The Bénabou-Roubaud result via string diagrams

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Abstract

We present the use of string diagrams for representation of the Grothendieck's cocycle condition in the special case of the basic fibration of a category and give a graphical proof of the Bénabou-Roubaud result in this setting. We also show how should our arguments be "lifted" in order to derive the proof for the general case of an arbitrary bifibration.

1 Preliminaries

Let $\mathbf{C}$ be a category with pullbacks. Given $f : B \to A$, the idea is to recover a morphism $t : X \to A$ from $s = f^*t$ by "descending" along f. We begin by characterising three different structures induced by f.

1.1 T-algebra structure

Let's start by recalling that the morphism $f : B \to A$ induces an adjunction

$$f_* \dashv f^* : \mathbf{C}/B \to \mathbf{C}/A,$$

in which

- $f_* : \mathbf{C}/B \to \mathbf{C}/A$ is the composition functor, defined by $f_*(s) = f \circ s$, and
- $f^* : \mathbf{C}/A \to \mathbf{C}/B$ is the pullback functor, defined by $f^*(s) = f^*s$, where f^*s is a pullback of s along f,

with unit $\eta : 1_{\mathbf{C}/B} \to f^*f_*$ and counit $\varepsilon : f_*f^* \to 1_{\mathbf{C}/A}$.

Let $T = (T, \eta, \mu)$ be the monad on $\mathbf{C}/B$ corresponding to this adjunction. Hence, the functor is given by $T = f^*f_* : \mathbf{C}/B \to \mathbf{C}/B$, the unit η is the unit of the adjunction, and the multiplication μ is defined in the usual way as $\mu = f^*\varepsilon f_* : T^2 \to T$.

Recall that a *T-algebra structure on s* is given by a morphism $\alpha : T(s) \to s$ in $\mathbf{C}/B$, which then satisfies

(TA1) $\alpha \circ \eta_s = 1_{\mathbf{C}/B}$, and

(TA2) $\alpha \circ \mu_s = \alpha \circ T\alpha$.

Let $(\mathbf{C}/B)^T$ be the Eilenberg–Moore category for the monad T.

Finally, recall that an adjunction $F \dashv G : \mathbf{C} \to \mathbf{D}$ is called monadic when $\mathbf{D}$ is equivalent to the category of T-algebras, where $T = GF$ is the monad induced by the adjunction.

1.2 Descent data

Next, let (d_0, d_1) be the kernel pair of f and denote $e = \langle id_B, id_B \rangle : B \to B \times_A B$. Note that whenever $s = f^*t$ (since $fd_1 = fd_0$), there is a canonical isomorphism $d_0^*s \cong d_1^*s$ in $\mathbf{C}/B \times_A B$.

We call an isomorphism $\varphi : d_0^*s \to d_1^*s$ a *descent datum for* s if it satisfies the following cocycle conditions:

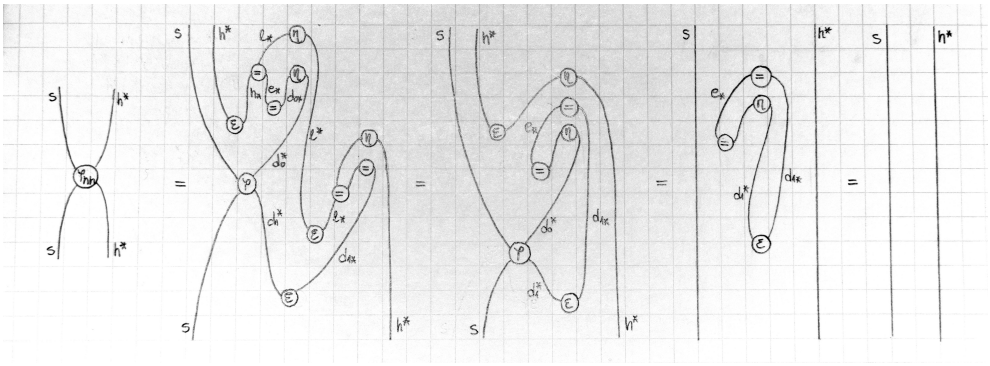
(DD1) $\varphi \circ \delta_0 = \delta_1$, and

(DD2) $\pi_0^*\varphi \circ \omega \circ \pi_1^*\varphi = \omega_2^{-1} \circ \pi^*\varphi \circ \omega_1$,

in which

- $\delta_i : e_* \to d_i^*$, $i = 1, 2$, is a natural transformation defined as $\delta_i = \eta_{d_i} \circ \iota_i$, where $\iota_i : d_{i*}e_* = 1_{\mathbf{C}/B}$, $i = 1, 2$, is the identity natural transformation (note that $d_i \circ e = id_B$),
- π_0 and π_1 are the pullback projections for the pair (d_0, d_1), such that $d_1\pi_1 = d_0\pi_0$,
- $\pi = \langle d_0\pi_1, d_1\pi_0 \rangle : (B \times_A B) \times_B (B \times_A B) \to (B \times_A B)$, and
- the canonical isomorphisms ω, ω_1 and ω_2 arise from the equalities $d_1\pi_1 = d_0\pi_0$, $d_0\pi_1 = d_0\pi$ and $d_1\pi_0 = d_1\pi$, respectively.

One may rightly wonder about the role of these conditions in the descent setting, i.e. from the point of view of the properties satisfied by an s of the form f^*t. The answer begins with the observation that an isomorphism $\varphi_{h,k} : h^*s \to k^*s$ exists given *any* commutative square $fh = fk$, not necessarily a pullback, and we could have defined a descent data for s to be the family of all such isomorphisms, i.e. the family indexed by all these commutative squares. Chosing this approach would require us to specify certain naturality conditions for this family, which in turn come down to the fact that it is determined by its instance $\varphi_{d_0,d_1} = \varphi$. Coherence condition (DD1) now ensures that the natural requirement $\varphi_{id,id} = id_s$ holds:



Similarly, (DD2) should be perceived with respect to *any* collection of morphisms that make a commutative cube with the three axes being f, pointing to the origin of the cube, and it accounts for preserving composition (up to canonical isomorphisms). It is not difficult (but it is cumbersome) to show that the coherence law for general cubes follows from the instance in which all the faces are pullbacks.

Let us denote the category of descent data for f (whose objects are pairs (s, φ) of an object s from $\mathbf{C}/B$ and a descent datum φ for s) with $\mathbf{Des}(f)$.

