The Shuffle Bialgebra

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Abstract

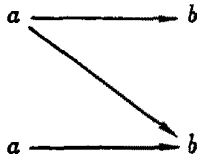
The *shuffle* multiplication and the *cut* comultiplication, a generalized car–cdr pairing, form a bialgebra. The *concatenation* multiplication, sometimes called tensor product, and the *spray* comultiplication form another bialgebra. The concatenation–spray bialgebras are the free bialgebras in the category of precise, graded bialgebras over a semiadditive symmetric monoidal category. The shuffle–cut bialgebras are the cofree bialgebras in the same category of bialgebras. These categories include many of the settings of interest in the theories of formal languages and the theories of distributed, concurrent and parallel computation. We analyze the *marked shuffle*, of interest in theories of distributed computing, in terms of its resolutions into the cofree shuffle–cut bialgebra.

1. Introduction

Shuffling, in the sense of preparing a deck of playing cards for a round of play, is a combinatorial process with uses in theory of bialgebras, [Sweedler]. Shuffling is important in our current understanding of many aspects of formal languages and in theories of distributed and concurrent processes. The shuffle operation is also called the *Hurwitz product* in [Salomaa & Soittola]. The dissertation [Gischer] attributes the shuffle operation on formal languages to S. Ginsburg's 1966 monograph. But [Ginsburg] attributes formal language shuffle to [Ginsburg & Spanier]. Without surprise then, [Spanier], shuffling in vector spaces is related to homology. Perhaps the earliest homological paper mentioning shuffle is [Eilenberg & MacLane]. Also see [MacLane 1963]. While I have not surveyed the older literature, surely there are more ancient mathematical papers on shuffling

decks of cards.

V. Pratt has advocated modeling concurrency with partial orders, for which see [Pratt]. Also see Gischer's dissertation and [Gaifman & Pratt] for other works in this development. Here I follow the older tradition of representing a partial order by a distribution of the total orders respecting the given partial order. The term *distribution* has a particular technical meaning in this paper, related to one's knowledge of a particular situation. For the moment we may consider a distribution to be a set. For example, the following labeled Hasse diagram denotes a pomset (partially ordered multiset) in Pratt's terminology.



In this paper we simply consider the distribution consisting of the two words,

$$aabb$$
$$abab$$

which are all and only the total orders respecting this particular pomset.

Another motivating example is the data flow process server called a *merger*, which nondeterministically shuffles together *datons* from two streams of these datons. In the picture



there are three datons on the red (r) input line and two datons on the blue (b) input line. The full distribution of nondeterministic output possibilities from the merger is

$$r_3r_2r_1b_2b_1$$
$$r_3r_2b_2r_1b_1$$
$$\vdots$$
$$r_3b_2r_2r_1b_1$$
$$\vdots$$
$$b_2b_1r_3r_2r_1.$$

That is, the full distribution is all possible shuffles of the red datons with the blue datons.

So long as the streams are finite, this process is isomorphic to shuffling cards. We must, of course, include the most imperfect shuffles in which all cards of one color either precede or else follow the cards of the other color. Further, we must consider strings of cards rather than decks of cards. A string is obtained by selection with replacement. Strictly speaking, selecting a card from a deck is selection without replacement. In this paper, "deck" is simply a synonym for "string."

I shall note here that so–called fairness considerations, for which see [Francez], have no direct role in this paper. An analysis of finite fairness is perhaps possible in this framework.

Whatever the particular application may be, *shuffle* is to be viewed as a binary operation in an appropriate algebra, where here I use the term "algebra" in the sense of universal algebra. At the least, the algebra needs to be an algebra of distributions. While we will become more abstract later on, one should have in mind the primary examples of distributional algebras as abelian monoids, also called commutative monoids, written additively as $(M, +, 0)$. A fundamental computer science example of an abelian monoid is obtained when the carrier is the power set of some given set, "+" is set union, and "0" is the empty set.

The tensor product of abelian monoids always exists by the universal considerations of bilinear maps given, for example, in [MacLane 1971]. We denote the tensor product of $(M, +, 0)$ and $(N, +, 0)$ by $M \otimes N$. When M and N are freely generated by X and Y respectively, one may view $M \otimes N$ as freely generated by the product set $X \times Y$. The collection of all finite subsets of a given set X will be called the *finitepowerset* of X. Thus when M and N are finitepowersets of X and Y respectively, $M \otimes N$ is the finitepowerset of $X \times Y$.

For additional mathematical background in the case of finitely generated abelian monoids, see [Redei]. In general, algebraic categories of abelian monoids are commutative algebraic categories. In such categories a tensor product, universal for bilinear maps, always exists and has the expected properties. See any of [Manes 1976, Eilenberg & Kelly 1966 pp. 548–550, Pfender 1974] for further discussion of this setting, attributed by Eilenberg and Kelly to Linton. Here we only mention that if the sum of every cardinality is defined and homomorphisms are completely additive in the sense of respecting sums of every cardinality, then when M and N are in fact the power sets of X and Y respectively, in this setting $M \otimes N$ is the power set of $X \times Y$.

Thus various notions of infinite nondeterminism may be obtained as well through considerations of infinite addition. One approach to infinite additions to serve as nondeterministic distributions may be found in [Manes 1986a, 1986b and *these proceedings*]. As of this writing there appear to be several different approaches to the questions of infinite nondeterminism. All of these must have the same finite structure, and it is therefore this case of finite nondeterminism which we consider in this paper.

The distributional algebras of abelian monoids with *finite* sum capture every notion of finite nondeterminism known to me. For example, the distribution of the pomset previously noted would be written

$$aabb + abab$$

as an element of the finitepowerset of strings over the alphabet $\{a, b\}$.

All this leads to defining shuffle as a particular *linear* function

$$\sigma\colon T \otimes T \to T$$

when the distributional algebra T is equipped with an appropriate notion of multiplication. Shuffling only makes sense in the context of an appropriately free multiplication and an appropriately cofree comultiplication which together describes total orderings in T.

The multiplication needed is exactly the tensor algebra over a distributional algebra M. In formal language theoretic terms, T_0 is the collection of distributions over the null word, $T_1 \cong M$ is the collection of distributions over words of length one, and in general $T_{n+1} = M \otimes T_n$ is the collection of distributions over words of length $n > 0$. The multiplication in the tensor algebra respects word length,

$$\kappa\colon T_m \otimes T_n \to T_{m+n},$$

with $(\alpha \otimes \beta)\kappa = \alpha\beta$, using the formal language notation for concatenation on the right side of the equation. Notice also that we write function application on the right. Now let $T = \bigsqcup_{n\geq 0} T_n$, where "$\sqcup$" denotes the categorical coproduct in some category of distributional algebras. The foregoing provides a multiplication

$$\kappa\colon T \otimes T \to T$$

which we might as well call *concatenation.*

There is one more operation we need to complete the picture, a comonoid operation, herein called a coalgebra operation. Using formal language terminology, let α be a distribution of strings. We may decompose α into all possible prefixes and corresponding suffixes to form the distribution of tensor pairs,

$$\alpha\Gamma = \sum_{\beta\gamma=\alpha} \beta \otimes \gamma$$

where $\beta\gamma = (\beta \otimes \gamma)\kappa$. Thus we have the linear map

$$\Gamma\colon T \to T \otimes T$$

which we shall call *cut*, although the cut in card play according to Hoyle requires reversing the order of the prefix and suffix. The cut operation used here may be viewed as a generalized car-cdr pair.

Shuffle and cut form a bialgebra with a certain universal embedding property. This property appears to be of value in understanding distributed and concurrent computation. The essence of the argument is obtained by thinking about card shuffling.

Imagine having two decks of cards of any finite size. The deck in the left hand is red backed; the deck in the right hand is blue backed. The goal is to shuffle the two decks together, preserving the original order of each deck separately, as is traditionally done by card players. If the decks are large, shuffling the decks may have to be done in parts. Set down both decks. Take any number of cards from the top of the red deck into the left hand. Similarly, take any number of cards from the blue deck into the right hand. Shuffle together these prefixes of the two decks. Lay the shuffled deck carefully to the side. Now shuffle together the remainder of the two original decks. Place the previously shuffled portion back on top of the shuffled remainders.

