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Free skew monoidal categories

John Bourke *, Stephen Lack

Department of Mathematics, Macquarie University, NSW 2109, Australia

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ABSTRACT

In the paper *Triangulations, orientals, and skew monoidal categories*, the free skew monoidal category **Fsk** on a single generating object was described. We sharpen this by giving a completely explicit description of **Fsk**, and so of the free skew monoidal category on any category. As an application we describe adjunctions between the operad for skew monoidal categories and various simpler operads. For a particular such operad $\mathcal{L}$, we identify skew monoidal categories with certain colax $\mathcal{L}$-algebras.

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1. Introduction

A skew monoidal category is a category $\mathcal{C}$ equipped with a functor $\mathcal{C}^2 \to \mathcal{C}$ whose effect on objects we write as $(a, b) \mapsto ab$, an object $i \in \mathcal{C}$, and natural transformations

$$(ab)c \xrightarrow{\alpha} a(bc) \qquad ia \xrightarrow{\lambda} a \qquad a \xrightarrow{\rho} ai$$

satisfying five coherence conditions. When the maps α, ρ, and λ are invertible, we recover the usual notion of monoidal category.

While this might seem like a mindless generalisation, it turns out that there are important examples of skew monoidal categories which are not monoidal. The first such class of examples arises from quantum algebra, and is due to Szlachányi [9]: he realised that bialgebroids can be described using skew monoidal categories. Specifically, a bialgebroid with base ring R (not necessarily commutative) is the same thing as a skew monoidal closed structure on the category R-**Mod** of R-modules.

A second class of examples arises from the intersection of homotopical algebra and 2-category theory: a host of naturally occurring skew monoidal closed structures on Quillen model categories that arise in

* Corresponding author.
E-mail addresses: john.d.bourke@mq.edu.au (J. Bourke), steve.lack@mq.edu.au (S. Lack).

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2-dimensional universal algebra were described in [1]. These examples are monoidal in a homotopical sense, in that they yield genuine monoidal closed structures on the associated homotopy categories.

Unlike the situation for monoidal categories, it is not the case for skew monoidal categories that all diagrams built up out of the structure maps commute: for example, the composite

$$ii \xrightarrow{\lambda} i \xrightarrow{\rho} ii$$

is not the identity, and so the "coherence problem" for skew monoidal categories is not a trivial one. One way to formulate this coherence problem is to ask what is the *free skew monoidal category* on a given category. An answer to this question was given in [8].

As was observed in [8], the structure of a skew monoidal category is *clubbable*, in the sense of [6]; equivalently, it can be given in terms of a *plain operad in* **Cat**, where by "plain", we mean that there are no actions of the symmetric groups. It then follows that in order to describe the free skew monoidal category on a general category $\mathcal{C}$ it suffices to do it on the terminal category, and in fact this is what is done in [8].

The free skew monoidal category on an object, called **Fsk** in [8], is determined by the following universal property. There is a designated object $X \in \mathbf{Fsk}$ ("the generator") and for any skew monoidal category $\mathcal{C}$, evaluation at X determines a bijection between the set of (strict) monoidal functors from **Fsk** to $\mathcal{C}$ and objects of $\mathcal{C}$.

An example of a skew monoidal category is the category $\mathbf{Ord}_\perp$ of finite non-empty ordinals, with morphisms the functions which preserve both order and bottom element. The product is given by ordinal sum, and the unit object is the ordinal $\mathbf{1} = \{0\}$. This is strictly associative, but the maps λ and ρ are non-invertible. By its universal property, the free skew monoidal category **Fsk** on one object has a unique structure-preserving functor to $\mathbf{Ord}_\perp$ which sends the generator to $\mathbf{1}$. A key result of [8] was that this functor is faithful, so that the morphisms of **Fsk** can be represented as certain functions between finite sets.

While the objects of **Fsk** were described in an entirely explicit way, the morphisms were not. The main goal of this paper is to remedy this, by giving a completely explicit condition characterising the morphisms.

As an application, we construct various adjunctions between the operad $\mathcal{S}$ for skew monoidal categories and various simpler operads $\mathcal{T}$. In each case we have an operad map $F\colon \mathcal{S} \to \mathcal{T}$ and we show that F has a left or right adjoint in each component. By the usual "doctrinal adjunction" results [5] this enables us to view skew monoidal categories as colax/lax $\mathcal{T}$-algebras. When $\mathcal{T}$ is the terminal operad, the unique map $F\colon \mathcal{S} \to \mathcal{T}$ has both adjoints, and so any skew monoidal category yields both a colax monoidal category and a lax monoidal category. These processes lose structure, but choosing for $\mathcal{T}$ an only slighter more complex operad $\mathcal{L}$, we find that colax $\mathcal{L}$-algebras encode the skew monoidal structure entirely. These results are used in our companion paper [2] which introduces and studies *skew multicategories*, the multicategorical analogue of skew monoidal categories.

Acknowledgements

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2. Background on clubs and operads

In this section we group together various facts about plain **Cat**-operads and clubs over $\mathbb{N}$. There is nothing particularly original here, but we could not find any convenient reference containing everything we need, which largely amounts to combining aspects of [6], [7], and [4].

Let $\mathbb{N}$ denote the discrete category with objects the natural numbers. The functor 2-category $[\mathbb{N}, \mathbf{Cat}]$ has a monoidal structure with tensor product $\circ$ given by $(A \circ B)_n = \sum_{n=n_1+\ldots+n_k} A_k \times B_{n_1} \times \ldots \times B_{n_k}$ and with unit J given by $J_1 = 1$ and $J_n = 0$ if $n \neq 1$.

A monoid in $[\mathbb{N}, \mathbf{Cat}]$ is called a *plain* $\mathbf{Cat}$*-operad* [7]; here the epithet "plain" serves to distinguish these operads from the variant involving actions of the symmetric groups. Since plain $\mathbf{Cat}$-operads are the only operads which appear in this paper, we may sometimes simply call them operads.

We generally write $\mu\colon \mathcal{T} \circ \mathcal{T} \to \mathcal{T}$ for the multiplication and $\eta\colon J \to \mathcal{T}$ for the unit of an operad $\mathcal{T}$. Explicitly, the components of the multiplication are "substitution" maps

$$\begin{aligned} \mathcal{T}_k \times \mathcal{T}_{n_1} \times \ldots \times \mathcal{T}_{n_k} &\longrightarrow \mathcal{T}_{n_1+\ldots+n_k} \\ (g, f_1, \ldots, f_k) &\longmapsto g(f_1, \ldots, f_k) \end{aligned}$$

while the unit amounts to an object of $\mathcal{T}_1$.

In the special case where all the fs are identities except (possibly) for f_i, we sometimes write $g \circ_i f_i$ for $g(f_1, \ldots, f_k)$. It is possible to reformulate the definition of operad using only the operations $\circ_i$; for instance $g(f_1, \ldots, f_k)$ can be constructed as

$$g(f_1, \ldots, f_k) = (((g \circ_k f_k) \ldots \circ_2 f_2) \circ_1 f_1).$$

The 2-category $[\mathbb{N}, \mathbf{Cat}]$ is equivalent to the 2-category $\mathbf{Cat}/\mathbb{N}$ of categories over $\mathbb{N}$, with the equivalence sending a functor $\mathcal{T}\colon \mathbb{N} \to \mathbf{Cat}$ to the coproduct $\sum_{n\in\mathbb{N}} \mathcal{T}_n \to \mathbb{N}$. The monoidal structure on $[\mathbb{N}, \mathbf{Cat}]$ can be transported across the equivalence to obtain a monoidal structure on $\mathbf{Cat}/\mathbb{N}$ (although it was first defined independently of that on $[\mathbb{N}, \mathbf{Cat}]$). A monoid in $\mathbf{Cat}/\mathbb{N}$ is called a *club over* $\mathbb{N}$ [6]; once again, in this paper no other clubs are considered so we may simply call it a club.

There is a functor $E\colon \mathbb{N} \to \mathbf{Cat}$ sending the natural number n to the discrete category with n objects. Left Kan extension along E determines a functor $[\mathbb{N}, \mathbf{Cat}] \to [\mathbf{Cat}, \mathbf{Cat}]$ which is strong monoidal and so sends monoids to monoids; that is, it sends plain $\mathbf{Cat}$-operads to 2-monads on $\mathbf{Cat}$.

An algebra for an operad is an algebra for the corresponding 2-monad, but these can also be described directly: a $\mathcal{T}$-algebra is a category A equipped with functors $\mathcal{T}_n \times A^n \to A$ for each n, satisfying associativity and unit conditions.

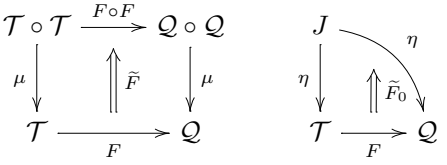
There is also another approach. For a category A there is an operad $\mathrm{End}(A)$ with $\mathrm{End}(A)_n = \mathbf{Cat}(A^n, A)$ and the substitution maps for $\mathrm{End}(A)$ are given by actual substitution

$$\mathbf{Cat}(A^k, A) \times \mathbf{Cat}(A^{n_1}, A) \times \ldots \times \mathbf{Cat}(A^{n_k}, A) \to \mathbf{Cat}(A^{n_1+\ldots+n_k}, A).$$

For an operad $\mathcal{T}$, an algebra structure on A is the same as an operad map $a\colon \mathcal{T} \to \mathrm{End}(A)$.

Because of the 2-category structure of $[\mathbb{N}, \mathbf{Cat}]$ it would be possible to consider various weakenings of the notion of operad. We do not do this, but we do consider weak morphism of operads. Specifically, we consider colax morphisms of operads; formally, these are analogous to opmonoidal functors between monoidal categories.

For operads $\mathcal{T}$ and $\mathcal{Q}$, then, a *colax morphism of operads* from $\mathcal{T}$ to $\mathcal{Q}$ is a morphism $F\colon \mathcal{T} \to \mathcal{Q}$ in $[\mathbb{N}, \mathbf{Cat}]$ equipped with 2-cells



satisfying one coassociativity and two counit conditions, analogous to those for opmonoidal functors. Such a colax morphism is said to be *normal* if $\widetilde{F}_0$ is an identity, and it is this normal case in which we are primarily interested. Of course if $\widetilde{F}$ is also an identity we recover the usual (strict) notion of morphism of operads.

The components of $\widetilde{F}$ have the form

$$F(g(f_1,\dots,f_n)) \xrightarrow{\widetilde{F}} F(g)(F(f_1),\dots,F(f_n))$$

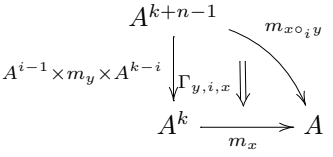
which need to be natural in g and the f_i, as well as satisfying the coassociativity and counit conditions. By coassociativity, these components can all be recovered from the special cases where all but one of the f_i is an identity, which then look like

$$F(g \circ_i f) \xrightarrow{\widetilde{F}} F(g) \circ_i F(f).$$

If $F\colon \mathcal{T} \to \mathcal{Q}$ and $G\colon \mathcal{Q} \to \mathcal{P}$ are colax morphisms of operads, then we may paste $\widetilde{F}$ and $\widetilde{G}$ to give $G \circ F$ the structure of colax morphism from $\mathcal{T}$ to $\mathcal{P}$, normal if F and G are so.

Definition 2.1. A *colax algebra* for an operad $\mathcal{T}$ is a category A equipped with a colax morphism $\mathcal{T} \to \mathrm{End}(A)$. The colax algebra is *normal* if the colax morphism if so.

More explicitly, these involve functors $m\colon \mathcal{T}_n \times A^n \to A$ for each n, and so in particular functors $m_x\colon A^n \to A$ for each n and each object $x \in \mathcal{T}_n$. Then the colax structure involves natural transformations



or in other words $\Gamma_{x,i,y}\colon m_{x\circ_i y} \to m_x \circ_i m_y$.

Whether or not our algebras are strict or colax, we need to consider various flavours of weak morphism between them.

We defined $\mathrm{End}(A)$ so that $\mathrm{End}(A)_n = \mathbf{Cat}(A^n, A)$. More generally, if A and B are categories, we may define $\langle A, B\rangle \in [\mathbb{N}, \mathbf{Cat}]$ by $\langle A, B\rangle_n = \mathbf{Cat}(A^n, B)$; thus $\langle A, A\rangle$ is the underlying object in $[\mathbb{N}, \mathbf{Cat}]$ of the operad $\mathrm{End}(A)$. This construction is functorial in both arguments, covariant in B and contravariant in A.