1.3 Internal presheaf structure

As a third construction induced by the morphism $f : B \to A$ we have a category C_f internal to $\mathbf{C}$, by setting $C_0 = B$, $C_1 = B \times_A B$, $s = d_0$, $t = d_1$ and taking

- $e = \langle id_B, id_B \rangle : B \to B \times_A B$ for the identity-assigning morphism, and
- $\pi = \langle d_0\pi_1, d_1\pi_0 \rangle : (B \times_A B) \times_B (B \times_A B) \to (B \times_A B)$ for the composition morphism, where π_0 and π_1 are the same pullback projections as before.

Recall that the "morphism part" of an *internal (covariant) presheaf* over the internal category $(B, B \times_A B)$, whose "object part" is given by s, is defined as a morphism $\beta : X \times_B (B \times_A B) \to X$ in $\mathbf{C}$, such that

(IP1) $s \circ \beta = d_1 \circ d_0^* s$,

(IP2) $\beta \circ \langle id_X, e \circ s \rangle = id_X$, and

(IP3) $\beta \circ (id_X \times \pi) = \beta \circ (\beta \times id_{B \times_A B})$.

Internal presheaves on C_f in $\mathbf{C}$ form a category denoted by $\mathbf{C}^{C_f}$.

We can now state the problem more precisely: our basic objective consists in stating the condition(s) to be fullfilled by f in order for it to induce a canonical equivalence between categories $\mathbf{C}/A$ and $\mathbf{Des}(f)$. Such a morphism f is called an *effective descent morphism*.

1.4 Coherence of equalities

We finish the introductory section by stating one of the explicit equality rules for string diagrams, which will be used on multiple occasions during the course of the proof. Namely, it is the principle that allowes us to replace a diagram consisting only of equality nodes by any other diagram of equalities with the same input and output interface.

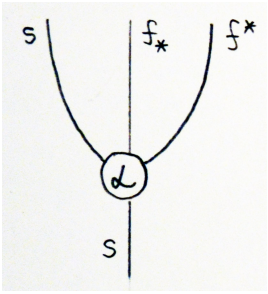
2 The Bénabou-Roubaud result

Theorem 2.1. *For a morphism $s : X \to B$, the following are equivalent:*

1. *a morphism exibiting a T-algebra structure on s, where T is the monad $f^* f_*$,*
2. *a descent datum for s, and*
3. *a "morphism part" of the internal presheaf whose "object part" is given by s.*

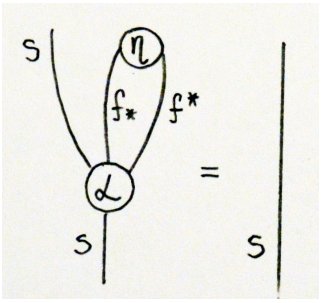
Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) : Let $\alpha : T(s) \to s$ be the T-algebra morphism induced by the monad $T = f^* f_*$. Graphically, α is represented as

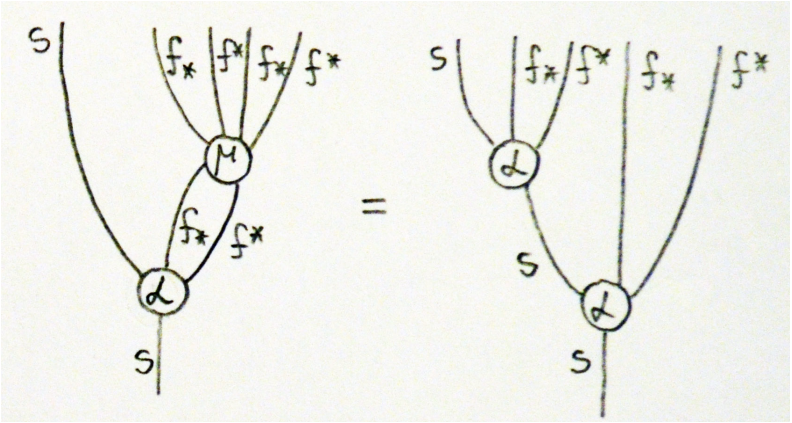


and the two axioms relating α with η and μ are given by the following equlalities between string diagrams:

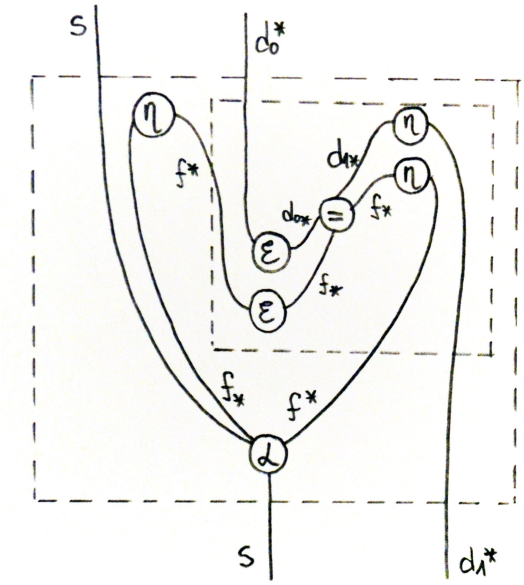
· EQUATION (TA1):



· EQUATION (TA2):

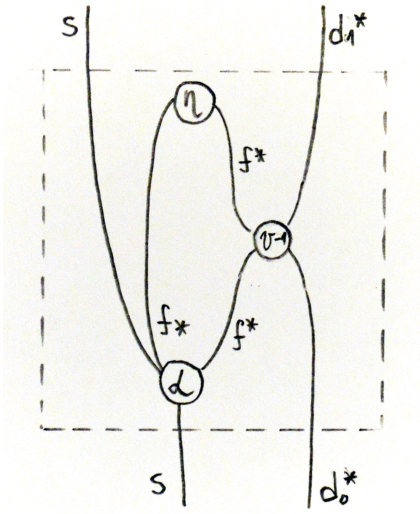


We define the isomorphism $\varphi : d_0^* s \to d_1^* s$ as follows:



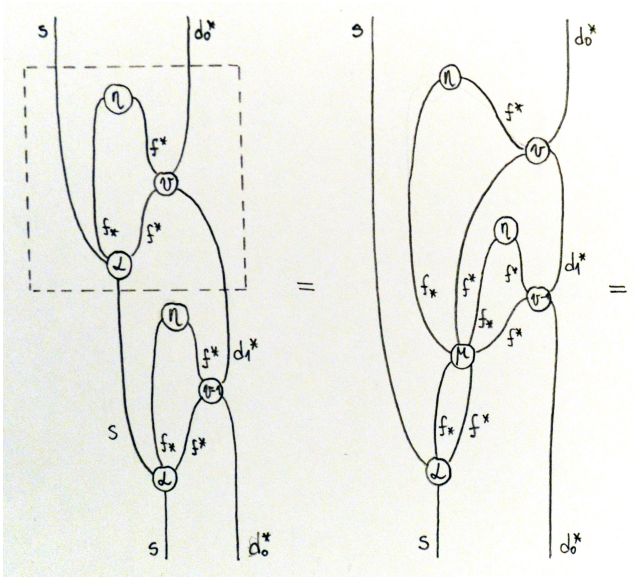
which, by forgetting the details in the outer box, is an arrow from d_0^*s to d_1^*s as required. Note that the content of the inner box represents the canonical isomorphism $v : d_0^*f^* \to d_1^*f^*$, so for the simplicity we will sometimes represent it just as a "black box" v.

