, of course, a father's child may not be the same as a mother's child if, for example, the father had children from a previous marriage. However, the almost universal fact that these half siblings are not terminologically distinguished makes it notationally convenient to denote either +♂− or +♀− by the symbol *.† Thus 'mother's brother' is denoted by +♀*♂. Similarly, 'husband' and 'wife' can be defined as ♀−*a*+♂ and ♂−*a*+♀, respectively, where *a* can be either ♂ or ♀. Thus affinal relatives can be defined from this set of primitive terms.

Now, a *kinship terminology* is sometimes considered to be the assignment of names to at least some of the strings over the set of kinship primitives, ♂, ♀, + and −. However, we want to consider a weaker type of isomorphism that equates systems that induce the same partition on the set of kin strings even though these systems may use different words. That is, we consider the English and Eskimo systems to be isomorphic since two strings are given the same English word if and only if they are given the same Eskimo word. Lounsbury (1964) shows that much of this partition aspect of kinship systems can be generated by 'grammars.' These grammars are really formal systems with grammatical rules, or axioms, which are different for different terminologies, together with two rules of inference, which are presumed to be universal over all kinship systems. The axioms are pairs of strings which are to be considered equivalent. For example, the Fox kinship has the following axioms:

(Half sibling) $+♂- = +♀- = *$,

(Merging) $a*a = a$ where a is ♂ or ♀,

(Skewing I) $+♀*♂- = +♀*$.

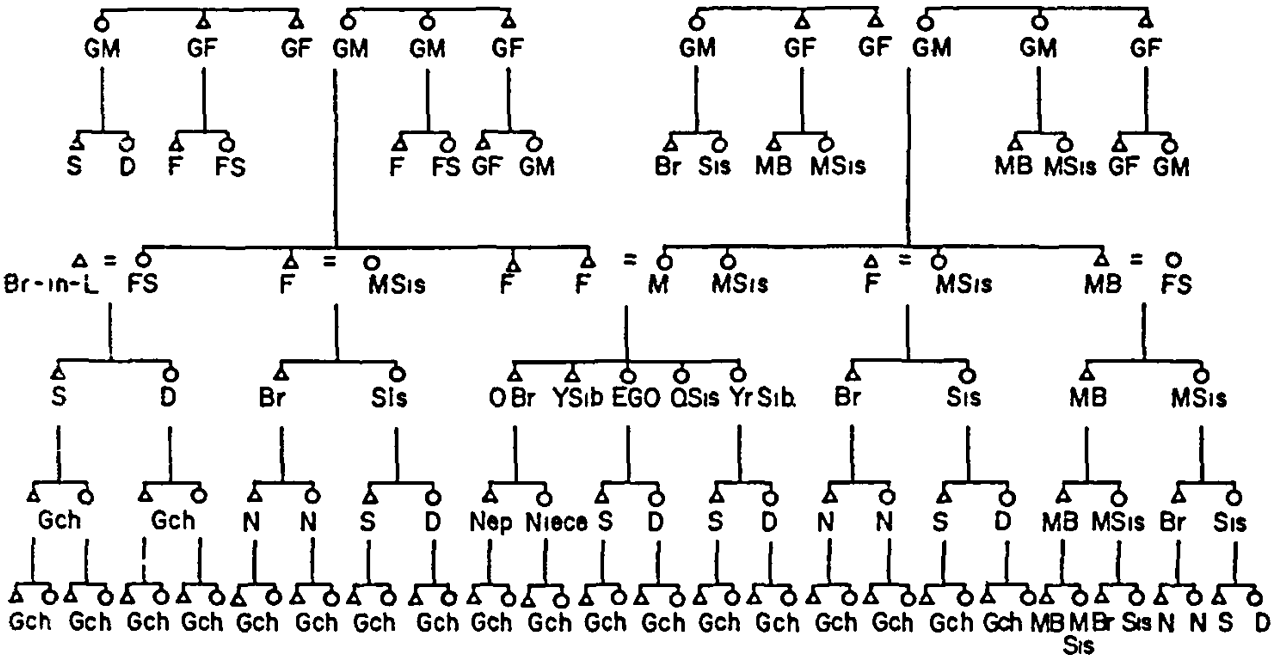
The Fox kinship system was described by Tax (1937) and is shown in Figure 1. We shall follow Lounsbury (1964) in referring to any real kinship system, such as the Fox, which satisfy the above axioms as an *Omaha I* system.

Axioms, of course, are useless without rules of inference to tell you how to use the axioms. In the case of kinship axioms, we have two rules of inference, substitution and reciprocity. The *substitution* rule is that if u, v, s and t are kinship strings and if $u = v$, then we also have $sut = svt$. In more colorful language, the substitution rule states that equals may be substituted for equals. For example, we can show that ♂*MoMoSiSoSo* = ♂*MoBr* by the following derivation:

† This notation differs from Romney's in that the traditional Mars and Venus symbols are used instead of 'm' and 'f' for male and female and '*' replaces his '0'. This was done because of the dissonance caused by the conflict with the standard notation m for mother, f for father and 0 for zero. Although treating * as a primitive symbol is redundant, this simplifies the presentation.

$$
\begin{aligned}
♂MoMoSiSoSo &= ♂+♀+♀*♀-♂-♂ \\
&= ♂+♀+♀-♂-♂ && \text{(by Merging axiom)} \\
&= ♂+♀*♂-♂ && \text{(half-sibling)} \\
&= ♂+♀*♂ && \text{(skewing)} \\
&= ♂MoBr
\end{aligned}
$$

Note that this formal system differs from a phrase structure grammar as defined by Chomsky (1963) in at least two respects: (1) strings are related by an equal sign



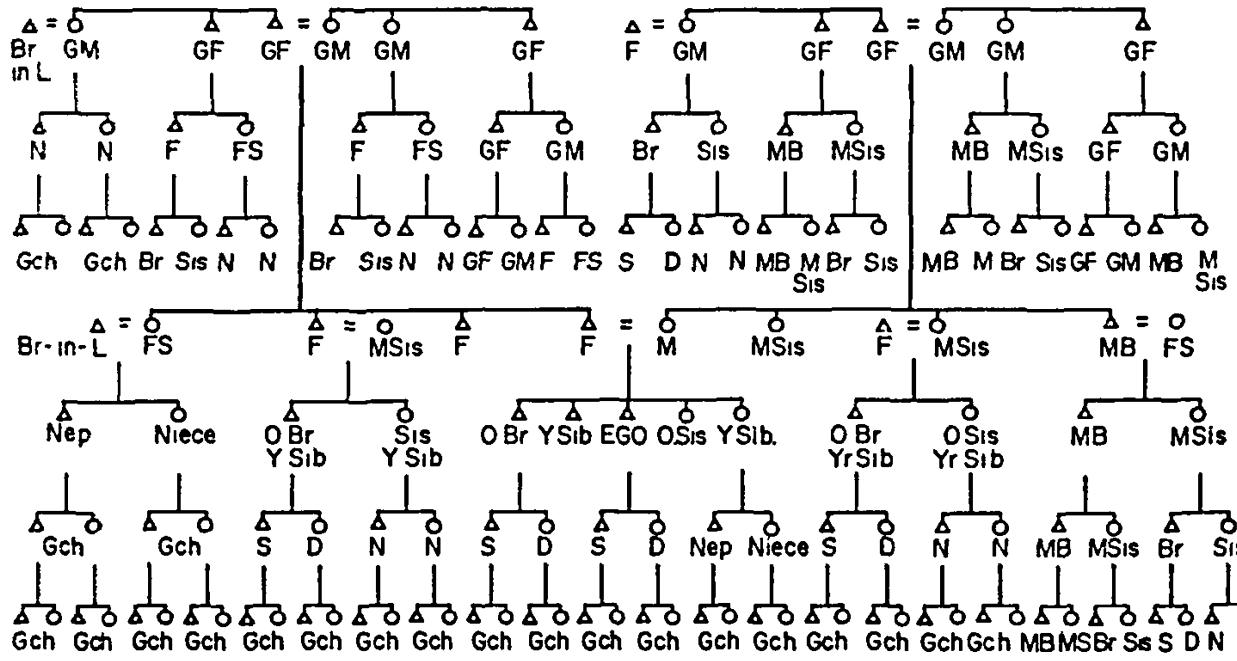


FIGURE 1 Chart of the Fox kinship system, taken from Tax in Eggan (1937). Ego is male in the lower and female in the upper chart.

which is taken to be reflexive, transitive, and symmetric instead of the Chomsky arrow, which is irreflexive and antisymmetric, and (2) derivations can start from any string instead of only from the initial *S* (for sentence). A third way in which this formal kinship system differs from a grammar is that it contains a new rule of inference called the *principle of reciprocity*, which states that if strings x and y are equivalent, so are their reciprocals, x^{-1} and y^{-1}, where the reciprocal of a string is found by writing the string backwards but substituting $-$ for $+$ and *vice versa.* For example, the reciprocal of ♂+♀*♀−♀ is ♀+♀*♀−♂. The principle of reciprocity is illustrated by taking two mothers' brothers of a male ego, say, ♂+♀*♂ and ♂+♀*♂−♂, and noting in Figure 1 that their respective reciprocals, ♂*♀−♂ and ♂+♂*♀−♂, are both labeled *N* and thus are equivalent. If these strings x are considered in their normal interpretations as binary relations, R_x. then the reciprocal x^{-1} is to be interpreted as the *converse* of R_x. That is, if person a is R_x related to person b, then b is R_x^{-1} related to a.

