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Categories, Toposes and Sets
Author(s): J. L. Bell
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J. L. BELL

CATEGORIES, TOPOSES AND SETS*

In recent years the mathematical discipline of *category theory* has emerged to play a growing role in the foundations of mathematics. Perhaps the most significant development in this respect is the invention by Lawvere and Tierney of the concept of elementary *topos*. This concept units in a simple, elegant way a number of important, but seemingly diverse, notions from algebraic geometry, set theory, and intuitionistic logic and has forged a new link between classical and constructive mathematics.

My purpose here is to furnish an introduction to the theory of toposes without assuming any prior knowledge of category theory on the part of the reader. I have arranged the material in such a way as to give the impression that the primary task is to find an axiomatic characterization of the usual category of sets, and that the notion of a topos is the (inevitable?) result of attempting to carry out this task. (This is essentially the way the idea of a topos actually *did* emerge!) In the final section I discuss the notions (from algebraic geometry) of a *topology* and a *sheaf* in a topos and describe how topos theory enables these ideas to illuminate and extend such apparently unrelated matters as Cohen-style independence proofs in set theory.

1. THE NOTION OF A CATEGORY

A *category* **E** consists of a collection of *objects* X, $Y, \ldots$ and a collection of *arrows* f, $g, \ldots$. Each arrow f has an object X as *domain* and an object Y as *codomain*, indicated by writing

$$X \xrightarrow{f} Y \text{ or } f\colon X \to Y.$$

If g is an arrow $Y \xrightarrow{g} Z$ with domain Y, the codomain of f, there is an arrow $fg\colon X \to Z$ (sometimes written $f.g$) called the *composition* of f with g. For each object Y there is an arrow $\mathrm{id}_Y\colon Y \to Y$ called the *identity arrow* of Y. These notions are assumed to satisfy identity and associativity axioms:

$$f.\mathrm{id}_Y = f;\ \mathrm{id}_Y.g = g;\ f(gh) = (fg)h,$$

for all arrows $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$, $Z \xrightarrow{h} W$. Finally we shall assume that,

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although **E** itself may not necessarily be a set, the collection – denoted by $\mathbf{E}(X, Y)$, $\hom_{\mathbf{E}}(X, Y)$ or plain $\hom(X, Y)$ – of arrows between any pair X, Y of its objects always is.

Examples

1. The category **S** of *sets*. Here the objects are all sets; the arrows, all functions between sets. Composition of arrows and identity arrows are as usual.

Our primary object will be to find a category-theoretic characterization of **S**, and to explore the category-theoretic generalization of **S** known as an (elementary) *topos*.

2. The category of *sets and relations*. Here the objects are all sets; the arrows, all (binary) relations between sets. Composition of arrows is just composition of relations in the usual sense.

3. The category of *groups*. Here the objects are all groups; the arrows, all group homomorphisms. Similarly for the categories of rings, fields, vector spaces, modules, near groupoids, etc.

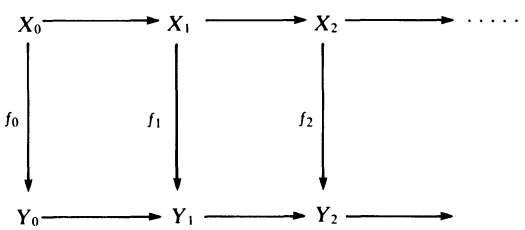
4. The category **Top** of topological spaces. Here the objects are all topological spaces; the arrows, all continuous maps.

5. Any partially (or preordered) ordered set $(P, \leq)$ may be regarded as a category in which the objects are the elements of P and there is exactly one arrow between elements p, q of P just in case $p \leq q$. In particular, P may be the family of open sets in a topological space, ordered by inclusion (or reverse inclusion).

6. The category $\mathbf{S}^{\mathbf{N}}$ of *sets varying over discrete time*. Here the objects are all countable sequences of maps

$$X_0 \to X_1 \to X_2 \to \cdots\cdots$$

and the arrows are all strings of maps $f_0, f_1, f_2, \ldots$ such that all squares in the diagram



commute.