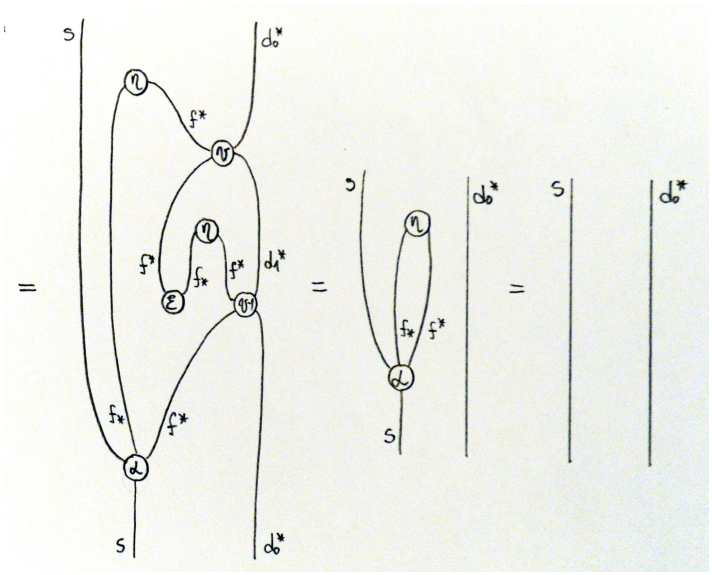
Let us now show that φ is indeed an isomorphism, with the inverse $\psi : d_1^*s \to d_0^*s$ defined in the same way as φ, only with v replaced by its inverse v^{-1}:



Let us prove, for example, that $\psi(\varphi(d_0^*s)) = d_0^*s$. By attaching the morphism ψ on the

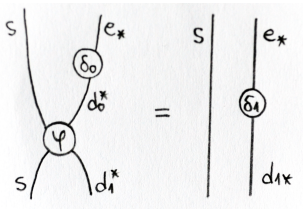
output of φ, we obtain the subregion with two instances of α, to which we then apply the equation (TA2). The second string diagram equality is obtained by substituting the "black box" μ with its explicit definition, and the third one holds thanks to the $\varepsilon - \eta$ rule for the adjunction $f_* \dashv f^*$. The conclusion finally follows from the (TA1) rule for algebra α:



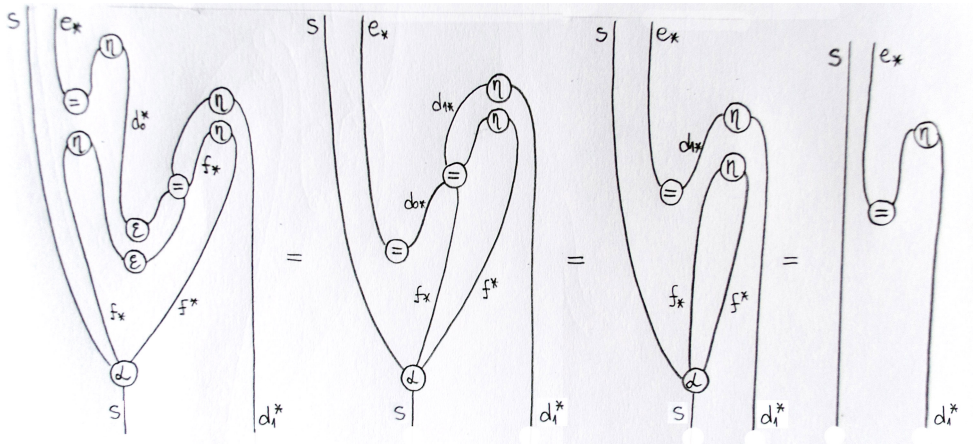


Our next step is to show that φ satisfies the condition (DD1), i.e. that the following string diagram equality holds:

· EQUATION (DD1):

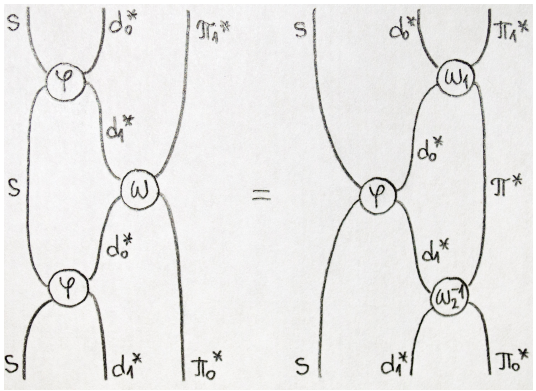


We start by unfolding all the definitions contained in the left-hand side of the equality, which enables us to remove $\eta - \varepsilon$ redexes for the adjunctions $d_{0*} \dashv d_0^*$ and $f_* \dashv f^*$. Then, using the coherence of equalities principle, we replace the subregion containing two equality nodes with the equal one, which provides us with isolated string diagram from the left hand side of the equation (TA1), so the last equality finally follows by aplying that rule:

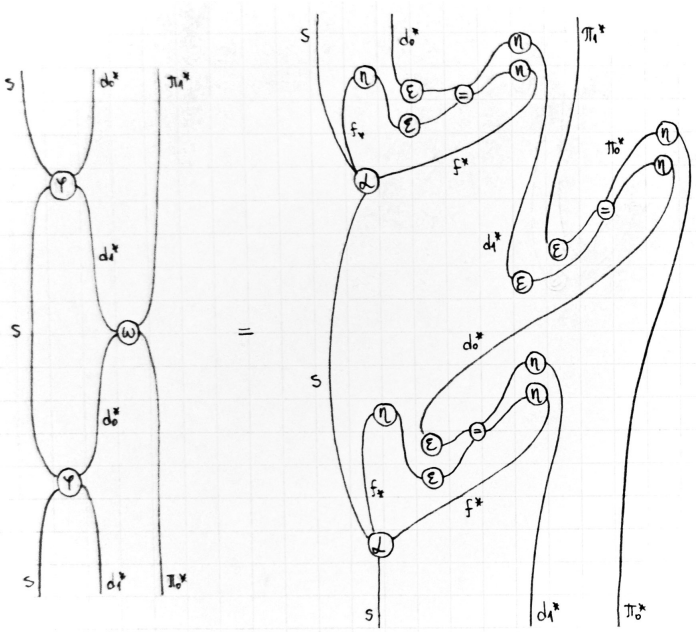


Finally, we need to show that φ satisfies the coherence condition given by (DD2):