If now $f\colon A \to B$ is a functor, we may form the comma object

$$\begin{array}{ccc} \{f,f\}_\ell & \xrightarrow{c} & \langle B, B\rangle \\ {\scriptstyle d}\downarrow & \Downarrow & \downarrow{\scriptstyle \langle f,B\rangle} \\ \langle A, A\rangle & \xrightarrow[\langle A,f\rangle]{} & \langle A, B\rangle \end{array}$$

in $[\mathbb{N}, \mathbf{Cat}]$ and $\{f, f\}_\ell$ inherits a unique operad structure for which d and c are morphisms of operads. An object of $(\{f, f\}_\ell)_n$ consists of functors $x\colon A^n \to A$ and $y\colon B^n \to B$, along with a natural transformation $y.f^n \to f.x$.

Definition 2.2. If A and B are colax $\mathcal{T}$-algebras, with $F\colon \mathcal{T} \to \mathrm{End}(A)$ and $G\colon \mathcal{T} \to \mathrm{End}(B)$ the corresponding colax morphisms of operads, a *lax morphism* from A to B is a functor $f\colon A \to B$ equipped with a colax morphism $H\colon \mathcal{T} \to \{f, f\}_\ell$ for which $dH = F$ and $cH = G$.

To give such an H is to give natural transformations

$$\begin{array}{ccc} \mathcal{T}_n \times A^n & \xrightarrow{1\times f^n} & \mathcal{T}_n \times B^n \\ \downarrow & \Downarrow \widetilde{f} & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

satisfying associativity and unit conditions.

In the case where A and B are strict algebras, so that F and G are strict morphisms of operads, such an H will itself be a strict morphism of operads. This is not of course to say that f is strict as a morphism of algebras; its lax nature has been incorporated already in our construction of $\{f, f\}_\ell$ as a comma object.

Definition 2.2 can be modified to give other flavours of weak morphism by replacing the operad $\{f, f\}_\ell$ by analogous operads.

If we formed an *iso-comma object* $\{f, f\}_{\mathrm{ps}}$ rather than the comma object $\{f, f\}_\ell$ we would still obtain an operad, and colax morphisms $\mathcal{T} \to \{f, f\}_{\mathrm{ps}}$ would then correspond to pseudomorphisms of algebras, which are just the special case where $\widetilde{f}$ is invertible.

If instead we formed the *pullback* $\{f, f\}_s$, then we would obtain the strict morphisms, corresponding to the case where $\widetilde{f}$ is an identity.

Alternatively, we could form the comma object $\{f, f\}_c$ in which the direction of the 2-cell is reversed, and this would correspond to reversing the direction of the $\widetilde{f}$, and so to colax morphisms.

For each of these notions of morphism there is a corresponding notion of 2-cell; here we only describe the case of 2-cells between lax morphisms. Suppose then that $\varphi\colon f \to g$ is a natural transformation. As well as $\{f, f\}_\ell$ we may form the operad $\{g, g\}_\ell$ and also the object $\{f, g\}_\ell$ of $[\mathbb{N}, \mathbf{Cat}]$ appearing in the comma object

$$\begin{array}{ccc} \{f, g\}_\ell & \xrightarrow{c} & \langle B, B\rangle \\ {\scriptstyle d}\downarrow & \Downarrow & \downarrow{\scriptstyle \langle f, B\rangle} \\ \langle A, A\rangle & \xrightarrow[\langle A, g\rangle]{} & \langle A, B\rangle \end{array}$$

and now the pullback

$$\begin{array}{ccc} [\varphi, \varphi] & \xrightarrow{q} & \{g, g\}_\ell \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \{\varphi, g\}_\ell} \\ \{f, f\}_\ell & \xrightarrow[\{f, \varphi\}_\ell]{} & \{f, g\}_\ell \end{array}$$

in which $[\varphi, \varphi]$ has a unique operad structure for which p and q are morphisms of operads. A colax morphism $\mathcal{T} \to [\varphi, \varphi]$ corresponds to a 2-cell between lax morphisms of colax algebras.

Definition 2.3. We write Colax-$\mathcal{T}$-Alg$_\ell$ for the 2-category of colax $\mathcal{T}$-algebras, lax morphisms, and algebra 2-cells, and we write nColax-$\mathcal{T}$-Alg$_\ell$ and $\mathcal{T}$-Alg$_\ell$ for the full sub-2-categories of normal colax algebras and strict algebras. Similarly we write Colax-$\mathcal{T}$-Alg$_{\mathrm{s}}$, nColax-$\mathcal{T}$-Alg$_{\mathrm{s}}$ and $\mathcal{T}$-Alg$_{\mathrm{s}}$ for the (non-full) sub-2-categories of strict morphisms.

There are analogous 2-categories with pseudomorphisms, designated with a subscript ps, and of colax morphisms, with a subscript c. In many cases there are parallel results for each such flavour of weak

morphism, and we write Colax-$\mathcal{T}$-Alg$_{\mathrm{w}}$, nColax-$\mathcal{T}$-Alg$_{\mathrm{w}}$, or $\mathcal{T}$-Alg$_{\mathrm{w}}$ if we do not need to specify a particular choice.

Before leaving this section we record the following standard fact.

Proposition 2.4. *A morphism of operads $F\colon \mathcal{T} \to \mathcal{Q}$ induces a 2-functor $F^*\colon \mathcal{Q}$-Alg$_{\mathrm{w}} \to \mathcal{T}$-Alg$_{\mathrm{w}}$. Similarly, a normal colax morphism $F\colon \mathcal{T} \to \mathcal{Q}$ induces a 2-functor $F^*\colon$ nColax-$\mathcal{Q}$-Alg$_{\mathrm{w}} \to$ nColax-$\mathcal{T}$-Alg$_{\mathrm{w}}$, and a colax morphism $F\colon \mathcal{T} \to \mathcal{Q}$ induces a 2-functor $F^*\colon$ Colax-$\mathcal{Q}$-Alg$_{\mathrm{w}} \to$ Colax-$\mathcal{T}$-Alg$_{\mathrm{w}}$.*

3. Background on the free skew monoidal category

Skew monoidal structure on a category $\mathcal{C}$ involves certain basic functors $\mathcal{C}^n \to \mathcal{C}$ and natural transformations between them, as well as equations asserting that various functors and natural transformations, derived from the basic ones via substitution, are equal.

Thus there is an operad $\mathcal{S}$ whose (strict) algebras in **Cat** are the skew monoidal categories; more precisely, the corresponding 2-category $\mathcal{S}$-Alg$_{\mathrm{s}}$ is isomorphic to the 2-category **Skew**$_{\mathrm{s}}$ of skew monoidal categories, strict monoidal functors between them, and monoidal transformations between these.

Furthermore, it is not hard to check that the lax morphisms of $\mathcal{S}$-algebras are the (lax) monoidal functors between skew monoidal categories, the colax morphisms are the opmonoidal functors, and the pseudo morphisms are the strong monoidal functors. Thus for each flavour w of weakness, there is an isomorphism **Skew**$_{\mathrm{w}} \cong \mathcal{S}$-Alg$_{\mathrm{w}}$.

As described in Section 2 above, the operad $\mathcal{S}$ can equivalently be described as a club $\mathbb{S}$. As an object of **Cat**/$\mathbb{N}$, this consists of the free $\mathcal{S}$-algebra on the category 1, equipped with the unique strict morphism to $\mathbb{N}$ sending the generator to $1 \in \mathbb{N}$.

In [8], a specific construction of the free $\mathcal{S}$-algebra was given, under the name **Fsk**. As explained in the introduction this construction was not fully explicit. The goal of the present section is to recall the construction of [8], before giving a fully explicit description of **Fsk** in Section 4.

3.1. Ordinals

We write $\mathbf{m}$ for the ordinal $\{0, 1, \ldots, m-1\}$. We can regard $\mathbf{m}$ as a poset and hence as a category.

A function $\varphi\colon \mathbf{m} \to \mathbf{n}$ is order-preserving if $i \leq j$ implies that $\varphi(i) \leq \varphi(j)$. Thus the order-preserving functions between ordinals are the functors.

We can ask whether such functors have adjoints. Such a functor $\varphi\colon \mathbf{m} \to \mathbf{n}$ has a right adjoint if and only if it preserves the least element: $\varphi(0) = 0$, as is always the case if φ is surjective. When $\varphi(0) = 0$, the right adjoint φ^* is given by

$$\varphi^*(j) = \max\{i \mid \varphi(i) \leq j\}$$

and may also be characterised by the fact that

$$\varphi(\varphi^*(j)) \leq j < \varphi(\varphi^*(j) + 1).$$

In the context of ordinals, the usual adjointness property

$$\varphi(i) \leq j \Leftrightarrow i \leq \varphi^*(j)$$

can be expressed as

$$j < \varphi(i) \Leftrightarrow \varphi^*(j) < i.$$

It is useful to record:

Proposition 3.1. *A right adjoint $\varphi^*\colon \mathbf{n} \to \mathbf{m}$ itself has a right adjoint φ_* if and only if $\varphi(1) \neq 0$. In this case*

$$\varphi_*(i) = \begin{cases} \varphi(i+1) - 1 & \text{if } \varphi(i) < \varphi(i+1) \\ \varphi(i) - 1 & \text{otherwise.} \end{cases}$$

Proof. We know that $\varphi(0) = 0$ since φ has a right adjoint. Now φ^* will have a right adjoint if and only if $\varphi^*(0) = 0$; that is, if $j > 0$ implies that $\varphi(j) > 0$. But this will clearly follow from the special case $\varphi(1) > 0$.

Now

$$\varphi_*(i) = \max\{j \mid \varphi^*(j) \leq i\}.$$

If $\varphi^*(j) = i$ for some j, then the greatest such j will clearly be $\varphi_*(i)$. We know that $\varphi^*(j) = i$ if and only if $\varphi(i) \leq j < \varphi(i+1)$, which is possible if and only if $\varphi(i) < \varphi(i+1)$, and in that case $\varphi(i+1) - 1$ will clearly be the greatest j.

If $\varphi(i) = \varphi(i+1)$ then there is no j with $\varphi^*(j) = i$, so we must settle for the greatest j with $\varphi^*(j) < i$, or equivalently with $j < \varphi(i)$. But the greatest such j is clearly $\varphi(i) - 1$. (Note that $\varphi(0) < \varphi(1)$, so $i = 0$ is impossible, and so $\varphi(i) > 0$ and $\varphi(i) - 1$ does exist.) □

3.2. Left and right bracketing functions

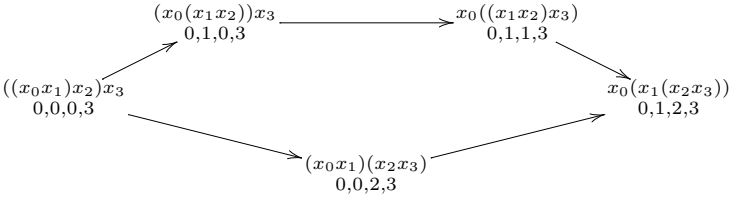
The starting point for the description of the objects of **Fsk** is the Tamari lattice, which consists of all possible bracketings of an n-fold product. These can be described explicitly using the idea of a left bracketing function, given in [3].

Let $\mathbf{m} = \{0, 1, \ldots, m-1\}$ be a non-empty finite ordinal. A *left bracketing function*, or lbf, on $\mathbf{m}$ is a function $\ell\colon \mathbf{m} \to \mathbf{m}$ satisfying three conditions:

(i) $\ell(j) \leq j$ for all $j \in \mathbf{m}$
(ii) if $\ell(j) \leq i < j$ then $\ell(j) \leq \ell(i)$
(iii) ℓ preserves the top element $\top$ of $\mathbf{m}$.

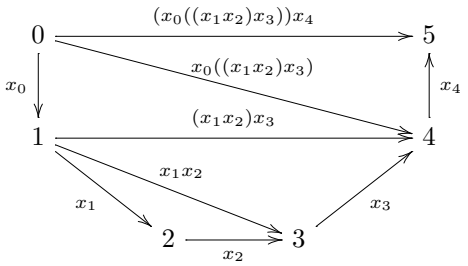
These are given the pointwise ordering: $\ell \leq \ell'$ if and only if $\ell(j) \leq \ell'(j)$ for all j.

For example, in the case $\mathbf{m} = 4$, this corresponds to the bracketings of a 4-fold product, as in the diagram below



where, for example, the list $0, 1, 0, 3$ denotes the lbf with $\ell(0) = \ell(2) = 0$, $\ell(1) = 1$, and $\ell(3) = 3$.

There are various ways to see the correspondence between bracketings and lbfs. Though not required in what follows let us give an example illustrating one such way, which passes through the intermediate step of a triangulation. The bracketing $(x_0((x_1x_2)x_3))x_4$ of five elements corresponds to the triangulation of the 6-gon as below.