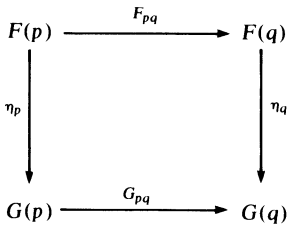
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7. (A generalization of 6). Given any preordered set $(P, \leq)$, a *set varying over P* is a map F which assigns:

•to each $p \in P$, a set $F(p)$;

•to each pair $p, q \in P$ such that $p \leq q$ a map F_{pq}: $F(p) \to F(q)$ such that if $p \leq q \leq r$, then $F_{pr} = F_{pq}F_{qr}$ $(= F_{qr} \circ F_{pq}$ in the customary set-theoretic notation for function composition), and F_{pp} is the identity map on $F(p)$ for each $p \in P$.

The category $\mathbf{S}^P$ of *sets varying over P* has as objects all sets varying over P in the above sense, and an arrow η: $F \to G$ in $\mathbf{S}^P$ is a collection of maps η_p: $F(p) \to G(p)$ for $p \in P$ such that whenever $p \leq q$ the diagram



commutes. Composition of arrows and the identity arrows are defined in the obvious way. *If* $a \in F(p)$ *we sometimes write* $a \restriction_F q$ *for* $F_{pq}(a)$, and call $a \restriction_F q$ the *restriction* of a to q.

8. Let X be a topological space and let T be the topology in X. Partially order T by *reverse* inclusion. A set varying over T is called a *presheaf* on X. A presheaf F on X is called a *sheaf* if it satisfies the "gluing" condition:

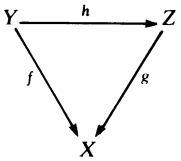
•Given any open cover $\{U_i\colon i \in I\}$ of an open set U in X and any family $\{\sigma_i\colon i \in I\}$ such that $\sigma_i \in F(U_i)$ and $\sigma_i \restriction_F (U_i \cap U_j) = \sigma_j \restriction_F (U_i \cap U_j)$ – such a family is said to be *F-compatible* – then there exists a *unique* $\sigma \in F(U)$ such that $\sigma \restriction_F U_i = \sigma_i$ for all $i \in I$.

The category $\mathbf{Shv}(X)$ of *sheaves* on X has as objects all sheaves on X and as arrows all arrows between them *qua* sets varying over T (see 7). (As a basic example of a sheaf on X, define $F(U) =$ all continuous real-valued functions on U; $F_{UV}(a) =$ restriction of a to U.)

9. Again let X be a topological space. A *space over X* is a space Y equipped with a local homeomorphism f: $Y \to X$. An *arrow* between

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spaces $Y \xrightarrow{f} X$ and $Z \xrightarrow{g} X$ over X is a continuous map h: $Y \to Z$ such that the diagram



commutes. The category **Spo**(X) of *spaces over* X has as objects all spaces over X and as arrows all arrows between them as defined above. It is well known that **Spo**(X) and **Shv**(X) are equivalent as categories in a sense to be defined later.

10. Again let X be a topological space with topology T. A *Scott algebra* (over X) is a set A equipped with two maps D: $A \to T$ and $|$: $A \times T \to A$ such that for all $a \in A$, $U, V \in T$,

$$D(a \mid U) = Da \cap U, \qquad (a \mid U) \mid V = a \mid (U \cap V),$$
$$a \mid Da = a.$$

A *morphism* of Scott algebras A, B is a map f: $A \to B$ such that $D(fa) = Da$ and $f(a \mid U) = fa \mid U$, for all $a \in A$, $U \in T$. Scott algebras over X and their morphisms form a category **Sco**(X) in the obvious way; we shall see that **Sco**(X) and the category $\mathbf{S}^T$ of presheaves on X are isomorphic in a natural sense.

Given a Scott algebra A over X, we define the *associated presheaf* $\tilde{A}$ on X by $\tilde{A}(U) = \{a \in A: Da = U\}$ and, for $U \supseteq V$, $\tilde{A}_{UV}(a) = a \mid V$.

Given a morphism of Scott algebras f: $A \to B$ we get an arrow $\tilde{f}$ between the corresponding presheaves $\tilde{A}$, $\tilde{B}$ by putting[1] $\tilde{f}_U = f \restriction \tilde{A}(U)$.

Conversely, given a presheaf F on X, the associated Scott algebra F^* is defined by $F^* = \coprod_{U \in T} F(U)$ (=disjoint union of the $F(U)$); for $a \in F^*$, Da = unique U such that $a \in F(U)$, and $a \mid V = a \restriction_F (Da \cap V)$.

