JOURNAL OF COMBINATORIAL THEORY 8, 115–133 (1970)

Heterogeneous Algebras*

GARRETT BIRKHOFF AND JOHN D. LIPSON†

Department of Mathematics, Havard University, Cambridge, Massachusetts 02138

Received December 3, 1968

ABSTRACT

Many of the basic theorems about general "algebras" derived in [1, Ch. 6] are extended to a class of *heterogeneous algebras* which includes automata, state machines, and monoids acting on sets. It is shown that some algebras can be fruitfully studied, using different interpretations, both as (homogeneous) algebras and as heterogeneous algebras, and a non-trivial "free machine" is constructed as an application. The extent of the overlap with previous work of Higgins [9] is specified.

1. INTRODUCTION

We generalize in this paper the usual [1, p. 132] notion of an "algebra", so as to include as special cases the notions of a monoid acting on a set, "automaton" (sequential machine) and "state machine" (semi-automaton). We show that most of the general theory of algebraic systems of [1, Ch. 6] applies with undiminished force. Our basic definition is the following; it is essentially equivalent to that of an "algebra with a scheme of operators" (or "$\sum$-algebras") of Higgins [9]; see also [14].

DEFINITION. An *algebra* is a system $A = [\mathscr{S}, F]$ in which:

1. $\mathscr{S} = \{S_i\}$ is a family of non-void sets S_i of different types of elements, each called a *phylum* of the algebra A. The phyla S_i are indexed by some set I; i.e., $S_i \in \mathscr{S}$ for $i \in I$ (or are called by appropriate names).

2. $F = \{f_\alpha\}$ is a set of finitary operations, where each f_α is a mapping

$$f_\alpha : S_{i(1,\alpha)} \times S_{i(2,\alpha)} \times \cdots \times S_{i(n(\alpha),\alpha)} \to S_{r(\alpha)} \tag{1}$$

* Presented at the Yale University Conference on Combinatorial Theory in honor of Professor Oystein Ore (May, 1968).

† This work was partly supported by the Office of Naval Research.

for some non-negative integer $n(\alpha)$, function $i_\alpha : k \to i(k, \alpha)$ from $n(\alpha) = \{1, 2,..., n(\alpha)\}$ to I, and $r(\alpha) \in I$. The operations f_α are indexed by some set Ω; i.e., $f_\alpha \in F$ for $\alpha \in \Omega$ (or are called by appropriate names).

Thus each operation f_α assigns to each $n(\alpha)$-tuple $(x_1 ,..., x_{n(\alpha)})$, where $x_j \in S_{i(j,\alpha)}$, some value $f_\alpha(x_1 , x_2 ,..., x_{n(\alpha)})$ in $S_{r(\alpha)}$. The operation f_α is said to be $n(\alpha)$-ary: unary when $n(\alpha) = 1$, binary when $n(\alpha) = 2$, ternary when $n(\alpha) = 3$, etc. When $n(\alpha) = 0$ the operation f_α is called "nullary"; it selects a fixed element (distinguished constant) of $S_{r(\alpha)}$.

An algebra which has only one phylum will be called a *homogeneous* algebra; an algebra having more than one phylum, a *heterogeneous algebra.* The discussion of "universal algebra" in [1, Ch. 6] assumes homogeneity, but we shall show below that this assumption is unnecessary.

For concreteness, we first give some familiar examples of heterogeneous algebras.

EXAMPLE 1. A *module* M over a ring R is a heterogeneous algebra $M = [\mathscr{S}, F_1 \sqcup F_2 \sqcup \{\nu\}]$ where[1] $\mathscr{S} = \{V, R\}$ consists a of phylum V of vectors and a phylum R of scalars. The vectors form an Abelian group under the 3 operations of $F_1 = \{+, -, 0\}$. The scalars from the ring R (with unity) under the 5 operations of $F_2 = \{+, -, 0; \cdot, 1\}$. In addition there is a 9th operation $\nu : R \times V \to V$ ("scalar multiplication"). These operations are made to satisfy the familiar module postulates [10, p. 190] $\lambda(a + b) = \lambda a + \lambda b$, $(\lambda\mu)\, a = \lambda(\mu a)$, $(\lambda + \mu)\, a = \lambda a + \mu a$, $1a = a$.

Observe that, when the module M is considered as a *heterogeneous* algebra, the ring R is explicitly defined and enjoys the status of a phylum along with V. Scalar multiplication is defined as a single operation $\nu : R \times V \to V$, and the class of all modules forms a single "family" of algebras, in the following sense.

DEFINITION. Two algebras $A = [\{S_i\}, F]$ and $B = [\{T_i\}, F]$ are said to be *similar*, or to belong to the *same family*, when they have *phyla and operations having the same names* (or index-sets).

Contrast the above definition of a module with the more sophisticated definition of a R-module as a homogeneous algebra $\mathscr{R} = [\{V\}, \{+, -, 0, R\}]$ which is normally invoked in universal algebra [1, p. 132]. In the latter definition, each "element" $\lambda \in R$ is considered as a different unary *operation* $f_\lambda : v \mapsto \lambda v$.

EXAMPLE 2. A monoid M acting on a set S is a heterogeneous algebra $[\{M, S\}, \{*, 1_M; \cdot\}]$, where the operations $* : M \times M \to M$ and 1_M of M

[1] Here and elsewhere, $\sqcup$ denotes *disjoint union.*

and $\cdot : M \times S \to S$ satisfy the identity and associativity laws $1_M \cdot s = s$ and $(m * n) \cdot s = m \cdot (n \cdot s)$, respectively.

In this paper, we shall be especially interested in the following kind of heterogeneous algebra.

EXAMPLE 3. A (*sequential*) *machine* or *automaton* is a heterogeneous algebra $M = [\mathscr{S}, F]$ in which $\mathscr{S} = \{S, A, Z\}$ contains three phyla; a non-void set S of "states," a non-void set A of input symbols (the "input alphabet"), and a finite set Z of output symbols (the "output alphabet"). There are two binary operations in F:

$$\nu : S \times A \to S \qquad \text{("change of state")}, \tag{2a}$$

$$\zeta : S \times A \to Z \qquad \text{("output function")}. \tag{2b}$$

A machine may be conveniently specified by a *state diagram* [8, p. 17]. A state diagram is a labeled graph in which the nodes represent states, and the label a/z associated with the arc from node (state) s to t indicates that $\nu(s, a) = t$ and $\zeta(s, a) = z$. For example, the machine M specified by the state diagram of Figure 1 below has $S = \{1, 2, 3\}$, $A = \{a, b, c\}$, and $Z = \{0, 1\}$. The label $a/0$ on the arrow from node 1 to 2 indicates that $\nu(1, a) = 2$ and $\zeta(1, a) = 0$, and similarly for the rest of the state diagram.