This system departs from the 'data' of Figure 1 in several relatively minor ways. First, note that *Mo* and *MoSi* violate the principle of reciprocity since their reciprocals are classed together together under *D*. It is for this reason that the merging axiom has been written so as to class *Mo* and *MoSi* together. An even less important failing is the inability to account for the elder-younger distinction among the siblings. A more important fact, however, is that we are occasionally unable to show that two strings with the same label are equivalent, though if strings are equivalent they generally have the same label. The trouble is that there are many different equivalence classes that are labeled *GF*'s, *GM*'s or *GCh*'s. For example, there is no way ♂+♂+♂ and ♂+♀+♂ can be proven equivalent. This could be taken care of by a set of special rules that has to start from a specific string, say, ♂+♂+♂ and generates all the *GF*'s. This, of course, detracts from the beauty of having one set of rules for all relatives and misses as we shall see in the next section, a vital structural fact about kinship systems.

3. INVERSE SEMIGROUPS

The structure we propose to account for some of the features of kinship systems is the inverse semigroup. This mathematical structure captures, or mirrors, many of the salient features we have talked about. In a sense the inverse semigroup behaves like a 'deep structure' in that it reflects underlying aspects of human behavior which is usually observed only after the 'deep structure' has been transformed into a 'surface structure.' In this case the 'surface structure' corresponds to a partition on the inverse semigroup in the 'deep structure.' This layering is reflected by the fact that some strings which are equivalent 'on the surface,' since they are named the same, are not equivalent 'in deep' since they belong to different congruence classes of the inverse semigroup.

We now will discuss the mathematical concepts necessary for further understanding of our model. The actual mathematical tools are more completely described in Clifford and Preston (1967). The basic concept is that of a *semigroup*, which is a set *S* together with an associative binary operation. An *inverse semigroup S* is a

semigroup with the property that for every element a in S there exists a unique element a^{-1} in S such that

$$\text{(Reciprocals)} \quad aa^{-1}a = a \text{ and}$$
$$a^{-1}aa^{-1} = a^{-1},$$

where a^{-1} is called the *inverse*, or *reciprocal*, of a. Inverse semigroups are very closely related to groups. To see this, recall that a *group* is an associative binary operation on a set S such that

(Identity) There is an element 1 in S such that $1a = a1 = a$ for all a in S, and

(Inverses) For every a in S there is an a^{-1}in S such that $aa^{-1} = 1$.

It is easy to show that inverses in a group are unique. Therefore every group is also an inverse semigroup since by multiplying the group Inverse equation on the right by a and on the left by a^{-1} both Reciprocal equations are obtained. Thus inverse semigroups occupy an *intermediate* position in level of generality between groups and semigroups. Because inverse semigroups are quite similar to groups, many group theoretic results can be generalized so as to apply to them. See, for example, Chap. 7 of Clifford and Preston (1967). The connection between inverse semigroups and kinship systems will be made in the same two stages as was the definition. That is, the semigroup structure will be introduced and then inverses brought in. These two stages are closely related to the two rules of inference for kinship grammars, substitution and reciprocity.

3.1 *Semigroup Structure from Substitution*

A semigroup was defined as an associative binary operation on a set S. where a *binary operation* is a function from the set of all pairs $S \times S$ into S. A binary operation is said to be *associative* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, c in S. Thus associative binary operations, such as plus and times on the integers, can be written unambiguously without any parentheses at all (so long as only one kind of operation at a time is considered), so that $(1+2)+3$ and $1+(2+3)$ is usually written $1+2+3$. Now, the fact that kin strings such as ♂+♂−♀ are usually written without parentheses is of itself evidence for semigroup structure. It must be emphasized at this point, however, that the operation is neither + nor − but rather is the concatenation of the primitive symbols ♂, ♀, +, − and ∗. If we denote this operation by · instead of leaving it out as is customary, the string above would appear as ♂·+·♂·−·♀. Another piece of evidence for semigroup structure is that the semantic interpretation of these kinship symbols is to interpret the strings as relations between people, and the operation as composition. Since it is well known that the composition of relations is an associative operation, a semigroup structure is indicated.

The only difficulty seems to be that not all strings are well defined. That is, ♂ ♂, + + and ♂ ♀ do not usually appear together and their appearance is unpleasant. If there are such undefined strings. then there can be no semigroup, for the operation

between strings must be everywhere defined. In this case the difficulty can be solved by introducing a zero element 0 and a few equations,† namely,

$$u\cdot 0 = 0\cdot u = 0\cdot 0 = 0, \quad u\epsilon\{♂, ♀, +, -, *\} \tag{1}$$

$$xy = 0, \quad x, y\epsilon\{+, -\}, \tag{2}$$

$$aa = a, \quad ♂♀ = ♀♂ = 0, \tag{3}$$

$$+a- = *, \quad a\epsilon\{♂, ♀\}. \tag{4}$$

Equation (1) merely states that 0 is a zero element. Equation (2) insures that pluses and minuses are always separated by a sex symbol in any non-zero string. The first and second parts of (3) repress in a natural way redundant and contradictory specifications of sex. Equation (4) is called by Lounsbury (1964) the 'half sibling rule' and is listed as a separate axiom for each kinship system. However, since it is regarded as a universal property of kinship systems, it seems more efficient to include it just once in our basic system. Thus given equations (1)–(4) the Fox system is specified by the Merging and Skewing axioms alone.

As with any set of generating relations, a semigroup is defined from (1)–(4) in the following stages. First, a special kind of equivalence relation $\equiv$, called a *congruence*, is defined by $x \equiv y$ if and only if there is a sequence of strings $x_1, x_2, \ldots, x_n$ where $x = x_1$, $y = x_n$ and for each i, for $1 \leqslant i < n$, we have $x_i = s_i u_i t_i$ and $x_{i+1} = s_i v_i t_i$ where u_i and v_i are equated by the generating equations. A congruence is so constructed that we can unambiguously define an associate binary operation on its equivalence classes. Each equivalence class can be denoted by any representative string, which we take in this case to be one of the strings with fewest symbols in it. These representative strings are said to be 'canonical forms'. Thus when we write the string ♂+♀ we are really referring to the equivalence class containing ♂+♀, ♂♂+♀, ♂♂+♀♀, and so on. In addition. we will for convenience adjoin to our kinship semigroup an identity element 1, which is defined by $x1 = 1x = x$. for all strings x. The identity element can be interpreted as the *empty string*, the string with no letters.

Now, we can use generating equations on more than just semigroups of strings; we can take arbitrary semigroups and define other semigroups on the equivalence classes of congruences generated by more generating equations. It seems clear that the rule of substitution for kinship grammars is just the same as the method described above for generating congruence relations from a set of generating equations, where the underlying semigroup is generated by (1)–(4).

3.2 *Inverses from Reciprocity*

Now that it has been shown that it is possible to consider a kinship system as a semigroup, let us try to narrow our attention even further to inverse semigroups. To

† Of course, a partial associative operation cannot always be converted into a semigroup merely by adjoining a zero. This is one of the reasons Lorrain and White (1971) introduced partial associative operations as a model of social relations. However, the operations which they study have the property that if $a\cdot b$ and $b\cdot c$ is defined, then $(a\cdot b)\cdot c$ and $a\cdot(b\cdot c)$ are also both defined and are equal to each other. Schmidt (1968) has shown that such systems can always be embedded in a semigroup in a unique way.

do this, several properties will be discussed which will show similarities between the reciprocal in kinship systems and inverses in inverse semigroups.

First, note that both reciprocals and inverses are preserved by congruences, which means that if x and y are equivalent with respect to a congruence, then so are their reciprocals and inverses. The preservation of reciprocals is just a restatement of the principle of reciprocity. The proof that inverses are preserved can be found in Clifford and Preston (1967), p. 57.

A stronger type of evidence for inverses as a model for reciprocals is given by the fact that the two mappings of elements onto their reciprocals and inverses are both involutionary anti-isomorphisms. An *anti-isomorphism* of a semigroup S is a one-to-one mapping $x \to x'$ of S onto itself such that $(xy)' = y'x'$. That is, it rewrites strings backwards. A mapping is *involutionary* if when applied twice it yields the identity relation; that is, when $x'' = x$ for all x in S. Obviously, if you turn a string around twice, you end up with what you started with. The proof for inverses can be found in Clifford and Preston (1961).