Given an arrow $F \xrightarrow{\eta} G$ between presheaves we get a morphism η^* between the corresponding Scott algebras F^*, G^* by "gluing" the maps η_U for $U \in T$.

It is now easy to see that the operations $\tilde{}$ and * are mutually inverse and establish an isomorphism between **Sco**(X) and $\mathbf{S}^T$.

We now see what objects in **Sco**(X) correspond to *sheaves* in $\mathbf{S}^T$.

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Given a presheaf F on X, an open cover $\{U_i\colon i \in I\}$ of an open set U, and a family $\{\sigma_i\colon i \in I\}$ such that $\sigma_i \in F(U_i)$ for each $i \in I$, we see that $\{\sigma_i\colon i \in I\}$ is F-compatible iff in the associated Scott algebra F^* we have $\sigma_i \mid D\sigma_j = \sigma_j \mid D\sigma_i$ for all i, j, and to say that F is a sheaf is then to say that for any such family there is a unique $\sigma \in F(U)$ – i.e. $\sigma \in A$ such that $D(\sigma) = U$ – for which $\sigma \mid U_i = \sigma_i$.

This leads to the *definition*: a *join* of a subset X of a Scott algebra A is an element $a \in A$ such that $Da = \bigcup_{x \in X} Dx$ and $a \mid Dx = x$ for each $x \in X$. A subset X of A is *compatible* if $x \mid Dy = y \mid Dx$ for all $x, y \in X$. A is said to be *complete* if each compatible subset has a unique join.

Then: *F is a sheaf iff its associated Scott algebra F^* is complete.* This is Scott's reformulation of sheaf theory in algebraic form (see [2]).

11. Let $(H, \leq)$ be a complete lattice. H is called a *complete Heyting algebra* (cHa, frame, locale) if it satisfies the distributive law

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} x \wedge y_i.$$

(This law corresponds to the intuitionistically correct rule: $A \wedge \exists x B(x) \leftrightarrow \exists x[A \wedge B(x)]$.) Notice that any topology on a set (ordered by *inclusion*) is always a cHa. Any complete Boolean algebra is a cHa but not conversely.

An *H-valued set* is a set X equipped with an "H-valued equality relation" $\delta\colon X \times X \to H$ satisfying

$$\delta(x, x') = \delta(x', x) \qquad \text{(symmetry)}$$
$$\delta(x, x') \wedge \delta(x', x'') \leq \delta(x, x'') \qquad \text{(transitivity)}.$$

The *category* $\mathbf{S}_H$ of H-valued sets is defined as follows. Its objects are all H-valued sets. An *arrow* f between two H-valued sets (X, δ) and (Y, ϵ) is an "H-valued functional relation between (X, δ) and (Y, ϵ)", i.e. a function $f\colon X \times Y \to H$ satisfying

$$\left.\begin{aligned} &\delta(x, x') \wedge f(x, y) \leq f(x', y) \\ &f(x, y) \wedge \epsilon(y, y') \leq f(x, y') \end{aligned}\right\} \text{substitutivity}$$
$$f(x, y) \wedge f(x, y') \leq \epsilon(y, y') \quad \text{functionality}$$
$$\bigvee_{y \in Y} f(x, y) = \delta(x, x) \qquad \text{definition on } X.$$

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Given two arrows $(X, \delta) \xrightarrow{f} (Y, \epsilon)$ and $(Y, \epsilon) \xrightarrow{g} (Z, \eta)$, the *composition* $(X, \delta) \xrightarrow{fg} (Y, \epsilon)$ is defined by

$$(fg)(x, z) = \bigvee_{y \in Y} f(x, y) \wedge g(y, z)$$

and the identity arrow on (X, δ) is just δ.

Evidently $\mathbf{S}_2$ is essentially $\mathbf{S}$ (where 2 is the complete Boolean algebra $\{0, 1\}$).

12. If B is a complete Boolean algebra, the *Boolean extension* $V^{(B)}$ of the universe of sets in the sense of Scott-Solovay (see [1]) may be regarded as a category in the following way. We identify elements of $V^{(B)}$ which are equal with probability 1. Then the objects of the category $V^{(B)}$ are the (thus identified) elements of $V^{(B)}$ and the arrows are those elements which are functions with probability 1.

It can be shown that $V^{(B)}$ and S_B are equivalent.

