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Bisimulation of Automata*

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We view CCS terms as defining nondeterministic automata. An algebraic representation of automata is given, and categories of automata and simulations between them are defined. The crucial feature is the consideration of only the *pure* simulations which carry the pure (actual, determined) states of the domain automation to the pure states of the codomain automaton. The pure epimorphisms between the automata partition the category into bisimulation equivalence classes. There is a unique canonical representative for each bisimulation equivalence class. These results hold for weak bisimulation and hence for strong bisimulation. Essentially the same results are obtained with regard to rooted bisimulation equivalence classes of automata with start states. © 1988 Academic Press, Inc.

INTRODUCTION

CCS is defined via bisimulation equivalence classes on the term algebra generated by applying the operations of prefix, alternative choice, and some other operations on an alphabet of event types, sometimes called communication symbols or actions. One desires an algebra convenient for describing all and only the unique processes. For a criticism of this goal, see (Hoare, 1985).

Another view of CCS is obtained by considering CCS terms as defining automata and the CCS equivalences as defining classes of automata with the same behavior. This is an implementer's view, in that a programming language expression is considered to define a nondeterministic automaton which is capable of the communications specified in CCS or other notations.

As is well understood, the CCS equivalences are a highly conservative notion of similarity of behavior. This behavior is not the output behavior traditionally studied in automata theory, but is a state transition behavior given by the CCS equivalences. In this note we study the notion of *strong bisimulation* and *weak bisimulation* advanced in (Milner, 1983) as

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appropriate for synchronous and asynchronous communicating systems, SCCS and ASCCS, respectively. Our approach has certain similarities to (Castellani, 1985), but is mathematically more elegant and produces stronger results.

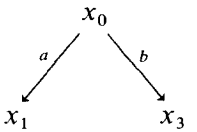
As we show, bisimulation equivalence classes are the epi-connected equivalence classes in appropriate categories of automata and simulations between them. Thus general categorical constructions guarantee the existence of minimal automata in each epi-connected equivalence class. The major portion of the technical effort is establishing that bisimulation equivalences are in fact epi-connected equivalences. The rest is the application of ideas which have been in the literature on categorical automata theory for some time. From the generality of the situation, this work may be applied in many concrete situations of practical import. We briefly give some instances in the conclusion.

To apply this general theory, we need incompletely specified nondeterministic input-only automata with inputs the event types or actions of CCS. This concept is sometimes called a nondeterministic semiautomaton and sometimes a labeled transition system.

As examples, the CCS expression $a \cdot b$ is to be thought of as denoting some automaton

$$x_0 \xrightarrow{a} x_1 \xrightarrow{b} x_2$$

and the alternative choice of a, b, sometimes denoted by $a + b$, as representing an automaton



The phrase "incompletely specified" is technically correct but misleading in this situation. The automata are intended to remain incompletely specified; indeed an important feature of these devices is the possibility of deadlock. We will use a zero to denote "impossible" and write something similar to $x_0 \cdot b = 0$ to mean that in state x_0, it is impossible for the automaton to read the input b. This is the case for the first of the two automata exemplified above.

As bisimulation is not a congruence, we define a refinement of the bisimulation relation, called the rooted bisimulation relation. See (Bergstra and Klop, 1984, 1985a, 1985b; Benson and Tiuryn, 1989) for additional results on rooted bisimulation. We show that rooted bisimulation is a congruence with respect to a conservative way of joining processes, represented by automata.

The presentation is algebraic, using notation and concepts from linear and multilinear algebra and from category theory. The linear and multilinear algebra is abstracted from the usual modules over a ring to the modules over a semiring appropriate to automata theory, (Give'on, 1967; Johnson and Manes, 1970; Kuich and Salomaa, 1986). While this concept is often called a semimodule, we prefer to call this idea a module over a semiring and reserve the term "semimodule" for a weakening of the module laws.

In contradistinction to the usual mathematical practice, we consistently write function application on the right, i.e., xf rather than $f(x)$. Further, we shall use addition and the plus sign, "+," to denote nondeterministic choice rather the alternative choice of CCS. These two concepts are related but not identical. The alternative choice of CCS we view as the join of two automata by merging the start states of the automata into a single start state, this combinator denoted by "$\vee$." The nondeterministic choice denoted herein by the plus sign, "+," is a purely internal choice, in a sense made precise in the body of this paper. This nondeterminism is related to the *kind* nondeterminism traditionally used in automata theory. Kind nondeterminism is also known as *may*, *angelic*, or *Hoare* nondeterminism in various studies.

1. Automata and Simulations

The algebraic automata used here consists of a set of *pure states* X and a *run map*, ζ, which gives all the data about the state transitions which occur when finite sequences of input symbols are read. Thus the automata make progress when a communication symbol is agreed upon by the automaton and the other party to the communication, in the manner often studied for communication protocols. For example, see (Gouda, Chow, and Lam, 1985).

As the automata are nondeterministic, the concept of semiring is central to the discussion.

Semirings

A set S equipped with a distinguished element $0 \in S$, a binary addition, and a binary multiplication is called a *semiring* when for all x, y, z in S we have

addition laws:

$$x + (y + z) = (x + y) + z,$$

$$x + y = y + x,$$

$$x + 0 = x,$$

multiplication laws:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

$$x \cdot 0 = 0 \cdot x = 0,$$

and distributive laws connecting addition and multiplication:

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

$$(x + y) \cdot z = x \cdot z + y \cdot z.$$

A semiring S is *unital* if there is a distinguished element $1 \in S$ called the *unit* of S such that $1 \cdot x = x = x \cdot 1$ for all $x \in S$. A semiring S is *commutative* if $x \cdot y = y \cdot x$ for all $x, y \in S$. A semiring S is *zerosumfree* if for all $x, y \in S$, $x + y = 0$ implies $x = y = 0$.

In this paper, a *scalar* semiring is a zerosumfree commutative unital semiring. The notation "$\flat$" always refers to a scalar semiring. The usual scalar semiring is the Boolean $\mathbf{B} = \{0, 1\}$ with "and" as multiplication and "inclusive or" as addition. There are many other scalar semirings of interest in computer science; see (Kuich and Salomaa, 1986; Mahr, 1982; Benson, 1987) for examples.

The multiplication symbol "$\cdot$" is frequently elided within expressions denoting elements of a semiring.

Right S-Modules

There are two right S-module structures of interest here. First, the nondeterministic states of an automaton form a right module over the scalar semiring $\flat$, these nondeterministic states being freely generated by the pure, or actual, states of the device. Then we will see that the nondeterministic states also have a right module structure over the nondeterministic inputs, this structure given by the run map of the automaton. This view goes back, at least, to (Meseguer and Sols, 1975).

An abelian monoid $(M, +, 0)$ is a *right S-module* over a semiring S when there is a scalar multiplication,

$$\mu\colon M \times S \to M$$

satisfying the laws of scalar multiplication given below. We write $m \cdot s$ for $(m, s)\mu$ with m in M and s in S in writing the laws

$$(m + n) \cdot s = m \cdot s + n \cdot s,$$

$$m \cdot (r + s) = m \cdot r + m \cdot s,$$

$$m \cdot 0 = 0 = 0 \cdot s,$$

$$(m \cdot r) \cdot s = m \cdot (rs),$$

for all $m, n \in M$ and $r, s \in S$.