The final property that reciprocals and inverses share is that the way in which they are usually interpreted as semigroups of relations is the same. That is, both reciprocals and inverses turn out to be relational *converses*, where the converse of a relation R is defined to be the relation R^c given by yR^cx holding if and only if xRy holds. The claim that reciprocals are converses when these terms are applied to the real world can be empirically, but not mathematically, checked by considering the meaning of the kinship terms in question. For example, the reciprocal of the English word 'aunt' for a male speaker is a woman speaking of a 'nephew.' Also, the converse of the relation, 'man's aunt' is the set of pairs, women and their nephews. On the other hand, the most common representation of inverse semigroups is to associate each semigroup element with a one-to-one relation of some set E. Any inverse semigroup S can be so represented and in each such representation inverses are mapped onto relational converse. That is, if x is in S and x' is the corresponding one-to-one relation on E, then $(x^{-1})' = (x')^c$. Again see Clifford and Preston (1961) for the proofs.

Thus in addition to equations (1)–(4) we confidently assume that for every string x of primitive kinship symbols, there is a unique reciprocal, or inverse x^{-1}, such that

$$\left.\begin{aligned} xx^{-1}x &= x \text{ and} \\ x^{-1}xx^{-1} &= x^{-1} \end{aligned}\right\} \tag{5}$$

The resulting inverse semigroup is called the *free inverse semigroup* for *kinship* and is abbreviated *FISK*. The adjective 'free' is appropriate in analogy with other 'free' algebraic systems such as free groups, etc., in that only the universal relations hold in *FISK* and every other kinship system can be considered as a congruence on *FISK*.†

† The author was sidetracked for several years by the seductive assumption that $a-b+a = a$ for all primitive symbols a, b. It can be shown that this implies (5) but not *vice versa*. The attraction of this equation is that it is always true in an empirical sense. That is, a man's child's father is always himself. This principle is in a sense even more true for female speakers and can be given the convincing name of 'the unique parenthood principle.' But notice that if the unique parenthood principle holds, we get into immediate trouble with the MoBr's of the Fox in Figure 1. For example, we have ♂+ ♀+ ♂− ♂ = ♂+ ♀+ ♂− ♂− ♂ and when we multiply both sides on the right by ♂+ ♂

4. HOMOMORPHISMS AND CONGRUENCES

A *homomorphism* of the semigroup S into the semigroup T is a function f from S into T such that for every x, y in S we have $f(xy) = (fx)\,(fy)$. Notice that the multiplication between x and y on the left-hand side of the equation takes place in S, whereas the multiplication between fx and fy on the right-hand side is the operation for T. Thus a semigroup homomorphism is said to 'preserve products.' More generally, the idea of a homomorphism is any 'structure preserving' function. The image $f(S)$ of a semigroup under a homomorphism f is a semigroup which in some sense 'approximates' S even though it may have many fewer elements. Thus the concept of homomorphism can be used to study relationships between systems. For example, if S is homomorphically mapped onto T and T is homomorphically mapped onto U, then in an algebraic sense, T is 'in between' S and U and 'U is a coarser approximation of S than is T.' Thus a sort of directional metric is introduced which can be used to study evolutionary sequences of such systems. See Boyd (1969) for an application of the homomorphism concept to marriage class systems.

A *congruence* on a semigroup S with an identity is defined to be an equivalence relation $\equiv$ on S such that for all x, y, s, t in S, $x \equiv y$ implies $sxt \equiv syt$. Congruences are closely related to homomorphisms. In fact, the set of congruences on a semigroup S is a sort of internal list of all the possible homomorphic images that S can have. The connection between congruences and homomorphisms is described in the Main Homomorphism Theorem in Clifford and Preston (1961).

4.1 *The Maximum Congruence of an Equivalence*

A *kinship terminology* can be defined as a function f from the free inverse semigroup *FISK* onto the set of kinship terms or labels, which must include an 'undefined' label. The equivalence on *FISK* associated with f, namely, $f_0^{-1}f$, is called *terminological equivalence* and is not in general a congruence. For example, in Figure 1 we see that ♂ + ♂ + ♂ * ♀ and ♂ + ♀ + ♂ * ♀ are terminologically equivalent by virtue of both being labeled *GM*. If we multiply both expressions on the right by ♀−♂, however, we find that the resulting strings are labeled *N* and *Br*, respectively. Therefore, terminological equivalence is not a congruence, but must be considered as a 'surface structure' which must be explained by the underlying congruence, which acts as a 'deep structure.' In this case the 'deep structure' is uniquely specified from the 'surface structure' as the largest possible congruence that still fits inside the relation of terminological equivalence. More precisely, if τ is terminological equivalence, then we are looking for a congruence $C\tau$ that is contained in τ such that if σ is any other congruence

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we get ♂+ ♀+ ♂ = ♂+ ♀+ ♂− ♂ which is wrong since one is GF and the other is MoBr still. When I first noticed this, I thought I had to throw out the baby, the concept of inverse semigroups, with the bath of the unique parenthood principle. The answer to this problem was to write the inverse equations in their full generality as they appear in (5) and to let ♂ and ♂− ♂+ ♂ stay in separate congruence classes, representing a true but unprovable equivalence. Thus kinship systems seem to have a sort of semantic incompleteness reminiscent of number theory.

contained in τ, then σ is also contained in $C\tau$. See Figure 2. Following Clifford and Preston (1967), p. 176, we define the congruence, $C\tau = \{(x, y)$: $(sxt, syt) \in \tau$ for all $s, t \in FISK\}$. The concept $C\tau$ is the dual of the idea of a semigroup generated by a relation ρ, where the ordered pairs of ρ correspond to generating equations. The duality refers to the fact that while $C\tau$ is the *largest*, or *coarsest*, congruence contained in τ, the congruence obtained from ρ as a set of generating equations is the *smallest*, or *finest*, congruence that contains ρ. However, even when, as in the case of kinship systems, the final goal is the maximal congruence $C\tau$ contained in a partition τ, this congruence must be specified by a generating relation ρ. The reason is that ρ is finite, whereas the definition of $C\tau$ requires the examination of *all* strings s, t to determine whether or not two strings x, y and equivalent.

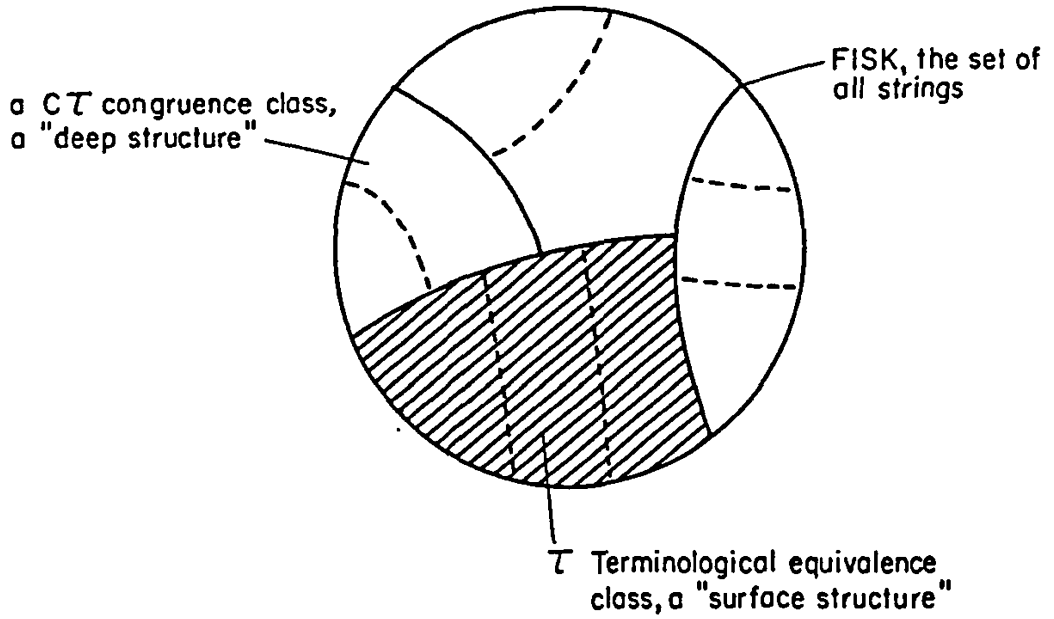


FIGURE 2 The 'deep structure' shown as the largest, or coarsest, possible congruence $C\tau$ that is still a subpartition of the 'surface structure,' the terminological equivalence partition τ. $C\tau$ is indicated by dotted boundaries and τ by solid boundaries. The shaded region is a single τ class containing three $C\tau$ classes.

If we examine the generating equations for the Omaha I kinship systems, we see that the resulting congruence appears to be $C\tau$, the maximal congruence contained in the terminology. There is no way to prove this to be true, since the partition τ is not completely specified. That is, we are given a few examples by Figure 1 of how the terminology is applied and are expected to deduce the rest. That is, a kinship chart presents a partition τ_0 on some proper subset, E, of *FISK*. In general, there is no inique largest congruence ρ whose restriction to E is contained in τ_0. For example, if x and y are generators of some free semigroup, and if $\tau_0 = \{\{x\}, \{y\}\}$, then the congruences generated by mapping x and y onto distinct elements of any of the several two-element semigroups are all maximal congruences separating x and y. Thus the problem is to search heuristically for a pleasing set of equations generating a very large congruence whose restriction to the data does not equate anything that the data separates.