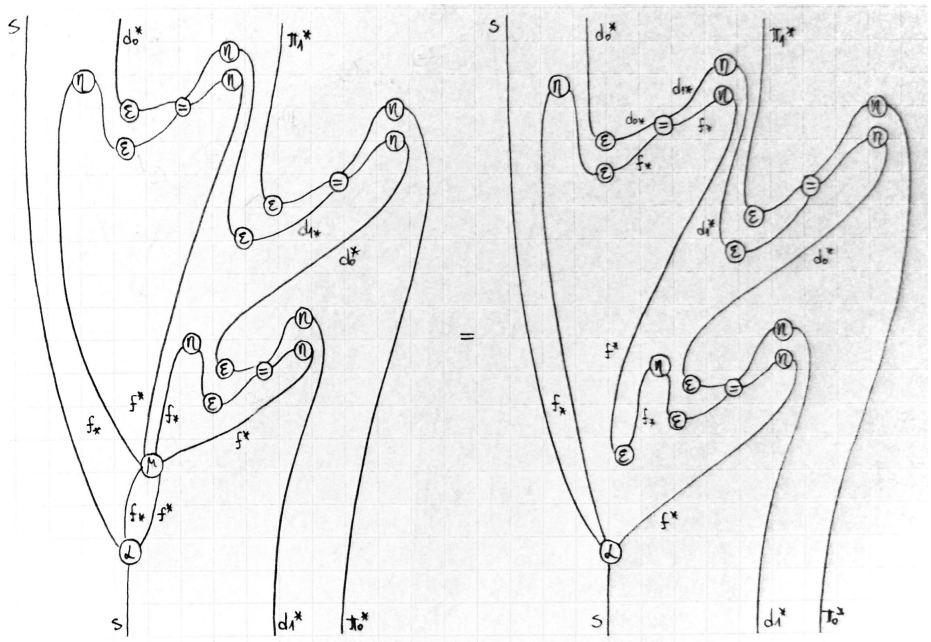
· EQUATION (DD2):



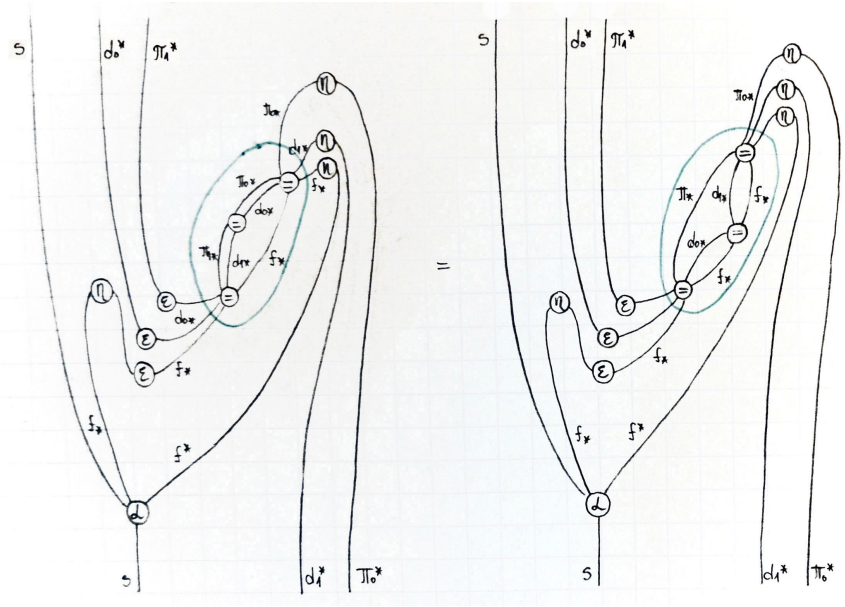
We transform $\pi_0^* \varphi \circ \omega \circ \pi_1^* \varphi$ as follows. We start by "unpacking" the definitions of φ and ω:



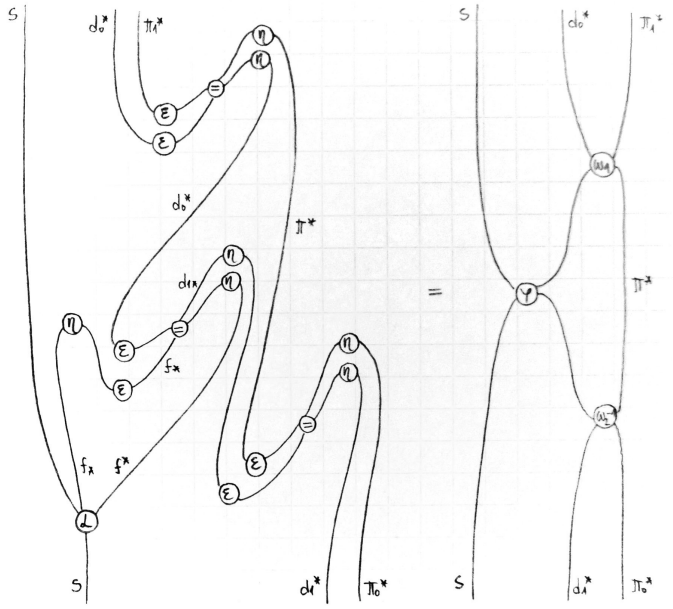
After applying the equation (TA2) and "unpacking" the definition of μ we obtain:



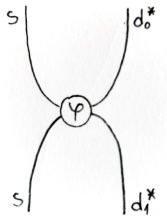
Then we eliminate two $\eta - \varepsilon$ redexes of the adjunction $f_* \dashv f^*$, one of the adjunction $d_{0*} \dashv d_0^*$, and one of the adjunction $d_{1*} \dashv d_1^*$. After that we apply the explicit equality rule on the delimited region, so we get:



We finish by plugging in $\eta - \varepsilon$ redexes, one for each of the adjunctions $d_{0*} \dashv d_0^*$, $d_{1*} \dashv d_1^*$ and $\pi_* \dashv \pi^*$, and "wrapping up" the definitions of φ, ω_1 and ω_2^{-1}:

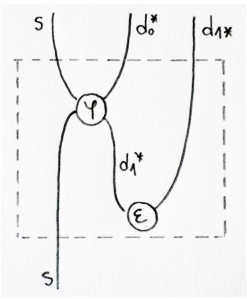


(2) ⇒ (3) : Suppose given an isomorphism $\varphi : d_0^* s \to d_1^* s$ which satisfies the cocycle conditions for the descent datum. Hence, we are provided with the string diagram



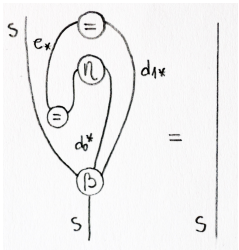
which satisfies the string diagram equations (DD1) and (DD2).