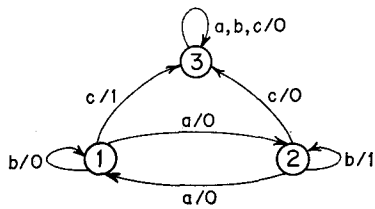


FIGURE 1

EXAMPLE 4. Associated with every machine or automaton is a state machine [8, p.18] or semiautomaton [6, p. 40] obtained by "forgetting" about Z and ζ. Thus a state machine is a heterogeneous algebra $M = [\{S, A\}, \{\nu\}]$. Evidently the class of all state machines constitutes a family of algebras.

A state machine can also be defined as a *homogeneous* algebra $[\{S\}, F]$, where F contains a unary operation $f_a : S \to S$ defined by $s \mapsto \nu(s, a)$ for each letter a of the input alphabet A. The class of all state machines does *not* constitute a family of algebras as homogeneous algebras because the set F of operations depends on the input alphabet A.

2. SUBALGEBRAS

DEFINITION. A *subalgebra* $[\{T_i\}, F]$ of an algebra $A = [\{S_i\}, F]$ is a family of subsets $T_i \subset S_i$ $(i \in I)$ closed under the operations of F. More precisely, the condition is that for each $f_\alpha \in F$, if $t_k \in T_{i(k,\alpha)}$ for $k = 1, 2, \ldots, n = n(\alpha)$ then $f_\alpha(t_1, t_2, \ldots, t_n) \in T_{r(\alpha)}$.

Note that the possibility that some T_i are void is not excluded by the above definition. However, if F contains any nullary operation, say f_α, which selects an element $t \in S_{r(\alpha)}$, then we require every subalgebra $[\{T_i\}, F]$ of A to contain this t. A subalgebra $[\{T_i\}, F]$ with one or more T_i void is said to be *degenerate*.

When applied to machines, the above definition of subalgebra yields a notion of a *submachine* which is equivalent to that in [8, *p.* 19]. Given a machine $M = [\{S, A, Z\}, \{\nu, \zeta\}]$ and subsets $S_1 \subset S, A_1 \subset A, Z_1 \subset Z$, then $M_1 = [\{S_1, A_1, Z_1\}, \{\nu, \zeta\}]$ is a submachine of M when (i) S_1 contains with each of its states s all those states $\nu(s, a)$ accessible from s via inputs a of A_1, and (ii) for every state s of S_1 and input symbol a of A_1, Z_1 contains the output symbol $\zeta(s, a)$.

Diagrams of three submachines of the machine M of Figure 1, Example 3, are given in Figure 2.

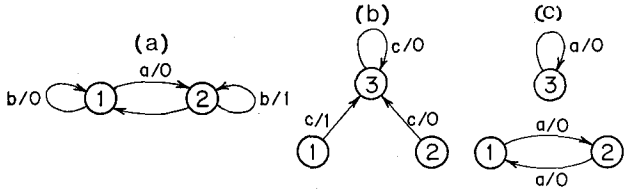


FIG. 2. Sample submachines.

Let $B_\nu = [\{T_{i,\nu}\}, F]$ be a family of subalgebras of $A = [\{S_i\}, F]$. Then the *intersection* $\bigcap_\nu B_\nu$ is the system $[\{\bigcap_\nu T_{i,\nu}\}, F]$, and we have, as in [9, p. 120],

PROPOSITION 1. *Any intersection* $\bigcap_\nu B_\nu$ *of subalgebras* $B_\nu = [\{T_{i,\nu}\}, F]$ *of* A *is a subalgebra.*

Let $A = [\{S_i\}, F]$ be an algebra, and let $\mathscr{T} = \{T_i\}$ be contained in A; i.e., $T_i \subset S_i$ for all $i \in I$. Denote by $\bar{\mathscr{T}} = [\{\overline{T}_i\}, F]$ the intersection of all subalgebras of A containing $\mathscr{T}$. Then $\bar{\mathscr{T}}$ will be called the subalgebra *generated* by $\mathscr{T}$, and $\mathscr{T}$ a "generating family of sets" for $\bar{\mathscr{T}}$.

For families $\mathscr{T} = \{T_i\}$ and $\mathscr{U} = \{U_i\}$ we clearly have: (i) $\mathscr{T} \subset \bar{\mathscr{T}}$, (ii) $\overline{\bar{\mathscr{T}}} = \bar{\mathscr{T}}$, and (iii) if $\mathscr{T} \subset \mathscr{U}$, then $\bar{\mathscr{T}} \subset \bar{\mathscr{U}}$. In other words [1, p. 112], we have

PROPOSITION 2. *The correspondence* $\mathscr{T} \to \bar{\mathscr{T}}$ *is a closure operation* [1, p. 111] *on the set* $\bigsqcup_{i \in I} S_i$.

As another consequence, we have (cf. [1, p. 111])

PROPOSITION 3. *The subalgebras of an algebra form a complete lattice.*

EXAMPLE 5. Let M be a monoid of functions acting on a set S. Then $U = [\{M, \mathscr{P}(S)\}, F]$ is a heterogeneous algebra, where $F = \{\cdot, 1, \bigcap, \bigcup, ', *\}$ contains besides the monoid operations of M and Boolean operations on $\mathscr{P}(S)$, the binary cross-operation $* : M \times B \to B$ which assigns to each $f \in M$ and $T \in \mathscr{P}(S)$ the *image* of T under f.

This example is closely related to the *homogeneous* algebra $V = [\mathscr{P}(S), F]$, where $F = \{\bigcap, \bigcup, ', M\}$ consists of Boolean operations and the "actions" of $f \in M$ on the $T \in \mathscr{P}(S)$. Note that U and V have very different subalgebra lattices. The least subalgebra of V consists of the field of sets generated (as a Boolean subalgebra of $[\mathscr{P}(S), \bigcap, \bigcup, ']$) by the set

$$\Gamma = \{T \in \mathscr{P}(S) \mid f(T) \subset T \quad \text{all} \quad f \in M\},$$

whereas U has many smaller subalgebras.

3. MORPHISMS

For similar algebras $A = [\{S_i\}, F]$ and $B = [\{T_i\}, F]$, we have the following

DEFINITION. A *morphism* Φ from A to B is a set of functions

$$\phi_i : S_i \to T_i\,, \tag{3}$$

one for each $i \in I$, such that for any $f_\alpha \in F$, $n = n(\alpha)$,

$$f_\alpha \circ (\phi_{i(1,\alpha)} \times \cdots \times \phi_{i(n,\alpha)}) = \phi_{r(\alpha)} \circ f_\alpha\,. \tag{4}$$

As usual, morphisms which are injective (ϕ_i injective for all i), surjective (ϕ_i surjective for all i), or bijective (ϕ_i bijective for all i), are called *mono*morphisms, *epi*morphisms, and *iso*morphisms, respectively.