The corresponding lbf $l\colon \mathbf{5} \to \mathbf{5}$ is obtained by defining $l(i)$ to be the least vertex of the triangle with middle $i+1$. This only makes sense for $i \leq 3$; for the top element we are forced to define $l(4) = 4$ so that the corresponding lbf is $0, 1, 1, 0, 4$.

We write Tam_m for the resulting poset. Its elements specify bracketings of $m+1$-fold products. One advantage of the lbfs as a description of the elements of Tam_m is that it makes it easy to construct joins in Tam_m. For if ℓ and ℓ' are lbfs then so is the function $\ell \vee \ell'$ given by $(\ell \vee \ell')(i) = \ell(i) \vee \ell'(i)$, where $\vee$ denotes the join (maximum). This $\ell \vee \ell'$ is clearly the join of ℓ and ℓ'; thus Tam_m has binary joins. The function $\mathbf{m} \to \mathbf{m}$ which has constant value 0 is an lbf, and is clearly the least element of Tam_m. Thus Tam_m has finite joins; but it is a finite poset, so therefore has all joins and all meets.

Every lbf $\ell\colon \mathbf{m} \to \mathbf{m}$ determines, and is determined by, a function $r\colon \mathbf{m} \to \mathbf{m}$, connected via the relationships

$$r(i) = \min\{j \mid \ell(j) < i \leq j\}$$
$$\ell(i) = \max\{i \mid i \leq j < r(i)\}$$

Functions r of this type are called *right bracketing functions* or rbfs. It turns out that if also r' corresponds to ℓ' then $\ell \leq \ell'$ if and only if $r \leq r'$.

Since we often go back and forth between lbfs and rbfs, it is convenient to introduce notation which is independent of this choice. We therefore write S for a particular element of the Tamari poset, ℓ_S for the corresponding lbf and r_S for the corresponding rbf.

3.3. Change of base for lbf's

If $\sigma\colon \mathbf{n} \to \mathbf{m}$ is a surjective order preserving map it has a right adjoint σ^*. Proposition 5.6 of [8] shows that if l_S is an lbf on $\mathbf{n}$ the function $\sigma l_S \sigma^*\colon \mathbf{m} \to \mathbf{m}$ is an lbf on $\mathbf{m}$. We write $\sigma S \sigma^*$ for the corresponding element of Tam_m.

3.4. The free skew monoidal category **Fsk** *on* 1

We now turn to the construction of **Fsk** given in [8].

Definition 3.2. An object of **Fsk** is a triple $(\mathbf{m}, u, S)$ where $\mathbf{m}$ is a non-empty finite ordinal, u is a subset of $\mathbf{m}$, and $S \in \mathrm{Tam}_m$, with corresponding lbf $\ell_S\colon \mathbf{m} \to \mathbf{m}$.

This is thought of as an m-fold product, bracketed according to S, with the generator X in the positions specified by u, and the unit i elsewhere.

The generating object is $(\mathbf{1}, \mathbf{1}, S)$ for the unique $S \in \mathrm{Tam}_1$. For an arbitrary skew monoidal category $\mathcal{C}$ and object $X \in \mathcal{C}$, there is a unique strict monoidal functor $\mathbf{Fsk} \to \mathcal{C}$ sending $(\mathbf{1}, \mathbf{1}, S)$ to X: this is what is meant by saying that **Fsk** is the free skew monoidal category on one object.

To motivate the definition of the morphisms of **Fsk** recall the category $\mathbf{Ord}_\perp$ of finite non-empty ordinals and functions preserving both order and bottom element. This admits a strictly associative skew monoidal structure, with ordinal sum for tensor product and unit 1.

Accordingly the unit $\mathbf{1} \in \mathbf{Ord}_\perp$ determines a canonical map $\mathbf{Fsk} \to \mathbf{Ord}_\perp$ preserving the skew monoidal structure strictly, and sending $(\mathbf{m}, u, S)$ to $\mathbf{m}$. One of the main results of [8] is that this $\mathbf{Fsk} \to \mathbf{Ord}_\perp$ is faithful; whereby morphisms $(\mathbf{m}, u, S) \to (\mathbf{m}, u, S)$ of **Fsk** can be identified with certain morphisms $\mathbf{m} \to \mathbf{m}$ of $\mathbf{Ord}_\perp$.

The question, then, is to identify which ones. In [8] this was done in stages, starting with various special classes of morphism. We begin with those corresponding to the associators α.

Definition 3.3. If $(\mathbf{n}, u, S)$ and $(\mathbf{n}, u, T)$ are objects of **Fsk**, we say that the identity $1\colon \mathbf{n} \to \mathbf{n}$ defines a *Tamari morphism* $1\colon (\mathbf{n}, u, S) \to (\mathbf{n}, u, T)$ if $S \leq T$.

Next we turn to those corresponding to applications of λ.

Definition 3.4. A *shrink morphism* from $(\mathbf{m}, u, S)$ to $(\mathbf{n}, v, T)$ is an order-preserving surjection $\sigma\colon \mathbf{m} \to \mathbf{n}$ satisfying the following conditions:

(i) σ and σ^* restrict to mutually inverse bijections between u and v
(ii) $\sigma \ell_S \sigma^* = \ell_T$
(iii) if $\sigma(j) = \sigma(j+1)$ then $\sigma(\ell_S(j)) = \sigma(j)$.

Remark 3.5. As observed in [8], if σ^* restricts to a bijection $v \to u$, the inverse is necessarily given by (the restriction of) σ. But to say that σ restricts to a bijection $u \to v$ is not enough: we should also insist that $\sigma^*\sigma j = j$ for any $j \in u$; in other words, if $j \in u$ then j is maximal in the fibre $\sigma^{-1}\sigma(j)$.

Remark 3.6. For an order-preserving surjection σ, to say that $\sigma(j) = \sigma(j+1)$ is to say that $\sigma(j+1) \leq \sigma(j)$, or equivalently $j + 1 \leq \sigma^*\sigma(j)$, or equivalently $j < \sigma^*\sigma(j)$. Thus we can reformulate (iii) as

(iii) if $j < \sigma^*\sigma(j)$ then $\sigma(\ell_S(j)) = \sigma(j)$.

Combining the two classes yields a class of morphism named after the fact that they are precisely those morphisms of **Fsk** sent to surjections by the canonical map $\mathbf{Fsk} \to \mathbf{Ord}_\perp$.

Definition 3.7. An **Fsk**-*surjection* $(\mathbf{m}, u, S) \to (\mathbf{n}, v, T)$ is an order-preserving surjection $\sigma\colon \mathbf{m} \to \mathbf{n}$ that factorizes as a Tamari morphism $1\colon (\mathbf{m}, u, S) \to (\mathbf{m}, u, S')$ followed by a shrink morphism $(\mathbf{m}, u, S') \to (\mathbf{n}, v, T)$.

Corresponding to the application of ρ there is the notion of a swell morphism. Combining these with the Tamari morphisms yields the **Fsk**-injections, so named since they are precisely the maps sent to injections by the canonical $\mathbf{Fsk} \to \mathbf{Ord}_\perp$.

These are defined in [8] using duality, as below, but an elementary description can also be given – see Section 10 of [8]. For the definition using duality observe that given $(\mathbf{n}, v, T)$ of **Fsk** we can form the object $(\mathbf{n}^{\mathrm{op}}, u, T^{\mathrm{op}})$ of **Fsk** in which T^{op} corresponds to the lbf r_T on $\mathbf{n}^{\mathrm{op}}$.

Definition 3.8. An **Fsk**-injection $(\mathbf{n}, v, T) \to (\mathbf{m}, u, S)$ is an order-preserving left adjoint $\delta\colon \mathbf{n} \to \mathbf{m}$ for which $\delta^*\colon \mathbf{m} \to \mathbf{n}$ defines an **Fsk**-surjection $(\mathbf{m}^{\mathrm{op}}, u, S^{\mathrm{op}}) \to (\mathbf{n}^{\mathrm{op}}, v, T^{\mathrm{op}})$. Such a δ is a swell morphism if and only if δ^* is a shrink morphism.

Finally we are in a position to describe the general case.

Definition 3.9. A morphism in **Fsk** from $(\mathbf{m}, u, S)$ to $(\mathbf{n}, v, T)$ is an order-preserving map $\varphi\colon \mathbf{m} \to \mathbf{n}$ with a right adjoint that can be factorized as an **Fsk**-surjection followed by an **Fsk**-injection.

4. Fsk revisited

Fsk-surjections, **Fsk**-injections, and general morphisms of **Fsk** were defined in terms of the existence of certain factorisations, which need not be unique. In the present section we revisit each class, giving completely explicit descriptions of them.

4.1. **Fsk** *surjections revisited*

In the definition of **Fsk**-surjection the object S' is not uniquely determined. The first step will be to describe a canonical choice for S'. It was shown in [8, Proposition 9.1] that there is a maximal choice of S'; here we describe it more explicitly.

Lemma 4.1. *Let $\sigma\colon \mathbf{m} \to \mathbf{n}$ be an order-preserving surjection and $\ell\colon \mathbf{n} \to \mathbf{n}$ an lbf. Consider the function $\ell^\sigma\colon \mathbf{m} \to \mathbf{m}$ given by*

$$\ell^\sigma(j) = \begin{cases} \sigma^*\ell\sigma(j) & \text{if } j = \sigma^*\sigma j \\ j & \text{otherwise.} \end{cases}$$

Then ℓ^σ is an lbf and $\ell^\sigma\sigma^* = \sigma^*\ell$; thus also $\sigma\ell^\sigma\sigma^* = \ell$.

Proof. First observe that if $j = \sigma^*h$, then $\sigma^*\sigma j = \sigma^*\sigma\sigma^*h = \sigma^*h = j$, thus $j = \sigma^*\sigma j$ if and only if $j = \sigma^*h$ for some h. Thus $\ell^\sigma\sigma^*h = \sigma^*\ell\sigma\sigma^*h = \sigma^*\ell h$ for all h, and so $\ell^\sigma\sigma^* = \sigma^*\ell$ and therefore $\sigma\ell^\sigma\sigma^* = \sigma\sigma^*\ell = \ell$.

Thus we need only show that ℓ^σ is an lbf. If $j = \sigma^*\sigma j$ then $\ell^\sigma(j) = \sigma^*\ell\sigma(j) \leq \sigma^*\sigma(j) = j$, while otherwise $\ell^\sigma(j) = j$. Thus $\ell^\sigma(j) \leq j$ for all j.

Since σ, σ^*, and ℓ all preserve top elements, so does ℓ^σ.

Finally, suppose that $\ell^\sigma(j) \leq i < j$. Then $\ell^\sigma(j) \neq j$, so we must have $\sigma^*\sigma j = j$ and $\ell^\sigma(j) = \sigma^*\ell\sigma j$.

If $i = \sigma^*\sigma i$ then $\ell^\sigma(j) = \sigma^*\ell\sigma j \leq \sigma^*\sigma i \leq j = \sigma^*\sigma j$, and so applying σ gives $\ell\sigma j \leq \sigma i \leq \sigma j$, and ℓ is an lbf so $\ell\sigma j \leq \ell\sigma i$, and

$$\ell^\sigma(j) = \sigma^*\ell\sigma(j) \leq \sigma^*\ell\sigma(i) = \ell^\sigma(i).$$

If $i \neq \sigma^*\sigma i$ then $\ell^\sigma(i) = i$ and so $\ell^\sigma(j) \leq i = \ell^\sigma(i)$. □

If T is the element of the Tamari lattice corresponding to ℓ, it is convenient to write T^σ for the element of the Tamari lattice corresponding to ℓ^σ.

Proposition 4.2. *Any* **Fsk**-*surjection $\sigma\colon (\mathbf{m}, u, S) \to (\mathbf{n}, v, T)$ factorises as*

$$(\mathbf{m}, u, S) \xrightarrow{1_{\mathbf{m}}} (\mathbf{m}, u, S \vee T^\sigma) \xrightarrow{\sigma} (\mathbf{n}, v, T)$$

where the second factor is a shrink morphism. Furthermore, $S \vee T^\sigma$ is the greatest S' for which $\sigma\colon (\mathbf{m}, u, S') \to (\mathbf{n}, v, T)$ is a shrink morphism.

Proof. Since $S \leq S \vee T^\sigma$, the first factor is an **Fsk**-surjection. We need to show that the second factor is a shrink morphism and that $S \vee T^\sigma$ is maximal.

Since $\sigma\colon (\mathbf{m}, u, S) \to (\mathbf{n}, v, T)$ is an **Fsk**-surjection there is an $S' \geq S$ for which $\sigma\colon (\mathbf{m}, u, S') \to (\mathbf{n}, v, T)$ is a shrink morphism.