A right S-module is *unital* if the semiring S is unital with unit $1 \in S$ and $m \cdot 1 = m$ for all $m \in M$. A *free* unital right S-module freely generated by X may be identified with the collection of all the functions from X to S having finite support. Such functions are denoted by $\sum_{x \in X} x \cdot s_x$ with only a finite number of the $s_x \in S$ different from zero. When X freely generates a unital right S-module, we call X the *pure states* of the unital right S-module. Each pure state $x \in X$ is thus associated with the sum $\sum_{y \in X} y \cdot \delta_{xy}$ with $\delta_{xy} = 0$ when $x \neq y$ and $\delta_{xx} = 1$.

If the semiring S is equipped with an infinite addition, the finite support requirement may be dropped in certain circumstances. The paper (Benson and Manes, 1985) suggests wide reaching possibilities. Be this as it may, finite sums are sufficient to establish the connection between CCS and the classes of automata of current interest.

Note that the only annihilator of a free unital right S-module is $0 \in S$. That is, for all $m \in M$ and all $s \in S$, $m \cdot s = 0$ implies $m = 0$ or $s = 0$. This property is needed for the proof of Theorem 4.

When $S = \flat$ is a scalar semiring, we abbreviate the phrase "free unital right $\flat$-module" to just "$\flat$-module." The $\flat$-module freely generated by pure states X is to be viewed as the collection of all $\flat$-nondeterministic distributions over the pure states X and is denoted $X \cdot \flat$. Each element of $X \cdot \flat$ is called a *state*.

If the scalars are Boolean a **B**-module is a unital semilattice and may be viewed as the collection of all finite subsets over pure states X. The addition in this case is simply set union. Additional facts about unital modules over unital semirings may be found in (Johnson and Manes, 1970).

Nondeterministic Event Types

Let E be the alphabet of event types, and let E^* denote the set of all strings over E. Now E^* is the free monoid freely generated by E. The binary operation of string concatenation is written multiplicatively. The unit or null string is denoted by $1 \in E^*$. The unit of E is sometimes called the unobservable event type or the silent action.

Let A be the $\flat$-module freely generated by the set E^*. The multiplication of string concatenation on E^* is extended to all of A by

$$\left(\sum_{\alpha \in E^*} \alpha \cdot k_\alpha\right) \cdot \left(\sum_{\beta \in E^*} \beta \cdot k'_\beta\right) = \sum_{\alpha, \beta \in E^*} (\alpha\beta) \cdot k_\alpha k'_\beta$$

for all $\sum_{\alpha \in E^*} \alpha \cdot k_\alpha$, $\sum_{\beta \in E^*} \beta \cdot k'_\beta \in A$. The $\flat$-module A equipped with this multiplication is called the monoid $\flat$-algebra generated by the monoid E^*. The multiplication is bilinear, so we may view the multiplication as a linear map on the tensor product $A \otimes A$, where the tensor is taken with respect

to $\flat$. The multiplication then has the form $A \otimes A \to A$. Note in this regard that if $X \cdot \flat$, $Y \cdot \flat$ are $\flat$-modules freely generated by X, Y, respectively, then $X \cdot \flat \otimes Y \cdot \flat$ is the $\flat$-module freely generated by the cartesian product $X \times Y$ (Main and Benson, 1984).

Clearly A is a unital semiring and we identify the unit of A with the null string of E^*. Further, A is zerosumfree since the underlying scalar semiring $\flat$ is zerosumfree by hypothesis. When the scalars are Boolean the monoid **B**-algebra A is the collection of all finite subsets of strings with set union as addition and set concatenation as multiplication.

Automata

An incompletely-specified $\flat$-nondeterministic input-only *automaton* with pure states X and inputs E is a linear map, called the *run map* or *reachability map* of the automaton

$$\zeta: X \cdot \flat \otimes A \to X \cdot \flat,$$

where A is the monoid $\flat$-algebra generated by E^*, satisfying the following law for all u in $X \cdot \flat$ and $\alpha, \beta \in A$:

$$(u \otimes (\alpha\beta))\zeta = ((u \otimes \alpha)\zeta \otimes \beta)\zeta.$$

The three equalities

$$(u \otimes (\alpha + \beta))\zeta = (u \otimes \alpha)\zeta + (u \otimes \beta)\zeta,$$
$$((u + v) \otimes \alpha)\zeta = (u \otimes \alpha)\zeta + (v \otimes \alpha)\zeta,$$
$$(u \otimes 0)\zeta = (0 \otimes \alpha)\zeta = 0,$$

hold simply because the domain of the transition function ζ is the tensor product $X \cdot \flat \otimes A$. Thus an incompletely-specified $\flat$-nondeterministic input-only automaton is a right A-module, made more obvious by writing the scalar multiplication form for ζ in the above laws:

$$u \cdot (\alpha\beta) = (u \cdot \alpha) \cdot \beta,$$
$$u \cdot (\alpha + \beta) = u \cdot \alpha + u \cdot \beta,$$
$$(u + v) \cdot \alpha = u \cdot \alpha + v \cdot \alpha,$$
$$u \cdot 0 = 0 \cdot \alpha = 0.$$

These right A-modules are in general neither free with respect to the semiring A nor unital with respect to the unit of A.

When the scalars are Boolean, $X \cdot \mathbf{B}$ is the collection of all finite subsets of the pure states, and again A is the collection of all finite subsets of

strings. Hence ζ is specified, or generated, by a function from the cartesian product of the set of pure states with the set of event types to the set of all states. Thus $\zeta: X \cdot \mathbf{B} \otimes A \to X \cdot \mathbf{B}$ is the run map of a traditional nondeterministic input-only automata. Such automata are sometimes called *labeled transition systems.*

Morphisms

Let $\zeta: X \cdot \mathfrak{b} \otimes A \to X \cdot \mathfrak{b}$ and $\xi: Y \cdot \mathfrak{b} \otimes A \to Y \cdot \mathfrak{b}$ be right A-modules with $X \cdot \mathfrak{b}$ freely generated by X and $Y \cdot \mathfrak{b}$ freely generated by Y, X and Y the pure states of the respective automata.

A morphism, or *simulation*, of automata from ζ to ξ is an A-linear transformation of the right A-modules, $h: X \cdot \mathfrak{b} \to Y \cdot \mathfrak{b}$. That is,

$$(u \cdot \alpha + v \cdot \beta)h = uh \cdot \alpha + vh \cdot \beta$$

for all u, v in $X \cdot \mathfrak{b}$ and all α, $\beta \in A$.

Having fixed some scalars $\mathfrak{b}$ and a $\mathfrak{b}$-algebra A, we have a category of A-automata. The objects are the automata viewed as right A-modules as above and the morphisms are all the A-linear transformations. This category is a setting for traditional nondeterministic automata theory. Variations may be obtained by introducing the unital law $(u \otimes 1)\zeta = u$ to exclude silent or autonomous moves, these sometimes viewed as moves not caused by an input. For additional results on unital algebraic automata, see (Arbib and Manes, 1974; Manes, 1976) as well as many other papers and monographs.

However, as each bisimulation relation is a relation between the pure states of automata, and not between the arbitrary $\mathfrak{b}$-nondeterministic states, we must consider a restricted notion of morphism and therefore a subcategory of A-automata.