Thus in Example 1, the above definition makes a morphism of vector spaces $[\{R^m, R\}, F] \to [\{S^m, S\}, F]$, R a field, mean what is usually called a *semilinear* transformation. Again, if $\mathbf{Z}$ and $\mathbf{Z}_n$ have their usual meanings

(ring of integers and integers mod n), and we consider the modules $\mathbf{Z}^2$ and $\mathbf{Z}_n{}^2$ as heterogeneous algebras,

$$M = [\{\mathbf{Z} \times \mathbf{Z}, \mathbf{Z}\}, F], \qquad M_n = [\{\mathbf{Z}_n \times \mathbf{Z}_n, \mathbf{Z}_n\}, F],$$

as in Example 1, then the assignments

$$\phi_1 : (a, b) \to ([a]_n, [b]_n), \ \phi_2 : a \mapsto [a]_n$$

define an epimorphism of M onto M_n.

In order to obtain the usual definition of morphism of R-modules [10, p. 193], in which the domain and codomain modules are assumed to have the same ring R of scalars and $\phi_2 : R \to R$ to be the identity morphism on R, one must consider the modules as homogeneous algebras [1, p. 132]. This interpretation has the advantage that the sum of any two linear transformations is itself a linear transformation.

When applied to automata (Example 3), the above definition of morphism specializes to the following definition [8, p. 20]:

Let $M_1 = [\{S_1, A_1, Z_1\}, \{\nu_1, \zeta_1\}]$ and $M_2 = [\{S_2, A_2, Z_2\}, \{\nu_2, \zeta_2\}]$ be machines. Then the set of functions

$$\phi_i : S_1 \to S_2, \quad \phi_2 : A_1 \to A_2, \quad \phi_3 : Z_1 \to Z_2, \tag{5}$$

is a *morphism of machines* when

$$\nu_2(\phi_1(s), \phi_2(a)) = \phi_1(\nu_1(s, a)), \tag{6a}$$

$$\zeta_2(\phi_1(s), \phi_2(a)) = \phi_3(\zeta_1(s, a)) \tag{6b}$$

are identically satisfied.

As an illustration of a morphism of machines, consider the machines M_1 and M_2 specified by the state diagrams of Figure 3. It is readily checked that the mappings

$$\begin{array}{lll} \phi_1 : p \mapsto s & \phi_2 : a \mapsto x & \phi_3 : 1 \mapsto 0 \\ \quad q \mapsto t & \quad b \mapsto y & \quad 0 \mapsto 1 \\ \quad r \mapsto t & \quad c \mapsto y & \end{array}$$

constitute a morphism from machine M_1 to machine M_2.

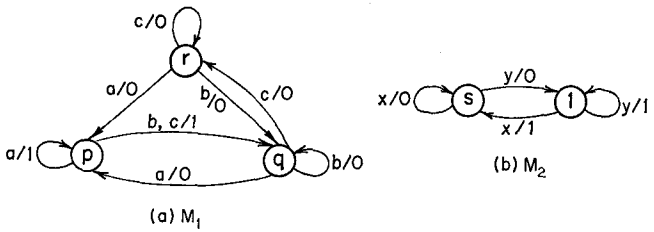


FIG. 3. Morphism of machines.

If $\Phi = \{\phi_i\}$ and $\Psi = \{\psi_i\}$ are morphisms of algebras $A \to B$ and $B \to C$, respectively, their *composite*, denoted $\Phi \circ \Psi$, is the set of mappings $\{\phi_i \circ \psi_i\}$.

PROPOSITION 4. *Any family of similar algebras forms a* concrete *category* [10, p. 64] *with respect to its morphisms.*

COROLLARY. *The endomorphisms of any algebra form a monoid.*

The preceding proposition relates our general algebraic approach to automata to that of Eilenberg and Wright [5]. However, their technical definitions [5, §12] are quite different from ours, being much more general. (Ours correspond much more closely to those usually used.)

It is easily verified that, if $\Phi = \{\phi_i : S_i \to T_i\}$ is an isomorphism from an algebra A to an algebra B, then $\Phi^{-1} = \{\phi_i^{-1} : T_i \to S_i\}$ is an isomorphism from B to A, and from the above corollary we conclude

PROPOSITION 5. *The automorphisms of any algebra A form a group* Aut A.

A generalization of [1, p. 135, Theorem 3] is stated in

PROPOSITION 6. *For any morphism* $\Phi = \{\phi_i\} : A \to B$:

(i) *if* $A' = [\{S_i'\}, F]$ *is a subalgebra of A then* $\Phi(A') = [\{\phi_i(S_i')\}, F]$ *is a subalgebra of B*;

(ii) *if* $B' = [\{T_i'\}, F]$ *is a subalgebra of B then* $\Phi^{-1}(B') = [(\phi_i^{-1}(T_i')\}, F]$ *is a subalgebra of A.*

PROPOSITION 7. *Let* $\Phi = \{\phi_i\}$ *be a morphism* $A \to B$. *If* $\mathscr{I} = \{T_i\}$ *is contained in* $A = [\{S_i\}, F]$ *(i.e.,* $T_i \subset S_i$ *for all* $i \in I$*), then* $\Phi(\bar{\mathscr{I}}) = \overline{\Phi(\mathscr{I})}$; *in words, the image of the subalgebra of A generated by* $\mathscr{I}$ *is equal to the subalgebra of B generated by* $\Phi(\mathscr{I}) = \{\phi_i(T_i)\}$.

COROLLARY. *If* $\Phi : A \to B$ *is an epimorphism with* $A = \bar{\mathscr{I}}$, *then* $B = \overline{\Phi(\mathscr{I})}$; *i.e., B is generated by* $\Phi(\mathscr{I}) = \{\phi_i(T_i)\}$.

PROPOSITION 8. *Let* $\Phi = \{\phi_i\}$ *and* $\Psi = \{\psi_i\}$ *be morphisms* $A \to B$, *and suppose that* $A = [\{S_i\}, F]$ *is generated by* $\mathscr{I} = \{T_i\}$; *i.e.,* $A = \bar{\mathscr{I}}$. *If* $\phi_i(x) = \psi_i(x)$ *for all* $x \in T_i$ *and all* $i \in I$ *then* $\Phi = \Psi$ *(i.e.,* $\phi_i = \psi_i$ *for all* $i \in I$*).*

4. Congruences

A standard theorem on (homogeneous) algebras asserts that the epimorphic images of any algebra. A are all determined, up to isomorphism, by the congruence relations on A. We now generalize this theorem to heterogeneous algebras; since one equivalence relation on each phylum is involved in the generalization, we replace the phrase "congruence relation" by the single word congruence.