The fact that σ^* restricts to a bijection $v \to u$ is unchanged by passing from S' to $S \vee T^\sigma$.

For the second condition we have

$$\begin{aligned}
\sigma\ell_{S\vee T^\sigma}\sigma^* j &= \sigma(\ell_S\sigma^* j \vee \ell_{T^\sigma}\sigma^* j)\\
&= \sigma(\ell_S\sigma^* j \vee \sigma^*\ell_T j)\\
&= \sigma\ell_S\sigma^* j \vee \sigma\sigma^*\ell_T j\\
&= \sigma\ell_S\sigma^* j \vee \ell_T j\\
&= \ell_T j
\end{aligned}$$

since $\sigma\ell_S\sigma^* \leq \sigma\ell_{S'}\sigma^* = \ell_T$. Thus $\sigma\ell_{S\vee T^\sigma}\sigma^* = \ell_T$.

It remains to show that if $j < \sigma^*\sigma(j)$ then $\sigma(\ell_{S\vee T^\sigma}(j)) = \sigma(j)$. But if $j < \sigma^*\sigma(j)$ then $\ell^\sigma_T(j) = j$ and so $\ell_{S\vee T^\sigma} j = j$, and so $\sigma(\ell_{S\vee T^\sigma} j) = \sigma(j)$.

Now we show that $S \vee T^\sigma$ is the greatest S' as in the proposition. First observe that if S_1 and S_2 are any two such, then $S_1 \vee S_2$ is another, thus it will suffice to show that if $S \vee T^\sigma < S'$ then S' is not such an element.

If $S \vee T^\sigma < S'$ then for some j we have both $\ell_S(j) < \ell_{S'}(j)$ and $\ell_{T^\sigma}(j) < \ell_{S'}(j)$. Clearly this is impossible if $\ell_{T^\sigma}(j) = j$, so we must have $j = \sigma^*\sigma(j)$ and $\ell_{T^\sigma}(j) = \sigma^*\ell_T\sigma(j)$. Now $\sigma^*\ell_T\sigma(j) < \ell_{S'}(j)$ and so

$$\begin{aligned}
\ell_T(\sigma(j)) &< \sigma\ell_{S'}(j) && \text{(adjointness)}\\
&= \sigma\ell_{S'}\sigma^*\sigma(j) && (j = \sigma^*\sigma(j))\\
&= \ell_T\sigma(j) && \text{(shrink morphism)}
\end{aligned}$$

giving a contradiction. □

We can now use this last result to provide a more explicit description of **Fsk**-surjections:

Proposition 4.3. *An order-preserving surjection $\sigma\colon \mathbf{m} \to \mathbf{n}$ defines an* **Fsk***-surjection $(\mathbf{m}, u, S) \to (\mathbf{n}, v, T)$ if and only if*

(i) *σ and σ^* restrict to mutually inverse bijections between u and v*
(ii) *$\sigma\ell_S\sigma^* \leq \ell_T$.*

Proof. By the previous result, σ will define an **Fsk**-surjection if and only if $\sigma\colon (\mathbf{m}, u, S \vee T^\sigma) \to (\mathbf{n}, v, T)$ is a shrink morphism.

Condition (i) in the definition of shrink morphism is condition (i) in the proposition. Condition (ii) in the definition of shrink morphism says that $\sigma(\ell_S \vee \ell^\sigma_T)\sigma^* = \ell_T$. Now

$$\begin{aligned}
\sigma(\ell_S \vee \ell^\sigma_T)\sigma^* &= \sigma\ell_S\sigma^* \vee \sigma\ell^\sigma_T\sigma^*\\
&= \sigma\ell_S\sigma^* \vee \sigma\sigma^*\ell_T\\
&= \sigma\ell_S\sigma^* \vee \ell_T
\end{aligned}$$

which is equal to ℓ_T if and only if condition (ii) in the proposition holds.

Finally condition (iii) in the definition of shrink morphism says that if $j < \sigma^*\sigma j$ then $\sigma(\ell_S \vee \ell_T^\sigma)j = \sigma j$. But if $j < \sigma^*\sigma j$ then

$$\begin{aligned}\sigma(\ell_S \vee \ell_T^\sigma)j &= \sigma(\ell_S j \vee \ell_T^\sigma j)\\ &= \sigma(\ell_S j \vee j)\\ &= \sigma(j)\end{aligned}$$

and so this is automatic. □

This in turn gives another factorisation:

Proposition 4.4. *Any* **Fsk***-surjection* $\sigma\colon (\mathbf{m}, u, S) \to (\mathbf{n}, v, T)$ *factorises as*

$$(\mathbf{m}, u, S) \xrightarrow{\sigma} (\mathbf{n}, v, \sigma S\sigma^*) \xrightarrow{1} (\mathbf{n}, v, T).$$

Proof. The first factor satisfies the characterisation in Proposition 4.3, so is an **Fsk**-surjection. By that same characterisation, $\sigma S\sigma^* \leq T$, and so the second factor is a Tamari morphism. □

4.2. **Fsk** *injections revisited*

We can deal quickly with **Fsk** injections using duality. First we dualise Lemma 4.1.

Lemma 4.5. *Let* $\delta\colon \mathbf{n} \to \mathbf{m}$ *be a bottom-preserving injection with right adjoint* δ^**, and let* $r\colon \mathbf{n} \to \mathbf{n}$ *be an rbf. Consider the function* $r^\delta\colon \mathbf{m} \to \mathbf{m}$ *given by*

$$r^\delta(j) = \begin{cases} \delta r\delta^*(j) & \text{if } j = \delta\delta^*(j)\\ j & \text{otherwise.}\end{cases}$$

Then r^δ *is an rbf and* $r^\delta\delta = \delta r$*.*

Proof. We can think of r as an lbf on $\mathbf{n}^{\mathrm{op}}$, and think of δ^* as an order-preserving surjection $\mathbf{m}^{\mathrm{op}} \to \mathbf{n}^{\mathrm{op}}$ in which case δ becomes its right adjoint. Now apply Lemma 4.1. □

Dualising the other results similarly, we obtain

Proposition 4.6. *Any* **Fsk***-injection* $\delta\colon (\mathbf{n}, v, T) \to (\mathbf{m}, u, S)$ *factorises as*

$$(\mathbf{n}, v, T) \xrightarrow{\delta} (\mathbf{m}, u, T^\delta \wedge S) \xrightarrow{1} (\mathbf{m}, u, S)$$

where the first factor is a swell morphism. (Furthermore $T^\delta \wedge S$ *is minimal with this property.)*

Proposition 4.7. *An order- and bottom-preserving injection* $\delta\colon \mathbf{n} \to \mathbf{m}$ *defines an* **Fsk***-injection* $(\mathbf{n}, v, T) \to (\mathbf{m}, u, S)$ *if and only if*

(i) δ *and* δ^* *restrict to mutually inverse bijections between* u *and* v
(ii) $r_T \leq \delta^* r_S \delta$*.*

Proposition 4.8. *Any* **Fsk***-injection* $\delta\colon (\mathbf{n}, v, T) \to (\mathbf{m}, u, S)$ *factorises as*

$$(\mathbf{n}, v, T) \xrightarrow{1} (\mathbf{n}, v, \delta^* S\delta) \xrightarrow{\delta} (\mathbf{m}, u, S).$$

4.3. General **Fsk** *morphisms revisited*

In the definition of a general **Fsk**-morphism, the underlying factorisation $\varphi = \delta \circ \sigma$ in $\mathbf{Ord}_\perp$ must be the unique epi-mono factorisation. So the definition can be reformulated as follows.

Definition 4.9. A morphism in **Fsk** from $(\mathbf{m}, u, S)$ to $(\mathbf{n}, v, T)$ is an order-preserving map $\varphi\colon \mathbf{m} \to \mathbf{n}$ with a right adjoint, such that there exist an **Fsk**-surjection

$$(\mathbf{m}, u, S) \xrightarrow{\sigma} (\mathrm{im}(\varphi), \varphi(u), R)$$

and an **Fsk**-injection

$$(\mathrm{im}(\varphi), \varphi(u), R) \xrightarrow{\delta} (\mathbf{n}, v, T)$$

with $\varphi = \delta \circ \sigma$ for some $R \in \mathrm{Tam}_{\mathrm{im}(\varphi)}$.

The R appearing in the factorization need not be given explicitly. In the following theorem we show that there is a canonical choice for R, and use this to give the promised explicit description of the morphisms of **Fsk**.

Theorem 4.10. *An order-preserving morphism* $\varphi\colon \mathbf{m} \to \mathbf{n}$ *defines an* **Fsk***-morphism* $(\mathbf{m}, u, S) \to (\mathbf{n}, v, T)$ *if and only if*

(a) φ *has a right adjoint* φ^*
(b) φ *and* φ^* *restrict to mutually inverse bijections between* u *and* v
(c) $\sigma\ell_S\sigma^* \leq \delta^*\ell_T\delta_*$

where $\sigma\colon \mathbf{m} \to \mathrm{im}(\varphi)$ *and* $\delta\colon \mathrm{im}(\varphi) \to \mathbf{n}$ *are the induced maps.*

Proof. Given any $\varphi\colon \mathbf{m} \to \mathbf{n}$ we may factorise it as a surjection $\sigma\colon \mathbf{m} \to \mathrm{im}(\varphi)$ followed by an injection $\delta\colon \mathrm{im}(\varphi) \to \mathbf{n}$.

To say that σ and σ^* restrict to mutually inverse bijections between u and $\sigma(u)$ is to say that if $j \in u$ then $\sigma^*\sigma(j) = j$, but $\sigma^*\sigma(j) = \sigma^*\delta^*\delta\sigma(j) = \varphi^*\varphi(j)$, so this says that if $j \in u$ then $\varphi^*\varphi(j) = j$.

Suppose that this is the case. Then to say that δ and δ^* restrict to mutually inverse bijections between $\sigma(u)$ and v is then to say that δ maps $\sigma(u)$ to v, and δ^* maps v to $\sigma(u)$, and if $i \in v$ then $\delta\delta^*(i) = i$.

Now δ maps $\sigma(u)$ to v if and only if φ maps u to v. And δ^* maps v to $\sigma(u)$ if and only if $\sigma^*\delta^*$ maps v to $\sigma^*\sigma(u)$; but $\sigma^*\delta^* = \varphi^*$ and $\sigma^*\sigma(u) = u$, thus this says that φ^* maps v to u. Also $\delta\delta^* = \delta\sigma\sigma^*\delta^* = \varphi\varphi^*$.

Thus to say that there are mutually inverse bijections

$$u \underset{\sigma^*}{\overset{\sigma}{\rightleftarrows}} \varphi(u) \underset{\delta^*}{\overset{\delta}{\rightleftarrows}} v$$

is just to say that condition (b) holds.

Suppose now that R is given as in Definition 4.9. Since δ is an **Fsk**-injection, we may use Proposition 4.8 to obtain a factorisation

$$(\mathbf{m}, u, S) \xrightarrow{\sigma} (\mathrm{im}(\varphi), \varphi(u), R) \xrightarrow{1} (\mathrm{im}(\varphi), \varphi(u), \delta^* T\delta) \xrightarrow{\delta} (\mathbf{n}, v, T)$$

where $r_{\delta^* T\delta} = \delta^* r_T \delta$. As observed in [8, Proposition 5.7], the corresponding lbf $\ell_{\delta^* T\delta}$ is $\delta^* \ell_T \delta_*$ where δ_* is the *right* adjoint of δ^*. In this new factorisation, the middle factor is also an **Fsk**-surjection. Thus the composite of the first two factors is an **Fsk**-surjection and so

$$(\mathbf{m}, u, S) \xrightarrow{\sigma} (\mathrm{im}(\varphi), \varphi(u), \delta^* T\delta) \xrightarrow{\delta} (\mathbf{n}, v, T)$$

is also a factorization as in Definition 4.9.

Thus we have proved that φ is a morphism in **Fsk** if and only if, in this last displayed composite, σ is an **Fsk**-surjection and δ is an **Fsk**-injection. By Proposition 4.3 this is equivalent to conditions (a), (b), and (c). □

By adjointness (c) above is equivalent to $\varphi\ell_S\sigma^* \leq \ell_T\delta_*$. One should resist the temptation to use adjointness once again to transform the inequality $\varphi\ell_S\sigma^* \leq \ell_T\delta_*$ to $\varphi\ell_S \leq \ell_T\delta_*\sigma$. This would be valid if we knew that ℓ_S and ℓ_T were functors (order-preserving), but this need not be the case.