4.2 *Homomorphic Images of FISK*

If we remove the + from the Skewing I axiom, we obtain the equation

$$\text{(Skewing II)}\quad ♀* ♂- = ♀*.$$

The Skewing II axiom, together with the Half-Sibling and Merging axioms, forms a new system called by Lounsbury (1964) an Omaha II system. Skewing II implies Skewing I by merely multiplying both sides on the left by a +. Thus, equivalence in Omaha I implies Omaha II equivalence. The converse does not hold, however. That is, there are strings which Omaha II equates but which are separated in the Omaha I, or Fox, system. For example, ♂+♂*♀ equals ♂*♀ in Omaha II, but not in Omaha I. Thus, the congruence generated by Omaha I is properly contained in the congruence given by Omaha II. This implies that the Omaha II semigroup is a homomorphic image of the Omaha I semigroup under the 'inclusion' mapping φ. That is, if A and B are congruence classes generated by Omaha I and Omaha II, respectively, then $\varphi A = B$ if and only if $A \subseteq B$.

There exist homomorphisms between *FISK* congruences that are not inclusion mappings. For example, let σ be the 'sex change' mapping that interchanges males and females in strings. Thus, σ maps *FISK* onto itself and induces one to one onto homomorphisms, that is, isomorphisms, between congruences on *FISK*. The sex change image of an Omaha system is called a *Crow* system, where the Half-Sibling and Merging axioms are the same but where the Crow Skewing I rule, for example, is found by changing the sexes in the Omaha rule:

$$\sigma\text{(Skewing I)}\quad +♂*♀- = +♂*.$$

It would seem to be more likely for a society to change via an inclusion mapping rather than along some other type of homomorphism. For example, a change from Omaha I to Omaha II seems more plausible than from Omaha I to Crow I since the former is an inclusion mapping and the latter, a sex change isomorphism. Although more empirical work is needed, we shall tentatively assume, then, that historical changes will tend to follow inclusion mappings. This is equivalent to saying that historical distance is reflected in the *lattice distance*, which is defined as the minimum number of steps between two *FISK* congruences, where a *step* is a pair of congruences, one included in the other, such that no other congruence is in between this pair.

There are several possible ways in which this 'lattice distance' hypothesis could be tested. One test would be to observe, directly or indirectly, historical changes in kinship systems. The prediction to be tested would be that historically close systems are close with respect to the lattice distance. For example, following Lounsbury's (1964) terminology, we have the lattice relationships indicated by arrows in Figure 3. If one could do a detailed historical analysis, one would expect that if a society began at O_{II} that it would more easily changed to O_{IV} or O_I than to O_{III} because moving to O_{III} would involve two steps and to the others, only one step. Similarly, if the society started at O_{IV}, one could more easily imagine a change to O_{II} or O_{III} than to O_I.† Another test would be to look for variations in the kinship systems between

† The formal basis for obtaining similarity from homomorphisms is the fact that every preorder determines a topology. See Lorrain (1969).

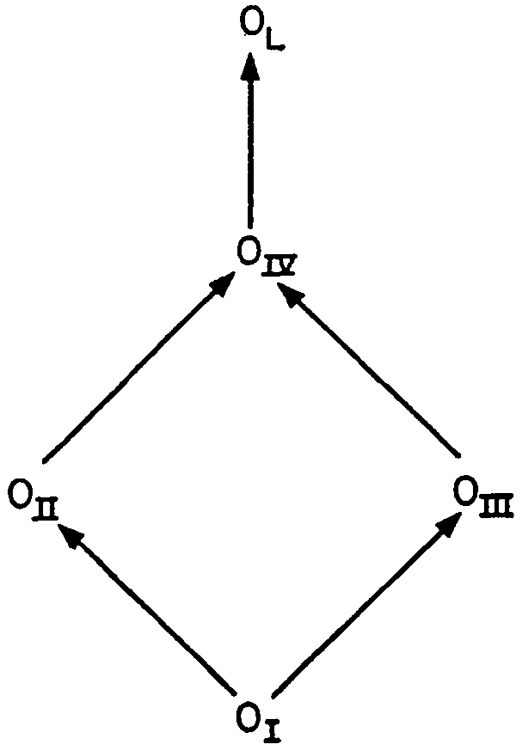


FIGURE 3 Arrows indicate inclusion maps among the Omaha systems.

and within individuals. That is, individual differences in the kinship systems ought to be explicable as close to the standard system in the lattice. In addition, one might expect that as a person's kinship system is traced back into his childhood, one would find a series of steps up the lattice.

Another use of congruences is to use the fact that a congruence relation on a system is an approximation of that system. A congruence is often more easily interpreted than the original system since it lumps together many of the congruence classes of the original system. For example, if the 'unprovable but true' Unique Parenthood axiom, $a-b+a = a$, for $a, b = \sigma, \varphi$, is added to the Omaha I axioms, then we get the 'Omaha Lineage' system, O_L, diagrammed in Figure 4. The Omaha Lineage semigroup can be efficiently described by a set of *canonical forms*,† which is a set F of strings such that F has exactly one string in each congruence class. That is, if E is any congruence class, then $F \cap E$ contains one and only one string. The particular canonical form which interests us for O_L is the set F of strings in one of the forms $a(+\varphi)^n + b$ or $a(-\varphi)^n + b$, where a, b are φ or σ and $n = 0, 1, \ldots$ Actually, if we interpret $(+\ \varphi)^{-n}$, as $(-\varphi)^n$ we get the following simple formula for the multiplication of these canonical forms

$$(a(+\varphi)^m+b)\,(c(+\varphi)^n+d) = \begin{cases} 0 \text{ if } b \neq c \\ a(+\varphi)^{m+n}+d \text{ if } b = c, \end{cases}$$

where a, b, c, d are φ or σ and m, n are integers. A set of canonical forms for an important subset of the Omaha I semigroup will be found in Section 4.3 using special

† 'Canonical forms' is the mathematical term for what Lounsbury (1964) and Romney (1965) call 'focal types' and 'kernels,' respectively. The word 'kernel' will be used here in a very different sense.

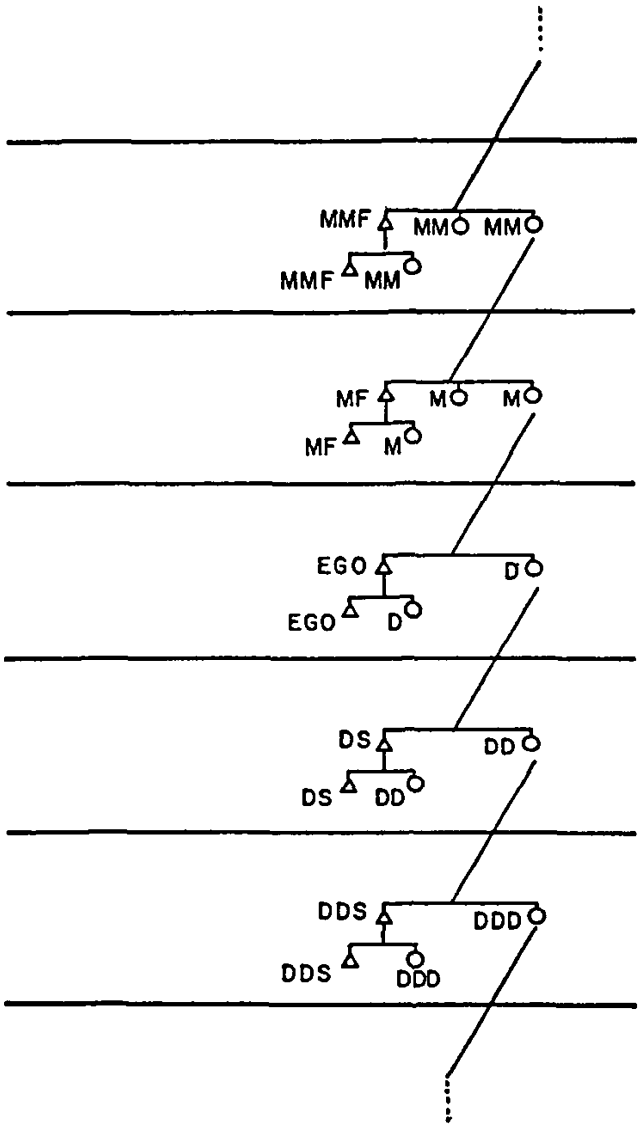


FIGURE 4 The Omaha Lineage System, O_L, with male ego. This system is a homomorphic image of all Omaha systems.

techniques. The Omaha lineage system is a homomorphic image under the inclusion map of all Omaha systems and thus supports Radcliff-Brown's (1965) 'unity of the lineage' theory, which attempts to relate Crow-Omaha systems to the existence of matri- or patrilineages, respectively. Lounsbury (1964) objects to this theory on the grounds that some men, e.g., *F* and *FF*, are terminologically separated within a single patrilineage and that other men who may belong to different patrilineages, e.g., *Br* and *MoSiSo*, are equated. The first objection is overcome by noting that although *F* and *FF* are named differently in all of the real Omaha systems, they are similar enough to be mapped onto Ego in the Omaha lineage system. The second objection is also eliminated if we assume a strong tendency for the prescriptive marriage between lineages. That is, if sisters marry into the same patrilineage, *Br* and *MoSiSo* do, in fact, belong to the same patrilineage. That this expectation exists is suggested by the affinal relative *MoSisHu* being equated with *F* in the Fox system.