Definition. A *congruence* on an algebra $A = [\{S_i\}, F]$ is a family $\mathscr{E} = \{E_i\}$ of equivalence relations, with E_i defined on S_i for each $i \in I$, which for each $f_\alpha \in F$ and x_j , $y_j \in S_{i(j,\alpha)}$ has the following *substitution property:*

$$x_j E_{i(j,\alpha)} y_j \quad (j = 1, 2,..., n(\alpha)) \quad \text{implies}$$

$$f_\alpha(x_1 ,..., x_{n(\alpha)}) \, E_{r(\alpha)} \, f_\alpha(y_1 ,..., y_{n(\alpha)}) \tag{7}$$

Proposition 9. *Let $\mathscr{E} = \{E_i\}$ be a congruence on an algebra $A = [\{S_i\}, F]$ and let $x \mapsto P_{E_i}(x)$ be the mapping which carries each $x \in S_i$ into its equivalence class in S_i/E_i . Then the operations f_α on $\{S_i/E_i\}$ given by, for $n = n(\alpha)$,*

$$f_\alpha(P_{E_{i(1,\alpha)}}(x_1),..., P_{E_{i(n,\alpha)}}(x_n)) = P_{E_{r(\alpha)}}(f_\alpha(x_1 ,..., x_n)) \tag{8}$$

define an algebra $B = [\{S_i/E_i\}, F]$ which is an epimorphic image of A under $\Phi = \{P_{E_i}\}$.

Proof: By the substitution property of the E_i , the operations f_α defined by (8) are single-valued. Also by (8), $\Phi = \{P_{E_i}\}$ is a morphism from A to B; moreover, each ϕ_i is clearly surjective so that Φ is an epimorphism.

The algebra B in the statement of Proposition 9 is called the *quotient algebra* of A relative to the congruence $\mathscr{E}$, and is denoted $A/\mathscr{E}$.

A converse to Proposition 9 is provided by

Proposition 10. *Let $\Phi = \{\phi_i\}$ be an epimorphism $A \to B$. Then a congruence $\mathscr{E} = \{E_i\}$ can be defined in A such that B is isomorphic to $A/\mathscr{E}$.*

As in [9, p.120], we have the following heterogeneous analog of Theorem 6 of [1, p. 137].

Proposition 11 (Morphism Theorem for Heterogeneous Algebras). *The epimorphic images of any algebra A are determined up to isomorphism by the quotient algebras $A/\mathscr{E}$ defined by the congruences $\mathscr{E}$ on A.*

The congruences $\mathscr{E} = \{E_i\}$ on an algebra $A = [\{S_i\}, F]$ are *partially ordered* as follows: for congruences $\mathscr{E} = \{E_i\}$, $\mathscr{F} = \{F_i\}$

$$\mathscr{E} \leqslant \mathscr{F} \text{ when } xE_i\,y \text{ implies } xF_i\,y \text{ for all } i \in I. \tag{9}$$

This poset has universal bounds $\mathscr{L} = \{L_i\}$, $\mathscr{U} = \{U_i\}$ defined by:

$$\left.\begin{array}{ll} xL_i\,y, & \text{if and only if } x = y, \\ xU_i\,y, & \text{for all } x, y \in S_i, \end{array}\right\} \text{ for all } i \in I. \qquad \begin{array}{l}(10a)\\(10b)\end{array}$$

As in [9, p. 121], sharpened by the argument of [1, Ch. 6, Theorem 8], we have also

PROPOSITION 12. *The poset $H(A)$ of all congruences on an algebra $A = [\{S_i\}, F]$ forms a complete lattice, which is lattice-monomorphic to the product $\Pi = \Pi(S_1) \times \Pi(S_2) \times \cdots \times \Pi(S_I)$ of the partition lattices $\Pi(S_i)$.*

PROPOSITION 13. *Let $B = A/\mathscr{E}$ be any epimorphic image of the algebra $A = [\{S_i\}, F]$. Then the congruences on B are the partitions of the phyla of B defined by the congruences $\mathscr{F} \geqslant \mathscr{E}$ on A.*

Proposition 13 also has the following immediate corollary.

COROLLARY. *Let $\mathscr{E} = \{E_i\}$ and $\mathscr{F} = \{F_i\}$ be congruences on an algebra $A = [\{S_i\}, F]$ with $\mathscr{E} \leqslant \mathscr{F}$. Then $A/\mathscr{F}$ is an epimorphic image of $A/\mathscr{E}$.*

5. DIRECT PRODUCTS AND POWERS OF ALGEBRAS

Let $A = [\{S_i\}, F]$ and $B = [\{T_i\}, F]$ be any two similar heterogeneous algebras. Then the *direct product* $A \times B$ of A and B is the algebra $[\{S_i \times T_i\}, F], f_\alpha$ defined for $n = n(\alpha)$ by:

$$\begin{aligned} f_\alpha((x_{i(1,\alpha)}\,, y_{i(1,\alpha)}),\ldots, (x_{i(n,\alpha)}\,, y_{i(n,\alpha)})) \\ = (f_\alpha(x_{i(1,\alpha)}\,,\ldots, x_{i(n,\alpha)}), f_\alpha(\,y_{i(1,\alpha)}\,,\ldots, y_{i(n,\alpha)})). \end{aligned} \tag{11}$$

Thus, in Example 1 of Section 1, the direct product of two modules $M_1 = [\{V_1\,, R_1\}, F]$ and $M_2 = [\{V_2\,, R_2\}, F]$, regarded as heterogeneous algebras, is $M_1 \times M_2 = [\{V_1 \times V_2\,, R_1 \times R_2\}, F]$ with the scalar multiplication in $M_1 \times M_2$ defined by $(\lambda_1\,, \lambda_2)(v_1\,, v_2) = (\lambda_1 v_1\,, \lambda_2 v_2)$.

Note that when $R = R'$ the direct product of modules as defined above does *not* agree with the (standard) definition of the direct sum of two R-modules as *homogeneous* algebras [10, p. 209]; in the latter case scalar multiplication is defined only when $\lambda_1 = \lambda_2$.

In Example 3 of Section 1, the direct product of machines $M_1 = [\{S_1\,, A_1\,, Z_1\}, \{\nu_1\,, \zeta_1\}]$ and $M_2 = [\{S_2\,, A_2\,, Z_2\}, \{\nu_2\,, \zeta_2\}]$, regarded as

heterogeneous algebras, is the machine $M_1 \times M_2 = [\{S_1 \times S_2, A_1 \times A_2, Z_1 \times Z_2\}, \{\nu, \zeta\}]$, whose next-state function ν and output function ζ are defined by:

$$\nu((s_1, s_2), (a_1, a_2)) = (\nu_1(s_1, a_1), \nu_2(s_2, a_2)), \tag{12a}$$

$$\zeta((s_1, s_2), (a_1, a_2)) = (\zeta_1(s_1, a_1), \zeta_2(s_2, a_2)). \tag{12b}$$

As usual, the corresponding definition for the direct product of state machines is obtained from the above by forgetting about outputs (Z, ζ, and (12b)).