Could this shuffle be obtained by just shuffling the two decks, assuming one has big enough hands to hold the cards? Yes, the two means of shuffling—either in parts or all together—can

always give identical results. For any shuffled result with a total of n cards, pick any prefix of p cards. This shuffled prefix of p cards can be obtained in exactly one way by, from the original two decks of red and blue cards, selecting a prefix of each deck of the appropriate number of cards and shuffling these in exactly one way to form the desired prefix of p cards.

Irrespective of the technicalities to follow, this is what one means by the shuffle and cut bialgebra. One may always use the intuition of decks of cards to aid in following the technical presentation.

This bialgebra is *graded* by length. Furthermore, cut is *precise*, which just means that single cards remain just exactly the same single cards and a deck of two or more cards has nontrivial cuts. The shuffle and cut bialgebra is couniversal in the sense that any other precise, graded bialgebra maps to the shuffle and cut bialgebra in just one way once the mapping into the individual cards is fixed.

The application in the ultimate section of the paper is *marked shuffle*. Marked shuffle arises in attempting to formalize the properties of three or more distributed processes intercommunicating. Rather than explain the origins of the problem, perhaps first explicated in [Riddle 1972, Riddle 1979], consider again the two decks of cards, one in each hand. This time, however, there are black marker cards at various positions throughout each of the red and blue decks. The marked shuffle rule requires shuffling the red and blue decks up to the first marker, keeping the markers, and repeating on each succeeding segment between markers. The resulting distribution of marked shuffles is a subdistribution of the distribution obtained if one forgot about the marker cards. In effect, but not in detail, this is the couniversal property of the shuffle and cut bialgebra.

As the patient reader will discover, this paper is written to preserve as much generality for as long as possible. This results in repeatedly returning to the various topics. Perhaps the paper should be subtitled "A Spiral Approach."

2. Shuffles as Permutations

A permutation on n letters is a bijection on a set of cardinality n. We take the sets of letters underlying permutations to be sets of positive integers equipped with the usual order on the integers. In the beginning, the permutations of n letters are taken as permuting the set $\{1, 2, \ldots, n\}$. A (k, ℓ)-*shuffle*, π, is a permutation on $k + \ell$ letters in which the first k letters and the last ℓ letters have stable orderings under the permutation:

$$i < j \leq k \text{ implies } i\pi < j\pi,$$
$$k < i < j \text{ implies } i\pi < j\pi.$$

Fix natural numbers k, ℓ and a (k, ℓ)-shuffle π to consider the first p letters of the result of shuffling by π for some $p \leq n = k + \ell$. For some k_1, ℓ_1 such that $k_1 + \ell_1 = p$, $\{1, 2, \ldots, k_1\} \cup \{k + 1, k +$

$2, \ldots, k+\ell_1\}$ is the set of letters with image $\{1, 2, \ldots, p\}$ under π. The values k_1, ℓ_1 may be zero depending upon π and the choice of p. Similarly, for some k_2, ℓ_2 such that $k_2 + \ell_2 = q$ with $p+q = n$, we have $k_1 + k_2 = k$, $\ell_1 + \ell_2 = \ell$ and the last q letters of the result of shuffling by π is the image of the set $\{k_1+1, k_2+2, \ldots, k\} \cup \{k+\ell_1+1, k+\ell_1+2, \ldots, n\}$ where $\{k_1+1, k_1+2, \ldots, k\}$ has cardinality k_2 and $\{k+\ell_1+1, k+\ell_1+2, \ldots, n\}$ has cardinality ℓ_2. Thus for each $p \leq n$ there is a (k_1, ℓ_1)–shuffle π_1, uniquely depending upon p and π, such that the first p letters of the result of shuffling by π is identical to the result of shuffling $\{1, 2, \ldots, k_1\} \cup \{k+1, k+2, \ldots, k+\ell_1\}$ by π_1. Similarly, for each q such that $p+q = n$ there is a (k_2, ℓ_2)–shuffle π_2 for which the last q letters of the result of shuffling by π is identical to the result of shuffling $\{k_1+1, k_1+2, \ldots, k\} \cup \{k+\ell_1+1, k+\ell_1+2, \ldots, n\}$ by π_2.

Let $\mathbf{k}_1$ denote the finite sequence $(1, 2, \ldots, k_1)$, $\mathbf{k}_2$ denote the finite sequence $(k_1+1, k_1+2, \ldots, k)$, ℓ_1 denote the finite sequence $(k+1, k+2, \ldots, k+\ell_1)$, and ℓ_2 denote the finite sequence $(k+\ell_1+1, k+\ell_2+2, \ldots, n)$. Using a now obvious notation, we have

$$(1, 2, \ldots, n) = \mathbf{k}_1 \cdot \mathbf{k}_2 \cdot \ell_1 \cdot \ell_2.$$

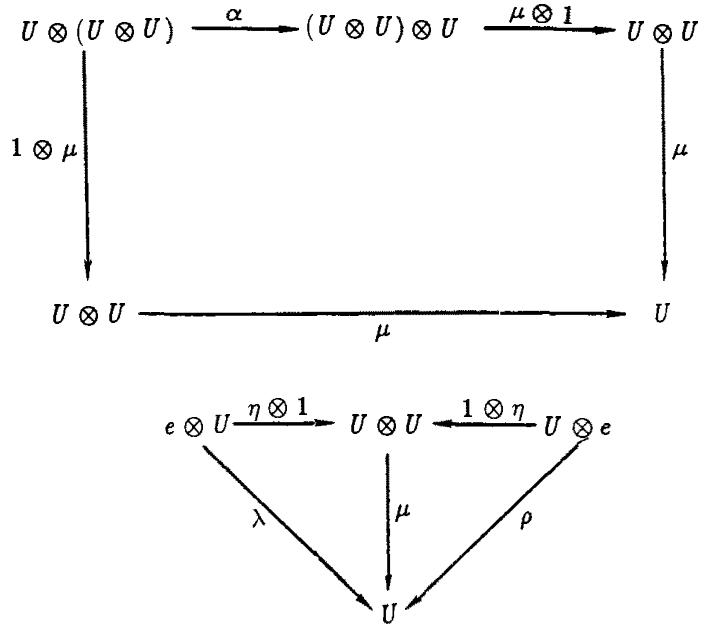
There is a composition of elementary transpositions t: $\mathbf{k}_1 \cdot \mathbf{k}_2 \cdot \ell_1 \cdot \ell_2 \rightarrow \mathbf{k}_1 \cdot \ell_1 \cdot \mathbf{k}_2 \cdot \ell_2$ which interchanges the positions of the subsequences $\mathbf{k}_2$ and ℓ_1. Write $\pi_1 \dot{+} \pi_2$ for the permutation on n letters which acts as π_1 on the first p letters and acts as π_2 on the last q letters. We have

$$\pi = t(\pi_1 \dot{+} \pi_2).$$

3. Algebras, Coalgebras and Bialgebras in a Symmetric Monoidal Category

Let $(\mathrm{M}, \otimes, e, \alpha, \lambda, \rho)$ be a monoidal category in which e is the unit object with respect to the family of left and right unitary isomorphisms, λ and ρ respectively, and α is the family of associativity isomorphisms. For details see [MacLane 1971]. We further posit that M is symmetric via the family of *twist* isomorphisms, $\tau_{U,V}$: $U \otimes V \rightarrow V \otimes U$, for all pairs of objects U, V of M. All these isomorphisms are *fixed*, once and for all, as the data of a symmetric monoidal category. While we shall endeavor to suppress this detail whenever possible, one understands that the choice of isomorphisms does make a difference. Thus this data always lies in the background of the presentation. For more about this technicality, see [MacLane 1971].

3.1 Definition. An M–*algebra* is a monoid in M, an object U of M equipped with two arrows μ: $U \otimes U \rightarrow U$ and η: $e \rightarrow U$ such that the diagrams



commute, where 1 denotes the appropriate identity arrow.