On the other hand, there is another possible reformulation:

Theorem 4.11. *An order-preserving morphism $\varphi\colon \mathbf{m} \to \mathbf{n}$ defines an* **Fsk***-morphism $(\mathbf{m}, u, S) \to (\mathbf{n}, v, T)$ if and only if*

(a) *φ has a right adjoint φ^**
(b) *φ and φ^* restrict to mutually inverse bijections between u and v*
(d) *if $\varphi(j) < \varphi(j+1)$ then $\varphi(\ell_S(j)) \leq \ell_T(\varphi(j+1) - 1)$.*

Proof. Factorise φ as $\sigma\colon \mathbf{m} \to \mathrm{im}(\varphi)$ and $\delta\colon \mathrm{im}(\varphi) \to \mathbf{n}$, as in Theorem 4.10. We need to show that (c) is equivalent to (d).

As mentioned above, (c) is equivalent to $\varphi\ell_S\sigma^* \leq \ell_T\delta_*$. If $j = \sigma^*(h)$ then $h = \sigma\sigma^*(h) = \sigma(j)$. Hence $\varphi\ell_S\sigma^* \leq \ell_T\delta_*$ says that if $j = \sigma^*(h)$ then $\varphi\ell_S(j) \leq \ell_T\delta_*\sigma(j)$. Now $j = \sigma^*(h)$ for some h if and only if $j = \sigma^*\sigma(j)$; and this is equivalent, as in Remark 3.6 to $\sigma(j) < \sigma(j+1)$. This in turn is clearly equivalent to $\varphi(j) < \varphi(j+1)$.

Thus (c) is equivalent to the condition that if $\varphi(j) < \varphi(j+1)$ then $\varphi\ell_S(j) \leq \ell_T\delta_*\sigma(j)$. So it will suffice to show that if $\varphi(j) < \varphi(j+1)$ then $\delta_*\sigma(j) = \varphi(j+1) - 1$.

Now $\delta_*\sigma(j) = \max\{i \mid \delta^*(i) \leq \sigma(j)\} = \max\{i \mid \delta^*(i) = \sigma(j)\}$, since δ^* is surjective and so there certainly exists an i with $\delta^*(i) = \sigma(j)$, and thus clearly the maximum must be of this type. To say that $\delta^*(i) = \sigma(j)$ is to say that

$$\delta(\sigma(j)) \leq i < \delta(\sigma(j)+1).$$

Since $\sigma(j) < \sigma(j+1)$ and σ is surjective, we must have $\sigma(j+1) = \sigma(j) + 1$ thus the displayed inequality becomes

$$\varphi(j) \leq i < \varphi(j+1)$$

and now the maximum value of i is clearly $\varphi(j+1) - 1$. □

5. Adjunctions of operads and skew monoidal categories as colax algebras

In this section we describe adjunctions between the operad for skew monoidal categories and other simpler operads $\mathcal{T}$. These adjunctions allow us to view skew monoidal categories as colax $\mathcal{T}$-algebras. We begin by taking $\mathcal{T}$ to be the terminal operad before passing to another operad $\mathcal{L}$ whose colax algebras fully capture skew monoidal structure.

5.1. Colax and lax monoidal structure

Let $\mathcal{S}$ be the operad for skew monoidal categories and $\mathcal{N}$ the operad for strict monoidal categories. Since $\mathcal{N}$ is the terminal operad, there is a unique (strict) operad morphism $P\colon \mathcal{S} \to \mathcal{N}$. The induced 2-functors $P^*\colon \mathcal{N}\text{-Alg}_{\mathrm{w}} \to \mathcal{S}\text{-Alg}_{\mathrm{w}}$ are the inclusions, for the various possible flavours of morphism, of strict monoidal categories in skew monoidal categories.

In this section we shall see that $P\colon \mathcal{S} \to \mathcal{N}$ has a *colax* left adjoint, which allows us to view each skew monoidal category as a colax $\mathcal{N}$-algebra – that is, a colax monoidal category.

The universal property of the free skew monoidal category **Fsk** on 1 gives a strict monoidal functor $\mathbf{Fsk} \to \mathbb{N}$, which sends $(\mathbf{n}, u, S)$ to the cardinality $|u|$ of the subset u. It follows that $\mathcal{S}_m$ is the full subcategory of **Fsk** consisting of all objects of the form $(\mathbf{n}, u, S)$ with $|u| = m$.

Theorem 5.1. *The map $P\colon \mathcal{S} \to \mathcal{N}$ has a left adjoint in $[\mathbb{N}, \mathbf{Cat}]$ with identity unit.*

Proof. This is equivalent to saying that $P\colon \mathcal{S}_m \to \mathcal{N}_m$ has a left adjoint in **Cat** with identity unit, for each $m \in \mathbb{N}$. Since $\mathcal{N}_m$ is the terminal category, this in turn is equivalent to saying that each $\mathcal{S}_m$ has an initial object.

The initial object is $(\mathbf{m+1}, \mathbf{m+1}/\{0\}, \bot)$, where the specified subset consists of all elements except 0, and $\bot \in \mathrm{Tam}_{m+1}$ is the bottom element of the Tamari lattice, corresponding to the lbf ℓ_L with $\ell_L(m) = m$ and $\ell_L(i) = 0$ if $i \neq m$.

We prove the universal property using the characterisation in Theorem 4.11. Suppose then that $(\mathbf{n}, v, T) \in \mathbf{Fsk}$ has $|v| = m$. There is a unique order-preserving bijection $\theta\colon \mathbf{m+1}/\{0\} \to v$. The only way to define a map $\varphi\colon \mathbf{m+1} \to \mathbf{n}$ which preserves order and the bottom element, as required to have a right adjoint, and which restricts to θ, is to define $\varphi(i) = \theta(i)$ if $i \in u$, and $\varphi(0) = 0$.

This proves uniqueness; it remains to verify the conditions (b) and (d) in Theorem 4.11. If $i \in \mathbf{m+1}/\{0\}$ then $\varphi^{-1}\varphi(i) = \{i\}$ so that φ^* restricts to give the inverse of θ, as required for (b). Finally we verify condition (d); that is, if $\varphi(j) < \varphi(j+1)$ then $\varphi(\ell_L(j)) \leq \ell_T(\varphi(j+1) - 1)$. But if $\varphi(j) < \varphi(j+1)$ then $j \neq m$ and so $\varphi(\ell_L(j)) = \varphi(0) = 0 \leq \ell_T(\varphi(j+1) - 1)$ as required. □

Given a strict morphism $U\colon \mathcal{T} \to \mathcal{Q}$ of **Cat**-operads whose underlying morphism in $[\mathbb{N}, \mathbf{Cat}]$ has a left adjoint $F\colon \mathcal{Q} \to \mathcal{T}$, the left adjoint F admits the structure of a colax morphism of operads: this is an instance of doctrinal adjunction [5]. To describe the structure, let η and ε denote the unit and counit of the adjunction. The components of the colax structure are given by

$$F(x \circ_i y) \xrightarrow{\eta_x \circ_i \eta_y} F(UFx \circ_i UFy) = FU(Fx \circ_i Fy) \xrightarrow{\varepsilon_{x \circ_i y}} Fx \circ_i Fy \tag{5.1}$$

and

$$Fe_{\mathcal{Q}} = FUe_{\mathcal{T}} \xrightarrow{\varepsilon_{e_{\mathcal{T}}}} e_{\mathcal{T}}, \tag{5.2}$$

where $e_{\mathcal{T}} \in \mathcal{T}_1$ and $e_{\mathcal{Q}} \in \mathcal{Q}_1$ are the units of the respective operads.

Corollary 5.2. *The left adjoint of Theorem 5.1 is a colax morphism of operads, and so sends skew monoidal categories to colax monoidal categories. More precisely, it defines a 2-functor* $J\colon \mathbf{Skew}_{\mathrm{w}} \to \text{Colax-}\mathcal{N}\text{-Alg}_{\mathrm{w}}$ *for each flavour* w *of weak morphism.*

Proof. The colax structure follows as above. Composition with the left adjoint therefore sends colax $\mathcal{S}$-algebras to colax $\mathcal{N}$-algebras, and so in particular sends strict $\mathcal{S}$-algebras to colax $\mathcal{N}$-algebras; that is, it sends skew monoidal categories to colax monoidal categories. □

We may describe this process more explicitly. Let $\mathcal{C}$ be a skew monoidal category. This becomes colax monoidal when we define the tensor product of the list $(a_1, \ldots, a_n)$ to be the tensor product

$$ia_1 \ldots a_n$$

in $\mathcal{C}$ bracketed to the left. Clearly this process loses structure: there is no way of recovering a general product ab in $\mathcal{C}$.

In a moment we shall describe another operad $\mathcal{L}$, only slightly more complex than $\mathcal{N}$, whose colax algebras do encode the entire skew monoidal structure. Before that, let us mention that there is a dual way of making a skew monoidal category into a lax monoidal category.

Theorem 5.3. *The map* $P\colon \mathcal{S} \to \mathcal{N}$ *has a right adjoint in* $[\mathbb{N}, \mathbf{Cat}]$ *with identity counit.*

Proof. This amounts to proving that each $\mathcal{S}_n$ has a terminal object. Explicitly, this will be given by $(\mathbf{n}+\mathbf{1}, v, \top)$ where v consists of all elements of $\mathbf{n}+\mathbf{1}$ except the top, and $\top$ is the greatest element of the Tamari lattice Tam_{n+1}, with lbf $\ell_\top$ given by $\ell_\top(i) = i$ for all $i \in \mathbf{n}+\mathbf{1}$.

But in fact there is no need to prove this separately; rather, we can use the following duality argument. For any skew monoidal category $\mathcal{C}$, the opposite category $\mathcal{C}^{\mathrm{op}}$ is also skew monoidal when we use the reverse tensor product: $a \otimes_{\mathcal{C}^{\mathrm{op}}} b = b \otimes_{\mathcal{C}} a$; this also interchanges the roles of λ and ρ. This means that there is an isomorphism $\mathcal{S}_n^{\mathrm{op}} \cong \mathcal{S}_n$, and the image under this isomorphism of the initial object of Theorem 5.1 will be terminal. □

The adjunction of Theorem 5.3 was established by Uustalu using a term rewriting approach in [10]. By doctrinal adjunction we obtain:

Corollary 5.4. *The right adjoint of Theorem 5.3 is a lax morphism of operads, and so sends skew monoidal categories to lax monoidal categories.*

This time the product in the lax monoidal category of the list $(a_1, \ldots, a_n)$ is the tensor product

$$a_1 \ldots a_n i$$

in the skew monoidal category, bracketed to the right.

5.2. Colax $\mathcal{L}$-algebras

As we have mentioned, the passage from a skew monoidal category to the associated (co)lax monoidal category loses information. In order to rectify this problem, we may consider intermediate structures between strict monoidal and skew monoidal categories, in the following sense. Suppose that $\mathcal{L}$ is an operad, and that $P\colon \mathcal{S} \to \mathcal{N}$ factorises as

$$\mathcal{S} \xrightarrow{Q} \mathcal{L} \xrightarrow{R} \mathcal{N}$$

so that $P^*\colon \mathcal{N}\text{-Alg}_{\mathrm{w}} \to \mathcal{S}\text{-Alg}_{\mathrm{w}}$ factorises as

$$\mathcal{N}\text{-Alg}_{\mathrm{w}} \xrightarrow{R^*} \mathcal{L}\text{-Alg}_{\mathrm{w}} \xrightarrow{Q^*} \mathcal{S}\text{-Alg}_{\mathrm{w}}$$

where w could be any of s, ℓ, c, or ps.

If each $Q\colon \mathcal{S}_m \to \mathcal{L}_m$ has a left adjoint F, then the various F inherit the structure of a colax morphism $\mathcal{L} \to \mathcal{S}$ of operads, and so composition with F sends skew monoidal categories to colax $\mathcal{L}$-algebras. We shall apply this for a specific choice of $\mathcal{L}$, whose algebras will be the following structures.

Definition 5.5. A λ-algebra is a skew monoidal category for which both the associativity maps α and the right unit maps ρ are identities.

This can be considered as a structure in its own right: it is a category $\mathcal{C}$ equipped with a strictly associative multiplication $\mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and a strict right unit i; there is also a natural transformation $\lambda\colon ia \to a$ satisfying three conditions: $\lambda_a \otimes b = \lambda_{a\otimes b}$, $a \otimes \lambda_b = 1$, and $\lambda_i = 1$. As such it is clear that λ-algebras are the algebras for an operad. This operad will be called $\mathcal{L}$ and can be described as follows.