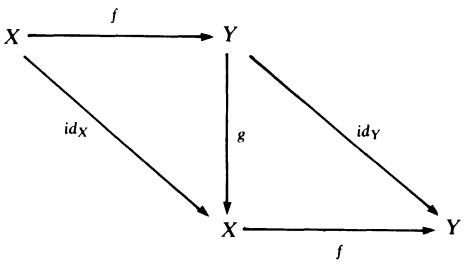
13. Let M be a monoid, i.e. a semigroup with identity. An *M-set* is a set on which M acts by transformations, i.e. a pair $(X, \cdot)$ in which X is a set and $\cdot$ is a map from $M \times X$ to X satisfying

$$\alpha.(\beta.x) = (\alpha\beta).x, \quad 1.x = x$$

for all $\alpha \in M$, $x \in X$. A *homomorphism* between two M-sets X and Y is a map $f: X \to Y$ such that $f(\alpha.x) = \alpha.f(x)$ for all $\alpha \in M$, $x \in X$. The category $\mathbf{S}^M$ of M-sets then has as objects all M-sets and as arrows all homomorphisms between them.

Now let $\mathbf{E}$ be an arbitrary category. A *diagram* in $\mathbf{E}$ is a set of objects of $\mathbf{E}$ together with a set of arrows between them. A diagram D is said to be *commutative* or to *commute* if the arrow obtained by composing along the arrows of any path in D depends only on the endpoints of the path.

An arrow $f: X \to Y$ in $\mathbf{E}$ is said to be an *isomorphism* if there is an arrow $g: Y \to Z$ such that the diagram



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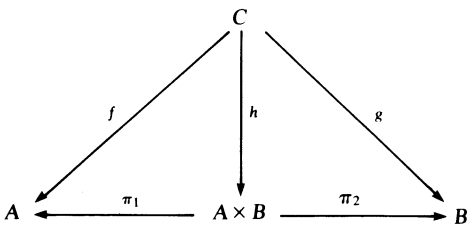
commutes, i.e. if $fg = \mathrm{id}_X$, $gf = \mathrm{id}_Y$. For example, the isomorphisms in S are just the bijections, in **Top**, the homeomorphisms. Two objects X, Y are *isomorphic*, written $X \cong Y$, if there is an isomorphism between them.

An arrow $X \xrightarrow{f} Y$ in **E** is said to be *monic* if it is *right cancellable*, i.e. if for any arrows g, h: $Z \to X$ such that $gf = hf$ we have $g = h$. We indicate the fact that f is monic by writing $X \overset{f}{\rightarrowtail} Y$. In **S**, the monic arrows are just the one-one maps. It is easily verified that a composition of monics is monic.

Similarly, an arrow $X \xrightarrow{f} Y$ in **E** is said to be *epic* if it is *left cancellable*, i.e. if for any arrows g, h: $Y \to Z$ such that $fg = fh$ we have $g = h$. We indicate the fact that f is epic by writing $X \twoheadrightarrow Y$. In **S**, the epic arrows are just the onto maps. It is easily verified that a composition of epics is epic.

An object A is *terminal* in **E** if for each object X there is exactly one arrow $X \to A$ and *initial* if for each object X there is exactly one arrow $A \to X$. It is easy to see that all terminal objects are isomorphic, as are all initial objects. In **S**, the empty set $\emptyset$ is initial and any singleton – in particular $1 = \{\emptyset\}$ – is terminal. If **E** has an initial (terminal) object, we shall denote it by 0 (1, respectively).

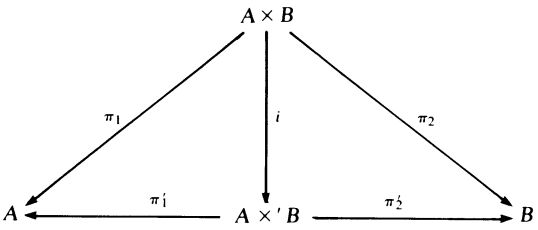
Let A, B be objects in **E**. A *product* of A and B is an object $A \times B$ together with arrows $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$, called *canonical projections*, such that, for each object C and each pair of arrows $A \xleftarrow{f} C \xrightarrow{g} B$ there is a *unique* arrow h: $C \to A \times B$ – which we shall write as $\langle f, g \rangle$ – such that the diagram



commutes. (Thus we may say that each diagram of the form $A \xleftarrow{f} C \xrightarrow{g} B$ "factors uniquely" through the diagram $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$.) It is easy to see that this condition specifies $A \times B$ "uniquely up to isomorphism" in the sense that, if $A \xleftarrow{\pi_1'} A \times' B \xrightarrow{\pi_2'} B$ is any other