We construct the "morphism part" β of an internal presheaf with "object part" s as follows:

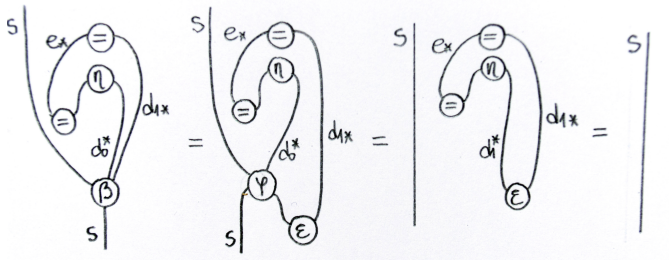


The condition (IP1) then follows directly from the construction of β.

String diagram equality corresponding to the condition (IP2) is given by:

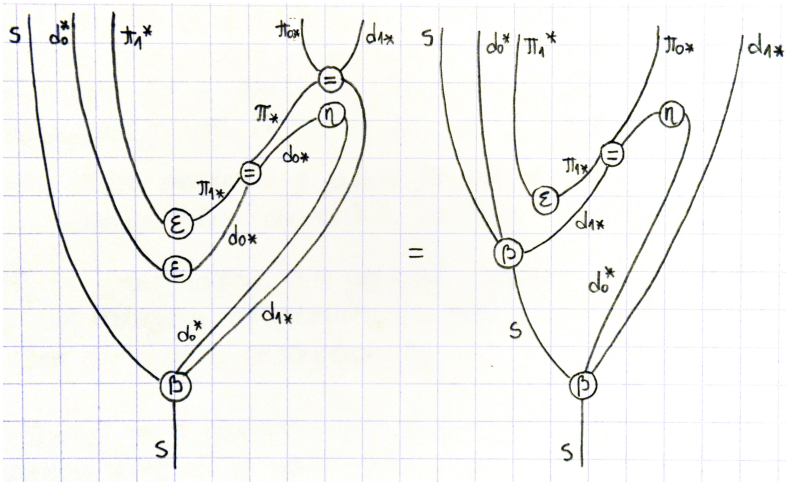


and is easily proved using the condition (DD1) satisfied by φ:

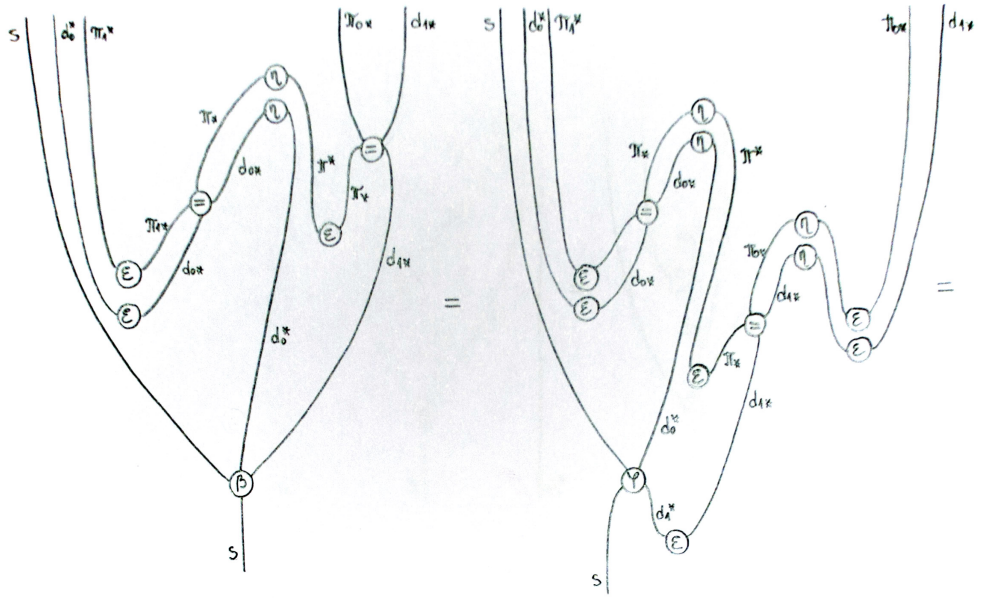


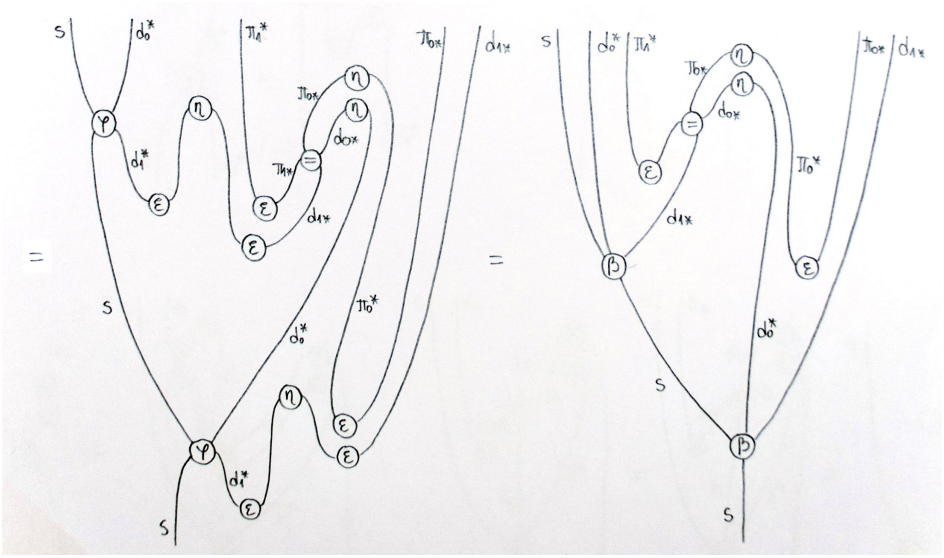
Again, we start by unfolding the definition of β. This immediately gives us the setting to apply the equality (DD1), after which we can remove $\eta - \varepsilon$ redex of the adjunction $d_{1*} \dashv d_1^*$, and, consequently, the remaining redex containing just two equality nodes.