A morphism of M–algebras is an arrow $h\colon U \to V$ such that

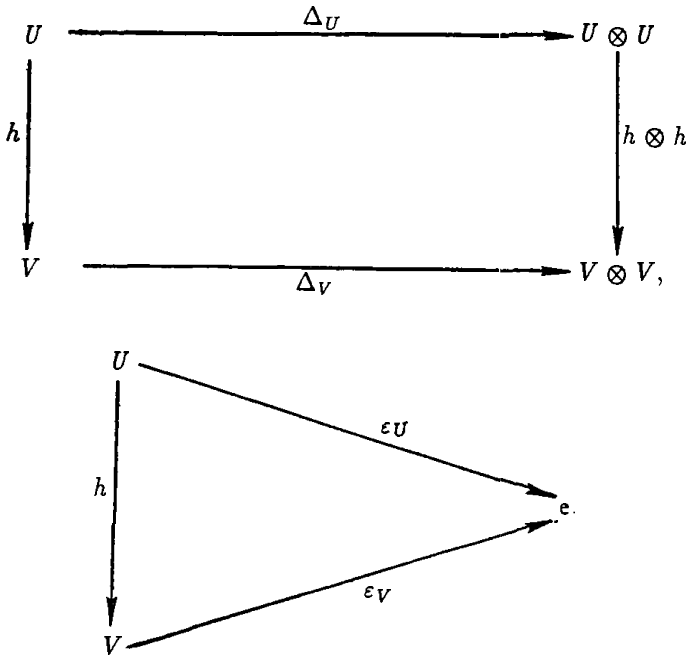
$$\mu h = (h \otimes h)\mu'\colon U \otimes U \to V$$

and

$$\eta h = \eta'\colon e \to V.$$

3.2 Definition. An M–*coalgebra* is a comonoid in M, obtained whenever the previous diagrams commute with the direction of the arrows reversed. We typically write triples such as $(U, \Delta_U, \varepsilon_U)$ for an M–coalgebra.

A morphism of M–coalgebras is an arrow $h\colon U \to V$ such that the diagrams



commute.

If U, V are M–coalgebras, the *tensor product* of U and V is the object $U \otimes V$ equipped with structure maps

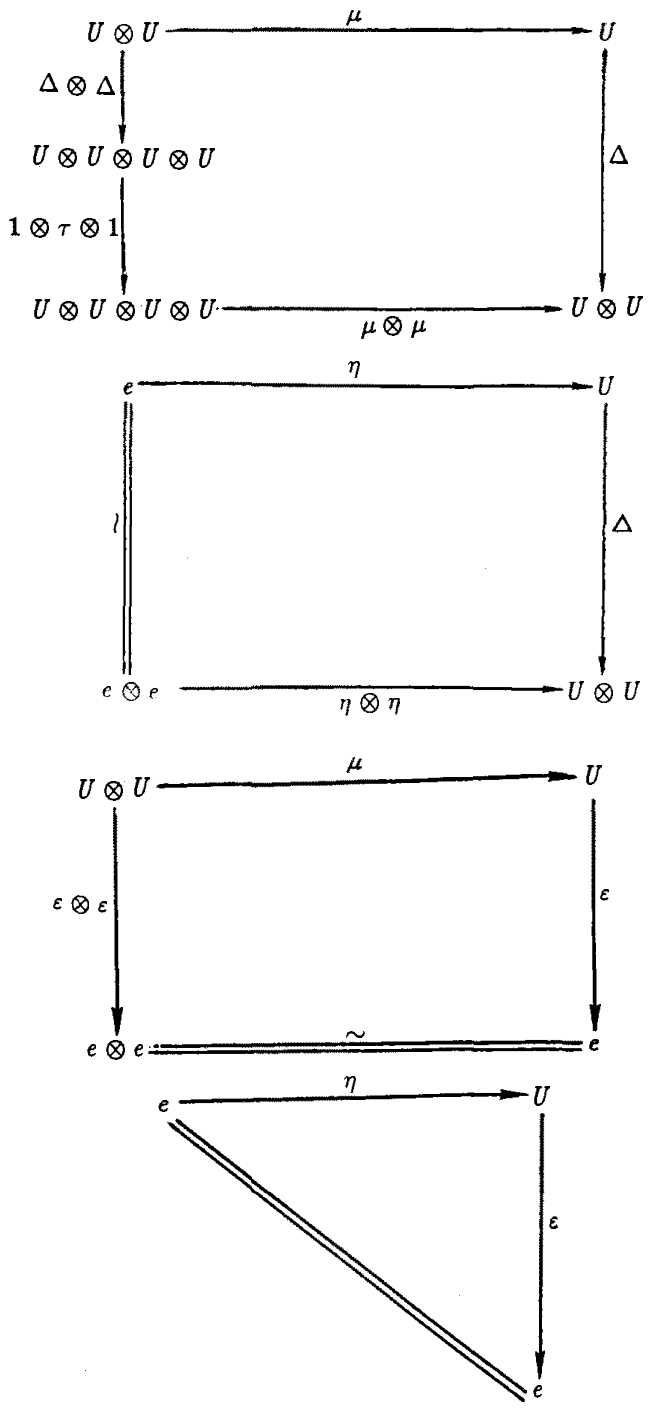
$$\Delta_{U\otimes V} = (\Delta_U \otimes \Delta_V)(1 \otimes \tau \otimes 1),$$

$$\varepsilon_{U\otimes V} = \varepsilon_U \otimes \varepsilon_V,$$

making $U \otimes V$ an M–coalgebra.

3.3 Example. For each set X write $X \cdot 2$ for the collection of all finite subsets over X. Let $\bullet$ be any singleton set to write 2 for $\bullet \cdot 2 = \{0, 1\}$. These are the objects of the category **FSub**, which is symmetric monoidal with 2 as the unit object. Let $T = \Sigma^* \cdot 2$, with Σ^* the collection of all finite strings over Σ, to observe that κ: $T \otimes T \to T$ as concatenation and $0\eta = \phi, 1\eta$ the singleton set containing the null string, also denoted by 1, provides T with an **FSub**–algebra structure. Further, observe that Γ: $T \to T \otimes T$ and ε: $T \to 2$ given by $1\eta\varepsilon = 1$, $\gamma\varepsilon = 0$ for $\gamma \neq 1\eta$, provides T with an **FSub**–coalgebra structure.

3.4 Definition. An M–*bialgebra* in a symmetric monoidal category M is an object U which is both an M–algebra and an M–coalgebra with the property that the monoid structure arrows are comonoid morphisms of the M–coalgebra structure and, equivalently, the comonoid structure arrows are monoid morphism of the M–algebra structure. Explicitly, the following diagrams must all commute for the M–algebra (U, μ, η) and the M–coalgebra (U, Δ, ε) to form an M–bialgebra:



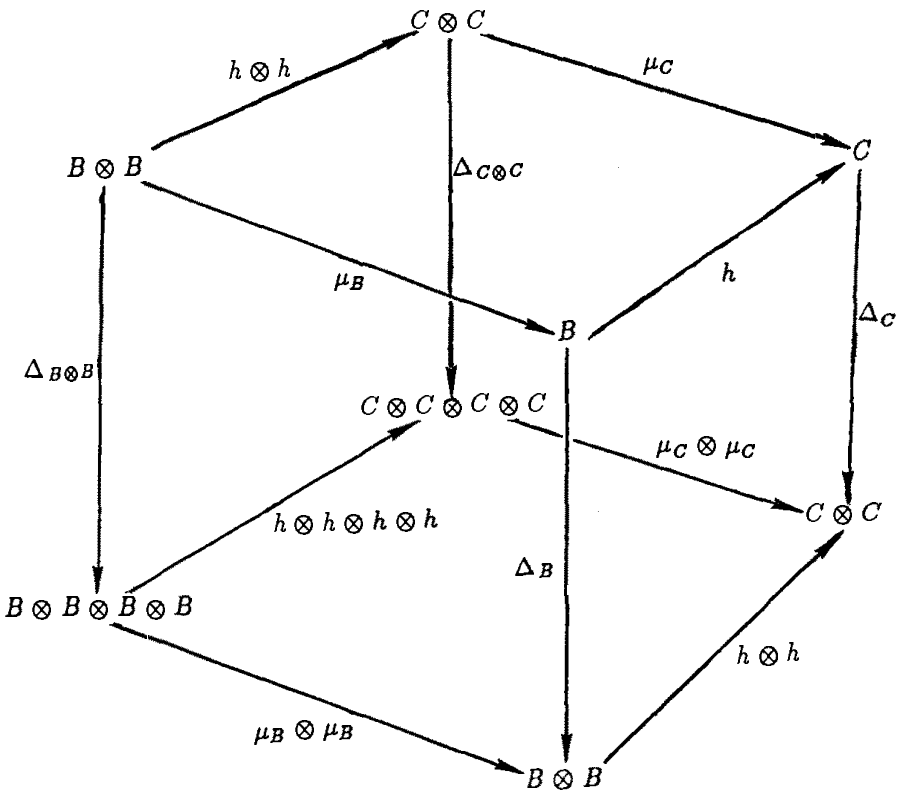
where we have denoted the isomorphism between $e \otimes e$ and e which is guaranteed by the left and right unitary isomorphisms as $e \otimes e \cong e$.

