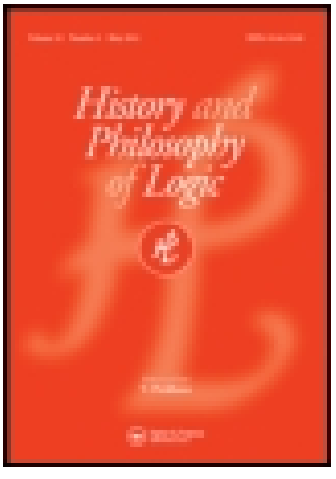
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Paul Rusnock [a] & Mark Burke [a]
[a] Department of Philosophy , University of Ottawa , Ottawa, Ontario, Canada , K1N 6N5
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Etchemendy and Bolzano on Logical Consequence

PAUL RUSNOCK and MARK BURKE
Department of Philosophy, University of Ottawa, Ottawa, Ontario, Canada K1N 6N5

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In a series of publications beginning in the 1980s, John Etchemendy has argued that the standard semantical account of logical consequence, due in its essentials to Alfred Tarski, is fundamentally mistaken. He argues that, while Tarski's definition requires us to classify the terms of a language as logical or non-logical, no such division is guaranteed to deliver the correct extension of our pre-theoretical or intuitive consequence relation. In addition, and perhaps more importantly, Tarski's account is claimed to be incapable of explaining an essential modal/epistemological feature of consequence, namely, its necessity and apriority.

Bernard Bolzano (1781–1848) is widely recognized as having anticipated Tarski's definition in his *Wissenschaftslehre* (or *Theory of Science*) of 1837. Because of the similarities between his account and Tarski's, Etchemendy's arguments have also been extended to cover Bolzano. The purpose of this article is to consider Bolzano's theory in the light of these criticisms. We argue that, due to important differences between Bolzano's and Tarski's theories, Etchemendy's objections do not apply immediately to Bolzano's account of consequence. Moreover, Bolzano's writings contain the elements of a detailed philosophical response to Etchemendy.

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1. Introduction

Long considered a settled matter, the nature of logical consequence is once again a topic of serious philosophical debate, and perhaps no one has done as much in recent years to re-open the question as the Stanford logician John Etchemendy. Beginning with his doctoral studies in the 1980s, Etchemendy first challenged the conventional, rather Whiggish history of the logic of textbooks and logicians' folklore. He showed that the usual accounts of the development of the concept of logical consequence were quite mistaken, that in particular the views habitually attributed to Tarski were incompatible with what Tarski actually maintained in his famous paper in 1936. But Etchemendy went much further, contending that both Tarski and the modern textbooks were mistaken. Neither, in his opinion, gave a full and satisfactory definition of logical consequence.[1]

It is well known that Bernard Bolzano (1781–1848), in his *Theory of Science*, formulated a definition of consequence (he called it *deducibility* [*Ableitbarkeit*]) similar in many respects to that given by Tarski in 1936.[2] Bolzano also makes an appearance in Etchemendy's reflections on logical consequence, but plays a decidedly secondary role. For Etchemendy's main focus is on the contemporary received view. Like many others, he mentions Bolzano as a precursor of Tarski, but does not give extended consideration to his views. This omission is fair enough given his aims, but also unfortunate, because Bolzano's approach to consequence differs from that of Tarski's in interesting and significant ways, and

[1] *Etchemendy 1990*; see also *Etchemendy 1983*, *1988a,b*.

[2] *Bolzano 1837*, §155. References to the *Wissenschaftslehre* will be by section number and, where appropriate, by volume and page numbers, e.g. §155 [II.113]. All translations are by R. George and P. Rusnock.

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his writings contain the elements of a detailed philosophical response to Etchemendy's criticisms. The purpose of this article is to develop such a response from Bolzano's principles.[3]

2. Historical preliminaries

It is sometimes said that Tarski's 1936 definition of semantic consequence[4] is substantially the same as those found in introductory textbooks today. As Etchemendy has pointed out, this could not possibly be right. For the advantage of a semantic notion, in Tarski's opinion, was that it captured genuine cases of consequence which syntactic notions inevitably missed. But in the usual first-order setting this cannot happen, since the syntactic and semantic consequence relations are coextensive.

Tarski's paper recalls a time when consequence, equivalence, logical truth, and other logical concepts were defined with reference to deductive systems, usually axiomatic in form. The problem that troubled him, briefly, was this: we may have an axiomatic theory $\mathcal{T}$ treating of some infinite domain (for example, the natural numbers), where we have names for every element in the domain (here, numerals), and where there is a propositional function $A(n)$ with one free variable for which we have:

- For every numeral $\bar{n}$, $\vdash_{\mathcal{T}} A(\bar{n})$
- Yet it is not the case that $\vdash_{\mathcal{T}} \forall n A(n)$

Such theories were called ω-incomplete, and the examples were well known to Tarski, prominent among them the theories discussed in Gödel's *1931* paper 'On formally undecidable propositions of *Principia Mathematica* and related systems'.[5]

Today, many people look upon ω-incompleteness as an interesting result of a correct definition of logical consequence (perhaps on this view we should understand it as bearing on the inevitable inadequacy of effectively specifiable axiomatizations of arithmetic and similar theories). But this is not at all the view Tarski takes in the 1936 paper. For him, the existence of ω-incomplete theories shows that the syntactic definition is woefully inadequate and indeed beyond saving:

> This fact seems to me to speak for itself. It shows that the formalized concept of consequence, as it is generally used by mathematical logicians, by no means coincides with the common concept. Yet intuitively it seems certain that the universal sentence A [i.e. $\forall n\ A(n)$] follows in the usual sense from the totality of particular sentences $A_0, A_1, A_2, \ldots$ [i.e. $A(\bar{\mathbf{0}}), A(\bar{\mathbf{1}}), A(\bar{\mathbf{2}}), \ldots$]. Provided all these sentences are true, the sentence A must also be true.[6]

[3] To forestall possible misunderstandings, let us remark that it is not our intention to argue for the correctness of Bolzano's views, but rather to present them as clearly as possible, so that others can make up their own minds about whether they contain anything of value.

[4] *Tarski 1983*.

[5] *Gödel 1931*.

[6] *Tarski 1983*, p. 410–411.

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Hence, he continues, we need to define a different notion of consequence, one that is not co-extensive with any syntactic one, and one that will allow us to declare that $\forall n A(n)$ is indeed a consequence of the infinite set of formulas[7]:

$$\{A(\bar{\mathbf{0}}),\ A(\bar{\mathbf{1}}), A(\bar{\mathbf{2}}), \ldots\}$$

The first definition that he considers, which we shall call *B**, tells us what it means for a sentence *X* to be a logical consequence of a class of sentences *K*. It runs as follows:

> (*B**) If, in the sentences of the class *K* and in the sentence *X*, the constants—apart from purely logical constants—are replaced by any other constants (like signs everywhere replaced by like signs), and if we denote the class of sentences thus obtained from *K* by '*K′*', and the sentence obtained from *X* by '*X′*', then the sentence *X′* must be true provided only that all the sentences of the class *K′* are true.[8]

Why *B**? A remarkably similar definition was proposed by Bolzano a century before Tarski (see section 4.4).[9]

We can illustrate this definition with a simple example. We will say that the sentences:

> No whales fly.

and

> Some mammals fly.

entail:

> Not all mammals are whales.

because when we uniformly substitute terms of the appropriate types for all the occurrences of 'whale', 'fly', and 'mammal', we find that in every case where the first two sentences come out true, the third does as well.

Tarski rejects his preliminary attempt *B** for the reason that it makes too much to depend upon the expressive resources of the language in question. Consider a simple case for the sake of illustration. Suppose that we have a language for arithmetic with

[7] Expanding the syntactic apparatus to deal with such cases (adding various versions of what logicians call ω-rules) turns out to be quite a complicated business, which, so long as the rules are still recursively specifiable, still falls short, since a Gödel sentence can be produced for each such extension. Cf. Tarski's remarks in *Tarski 1983*, p. 411 ff.

[8] *Tarski 1983*, p. 415.

[9] Tarski tells us he was unaware of Bolzano's work before it was pointed out to him by Heinrich Scholz (*Tarski 1983*, p. 417 note). As Peter Simons (*1992*, pp. 14–15) has pointed out, however, he might well have been indirectly aware of Bolzano's ideas: 'Tarski's teachers in philosophy and logic in Warsaw included Łukasiewicz, Leśniewski, and Kotarbiński, all of whom had studied under Twardowski in Lwów before the First World War. Twardowski, who had studied in Vienna under Brentano, had received his doctorate under Robert Zimmermann, himself a former pupil of Bolzano, and from this source, perhaps among others, he had picked up a fair amount of knowledge about Bolzano's logic.'