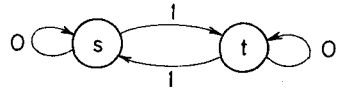


FIG. 4(a). State Machine M

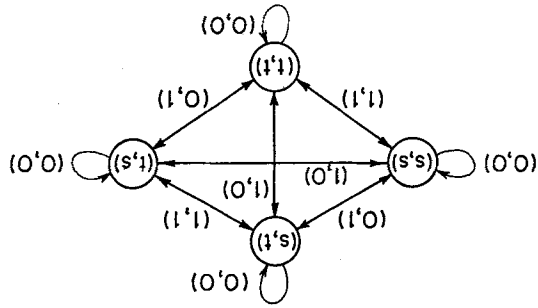


FIG. 4(b). *Heterogeneous* Direct Product $M \times M$

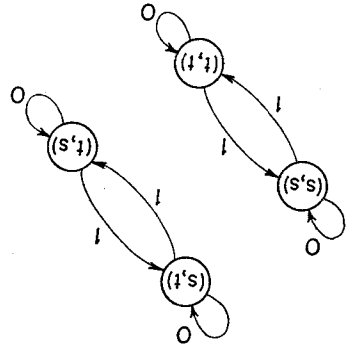


FIG. 4(c). *Homogeneous* Direct Product $M \times M$

A more restrictive definition of direct product for machines is obtained when they are considered as *homogeneous* algebras; this is the definition of Rabin and Scott [11, p. 74]; it is defined only for "similar" machines having fixed alphabets A, Z. Moreover, even in this case, the definitions are not equivalent.

Thus, let M have $A = Z = \{0, 1\}$. If M is considered as a *heterogeneous* algebra, the alphabet of $M \times \cdots \times M = M^r$ consists of all r-bit words. If, on the other hand, M is considered as a homogeneous algebra, the alphabet of M^r still consists of just 0 and 1.

The diagrams of Figure 4 will illustrate the distinction (for the case $r = 2$).

We now generalize the definition of direct product given above.

Let Γ be the index set of a family of similar algebras $A_\gamma = [\{S_{i,\gamma}\}, F]$, and define the (unrestricted) *direct product* $\prod_\Gamma A_\gamma$ as the algebra $[\{\mathcal{A}_i\}, F]$ where each $\mathcal{A}_i$ $(i \in I)$ is the set of all functions $a_i : \gamma \to a_i(\gamma) \in S_{i,\gamma}$ with

$$f_\alpha(a_{i(1,\alpha)}, \ldots, a_{i(n,\alpha)}) = a_{r(\alpha)} \in A_{r(\alpha)} ,$$

where

$$a_{r(\alpha)} : \gamma \mapsto f_\alpha(a_{i(1,\alpha)}(\gamma), \ldots, a_{i(n,\alpha)}(\gamma)). \tag{13}$$

As usual, the direct product $\prod_\Gamma A_\gamma$ is independent of the order and order of combination of the factors.

A special case of interest arises when $A_\gamma = A$ for all $\gamma \in \Gamma$. In this case we denote $\prod_\Gamma A_\gamma$ by A^Γ and refer to it as the Γ-th (direct) *power* of A.

Consider again the direct product $\prod_\Gamma A_\gamma = [\{\mathcal{A}_i\}, F]$ of the algebras $A_\gamma = [\{S_{i,\gamma}\}, F]$. For any $\delta \in \Gamma$ we define the *projection* P_δ as the set of mappings $\{p_{i,\delta}\}(i \in I)$

$$p_{i,\delta} : \mathcal{A}_i \to S_{i,\delta} : a_i \mapsto a_i(\delta). \tag{14}$$

As in [9, p. 120], where more is stated, we have

PROPOSITION 14. *For any direct product of heterogeneous algebras, P_δ is an epimorphism $\prod_\Gamma A_\gamma \to A_\delta$.*

6. WORD ALGEBRAS

In this section and the next, the concepts of *word algebra* and *free algebra* will be extended from homogeneous to heterogeneous algebras.

Consider the species of all algebras $A = [\{S_i\}, F]$ having a given set $F = \{f_\alpha\}$ of operations, with index-set I of phyla. For any given family

of non-void generator sets $X_i = (x_1^i, x_2^i, \ldots, x_{m_i}^i\}$, one for each $i \in I$, one can construct a (free) *word algebra* $W_{\mathscr{X}}(F)$, where the symbol $\mathscr{X}$ stands for the set of indexed sets X_i. If $\mathbf{m}$ stands for the "vector" of non-zero cardinal numbers $m_i = |X_i|$, then $W_{\mathscr{X}}(F)$, is determined up to isomorphism by $\mathbf{m}$; hence we can write $W_{\mathbf{m}}(F)$ for the isomorphism-type of the $W_{\mathscr{X}}(F)$ with given $\mathbf{m}$.

The construction of the word algebra $W_{\mathbf{m}}(F)$ is as follows [1, p. 141]: Define $W_0(i) = X_i$, and, recursively, $W_\rho(i) = \bigsqcup_{\alpha \in A_i} f_\alpha(u_1, u_2, \ldots, u_{n(\alpha)})$, where:

(i) A_i is the set of $f_\alpha \in F$ with values in the i-th phylum,

(ii) all $u_j = p_j(x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^i, \ldots, x_{m_i}^i, \ldots)$ are in $\bigsqcup_{k<\rho} W_k(i(j, \alpha))$,

and

(iii) some $u_j \in W_{\rho-1}(i(j, \alpha))$.

Then $\bigsqcup_{\rho \in N} W_\rho(i) = W_i$, and $W_{\mathbf{m}}(F)$ is the algebra $[\{W_i\}, F]$.

The elements of W_i are called W_i-polynomials (words), those of $W_\rho(i)$ are called W_i-polynomials *of rank* ρ. Equality in $W_{\mathbf{m}}(F)$ means formal identity: $x_j^i = x_k^l$ means $j = k$, $i = l$, and $f_\alpha(u_1, \ldots, u_{n(\alpha)}) = f_\beta(v_1, \ldots, v_{n(\beta)})$ means that $\alpha = \beta$ and $u_k = v_k$ for all $k = 1, 2, \ldots, n(\alpha) = n(\beta)$.

By construction, the sets X_i ($i \in I$) constitute a generating family of sets for $W_{\mathbf{m}}(F) \cong W_{\mathscr{X}}(F)$, where $\mathscr{X} = \{X_i\}$ and $m_i = |X_i|$.

EXAMPLE 6. We apply the above construction to the species $\mathscr{M}$ of state machines $\mathscr{M} = [\{S_1 = S, S_2 = A\}, \{\nu\}]$ of Example 4 of Section 1, and construct the (free) *word machine* generated by set of states $X_1 = \{s, t\}$ and set of input letters $X_2 = \{x, y, z\}$. In this example $\mathbf{m} = (2, 3)$, and we denote the word machine by $W_{\mathbf{m}}(\mathscr{M})$.