Pure Morphisms

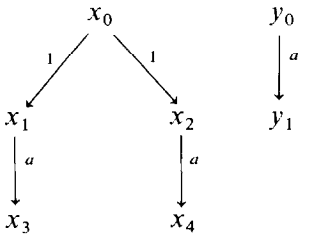
As before, let $\zeta: X \cdot \mathfrak{b} \otimes A \to X \cdot \mathfrak{b}$ and $\xi: Y \cdot \mathfrak{b} \otimes A \to Y \cdot \mathfrak{b}$ be right A-modules with $X \cdot \mathfrak{b}$, $Y \cdot \mathfrak{b}$ freely generated by X, Y, respectively. A *pure morphism* from ζ to ξ is a linear transformation of the right A-modules which is the unique extension of a function $f: X \to Y$. Otherwise stated, a pure morphism carries the pure states of the domain automaton to the pure states of the codomain automaton. We may say that $f: X \to Y$ generates a pure morphism. A pure morphism $f: X \cdot \mathfrak{b} \to Y \cdot \mathfrak{b}$ is a *pure epimorphism* if and only if the underlying function $f: X \to Y$ is surjective.

While each function $f: X \to Y$, indeed each function $f: X \to Y \cdot \mathfrak{b}$, uniquely generates a $\mathfrak{b}$-linear transformation from the $\mathfrak{b}$-module $X \cdot \mathfrak{b}$ to the $\mathfrak{b}$-module $Y \cdot \mathfrak{b}$, we require that the resulting function be an A-linear trans-

formation of right A-modules. So many functions $f: X \to Y$ fail to generate a pure morphism between fixed automata over the pure state sets X and Y.

We denote the category of all A-automata and pure morphisms between them by **PureAut**.

EXAMPLE. With the Boolean scalar semiring, let $E = \{a\}$, $X = \{x_0, x_1, x_2, x_3, x_4\}$, and $Y = \{y_0, y_1\}$. The following diagrams determine right A-modules $\zeta: X \cdot \flat \otimes A \to X \cdot \flat$ and $\xi: Y \cdot \flat \otimes A \to Y \cdot \flat$.



We take the left diagram to specify the right A-action

$$x_0 \cdot 1 = x_0 + x_1 + x_2,$$
$$x_0 \cdot a = x_3 + x_4,$$
$$x_1 \cdot 1 = x_1,$$
$$x_1 \cdot a = x_3,$$
$$x_2 \cdot 1 = x_2,$$
$$x_2 \cdot a = x_4,$$
$$x_3 \cdot 1 = x_3,$$
$$x_3 \cdot a = 0,$$
$$x_4 \cdot 1 = x_4,$$
$$x_4 \cdot a = 0,$$

and $x_i \cdot a^n = 0$ for $0 \leqslant i \leqslant 4$ and all $n > 1$. The right diagram is to specify the right A-action

$$y_0 \cdot 1 = y_0,$$
$$y_0 \cdot a = y_1,$$
$$y_1 \cdot 1 = y_1,$$
$$y_1 \cdot a = 0,$$

and $y_i \cdot a^n = 0$ for $0 \leqslant i \leqslant 1$ and all $n > 1$. Then the function $f: X \to Y$ given by the following table determines a pure morphism $\varphi: X \cdot \mathfrak{h} \to Y \cdot \mathfrak{h}$. The function mapping each x_i to y_0 does not.

Definition of the function $f: X \to Y$:

x	xf
x_0	y_0
x_1	y_0
x_2	y_0
x_3	y_1
x_4	y_1

Note that

$$(x_0\varphi) \cdot 1 = y_0 \cdot 1 = y_0 = (x_0 \cdot 1)\varphi$$

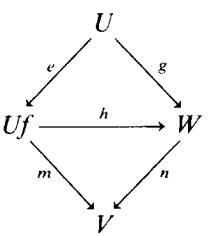
follows from the linearity requirement for morphisms.

2. PureAut is an (Epi, Mono)-Category and is Cocomplete

PureAut may be considered to be a subcategory of **Set** by letting a unique copy of the set of pure states X stand in place of each right A-module $\zeta: X \cdot \mathfrak{h} \otimes A \to X \cdot \mathfrak{h}$. Since each pure morphism is the unique extension of some function $f: X \to Y$ we can consider **PureAut** as a subcategory of **Set**. Note that **PureAut** is not a full subcategory of **Set**, as each hom-set **PureAut**(X, Y) consists only of those functions generating pure morphisms from $\zeta: X \cdot \mathfrak{h} \otimes A \to X \cdot \mathfrak{h}$ to $\xi: Y \cdot \mathfrak{h} \otimes A \to Y \cdot \mathfrak{h}$.

In the following proofs we will use the fact that **Set** is an (epi, mono)-category and is cocomplete. The results in this section are routine consequences of the uniform embedding of **PureAut** into **Set** and are included for the sake of completeness.

1. Proposition. **PureAut** *has unique* (*epi, mono*)*-factorizations.*



Proof. Let $U=\zeta\colon X\cdot\mathfrak{h}\otimes A\to X\cdot\mathfrak{h}$, $V=\xi\colon Y\cdot\mathfrak{h}\otimes A\to Y\cdot\mathfrak{h}$ be objects in **PureAut**, and let $f\colon X\to Y$ be a pure morphism. Define $Uf=\{uf\mid u\in X\cdot\mathfrak{h}\}$ with the induced transition map and define $e\colon X\cdot\mathfrak{h}\to X\cdot\mathfrak{h}f$ by $ue=uf$ for all u in $X\cdot\mathfrak{h}$. Define $m\colon Uf\to V$ by $vm=v$ for v in Uf. Clearly $em=f$ and (e,m) is an (epi, mono)-factorization of f. Now (e,m) is a unique (epi, mono)-factorization of f, for suppose W is an object in **PureAut** and $g\colon U\to W$, $n\colon W\to V$ are pure morphisms such that g is a surjection, n is an injection, and $gn=f$. Define $h\colon Uf\to W$ by $ueh=ug$. The underlying function $h\colon Xf\to Z$ is an isomorphism in **Set** because **Set** has unique (epi, mono)-factorizations. Hence, h is also an isomorphism in **PureAut**. ∎

Similarly, **PureAut** has coproducts if the coproduct of sets is in the subcategory **PureAut**.