Proposition 5.6. *The operad $\mathcal{L}$ for λ-algebras has $\mathcal{L}_0 = \{l\}$ whilst $\mathcal{L}_n$ is the two-element poset $\{\ell \leq t\}$ for $n > 0$. The multiplication $\mathcal{L}_n \times \mathcal{L}_{k_1} \times \ldots \times \mathcal{L}_{k_n} \to \mathcal{L}_{k_1+\ldots+k_n}$ is given by*

$$x(x_1 \ldots, x_n) = \begin{cases} t & \text{if } x, x_1 = t \\ \ell & \text{otherwise} \end{cases}$$

and the unit by $t \in \mathcal{L}_1$.

In what follows we shall often write ℓ_n,t_n to indicate that we are referring to $\ell, t \in \mathcal{L}_n$.

Proof. Let $\mathcal{C}$ be a category and consider $L\mathcal{C} = \sum_{n\in\mathbb{N}} \mathcal{L}_n \times \mathcal{C}^n$. We write $\overline{a} = (a_1, \ldots, a_n)$ for a typical element of $\mathcal{C}^n$, and $\overline{a}\overline{b}$ for the concatenation of lists $\overline{a}$ and $\overline{b}$. We equip $L\mathcal{C}$ with multiplication $(x_m, \overline{a}) \otimes (y_n, \overline{b}) = (x_{m+n}, \overline{a}, \overline{b})$ and unit $(l_0, -)$. The left unit maps are the morphisms $(\ell_0, -) \otimes (x_n, \overline{a}) = (\ell_n, \overline{a}) \to (x_n, \overline{a})$ induced by $l_n \leq x_n$.

Next we show that $L\mathcal{C}$ is the free λ-algebra on $\mathcal{C}$. To this end, consider a λ-algebra $\mathcal{D}$ and functor $F\colon \mathcal{C} \to \mathcal{D}$. We must show that there is a unique structure-preserving morphism $\widehat{F}\colon L\mathcal{C} \to \mathcal{D}$ sending each (t_1, a) to Fa. This is straightforward. We can and must define $\widehat{F}(t_n, \overline{a})$ as the n-fold tensor product $\otimes_{i=1}^n Fa_i$, with $\widehat{F}(\ell_n, \overline{a}) = i \otimes (\otimes_{i=1}^n Fa_i)$; while $\widehat{F}$ applied to a morphism $(\leq, \overline{a})\colon (\ell_n, \overline{a}) \to (t_n, \overline{a})$ is the left unit map $i \otimes (\otimes_{i=1}^n Fa_i) \to (\otimes_{i=1}^n Fa_i)$.

The unique λ-algebra map $L1 \to \mathbb{N}$ sending the generator to $1 \in \mathbb{N}$ produces the values of our operad $\mathcal{L}_i$ as its fibres. The components of the multiplication are calculated as the components of the counit $1^*_{L1}\colon LL1 \to L1$. □

There is a unique (strict) operad morphism $R\colon \mathcal{L} \to \mathcal{N}$ sending ξ_n to n, and the induced 2-functor $R^*\colon \mathcal{N}\text{-Alg}_{\mathrm{w}} \to \mathcal{L}\text{-Alg}_{\mathrm{w}}$ is the inclusion of strict monoidal categories in λ-algebras. Though we shall not use this fact, we note that $R\colon \mathcal{L} \to \mathcal{N}$ has both adjoints in $[\mathbb{N}, \mathbf{Cat}]$ since each $\mathcal{L}_n$ has both an initial and a terminal object.

Of more interest to us is the unique (strict) operad morphism $Q\colon \mathcal{S} \to \mathcal{L}$ for which the induced 2-functor $Q^*\colon \mathcal{L}\text{-Alg}_{\mathrm{w}} \to \mathcal{S}\text{-Alg}_{\mathrm{w}}$ is the inclusion of $\mathcal{L}$-algebras in skew monoidal categories. Explicitly, $Q\colon \mathcal{S}_n \to \mathcal{L}_n$ sends $(\mathbf{n}, u, S)$ to $t_{|u|}$ if the bottom element of $\mathbf{n}$ is in u, and $\ell_{|u|}$ otherwise.

Theorem 5.7. *The map $Q\colon \mathcal{S} \to \mathcal{L}$ has a left adjoint H in $[\mathbb{N}, \mathbf{Cat}]$ with identity unit.*

Proof. We need to show that each $Q\colon \mathcal{S}_m \to \mathcal{L}_m$ has a left adjoint with identity unit. Since $\ell_m \in \mathcal{L}_m$ is initial we define $H\ell_m$ to be the initial object $(\mathbf{m}, \mathbf{m}+\mathbf{1} \setminus \{0\}, \bot)$, constructed in Theorem 5.1. Then $H\ell_m$ has the correct universal property. By construction $QH\ell_m = \ell_m$ so that the unit component is the identity at ℓ_m.

That leaves the case $\xi = t_m$. In this case we shall show that $Ht_m = (\mathbf{m}, \mathbf{m}, \bot)$, where $\bot \in \mathrm{Tam}_m$ is the bottom element. If $(\mathbf{n}, v, T) \in \mathcal{S}_m$ then $|v| = m$, and $t \leq Q(\mathbf{n}, v, T)$ just when $0 \in v$. There is then a unique order-preserving bijection $\theta\colon \mathbf{m} \to v$, and composing this with the inclusion $v \to \mathbf{n}$ gives the order-preserving $\varphi\colon \mathbf{m} \to \mathbf{n}$ mapping $\mathbf{m}$ bijectively to v; furthermore, since $0 \in v$ it satisfies $\varphi(0) = 0$ and so has a right adjoint φ^*. This restricts to θ^{-1} since the given subset of $\mathbf{m}$ is its entirety. Finally it satisfies $\varphi(\ell_L(j)) \leq \ell_T(\varphi(j+1) - 1)$ for all j not equal to the top element of $\mathbf{m}$, and so defines a morphism in **Fsk**. The unit at t_m is once again the identity. □

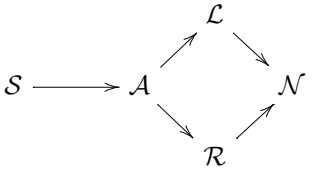
Remark 5.8. There is a little more we can say about morphisms $Hx_m \to (\mathbf{n}, v, T) \in \mathcal{S}_m$ in the context of the adjunction $H \dashv Q$. First, such a morphism is unique if it exists, since $\mathcal{L}_m$ is a poset. Second, tracing through the construction of the adjunction we see that the unique map $Hx_m \to (\mathbf{n}, v, T)$ in $\mathcal{S}_m$ corresponding to an identity $x_m = Q(\mathbf{n}, v, T)$ is an **Fsk**-injection. In other words, the components of the counit are **Fsk**-injections.

Corollary 5.9. *The left adjoint H of Theorem 5.7 is a normal colax morphism of operads, and so sends skew monoidal categories to normal colax $\mathcal{L}$-algebras. More precisely, it defines a 2-functor $H^*\colon \mathbf{Skew}_{\mathrm{w}} \to \mathrm{nColax}\text{-}\mathcal{L}\text{-}\mathrm{Alg}_{\mathrm{w}}$ for any flavour w of weak morphism.*

Proof. We just need to check that the colax morphism H is normal. H sends the unit $t_1 \in \mathcal{L}_1$ of $\mathcal{L}$ to the unit $(\mathbf{1}, \mathbf{1}, \bot)$ of $\mathcal{S}$, and this object has no non-identity endomorphisms, so the colax structure map $H(t_1) \to (\mathbf{1}, \mathbf{1}, \bot)$ can only be the identity. □

We shall see in Section 7 that this 2-functor is fully faithful, and we shall also characterise its image.

Remark 5.10. Between the operads $\mathcal{S}$ and $\mathcal{N}$ there are various other possible operads one may consider. Though not necessary in what follows let us briefly mention a fuller picture of such possibilities.



Here, in addition to $\mathcal{S}$, $\mathcal{L}$, and $\mathcal{N}$, are the operads $\mathcal{A}$ for skew monoidal categories in which α is an identity and $\mathcal{R}$ for skew monoidal categories in which both α and λ are identities. (In fact $\mathcal{R}$ is dual to $\mathcal{L}$, in the sense that $\mathcal{R}_n = \mathcal{L}_n^{op}$.) In this diagram all of the morphisms on or above the horizontal have left adjoints in $[\mathbb{N}, \mathbf{Cat}]$ with identity unit whilst all those on or below the horizontal have right adjoints in $[\mathbb{N}, \mathbf{Cat}]$ with identity counit.

6. LBC-algebras and LBC-morphisms

We have seen that each skew monoidal category gives rise to a normal colax $\mathcal{L}$-algebra. In this section we identify the property that characterises the colax $\mathcal{L}$-algebras arising in this way: we call such objects LBC-algebras; the "LB" stands for left-bracketed and the "C" for colax.

We give a detailed analysis of the corresponding notion of LBC-morphism of operads, which we shall use in the following section to establish the correspondence with skew monoidal categories.

Recall that $\mathcal{L}_n = \{t_n, \ell_n\}$ for each $n \geq 0$ and $\mathcal{L}_0 = \{\ell_0\}$. The multiplication for $\mathcal{L}$ satisfies $t_2 \circ_1 x_n = x_{n+1}$ for all $x_n \in \mathcal{L}_n$.

Consider a normal colax $\mathcal{L}$-algebra. This consists of a category A equipped with functors $\ell = \ell_n \colon A^n \to A$ for each $n \geq 0$ and $t = t_n \colon A^n \to A$ for each $n > 0$, a natural transformation $\lambda = \lambda_n \colon \ell_n \to t_n$ for each $n > 0$, and suitably coherent natural transformations $\Gamma_{x,i,y} \colon m_{x \circ_i y} \to m_x \circ_i m_y$ for each $x \in \mathcal{L}_k$, $y \in \mathcal{L}_n$, and $i \in \{1, \ldots, k\}$.

Definition 6.1. We say that a normal colax $\mathcal{L}$-algebra A is an LBC-*algebra* if each $\Gamma_{t_2,1,x} \colon m_x \to m_{t_2} \circ_1 m_x$ is an identity.

In particular, there are equalities

$$m_x(a_1, \ldots, a_{n+1}) = m_{t_2}(m_x(a_1, \ldots, a_n), a_{n+1}).$$

We write LBC-$\mathrm{Alg}_{\mathrm{w}}$ for the full sub-2-category of nColax-$\mathcal{L}$-$\mathrm{Alg}_{\mathrm{w}}$ consisting of the LBC-algebras.

Since normal colax algebra structure on A amounts to a normal colax morphism $\mathcal{T} \to \mathrm{End}(A)$, there is a natural extension of the previous definition.

Definition 6.2. For an operad $\mathcal{T}$, an LBC-*morphism* from $\mathcal{L}$ to $\mathcal{T}$ is a normal colax morphism $F \colon \mathcal{L} \to \mathcal{T}$ for which each

$$\widetilde{F} \colon F(t_2 \circ_1 x_n) \to F(t_2) \circ_1 F(x_n)$$

is an identity.

Thus an LBC-morphism $\mathcal{L} \to \mathrm{End}(A)$ is the same as an LBC-algebra structure on A.

Example 6.3. The normal colax morphism $H \colon \mathcal{L} \to \mathcal{S}$ of Corollary 5.9 is an LBC-morphism. First observe that

$$\begin{aligned} H(t_2) \circ_1 H(\ell_n) &= (\mathbf{2}, \mathbf{2}, \bot_2) \circ_1 (\mathbf{n}, \mathbf{n} \setminus \{0\}, \bot_n) \\ &= (\mathbf{n+1}, \mathbf{n+1} \setminus \{0\}, \bot_{n+1}) \\ &= H(\ell_{n+1}) \\ &= H(t_2 \circ_1 \ell_n). \end{aligned}$$

The colax structure map $H(t_2 \circ_1 \ell_n) \to H(t_2) \circ_1 H(\ell_n)$ is given by the composite

$$H(t_2 \circ_1 \ell_n) = H(QHt_2 \circ QH\ell_n) = HQ(Ht_2 \circ_1 H\ell_n) \longrightarrow Ht_2 \circ_1 H\ell_n$$

where the first equality is because the unit of $H \dashv Q$ is an identity, and the second because Q is a (strict) morphism of operads, while the unnamed arrow is the component at $Ht_2 \circ_1 H\ell_n$ of the counit. But this is also the component of the counit at $H(\ell_{n+1})$, which is an identity by one of the triangle equations. The proof that the colax structure map $H(t_2 \circ_1 t_n) \to H(t_2) \circ_1 H(t_n)$ is an identity is similar.

Note that unlike colax morphisms, LBC-morphisms are only defined when the domain is $\mathcal{L}$. However, if $F \colon \mathcal{L} \to \mathcal{T}$ is an LBC-morphism and $G \colon \mathcal{T} \to \mathcal{Q}$ is a (strict) morphism of operads, then the composite $G \circ F$ is also an LBC-morphism.