Finally, we use the coherence condition (DD2) to prove that β also satisfies the condition (IP3) for internal presheaves, which, once translated into the ambient slice category setting, is given by the following string diagram equality:



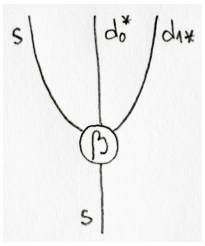
The equality follows easily from (DD2) and simple manipulations with $\eta - \varepsilon$ redexes of the adjunctions $\pi_* \dashv \pi^*$, $\pi_{0*} \dashv \pi_0^*$ and $d_{1*} \dashv d_1^*$:





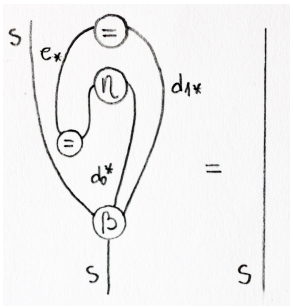
(3) $\Rightarrow$ (1): The last part of our proof consists of showing that, given an internal presheaf (s, β) over the internal category determined by f, we are also provided with a T-algebra structure on s, with T being the monad f^*f_*.

It is immediate to realize that the structure of the "morphism part" β itself induces an endofunctor on the category $\mathbf{C}/B$, $T' = d_{1*}d_0^*$:

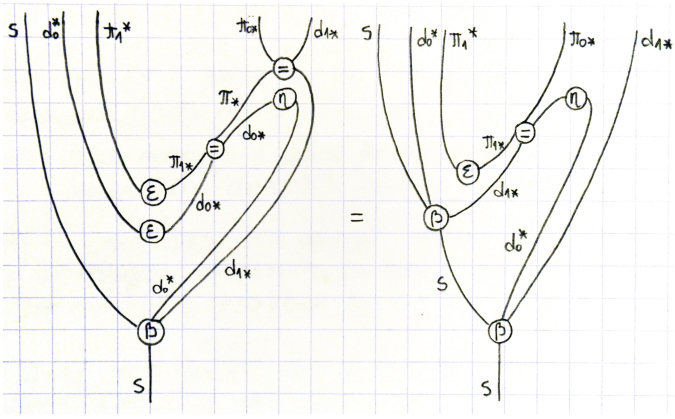


Together with the conditions satisfied by β:

· EQUATION (IP2):



· EQUATION (IP3):

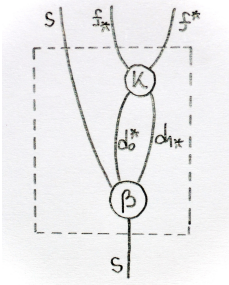


this suggests us to consider the structure $T' = (T', \eta', \mu')$, where η' comes from (IP2) and μ' from (IP3), as a tentative monad structure on $\mathbf{C}/B$, and furthermore, the morphism $\beta : d_{1*}d_0^* s \to s$ as a T'-algebra structure on s.

The T-algebra structure on s, where $T = f^* f_*$, is obtained from $T' = d_{1*}d_0^*$ immediately - these two functors are isomorphic with respect to the canonical isomorphism between pullbacks. However, the natural isomorphism

$$\kappa : f^* f_* \to d_{1*}d_0^*,$$

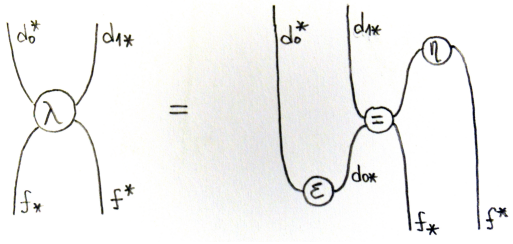
which is an integral part in the construction of $\alpha : T(s) \to s$ from $\beta : T'(s) \to s$:



does not have its explicit string diagram definition. The cause of this issue lies in the fact that defining κ requires to look at $fd_0 = fd_1$ not just as a commutative square, but moreover a pullback. And from this point of view, d_0 would be a morphism in $\mathbf{C}/A$ and not in $\mathbf{C}$, as we would want it. The remedy for this situation is to work with a transition in other direction, i.e. to consider a natural transformation

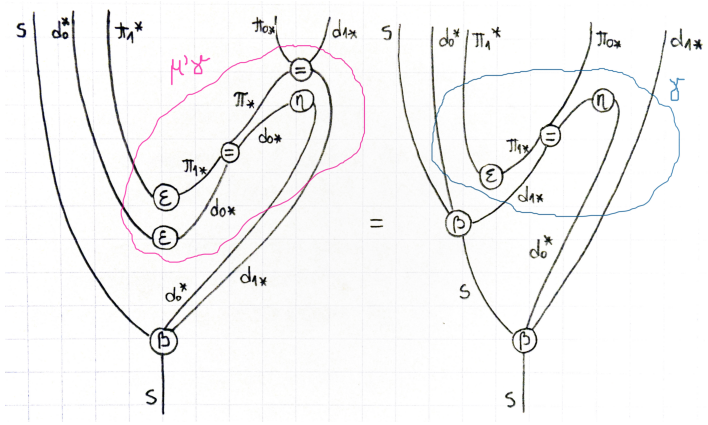
$$\lambda : d_{1*}d_0^* \to f^* f_*,$$

which has a string diagram representation:



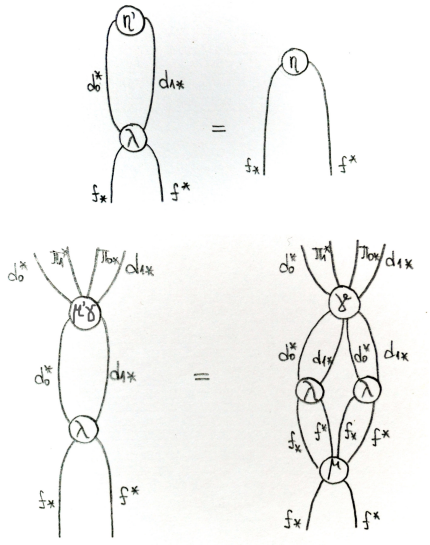
(since its construction does not require the commutative square $fd_0 = fd_1$ to be a pullback). Then, since $fd_0 = fd_1$ *is* a pullback, by the universality of pullbacks, we can conclude that λ must be an isomorphism, and axiomatise κ as its inverse.