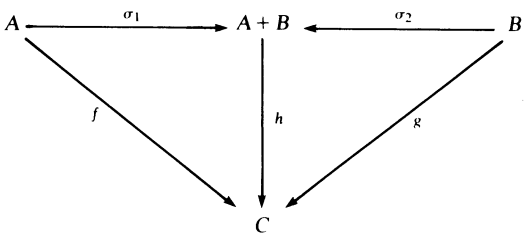
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diagram satisfying the specified conditions, then there is a unique isomorphism $i\colon A \times B \to A' \times B'$ such that the diagram



commutes. Accordingly we shall refer to $A \times B$ as *the* product of A and B. In **S**, $A \times B$ is the usual Cartesian product of the sets A, B; π_1, π_2 are the projections onto the factors; and $\langle f, g\rangle$ is given by $\langle f, g\rangle (c) = (f(c), g(c))$.

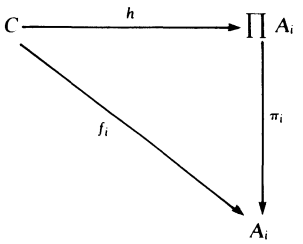
Similarly, a *coproduct* of A and B is an object $A + B$ together with arrows $A \xrightarrow{\sigma_1} A + B \xleftarrow{\sigma_2} B$ called *canonical injections* such that, for each object C and each pair of arrows $A \xrightarrow{f} C \xleftarrow{g} B$ there is a unique arrow $h\colon A + B \to C$ – which we shall write as $\binom{f}{g}$ – such that the diagram



commutes. Again, this condition determines $A + B$ "uniquely up to isomorphism". In **S**, $A + B$ is the disjoint union of A and B.

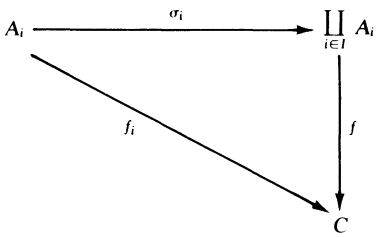
More generally, suppose $\{A_i\colon i \in I\}$ is an indexed family of objects of **E**. A *product* of the family is an object $\Pi_{i\in I} A_i$ together with a family of arrows $\pi_i\colon \Pi_{i\in I} A_i \to A_i$ $(i \in I)$ such that, for any object C and any family of arrows $f_i\colon C \to A_i$ $(i \in I)$ there is a unique arrow $h\colon C \to \Pi_{i\in I} A_i$ such that, for each $i \in I$, the diagram

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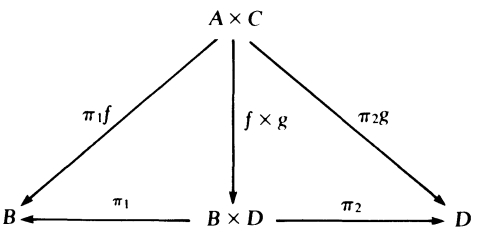
commutes. In **S**, $\Pi_{i\in I} A_i$ is the usual Cartesian product of the sets A_i.

Similarly, a *coproduct* of the family $\{A_i: i \in I\}$ is an object $\amalg_{i\in I} A_i$ together with a family of arrows σ_i: $A_i \to \amalg_{i\in I} A_i$ $(i \in E)$ called *canonical injections* such that, for each object C and arrows f_i: $A_i \to C$ $(i \in I)$ there is a unique arrow f: $\amalg_{i\in I} A_i \to C$ such that for each $i \in I$ the diagram



commutes. We often write $[f_i]_{i\in I}$ for f. If each A_i is a fixed object A, the coproduct is written $I.A$ and is called the *I-fold copower* of A. In **S**, A_i is the disjoint union of the A_i and $[f_i]_{i\in I}$ is the map obtained by "gluing" the f_i.

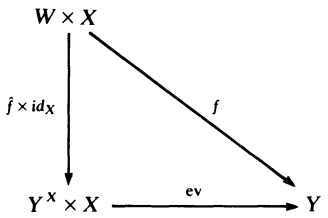
We define the *product* $f \times g$ of arrows $A \xrightarrow{f} B$, $C \xrightarrow{g} D$ to be the arrow $\langle \pi_1 f, \pi_2 g\rangle$: $A \times C \to B \times D$. Thus $f \times g$ is the unique arrow making the diagram



commute. In **S**, $f \times g$ is given by $(f \times g)(x, y) = (f(x), g(y))$.