We now analyse what exactly is involved in giving an LBC-morphism.

Proposition 6.4. *For an* LBC*-morphism* $F\colon \mathcal{L} \to \mathcal{T}$*, the maps*

$$\widetilde{F}_{t_n,1,x_k}\colon F(x_{n+k-1}) \to F(t_n) \circ_1 F(x_k)$$

are also identities for all n*.*

Proof. The case $n = 1$ holds by counitality, and the case $n = 2$ by the LBC condition. For $n > 2$, use the coassociativity condition

$$\begin{array}{ccc} F(x_{n+k-1}) & \xrightarrow{\widetilde{F}_{t,1,x}} & F(t_n) \circ_1 F(x_k) \\ {\scriptstyle \widetilde{F}_{t,1,x}}\big\| & & \big\|{\scriptstyle \widetilde{F}_{t,1,t}\circ_1 1} \\ F(t_2) \circ_1 F(x_{n+k-2}) & \xrightarrow[1\circ_1 \widetilde{F}_{t,1,x}]{} & F(t_2) \circ_1 F(t_{n-1}) \circ_1 F(x_k) \end{array}$$

and induction. □

Proposition 6.5. *In an* LBC*-morphism* $F\colon \mathcal{L} \to \mathcal{T}$*, all of the functors* $\mathcal{L}_n \to \mathcal{T}_n$ *are determined by* $F(t_2) \in \mathcal{T}_2$*,* $F(\ell_0) \in \mathcal{T}_0$*, and* $F(\lambda_1)\colon F(\ell_1) \to F(t_1) \in \mathcal{T}_1$*.*

Proof. This follows by a straightforward induction using the fact that $t_{n+1} = t_2 \circ_1 t_n$, $\ell_{n+1} = t_2 \circ_1 \ell_n$, and $\lambda_{n+1} = t_2 \circ_1 \lambda_n$. □

Lemma 6.6. *For an* LBC*-morphism* $F\colon \mathcal{L} \to \mathcal{T}$*, all colax structure maps* $\widetilde{F}_{x,j,y}\colon F(x \circ_j y) \to F(x) \circ_j F(y)$ *are determined by those for which* $x = t \in \mathcal{L}_n$ *and* $j = n$*.*

Proof. Let $x \in \mathcal{L}_n$ and $y \in \mathcal{L}_k$ be given and consider $\widetilde{F}_{x,j,y}\colon F(x \circ_j y) \to F(x) \circ_j F(y)$. If $j = 1$ and $x = t$ then this is an identity.

Step 1: $j > 1$. In this case $x \circ_j y = x$, regardless of the value of y. By coassociativity, the diagram

$$\begin{array}{ccc} F(x_{n+k-1}) & \xrightarrow{\widetilde{F}_{x_n,j,y_k}} & F(x_n) \circ_j F(y_k) \\ \Big\| & & \big\|{\scriptstyle \widetilde{F}_{(t_j,1,x_{n-j+1})}\circ_j 1} \\ {\scriptstyle \widetilde{F}_{(t_{j+k-1},1,x_{n-j+1})}}\Big\| & & (F(t_j) \circ_1 F(x_{n-j+1})) \circ_j F(y_k) \\ \Big\| & & \Big\| \\ F(t_{j+k-1}) \circ_1 F(x_{n-j+1}) & \xrightarrow[\widetilde{F}_{t_j,j,y_k}\circ 1]{} & (F(t_j) \circ_j F(y_k)) \circ_1 F(x_{n-j+1}) \end{array}$$

commutes and by the LBC property the verticals are identities. Thus the upper horizontal is equal to the lower horizontal, which depends only on the $\widetilde{F}$ of the given form.

Step 2: $j = 1 < n$. If $x = t$ there is nothing to prove, so suppose that $x = \ell$. In this case $x_n \circ_j y_k = \ell_{n+k-1}$ regardless of the value of y. If $n > 1$ then by coassociativity, the diagram

$$\begin{array}{ccc} F(\ell_{n+k-1}) & \overset{\widetilde{F}_{t_2,1,\ell_{n+k-2}}}{=\!=\!=} & F(t_2) \circ_1 F(\ell_{n+k-2}) \\ {\scriptstyle \widetilde{F}_{\ell_n,1,y_k}}\downarrow & & \downarrow{\scriptstyle 1\circ_1 \widetilde{F}_{\ell_{n-1},1,y_k}} \\ F(\ell_n) \circ_1 F(y_k) & \underset{\widetilde{F}_{t_2,1,\ell_{n-1}}\circ_1 1}{=\!=\!=} & F(t_2) \circ_1 F(\ell_{n-1}) \circ_1 F(y_k) \end{array}$$

commutes. We may now use induction to reduce to the case $n = 1$. In that case use coassociativity as in

$$\begin{array}{ccc} F(\ell_k) & \xrightarrow{\widetilde{F}_{\ell,1,y}} & F(\ell_1) \circ_1 F(y_k) \\ \| & & \| {\scriptstyle \widetilde{F}_{t_2,1,\ell_0}\circ_1 1} \\ {\scriptstyle \widetilde{F}_{t_{k+1},1,\ell_0}}\| & & (F(t_2) \circ_1 F(\ell_0)) \circ_1 F(y_k) \\ \| & & \| \\ F(t_{k+1}) \circ_1 F(\ell_0) & \xrightarrow[\widetilde{F}_{t_2,2,y_k}\circ_1 1]{} & (F(t_2) \circ_2 F(y_k)) \circ_1 F(\ell_0) \end{array}$$

to reduce to $\widetilde{F}_{t_2,2,y_k}$. □

Proposition 6.7. *The $\widetilde{F}$'s are determined by $\widetilde{F}_{t_2,2,t_2}$ and $\widetilde{F}_{t_2,2,\ell_0}$.*

Proof. We have already reduced to the case of $\widetilde{F}_{t_n,n,y_k}$; and the case $n = 1$ is already covered by counitality. If $n > 2$ use coassociativity as in

$$\begin{array}{ccc} F(t_{n+k-1}) & \xrightarrow{\widetilde{F}_{t_n,n,y_k}} & F(t_n) \circ_n F(y_k) \\ \| & & \| {\scriptstyle \widetilde{F}_{t_2,1,t_{n-1}}\circ_n 1} \\ {\scriptstyle \widetilde{F}_{t_{k+1},1,t_{n-1}}}\| & & (F(t_2) \circ_1 F(t_{n-1})) \circ_n F(y_k) \\ \| & & \| \\ F(t_{k+1}) \circ_1 F(t_{n-1}) & \xrightarrow[\widetilde{F}_{t_2,2,y_k}\circ_1 1]{} & (F(t_2) \circ_2 F(y_k)) \circ_1 F(t_{n-1}) \end{array}$$

to reduce to the case where $n = 2$.

If now $y_k \in \{\ell_0, t_2\}$ there is nothing to prove; if $y_k = t_1$ use counitality; otherwise use coassociativity as in

$$\begin{array}{ccc} F(t_{k+1}) & \xrightarrow{\widetilde{F}_{t_2,2,y_k}} & F(t_2) \circ_2 F(y_k) \\ \downarrow & & \| {\scriptstyle 1\circ_2 \widetilde{F}_{t_2,1,y_{k-1}}} \\ {\scriptstyle \widetilde{F}_{t_3,2,y_{k-1}}}\downarrow & & F(t_2) \circ_2 (F(t_2) \circ_1 F(y_{k-1})) \\ \| & & \| \\ F(t_3) \circ_2 F(y_{k-1}) & \xrightarrow[\widetilde{F}_{t_2,2,t_2}\circ_2 1]{} & (F(t_2) \circ_2 F(t_2)) \circ_2 F(y_{k-1}) \end{array}$$

to reduce to $\widetilde{F}_{t_2,2,t_2}$ and $\widetilde{F}_{t_3,2,y_{k-1}}$, and now repeat the process in Step 1 of Lemma 6.6 to deal with $\widetilde{F}_{t_3,2,y_{k-1}}$. □

Proposition 6.8. *All of the structure of an* LBC*-morphism* $F\colon \mathcal{L} \to \mathcal{T}$ *is determined by* $F(t_2)$, $F(\ell_0)$, $F(\lambda_1)$, $\widetilde{F}_{t_2,2,t_2}$, *and* $\widetilde{F}_{t_2,2,\ell_0}$.

Proof. This follows directly from the previous results. □

7. Skew monoidal categories as LBC-algebras

In this section we describe the perfect correspondence between skew monoidal categories and LBC-algebras.

7.1. From an LBC*-algebra to a skew monoidal category*

In Proposition 6.8, we saw that a special morphism $F\colon \mathcal{L} \to \mathcal{T}$ is determined by a small amount of data. Next we apply this to the case where T is $\mathrm{End}(A)$, or $\{F, F\}_{\mathrm{w}}$, or $[\rho, \rho]$.

Suppose that A is a normal colax $\mathcal{L}$-algebra satisfying Property LBC. The corresponding LBC-morphism $F\colon \mathcal{L} \to \mathrm{End}(A)$ is determined by:

- $m = F(t_2)\colon A^2 \to A$
- $i = F(\ell_0) \in A$
- $\lambda = F(\lambda_1)\colon m \circ_1 i \to 1$
- $\alpha = \widetilde{F}_{t_2,2,t_2}\colon m \circ_1 m \to m \circ_2 m$
- $\rho = \widetilde{F}_{t_2,2,\ell_0}\colon 1 \to m \circ_2 i$.

We shall show that $(m, i, \alpha, \lambda, \rho)$ satisfy the five axioms needed to define a skew monoidal structure on A.

Example 7.1. Consider $\widetilde{F}_{t_2,2,\ell_1}$. Observe that $\ell_1 = t_2 \circ_1 \ell_0$. By coassociativity the diagram

$$\begin{array}{ccc} F(t_2) & \xrightarrow{\widetilde{F}_{t_2,2,\ell_1}} & F(t_2) \circ_2 F(\ell_1) \\ {\scriptstyle \widetilde{F}_{t_3,2,\ell_0}}\downarrow & & \Vert\, {\scriptstyle 1\circ_2 \widetilde{F}_{t_2,1,\ell_0}} \\ F(t_3) \circ_2 F(\ell_0) & \xrightarrow[\widetilde{F}_{t_2,2,t_2}\circ_2 1]{} & (F(t_2) \circ_2 F(t_2)) \circ_2 F(\ell_0) = F(t_2) \circ_2 (F(t_2) \circ_1 F(\ell_0)) \end{array}$$

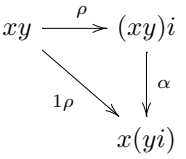
commutes. The lower horizontal is $\alpha\colon (xi)y \to x(iy)$. The left vertical is $m \circ_1 \widetilde{F}_{t_2,2,\ell_0}$, which is $\rho 1\colon xy \to (xi)y$.

Example 7.2. Consider $\widetilde{F}_{t_3,2,t_2}$. By coassociativity the diagram

$$\begin{array}{ccc} F(t_4) & \xrightarrow{\widetilde{F}_{t_2,2,t_3}} & F(t_2) \circ_2 F(t_3) \\ {\scriptstyle \widetilde{F}_{t_3,2,t_2}}\downarrow & & \Vert\, {\scriptstyle 1\circ_2 \widetilde{F}_{t_2,1,t_2}} \\ F(t_3) \circ_2 F(t_2) & \xrightarrow[\widetilde{F}_{t_2,2,t_2}\circ_2 1]{} & (F(t_2) \circ_2 F(t_2)) \circ_2 F(t_2) = F(t_2) \circ_2 (F(t_2) \circ_1 F(t_2)) \end{array}$$

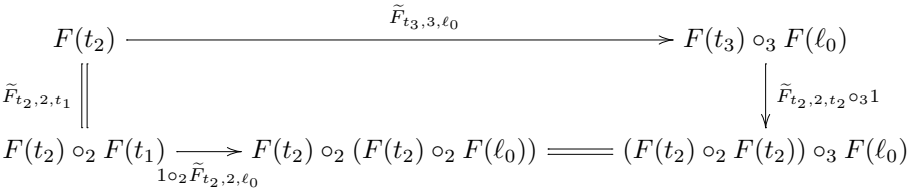
commutes. The lower horizontal has the form $a\colon (w(xy))z \to w((xy)z)$, and the left vertical is $a1\colon ((wx)y)z \to (w(xy))z$.

Proposition 7.3. *The diagram*



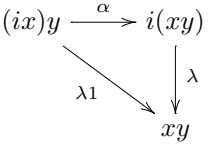
commutes.