The discussion becomes clearer if we examine an even simpler homomorphic image of the Omaha lineage system. This image is called the *patrilineal moiety system* and is formed from the lineage system by equating every other patrilineage of Figure 4.

That is, we add the equation ♀+♀+♀ = ♀ to the axioms for the Omaha lineage system. A society based on patrilineal moieties is formed by assigning everyone in the society to patrilineal moieties is formed by assigning everyone in the society to one of two groups, or moieties, such that each person belongs to his father's moiety and obtains his spouse from the other. *Matrilineal* moiety systems are defined in the same way as patrilineal systems, except that people belong to the moiety of their mothers rather than of their fathers. It can easily be verified that Omaha systems cannot be mapped onto matrilineal moiety systems with an inclusion mapping. Crow systems, however, can be mapped via an inclusion map onto matrilineal moiety systems but not onto the patrilineal kind. Although a moiety system is consistent with Crow-Omaha systems they are not necessitated by this theory. However, if moieties are present, one would certainly expect to find most of the matrilineal moiety systems among the Crow systems, and the patrilineal among the Omaha. Indeed, Murdock (1949), pp. 166–167, finds this to be the case.

The Omaha lineage system can be represented by a set of matrices whose entries are taken from a group with zero. This is called the Schützenburger representation and is described in Clifford and Preston (1961). In this case we assign each element of the Omaha lineage system, which is of the form $a(+\ ♀)^n+\ ♂$, $a(+\ ♀)^n$ or reciprocals of these, where $a = ♂, ♀$ and n is an integer, the two by two matrix which has M^n in its a^{th} row and $♂^{th}$ or $♀^{th}$ column, respectively, and zero elsewhere. For example, if we assume males come first in the rows and columns, the Schützenburger representation maps the strings on the left onto the matrices on the right:

$$
\begin{aligned}
♂ &\rightarrow \begin{pmatrix} M^0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
♂+♀ &\rightarrow \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \\
♀+♀ &\rightarrow \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \\
♂+♀+♀ &\rightarrow \begin{pmatrix} 0 & \mathrm{M}^2 \\ 0 & 0 \end{pmatrix} \\
♂(+♀)^n &\rightarrow \begin{pmatrix} 0 & M^n \\ 0 & 0 \end{pmatrix} \\
(♀-)^n♂ &\rightarrow \begin{pmatrix} 0 & 0 \\ M^{-n} & 0 \end{pmatrix}
\end{aligned}
$$

where the group is just the infinite cyclic group generated by M, i.e., the additive group of the integers.

Also, we require that the zero element in the lineage system be mapped onto the matrix with all zeroes. In order to be a representation, a mapping must be a homomorphism. That is, products must be preserved. To illustrate that this is the case note that $(♂ + ♀)\ ♂ = 0$ and $(♂ + ♀)\ (♀ + ♀) = ♂ + ♀ + ♀$. Then carry out the corresponding operations with the matrices and note that the product of the

images equals the image of the product. That is, the following diagrams 'commutes':

$$\begin{array}{ccccc} (♂+♀) & \cdot & ♂ & = & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} & \cdot & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & = & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

and

$$\begin{array}{ccccc} (♂+♀) & \cdot & (♀+♀) & = & ♂+♀+♀ \\ \downarrow & & \downarrow & & \downarrow \\ \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} & \cdot & \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} & = & \begin{pmatrix} 0 & M^2 \\ 0 & 0 \end{pmatrix} . \end{array}$$

Since this mapping is one-to-one, the representation is said to be *faithful.* It can easily be seen by looking at the matrices that all non-trivial homomorphic images of the Omaha lineage system are found by mapping the group onto finite cyclic groups modulo some integer k. Thus if $k = 2$ we get the patrilineal moiety system discussed above. If $k = 3$, we get a three class system.† Besides specifying all possible homomorphic images, the Schützenburger representation can be interpreted in the light of the 'unity of the lineage' theory to mean that the nonzero entry in the matrix of any relative corresponds to the lineage to which that relative belongs.

4.3 *Congruences and Kernels*

It may be recalled from elementary group theory that the *kernel* of a homomorphism of a group G is the set of elements of G that are mapped onto the identity element of the image group. In terms of a congruence, the kernel is the congruence class containing the identity element. The most important property possessed by a group kernel is that the kernel uniquely determines its congruence. That is, if K is a kernel of a group G, then there is one and only one congruence ρ such that K is the kernel of ρ. This congruence ρ is defined by $x\rho y$ if and only if xy^{-1} is in K. Furthermore, kernels can be characterized independently of congruences or homomorphisms as normal subgroups, where a subgroup N is said to be *normal* if $xN = Nx$ for all x in G.

Now, the concept of kernel can be generalized to inverse semigroups so as to preserve the fundamental property that every congruence is uniquely determined by its kernel. The analog of the identity element is the set of idempotents, where an *idempotent* of any semigroup is defined to be any element e such that $ee = e$. Clearly, any identity element or zero element is an idempotent. In addition to the identity and

† In terms of the marriage class systems discussed in Boyd (1969) the group used as matrix entries is the group of the marriage class systems. Thus the Omaha rule that ♂− ♀ = ♂* ♀ translates in group systems to the equation $C = 1$, so that Omaha systems can never be associated with non-cyclic groups.

the zero, the Schützenburger representation of the Omaha lineage system has two other idempotents:

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \text{ and } \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} .$$

More generally, idempotents in an inverse semigroup are of the form xx^{-1} since $xx^{-1}xx^{-1} = xx^{-1}$ from the Reciprocal equation.

The *kernel* of a homomorphism between inverse semigroups is the set of elements mapped onto idempotents. In terms of a congruence ρ, the kernel is the set of ρ classes containing idempotents. As with groups, kernels of an inverse semigroup S can be defined independently of congruences as a *kernel normal system*, which is, still following Clifford and Preston (1967), a collection $A = \{A_i : i \epsilon I\}$ of subsets of S such that

($K1$) each A_i is an inverse subsemigroup of S;
($K2$) $A_i \cap A_j = \varphi$ if $i = j$;
($K3$) each idempotent in S is contained in some element of A;
($K4$) for each a in S and A_i in A, $a^{-1}A_i a \subseteq A_j$ for some A_j in A;
($K5$) if a, ab and bb^{-1} are all in some one A_i, then b is in A_i also.

Given a kernel normal system A, its unique congruence ρ_A is defined by (K) $x \rho_A y$ if any only if aa^{-1}, bb^{-1} and ab^{-1} are in some A_i in A.

An example of a kernel normal system on the Omaha lineage system is given by the sets

$$A_1 = \left\{ \begin{pmatrix}M^k & 0\\ 0 & 0\end{pmatrix} : \text{for all even integers } k, \right\}$$

$$A_2 = \left\{ \begin{pmatrix}0 & 0\\ 0 & M^k\end{pmatrix} : \text{for all even integers } k, \right\}$$

$$A_3 = \{0\} \text{ and}$$

$$A_4 = \left\{ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \right\} .$$

It can easily be verified that this is, in fact, a kernel normal system and that it is the kernel of the homomorphism onto patrilineal moieties. It is easy to verify that inverse semigroup kernels of groups are the same as group kernels.

We shall apply equation (K) to the problem of finding a set of canonical forms for a subset of the Omaha I inverse semigroup. The subset which we will consider is the set of all consanguineals, strings of the form $a_0 + \ldots + a_m - \ldots - a_{m+n}$, $m, n \geqslant 0$. Consanguineals contrast with *affineals*, which are strings containing 'Husband,' $\female - a + \male$, or 'Wife,' $\male - a + \female$, as a substring, but which contains no 'uncancelled' substrings of the form $a - b + a$. See footnote 3, which explains why these strings cannot be cancelled. Now, it is obvious that by repeated applications of the Omaha I Merging and Skewing rules every consanguineal can be reduced to a consanguineal where $m \leqslant 1$ or $n \leqslant 1$. A stronger result holds, however: Omaha I consanguineals can

be reduced to consanguineals $a_0+ \ldots +a_m- \ldots -a_{m+n}$ where $m \leqslant 1$ or $n \leqslant 1$ and where $a_i = ♀$ implies $a_j = ♀$ for $0 < i < j < m$ and $a_i = ♂$ implies $a_j = ♂$ for $m < i < j < m+n$. In other words, each Omaha I consanguineal is equivalent to an 'ascending' string $a(+♂)^p(+♀)^q+b$ or $a(+♂)^p(+♀)^q*b$ where in the latter case $b = ♂$ if $q > 0$ and $b = ♀$ if $q = 0$ for $p, q = 0, 1, \ldots$, or to a 'descending' string, which is the inverse of an ascending string. To prove this, it is sufficient to show that any string $a+♀+♂+b = x$ is equivalent modulo Omaha I to $a+♀+b = by$. Now, $xx^{-1} = yy^{-1} = a$ by a sufficient number of applications of the Merging relation. In addition, $xy^{-1} = a$ using both the Merging and Skewing relations. The element a is a member of some A_i because a is an idempotent, i.e., $aa = a$. Thus, by (K) $a+♀+♂+b = a+♀+b$ modulo Omaha I, as required.