3.5 Definition. An M-*bialgebra morphism*

$$h\colon (B, \mu_B, \Delta_B, \eta_B, \varepsilon_B) \to (C, \mu_C, \Delta_C, \eta_C, \varepsilon_C)$$

is a morphism of M which is simultaneously an M–algebra morphism $h\colon (B, \mu_B, \eta_B) \to (C, \mu_C, \eta_C)$ and an M–coalgebra morphism $h\colon (B, \Delta_B, \varepsilon_B) \to (C, \Delta_C, \varepsilon_C)$. When and only when h is an

M–bialgebra morphism the cube diagram



commutes. We refer to this cube diagram later in the paper.

3.6 Definition. Let N denote the natural numbers, $\mathrm{N} = \{0, 1, 2, \ldots\}$. An N–*graded object* over M is a function from N to the objects of M. A *morphism of* N–*graded objects* is a family of M–arrows indexed by N so as to preserve the N–grading. Explicitly, if $\{U_n | n \in \mathrm{N}\}$ and $\{V_n | n \in \mathrm{N}\}$ are N–graded objects over M, a family of M–arrows $\{f_n \colon U_n \to V_n | n \in \mathrm{N}\}$ is a morphism of these N–graded objects.

3.7 Definition. The M–(*tensor algebra*) over object V in M is the free monoid over V in M, when it exists. That is, the M–(tensor algebra) is an M–algebra freely generated by V. The recursion equations

$$T_0 = e,$$

$$T_{n+1} = V \otimes T_n,$$

define the N–*graded tensor* of V as an N–graded object over M. For each $m, n \in \mathrm{N}$ there is a unique associativity isomorphism given by the data of the monoidal categoy M from $T_m \otimes T_n$ to T_{m+n}, unique via the coherence conditions on the associativity isomorphisms. Denote these particular

associativity isomorphisms by

$$\kappa_{m,n}\colon T_m \otimes T_n \to T_{m+n}.$$

3.8 Proposition. If the monoidal category M has countable coproducts, there is a tensor algebra over every object in M.

Proof: Let $T = \bigsqcup_{n\geq 0} T_n$ be the countable coproduct of the N–graded tensor of V. Let $\kappa\colon T\otimes T \to T$ be the coproduct of the morphisms $\kappa_{m,n}$. The unit structure map $\eta\colon e \to T$ is simply the injection of $e = T_0$ into T. The proof that T is a free monoid in M is standard. □

When M has countable coproducts, the tensor algebra $T = \bigsqcup_{n\geq 0} T_n$ has graded components T_n which inject into T but cannot, in general, be retrieved. Despite this handicap, we may define the graded components of $T \otimes T$ in the usual fashion as

$$(T \otimes T)_n = \bigsqcup_{p+q=n} (T_p \otimes T_q)$$

which injects into $T\otimes T$ provided tensor ($\otimes$) distributes over coproduct ($\sqcup$). When M has countable coproducts and tensor distributes over these coproducts the M–(tensor algebra) may be viewed as the N–*graded tensor algebra* with components of the algebra structure map the coproducts of the $\kappa_{p,q}$ with $p + q = n$,

$$\kappa_n\colon (T \otimes T)_n \to T_n,$$

$$\kappa_n = \bigsqcup_{p+q=n} \kappa_{p,q}.$$

Now the monoidal category M we are considering is also symmetric. Therefore, for each permutation π on n letters, as π is a composition of transpositions, there is a natural isomorphism on T_n formed solely of identity arrows and twist isomorphisms by composition and tensor,

$$\pi_n\colon T_n \to T_n.$$

For example, $(1 \otimes \tau \otimes 1)\colon T_4 \to T_4$ transposes the two center factors and is induced by the transposition (2 3).

3.9 Definition. A *shuffle* $\sigma_\pi\colon T_p \otimes T_q \to T_{p+q}$ induced by the (p,q)–shuffle π is the composition of natural isomorphisms

$$T_p \otimes T_q \xrightarrow{\kappa_{p,q}} T_{p+q} \xrightarrow{\pi} T_{p+q},$$

$\sigma_\pi = \kappa_{p,q}\pi$. Since we want to form the distributions over all the possible shuffles induced by all the (p,q)–shuffles, we must now specialize slightly by considering only the objects of M which are abelian monoids. Furthermore, we will need an addition of the morphisms between the abelian monoids, and this addition of morphisms will have to have the usual properties.

We leave this ramification to the next section. Instead we conclude this section by noting that we can obtain the sections necessary to construct the cut coalgebra from the family of natural associativities.

The N–graded tensor over object V in M is equipped with a subfamily of the associativity isomorphisms,

$$\Gamma_{m,n}\colon T_{m+n} \to T_m \otimes T_n.$$

3.10 Proposition. If the monoidal category **M** has countable products, there is a cut coalgebra over every object in **M**.

Proof: Let $C = \prod_{n\geq 0} T_n$ be the countable product of the **N**–graded tensor over V. The cut structure map $\Gamma\colon C \to C \otimes C$ is the product of the $\Gamma_{m,n}$. The counit structure map $\varepsilon\colon C \to e$ is the projection to T_0. □

4. Semiadditive Symmetric Monoidal Categories

From [Herrlich & Strecker], a category **A** is semiadditive if for each pair of objects U, V in **A**, the hom set $\mathbf{A}(U, V)$ is an abelian monoid with respect to $+$, morphism composition is left and right distributive over $+$, whenever morphism composition is defined on either side, and **A** has zero morphisms which are the monoid identities with respect to $+$. If **A** also has finite coproducts, then the semiadditive structure is unique and **A** has finite biproducts. This last means that each pair of **A**–objects U, V possess a single object $U \oplus V$, which is both the categorical coproduct and the categorical product of the pair U, V.

Now the category **AbMon** of abelian monoids and morphisms between them is semiadditive as well as symmetric monoidal. Further, tensor ($\otimes$) distributes over biproduct ($\oplus$). The only other examples of semiadditive symmetric monoidal categories I know about are well-known symmetric monoidal subcategories of **AbMon** such as *R*–**Mod**. In all these examples tensor distributes over biproduct, the category is closed monoidal and the tensor may be formed by universal bilinear considerations. Each of these properties naturally arises in considering nondeterminism. I believe there is a structure theorem to the effect that any category with all these properties is embeddable in **AbMon**, but I am unable to locate this result just now.