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the peculiarity that every one-place predicate which applies to 1, 2, and 3 also applies to all natural numbers. On the substitutional account, the following rule of hasty inference:

$$\frac{P(1), P(2), P(3)}{\forall x Px}$$

would be declared valid. It seemed obvious to Tarski that this was not a happy result.

The problem Tarski points out here is by no means found only in such degenerate cases. Related problems are liable to occur with any theory involving the real numbers, since there are non-denumerably many of them, while as a rule scientific languages contain at most denumerably many expressions. This is why he has recourse to the concept of the *satisfaction* of a sentential function by a sequence of objects, which can be defined in set-theoretic terms, and which avoids the problems connected with objects or properties for which no designations exist. Even if we have no expression designating the set {1, 2, 3}, for instance, we can still say that this set, as an interpretation of 'P', satisfies '$P(1)$', '$P(2)$', and '$P(3)$', but not '$\forall x P(x)$'.

Now, given a set of sentences, Tarski tells us to uniformly replace all the *non-logical* constants occurring in them by variables of the appropriate types to obtain a set of sentential functions; when a sequence of objects satisfies every one of these sentential functions, we call it a *model* of the set of sentences. He can then state the following definition:

> (T) The sentence X follows logically from the sentence of the class K if and only if every model of the class K is also a model of the sentence X.[10]

A sentence is said to be *analytic* if every sequence of objects is a model of it, and *contradictory* if it has no model.[11]

Now Tarski, like Bolzano before him, finds that these characterizations cannot be made perfectly precise, because he does not think it is entirely clear just which terms should be considered logical:

> Underlying our whole construction is the division of all terms of the language discussed into logical and extra-logical. This division is certainly not quite arbitrary. If, for example, we were to include among the extra-logical signs the implication sign, or the universal quantifier, then our definition of the concept of consequence would lead to results which obviously contradict ordinary usage. On the other hand, no objective grounds are known to me which permit us to draw a sharp boundary between the two groups of terms.[12]

[10] *Tarski 1983*, p. 417.

[11] *Tarski 1983*, pp. 417–418.

[12] *Tarski 1983*, pp. 418–419. For Bolzano's remarks, see below, section 4.4. In a talk delivered in 1966 (published posthumously as *Tarski 1986*), Tarski suggested that the approach of Klein's *Erlanger Programm* might provide a method that would allow us to determine what counts as a logical notion in a non-arbitrary manner. Just as the basic notions of Euclidean geometry can be characterized as invariants under the Euclidean group of transformations, and those of affine geometry as the invariants under the affine group of transformations, and so on, so too, he suggests, logical notions might be characterized as invariants under the most general group of transformations, namely, the group of all permutations of a domain.

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It is important to note that in the 1936 paper, in marked contrast to the way most introductory textbooks present things today, there is no variation of the domain over which quantifiers range. That is, the arithmetical theory under consideration is held actually to treat of the natural numbers, and not merely to be intended to do so. So, too, the numerals are held to designate the appropriate natural numbers, and not merely to be intended to do so. It seems, too, that Tarski intended the numerals to be treated as logical constants (otherwise, the phenomenon of ω-incompleteness would still be present), another striking difference from contemporary practice.

Thus, we should recognise a fourth definition in the above sequence, namely, the standard modern definition, where the logical constants are limited to the usual set (in first-order logic with identity, $\{\neg, \wedge, \vee, \rightarrow, \exists, \forall, =\}$) and where the domain of quantification is allowed to vary. On this account, it is perhaps helpful to think of the quantifiers as expressing a two-fold meaning, part logical, part non-logical. The universal quantifier of first-order logic, for example, could be taken to mean 'all of the S', where S is a (tacit) non-logical parameter.[13]

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3. Etchemendy's objections

In *The Concept of Logical Consequence* and elsewhere, Etchemendy has argued that standard accounts of logical consequence are inadequate. This holds for Tarski's definition in 1936, for the textbook version often said to be Tarski's definition of 1936, and for B^*, which he associates with Bolzano.

Etchemendy has two main criticisms, plus some minor ones. We will confine our attention to the former:

(1) These accounts require us to specify once and for all the set of logical constants. But no matter how we do this, the class of consequences thus determined will (or at least can) be both too wide and too narrow compared to the extension fixed by our pre-theoretic notion of logical consequence.

(2) What Etchemendy calls our ordinary concept of logical consequence contains a modal/epistemological element. But the above accounts have nothing modal about them – they say nothing about possibility, about necessity, or about epistemic justification, but rather only speak about what is. Thus, they leave out something that is essential to the notion of logical consequence, or, insofar as they claim that such elements are still present, do so without justification. In fact, Etchemendy claims the latter, reproaching both Tarski and Bolzano with an error in reasoning which he called, on different occasions, *Bolzano's* and *Tarski's fallacy* – namely, inferring necessity from mere universality.[14]

Let us begin with the first claim. For the sake of simplicity, we will follow Etchemendy in talking about logical truths (or consequences of the empty set of sentences) here. The usual specification of logical constants (in first order logic with

[13] This was noted by Graham Priest (*1995*, p. 286).

[14] In *Etchemendy 1983*, Etchemendy called it Bolzano's fallacy (p. 330), while in *Etchemendy 1990* it is called Tarski's.

identity, $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $=$), he claims, results in a definition which is too wide. For, consider the formulas:

$$\exists x(x = x)$$
$$\exists x \exists y(x \neq y)$$
$$\exists x \exists y \exists z(x \neq y \wedge x \neq z \wedge y \neq z)$$
$$\exists w \exists x \exists y \exists z(w \neq x \wedge w \neq y \wedge w \neq z \wedge x \neq y \wedge x \neq z \wedge y \neq z)$$
$$\vdots$$

On Etchemendy's understanding of the standard account, all the constants in these formulas are logical constants.[15] Thus, when we vary all the non-logical constants (that is, all zero of them), we end up exactly where we started. Applying the definition, we find that these statements are logically true provided that they are true. (They will also be consequences of any other sentences, etc., provided they are true.) But this cannot be right, according to Etchemendy. Surely it cannot be a truth of logic that more than one thing exists. Parmenides might have been right, after all, and this does not seem to be a matter for logicians to decide.[16]

The usual choice of constants also gives rise to a definition of consequence which is too narrow, Etchemendy argues. For surely, he claims, we are right in saying that

a is a triangle. (i)

follows (logically) from

b is a triangle. (ii)

and

a has the same shape as *b*. (iii)[17]

[15] Note that if we suppose the quantifiers to contain a tacit non-logical parameter, this would no longer be the case.

[16] Etchemendy (*1990*, p. 117) writes: 'The claim that there are at least twenty-seven objects is not a logical truth, by anyone's lights'. Stephen Read (*1994*, p. 41) takes a similar position:

> Consider the proposition, 'there are at least two things'. It is not a matter of logic that there are at least two things. None the less, the Bolzano or purely substitutional criterion characterizes this as a logical truth, given the usual acceptance of the quantifier 'some' or 'there are', negation, and identity as logical expressions. [. . .] Since there are in the world at least 10^{80} atoms, the proposition is true—and similarly, arguments such as 'There are two things, so there are 76 things', 'It is raining, so there are 10^{26} things, and so on, turn out valid. This is clearly absurd.
>
> Two points might be made here. First, Carnap, at least in the case of his Language II, does seem to have held that 'there are at least twenty-seven objects' was a logical truth (*Carnap 1937*, §38a; the objects in question are 'positions'). Second, a case might be made for the claim that many, perhaps most, logicians accept the logical truth of such claims, if only implicitly. For in the specification of formal languages it is customary to assume that there are infinitely many entities available to serve as individual constants, predicate and relation symbols, etc. At the same time, these entities seem, at least in certain cases, to be treated as objects on a par with any others. In Henkin's completeness proof for first-order logic, for example, we are told to introduce denumerably many new constants to the language under consideration and, afterwards, to let the set of these constants form the domain of an interpretation of the language. Now one might ask: if it is not a truth of *logic* that there are infinitely many objects available to serve as elements of formal languages, on what basis do *logicians* make this assumption?

[17] The example, chosen for the sake of simplicity, is taken from an unpublished article by Etchemendy. For other examples, see *Etchemendy 1990*, p. 130 ff.