5. COMPONENTIAL REPRESENTATIONS

The general problem of assigning coordinates to the elements of a system is ubiquitous in science. For example, any 'conservation' law in physics depends upon a cleverly chosen coordinate description of the state space. Thus, to say that 'energy is conserved' in some system implies that the energy coordinate of the state space is left invariant by the inputs under consideration. Krohn, *et al.* (1967), discuss the problem of coordinatization for system in general. In social science this problem arises in the attempt to define 'meaningful' factors when items are imbedded in some n-dimensional Euclidean space. In semantic analysis the values on the various dimensions are often discrete or even binary. In this discrete form the coordinates are usually called 'components' or 'features,' and a particular coordinatization is called a 'componential analysis.' For example, the English word 'mother' can be analyzed by assigning it the components 'female,' 'lineal,' and 'first ascending generation.' A small number of such dimensions with a small finite number of values on each dimension may serve to distinguish kinship terms from each other. Of course, to relate kinship terms to a larger semantic domain so as to include feelings, attitudes and social obligations toward and between these relatives, it may be necessary to add dimensions with a continuum of values. For example, 'amount of authority' might be added. However, for the purpose of describing interrelationships between kin terms, as well as relationships to other systems, it seems to be more instructive to view kinship systems as fundamentally combinatorial and discrete in nature. In this respect kinship systems are like DNA molecules: they may be imbedded in some continuous space and though the data, x-ray photographs, may look continuous, the fundamental problem in understanding their genetic implications consists of decoding the discrete sequences in these double helixes.

A *componential analysis* of some semantic domain D can be defined as a function onto D from a subset of some Cartesian product $C_1 \times C_2 \times \ldots \times C_n$, where each C_i is interpreted as a 'component' or 'dimension.' The elements of each C_i are called the 'values' of that component. There are, of course, many such functions even after lumping together functions that are equivalent under some permutation of the components and values within components. For example, if the domain D contains 8 elements, and we consider only three binary dimensions, there are $8! = 40{,}320$

functions and 840 equivalence classes of functions under the group of complements of values and permutations of variables. In order to select out which componential analyses are best, a 'theory' of the domain D is needed. Different theories will, in general, give different componential analyses. The next section will discuss some ways in which one can empirically compare componential analyses, thus producing a critical test of the theories involved. The theory presented in this paper is based on the algebraic theory of finite automata as developed principally for the analysis of computers. One of the main concepts in this algebraic theory is that of the 'semidirect product,' which is associated with the serial processing of information.

The remainder of this section will be devoted to devising a componential analysis based on this algebraic theory for two closely related Omaha systems: the Fox and the Tenejapanecos. The Fox is included because their system is classic and well elaborated. The Indians of Tenejapa were chosen because they were available for psychological tests of the analysis.

5.1 *Semidirect Products of Semigroups.*

If a semigroup is to be coordinatized in terms of other semigroups, it is reasonable to require the multiplication to have a 'natural' form with respect to this coordinate system. That is, if the elements of a semigroup are to be represented as the Cartesian product $T \times S$ of the elements of the semigroups T and S, then one wants the product $(t_1, s_1)(t_2, s_2)$ to be 'simply' or 'naturally' related to the products $t_1 t_2$ in T and $s_1 s_2$ in S. The most obvious way of defining this multiplication is to let

$$(t_1, s_1)(t_2, s_2) = (t_1 t_2, s_1 s_2),$$

for all t_1, t_2 in T and s_1, s_2 in S. With this operation $T \times S$ is called the *direct product of T and S*. For example, the additive group of points on the plane is the direct product of two copies of the additive groups of the real numbers, given a fixed system of coordinates.

An example from kinship of a direct product is the Omaha Lineage system O_L, which can be represented as a homomorphic image of $Z \times S$, where Z is the set of integers, which keep track of lineages, and S is the five element inverse semigroup of 2 by 2 matrices that are either all zeros or else have three zeros and one 1. S does nothing but keep track of the Sex of speaker and referent, and whether or not the string is well defined. The location of the 1 *vis-à-vis* the rows determines the sex of the speaker, and the columns determine the sex of the referent. The all zero matrix indicates 'not defined' and results from the multiplication of two matrices when the sex of referent of the first is not the same as the sex of speaker of the second. The homomorphism onto 0_L as represented by these matrices from $Z \times S$ is given by

$$\left(M^k, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left(M^k, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \rightarrow \begin{pmatrix} M^k & 0 \\ 0 & 0 \end{pmatrix} \quad \text{etc.,}$$

where k is an integer. In other words, elements of the direct product $Z \times S$ are lumped if the second coordinate is zero.

For kinship semigroups, however, a generalization of direct products is needed. First, we must define an *endomorphism* of a semigroup T to be a homomorphism from T into itself. The set of all endomorphisms of a semigroup T is a semigroup under composition of functions, and is denoted by Endo(T). Let S be another semigroup and let φ be a homomorphism from S into Endo (T). This, we can define $T \times_{\varphi} S$, the *semidirect product of T by S with connecting homomorphism* φ, to be the set $T \times S$ with the operation

$$(t_1, s_1)(t_2, s_2) = (t_1(\varphi s_1)t_2, s_1 s_2).$$

Associativity is easily verified. If we write s_t for $(\varphi s)t$, then since φ is a homomorphism

$$(s_1 s_2)_t = s_1(s_{2_t}).$$

Since φs is an endomorphism for each s in S, we also have

$$s_{(t_1 t_2)} = (s_{t_1})(s_{t_2}).$$

Direct products are a special case of semidirect products if φs is the identity endomorphism for all s in S; that is, if ${}^s t = t$ for all s and t. From the semidirect product $T \times_{\varphi} S$, the projection π onto S is a homomorphism since by definition multiplication on the S component is carried out normally, as in S itself. On the other hand, the projection onto T is not a homomorphism since multiplication of the T component has a 'twist' determined by φs_1.

The Omaha I system provides the example of a semidirect product that motivated its definition here. Recall from Section 4.3 that Fox consanguineal relatives can be reduced to one of the canonical forms $a(+♂)^m(+♀)^n + b$, $a(+♂)^m(+♀)^n * b$, $m, n \geq 0$, $a, b = ♂, ♀$ or reciprocals of these. Consider the subsemigroup† of zero plus the first type of these forms: $a(+♂)^m(+♀)^n$ where $m, n \geq 0$ and $a, b = ♂, ♀$, and $a(+♂)^m(+♀)^n + ♂$ where $m \geq 0$, $n \geq 1$ and $a = ♂, ♀$. If we examine the relevant products

$$a(+♂)^m(+♀)^n(+♂)^p(+♀)^q = a(+♂)^m(+♀)^{n+q}, m, q \geq 0; n, q \geq 1,$$

$$a(+♂)^m(+♀)^n + ♂(+♂)^p(+♀)^q = a(+♂)^m(+♀)^{n+q}, m, p \geq 0; n, q \geq 1,$$

$$a(+♂)^m(+♂)^p(+♀)^q = a(+♂)^{m+p}(+♀)^q, m, p, q \geq 0,$$

where $a = ♂, ♀$, we see that each form, $a(+♂)^m(+♀)^n$ and $a(+♂)^m(+♀)^n + ♂$ can be represented by the pairs $((+♂)^m, a(+♀)^n)$ and $((+♂)^m, a(+♀)^n + ♂)$, respectively. The second coordinate is isomorphic to a subsemigroup of the Omaha lineage system 0_L and the first is isomorphic to the non-negative integers P. Thus, these 'pairs form a semidirect product if we define the connecting homomorphism $\varphi: 0_L \to$ Endo (P) as follows:

$$\left.\begin{array}{l} \varphi(a(+♀)^n)(+♂)^p \\ \varphi(a(+♀)^n + ♂)(+♂)^p \end{array}\right\} = 1 \text{ if } n \geq 1, \text{ but}$$

$$\varphi(a)(+♂)^p = (+♂)^p, \text{ the identity endomorphism.}$$

† This is not an inverse subsemigroup, however, since it is not closed under inverse. In fact, *no* inverse of a member of this subsemigroup is also a member, except for zero.

The first component P is interpreted as a patrilineal generational system. The connecting homomorphism φ has the effect of wiping out this distinction in all patrilines except ego's.

If we now take reciprocals of $P \times_{\varphi} O_L$ in the Fox system, all consanguineal relatives are accounted for except *MoBr* and *SiSo*. *MoBr* is most closely associated with *MoFa* since they are lumped together in O_{III} and O_{IV}. Thus, it is reasonable to assign *MoBr* all the same components as *MoFa* and to add a new component, which we may call the *linear-collateral* distinction, to separate them. This distinction also separates *SiSo* from *DaSo*.