For the above reasons, the rest of the work is effectively within **AbMon**. Now **AbMon** is a lovely category which was used in [Lorentz & Benson] to explore certain issues of nondeterminism in programming language semantics. Each object V of **AbMon** has an internal addition which is compatible with the external addition provided by the coproducts. Thus we may freely write finite sums, as in

$$\sum_{i\in I} v_i$$

where while the various v_i may lie in different objects, the sum lies in the coproduct of all finitely many objects. The **AbMon**–(tensor algebra) may be viewed as the **N**–graded **AbMon**–(tensor algebra) and we shall do so as this avoids an annoying technicality which arises: The countable coproduct $T = \bigsqcup_{n\geq 0} T_n$ is in general a different object than the countable product $C = \prod_{n\geq 0} T_n$. If

one wishes, one may identify the N-graded **AbMon**-(tensor algebra) object with C. In any case, concatenation is the sequence of **AbMon**-arrows

$$\kappa_n\colon (C \otimes C)_n \to C_n$$

where $(C \otimes C)_n$ denotes $\bigoplus_{p+q=n} (T_p \otimes T_q)$ and $C_n = T_n$.

In general we have the following definitions.

4.1 Definition. Let **A** be a semiadditive symmetric monoidal category with biproducts and in which tensor distributes over biproduct. Such categories will be called *abmon* categories. Only the properties of abmon categories are required for the remainder of the paper. An N-*graded* **A**-*algebra* (A, μ, η) consists of an N-graded object $A = \{A_n | n \in \mathbf{N}\}$ in **A** and N-graded morphisms $\mu = \{\mu_n | n \in \mathbf{N}\}$, $\eta = \{\eta_n | n \in \mathbf{N}\}$ called the *structure maps* on the N-graded **A**-algebra. Each grade n component of η is an **A**-morphism from e to A_n such that η_n is the zero morphism from e to A_n for $n > 0$, $\eta_n = 0_{e,A_n}$. Thus we may view η as consisting solely of its interesting component $\eta_0\colon e \to A_0$. Each grade n component of μ is an **A**-morphism from $(A \otimes A)_n = \bigoplus_{p+q=n} A_p \otimes A_q$ to A_n, $\mu_n\colon (A \otimes A)_n \to A_n$. Clearly, μ is an N-graded morphism from the N-graded object $A \otimes A = \{(A \otimes A)_n | n \in \mathbf{N}\}$ to the N- graded object A. Dually, an N-*graded* **A**-*coalgebra* (C, Δ, ε) is an N-graded object C with N-graded structure maps Δ, ε such that

$$\varepsilon_n\colon C_n \to e = 0_{C_n,e} \text{ for } n > 0,$$

$$\Delta_n\colon C_n \to (C \otimes C)_n,\ n \geq 0.$$

An N-*graded* **A**-*bialgebra* $(B, \mu, \Delta, \eta, \varepsilon)$ consists of an N-graded **A**-algebra (B, μ, η) and an N-graded **A**-coalgebra (B, Δ, ε) such that μ is an N-graded coalgebra morphism, Δ is an N-graded algebra morphism, and similarly for η, ε.

We are now in the position to define the N-graded shuffle operation in abmon category **A**. First, the components

$$\sigma_{p,q}\colon T_p \otimes T_q \to T_{p+q}$$

are given as the sum

$$\sigma_{p,q} = \sum_{\pi \in S_{p,q}} \sigma_\pi = \kappa_{p,q} \sum_{\pi \in S_{p,q}} \pi$$

where $S_{p,q}$ is the set of all (p,q)-shuffles. Now define

$$\sigma_n\colon (C \otimes C)_n \to C_n$$

as the coproduct of all the appropriate $\sigma_{p,q}$,

$$\sigma_n = \bigsqcup_{p+q=n} \sigma_{p,q}.$$

In the usual way we extend the domain of each $\sigma_{p,q}$ to all of $(C \otimes C)_n$ by writing

$$\widehat{\sigma}_{p,q} = \bigsqcup_{i+j=n} \delta_{i,p;j,q} \cdot \sigma_{p,q}$$

where the δ morphisms are the Kronecker deltas: the zero morphism from $C_i \otimes C_j$ to C_n unless $i = p$ and $j = q$. In this latter case, $\delta_{p,p;q,q}$ is the identity morphism. Now we slightly abuse the notation to set $\sigma_{p,q} = \widehat{\sigma}_{p,q}$ and may now write

$$\sigma_n = \sum_{p+q=n} \sigma_{p,q}.$$

Note that $\sigma_n = \kappa_n \cdot \sum_{p+q=n} \sum_{\pi \in S_{p,q}} \sigma_\pi$ by the distributivity of composition over addition. It is both routine and tedious to verify that the family $\sigma = \{\sigma_n | \in \mathbf{N}\}$ is an $\mathbf{N}$-graded $\mathbf{A}$-algebra structure map with the unit of the shuffle algebra given by the identification of $C_0 = T_0$ with the unit e of monoidal category $\mathbf{A}$. In **AbMon** this is the free abelian monoid over one generator, $(\mathbf{N}, +, 0)$. (Of course in R-**Mod**, one uses R for $C_0 = T_0$, and similarly for other subcategories of **AbMon**.) Thus η_0 is the identity while η_n for $n > 0$ is the zero map.

Similarly, we may define the $\mathbf{N}$-graded cut coalgebra in abmon category $\mathbf{A}$ with structure map components

$$\Gamma_n \colon C_n \to (C \otimes C)_n$$

by

$$\Gamma_n = \sum_{p+q=n} \Gamma_{p,q}.$$

The counit component $\varepsilon_0 \colon C_0 \to \mathbf{N}$ is simply the identity, while the remaining counit components, ε_n for $n > 0$, are all zero maps. Let $\Gamma = \{\Gamma_n | n \in \mathbf{N}\}$ and $\varepsilon = \{\varepsilon_n | n \in \mathbf{N}\}$.

4.2 Theorem. With the notation from above, $(C, \sigma, \Gamma, \eta, \varepsilon)$ is an $\mathbf{N}$-graded bialgebra.

Proof: This is a combinatorial exercise to show that every shuffle permutation is accounted for once and only once in the diagram we may abbreviate as

$$\sigma\Gamma = (\Gamma \otimes \Gamma)(1 \otimes \tau \otimes 1)(\sigma \otimes \sigma).$$

Explicitly, we must show that the diagram

(*)

$$\begin{array}{ccc}
(C \otimes C)_n & \xrightarrow{\sigma_n} & C_n \\
\downarrow{\scriptstyle (\Gamma \otimes \Gamma)_n} & & \\
(C \otimes C \otimes C \otimes C)_n & & \downarrow{\scriptstyle \Gamma_n} \\
\downarrow{\scriptstyle (1 \otimes \tau \otimes 1)_n} & & \\
(C \otimes C \otimes C \otimes C)_n & \xrightarrow[(\sigma \otimes \sigma)_n]{} & (C \otimes C)_n
\end{array}$$

commutes for every $n \epsilon \mathbf{N}$.

Fix some k, ℓ, p, q such that $k + \ell = p + q = n$ to consider a component of the previous diagram,

$$
\begin{array}{ccc}
C_k \otimes C_\ell & \xrightarrow{\sigma_{k,\ell}} & C_n \\
\Gamma_{k_1,k_2} \otimes \Gamma_{\ell_1,\ell_2} \downarrow & & \\
C_{k_1} \otimes C_{k_2} \otimes C_{\ell_1} \otimes C_{\ell_2} & & \Gamma_{p,q} \downarrow \\
1 \otimes \tau \otimes 1 \downarrow & & \\
C_{k_1} \otimes C_{\ell_1} \otimes C_{k_2} \otimes C_{\ell_2} & \xrightarrow[\sigma_{k_1,\ell_1} \otimes \sigma_{k_2,\ell_2}]{} & C_p \otimes C_q.
\end{array}
\qquad (**)
$$

We have $\sigma_{k,\ell}\Gamma_{p,q} = \kappa_{k,\ell} \sum_{\pi \in S_{k,\ell}} \sigma_\pi \Gamma_{p,q} = \sum_{\pi \in S_{k,\ell}} \kappa_{k,\ell}\sigma_\pi \Gamma_{p,q}$. By the earlier section on shuffles as permutations, for each $\pi \in S_{k,\ell}$ and each pair (p,q) there exist unique choices of k_1, k_2, ℓ_1, ℓ_2 with $k_1 + k_2 = k$, $\ell_1 + \ell_2 = \ell$, $k_1 + \ell_1 = p$, $k_2 + \ell_2 = q$, a unique choice of (k_1, ℓ_1)-shuffle π_1 and a unique choice of (k_2, ℓ_2)-shuffle π_2 such that the above diagram commutes for σ_{π_1} in place of σ_{k_1,ℓ_1} and σ_{π_2} in place of σ_{k_2,ℓ_2}. Thus each σ_π determines the component of the down-and-right side of the diagrams with which it commutes.