The states of this machine are the set of W_1-polynomials s, t, $\nu(s, x)$, $\nu(s, y)$, $\nu(s, z)$, $\nu(\nu(s, x), x)$, $\nu(\nu(s, x), y)$, etc.

NOTATION. For convenience we denote $\nu(\nu(\cdots(\nu(s, x_1), x_2), \ldots), x_n)$ in abbreviated form by $s \cdot x_1 x_2 \cdots x_n$.

Returning to Example 6, we can now give an explicit description of the states W_1 of $W_{\mathbf{m}}(\mathscr{M})$, namely,

$$W_1 = \{u\alpha \mid u \in X_1, \alpha \in X_2^*\},$$

where X_2^* is the free monoid generated by $X_2 = \{x, y, z\}$ under concatenation. We define $u \cdot e$, where e is the identity element of X_2^* (or "empty tape"), by $u \cdot e = u$. The input symbols of $W_{\mathbf{m}}(\mathscr{M})$ are the W_2-polynomials, namely, x, y, and z.

Note that for any (non-void) X_1 and X_2, W_1 is *never finite*, so that $W_{\mathfrak{m}}(\mathscr{M})$ is *never* a *finite* state machine.

PROPOSITION 15. *Let $A = [\{S_i\}, F]$ be an arbitrary F-algebra, and $W_{\mathscr{X}}(F) = [\{W_i\}, F]$ the word algebra generated by $\mathscr{X} = \{X_i\}$. Then any family $\Phi = \{\varphi_i\}$ of mappings $\varphi_i : X_i \to S_i$ $(i \in I)$ extends uniquely to a morphism $\bar{\Phi} : W_{\mathscr{X}}(F) \to A$.*

IDEA OF PROOF: Intuitively speaking, $\bar{\Phi}$ is the family of valuation maps which carries each W_i-polynomial

$$p(x_1^1,\ldots, x_{m_1}^1,\ldots, x_i^i,\ldots, x_{m_i}^i,\ldots)$$

into what one gets by replacing the (free) generators x_j^k by their values $\phi_k(x_j^k) \in S_k$. This results in an element

$$p(\phi_1(x_1^1),\ldots, \phi_1(x_{m_1}^1),\ldots, \phi_i(x_1^i),\ldots, \phi_i(x_{m_i}^i),\ldots) \tag{15}$$

of the phylum S_i.

As in [9, p. 122], we also have

PROPOSITION 16. *Any F-algebra $A = [\{S_i\}, F]$ generated by $\mathscr{X} = \{X_i\}$ is an epimorphic image of the word algebra $W_{\mathscr{X}}(F)$.*

PROOF: By Proposition 15, there is a morphism Φ: $W_{\mathscr{X}}(F) \to A$ with $\varphi_i(x_j^i) = x_j^i$ for all $x_j^i \in X_i$ and $i \in I$. Its image $\Phi(W_{\mathscr{X}}(F))$ is a subalgebra which contains all X_i and hence A (since $\mathscr{X}$ generates A). Therefore Φ is onto, completing the proof.

7. FREE ALGEBRAS

We now define and construct free *heterogeneous algebras*, proving an existence and uniqueness theorem which generalizes the results of [1, Ch. 6, §7], from homogeneous to general heterogeneous algebras. This generalizes a result of Higgins [9, §5], who considered the special case that Γ was a variety, so that $F_{\mathscr{X}}(\Gamma) \in \Gamma$. In this special case, condition F3 below is redundant, but it is needed to ensure uniqueness in our more general case. See also Grätzer [14].

Let Γ be a class of similar algebras $A = (\{S_i\}, F]$, and let $\mathscr{X} = \{X_i\}$ be a family of sets, each X_i having non-zero cardinality m_i. We then have the following

DEFINITION. An algebra $C = [\{U_i\}, F]$ is said to be the *free* Γ-algebra generated by $\mathscr{X}$, and denoted $C_{\mathscr{X}}(\Gamma)$, when

F1. C is generated by $\mathscr{X}$.

F2. For any algebra $A \in \Gamma$ and set of mappings $\phi_i : X_i \to S_i$ $(i \in I)$, each ϕ_i extends (uniquely) to a mapping $\bar{\phi}_i : U_i \to S_i$ such that $\bar{\Phi} = \{\bar{\phi}_i\}$ is a morphism $C \to A$.

F3. C is *minimal* subject to F1 and F2 in the following sense: if B is any other algebra satisfying F1 and F2, then C is an epimorphic image of B.

REMARK. Since $\mathscr{X}$ generates $C_{\mathscr{X}}(\Gamma)$, the extension $\bar{\Phi}$ postulated in F2 is unique by Proposition 8.

Note that the word algebra $W_{\mathscr{X}}(F)$ always satisfies F1 and F2; F3 is needed to eliminate this and other algebras that are "too large" and fail to satisfy the identities holding in all algebras of Γ. We now show how to construct $C_{\mathbf{m}}(\Gamma)$, beginning with the special case that $\Gamma = \{A\}$ is a single algebra.

We first fix generator sets $X_i = \{x_j{}^i\}$ of cardinalities $m_i = |X_i|$, and let $\mathscr{X} = \{X_i\}$. We then construct $C_{\mathbf{m}}(A)$, we consider the set Ψ of all phylum-preserving assignments $\Phi : x_j{}^i \mapsto a_j{}^i \in S_i$. We can think of Φ as having as "components" arbitrary mappings $\varphi_i : X_i \to S_i$, one for each phylum $i \in I$; this suggests writing $\Phi : \mathscr{X} \to S$. Let $A^{|\Psi|}$ be the direct product ($|\Psi|$-th power of A) of $|\Psi|$ copies of A, and let $\mu : \mathscr{X} \to A^{|\Psi|}$ map each $x_j{}^i \in X_i$ into the element $\mathbf{a}_j{}^i \in A^{|\Psi|}$ whose Φ-component is $a_j{}^i = \varphi_i(x_j{}^i)$; let $\mathscr{A} = \{\mathbf{a}_j{}^i\}$

PROPOSITION 17. *The subalgebra* $\bar{\mathscr{A}} = [\bar{X}_i, F]$ *of* $A^{|\Psi|}$ *generated by the* $\mathbf{a}_j{}^i$ *above is the free* A*-algebra* $C_{\mathbf{m}}(A)$ *with* m_i *generators in the* i*-th phylum.*

PROOF: By construction, F1 holds. Moreover the proof of F2 is only a little more difficult; the morphism desired is obtained by fixing attention on the appropriate Φ-component of $C_{\mathbf{m}}(A)$. Finally, if B is any other F-algebra which satisfies F1 and F2 for $\Gamma = \{A\}$, then the $|\Psi|$ morphisms $\bar{\Phi} : B \to A$ whose existence is asserted in F1 and F2 define an epimorphism $\bar{\Psi} : B \to C_{\mathbf{m}}(A)$ as constructed above.