2. PROPOSITION. **PureAut** *has arbitrary small coproducts.*

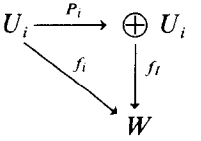
Proof. Let I be an index set and let $\{U_i=\mu_i\colon X_i\cdot\mathfrak{h}\otimes A\to X_i\cdot\mathfrak{h}\mid i\in I\}$ be an I-indexed family of A-automata. Define

$$\bigoplus X_i\cdot\mathfrak{h}=\left\{\sum_{i\in I}u_i\cdot k_i\,\middle|\,u_i\in X_i\cdot\mathfrak{h},\ k_i=0\text{ almost everywhere}\right\}.$$

Define $\bigoplus U_i=\rho\colon\bigoplus X_i\cdot\mathfrak{h}\otimes A\to\bigoplus X_i\cdot\mathfrak{h}$ by

$$\left(\sum_{j=1}^{n}u_{i_j}\otimes\alpha\right)\rho=\sum_{j=1}^{n}(u_{i_j}\otimes\alpha)\,\mu_{i_j}$$

for all $\alpha\in A$, $u_{i_j}\in X_{i_j}\cdot\mathfrak{h}$. Hence $\bigoplus U_i$ is an object in **PureAut**. Define $P_i\colon U_i\to\bigoplus U_i$ by $u_iP_i=u_i$ for all $u_i\in X_i\cdot\mathfrak{h}$ and for each $i\in I$. Clearly each P_i is a pure morphism:



Now $(\bigoplus U_i, I_i)_{i\in I}$ is a coproduct of $\{U_i\mid i\in I\}$, for given an A-automaton $W=\omega\colon Z\cdot\mathfrak{h}\otimes A\to Z\cdot\mathfrak{h}$ and an I-indexed family of pure morphisms $\{f_i\colon U_i\to W\mid i\in I\}$, define $f_I\colon\bigoplus X_i\cdot\mathfrak{h}\to Z\cdot\mathfrak{h}$ by

$$(u_{i_1}+\cdots+u_{i_n})f_I=u_{i_1}f_{i_1}+\cdots+u_{i_n}f_{i_n},$$

hence $P_if_I=f_i$ for all $i\in I$. Now the uniqueness of f_I is a consequence of the uniqueness of the underlying function $f_I\colon\bigoplus X_i\to Z$ in **Set**. ∎

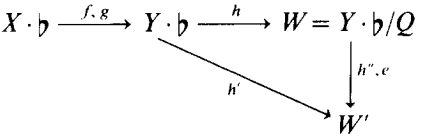
Finally, **PureAut** has coequalizers if the coequalizers of two functions in the subcategory is in fact in the subcategory.

3. PROPOSITION. **PureAut** *has coequalizers.*

Proof. Let $U=\mu: X\cdot\mathfrak{h}\otimes A\to X\cdot\mathfrak{h}$ and $V=\nu: Y\cdot\mathfrak{h}\otimes A\to Y\cdot\mathfrak{h}$ be A-automata, let $f, g: U\to V$ be pure morphisms. Let Q be the smallest equivalence relation on Y that contains all pairs (xf, xg) for all $x\in X$. Define $Z=Y/Q$. For all $y\in Y$ let $[y]$ denote the equivalence class that includes y and define

$$[v]=\left[\sum_{i=1}^{n} x_i k_i\right]=\sum_{i=1}^{n}[x_i]\,k_i.$$

Define $\psi: Z\cdot\mathfrak{h}\otimes A\to Z\cdot\mathfrak{h}$ by: $([v]\otimes a)\psi=[v\cdot a]$ for all $[v]\in Z\cdot\mathfrak{h}$. Suppose we consider $uf, ug\in[v]$, $a\in A$. As $uf\cdot a=u\cdot af$, $ug\cdot a=u\cdot ag$, $u\cdot af$ and $u\cdot ag$ are in $[v\cdot a]$, ψ is well defined. Clearly $\psi: W\otimes A\to W$ is an A-automata. Let $h: Y\cdot\mathfrak{h}\to W$ map each $v\in Y\cdot\mathfrak{h}$ into its equivalence class, then h is a pure morphism because $vh\cdot\alpha=[v]\cdot\alpha=[v\cdot\alpha]=v\cdot\alpha h$:



Now $(h, \psi: Y\cdot\mathfrak{h}\otimes A\to Z\cdot k)\approx$ Coequalizer(f, g) since it is a pre-coequalizer, $fh=gh$, while the uniqueness property is derived from the uniqueness of the underlying function in **Set**. ∎

PureAut has arbitrarily small coproducts and coequalizers, so **PureAut** has all small colimits, including pushouts.

3. WEAK BISIMULATION

Define the relation $\leqslant$ between X and the $\mathfrak{h}$-module $X\cdot\mathfrak{h}$ freely generated by X as follows: $x_0\leqslant u$ for $x_0\in X$ and $u=\sum_{x\in X} x\cdot k_x$ if and only if $k_{x_0}\neq 0$. One might say x_0 *participates in* u when $x_0\leqslant u$.

As before, let $\zeta: X\cdot\mathfrak{h}\otimes A\to X\cdot\mathfrak{h}$ and $\xi: Y\cdot\mathfrak{h}\otimes A\to Y\cdot\mathfrak{h}$ be A-automata with $X\cdot\mathfrak{h}$, $Y\cdot\mathfrak{h}$ freely generated by pure states X, Y, respectively.

The relation $R\subseteq X\times Y$ is a *weak bisimulation* if the following conditions hold:

(1) $R\pi_X=X$, $R\pi_Y=Y$, where π_X is the projection onto X and π_Y is the projection onto Y;

(2) whenever xRy and $a \in E$ then:

(i) $x' \in X$ and $x' \leqslant x \cdot a$ imply there exists $y' \in Y$ for which $y' \leqslant y \cdot a$ and $x'Ry'$; and

(ii) $y' \in Y$ and $y' \leqslant y \cdot a$ imply there exists $x' \in X$ for which $x' \leqslant x \cdot a$ and $x'Ry'$.

As $\alpha = 1^i \alpha 1^j$ for $\alpha \in A$ and all $i, j \in \{0, 1, 2, \ldots\}$, this is equivalent to the definition of weak bisimulation ($\approx$) in (Milner, 1983), rewritten to accommodate our algebraic automata style. Note that in the above $x \cdot \alpha = (x \otimes \alpha)\zeta$ and $y \cdot \alpha = (y \otimes z)\xi$.

4. THEOREM. *Let* $\mu: X \cdot \mathfrak{b} \otimes A \to X \cdot \mathfrak{b}$, $\nu: Y \cdot \mathfrak{b} \otimes A \to Y \cdot \mathfrak{b}$, *and* $\omega: Z \cdot \mathfrak{b} \otimes A \to Z \cdot \mathfrak{b}$ *be A-automata with* $X \cdot \mathfrak{b}$, $Y \cdot \mathfrak{b}$, $Z \cdot \mathfrak{b}$ *freely generated by pure states X, Y, Z, respectively. Let* $f: X \cdot \mathfrak{b} \to Z \cdot \mathfrak{b}$, $g: Y \cdot \mathfrak{b} \to Z \cdot \mathfrak{b}$ *be pure epimorphisms. Define* $R = \{(x, y) \mid x \in X, y \in Y, xf = yg\}$. *Then R is a weak bisimulation.*