In the other direction, fix k_1, k_2, ℓ_1, ℓ_2 with $k_1 + k_2 = k$, $\ell_1 + \ell_2 = \ell$. Then $k_1 + \ell_1 = p$ and $k_2 + \ell_2 = q$ is required. Now each pair of shuffles π_1, π_2 determining some $\kappa_{k_1,\ell_1}\sigma_{\pi_1} \otimes \kappa_{k_2,\ell_2}\sigma_{\pi_2}$ uniquely determines a (k,ℓ)-shuffle π and hence uniquely some $\kappa_{k,\ell}\sigma_\pi$.

Since the correspondence is a bijection in both directions, forming all the sums still leaves a bijective correspondence. Therefore, the diagram $(**)$ commutes for each k, ℓ, p, q. A summation argument shows that the diagram $(*)$ commutes for each n. □

The *spray* operation is in a sense dual to shuffle, being determined by the collection of inverses to the shuffle permutations.* Let $R_{p,q} = \{\pi^{-1} | \pi \in S_{p,q}\}$ and let the subcomponents of spray be determined on the N-graded tensor over C_1 by

$$\chi_\pi \colon C_{p+q} \to C_p \otimes C_q = \pi\Gamma_{p,q} \colon C_{p+q} \to C_{p+q} \to C_p \otimes C_q$$

for each $\pi \in R_{p,q}$. Then

$$\chi_{p,q} = \sum_{\pi \in R_{p,q}} \chi_\pi = \sum_{\pi \in R_{p,q}} \pi\Gamma_{p,q}.$$

Finally, the graded component of χ at $n \in \mathbf{N}$,

$$\chi_n \colon C_n \to (C \otimes C)_n$$

is given by the product map

$$\chi_n = \prod_{p+q=n} \chi_{p,q}$$

---

* The term *spray* is from [Pratt].

which by the usual techniques we view as the sum,

$$\chi_n = \sum_{p+q=n} \chi_{p,q}.$$

With $\chi = \{\chi_n | n \in \mathbf{N}\}$ the spray morphism, κ concatenation, and η,ε as before, we obtain another N-graded bialgebra.

4.3 Proposition. $(C, \kappa, \chi, \eta, \varepsilon)$ is an N-graded **A**-bialgebra.

Another piece of our apparatus is the observation that the components of concatenation, $\kappa_{p,q}$, and the components of cut, $\Gamma_{p,q}$, are purely structural on an N-graded tensor. Specifically,

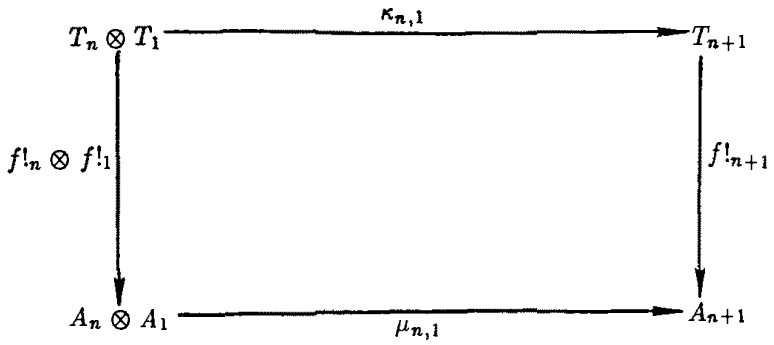
4.4 Proposition. Let C be the N-graded tensor over C_1. For each $p, q \in \mathbf{N}$, $\kappa_{p,q}\Gamma_{p,q}$ is the identity morphism on $C_p \otimes C_q$ and $\Gamma_{p,q}\kappa_{p,q}$ is the identity morphism on C_{p+q}.

If V is an object of **A**, let $V \approx T_1$ and let (T, κ, η) be the N-graded tensor algebra over V. For each N-graded **A**-algebra (A, μ, η_A) and each **A**-morphism $f\colon V \to A_1$ there is a unique N-graded **A**-morphism $f!$ extending f to an N-graded **A**-algebra morphism. In detail,

$$f!_0\colon T_0 \to A_0 = \varepsilon\eta_A,$$

$$f!_{n+1}\colon T_{n+1} \to A_{n+1} = \Gamma_{n,1}(f!_n \otimes f)\mu_{n,1}$$

renders commutative



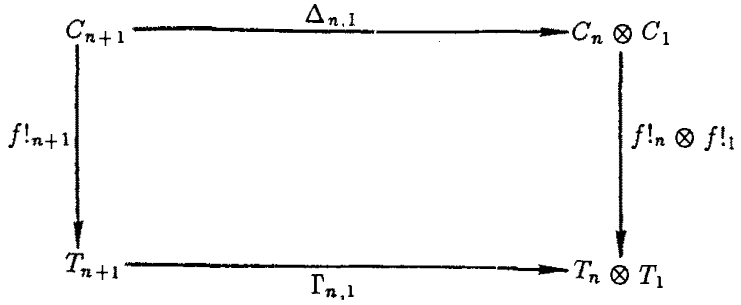
since $f!_1 = \Gamma_{0,1}(\varepsilon\eta_A \otimes f)\mu_{0,1} \approx f$. Simple induction arguments then finish the proof.

Similarly, let (T, Γ, ε) be the N-graded cut coalgebra over **A**-object V. For each N-graded **A**-coalgebra $(C, \Delta, \varepsilon_C)$ and each **A**-morphism $f\colon C_1 \to V$ there is a unique N-graded **A**-morphism $f!\colon C \to T$ extending f to an N-graded **A**-coalgebra morphism. Define

$$f!_0\colon C_0 \to T_0 = \varepsilon_C\eta,$$

$$f!_{n+1}\colon C_{n+1} \to T_{n+1} = \Delta_{n,1}(f!_n \otimes f)\kappa_{n,1}$$

to observe that



commutes as $f!_1 = \Delta_{0,1}(\varepsilon_C \eta \otimes f)\kappa_{0,1} \approx f$ so that

$$\begin{aligned} f!_{n+1}\Gamma_{n,1} &= \Delta_{n,1}(f!_n \otimes f)\kappa_{n,1}\Gamma_{n,1} \\ &= \Delta_{n,1}(f!_n \otimes f!_1). \end{aligned}$$

Thus (T, κ, η) is the free N-graded **A**-algebra over $V \approx T_1$ and (T, Γ, ε) is the cofree N-graded **A**-coalgebra over V.

5. Precise N-Graded Bialgebras

5.1 Definition. An N-graded **A**-bialgebra $(B, \mu, \Delta, \eta, \varepsilon)$ is *precise* if $B_0 \approx e$ so that $\eta_0\colon e \to B_0$ and $\varepsilon_0\colon B_0 \to e$ are isomorphisms and the components $\mu_{n,0}$, $\mu_{0,n}$, $\Delta_{n,0}$ and $\Delta_{0,n}$ are the specific isomorphisms

$$\mu_{n,0} = 1_n \otimes \varepsilon_0\colon B_n \otimes B_0 \to B_n \otimes e \cong B_n,$$

$$\mu_{0,n} = \varepsilon_0 \otimes 1_n\colon B_0 \otimes B_n \to e \otimes B_n \cong B_n,$$

$$\Delta_{n,0} = 1_n \otimes \eta_0\colon B_n \cong B_n \otimes e \to B_n \otimes B_0,$$

$$\Delta_{0,n} = \eta_0 \otimes 1_n\colon B_n \cong e \otimes B_n \to B_0 \otimes B_n,$$

where 1_n denotes the identity on B_n and the left and right unitary isomorphisms specified as part of the monoidality data of **A** are denoted by "$\cong$". We simply say $(B, \mu, \Delta, \eta, \varepsilon)$ is a *precise bialgebra.**

The shuffle-cut bialgebra and the concatenation-spray bialgebra are both precise bialgebras.