REMARK. The fact that $\mathbf{m}$ determined $C_{\mathbf{m}}(A)$ up to isomorphism was evident above. We now show that it follows abstractly from conditions F1-F3. Indeed, let C and $\tilde{C}$ be free F-algebras generated by $\mathscr{X}$ and $\tilde{\mathscr{X}}$ whose corresponding phyla X_i and $\tilde{X}_i$ are bijective for bijections β_i (have the same cardinality, so that $\mathbf{m} = \tilde{\mathbf{m}}$), then the extension of $\{\beta_i\}$ will be an isomorphism $\beta : C \to \tilde{C}$. Thus we may also refer the isomorphism-type of $C_{\mathscr{X}}(\Gamma)$ as *the free* Γ-algebra with m_i generators in its i-th phylum, and denote it by $C_{\mathbf{m}}(\Gamma)$, where $\mathbf{m}$ is the "vector" whose i-th component is the non-zero cardinal number m_i.

By Proposition 16, $\mathscr{X}$ is an epimorphism image of the word algebra $W_{\mathscr{X}}(F) = [\{W_i\}, F]$, whence by Proposition 11 $\mathscr{X} = W_{\mathscr{X}}(F)/\mathscr{E}$ for some congruence $\mathscr{E}$ of W_i-polynomials defined in Section 6

$$p(x_1^1,\ldots, x_{m_i}^1,\ldots, x_1^i,\ldots, x_{m_i}^i,\ldots),$$

where the $x_j^i \in X_i$ ($j = 1, 2,\ldots, m_i$) freely generate the i-th phylum. For polynomials p_1, $p_2 \in W_{\mathscr{X}}(F)$, we see from definition of $\mathscr{X}$ that p_1 and p_2 are *equal as functions* $\prod_I S_i^{m_i} \to S_i$; i.e., whenever p_1 and p_2 have the same value for *every* family of valuations $\Phi = \{\phi_i : X_i \to S_i\}$ of the generators $x_j^i \in X_i$ to the $a_j^i \in S_i$. By definition, these equivalences $p_1 \equiv p_2 \pmod{\mathscr{E}}$ are *precisely the identities valid in* A.

Proposition 17 represents $C_{\mathbf{m}}(A)$ as a subalgebra of a power (product of copies) of A, one for each set of valuations $\Phi = \{\phi_i\}$. We thus have the following

COROLLARY. *If* $A = [\{S_i\}, F]$ *is a* "finite" *algebra (i.e., A has a* "finite" *number of phyla* S_i *each of which is of* "finite" *order* $|S_i|$) *and* $\mathbf{m} = (m_i)$ *is a finite vector of* $m_i \in P$, *then the free A-algebra* $C_{\mathbf{m}}(A)$ *is a (finite) subalgebra of the* "finite" *algebra* $A^{[\Pi_I |S_i|^{m_i}]}$.

We shall now generalize Proposition 17 to any set Γ of F-algebras $A_\gamma = A(\gamma)$. We first fix generator sets $X_i = \{x_j^i\}$ of cardinalities $m_i = |X_i|$, and let $\mathscr{X} = \{X_i\}$ as before. Proposition 17 associates with $\mathscr{X}$ and each $A(\gamma) \in \Gamma$ a free algebra $C_\gamma = C_{\mathscr{X}}(A(\gamma))$ with generator sets X_i, and a monomorphism, $\bar{\Phi}_\gamma$ from C_γ to a power (product of copies) of $A(\gamma)$. We now let γ range over Γ; the result is a mapping Φ from $\mathscr{X}$ to $\prod_\Gamma A(\gamma)^{\mathbf{m}}$. Let $\mathscr{T} = \{T_i\}$ be the *F-subalgebra* generated by $\Phi(\mathscr{X})$ in this product of copies of the $A(\gamma)$; we shall call this $C_{\mathscr{X}}(\Gamma)$, and designate its isomorphism-type by $C_{\mathbf{m}}(\Gamma)$. We then have

PROPOSITION 18. *Let there be given a set* Γ *of F-algebras* $A(\gamma)$, *and indexed generator sets* X_i *of cardinalities* m_i ($i \in \Gamma$), *one for each phylum. Then there exists an F algebra* $C_{\mathbf{m}}(\Gamma)$ *which satisfies conditions* F1-F3; *it is a subalgebra of products of copies of the* $A(\gamma) \in \Gamma$.

We now ask: given a class Γ of similar algebras, when will the free algebra $C_{\mathbf{m}}(\Gamma)$ be a member of Γ? Proposition 18 establishes $C_{\mathbf{m}}(\Gamma)$ as a subalgebra of a direct product of powers of the $A(\gamma) \in \Gamma$, as in [9, p. 124],

COROLLARY 1. *A sufficient condition for* $C_{\mathbf{m}}(\Gamma) \in \Gamma$ *is that* Γ *be closed under the taking of subalgebras and direct products.*

COROLLARY 2. *The identities valid in* $C_{\mathbf{m}}(\Gamma)$ *are precisely the identities in* m_i *or fewer symbols for elements from the i-th phylum which are valid in all* $A_\gamma \in \Gamma$.

In the more general context of Σ-algebras, Grätzer [14] has proved a much stronger result: if J is any set of identities meaningful for the operations of F, then among all F-algebras satisfying all identities of J and having m_i or fewer generators of the i-th phylum, $C_{\mathbf{m}}(\Gamma)$ is one of which all other are subalgebras of epimorphic images.

8. FREE MACHINES

In this section we use Proposition 17 to construct the *free* (state) *machine* $F_{\mathscr{X}}(M)$ generated by one state letter s, and two input letters x and y (i.e., $\mathscr{X} = \{X_1, X_2\}$ where $X_1 = \{s\}$, and $X_2 = \{x, y\}$), where $M = [\{S, A\}, \{\nu\}]$ is the state machine specified by the state diagram of Figure 5.

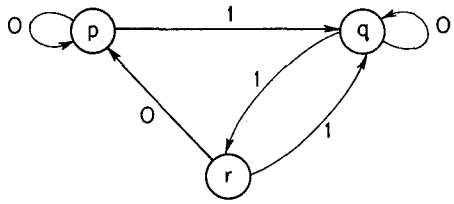


FIG. 5. Another state machine.

By Proposition 17, $F_{\mathscr{X}}(M)$ is a subalgebra of $M^{M^{\mathbf{m}}}$, where the vector $\mathbf{m} = (1, 2)$ for our example. Thus $F_{\mathscr{X}}(M)$ is a subalgebra of the dirct power M^{12}.

We now determine the state and input phyla of $F_{\mathscr{X}}(M)$ by computing a *truth-table* (Table I) which displays the values in M of the polynomials of the word machine generated by $\mathscr{X}$ (Example 6) for all possible assignments (valuations) Φ of the free input symbols x and y to the input symbols $\{0, 1\}$ of M and the free state symbol s to the states $\{p, q, r\}$ of M.