Proof. As f, g are pure epimorphisms, for each $x \in X$ there exists some $y \in Y$ such that $xf = yg$, hence $(x, y) \in R$. For $a \in E$ we have $x \cdot a = \sum_{x' \in X} x' \cdot k_{x'}$, for some collection of scalars $k_{x'} \in \mathfrak{b}$, and $y \cdot a = \sum_{y' \in Y} y' \cdot l_{y'}$, for some collection of scalars $l_{y'} \in \mathfrak{b}$. Hence

$$\sum_{x' \in X} (x'f) \cdot k_{x'} = \left(\sum_{x' \in X} x' \cdot k_{x'} \right) f = (x \cdot a) f = (xf) \cdot a = (yg) \cdot a$$

$$= (y \cdot a) g = \left(\sum_{y' \in Y} y' \cdot l_{y'} \right) g = \sum_{y' \in Y} (y'g) \cdot l_{y'},$$

with $x'f$, $y'g \in Z$. If $u = \sum_{x \in X} x \cdot k_x$ is a pure state then $k_x = 0$ for all $x \in X$, except for k_u, and at $u \in X$ one has $k_u = 1$. As f, g are pure morphisms, the values $x'f$, $y'g$ are pure states for pure states x', y', hence

$$\sum_{x'f = z_0} (x'f) \cdot k_{x'} = \sum_{y'g = z_0} (y'g) \cdot l_{y'} \qquad (*)$$

for each $z_0 \in Z$. Now if $(*) \neq 0$ for some z_0 then for each x' such that $x'f = z_0$ there exist y' such that $y'g = z_0 = x'f$ as $\mathfrak{b}$ is zerosumfree. Similarly in the other direction. For each $w \in Z \cdot \mathfrak{b}$, $w \neq 0$ implies that whenever $w \cdot k_w = 0$ for $k_w \in \mathfrak{b}$ we may conclude that $k_w = 0$ since $Z \cdot \mathfrak{b}$ is freely generated and thus has no nonzero annihilators in $\mathfrak{b}$. So if $(*) = 0$ for some $z_0 \in Z$, we have

$$\sum_{x'f = z_0} k_{x'} = \sum_{y'g = z_0} l_{y'} = 0$$

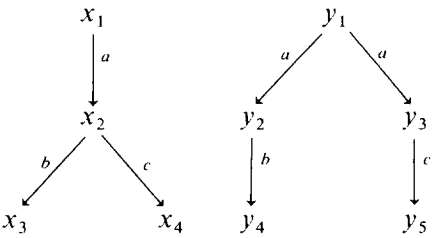
and as $\mathfrak{h}$ is zerosumfree, $k_{x'}=0$ for all $x' \in X$ such that $x'f = z_0$ and $l_{y'}=0$ for all $y' \in Y$ such that $y'g = z_0$. Hence for all $x' \leqslant x \cdot a$ there exists y' such that $y' \leqslant y \cdot a$, and $x'f = y'g$. Similarly, for all $y' \leqslant y \cdot a$ there exist x' such that $x' \leqslant x \cdot a$, $x'f = y'g$. This establishes that $(x', y') \in R$. ∎

Theorem 4 provides the connection between bisimulation relations and pure epimorphisms. From here on we use category theory to obtain the main results.

5. COROLLARY. *If there is a pure epimorphism between two A-automata then there is a weak bisimulation between them.*

Proof. Let $\mu: X \cdot \mathfrak{h} \otimes A \to X \cdot \mathfrak{h}$, $\nu: Y \cdot \mathfrak{h} \otimes A \to Y \cdot \mathfrak{h}$ be A-automata, and let $f: X \cdot \mathfrak{h} \to Y \cdot \mathfrak{h}$ be a pure epimorphism. The identity morphism $\mathbf{1}_Y \cdot \mathfrak{h}$: $Y \cdot \mathfrak{h} \to Y \cdot \mathfrak{h}$ is a pure epimorphism, hence according to the previous theorem there is a weak bisimulation between $X \cdot \mathfrak{h}$ and $Y \cdot \mathfrak{h}$. ∎

For example consider the following two automata:



The above automata are not bisimulated and there is no pure epimorphism between them. For suppose to the contrary there was an pure epimorphism generated by a function $f: X \to Y$ then:

$$(y_2 \cdot c)f = 0 \neq (y_2 f) \cdot c = x_3$$
$$(y_3 \cdot b)f = 0 \neq (y_3 f) \cdot b = x_4.$$

Hence f is not a morphism. These inequalities are exactly the reasons that inhibit a bisimulation between these two automata.

6. *Observation.* Let R be the union of all weak bisimulations between two A-automata U and V. It follows that R is the maximum weak bisimulation between U and V.

Let $B = \{U_i \mid i \in I\}$ be a system of representatives of a weak bisimulation equivalence class. Technically, each V in the class is isomorphic to some U_i and the U_i are non-isomorphic. We denote the run map for U_i by

$\mu_i: X_i \cdot \mathfrak{b} \otimes A \to X_i \cdot \mathfrak{b}$ for each $i \in I$. Note that each $X_i \cdot \mathfrak{b}$ is freely generated by X_i.

Let R_{ij} denote the maximum weak bisimulation relation between U_i and U_j, for each $i, j \in I$. Let π_i denote the projection from $X = \prod_{i \in I} X_i$ to X_i. Define $R = \{x \in X \mid \text{for all } i, j \in I, (x\pi_i, x\pi_j) \in R_{ij}\}$.

7. LEMMA. *For each $i \in I$ and each $x_i \in X_i$ there is an $r \in R$ such that $r\pi_i = x_i$.*

Proof. Fix $i_0 \in I$ and $x_{i_0} \in X_{i_0}$. For each $i \in I$ there exists $x_i \in X_i$ such that $(x_{i_0}, x_i) \in R_{i_0 i}$, since B is a system of representatives of some weak bisimulation equivalence class. This establishes $x = (x_0, x_1, \ldots)$ in X with the property that $(x\pi_{i_0}, x\pi_i) \in R_{i_0 i}$ for all $i \in I$. As weak bisimulation is symmetric and transitive, $(x\pi_i, x\pi_j) \in R_{ij}$ for all $i, j \in I$, hence $x \in R$. ∎

Let $R \cdot \mathfrak{b}$ be the right $\mathfrak{b}$-module freely generated by R. Define $P_i: R \cdot \mathfrak{b} \to X_i \cdot \mathfrak{b}$ for $i \in I$ by $(x_0, x_1, \ldots) P_i = x_i$. Clearly P_i is a pure epimorphism for all $i \in I$. Define the $\mathfrak{b}$-linear map $\rho: R \cdot \mathfrak{b} \otimes A \to R \cdot \mathfrak{b}$ by $(v \otimes a) \rho P_i = (vP_i \otimes a) \mu_i$ for all $i \in I$.

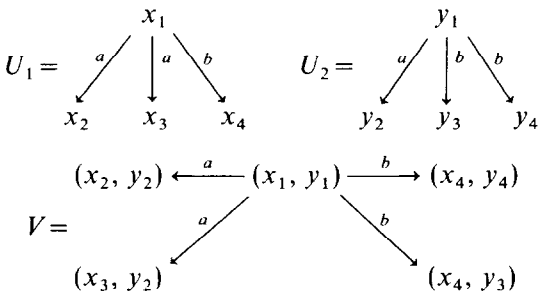
8. LEMMA. *$V = \rho: R \cdot \mathfrak{b} \otimes A \to R \cdot \mathfrak{b}$ is an A-automata.*

Proof. Because $\mu_i: X_i \cdot \mathfrak{b} \otimes A \to X_i \cdot \mathfrak{b}$ are all A-automata, it suffices to show that if $r \in R$ then $r \cdot a \in R \cdot \mathfrak{b}$. Suppose $r \cdot a = \sum_{j=1}^{m} r_j \cdot k_j$ with $k_j \in \mathfrak{b}$, $r_j \in R$. If (x_i, x_j) is in R_{ij}, then for every $x_{i'}$ such that $x_{i'} \leqslant x_i \cdot a$, there exist $x_{j'}$ such that $x_{j'} \leqslant x_j \cdot a$ and $(x_{i'}, x_{j'})$ is in R_{ij} hence $r_j \in R$ for all $j \in I$, hence $r \cdot a \in R \cdot \mathfrak{b}$. ∎

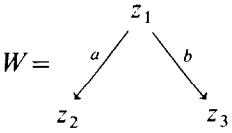
Let $(\{f_i\}_{i \in I}, Z \cdot \mathfrak{b}) \approx \text{Pushout}(V, \{P_i\}_{i \in I})$. As the P_i are pure epimorphisms, by the general fact that the pushout of an epimorphism is always an epimorphism, (Herrlich and Strecker, 1979, Section 21.13-dual), the f_i are also pure epimorphisms.