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Given two objects X, Y in $\mathbf{E}$, an *exponential* is an object Y^X together with an arrow ev: $Y^X \times X \to Y$ called *evaluation* satisfying the following condition: for each arrow f: $W \times X \to Y$ there is a unique arrow $\hat{f}$: $W \to Y^X$ such that the diagram



commutes.

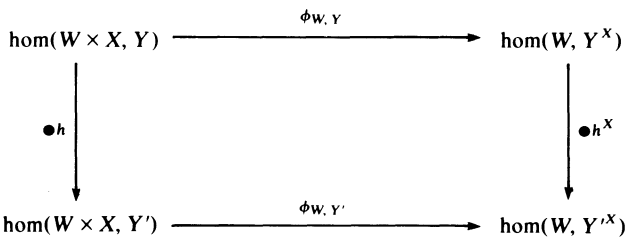
In $\mathbf{S}$, the exponential is the usual set $\hom(X, Y)$ of all functions from X to Y and ev is that map which takes a function $t \in \hom(X, Y)$ and $x \in X$ to the value $\mathrm{ev}(t, x) = t(x)$ of t at x. Given any function f: $W \times X \to Y$ of two variables w and x, the corresponding function $\hat{f}$ is the function of one variable to Y^X given by

$$\hat{f}(w)(x) = f(w, x).$$

Thus the correspondence $f \mapsto \hat{f}$ is a bijection

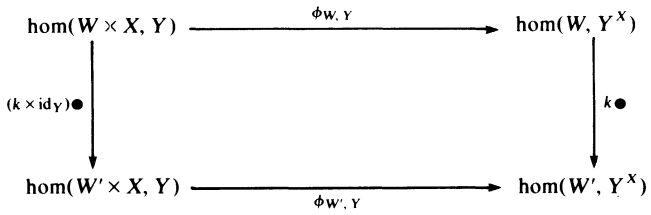
(*) $\qquad \phi_{W,Y}$: $\hom(W \times X, Y) \cong \hom(W, Y^X)$.

Moreover, the bijection ϕ is *natural* in W and Y in the sense that, for each h: $Y \to Y'$ and k: $W' \to W$, the diagrams

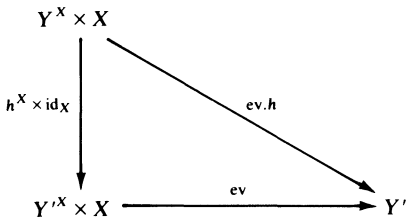


and

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both commute, where $h^X\colon Y^X \to Y'^X$ is the unique arrow such that the diagram

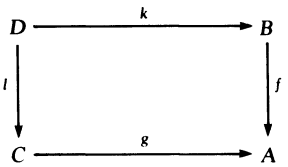


commutes; and $\bullet p$, $q\bullet$ are the maps $f \mapsto f.p$, $g \mapsto q.g$ respectively.

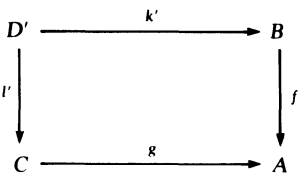
The existence of a natural bijection (*) is actually *characteristic* of exponentials in an arbitrary category **E**.

A commutative diagram

(**)



is called a *pullback diagram* in **E** and D is called the *pullback* of C against B or of C along f, or of B along k, if for any commutative diagram of the form



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there is a unique arrow $D' \xrightarrow{h} D$ such that $l' = hl$, $k' = kl$. If a pullback exists, it is "unique up to isomorphism" as usual.

In **S**, the pullback is given by

$$D = \{(x, y) \in B \times C\colon f(x) = g(y)\}$$

while h is the unique function such that

$$h(z) = (l'(z), k'(z))$$

for $z \in D$. In **S** there are several useful special cases of pullbacks:

•If $A = 1$, then the square (**) automatically commutes and D is $B \times C$.

•If B is a subset $B \subseteq A$ and f is the inclusion map, then D is (bijective with) $g^{-1}[B]$. More generally, if f is monic (i.e., one-one), then D is (bijective with) $g^{-1}[f[B]]$.