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To say this on Tarski's account, however, we would have to expand the set of fixed terms – for instance, by including 'triangle' and 'has the same shape as' among the fixed, logical vocabulary. Then, sure enough, we could say that (i) follows from (ii) and (iii). But by expanding the logical vocabulary, we would generate too many logical consequences, too many logical truths. For now, the sentence:

$$\exists x \exists y (x \text{ is a triangle } \wedge \ y \text{ is a traingle } \wedge x \neq y)$$

along with the other similar ones, would contain only logical (fixed) vocabulary, and so have to be declared *logically* true provided only they are true. But surely, Etchemendy argues, it is not a *logical* truth that there are more or fewer than two triangles.[18] The phenomenon he points out here is, he claims, quite general:

> It is clear that any instance of the intuitive consequence relation can be made out to be a 'Tarskian' consequence, at least on some selection of logical constants. But as soon as we extend this selection beyond the standard constants, we also introduce many Tarskian consequences that are *not* instances of the intuitive relation. Thus although we can avoid, in a piecemeal way, any specific objection that the definition is too restrictive—by just adding more expressions to the list of logical constants—the effect of doing so is invariably a relation far more extensive than the intuitive consequence relation. There seems to be no *single* choice of logical constants that is neither too restrictive nor too permissive.[19]

In short, Etchemendy thinks he can show that no choice of fixed terms will necessarily deliver the right extension for the consequence relation. This is no accident, in his view. Rather, it is the result of a fundamentally mistaken approach to the question, one that from the start neglects the essential features of logical consequence.

It is here that we make contact with Etchemendy's second main criticism. For logical consequence has, he claims, always been thought to have some sort of modal component. Aristotle, for one, has:

> A syllogism is a discourse in which, certain things being stated, something other than what is stated *follows of necessity* from their being so.[20]

And one can point to a variety of introductory textbooks where the same sort of language is used. Here Etchemendy:

> The most important feature of logical consequence, as we ordinarily understand it, is a modal relation that holds between implying sentences and sentence implied. The premises of a logically valid argument cannot be true if the conclusion is false; such conclusions are said to 'follow necessarily' from their premises.[21]

[18] The case of arithmetic is similar – cf. *Etchemendy 1988b*, p. 73. Etchemendy suggests that Tarski meant to count the numerals as logical constants in order to avoid the phenomenon of ω-incompleteness. But then a variety of arithmetical truths will have to be said to be *logically* true.

[19] *Etchemendy 1988b*, p. 73.

[20] *Aristotle 1941*, I, 1 [24^b 18 ff].

[21] *Etchemendy 1990*, p. 81.

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Etchemendy explains elsewhere that he is inclined to interpret the modality in question as being fundamentally epistemic.

> If you accept the premises of a valid argument, you must also accept the conclusion (to which we sometimes add 'on pain of irrationality'). This epistemic characteristic is sometimes thought to be more important than, and perhaps to underlie, our intuitions about the alethic modality involved in valid arguments. For example, some would claim, not implausibly, that it is only due to the *a priori* relation between the premises and conclusion of a valid argument that we judge the latter to follow necessarily from the former, and hence we judge the argument valid. On this view, a necessary consequence that could not be recognised as such *a priori* would never qualify as a logical consequence. And this certainly seems right.[22]

Now Tarski (and Bolzano) fail in his opinion because they speak of what is instead of what might or what must be. The man in the street says that if Jones is a politician, and no politicians are honest; then it *must* be the case that Jones is not honest; indeed we may be *certain a priori* that it must be so. Yet this is not what we hear from the usual account of logical consequence. Instead, we get the following:

> Every model of Pj and $\neg\exists x(Px \wedge Hx)$ *is* also a model of $\neg Hj$.

Perhaps that it is so is just a remarkable coincidence, completely accidental. Perhaps it might have been otherwise. As far as the Bolzano/Tarski account is concerned, Etchemendy contends, we have no way of telling. For what difference is there, really, between saying that the above argument is valid and saying that the following one is:

> George H. W. Bush was president of the US in 1990. Therefore, George H. W. Bush's eldest son was president of the US in 2003.

We can certainly say that a relation of consequence is obtained here on accounts such as B^* or T, for, if only 'George H. W. Bush' is treated as variable, we obtain the following schema:

$$\frac{\text{x was president of the US in 1990.}}{\therefore \text{x's eldest son was president of the US in 2003.}}$$

And anything satisfying the first sentential function will also satisfy the second. But despite what one might think about the Supreme Court or the electoral system in Florida, who would want to say that the latter *follows of necessity* from the former? Surely, we have to do with pure contingency here. Now the accounts of Bolzano and Tarski, according to Etchemendy, do not and cannot tell the difference between such accidental consequences and the truly logical ones. Their mistake, Etchemendy says, is that they think they can derive a 'must' from an 'is': they confuse mere universality with necessity. But clearly the former can exist without the latter. Hence, this

[22] *Etchemendy 1990*, p. 89.

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approach is fundamentally flawed, and the search for the 'right' set of logical constants, the ones that will yield precisely the genuine logical consequences, is a wild goose chase. This is so because any set of logical constants we choose will simply give us a class of universally satisfied forms – but such mere universality can never support a claim of necessity.

Interestingly, Etchemendy was preceded in this very criticism by William Kneale. In a 1961 paper entitled 'Universality and Necessity,' Kneale criticizes Tarski's and Bolzano's characterizations of analyticity and deducibility precisely because they lack what he takes to be an essential modal/epistemological component.[23] He writes that:

> Just as according to [Bolzano's] definitions a proposition can be analytically true by accident, so too one proposition may follow from another by accident, that is to say in such a way that the truth of the universal proposition about the results can be known only by an examination of the individual results.[24]

Kneale takes this to show that there is 'something wrong in the use of the word 'derivability' [or 'deducibility'] for the relation defined by Bolzano' insofar as it diverges widely from the accepted concept attached to that word:

> For a proposition cannot properly be said to be derivable from a set of premises unless it is possible to establish that if the premises are true the proposition is also true without first establishing whether or not the premisses and the proposition are true.[25]

As we shall see below, however, this view of Bolzano overlooks certain key elements of his system of logic, to which we now turn our attention.

4. Bolzano's fundamental notions

As noted above, because Etchemendy's main concern is the received modern view of logical consequence, he does not enter into a detailed discussion of Bolzano. Insofar as the name 'Bolzano' occurs in his book, it is not so much as a proper name of the philosopher, but rather as part of a convenient label for a well-known position (Tarski's rejected substitutional definition B^*). Etchemendy is quite explicit about this,[26] and so the careful reader has no difficulty distinguishing general philosophical points from historical claims. One might wish for the sake of the not-so-careful reader, however, that he had chosen another label for the position in question. So (with a few exceptions) the following section should not be understood as a rebuttal of Etchemendy's historical claims about Bolzano – for as far as we can tell, he makes very few. The goal is simply to present the relevant points of Bolzano's theories as briefly as possible.

[23] *Kneale 1961*. It is clear that Kneale considers the two theories to be essentially the same in the relevant respects. See, e.g. p. 95: '... Bolzano's theory has been restated independently in our time by Tarski.'

[24] *Kneale 1961*, p. 94.

[25] *Kneale 1961*, p. 94.

[26] See, e.g. *Etchemendy 1990*, p. 163 note 5.

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4.1. Propositions and ideas

Perhaps the most fundamental notion for Bolzano's system is that of the proposition in itself [*Satz an sich*]. Bolzano introduces this term early in the *Theory of Science* (§19) by contrasting it with the more commonly understood notion of an actual proposition, i.e. a thought or an expression in language which is either true or false. Bolzanian propositions in themselves are similar in some respects to Frege's 'thoughts'.[27] Propositions in this sense possess no actual existence, in the way that expressed propositions do, either in symbolic form (written or spoken) or in the processes of thought. We might call propositions in themselves 'objective' (insofar as they, like objects, possess a particular structure independently of our apprehension) and thoughts or linguistic propositions 'subjective' (insofar as they depend upon the thoughts of an individual or the inter-subjective basis of language). Objective propositions, like Frege's 'thoughts' occupy a sort of third realm, and comprise the matter of subjective propositions. This distinction is crucial for Bolzano, as the objective aspect of propositions in themselves enables him to clearly separate, for example, objective relations of consequence from the mind's inferential activities. It will become clearer below that strictly maintaining this division is of prime importance for understanding Bolzano's notions of necessity, apriority, and deducibility.

An *idea* [*Vorstellung*] is any part of a proposition which is not itself a proposition.[28] Ideas, like propositions, can be either objective or subjective. Objective ideas are the components of objective propositions, while subjective ideas are the components of subjective propositions. Some ideas can be further subdivided into parts (e.g. 'wise man') while others (the simple ideas) have no parts whatsoever. While it is often apparent linguistically when a given subjective idea is complex (as the two terms used to represent the idea 'wise man' illustrate), this is not so clear in the case of simple ideas insofar as a linguistically simple term like 'shoe' may designate an idea containing, e.g. the ideas 'foot' and 'covering'. At various places, Bolzano suggests that the ideas 'something', 'has', 'and', and 'not' may be simple, though he rarely seems to hazard anything more definite than conjecture on this score.

Another important division is between ideas which have or represent objects (e.g. 'dog', 'number') and those (e.g. 'round square', 'and') which do not. Bolzano calls ideas which represent objects 'objectual' [*gegenständlich*] and those which do not represent objects 'objectless' [*gegenstandlos*].[29] Bolzano's term 'objectual' does not, however, imply that the objects possessed by a given idea are actual, or that they possess actual existence. An idea may be objectual even if the objects that it represents are abstract. The geometrical idea of a point, for example, is objectual even though its objects possess no actuality.