From Table I we see that $F_{\mathscr{X}}(M)$ has 9 states, which we can conveniently designate by s, $s.x$, $s.y$, $s.xx$, $s.xy$, $s.yx$, $s.yy$, $s.xyx$, $s.xyy$; and two input letters x and y. The state diagram for $F_{\mathscr{X}}(M)$ is given in Figure 6.

From either the truth-table of the state diagram of $F_{\mathscr{X}}(M)$, we can ascertain by inspection all *identities* holding in the given machine M. One such identity is

$$s.xxx = s.x;$$

TABLE I

TRUTH-TABLE FOR $F_{\mathfrak{X}}(M)$

Valuations s	p				q				r			
(x, y)	00	01	10	11	00	01	10	11	00	01	10	11
State Polynomials												
s	p	p	p	p	q	q	q	q	r	r	r	r
$s.x$	p	p	q	q	q	q	r	r	p	p	q	q
$s.y$	p	q	p	q	q	r	q	r	p	q	p	q
$s.xx$	p	p	r	r	q	q	q	q	p	p	r	r
$s.xy$	p	q	q	r	q	r	p	q	p	q	q	r
$s.yx$	p	q	q	r	q	p	r	q	p	q	q	r
$s.yy$	p	r	p	r	q	q	q	q	p	r	p	r
$s.xxx = s.x$	p	p	q	q	q	q	r	r	p	p	q	q
$s.xxy = s.y$	p	q	p	q	q	r	q	r	p	q	p	q
$s.xyx$	p	q	r	r	q	p	q	q	p	q	r	r
$s.xyy$	p	r	q	q	q	q	p	r	p	r	q	q
$s.yxx = s.xyy$	p	r	q	q	q	q	p	r	p	r	q	q
$s.yxy = s.xyy$	p	r	q	q	q	q	p	r	p	r	q	q
$s.yyx = s.x$	p	p	q	q	q	q	r	r	p	p	q	q
$s.yyy = s.y$	p	q	p	q	q	r	q	r	p	q	p	q
$s.xyxx = s.yx$	p	q	q	r	q	p	r	q	p	q	q	r
$s.xyxy = s.yy$	p	r	p	r	q	q	q	q	p	r	p	r
$s.xyyx = s.xx$	p	p	r	r	q	q	q	q	p	p	r	r
$s.xyyy = s.xy$	p	q	q	r	q	r	p	q	p	q	q	r
Input Polynomials												
x	0	0	1	1	0	0	1	1	0	0	1	1
y	0	1	0	1	0	1	0	1	0	1	0	1

this identity states that, no matter what state the machine M is in initially, the application of a tape consisting of 3 repetitions of either one of the input symbols 0, 1 results in the same final state as applying a tape consisting of a single occurrence of that same symbol.

As another example, consider the identity

$$s.yyx = s.x,$$

which holds in M (and hence in $F_{\mathfrak{X}}(M)$). This identity states that, for any initial state, the processing of a tape consisting of any input symbol repeated twice followed by any other input symbol (perhaps the same one) results in the same final output symbol.

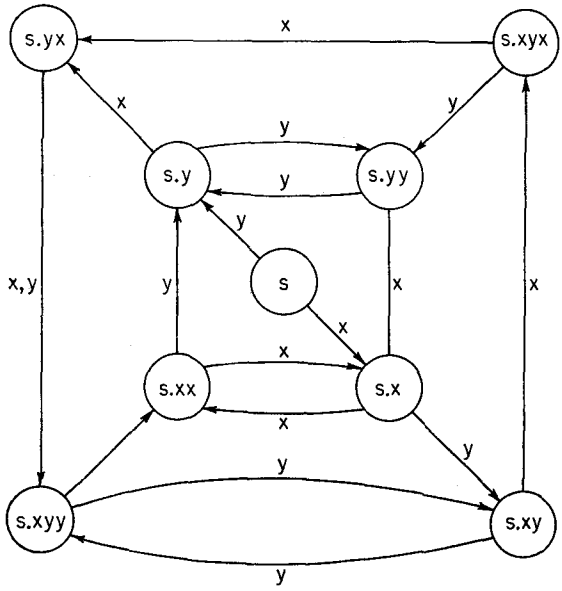


FIG. 6. State diagram of $F_{\mathfrak{X}}(M)$.

In discussing Example 6, we observed that the word machine is never a *finite* state machine. For the case of free machines the situation is quite different. Specifically, from the representation of the free algebra $F_{\mathfrak{m}}(\Gamma)$ provided by Proposition 17, we have the following

PROPOSITION 19. *Let Γ be a finite set of finite state machines $M(\gamma) = [\{S(\gamma), A(\gamma)\}, \{\nu, \zeta\}]$. Then the free machine $F_{\mathfrak{m}}(\Gamma)$ with m generators of the state phylum, and m' generators of the input phylum (m, m' finite) is a (finite) submachine of the finite state machine*

$$\prod_{\Gamma} A(\gamma)^{|S(\gamma)|^m |A(\gamma)|^{m'}}.$$

REFERENCES

1. G. BIRKHOFF, "Lattice Theory", 3rd ed., American Mathematical Society, Providence, R. I., 1967.
2. G. BIRKHOFF, On the Structure of Abstract Algebras, *Proc. Cambridge Philos. Soc.* **31** (1935), 433–54.
3. G. BIRKHOFF, Universal Algebra, *Proc. First Canad. Math. Congr.* (1945), 310–26.
4. P. M. COHN, "Universal Algebra", Harper & Row, New York, 1965.
5. S. EILENBERG AND J. B. WRIGHT, Automata in General Algebras, *Information and Control* **11** (1967), 452–70.

6. A. GINZBURG, *Algebraic Theory of Automata*, Academic Press, New York, 1968.
7. G. GRÄTZER, *Universal Algebra*, Van Nostrand Reinhold, New York, 1969.
8. J. HARTMANIS AND R. E. STEARNS, *Algebraic Structure Theory of Sequential Machines*, Prentice-Hall, Englewood Cliffs, N. J., 1966.
9. P. J. HIGGINS, Algebras with a Scheme of Operators, *Math. Nachr.* **27** (1963), 115–32.
10. G. BIRKHOFF AND S. MACLANE, *A Survey of Modern Algebra*, 3rd ed., Macmillan, New York, 1965.
11. E. F. MOORE (ed.), *Sequential Machines: Selected Papers*, Addison-Wesley, Reading, Mass., 1964.
12. M. A. ARBIB (ed.), *The Algebraic Theory of Machines, Languages, and Semigroups*, *Academic Press*, New York, 1968.
13. E. R. CAIANIELLO, (ed.), *Automata Theory*, Academic Press, New York, 1966.
14. G. GRÄTZER, *Trans. Amer. Math. Soc.* **135** (1969), 517–42.