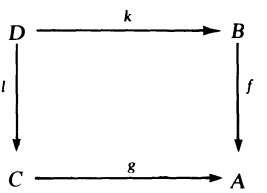
•If B and C are both subsets of A and f and g are inclusions, then D is (bijective with) $B \cap C$.

These same constructions, carried out in any category **E** which has pullbacks, yield *products* and *intersections* in **E**.

As an exercise we will prove that pullbacks of monics are monic.

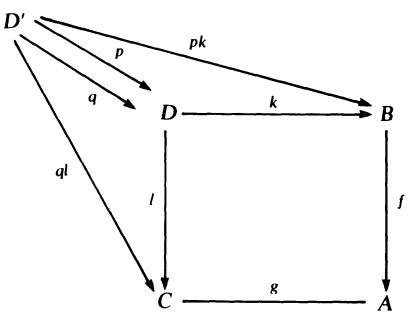
1.1 LEMMA. Let

(***)



be a pullback diagram. If g (or f) is monic, so is k (or l).

Proof. Suppose g is monic, and that $p, q\colon D' \to D$ satisfy $pk = qk$.

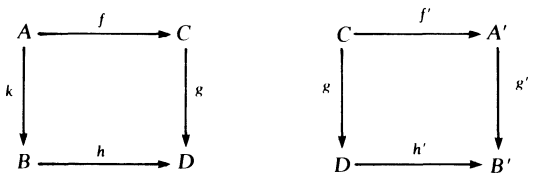


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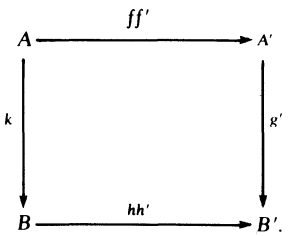
Since $pk = qk$, it follows that $pkf = qkf$, so that $plg = pkf = qkf = qlg$. Since g is monic, we may cancel it on the right, obtaining $ql = pl$. Since (***) is a pullback, there must be unique $h\colon D' \to D$ such that $pk = hk$ and $ql = hl$. But, by the above, both p and q satisfy these equations when substituted for h. It follows that $p = q$ and so k is monic. ■

The proof of the next result is left as an easy exercise for the reader.

1.2 LEMMA. If the commutative diagrams



are pullbacks, so is the "composed" diagram



■

2. ELEMENTARY TOPOSES

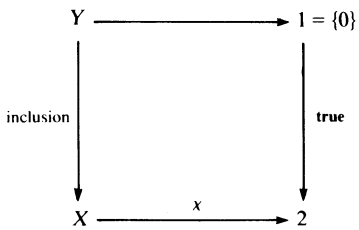
We observe that in **S** the inclusion map $m\colon Y \to X$ of a subset Y of a set X is always monic. And conversely, every monic $m'\colon Y' \rightarrowtail X$ is essentially of this form, in so far as there is always an isomorphism (i.e., a bijection) $\alpha\colon Y' \cong Y$ to a subset $Y \subseteq X$ such that $\alpha m = m'$.

Extending these ideas to an arbitrary category **E**, we call two monics $Y \overset{m}{\rightarrowtail} X$ and $Y' \overset{m'}{\rightarrowtail} X$ *equivalent*, and write $m \sim m'$, if there is an isomorphism $\alpha\colon Y' \cong Y$ such that $\alpha m = m'$. Clearly $\sim$ is an equivalence relation on the class of all monic arrows in E with codomain X; let $[m]$ be the equivalence class of any such monic m. An entity of the

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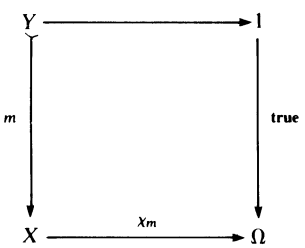
form $[m]$ is called a *subobject* of X. When no confusion is likely to occur, we shall identify $[m]$ with its representative m, and even sometimes with the domain Y of m. Thus a subobject of an object X may loosely be regarded as a monic with codomain X or as the domain of such a monic.

In **S**, we can describe subobjects in another way, namely, by *characteristic functions*. Recall that the characteristic function of a subset Y of a set X is the map χ from X to the set 2 of two "truth values" 0 and 1 with $\chi(x) = 1$ when $x \in Y$ and $\chi(x) = 0$ when $x \notin Y$. Now we can use the pullback technique to reconstruct a subset of X from its characteristic function χ. We take the special subset $1 = \{0\} \subseteq 2$ and the monic $1 \rightarrowtail 2$ – called *true* – which sends 0 to 1, and observe that since $Y = \chi^{-1}(1)$ the diagram



is a pullback. Thus we recover Y (up to bijective correspondence) as the pullback of 1 along χ.