So, at the most basic level of Bolzano's system we find propositions as well as ideas. Ideas can be further subdivided into the simple and the complex and also into the objectual and the objectless.

[27] There are, of course, also important differences between Bolzano's *propositions* and Frege's *thoughts*. For a detailed comparison of the two theories, see *Künne 1997*.

[28] *Bolzano 1837*, §48.

[29] *Bolzano 1837*, §66 [I.297].

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4.2. *Intuitions and pure concepts*

Bolzano draws a further important distinction between two sorts of ideas, which he calls *intuitions* and *pure concepts*. Intuitions are simple ideas which have exactly one object, while pure concepts, by contrast, are those ideas which are not intuitions and contain no intuitions among their parts.[30] A purely conceptual proposition is any proposition all of whose constituent ideas are pure concepts (i.e. a proposition which contains no intuitions among its parts). Other propositions are called *intuitional* or *empirical*.

Veterans of analytic philosophy will get a fairly good idea of what Bolzano means by an intuition when they recall Russell's notion of *logically proper names*,[31] indefinable expressions which denote exactly one object. Bolzano's intuitions are similar, though non-linguistic. They would be the *senses* of Russell's logically proper names had Russell, like Bolzano and Frege, allowed for such things. The kind of direct reference to individual entities we find here is limited, on both Russell's and Bolzano's views, to particulars with which we are acquainted (in Russell's sense of the word),[32] though Bolzano, in line with contemporary psychology, seems to have thought that such acquaintance is limited to our own states of mind, i.e. sensations, ideas, judgments, and the like.[33] Both, finally, single out certain occurrences of demonstratives ('this', 'that') as the best we can manage by way of finding expressions for these elements of meaning.[34]

Linguistically, Bolzano claims, intuitions make their presence felt in occurrences of indexicals, proper names, and even some occurrences of natural kind terms.[35] Propositions that can be expressed without using any of these kinds of words are thus good candidates for purely conceptual status. Since most if not all mathematical and logical propositions have this character, it is thus reasonable to suppose that they are purely conceptual.

4.3. *Experience,* a priori *knowledge, and necessity*

We now turn to epistemological matters. Recall that Bolzano held that the objects of all human intuitions are mental events. To form an intuition is thus to form an idea of a contingently existing particular (namely, the state of mind which is the unique object of the intuition). But to do this requires (a primitive kind of) experience.[36] Hence, a proposition containing an intuition cannot be thought, and still less known, without experience. It follows that any proposition which can be known *a priori* must contain no intuitions – that is, it must be purely conceptual. Note, however, that this would not entail that all purely conceptual propositions (e.g. all propositions of logic and mathematics) can be known *a priori*. For Bolzano

[30] *Bolzano 1837*, §§72–79. There are also several helpful discussions of Bolzano's understanding of intuitions in his correspondence with Franz Exner; see, e.g. *Bolzano 2004*, pp. 124–127.

[31] See, e.g., *Russell 1998*, p. 61 ff. For further discussion of Bolzano's concept of an intuition, see *Textor 1996*, especially part I; *Rusnock 2003*; *George 2004*.

[32] *Russell 1998*, p. 62: 'A name, in the narrow logical sense of the word, can only be applied to a particular with which the speaker is acquainted ….'

[33] *Bolzano 1837*, §286.8 [III.88]: '… each intuition of which man is capable has as its proper *object* a *change* which presently occurs in the mind ….'

[34] Bolzano, letter to Exner, 23 August 1833, in *Bolzano 2004*, p. 116; cf. 'On the Mathematical Method', §6 (*Bolzano 1975*, pp. 58–60; *Bolzano 2004*, pp. 50–51); and *Bolzano 1837*, §72; Russell *1998*, p. 62.

[35] Cf. *Bolzano 1837*, §75.

[36] Cf. *Bolzano 1837*, §294 [III.115]: '… judgments which contain an intuition, especially those which are held to be true, are usually called *experiences* or *judgments of experience* in a very broad sense of these terms.'

maintained only the converse: if a proposition can be known *a priori*, it is purely conceptual.[37]

To have *a priori* knowledge, however, it is not enough to simply judge that some purely conceptual proposition is true. Rather, the judgment (in case it is inferred from others) must derive from a series of judgments, all of which are purely conceptual themselves:

> If the propositions from which a judgment *M* is deduced, as well as those from which the former follow down to the immediate judgments are all purely conceptual propositions, then judgment *M* can be called a *judgment from pure concepts*, or *pure*, or *a priori*. In all other cases it could be said to be *drawn from experience* or *a posteriori*.[38]

To have such knowledge would be remarkable enough. To know distinctly that one has it would be even more so, since this would require nothing less than a definitive analysis of all the concepts occurring in a given proposition, the discovery of relevant primitive truths involving these concepts, and the deductive order linking these truths to the given proposition. As one of the first to look into the foundations of mathematics in detail, Bolzano knew better than most what a tall order this was; hence his repeated claims, usually directed at the Kantians, that our confidence in many mathematical and logical propositions is based upon experience rather than unerring *a priori* insight:

> The only reason why we are so certain that the rules *Barbara, Celarent*, etc., are valid is because they have been confirmed in thousands of arguments in which we have applied them. This also is the true reason why we are so confident, in mathematics, that factors in a different order give the same product, or that the sum of the angles in a triangle is equal to two right angles, or that the forces on a lever are in equilibrium when they stand in the inverse relation of their distances from the fulcrum, etc. But that $\sqrt{2} = 1.4142...$, that the volume of a sphere is exactly two-thirds of the circumscribed cylinder, that in each solid there are three free axes of revolution, etc., we assert mainly because they follow from propositions of the first kind by arguments which others have conducted hundreds of times and have found valid; an additional factor is that in all these matters we do not have the slightest advantage if the thing turns out to be otherwise. That the reason for our confidence really lies in these circumstances can be seen most clearly from the fact that our confidence rises and falls as these circumstances dictate. If we have not tested the truth of a proposition either by experiment, or by repeated checking of its derivation, we do not give it unqualified assent, if we are at all sensible, no matter what Critical Philosophy may say about the infallibility of pure intuition which is supposed to ground these judgments.[39]

[37] Bolzano was nevertheless confident that many purely conceptual propositions could be known *a priori*. In §133 of *Bolzano 1837* [II.36], he takes the optimistic view that *most* can.

[38] *Bolzano 1837*, §306 [III.202].

[39] *Bolzano 1837*, §315.4 [III.244–245]. Note that Bolzano is speaking here of the basis of our *confidence* in such propositions, not of what he calls their *objective ground*. For Bolzano, the reason for our acceptance of a purely conceptual proposition may well lie in experience. But the objective ground of a purely conceptual proposition, in case it has one, can only lie in other purely conceptual propositions, never in experience (*Bolzano 1837*, §133). Thus, the above remarks should not be interpreted as committing Bolzano to the view that the ground of logical and

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Closely related to the notion of apriority is that of necessity. Bolzano's most explicit account of this notion is offered in §182 of the *Theory of Science*. In the strictest sense, Bolzano thinks that the term 'necessity' only occurs in conjunction with the concept of actuality, that is, it always qualifies being. In a broader sense, however, Bolzano thinks that necessity sometimes qualifies propositions. We may say, for example, that the sum of two odd numbers is necessarily even, or that it is a necessary truth that every triangle has angles that sum to two right angles. What we mean by such assertions, Bolzano claims, is simply that these are *purely conceptual truths*, that is, propositions that contain no intuitions:

> There is a *second* sense of the words 'necessary', 'possible' and 'contingent', which I call *wider* or *improper*. In this sense they are not [. . .] applied to the actuality of things, but to truths in themselves. Whenever a proposition '*A* has *b*' is a purely conceptual truth, it is customary to say that the attribute *b* belongs *necessarily* to object *A* regardless whether or not this object, and hence that attribute, is actual. Thus we say that every equation of odd degree necessarily has a real root, although neither equations nor their roots are something that exists.[40]

> These definitions should make it obvious what I consider to be the sense of sentences which state a necessity, possibility, or contingency. Thus I believe, for example, that the assertion 'God exists with necessity' has no other sense than 'The proposition that God exists is a purely conceptual truth.' 'Every effect must necessarily have its cause' I take to mean nothing but 'It is a purely conceptual truth that every effect has its cause', etc.[41]

On this wider definition, if a proposition can be known *a priori*, then it is a purely conceptual truth, and hence necessary in the broad sense as well. Once again, however, Bolzano does not claim the converse, leaving open the possibility that there are necessary truths that cannot be known *a priori*.[42]

4.4. Bolzano's variation logic

Central to Bolzano's variation logic[43] is the notion of a propositional form. In the *Theory of Science*, he distinguishes several senses of the word 'form', two of them

mathematical truths is empirical. This point has been missed by some commentators, notably by the late J. A. Coffa. Coffa writes, for example, that Bolzano held that the *ground* of logical truth 'derives from below, from the facts.' (*Coffa 1991*, p. 38). Here, as elsewhere, Coffa seems to have overlooked the important distinction Bolzano draws between epistemic and objective grounds.