Proof. By coassociativity the diagram



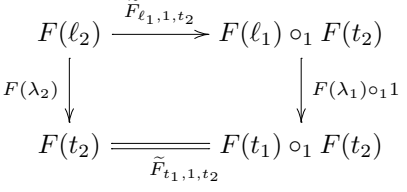
commutes. This agrees with the diagram in the statement (use Proposition 6.7 to identify the top row with $\rho\colon xy \to (xy)i$). □

Proposition 7.4. *The diagram*



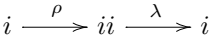
commutes.

Proof. This amounts to commutativity of



which follows by naturality of $\widetilde{F}$. □

Proposition 7.5. *The composite*



is the identity.

Proof. This amounts to commutativity of

$$\begin{array}{ccc} F(\ell_0) & \xrightarrow{\widetilde{F}_{\ell_1,1,\ell_0}} & F(\ell_1) \circ_1 F(\ell_0) \\ & \searrow^{\widetilde{F}_{t_1,1,\ell_0}} & \downarrow{\scriptstyle F(\lambda_1)\circ_1 1} \\ & & F(t_1) \circ_1 F(\ell_0) \end{array}$$

in which the diagonal is an identity by counitality. The diagram commutes by naturality of $\widetilde{F}$ once again. □

Proposition 7.6. *The composite*

$$xy \xrightarrow{\rho 1} (xi)y \xrightarrow{\alpha} x(iy) \xrightarrow{1\lambda} xy$$

is the identity.

Proof. By naturality of $\widetilde{F}$ once again, the diagram

$$\begin{array}{ccc} & F(t_2) & \\ {\scriptstyle \widetilde{F}_{t_2,2,\ell_1}}\downarrow & \searrow^{\widetilde{F}_{t_2,2,t_1}} & \\ F(t_2) \circ_2 F(\ell_1) & \xrightarrow[1\circ_2 F(\lambda_1)]{} & F(t_2) \circ_2 F(t_1) \end{array}$$

commutes, where the diagonal is an identity by counitality and the horizontal is $1\lambda\colon x(iy) \to xy$. The vertical is $\alpha.\rho 1$ by Example 7.1. □

Proposition 7.7. *The pentagon*

$$\begin{array}{ccccc} ((wx)y)z & \xrightarrow{\alpha 1} & (w(xy))z & \xrightarrow{\alpha} & w((xy)z) \\ {\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle 1\alpha} \\ (wx)(yz) & & \xrightarrow[\alpha]{} & & w(x(yz)) \end{array}$$

commutes.

Proof. By coassociativity, the diagram

$$\begin{array}{ccccc} F(t_4) & & \xrightarrow{\widetilde{F}_{t_2,2,t_3}} & & F(t_2) \circ_2 F(t_3) \\ {\scriptstyle \widetilde{F}_{t_3,3,t_2}}\downarrow & & & & \downarrow{\scriptstyle 1\circ_2 \widetilde{F}_{t_2,2,t_2}} \\ F(t_3) \circ_3 F(t_2) & \xrightarrow[\widetilde{F}_{t_2,2,t_2}\circ_3 1]{} & (F(t_2) \circ_2 F(t_2)) \circ_3 F(t_2) & = & F(t_2) \circ_2 (F(t_2) \circ_2 F(t_2)) \end{array}$$

commutes. The left vertical, right vertical, and lower path in this diagram coincide with those in the statement of the proposition; and the upper horizontal does too, by Example 7.2. □

7.2. The correspondence

With these preparations, we are now ready to prove the following result.

Theorem 7.8. *If w is any of the flavours ℓ, c, ps, s of weak morphism, the 2-functor $H^*\colon \mathbf{Skew}_{\mathrm{w}} \to \mathrm{nColax}\text{-}\mathcal{L}\text{-}\mathrm{Alg}_{\mathrm{w}}$ of Corollary 5.9 is injective on objects and fully faithful, with image given by the* LBC*-algebras.*

Proof. A skew monoidal category A, with structure corresponding to an operad morphism $F\colon \mathcal{S} \to \mathrm{End}(A)$, is sent to the normal colax algebra $H^*(A)$ given by the composite $F \circ H\colon \mathcal{L} \to \mathrm{End}(A)$. This is indeed an LBC-algebra by Example 6.3.

It is not hard to see that this is injective. The multiplication of the skew monoidal category is encoded by $F(\mathbf{2}, \mathbf{2}, \bot_2) = FH(t_2)$, and the unit by $F(\mathbf{1}, \emptyset, \bot_1) = FH(\ell_0)$, thus these are both determined by the LBC-algebra. Similarly the left unit map is given by $FH(\lambda_1)$, the right unit map by $F\widetilde{H}_{t_2,2,\ell_0}$, and the associativity map by $F\widetilde{H}_{t_2,2,t_2}$.

This proves that H^* is injective on objects, and injectivity on morphisms and 2-cells is similar but easier.

Suppose conversely that A is an LBC-algebra, with corresponding LBC-morphism $F\colon \mathcal{L} \to \mathrm{End}(A)$. Then we may define, as at the beginning of the section, $m = F(t_2)\colon A^2 \to A$, $i = F(\ell_0) \in A$, and so on, and then by Propositions 7.3, 7.4, 7.5, 7.6, and 7.7 this defines a skew monoidal category. Furthermore, by Proposition 6.8, the resulting skew monoidal category is sent by H^* to the original LBC-algebra.

Now suppose that $f\colon A \to B$ is a lax morphism of LBC-algebras, and let $G\colon \mathcal{L} \to \{f, f\}_\ell$ be the corresponding colax morphism of operads. Since $d \circ G$ and $c \circ G$ are LBC-morphisms, and d and c are strict morphisms which jointly reflect identities, it follows that G is also an LBC-morphism. Now $G(t_2)$ has the form

$$\begin{array}{ccc} A^2 & \xrightarrow{f^2} & B^2 \\ {\scriptstyle m}\downarrow & \Downarrow & \downarrow{\scriptstyle m} \\ A & \xrightarrow[f]{} & B \end{array}$$

while $G(\ell_0)$ has the form $i \to fi$. Furthermore, $\widetilde{G}_{t_2,2,t_2}$ is determined by $G(t_2)$ and the LBC-algebra structures, and encodes the associativity condition for $G(t_2)$. Similarly $\widetilde{G}_{\ell_2,2,\ell_0}$ and $G(\lambda_1)$ are determined by the other data and encode the unit conditions for $G(t_2)$. This proves fullness of H^* on lax morphisms.

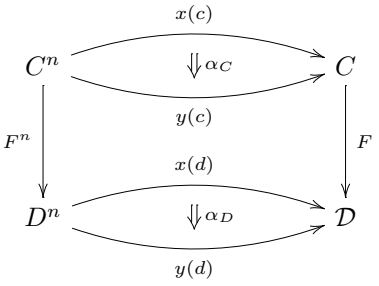
The cases of the other flavours of morphism and of 2-cells are similar and left to the reader. □

7.3. The colax L-algebra associated to a skew monoidal category

We have seen that skew monoidal categories correspond to LBC-algebras, and we have given a concrete description of the skew monoidal category associated to an LBC-algebra. We now match this with a concrete description of the inverse construction, given abstractly in Theorem 7.8. This will be used in the companion paper [2] to describe the *skew multicategory* associated to a skew monoidal category.

Given a skew monoidal C, the corresponding $\mathcal{S}$-algebra is specified by an operad morphism $c\colon \mathcal{S} \to \mathrm{End}(C)$, whose value at $x \in \mathcal{S}_n$ we write as $c(x)\colon C^n \to C$ and whose value at $\alpha\colon x \to y$ we write as $c(\alpha)\colon c(x) \Rightarrow c(y)$. At $(\mathbf{2}, \mathbf{2}, \bot) \in \mathcal{S}_2$ and $(\mathbf{0}, \mathbf{0}, \bot) \in \mathcal{S}_0$ the corresponding functors are $\otimes\colon C^2 \to C$ and $I\colon C^0 \to C$. Every element of $\mathcal{S}_n$ is obtained from the above elements of $\mathcal{S}_2$ and $\mathcal{S}_0$ by operadic substitution; it follows that the functors of the form $c(x)\colon C^n \to C$ are precisely those obtained from $\otimes\colon C^2 \to C$ and $I\colon C^0 \to C$ by substitution.

We note that the functors $c(x)\colon C^n \to C$ are 2-natural in C – more precisely, in the strict monoidal functors and monoidal natural transformations of $\mathbf{Skew}_s$. We mention this last abstract point because we would like to say something about certain components of the form $c(\alpha)\colon c(x) \to c(y)$ for $\alpha\colon x \to y \in \mathcal{S}_n$, whilst avoiding the syntax of $\mathcal{S}_n$ itself. To that end, we point out that for each family of natural transformations $\{\alpha_C\colon C \in \mathbf{Skew}_s\}$ natural in strict monoidal functors F in the sense of the following diagram



(7.1)

there exists a *unique* $\alpha\colon x \to y \in \mathcal{S}_n$ with $c(\alpha) = \alpha_C$ for each skew monoidal C. This is a general syntax/semantics fact that holds for any (plain) **Cat**-operad.

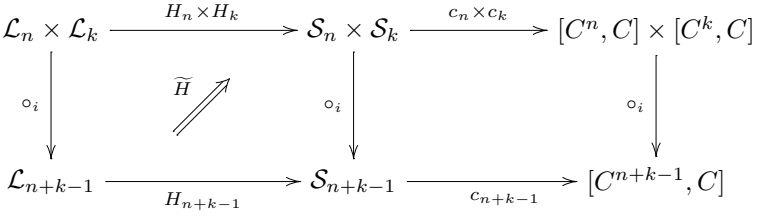
Now the corresponding colax $\mathcal{L}$-algebra $m\colon \mathcal{L} \to \mathrm{End}(C)$ is given by the composite colax morphism of operads

$$\mathcal{L} \xrightarrow{H} \mathcal{S} \xrightarrow{c} \mathrm{End}(C)$$

in which H is the colax morphism of operads of Corollary 5.9. This has components

$$\mathcal{L}_n \xrightarrow{H_n} \mathcal{S}_n \xrightarrow{c_n} [C^n, C]$$

and substitution maps given by the natural transformations below.



Let us write $a_1 \dots a_n$ for the left bracketed n-fold tensor product in C, so that $a_1 \dots a_n a_{n+1} = (a_1 \dots a_n) \otimes a_{n+1}$. By Theorem 5.7 $H_n(l) = (\mathbf{n+1}, \mathbf{n+1}/\{0\}, \bot_{n+1})$ is the *initial object* of $\mathcal{S}_n$, consisting of the ordinal $\mathbf{n+1}$ with 0 omitted, and $\bot_{n+1} \in T_{n+1}$ the least element of the Tamari lattice, corresponding to the left bracketing of $n+1$-elements. Accordingly $m_{l_n} = c_n \circ H_n(l)\colon C^n \to C$ has value

$$m_{l_n}(a_1, \dots, a_n) = i a_1 \dots a_n$$

the left bracketing of the $n+1$-tuple $(i, a_1, \dots, a_n)$.

By Theorem 5.7 we also have $H_n(t) = (\mathbf{n}, \mathbf{n}, \bot_n)$. It follows that

$$m_{t_n}(a_1, \dots, a_n) = a_1 \dots a_n$$

the leftmost bracketing. At $\lambda\colon l \to t \in \mathcal{L}_n$ for $n > 0$ the induced map from $m_{l_n}(\overline{a}) \to m_{t_n}(\overline{a})$ has component $\lambda a_1 \dots a_n \colon i a_1 \dots a_n \to a_1 \dots a_n$.

With regards substitution, it follows from Remark 5.8 that for any $(x, y) \in \mathcal{L}_n \times \mathcal{L}_k$ the morphism $\widetilde{H}_{x,y}\colon H_{n+k-1}(x \circ_i y) \to H_n(x) \circ_i H_k(y) \in \mathcal{S}_{n+k-1}$ is *unique* and, moreover, an **Fsk**-*injection*. Uniqueness allows us to say that the substitution component $m_{x \circ_i y} \to m_x \circ_i m_y$ is the unique natural transformation that exists for all skew monoidal categories C and is natural in the sense of Diagram (7.1). Furthermore, by Proposition 4.6 each **Fsk**-injection admits a canonical factorisation as a swell morphism (corresponding to applications of ρ) followed by a Tamari morphism (corresponding to applications of α). Accordingly each substitution is obtained by repeated applications of the right unit maps ρ followed by applications of associativity maps α, each possibly tensored on either side. For instance $m_l(a, b, c, d) \to m_l(m_t(a, b), m_l(c, d))$ is the map given by

$$(((ia)b)c)d \xrightarrow{((i(ab))\rho)d} (((ia)b)(ic))d \xrightarrow{((\alpha(ic))d} (((i(ab))(ic))d \xrightarrow{\alpha} (i(ab))((ic)d).$$

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