Here we need to be precise and also note that the input interface of the equation (IP3) does not have the expected shape: the sequence of morphisms for the multiplicative law of our potential T'-algebra structure should start from $d_{1*}d_0^*d_{1*}d_0^*s$. However, functors $d_{1*}\pi_{0*}\pi_1^*d_0^*$ and $d_{1*}d_0^*d_{1*}d_0^*$ are also isomorphic and passing from first to second functor can be seen in the delimited area of the right-hand side of the given equality:

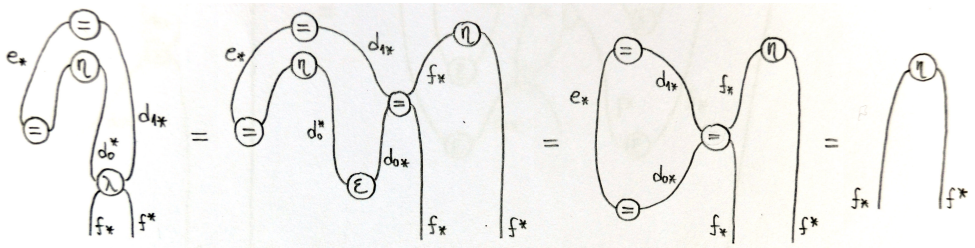


We introduce the notation $\gamma : d_{1*}\pi_0^*\pi_{1*}d_0^* \to d_{1*}d_0^*d_{1*}d_0^*$. Note that this transition is analogous to the one with λ: it is just a whiskering of the canonical morphism from $\pi_{0*}\pi_1^*$ to $d_0^*d_{1*}$ (and for the same reason as before it is not possible to draw the definition of γ^{-1}). Having this in mind, the delimited area on the left-hand side of the equality does not represent μ', but rather $\mu'\gamma$. The genuine multiplicative law for T' is then obtained by precomposing both sides of the equality with γ^{-1}.

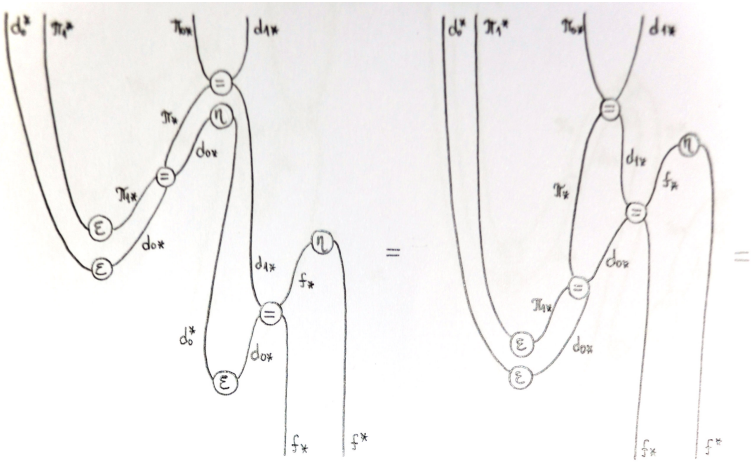
In order to prove that the structures given by (3) and (1) are indeed isomorphic, we must also prove that the rest of our T'-algebra structure behaves in the expected way when transported along the isomorphism λ. Namely, we need to show that our candidate unit η' and the candidate multiplication μ' are transported along the isomorphism $\lambda : T' \to T$ exactly to η and μ, yielding a genuine T-algebra structure on alpha (which will also mean that (T', η', μ') is indeed a monad). In the language of string diagrams, these transitions are given by the following equalities:

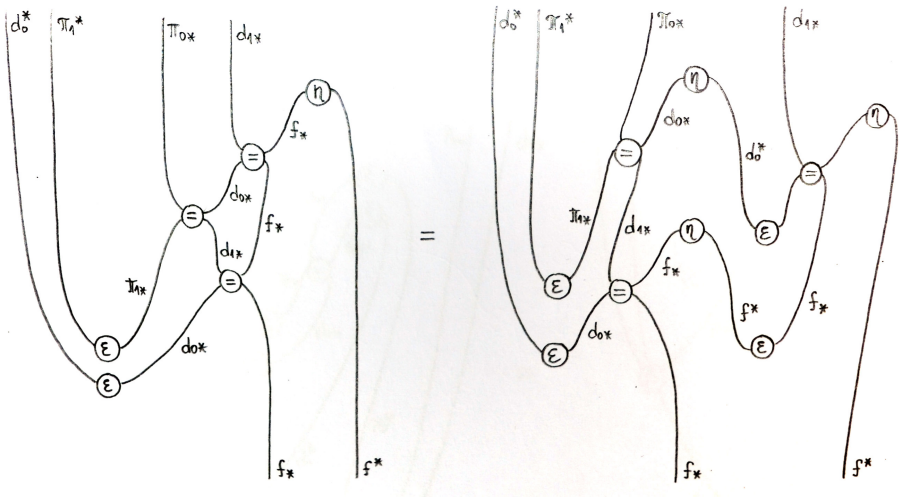


The first equality follows simply after "unfolding" all the definitions and removing the obtained $\eta - \varepsilon$ redex of the adjunction $d_{0*} \dashv d_0^*$, by the coherence of equalities principle:



As for the second part, we proceed analogously as follows:





This completes our proof. □

This theorem answers the initial question: the category $\mathbf{Des}(f)$ is equivalent to $\mathbf{C}/A$ exactly when the adjunction $f_* \dashv f^*$ is monadic.

3 Generalization

Let $\mathbf{C}$ be a category with pullbacks, $p : \mathbf{D} \to \mathbf{C}$ an arbitrary bifibration and $f : B \to A$ a fixed morphism in $\mathbf{C}$.

Since $p : \mathbf{D} \to \mathbf{C}$ is a fibration, for each object X of the fiber $\mathbf{D}(A) = p^{-1}(A)$ there exists a cartesian lifting $\hat{f}$ of f to X. Let us fix for each X from $\mathbf{D}(A)$ one cartesian lifting arrow and denote it with $\hat{f}_X$. We call this choise *a cleavage* for p with respect to f. Thanks to the universal property of cartesian arrows, the cleavage correspondence extends to the *inverse image functor*

$$f^* : \mathbf{D}(A) \to \mathbf{D}(B),$$

which sends each object X to the domain f^*X of $\hat{f}_X$, and each arrow $\alpha : X \to X'$ to the unique arrow $f^*\alpha : f^*X \to f^*X'$ making the diagram

$$\begin{array}{ccc} f^*X & \longrightarrow & X \\ {\scriptstyle f^*\alpha}\,\vdots & & \vdots\,{\scriptstyle \alpha} \\ f^*X' & \longrightarrow & X' \end{array}$$

commute.