The final componential analysis based on this algebraic approach is shown in Table 1. Actually, the m of $(+\delta)^m$ and the n of $a(+\varphi)^n$ have been combined into the new variable $m+n$ which can easily be interpreted as 'generation.'

5.2 *A Preliminary Experiment in Tenejapa*

Although there are many ways to evaluate a scientific theory using such criteria as beauty, consistency and naturalistic observation, the most desirable way of testing a theory against reality is by way of an experiment. In this section an experiment will be described which consists of correlating a theoretic similarity measure between Tenejapan kinship terms against an empirical measure of direct judged similarity.

Tenejapa is a Tzeltal-speaking tribe of Mayan Indians found in highland Chiapas, a Mexican state adjacent to Guatemala. Lounsbury (1964) classifies the Tenejapan kinship system as Omaha II, and this seems to be correct in our experience except that a weaker form of the merging rule must be used to distinguish *Fa* from *FaBr*, *Mo* from *MoSi* and also for their reciprocals. The new merging rule would have to be $+a*a- = *$. Figure 5 is a kinship diagram for Tenehapa. Note the interesting fact that the reciprocals, *MoBr* and *SisSo*, are called by the same term. This, together

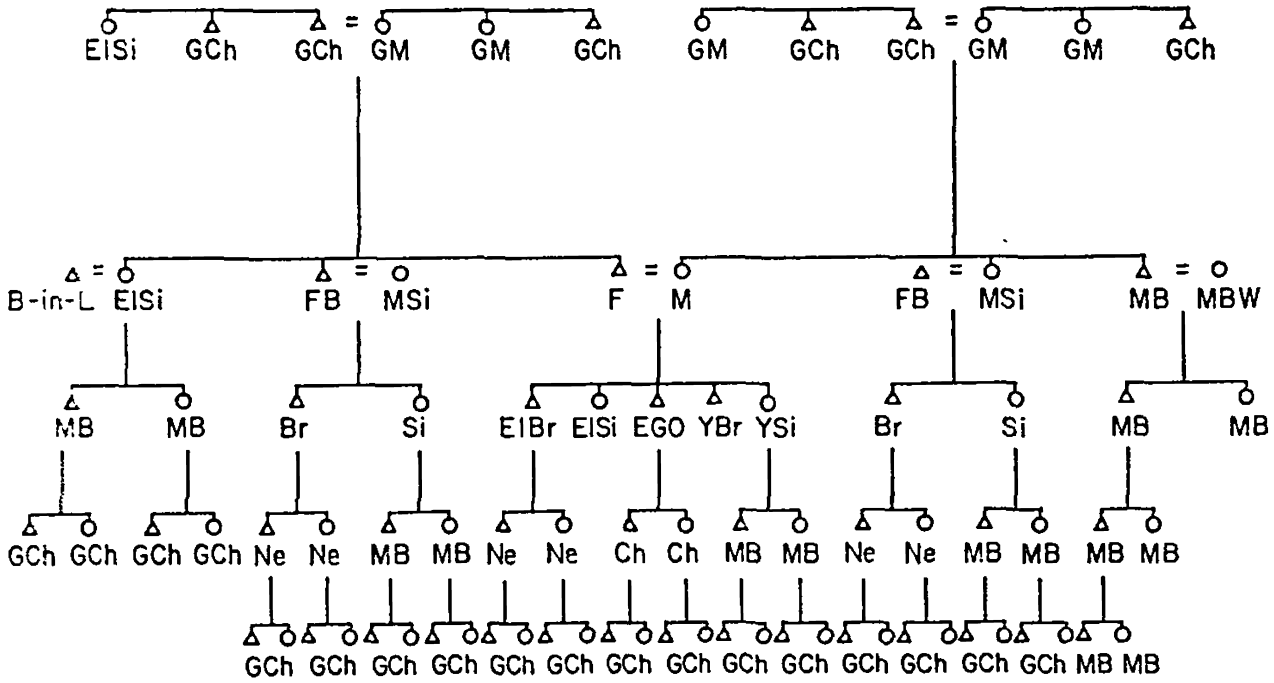
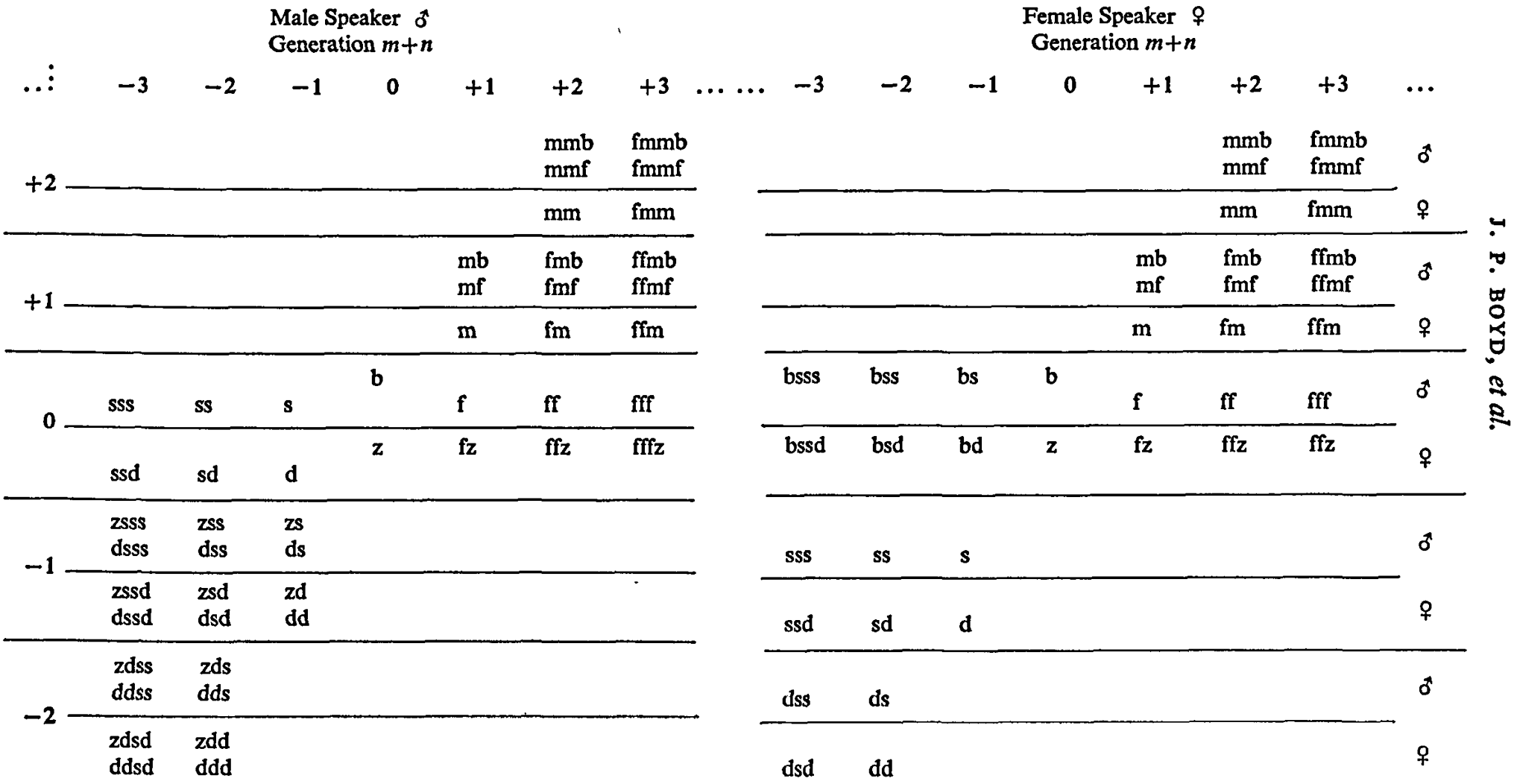


FIGURE 5 Kinship diagram of Tenejapa kinship, male ego.


Male Speaker ♂
Generation m+n
Female Speaker ♀
Generation m+n
... −3 −2 −1 0 +1 +2 +3 ... ... −3 −2 −1 0 +1 +2 +3 ...
+2
mmb
mmf
fmmb
fmmf
mm
fmm
+1
mb
mf
fmb
fmf
ffmb
ffmf
m
fm
ffm
0
b
sss
ss
s
f
ff
fff
z
fz
ffz
fffz
ssd
sd
d
−1
zsss
dsss
zss
dss
zs
ds
zssd
dssd
zsd
dsd
zd
dd
−2
zdss
ddss
zds
dds
zdsd
ddsd
zdd
ddd
bsss
bss
bs
b
bssd
bsd
bd
sss
ss
s
ssd
sd
d
dss
ds
dsd
dd
♂
♀

with the success achieved in Romney and D'Andrade (1964) by making reciprocals similar, has led us to do the same in our theoretical similarity measure.