Note that as f_i are pure epimorphisms, and thus surjective functions on generators, the cardinality of the state set of $Z \cdot \mathfrak{b}$ is less than or equal to the cardinality of the state set of each of $X_i \cdot \mathfrak{b} = R \cdot \mathfrak{b} P_i$.

Consider the following example of a pushout of A-automata. Let



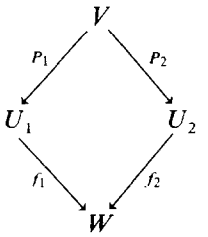
Note that the pure states of V are all of the form (x, y), with xRy for R the maximum bisimulation between U_1 and U_2. Let



Let $P_i: V \to U_i$, $i = 1, 2$, be the projection from V onto U_i. Let $f_i: U_i \to W$, $i = 1, 2$, be defined on the pure states as follows:

f_1		f_2	
x	xf_1	y	yf_2
x_1	z_1	y_1	z_1
x_2	z_2	y_2	z_2
x_3	z_2	y_3	z_3
x_4	z_3	y_4	z_3

Now the following diagram illustrates the fact that $(\{f_1, f_2\}, W) = \text{Pushout}(V, \{P_1, P_2\})$.



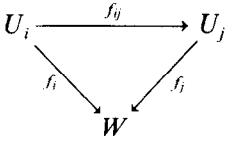
A weakly connected component of a category consists of all objects connected in the underlying *undirected* graph of the category. So ζ_0 and ζ_n are weakly connected in **PureAut** when there are pure morphisms $\zeta_0 \to \zeta_1 \leftarrow \zeta_2 \to \cdots \leftarrow \zeta_n$. An epi-connected component of a category is a weakly connected component in which all connecting morphisms are epimorphisms.

9. COROLLARY. *The epi-connected components of* **PureAut** *are exactly the weak bisimulation equivalence classes.*

Minimal Automata

10. LEMMA. *Let D be a diagram in* **PureAut** *with objects all the A-automata of a weak bisimulation equivalence class B and arrows the epimorphisms between objects in B. Let $(\{f_i \mid i \in I\}, W)$ be the colimit of D.*

Let $\{f_{ij}: U_i \to U_j \mid i, j \in I\}$ *be pure epimorphisms of* D, *making the following diagrams commute for all* $i, j \in I$:



Then f_i, f_j *are pure epimorphisms.*

Proof. Uniquely factor each morphism $f_i: X_i \cdot \flat \to Z \cdot \flat$ of the cocone, where $U_i = \mu_i: X_i \cdot \flat \otimes A \to X_i \cdot \flat$ and $W = \zeta: Z \cdot \flat \otimes A \to Z \cdot \flat$, as $e_i m_i = f_i$ with e_i a pure epimorphism and m_i a monomorphism. As f_{ij} is a pure epimorphism, $f_{ij} f_j$ uniquely factors as $f_{ij} f_j = f_{ij} e_j m_j$. But $f_i = f_{ij} f_j$ implies, by unique (epi, mono)-factorization, that $f_{ij} e_j = e_i$. Hence $(\{e_i \mid i \in I\}, Z \cdot \flat)$ is a sink. By the extremal condition of colimits (Herrlich and Strecker, 1979, Section 20.4-dual, Section 19.1(3)-dual), each m_i is an isomorphism. Hence $f_i = e_i$. ∎

The following theorem resembles similar results in (Goguen, 1973; Benson, 1974), for example. Those results have been generalized in (Goguen and Meseguer, 1982; Meseguer and Goguen, 1985). These generalizations do not appear to include the theory of this paper as corollaries, despite the substantial similarities in purpose and development.

11. THEOREM. *Every weak bisimulation equivalence class has a minimal representing automaton, minimal with respect to the cardinality of the pure states and unique up to isomorphism.*

Proof. Immediate consequence of the previous lemma. ∎

4. STRONG BISIMULATION

12. *Observation.* A weak bisimulation relation between automata with only trivial silent actions, so that the right A-modules are unital, is a strong bisimulation relation between these automata.

As a consequence of the above observation, or directly from (Milner, 1983), it is clear that strong bisimulation is a degenerate case of weak bisimulation. Hence if we restrict ourselves to automata that have only trivial silent actions, then all the theorems that hold for weak bisimulation also hold as a special case for strong bisimulation.

Here are the two most important results restated for strong bisimulation:

13. COROLLARY. *The epi-connected components of the automata having only trivial silent actions in* **PureAut** *are exactly the strong bisimulation equivalence classes.*

14. COROLLARY. *Every strong bisimulation equivalence class has a minimal representing automaton, minimal with respect to the cardinality of the pure states and unique up to isomorphism.*

5. ROOTED BISIMULATION IS A JOIN CONGRUENCE

The *join*, denoted by $\vee$, of two automata with start states is the automata-theoretic equivalent of the alternative choice operator of CCS, which is denoted by + in most CCS-related studies. As is well known from (Milner, 1983; Hennessy and Milner, 1985), bisimulation is not a congruence with respect to alternative choice. However, from (Bergstra and Klop, 1984, 1985a, 1985b) we know that *rooted bisimulation* is a congruence for context involving the alternative choice. In this section we show that rooted bisimulation is a congruence with respect to the join of automata with start states. In addition, we provide a theory for rooted bisimulation entirely analogous to that of previous sections for weak bisimulation.

DEFINITION. An *automaton with start state*, (X, ζ, x_0), consists of a set of pure states, X, a run map $\zeta: X \cdot \flat \otimes A \to X \cdot \flat$, and a distinguished element of X, x_0.

DEFINITION. The *join* of automata with start states $(X, \zeta: X \cdot \flat \otimes A \to X \cdot \flat, x_0)$ and $(Y, \xi: Y \cdot \flat \otimes A \to Y \cdot \flat, y_0)$ is the automaton with start state $(Z, \rho: Z \cdot \flat \otimes A \to Z \cdot \flat, z_0)$, where

$$Z = (X - \{x_0\}) \sqcup (Y - \{y_0\}) \sqcup \{z_0\}$$

and the run map ρ is defined as follows: First, let $j_X: X \to Z$ and $j_Y: Y \to Z$ be given by

$$xj_X = \begin{cases} x & \text{if } x \neq x_0 \\ z_0 & \text{if } x = x_0 \end{cases}$$

and

$$yj_Y = \begin{cases} y & \text{if } y \neq y_0 \\ z_0 & \text{if } y = y_0. \end{cases}$$

Let the unique extension to $\flat$-linear maps be denoted by $j_X: X \cdot \flat \to Z \cdot \flat$

and $j_Y: Y \cdot \mathfrak{h} \to Z \cdot \mathfrak{h}$, respectively. Now define the $\mathfrak{h}$-linear map $\rho: Z \cdot \mathfrak{h} \otimes A \to Z \cdot \mathfrak{h}$ by the action of ρ on the pure states Z as

$$(z \otimes \alpha)\rho = \begin{cases} (z \otimes \alpha)\,\zeta j_X & \text{if } z \in X \\ (z \otimes \alpha)\,\xi j_Y & \text{if } z \in Y \\ (x_0 \otimes \alpha)\,\zeta j_X + (y_0 \otimes \alpha)\,\xi j_Y & \text{if } z = z_0. \end{cases}$$

The join of automata with start states $U_i = (X_i, \zeta_i, x_i)$, $i = 1, 2$, is denoted by $U_1 \vee U_2$.