5.2 Definitions. When B is an N-graded **A**-object, the subobject of $(B \otimes B)_n$, $n > 1$,

$$(B \otimes B)^c_n = \bigoplus_{\substack{i+j=n \\ i,j>0}} B_i \otimes B_j$$

is called the *core* of $(B \otimes B)_n$. The core of $(B \otimes B)_1$ is $(B \otimes B)_1$ itself. If $\mu\colon B \otimes B \to B$ is an N-graded **A**-morphism the restriction of μ to the core,

$$\mu^c_n\colon (B \otimes B)^c_n \to B_n,$$

---

* [Sweedler] uses a related condition, called *strict*, in vector space categories.

is called the core of μ_n. Dually, if Δ: $B \to B \otimes B$ is an N-graded **A**-morphism the projection of Δ to the core,

$$\Delta_n^c\colon B_n \to (B \otimes B)_n^c,$$

is called the core of Δ_n. If h: $B \to C$ is an N-graded **A**-morphism between N-graded **A**-objects, the restriction of $(h \otimes h)_n$ to the core $(B \otimes B)_n^c$ followed by the projection to the core $(C \otimes C)_n^c$ is denoted by $(h \otimes h)_n^c$. Similar notations apply to obtain the core of $B \otimes B \otimes B \otimes B$ and $h \otimes h \otimes h \otimes h$. For example, the core of $(B \otimes B \otimes B \otimes B)_2$ is $(B_1 \otimes B_0 \otimes B_1 \otimes B_0) \oplus (B_0 \otimes B_1 \otimes B_0 \otimes B_1)$.

We say the N-graded **A**-morphism Δ: $B \to B \otimes B$ is a *monomorphism at the core* if for each $n \in \mathbf{N}$, the core of Δ_n is an **A**-monomorphism. Similarly, the N-graded **A**-morphism μ: $B \otimes B \to B$ is an *epimorphism at the core* if for each $n \in \mathbf{N}$, the core of μ_n is an **A**-epimorphism.

5.3 Lemma. Let $(B, \mu_B, \Delta_B, \eta_B, \varepsilon_B)$ and $(C, \mu_C, \Delta_C, \eta_C, \varepsilon_C)$ be precise bialgebras. Let h: $B \to C$ be an N-graded **A**-morphism. If h is an N-graded **A**-coalgebra morphism, h: $(B, \Delta_B, \varepsilon_B) \to (C, \Delta_C, \varepsilon_C)$, and Δ_C is a monomorphism at the core, then h is an **A**-bialgebra morphism from $(B, \mu_B, \Delta_B, \eta_B, \varepsilon_B)$ to $(C, \mu_C, \Delta_C, \eta_C, \varepsilon_C)$. Dually, if h is an N-graded **A**-algebra morphism from (B, μ_B, η_B) to (C, μ_C, η_C) and μ_B is an epimorphism at the core, then h is an **A**-bialgebra morphism from the given B bialgebra to the given C bialgebra.

Proof: Suppose h: $(B, \Delta_B, \varepsilon_B) \to (C, \Delta_C, \varepsilon_C)$ is an N-graded **A**-coalgebra morphism, where $(B, \mu_B, \Delta_B, \eta_B, \varepsilon_B)$ and $(C, \mu_C, \Delta_C, \eta_C, \varepsilon_C)$ are precise bialgebras. Clearly $\mu_{B,1,0}h_1 = (h_1 \otimes h_0)\mu_{C,1,0}$ as $\mu_{B,1,0}, h_0$, and $\mu_{C,1,0}$ are isomorphisms. Similarly when the 0 and 1 indicies are reversed. Therefore,

$$(\mu_{B,1,0} \otimes \mu_{B,1,0})(h_1 \otimes h_1) = (h_1 \otimes h_0 \otimes h_1 \otimes h_0)(\mu_{C,1,0} \otimes \mu_{C,1,0})$$

and similarly with the 0 and 1 indicies reversed. We now have the basis for an induction using the cube diagram of *3.5*, but with cores on the bottom face of the diagram. All four side faces commute by hypothesis. It remains, therefore, to show by induction that the top face commutes. The proof at each induction step is a diagram chasing argument. For simplicity of notation we have suppressed the induction index n. Chasing around the cube diagram,

$$\begin{aligned}
\mu_B h \Delta_C^c &= \mu_B \Delta_B^c (h \otimes h)^c \\
&= \Delta_{B\otimes B}^c (\mu_B \otimes \mu_B)^c (h \otimes h)^c \\
&= \Delta_{B\otimes B}^c (h \otimes h \otimes h \otimes h)^c (\mu_C \otimes \mu_C)^c \\
&= (h \otimes h) \Delta_{C\otimes C}^c (\mu_C \otimes \mu_C)^c \\
&= (h \otimes h) \mu_C \Delta_C^c,
\end{aligned}$$

show that as Δ_C is a monomorphism at the core we have the commutivity of the top face, $\mu_B h = (h \otimes h)\mu_C$. Since the bialgebras in question are precise, we further deduce that

$$\mu_B^c h = (h \otimes h)^c \mu_C^c$$

and this completes the induction step. The remaining assertion of the lemma is obtained by duality. □

5.4 Corollary. If $(B, \mu, \Delta, \eta, \varepsilon)$ is a precise bialgebra and $(C, \sigma, \Gamma, \eta, \varepsilon)$ is a shuffle–cut bialgebra, an **A**–morphism $f\colon B_1 \to C_1$ extends uniquely to an **N**–graded **A**–bialgebra morphism from the given bialgebra B to the shuffle–cut bialgebra. Dually, if $(A, \kappa, \chi, \eta, \varepsilon)$ is the concatenation–spray bialgebra, an **A**–morphism $g\colon A_1 \to B_1$ extends uniquely to an **N**–graded **A**–bialgebra morphism from the concatenation–spray bialgebra to the given precise bialgebra B.

To obtain any permutation of an ordinary deck of playing cards, first separately obtain the desired ordering among the black cards in one pile and the red cards in another pile. Then shuffle the two piles together. So when we consider the previous result for the unique morphism from the concatenation–spray bialgebra to the shuffle–cut bialgebra over the same grade 1 object, $A_1 \approx C_1$, it is immediate that the unique morphism at component n is the sum of all the permutations in S_n, the symmetric group on n letters. In symbols,

$$h_n\colon A_n \to C_n = \sum_{\pi \in S_n} \pi$$

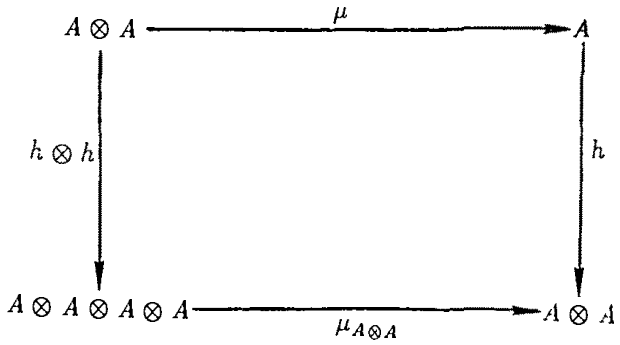
where A is the concatenation–spray bialgebra and C is the shuffle–cut bialgebra.

Let B be a precise **A**–bialgebra with grade 1 component B_1. Let $A_1 \approx B_1 \approx C_1$ and continue to use the notations of the previous lemma and corollary. There are unique **N**–graded **A**–bialgebra morphisms $f\colon A \to B$ and $g\colon B \to C$ uniquely determined by the isomorphisms $A_1 \approx B_1 \approx C_1$. Furthermore, $fg = h$. One suspects that a pleasant characterization of the precise bialgebra B in terms of f and g is available.