The theoretical components as derived from our algebraic theory, together with relative age distinction that is necessitated by the sibling terms, are shown in Table 2 as they apply to 11 sets of strings that represent the 11 terms that were used in the experiment. The way that the representatives were chosen out of the possibly infinite

TABLE 2

The eleven kinship words and their components used in the preliminary experiment. The transcription follows Romney (1967)

		Lineage	Genera-tion	Laterality	Relative age	sex of speaker	sex of referent
1. ElSi	/wiš/	0	0	1	e	♂	♀
2. BrCh	/nič'hun/	0	−1	c	—	♂	♂
3. Fa	/tat/	0	+1	1	—	♂	♂
4. ♂ElBr	/bankil/	0	0	1	e	♂	♂
5. MoBr	/ʔičan/	+1	+1	c	—	♂	♂
6. ♂YBr, ♀YSib	/ʔh'¢in/	0	0	1	y	♂	♂
7. GrFa	/mam/	0	+2	1	—	♂	♂
8. ♀ElBr	/ši ʔlel/	0	0	1	e	♀	♂
9. FaBr	/tahun/	0	+1	c	—	♂	♂
10. ♂Ch	/nič'an/	0	−1	1	—	♂	♂
11. ♂YSi	/ʔišlel/	0	0	1	y	♂	♀

alternatives, was on the basis of the following criteria: first, the shortest strings were selected; next, elder generation strings were chosen over younger; and finally, males were chosen over females. This was motivated by the argument that on deciding on the similarity between words, 'unmarked' representatives are chosen for comparison. Of course, additional assumptions are needed to transform the components of Table 2 into a matrix of similarity numbers. First, a measure of similarity was defined on each component by assigning a 1 if the two strings have the same value on that component, and a 0 if they have different values or if the component is 'irrelevant' to one or both strings. Next, it is assumed that the similarity between the two strings is the unweighted sum of the similarities on the components. In a more sophisticated theory one would want to weight the components differently. The algebraic theory would suggest that the lineage and sex components should be weighted more highly than the others, since they are preserved under the homomorphisms considered here. However, in the absence of a reasonable method for numerically estimating these parameters, we will take the usual finesse of assigning equal weights. Finally, to take account of the similarity between reciprocals, we shall take as our real similarity number, the

similarity between two strings as the similarity between one of the strings and the reciprocal of the other if that number is greater. In symbols, if x and y are two strings and $S_i(x, y)$ is the similarity on the i^{th} component, then the similarity between x and y is given by

$$S(x, y) = \max \left[\sum_i S_i(x, y), \sum_i S_i(x^{-1}, y)\right].$$

The theoretical similarity numbers $S(x, y)$ for the 11 terms of this experiment is found in Table 3.

TABLE 3

Theoretical similarity numbers, $S(x, y)$, between kin term x and kin term y.

	1	2	3	4	5	6	7	8	9	10	11
1	–	2	3	5	1	6	3	5	2	3	5
2	2	–	4	3	4	3	3	2	5	4	2
3	3	4	–	4	3	4	4	3	4	5	3
4	5	3	4	–	2	6	4	5	3	4	5
5	1	4	3	2	–	2	2	1	4	3	1
6	6	3	4	6	2	–	4	5	3	4	5
7	3	4	4	4	2	4	–	3	3	4	3
8	5	2	3	5	1	5	3	–	2	3	6
9	2	5	4	3	4	3	3	2	–	4	2
10	3	4	5	4	3	4	4	3	4	–	3
11	5	2	3	5	1	5	3	6	2	3	–

The subjects for this experiment were 74 Tenejapan adult males who were hired off the street in Tenejapa center or in the market town of San Cristobal de Las Casas. The experiment was conducted entirely in their native language, Tzeltal, although most of them also knew some Spanish. A preliminary free recall of kinship words was used as a warm-up. In the free recall tasks the subjects were asked to give us their words for kinds of relatives. Next, the judged similarity task was given. The subject was presented with the first word and was asked to give several other kinds of relatives that were similar to it. These responses were written down, and we went to the next word until the 11 words were completed. There were 4 different 'random' orders of presentation that were used.

When all the responses were accumulated and normalized, the result was an 11 by 29 probability matrix (P_{ij}) where P_{ij} is the number of times word j was said to be similar to the stimulus word i, divided by the total number of responses. If we define P_{i*} to be the row sum $\sum_j P_{ij}$ and similarly let P_{*j} be the column sum $\sum_i P_{ij}$, then a reasonable measure of empirical similarity $\bar{S}$ is given by

$$\bar{S}(i, j) = \frac{P_{ij}+P_{ji}}{P_{i*}P_{*j}+P_{j*}P_{*i}}.$$

Note that if i and j are statistically independent, then $\bar{S}(i, j) = 1$. The matrix $\bar{S}(i, j)$ is given in Table 4.

TABLE 4

The empirical similarity numbers, $S(x, y)$.

——	0.363	1.226	1.348	1.359	1.441	0.609	1.963	0.906	0.814	1.375
0.363	——	0.583	1.533	0.265	0.782	1.397	0.568	2.250	1.204	0.671
1.226	0.583	——	1.034	0.783	1.077	1.946	0.339	1.569	1.956	0.587
1.348	1.533	1.034	——	0.928	1.821	0.781	0.951	0.662	0.825	1.227
1.359	0.265	0.783	0.928	——	0.540	1.329	0.814	1.326	0.729	0.987
1.441	0.782	1.077	1.821	0.540	——	0.631	1.212	0.837	1.102	1.749
0.609	1.397	1.946	0.781	1.329	0.631	——	0.596	1.391	0.831	0.496
1.963	0.568	0.339	0.951	0.814	1.212	0.596	——	0.435	0.316	2.853
0.906	2.850	1.569	0.662	1.336	0.837	1.391	0.435	——	1.183	0.485
0.814	1.204	1.956	0.825	0.729	1.102	0.831	0.316	1.183	——	0.778
1.375	0.671	0.587	1.227	0.987	1.749	0.496	2.853	0.485	0.778	——

The product moment correlation between the theoretical similarity S and the empirical similarity $\bar{S}$ is 0.578, which is significantly positive and perhaps not too low, given the crudity of the model.

6. *Summary and Conclusions*

Inverse semigroups show promise in several different ways as a model of kinship systems. The most elemental justification for this algebraic model is that it reflects traditional anthropological principles of kinship organization. In particular, it has been shown in this paper how the semigroup structure is related to the principle of substitution, or uniform descent. The set of kinship models is narrowed down even further from semigroups to inverse semigroups by consideration of the principle of reciprocity. Although only Omaha-type systems were analyzed in this paper, it seems clear that inverse semigroups can be applied to any of the well known types of kinship systems. Omaha systems were discussed only because they seemed to be the most interesting and difficult systems to analyze.

The 'deep' and 'surface' structures, corresponding to the largest congruence contained in the relation of terminological equivalence, are interesting because of the obvious analogy to the situation in linguistics. Although we were forced by the data to introduce these two levels, we would have avoided it if possible. The larger the gap between unobserved 'deep' structures and 'surface' data, the weaker the theory. Of course, it would be possible experimentally to demonstrate the existence of the 'deep' level in kinship systems. This could be done by considering a single kinship term that includes several congruence classes of kin strings. If native subjects sort the kin strings into the congruence classes, then this would be strong evidence for the psychological reality of this 'deep structure.'

There is an interesting pattern in the relationship between the 'deep' congruence structure and the 'surface' terminological equivalence: as one moves away from ego's generation there are more congruence classes per terminological class. For example, in the Fox system there is only one congruence class per kin term on ego's, the first ascendant and the first descending generation. 'Grandfather,' however,

contains infinitely many congruence classes. In fact, if the older and younger distinction on siblings is considered, the effect is even stronger. This corresponds to the general semantic principle that more distinctions are made close in than further out.

Intuitively speaking, a homomorphism is a structure preserving map between two structures. We hypothesize that kinship systems that share a recent historical origin will be homomorphically related by an inclusion map. In other words, the congruence classes of one deep structure will be subsets of the congruence classes of the other. The set of kinship congruences has a lattice structure which in turn generates a natural distance measure. We claim that this lattice distance is correlated with historical relatedness.

The concept of congruences also relates kin systems to certain conceptually useful approximations, or distillations, of real kinship systems. For example, it has been shown here that all Omaha systems have a congruence called the Omaha lineage system. This agrees with Radcliff-Brown's (1965) 'unity of lineage' theory and is supported by empirical correlations between Omaha-type systems and patrilineages.

Some of the Omaha systems have been algebraically decomposed into direct and semidirect products. The result is a componential analysis whose components, or factors, have ready interpretations, such as sex, lineage and generation. With a number of intermediate assumptions, a theoretical similarity measure has been derived from this componential analysis which is then compared with a preliminary judged similarity experiment in Tenejapa. The results are encouraging, but a comparison with other theories is needed.

Lévi-Strauss (1962) has said that the most important line of future advance in the study of structures lies in the ability of the social scientist to formulate his structures algebraically. We agree and feel that the introduction of inverse semigroups as kinship models is an important advance over the previous group theoretic formulations since the latter can only be applied to a very limited number of societies, called 'marriage class systems.' We would go further than Lévi-Strauss, however, in urging not only more algebraic *formulations* but also more algebraic *analysis*, using the powerful concept of structure preserving map to relate different systems and to decompose single systems.

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