[40] *Bolzano 1837*, §182.4 [II.231–232].

[41] *Bolzano 1837*, §182.6 [II.233–34].

[42] A detailed examination of Bolzano's notion of necessity is beyond the scope of this article, and is the subject of a separate paper. For the time being, we note the following: Bolzano's definition was not intended to capture either the notion of metaphysical or logical necessity, but instead a wider notion which covers both metaphysical *and* moral necessities in Leibniz's sense (see, e.g. *Leibniz 1985*, §§174, 281, 367; cf. *Bolzano 1979*, especially p. 45.). That is, Bolzano's conception of necessity embraces not only what Leibniz called metaphysical or brute necessities but also (conceptually representable) features of the universe that are due to God's creation of the best among the metaphysically possible worlds – an act which is itself necessary, given God's necessary attributes of omniscience, omnipotence, and perfect goodness. (Interestingly, and in marked contrast with Leibniz, Bolzano thought that even on this broad definition there was still plenty of room for contingency.) Metaphysical, logical, physical, etc. necessity are special kinds of necessity in Bolzano's general sense. Thus, modern readers should not be surprised if Bolzano's definition of necessity turns out to be much wider than they expect.

[43] For more detailed accounts of Bolzano's variation logic, see *Berg 1962*, *Sebestik 1992*, *Siebel 1996*, *Rusnock 2000*.

particularly relevant for the logic of variation.[44] The first is extensional: if we have a logical object (e.g. an idea, a proposition, an argument) among whose parts are ideas $i, j, k, \ldots$, we may consider the class of objects which differ from the given one at most in that certain other ideas $i', j', k' \ldots$ occur in the places of $i, j, k, \ldots$. This class is then called a form. Thus for example, beginning with a proposition such as:

> Aristotle was a philosopher,

we might consider the idea 'Aristotle' as the only variable part. In this case, the propositional form in question would contain propositions such as:

> Socrates was a philosopher.
> Genghis Khan was a philosopher.
> Spiro Agnew was a philosopher.
> Etc.

But not:

> Aristotle was a biologist.

While if we consider 'philosopher' variable in the first proposition, 'Aristotle was a biologist' would belong to the form, while 'Spiro Agnew was a philosopher' would not.

Logicians generally deal with forms via linguistic expressions containing signs for variables: this is a second sense of 'form' singled out by Bolzano.[45] For instance, the two forms just mentioned could be picked out with the help of the expressions:

> *X* was a philosopher.
> Aristotle was a *Y*.

It is important to note that on Bolzano's understanding, every proposition belongs to many different forms. Hence, there is no question of a unique form of any proposition (or by extension, any argument). One cannot simply say that an argument's form is valid, or that a proposition is true in virtue of its form: one must also indicate *which* form one has in mind.[46]

In §147 of the *Theory of Science*, Bolzano introduces the concept of a *universally satisfiable/unsatisfiable* proposition as follows:

> Let the proposition *A* be such that all the propositions which can be generated from it are true, if the ideas *i*, *j*, ... alone are considered variable, and if only objectual propositions may be formed. Then [...] we can call the proposition *universally* or *fully valid*. If, on the other hand, all propositions developed from *A* are false, then [we can] say that it is a *universally* or *absolutely invalid* proposition. Universally valid propositions could also be said to be *true after their kind* or *form*,

[44] See *Bolzano 1837*, §81.
[45] *Bolzano 1837*, §12 [I.48]; §81, note 2 [I.393].
[46] *George 1983*.

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> universally invalid propositions *false* after their kind or form, where by *kind* is meant the sum of all propositions, which differ from A only in the ideas $i, j, \ldots$.[47]

He then proposes that we apply the term 'analytic' to any proposition that is universally valid or universally invalid with respect to at least one of its constituents.[48]

Bolzano's definitions were intended to improve upon Kant's well-known definitions of analyticity. Kant had singled out propositions (or rather judgments) of the form:

> An A, which is B, is B

for special attention. On the 'subject contained in predicate' understanding, the truth of such propositions was grounded in the content of the subject and predicate. Bolzano preferred to say that the truth of such propositions was *independent* of part of their content, in that any appropriate ideas, put in the places indicated by 'A' and 'B' would produce a true proposition. Once seen in this light, however, the more general definition given above seemed a natural extension of Kant's. Analytic propositions are those whose truth-value is invariant under an entire class of transformations.[49]

Having defined analyticity in this very general sense, Bolzano proceeds to define a narrower concept. He begins by contrasting the analyticity of propositions such as:

> Every object is either B or non-B

with that of propositions such as:

> A depraved *man* does not deserve respect.

He remarks that in order to appreciate the analyticity of the former, 'only logical knowledge is needed, since the concepts which form the invariable part of these propositions all belong to logic,' while with the latter 'a wholly different kind of knowledge is required, since concepts alien to logic intrude.'[50] To this he adds:

> This distinction, I admit, is rather unstable, as the whole domain of concepts belonging to logic is not circumscribed to the extent that controversies could not arise at times. Nevertheless, it might be profitable to keep this distinction in mind. Hence propositions like [the former] may be called *logically* analytic, or analytic in the narrower sense; [and the latter], analytic in the *broader* sense.[51]

[47] *Bolzano 1837*, §147 [II.82].

[48] *Bolzano 1837*, §148 [II.83].

[49] A small clarification is in order here. In line with logical tradition, Bolzano maintained that if the subject idea of a proposition represented no object, the proposition was false. Thus, for example, 'Round squares are square' would be false according to him. This being so, we cannot say that every proposition of the form 'A, which is B, is B' is true. To deal with this problem, Bolzano tells us to consider only the variants whose subject idea has an object (he calls these *objectual* [*gegenständlich*] propositions in the above definition).

[50] *Bolzano 1837*, §148.3 [II.84].

[51] *Bolzano 1837*, §148.3 [II.84]. As one might expect from an early nineteenth-century philosopher, Bolzano's list of logical constants is quite different from what one finds, e.g. in Quine or Tarski. On this point, see *Siebel 1996*, pp. 119 ff; *Sebestik 1999*, pp. 503–505; also *Siebel 2002*, pp. 589–590.

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Although Bolzano mentions an epistemological (only logical *knowledge* is required) as well as a logical criterion (the only invariable parts are logical concepts), it would seem appropriate to regard the logical distinction as the fundamental one here, given that this definition occurs in the Theory of Elements, which deals with propositions and ideas in themselves, independently of their being thought or expressed.

Deducibility is defined afterwards, in §155 of the *Theory of Science*. Here we cite the more compact version from the essay 'On the Mathematical Method':

> (B) If one or more propositions A, B, C, ... are compatible with one or more others M, N, ... with respect to the components i, j, ..., then there must be, as just said, at least some ideas that, when put in the places of i, j, ... make all of A, B, C, ... as well as all of M, N, ... true. One especially noteworthy case occurs, however, if not just some, but *all* of the ideas that, when substituted for i, j, ... in A, B, C, ... make all these true, also make all of M, N, ... true. In this case I say that the propositions M, N, ... stand in the relation of *deducibility* to the propositions A, B, C, ... with respect to the variable parts i, j, ...[52]

As was the case with analyticity, Bolzano also points to a narrower notion of *logical* deducibility, cases of deducibility where the only invariable elements are logical concepts.[53] Bolzano adds that, based on his observation of prevailing usage, people only call such inferences logical when they are blindingly obvious to everyone.[54] Indeed, he thought there was something pejorative in the contemporary use of the word 'logical' in such contexts:

> If the distinction which is drawn here between two kinds of deductions, one which can be judged using only *logical* knowledge, the other of which requires various kinds of other knowledge, is not found in previous treatises of logic, it is nevertheless so well known that inferences of the former kind are usually called *logical* in contrast to the latter, and are often looked upon with disdain precisely due to the fact that we do not require any further knowledge in order to be in a position to make them.[55]

Logicians sometimes distinguish between *formal* and *material* consequence, a distinction which nearly if not exactly coincides with Bolzano's distinction between logical and non-logical cases of deducibility.[56] At a couple of places in the *Theory of Science*, Bolzano shows his awareness of this usage[57] and he even seems prepared to accept it, up to a point – the point, namely, where the notion of form as used here is invested with any special explanatory force, as it was, notably, by Kant and his followers. Strictly speaking, it would not make sense to speak of material consequence on Bolzano's dispensation, since all cases of deducibility are formal in his broad sense. An argument such as 'Heifetz is a violinist. Therefore, Heifetz is a musician', for example, can be said to be formally valid, namely, in virtue of the form 'A is a violinist. Therefore, A is a musician.' Thus, the invocation of form as such did

[52] B. Bolzano, 'On the Mathematical Method,' §8 (*Bolzano 1975*, pp. 62–63; *Bolzano 2004*, p. 54).