6. Marked Shuffle

The marked shuffle operation shuffles together the strings between distinguished marked entities, but preserves the order at the marks. Let C_1 be some **A**–object which to aid intuition may be thought of as the collection of ordinary, unmarked cards. The shuffle–cut bialgebra $(C, \sigma, \Gamma, \eta, \varepsilon)$ describes the strings and the operations on the ordinary cards. We now view $C \otimes C$ as describing strings containing an intermediate marked card. Any notation for the intermediate marked card is suppressed, as we assume that the matching of the distinguished cards has actually taken place. Similarly, $C \otimes C \otimes C$ describes strings containing two intermediate marked cards and so on. Now each of $C \otimes C$, $C \otimes C \otimes C, \ldots$ inherits a bialgebra structure from $(C, \sigma, \Gamma, \eta, \varepsilon)$ by the very existence of the bialgebra structure on C.

To begin, when (A, μ, η) is an **A**–algebra, $(A \otimes A, \mu_{A\otimes A}, \eta \otimes \eta)$ is also an **A**–algebra when $\mu_{A\otimes A} = (1 \otimes \tau \otimes 1)(\mu \otimes \mu)$. The morphism $h\colon A \to A \otimes A$ is an **A**–algebra morphism from (A, μ, η) to $(A \otimes A, \mu_{A\otimes A}, \eta \otimes \eta)$ when the diagram



commutes. Since C is equipped with the shuffle-cut bialgebra structure, Γ: $C \to C \otimes C$ is an **A**-algebra morphism from (C, σ, η) to $(C \otimes C, \sigma_{C\otimes C}, \eta \otimes \eta)$. Dually, σ: $C \otimes C \to C$ is an **A**-coalgebra morphism from $(C \otimes C, \Gamma_{C\otimes C}, \varepsilon \otimes \varepsilon)$ to (C, Γ, ε).

Now $\sigma_{C\otimes C}$: $(C \otimes C) \otimes (C \otimes C) \to C \otimes C$ is the desired marked shuffle of two strings each containing one intermediate mark. Similarly, $\sigma_{C\otimes C\otimes C}$ is the desired marked shuffle of two sequences each containing two intermediate marks.

We now change notation. Let $M_{(0)} = e$, $M_{(n+1)} = M_{(n)} \otimes C$ be the sequence of precise bialgebras alluded to above. Let

$$\sigma_{(0)} = 1_e,$$
$$\sigma_{(n+1)} = (1 \otimes \tau \otimes 1)(\sigma_{(n)} \otimes \sigma)\colon\ M_{(n+1)} \otimes M_{(n+1)} \to M_{(n+1)}$$

be the sequence of marked shuffle algebra maps. Finally, let

$$\Gamma_{(0)} = 1_e,$$
$$\Gamma_{(n+1)} = (\Gamma_{(n)} \otimes \Gamma)(1 \otimes \tau \otimes 1)\colon\ M_{(n+1)} \to M_{(n+1)} \otimes M_{(n+1)}$$

be the sequence of cut coalgebra maps. As η,ε for each of the $M_{(n)}$ are just the η,ε of C, we have a sequence of bialgebras $M_{(n)}$ with connecting morphisms

$$M_{(0)} \underset{\varepsilon}{\overset{\eta}{\rightleftarrows}} M_{(1)} \underset{1_0 \otimes \sigma}{\overset{1_0 \otimes \Gamma}{\rightleftarrows}} M_{(2)} \underset{1_1 \otimes \sigma}{\overset{1_1 \otimes \Gamma}{\rightleftarrows}} \cdots$$

where 1_n denotes the identity morphism on $M_{(n)}$. Each $1_n \otimes \Gamma$ is to be thought of as introducing a new mark while each $1_n \otimes \sigma$ forgets the last intervening mark to shuffle the last two strings together.

The sequence of morphisms from $M_{(n)}$, $n > 0$, to $M_{(1)}$

$$\sigma^{(0)} = 1_1,$$
$$\sigma^{(n+1)} = (1_n \otimes \sigma)\sigma^{(n)}$$

provides the sequence of unique N-graded A-bialgebra morphisms to the shuffle-cut bialgebra. Each $\sigma^{(n)}$ is the unique extension of the isomorphism between the grade 1 component of $M_{(n)}$ and the grade 1 component of $M_{(1)}$, C_1.

Each $\sigma^{(n)}$ forgets all the marks, and shuffles all the strings together. In the context of distributed computing theory, this is a significant loss of information. Therefore, in this setting one may view a string containing a single intermediate marked card as an element of $C \otimes C_1 \otimes C$, with the central factor C_1 denoting the position of the marked card. Forgetting the distinguished status of the marked card is equivalent to forgetting the bialgebra structure necessary to shuffle the first factors and the last factors as described earlier. Forgetting the distinguished status of the central factor may now be viewed as the concatenation

$$(1 \otimes \kappa)\kappa\colon C \otimes C_1 \otimes C \to C.$$

Since the central factor is only C_1, this concatenation is not an algebra structure map. This concatenation is not an algebra morphism for shuffle nor is it a coalgebra morphism for cut. Thus further analysis appears desirable.

References

1. M. A. Arbib and E. B. Manes, Adjoint Machines, State-Behavior Machines, and Duality, J. Pure Appl. Alg. **6**(1975), 313–344.
2. M. Broy, Semantics of Communicating Processes, Inform. & Control **61**(1984), 202–241.
3. S. Eilenberg and G. M. Kelly, Closed Categories *in* Proc. Conf. Categorical Alg., La Jolla 1965, Springer-Verlag, New York, 1966, pp. 421–562.
4. S. Eilenberg & S. MacLane, On the Groups H($\prod$, n). I., Ann. of Math **58**(1953), 55–106.
5. N. Francez, *Fairness*, Springer-Verlag, New York, 1986.
6. H. Gaifman and V. Pratt, Partial Order Models of Concurrency and the Computation of Functions, Proc. IEEE Symp. Logic in Comput. Sci., Ithaca, NY, 1987.
7. S. Ginsburg, *The Mathematical Theory of Context-Free Languages*, McGraw-Hill, 1966.
8. S. Ginsburg and E. H. Spanier, Mapping of Languages by Two-tape Devices, *J. Assoc. Comput. Mach.* **12**(1965), 423–434.
9. J. L. Gischer, Partial Orders and the Axiomatic Theory of Shuffle, Stanford Univ. Report STAN-CS-84-1033.
10. H. Herrlich & G. Strecker, *Category Theory*, Heldermann-Verlag, Berlin, 1979.
11. R. J. Lorentz & D. B. Benson, Deterministic and Nondeterministic Flowchart Interpretations, J. Comput. Sys. Sci. **27**(1983), 400–433.
12. S. MacLane, *Categories for the Working Mathematician*, Springer-Verlag, New York, 1971.
13. S. MacLane, *Homology*, Academic Press, New York, 1963.

14. M. Main and D. B. Benson, Functional Behavior of Nondeterministic and Concurrent Programs, Inform. & Control **62**(1984), 144–189.
15. E. G. Manes, *Algebraic Theories*, Springer–Verlag, New York, 1976.
16. E. G. Manes, Additive Domains, Springer–Verlag LNCS 239, 1986, 184–195.
17. E. G. Manes, Weakest Preconditions: Categorical Insights, Springer–Verlag LNCS 240, 1986, 182–197.
18. E. G. Manes, Assertional Categories, Third Workshop on Math. Found. Program. Semantics, Tulane, April 1987, *These Proceedings.*
19. M. Pfender, Universal Algebra in S–Monoidal Categories, Bericht Nr. 20, Mathematisches Institut, Univ. München, 1974.
20. V. Pratt, Modeling Concurrency with Partial Orders, *International J. Parallel Programming*, **15**(1986), 33–72.
21. L. Redei, The Theory of Finitely Generated Commutative Semigroups, Pergammon Press, 1965.
22. W. E. Riddle, Modelling and Analysis of Supervisor Systems, Ph.D. Thesis, Stanford Univ., 1972.
23. W. E. Riddle, An Approach to Software System Behavior Description, *Computer Languages* **4**(1979), 29–47.
24. A. Salomaa and M. Soittola, *Automata–Theoretic Aspects of Formal Power Series*, Springer–Verlag, New York, 1978.
25. E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., New York, 1966.
26. M. Sweedler, *Hopf Algebras*, W. A. Benjamin, New York, 1969.