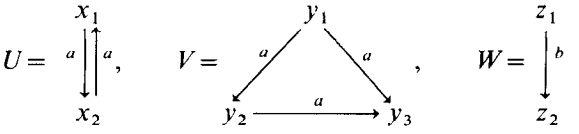
15. PROPOSITION. *The join of two automata with start states is an automaton with start states.*

Proof. Let $U = (X, \zeta: X \cdot \mathfrak{h} \otimes A \to X \cdot \mathfrak{h}, x_0)$, $V = (Y, \xi: Y \cdot \mathfrak{h} \otimes A \to Y \cdot \mathfrak{h}, y_0)$, and $W = U \vee V = (Z, \rho: Z \cdot \mathfrak{h} \otimes A \to Z \cdot \mathfrak{h}, z_0)$. For each $w \in Z \cdot \mathfrak{h}$, $w = uj_X + vj_Y$ for some $u \in X \cdot \mathfrak{h}$, $v \in Y \cdot \mathfrak{h}$. Hence for all $\alpha, \beta \in A$,

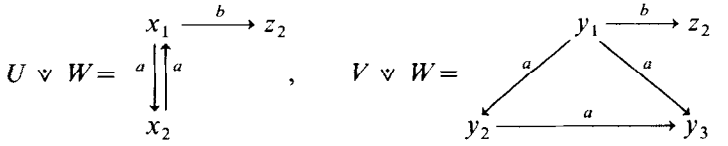
$$\begin{aligned} w \cdot (\alpha\beta) &= (uj_X + vj_Y) \cdot (\alpha\beta) = u(\alpha\beta) j_X + v(\alpha\beta) j_Y \\ &= ((u \cdot \alpha) \cdot \beta) j_X + ((v \cdot \alpha) \cdot \beta) j_Y = (uj_X \cdot \alpha) \cdot \beta + (vj_Y \cdot \alpha) \cdot \beta \\ &= ((uj_X + vj_Y) \cdot \alpha) \cdot \beta = (w \cdot \alpha) \cdot \beta. \quad \blacksquare \end{aligned}$$

The following two examples demonstrate that bisimulation is not a congruence with respect to join. Let x_1, y_1, z_1 be the start states of U, V, W, respectively.

(1) Let

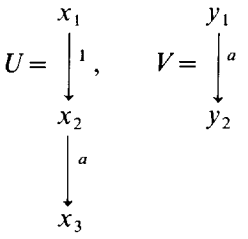


Now U is bisimilar to V but

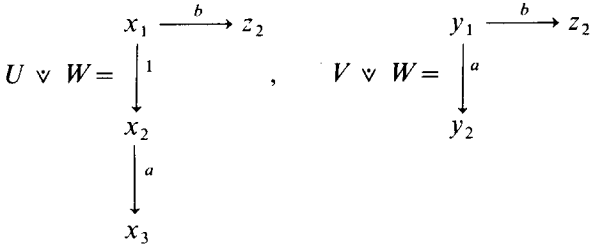


so that $U \vee W$ is not bisimilar to $V \vee W$.

(2) Let



With W as in the prior example,



We see that U is bisimilar to V but $U \vee W$ is not bisimilar to $V \vee W$.

In order to overcome this difficulty we will define a restriction of bisimulation which is a congruence for join.

As before, let $U_i = (X_i, \zeta_i : X \cdot \flat \otimes A \to X \cdot \flat, x_{0,i})$, $i = 1, 2$, be two automata with start states. The relation $R \subseteq X \times Y$ is a *rooted weak bisimulation* if the following conditions hold:

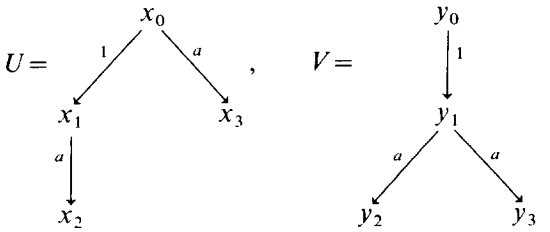
(1) R is a weak bisimulation,

(2) $(x_{0,1}, x_{0,2}) \in R$,

(3.1) for all $x \in X_1$, $x \neq x_{0,1}$ implies $(x, x_{0,2}) \notin R$,

(3.2) for all $x \in X_2$, $x \neq x_{0,2}$ implies $(x, x_{0,1}) \notin R$.

We write $U \approx_c V$ if there exists a rooted bisimulation relation between U and V. Notice that $U \not\approx_c V$ in the previous two examples. Here is an example of a rooted bisimulation. Let



and let $R \subseteq X \times Y$ be the collection of ordered pairs,

$$(x_0, y_0),$$
$$(x_1, y_1),$$
$$(x_2, y_2), \quad (x_2, y_3)$$
$$(x_3, y_2), \quad (x_3, y_3).$$

Clearly the start states, x_0, y_0, are related to each other and to no other states. One may, with some tedium, verify that R is a weak bisimulation.

16. THEOREM. *Rooted bisimulation is a congruence with respect to join; furthermore, rooted bisimulation is the coarsest bisimulation congruence with respect to join.*

Proof. Let U, V, W be objects in **PureAut**, let x_0, y_0 be the start states of U, V, respectively. Suppose there is a rooted bisimulation between U and V. We claim that $C_1 = U \vee W \approx_c V \vee W = C_2$ because, as the start states of U and V are bisimilar, the start states of C_1 and C_2 are also bisimilar. Furthermore, condition (3) of the rooted bisimulation relation guaranties that the non-start pure states in C_1, C_2 are also non-start pure states in U, V, W; so they can be related to the same states as in the bisimulation between U, V or as in the identity relation on W. Conversely, suppose there is a bisimulation relation between U, V but $U \not\approx_c V$, then there exist $y \neq y_0$ such that the only pure state in U that is bisimilar to y is x_0. Now let W, as before, be such that there is no pure state in W bisimilar to y. Now $U \vee W \not\approx V \vee W$, because there is no pure state in $U \vee W$ that is bisimilar to y, which is a pure state in $V \vee W$. ∎

The following provides results for rooted bisimulation which are entirely analogous to the case of bisimulation.

Let $\mu: X \cdot \flat \otimes A \to X \cdot \flat$, $\nu: Y \cdot \flat \otimes A \to Y \cdot \flat$ be A-automata with $X \cdot \flat$, $Y \cdot \flat$ freely generated by pure states X, Y, respectively. Suppose $x_0 \in X$, $y_0 \in Y$ are the start states of $X \cdot \flat$, $Y \cdot \flat$, respectively, then $f: X \cdot \flat \to Y \cdot \flat$ is a *rooted* morphism if f is a pure morphism and further, $xf = y_0$ iff $x = x_0$.