[53] *Bolzano 1837*, §223 [II.394–395]; 'On the Mathematical Method,' §8 (*Bolzano 1975*, p. 64; *Bolzano 2004*, p. 55).

[54] 'On the Mathematical Method,' §8 (*Bolzano 1975*, p. 64; *Bolzano 2004*, p. 55).

[55] *Bolzano 1837*, §223 [II.395].

[56] These matters are discussed at greater length in *Siebel 1996*, Ch. 3, sect. 4.

[57] *Bolzano 1837*, §29 [I.142]; §515 [IV.243].

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not impress Bolzano, who remained unpersuaded when the Kantians claimed that form was contributed by the mind, and hence necessary and knowable *a priori* with apodeictic certainty. Nor did he think they or anyone else had succeeded in drawing a sharp line separating form and matter.[58]

Bolzano-deducibility is clearly quite closely connected with universal validity. If a proposition Z is deducible from others A, B, C, ..., M with respect to variands i, j, k, ... then the conditional:

If (A and B and C, ... and M) then Z

is universally valid w.r.t. the same variands (and hence analytically true), and conversely. Indeed, Bolzano maintains that conditionals are often used to assert that a relation of deducibility is obtained between given propositions:

> When we claim that M, N, O, ... are deducible from A, B, C, ... with respect to the ideas i, j, ..., we at bottom really only say the following (cf. §155): 'Every collection of ideas which, when put in the places of i, j, ... in the propositions A, B, C, ..., M, N, O, ..., makes all of A, B, C, ... true has the attribute of making all of M, N, O, ... true.' The most common expression for this kind of proposition is well known: '*If* A, B, C, ... are true, *then* so too are M, N, O, ...'[59]

Bolzano's definition of deducibility differs from Tarski's *1936* definition of logical consequence in many respects.[60] To begin with, the relation of deducibility holds neither between linguistic entities (as in *B** and Tarski's definition), nor between mental entities (the most popular option during his lifetime, the heyday of psychologism), but rather between propositions in themselves and their parts, ideas in themselves. Because of this feature, Bolzano's definition is not subject to the same problems as *B**. In particular, there is no question of the expressive capacities of a language, for no language is involved. Nor would Bolzano have seen the need for Tarski's recourse to the concept of satisfaction, since he maintained that for every set of objects there is an idea in itself which represents the members of the set exclusively.[61]

In the second place, Bolzano's definition is deliberately more general than Tarski's, in that he is concerned in the first instance with defining deducibility in the broadest possible sense. Only afterwards does he gesture at a definition of *logical* deducibility.

Third, deducibility is a *triadic* relation, holding between premises, conclusion, *and* a set of parts of these propositions which are treated as variable. Unlike Tarski's definition, which requires us to specify a set of invariable terms once and for all, Bolzano's definition allows the selection of variable elements to shift from one

[58] See in particular *Bolzano 1837*, §§12, 116, 185–186, 254. These matters are discussed in greater detail in a separate paper.

[59] *Bolzano 1837*, §164 [II.198–199].

[60] For a detailed comparison of the two theories, see *Siebel 1997*.

[61] *Bolzano 1837*, §101 [I.469–470]. Given Bolzano's other assumptions, this may well lead to familiar types of contradictions (see *Simons 1992*, p. 39). In particular, Bolzano's assumption seems to make it possible to deduce Russell's paradox in his system. Thus, Bolzano's theory certainly has problems; but they are not the problems noted by Tarski with regard to the linguistic/substitutional definition *B**.

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context to the next. In a series of inferences, different sets of variands may be chosen on different occasions. This has important consequences for logical theory, duly noted by Bolzano: deducibility is only transitive, for example, under certain conditions.[62]

Fourth, Bolzano's notion of substitution differs from Tarski's in that he allows for variation of only some of the occurrences of a given idea. In the proposition:

$$\text{If } \frac{a^2}{\underline{2}} = b, \quad \text{then } a = \pm\sqrt{(\underline{2}b)}.$$

for example, Bolzano invites us to consider as variable only the occurrences of the idea '2' corresponding to the occurrences of the symbol '2' underlined in the above expression.[63] Thus, in the case of deducibility, it is misleading to speak of a set of fixed *terms* or a fixed *vocabulary*; it would be more accurate to speak of the variation of specified *occurrences* of terms (or rather ideas).[64]

5. A Bolzanian response to Etchemendy

With this sketch of the relevant points of Bolzano's system in hand, let us return to Etchemendy's criticisms of Tarski's (and, by extension, Bolzano's) conception of logical consequence.

5.1. *Finding the right set of logical constants*

Etchemendy's initial criticism of Tarski's conception of consequence is directed at that account's reliance on fixing the correct set of logical terms. He claims, as we have seen, that

> ...it is clear that no matter what language we may consider, any *given* valid argument will be declared valid on *some* selection of fixed terms. For at the very least, we can include in [the set of fixed terms] every atomic expression appearing in the particular argument. Likewise any given *invalid* argument will be declared such on some choice of fixed terms; excluding all expressions from [the set of fixed

[62] *Bolzano 1837*, §155.24 and §155.25; cf. §154.12.

[63] *Bolzano 1837*, §447. Bolzano's practice thus accords with Frege's: see *Begriffsschrift*, §9: 'If, in an expression (whose content need not be assertible), a simple or a complex symbol occurs in one or more places and we imagine it as replaceable by another (but the same one each time) at all or some of these places, then we call the part of the expression that shows itself invariant a function and the replaceable part the argument.' (*Frege 1972*, p. 127).

[64] We also note the following differences: 5) Bolzano's definition requires the premises to be compatible whenever a relation of deducibility holds, i.e. at least one substitution must make all the premises true. Consequently, nothing follows from incompatible premises. 6) In addition to the concept mentioned above, Bolzano defines a second concept, called *exact deducibility* or deducibility in the narrow sense. The most workable definition is found in the essay on mathematical method:

> In a *narrower* sense – the one in which I will henceforth understand this way of speaking – I say that a proposition M is *deducible* from the propositions $A, B, C, \ldots$ if each collection of ideas that, when substituted for $i, j, \ldots$ makes all of $A, B, C, \ldots$ true also makes the proposition M true, and when the same does not hold for any *part* of the [set of] propositions $A, B, C, \ldots$ – i.e. if it is not also the case that whenever only a part of this [set of] propositions becomes true, M does as well. ('On the Mathematical Method,' §8.2 (*Bolzano 1975*, p. 63; *Bolzano 2004*, p. 54; a different definition occurs in *Bolzano 1837*, §155.26.)

According to this narrower definition, it is not true to say that a universally satisfiable proposition is a consequence of any set of premises.

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> terms] will guarantee this. But we have no assurance that there will be any *one* selection of fixed or 'logical' terms that produces the right assessment for *every* argument expressible in the language.[65]

The problem which results is that, while the account depends upon fixing the proper set of logical constants, it seems that there can be no guarantee that any particular set of fixed terms will produce the correct evaluation for each argument of the language. And, even beyond the problems we are bound to encounter when actually attempting to properly select our logical constants, Etchemendy believes that Tarski's account is flawed precisely because it relies upon the proper selection of logical constants in the first place. He argues, that is, that any selection of constants is liable to end up extensionally incorrect because of this 'faulty dependence'[66] and that any account requiring such dependence is misguided and doomed to failure.

Here we shall reserve judgment regarding Etchemendy's assessment of Tarski and turn again to Bolzano. In general, the problem of fixing logical constants is one which makes no sense when applied to Bolzano's account of deducibility, since he, unlike Tarski, does not require us to specify, once and for all, a set of logical terms (or ideas), all occurrences of which are to remain fixed. For not only may the choice of variable ideas differ from one argument to the next, it can also be the case that some but not all occurrences of a given type of idea may be varied.

According to Bolzano's definition, for instance, one can maintain with Etchemendy that:

> *a* is a triangle. (i)

Does indeed follow from:

> *b* is a triangle. (ii)

and

> *a* has the same shape as *b*. (iii)

Without thereby being committed ever and anon to considering the terms 'triangle' and 'has the same shape as' fixed terms. So the criticism that no single choice of fixed terms will deliver the right consequence relation does not touch Bolzano's account at all.[67]

All the same, it might be thought that the problem Etchemendy points to will recur when it comes to be a question of *logical* analyticity and *logical* deducibility. For here, it might seem, we are forced even on Bolzano's account to specify a fixed set of logical notions, and will hence run into the problems Etchemendy described.