Analogously, since $p : \mathbf{D} \to \mathbf{C}$ is an opfibration, by choosing for each Y from $\mathbf{D}(B)$

a cocartesian lifting $\hat{f}_Y$, i.e. by fixing *a cocleavage* for p with respect to f, we obtain the *direct image functor*

$$f_* : \mathbf{D}(B) \to \mathbf{D}(A),$$

which assigns to each Y the codomain f_*Y of $\hat{f}_Y$, and with the obvious assignment of arrows.

The direct and the inverse image functor form an adjunction

$$f_* \dashv f^* : \mathbf{D}(B) \to \mathbf{D}(A).$$

The adjunction isomorphism

$$\mathrm{Hom}_{\mathbf{D}(A)}(f_*X, Y) \cong \mathrm{Hom}_{\mathbf{D}(B)}(X, f^*Y)$$

follows easily by the universal property of cartesian arrows. Namely, for each arrow $\alpha : f_*X \to Y$ from $\mathbf{D}(A)$, by precomposing with $\hat{f}_X$, we obtain a uniquely associated arrow $\alpha \circ \hat{f}_X : X \to Y$ in $\mathbf{D}$:

$$\begin{array}{ccc} X & \xrightarrow{\hat{f}_X} & f_*X \\ & & \downarrow{\scriptstyle \alpha} \\ f^*Y & \xrightarrow{\hat{f}_Y} & Y \end{array}$$

for which we have:

$$p(\alpha \circ \hat{f}_X) = p(\alpha) \circ p(\hat{f}_X) = id_A \circ f = f.$$

Hence, we have $id_B : p(X) \to B$ and $f \circ id_B = p(\alpha \circ \hat{f}_X)$ in $\mathbf{C}$, and since $\hat{f}_Y$ is cartesian, this gives us a unique arrow $\beta : X \to f^*Y$. The other direction follows analogously, using the fact that $\hat{f}_X$ is the cocartesian lifting of f to X.

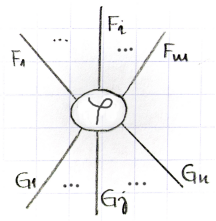
The generalization of our elementary approach to Bénabou-Roubaud result is quite straightforward: each structure we introduced before is now seen with respect to the generalized adjunction $f_* \dashv f^*$. However, there are two aspects in which our reasoning for special case must be specifically adapted to the general setting.

3.1 Coherence of canonical isomorphisms

The first issue we must address arises from the fact that, in the case of an arbitrary bifibration, the pseudofunctor $(-)_* : \mathbf{C} \to \mathbf{Cat}$ does not commute with composition anymore. More precisely, all our previous arguments about the coherence of *equalities* must be "lifted" to a coherence of *canonical isomorphisms*, which we are dealing with now. We state this principle in the following lemma.

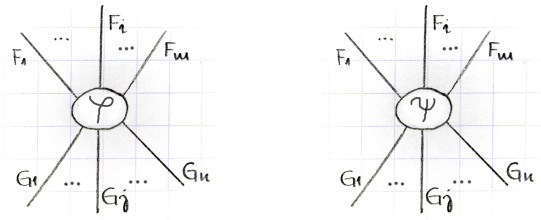
Lemma 3.1. *Any two connected string diagrams which have the same input and output interface and which consist only of canonical isomorphisms between compositions of direct image functors are equal.*

Proof. Let us first note that any connected string diagram with m inputs and n outputs, that consists only of nodes representing canonical isomorphisms between compositions of direct image functors, is equal to some diagram of the form



in which the node φ also represent an isomorphism. This observation follows easily by induction on the size of the diagram (we push upwards one of the top isomorphism nodes and apply induction hypothesis on the rest of the diagram) and the fact that all the manipulations we do afterwards preserve isomorphisms.

Now, let us suppose that we are given two string diagrams with the same input and output interface and which are reduced to this so-called normal form:



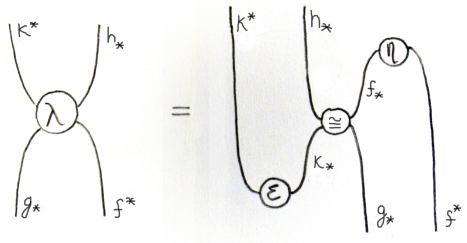
Suppose moreover that the source and target cateogories for the compositions $F_m \dots F_1$ and $G_n \dots G_1$ are given by the fibers $\mathbf{D}(X)$ and $\mathbf{D}(Y)$ respectively, and let s be an object from $\mathbf{D}(X)$. By instantiating our two sequences of functors to s we obtain two sequences of cocartesian arrows in $\mathbf{D}$, and since the composition of cocartesian arrows is again cocartesian, this gives us two cocartesian arrows with the same source. Hence, we are in a setting where we can utilize the fact that $p : \mathbf{D} \to \mathbf{C}$ is an opfibration, and obtain a unique arrow $u : F_m(\dots(F_1 s)) \to G_n(\dots(G_1 s))$ in $\mathbf{D}(Y)$. This amounts precisely to the fact that there can be only one isomorphism between $F_m \dots F_1$ and $G_n \dots G_1$. □

Consequently, this allowes us to switch between any two string diagrams of direct image functors which have the same input and output interface and which consist of canonical isomorphisms. As a matter of fact, thanks to the properties of opfibrations, this argument applies not only to the case when nodes are isomorphisms, but also to any two string diagrams of direct image functors with the same input and output structure, no matter which nautral transformations act in between! Hence, our lemma is just an instance of the universal property of cocartesian arrows. Of course, we could use the same reasoning to show that the analogous statement holds for inverse image functors, but in order to be consistent with our approach in the special case, we choose

to represent these canonical isomorphisms via the corresponding ones of the direct image functors.

3.2 Beck-Chevalley property

The requirement we have to add explicitly in order to be able to prove the sequence of equivalences in general case is that the bifibration $p : \mathbf{D} \to \mathbf{C}$ must satisfy the Beck-Chevalley property. Informally speaking, for each commutative square $fh = gk$ in $\mathbf{C}$, this condition guarantees that the canonical natural transformation $\lambda : h_* k^* \to f^* g_*$:



is moreover a natural isomorphism. This is something that does not come "in package" with an arbitrary bifibration! The codomain fibration satisfies this condition and we have used it implicitly when we axiomatized κ as the inverse of natural transformation $\lambda : d_{1*} d_0^* \to f^* f_*$.

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