17. THEOREM. *Let $\mu_i: X_i \cdot \flat \otimes A \to X_i \cdot \flat$, $i = 1, 2, 3$, be A-automata with $X_i \cdot \flat$ freely generated by pure states X_i, respectively. Let x_0, y_0, z_0 be the start states of $X_i \cdot \flat$, $i = 1, 2, 3$, respectively. Let $f: X_1 \cdot \flat \to X_3 \cdot \flat$, $g: X_2 \cdot \flat \to X_3 \cdot \flat$ be rooted pure epimorphisms. Define $R = \{(x, y) \mid x \in X, y \in Y, xf = yg\}$. Then R is a rooted bisimulation.*

Proof. From Theorem 4, we need only to show that $(x_0, y_0) \in R$ and that this is the only pair in R that contains either x_0 or y_0. Now $x_0 f =$

$z_0 = y_0 g$ as f is a rooted morphism, and also $z_0 f^{-1} = x_0$, $z_0 g^{-1} = y_0$, which concludes the proof. ∎

18. COROLLARY. *If there is a rooted pure epimorphism between two A-automata, then there is a rooted bisimulation between them.*

Let $B = \{U_i | i \in I\}$ be a system of representatives of a rooted bisimulation equivalence class. Let R_{ij} denote the maximum rooted bisimulation between U_i and U_j for each $i, j \in I$. That is, R_{ij} is the union of all rooted bisimulations between U_i and U_j. Let π_i denote the projection from $X = \prod_{i \in I} X_i$ to X_i. Define $R = \{x \in X |$ for all $i, j \in I$, $(x\pi_i, x\pi_j) \in R_{ij}\}$.

19. LEMMA. *For each $i \in I$ and $x_i \in X_i$ there is an $r \in R$ such that $r\pi_i = x_i$.*

Let $R \cdot \mathfrak{h}$ be the right A-module freely generated by R. Define $P_i: R \cdot \mathfrak{h} \to X_i \cdot \mathfrak{h}$ for $i \in I$ by $(x_0, x_1, \ldots) P_i = x_i$, P_i is a pure epimorphism for all $i \in I$. Define the $\mathfrak{h}$-linear map $\rho: R \cdot \mathfrak{h} \otimes A \to R \cdot \mathfrak{h}$ by $(v \otimes a) \rho P_i = (vP_i \otimes a) \mu_i$ for all $i \in I$.

20. LEMMA. *$V = \rho: R \cdot \mathfrak{h} \otimes A \to R \cdot \mathfrak{h}$ is an A-automata.*

Lemmata 19 and 20 are corollaries of Lemmata 7 and 8, respectively, as rooted bisimulation is itself a bisimulation.

Let $(\{f_i\}_{i \in I}, W) \approx \text{Pushout}(V, \{P_i\}_{i \in I})$, where P_i is the projection from $R \cdot \mathfrak{h}$ onto $X_i \cdot \mathfrak{h}$. As the P_i are rooted pure epimorphisms, by (Herrlich and Strecker, 1979, Section 21.13-dual) the f_i are also pure epimorphisms.

21. LEMMA. $\{f_i\}_{i \in I}$ *are rooted pure epimorphisms.*

Proof. Let $r_0 \in R$, $x_{i,0} \in X_i$ be the start states of $R \cdot \mathfrak{h}$, $X_i \cdot \mathfrak{h}$, respectively, then $r_0 P_i f_i = x_{i,0} f_i = z_0$. Suppose $x_j \in X_j$ such that $x_j \neq x_{j,0}$, $x_j f_j = z_0$, then according to Theorem 4, x_j, $x_{i,0}$ are bisimulated, but $(x_{i,0}, x_{j,0}) \in R_{ij}$, hence $(x_j, x_{i,0}) \notin R_{ij}$ as R_{ij} is a rooted bisimulation. ∎

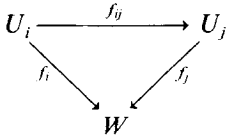
A rooted epi-connected component of a category consists of all objects weakly connected solely by rooted pure epimorphisms.

22. COROLLARY. *The rooted epi-connected components of* **PureAut** *are exactly the rooted bisimulation equivalence classes.*

Minimal Automata

Let D be a diagram in **PureAut** with objects all the A-automata of a rooted bisimulation equivalence class B and arrows the rooted epimorphisms between objects in B.

23. LEMMA. *Let* $(\{f_i | i \in I\}, W)$ *be the colimit of* D. *Let* $\{f_{ij}: U_i \to U_j | i, j \in I\}$ *be rooted pure epimorphisms in* D *making the following diagram commute for all* $i, j \in I$:



Then f_i, f_j *are rooted pure epimorphisms.*

Proof. As a result of Lemma 10, f_i are pure epimorphisms. Let x_{i_0} be the start state of $X_i \cdot \flat$, then $x_{i_0} f_{ij} = x_{j_0}$ and $x_{i_0} f_i = z_0$. Suppose there exists $x_{i_1} \in X_i$, where X_i freely generates $X_i \cdot \flat$, $x_{i_1} \neq x_{i_0}$, $x_{i_1} f_i = z_0$; now for all $j \in I$, $x_{i_1} f_{ij} = x_{j_1} \neq x_{j_0}$ because the f_{ij} are rooted pure epimorphisms. Hence for every X_i there exists $x_{i_1} \in X_i$ such that $x_{i_1} \neq x_{i_0}$ and $x_{i_1} f_i \neq z_0$. Let Z be the pure states of W. Define $Z' = Z \sqcup \{z'\}$, $z' \notin Z$, and let $Z' \cdot \flat$ be freely generated by Z'. Define $f_i': X_i \cdot \flat \to Z' \cdot \flat$ by

$$x_i f_i' = \begin{cases} x_i f_i & \text{if } x_i \neq x_{i_1} \\ z' & \text{if } x_i = x_{i_1}. \end{cases}$$

Now $f_{ij} f_i' = f_i'$, hence W' is a natural sink for D. However, there is no pure epimorphism $h: W \to W'$ such that $f_i h = f_i'$ because if there was such h then $z_0 h = z_0$ and $z_0 h = z'$. Hence contradiction to the fact that W is a colimit. ∎

24. THEOREM. *Every rooted bisimulation equivalence class has a minimal representing automaton, minimal with respect to the cardinality of the pure states and unique up to isomorphism.*

Proof. Immediate consequence of the previous lemma. ∎

Note that the above results hold for strong bisimulation congruence as well as for weak bisimulation congruence, in the same manner as in Section 4.

CONCLUSION

These results are categorical. Any subcategory of **PureAut** with a reasonable notion of bisimulation will have its epiconnected components as the bisimulation equivalence classes. Any subcategory of **PureAut** with coequalizers and with coproducts of sufficiently large cardinality will have unique minimal automata representing the bisimulation equivalence

classes. Typical examples are the category of finite A-automata and all pure morphisms between them and the category of recursively defined A-automata in which the pure states, transition functions and pure morphisms are given by total recursive functions.

In conclusion, we see that SCCS and ASCCS may be viewed as methods of expressing certain equivalence classes of automata, each with a unique minimal representing automaton.

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