In reply, let us note first that Bolzano did not set great store in the distinction between the logical and the non-logical. Unlike the Kantians, who claimed to have discovered a sharp boundary between logic and everything else, fixed by the very

[65] *Etchemendy 1990*, p. 80.
[66] *Etchemendy 1990*, p. 110.
[67] Cf. *Siebel 2002*, p. 589.

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architecture of the mind, Bolzano thought that the domain of logic, like that of every other science, was delimited on largely pragmatic grounds.[68] The limits of logic can and do change, as anyone familiar with the history of the subject will attest. Bolzano, who was as aware of this as anyone, did not attempt to draw any grand conclusions from his definitions of logical analyticity and logical deducibility. In the case of the latter, for instance, the main reason for introducing the distinction is to limit the size of his section on the theory of inferences:

> It is incumbent on logic to acquaint us with the most general rules according to which conclusions can be derived from premises. But since it is possible to deduce infinitely many propositions from any given proposition (provided only it is not false after its kind), and *a fortiori* from any (compatible) collection of propositions, it would be unreasonable to demand a listing of all propositions that can be deduced from one or several given propositions. Moreover, according to the very wide sense in which I have taken the word deducibility (§155) the validity or invalidity of some deductions can be assessed only if we have knowledge of matters outside logic. Thus from the proposition 'this is a triangle' we may deduce the proposition 'this is a figure the sum of whose angles equals two right angles' (with respect to the idea 'this'), and from the proposition 'Caius is a man', we can deduce the proposition 'Caius has an immortal soul' (with respect to the idea 'Caius'). For whenever we replace the indicated idea by some other idea, the conclusions become true whenever the premises are true. But to realize this, we must know two truths, namely that the sum of the angles in any triangle equals two right angles, and that the souls of all men are immortal. Since these are truths which are not at all concerned with logical objects, i.e., with the nature of concepts and propositions, or rules according to which we must proceed in scientific exposition, nobody will demand that logic should teach deductions of that sort.[69]

Thus not much seems to be riding on the distinction between logical and non-logical deducibility, and the charge that Bolzano failed to capture an important epistemological kind in framing his definitions might well have been met by him with a shrug.

Secondly, close attention to Bolzano's definitions reveals that he does not even have to fix a sharp boundary between logical and non-logical concepts in order to speak of logical analyticity. Recall that on Tarski's approach, we are to specify a logical vocabulary, and then consider what happens when *all and only* the non-logical terms are uniformly varied. Although Bolzano has been interpreted in this way,[70] he does not in fact say this. Rather, he says that a proposition is logically analytic if it is universally valid (or invalid) relative to a form where the only invariable parts are logical concepts. That is, while Tarski's definition identifies (logical) analyticity with the universal validity of a uniquely determined form (which one might call the *ultimate* or *finest* logical form of a sentence), Bolzano's does not.

Consider, for example, the proposition:

[68] *Bolzano 1837*, §395 [IV.26].

[69] *Bolzano 1837*, §223 [II.391–392].

[70] E.g. by Y. Bar-Hillel (*1950*, p. 41) and J. Berg (*1987*, p. 16).

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Either the King is a fink and the Prince is a schnook or it is not the case that the King is a fink and the Prince is a schnook.

For Bolzano, this proposition belongs to many forms, several of which only number logical concepts among their invariable parts, e.g.:

$$A \vee B$$
$$A \vee \neg A$$
$$(A \wedge B) \vee \neg(A \wedge B)$$
$$(Fa \wedge Gb) \vee \neg(Fa \wedge Gb).$$

For Tarski, by contrast, only the last of these forms comes into consideration. We see, then, that by Bolzano's lights a proposition may belong to several purely logical forms, and indeed may be universally valid with respect to some of them but not others.[71]

The above difference is especially important in light of Bolzano's view that the boundary between logical and non-logical concepts is unstable [*schwankend*]. For even if we are not sure that there is a sharp boundary, or (supposing there is one) exactly where it lies, and hence exactly what the ultimate 'Tarskian' logical form of a given proposition is, we may still be in a position to recognize that it is logically analytic on Bolzano's account.

Suppose, for example, that A is a proposition containing several concepts that, as far as we can tell, might or might not be logical. In this case, we cannot say with confidence what the ultimate logical form of this proposition is. Even so, we can still say, on Bolzano's account, that the proposition 'A or not A' is logically analytic, provided we are confident that 'or' and 'not' are both logical concepts.[72]

The upshot is that, with Bolzano's way of defining logical analyticity, we are not even obliged to specify a sharp boundary between logical and non-logical concepts. Thus, the claim that no such division of concepts into logical and non-logical necessarily delivers the right extension for logical analyticity once again does not seem to touch Bolzano's account at all.

5.2. *Did Bolzano commit Bolzano's fallacy?*

Still, one might wonder whether any of Bolzano's ingenious definitions even comes close to capturing what Etchemendy calls *the* concept of logical consequence. According to Etchemendy, Bolzano was (or at least should have been) trying to capture a pre-theoretic concept of consequence which involved some notion of necessity or apriority. If this were indeed the case, Etchemendy would have been quite right to say that Bolzano had failed, and was indeed guilty of committing a modal fallacy. Yet, Bolzano's texts speak quite clearly to the contrary on this point. The bug that Etchemendy points to is rather a feature in Bolzano's eyes.

From the beginning, Bolzano makes it clear that his purpose is to define *objective* relations between propositions and ideas in themselves:

[71] Cf. *Quine 1981*; *Føllesdal 1981*.

[72] Note that if we assume that there is a definitive distinction between logical and non-logical concepts, we could say in this case that 'A or not A' is analytic in Tarski's sense as well, since any finer variations within the parts considered variable in the 'grosser' form (i.e. within 'A') will simply produce results that were already in the range of possible substitutions.

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> The relation of deducibility between propositions which I have here described is too obvious and important for the discovery of new truths to have escaped logicians altogether. Rather, the development of this concept (in the usual chapter on inferences) forms the most important part of the elements of logic. On the other hand, it seems to me that the nature of this relation has not always been properly grasped. Where it was comprehended, it was not discussed with sufficient generality, or no precise definition was given. It seems to me that the deducibility of propositions from each other is a relation which holds *objectively*, i.e., regardless of our faculties of representation and understanding, and it should be discussed accordingly. This was not generally done; deducibility was described as a relation between *judgments* (i.e., thought and accepted propositions), and it was said that this relation consists in the fact that the acceptance of one proposition brought about the acceptance of another.[73]

For Bolzano, the relation of deducibility does not hold between thoughts, judgments, or even expressions, but between propositions in themselves. He would have had no difficulty in admitting that in some cases we can recognize *a priori* that this relation obtains. But he would hasten to add that in other cases, we could only recognize this *a posteriori*, and in many more cases, not at all.

Bolzano's awareness of these points is clearly visible throughout the *Theory of Science*. In §155, for example, he writes:

> [The relation of deducibility] is of particular importance, since *once we know that it obtains* it puts us in a position to infer the truth of *M, N, O*, ..., once we have recognized the truth of *A, B, C, D*, ...[74]

Here, it is clear that knowledge of deducibility relations is quite separate from the existence of such relations.

A particularly striking case is provided in §110, where Bolzano speaks of what he calls a *complete* idea of a given object:

> If *A* is a *correct* idea of object α, or, what comes to the same, if the proposition 'α is *A*' is true, then from this proposition either all, or only some of the attributes of α can be derived, where for this derivation nothing is used but truths of the form '*A* has the attribute *m*', '*A* has the attribute *n*', etc. If all attributes of α can thus be derived, then *A* is called a *complete* or *exhaustive* idea of its object; otherwise we say that *A* represents the object α only *incompletely*.[75]

After setting out this definition, Bolzano announces the theorem that any singular idea is a complete idea of its object – that is, if '*A*' is an idea that represents only one object, α, and *b* is any property of α whatsoever, then the proposition 'α has *b*' is deducible from the proposition 'α is *A*.' That this is so is easily verified from Bolzano's definition, since any substitution for *x* that makes '*x* is *A*' true will clearly also make '*x* has *b*' true.

[73] *Bolzano 1837*, §155, note 1 [II.128].

[74] *Bolzano 1837*, §155 [II.113], emphasis added. This observation is repeated in §223 [II.391].

[75] *Bolzano 1837*, §110 [I.517–518].

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For example, the proposition 'RMN was president of the US in 1970' has among its consequences the propositions 'RMN ordered the illegal bombing of Cambodia', 'RMN was overly fond of expletives', 'RMN gave a nauseating speech about his dog', 'RMN had difficulty finding competent plumbers', along with countless other true propositions with RMN as their subject, most of which will, perhaps thankfully, remain forever unknown to mankind. For any substitution for 'RMN' that makes the former true will also make all of the latter true. Clearly, Bolzano would not speak of consequence in Etchemendy's epistemological sense here.

Taking stock, then, we may say that for Bolzano, deducibility is an objective relation which has both necessary and non-necessary cases. Human beings are in a position to recognize some, but by no means all, cases of deducibility, and among these some but not all may be recognized *a priori*.

Yet, does not Bolzano still speak of necessity in cases of consequence? Etchemendy draws our attention to passages such as the following:

> If certain propositions M, N, O, ... are deducible from certain others A, B, C, D, ... and one of the former is false, then there *must* also be a false one among the latter. For if A, B, C, D, ... were all true, M, N, O, ... would also *have to* be true, since otherwise it would not be the case (namely for ideas i, j, ... themselves) that each collection of ideas whose substitution for i, j, ... makes A, B, C, D, ... true also makes M, N, O, ... true.[76]

It seems to us, however, that this is a very tenuous basis for Etchemendy's claim that '...Bolzano [...] explicitly offer[s] the modal version as justification for [his] definition of validity [i.e. deducibility].'[77] In this and similar passages, we do indeed find Bolzano using modal language in describing particular deductive relations. But this is not at all the same thing as claiming that the character of necessity attaches to *every* case of deducibility, still less that all cases of deducibility are recognizable *a priori*. On the contrary, Bolzano is exceptionally clear in emphasizing that the existence of deductive relations is one thing, recognizing that they obtain another thing altogether.

But what of Bolzano's use of the word 'must' in describing relations of consequence? Does this not indicate that he was at least dimly aware of the necessary character of some cases of consequence?

We saw above that, according to Bolzano, to say that a proposition is necessary amounts to saying that it is a purely conceptual truth. In a note to this section, he explains why he considers this sense improper. When people speak of necessity in such cases, he remarks, they do so merely for the sake of emphasis, especially in cases where there might be some doubt:

> I believe [...] that the words 'necessary' and 'possible' were only applied to things which do not have actual existence, e.g., to mere truths, because they were accorded an existence at least in the imagination—as one gathers from the fact that people also say that they are actual. Thus one says, e.g., (especially when someone believes the contrary) 'This is actually, or in fact, true.' But if one speaks

[76] *Bolzano 1837*, §155.5 [II.115], emphasis added.

[77] *Etchemendy 1983*, p. 330. We can find no other evidence for Etchemendy's claim on the page he cites apart from other examples of the same sort.

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> of things that have no actuality as if they were actual, then it is no wonder that when an attribute of these things follows from conceptual truths it is called necessary and if no purely conceptual truth implies that the attribute does not belong to them, it is called possible. It is obvious that this usage is dispensable, and at most serves in certain cases to make a stronger impression. For instead of saying of an assertion that it is actually and in fact true we could simply say that it is true; and instead of calling it necessary or saying that it follows necessarily from this or that other truth, or that it is merely possible, we could simply say, without qualification, that it is (or is not) a purely conceptual truth, or that it stands in the relation of deducibility to another truth, and so on.[78]

According to this, any occurrences of 'must' and the like in descriptions of logical relations may simply be replaced with appropriate non-modal terms without loss of content. In that case, the charge that Bolzano committed a modal fallacy would have to be dropped, though he might still be criticized for not even attempting to account for the alleged modal-epistemological character of consequence.

It seems to us, however, that on his own principles, Bolzano could and should have said more at this point. We saw before that he identified necessity in the broader, improper sense with purely conceptual truth. Yet in the case of the expression 'follows of necessity', he no longer speaks of purely conceptual truth, but rather only of truth. Why not speak of purely conceptual truth in this case as well? In the case of deducibility, when we are speaking of what *follows* of necessity, it seems to us that it is what Bolzano calls the 'rule of the deduction' which is of interest[79] – this because the necessity seems to attach to the following (the deductive relation), rather than to what follows (the conclusion). For example, the proposition 'Caius is mortal' is deducible from 'All men are mortal' and 'Caius is a man' when all occurrences of 'Caius', 'man', and 'mortal' are considered variable. Here the rule of the deduction is

> From two propositions of the forms '*A* is *B*' and '*B* is *C*', a third proposition of the form '*A* is *C*' follows.[80]

Or (equivalently in Bolzano's eyes):

> If Caius is a man and all men are mortal, then Caius is mortal (where 'Caius', 'man' and 'mortal' are considered variable).

i.e.:

> (For all appropriate *A*, *B*, *C*) If *A* is *B* and *B* is *C*, then *A* is *C*.[81]

It seems reasonable to suppose that this is a purely conceptual truth and hence necessary in Bolzano's wider sense. Could we not then say that 'Caius is mortal'

[78] *Bolzano 1837*, §182, Note [II.235]. Bolzano's deflationary remarks here bear a certain resemblance to views expressed in *Quine 1976*.

[79] *Bolzano 1837*, §199 [II.344], §315 [III.240].

[80] *Bolzano 1837*, §315.2 [III.240]; cf. §223.

[81] Cf. *Bolzano 1837*, §164 [II.198–199], quoted above, p. 24.

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follows *of necessity* from the other two propositions? And could we not also say that whenever substitutions for 'Caius', 'man' and 'mortal' make the premises true, the conclusion *must* become true as well?[82]

Later in the *Theory of Science*, Bolzano seems to recognise precisely this point:

> When the constituents of a hypothetical [. . .] judgment are pure concepts, or when at least the intuitions occurring in [it] belong to [its] variable parts, then [it] may be combined with a *must* [. . .] without destroying [its] truth: 'If *A* is the case, then *necessarily* so is *B*.'[83]

On the other hand, with a deduction such as:

$$\frac{\text{Napoleon once celebrated a famous victory in Pressburg.}}{\text{Napoleon visited a one-time seat of the Hungarian Royal Court.}}$$

The deduction rule would be:

> From a proposition of the form '*x* once celebrated a famous victory in Pressburg' another of the form '*x* visited a one-time seat of the Hungarian Royal Court' follows.

which is obviously not purely conceptual, and hence not necessary in Bolzano's sense.

Thus, we can think of the deductive 'guarantee' Etchemendy requires in the following Bolzanian terms. A given proposition follows necessarily from others just in case there is a purely conceptual truth which states that a relation of deducibility holds between them. Note that this definition would also allow us to say that necessary relations of consequence hold between empirical propositions, i.e. in cases where the rule of deduction is a purely conceptual truth.

The epistemological application of the above considerations will perhaps already be obvious to most readers. If necessity in the broad sense is just purely conceptual truth, and these are the only kind of truths that can be known *a priori*, we will be able to say that only necessary relations of consequence can be recognized *a priori*. At the same time, following Bolzano's usual prudence, we will not want to claim that *all* such relations can be known *a priori*, and may even be happy to admit that, among those that can, many are in fact known to us (or at least believed) only on empirical grounds. If a genuinely epistemological notion is desired, one might require not only that the relation of deducibility be necessary in Bolzano's broader sense, but also that it be recognizable *a priori*.

These distinctions still might not be enough to satisfy some philosophers. It might be objected, for one thing, that there are genuine relations of (logical) consequence

[82] Coffa (*1991*, p. 38) clearly thought that Bolzano made a slip in interpreting Aristotle's 'follows of necessity' in this way, since he placed a '*sic*.' after the 'must' in Bolzano's statement (*Bolzano 1837*, §155, note [II.129]). In our view, Bolzano would be guilty of nothing of the sort, since all of Aristotle's syllogisms would count as cases of necessary deducibility in Bolzano's sense.

[83] *Bolzano 1837*, §193 [II.312]: Bolzano's original statement also covers disjunctive propositions: 'When the constituents of a hypothetical or disjunctive judgment are pure concepts, or when at least the intuitions occurring in them belong to their variable parts, then both may be combined with a *must* (as Prof. Krug says) without destroying their truth: "If *A* is the case, then *necessarily* so is *B*" and "*Necessarily*, one of the propositions *A* and *B* is true."'

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that are not cases of necessary deducibility in Bolzano's sense. For another, the bare stipulation that the relation must be recognizable *a priori* might be found wanting. Kneale, for example, has written:

> [It is not appropriate] that a distinction of kinds of consequence should be made to depend upon a distinction between cases in which we can and cases in which we cannot gain knowledge *a priori*. On the contrary, the epistemological distinction should be explained by an account of the difference of the cases; and it is just this which is lacking so far.[84]

It seems to us that Bolzano has done considerable service by developing a theory which can explain, *on the basis of the objective content of propositions alone*, why only some of them might be known *a priori* and, by extension, why only some cases of consequence might be recognized *a priori*. This is at least part of what Kneale was asking for. And perhaps at this level of abstraction, in light of well-known results on the limitations of formal methods, it is simply not reasonable to ask for more.

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[84] *Kneale 1961*, p. 97.

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