We extend this idea to an arbitrary category as follows. A *subobject classifier* or *truth value object* in a category **E** is an object Ω of E together with a monic **true**: $1 \to \Omega$ from the terminal object such that for each monic m: $Y \rightarrowtail X$ there is a unique arrow χ_m – the *characteristic arrow* of m (or Y) – such that the diagram



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is a pullback. The object Y (or monic m) is called the subobject of X *classified* by χ_m.

Let us illustrate this idea in the case of the category $\mathbf{S}^P$ of sets varying over a preordered set P. Given an object G in $\mathbf{S}^P$, a subobject of G is an object F in $\mathbf{S}^P$ such that $F(p) \subseteq G(p)$ for all $p \in P$ and $F_{pq} = G_{pq} \restriction F(p)$ for $p \leq q$. Any $x \in G(p)$ may be "classified" by the set

$$\chi_p(x) = \{q \geq p\colon G_{pq}(x) \in F(q)\};$$

thus $\chi_p(x)$ is the set of "stages" $q \in P$ at which x has "become" a member of the subobject F. It is easy to see that $\chi_p(x) = Q$ has the property

$$q \in Q \;\&\; r \geq q \Rightarrow r \in Q.$$

A subset Q of P satisfying this condition is called a *filter* in P; a filter included in $\{q\colon q \geq p\}$ is called a *filter over* p. Accordingly the subobject classifier Ω in $\mathbf{S}^P$ has

$$\Omega(p) = \text{all filters over } p$$

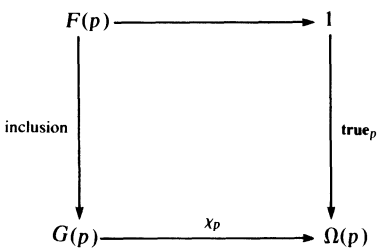
and, for $q \geq p$, $Q \in \Omega(p)$,

$$\Omega_{pq}(Q) = \{r \in Q\colon r \geq p\}.$$

The terminal object $\mathbf{1}$ in $\mathbf{S}^P$ is the function on P with constant value 1, and **true**: $\mathbf{1} \to \Omega$ is defined by

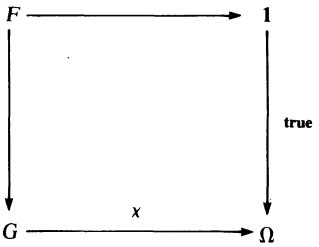
$$\mathbf{true}_p(0) = \{q\colon q \geq p\} = M_p.$$

Now the subobject F of G may be obtained by pulling back **true** along the "characteristic arrow" $\chi = \{\chi_p\colon p \in P\}$. For since $F(p) = \chi_p^{-1}(M_p)$ for each $p \in P$ by construction, it follows that



is a pullback for each $p \in P$, and so therefore is

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Several of the categories introduced in Section 1 can be shown to have subobject classifiers, in particular, $\mathbf{Shv}(X)$, $\mathbf{Spo}(X)$, $\mathbf{S}_H$, $V^{(B)}$, $\mathbf{S}^M$.

For example, in $\mathbf{Shv}(X)$, Ω is the sheaf defined by

$$\Omega(U) = \text{set of open subsets of } U,$$
$$\Omega_{UV}(W) = W \cap V.$$

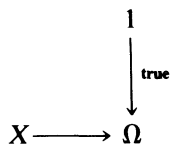
In $V^{(B)}$, Ω is just $\hat{2}$, the canonical image of the set 2 in $V^{(B)}$. In $\mathbf{S}^M$, where M is a monoid, Ω is the set of left ideals in M, where a subset of M is a *left ideal* if it is closed under multiplication on the left. The action of M on Ω is *division*: $\alpha.A = \{\beta\colon \beta\alpha \in A\}$. The arrow $1 \xrightarrow{\text{true}} \Omega$ is the function with value M, the unit ideal. In S_H, Ω is $(H, \Leftrightarrow)$ where $\Leftrightarrow$ is defined by $x \Leftrightarrow y = \{z\colon z \wedge x = z \wedge y\}$.

We can now define the categories to which we are going to devote special attention.

DEFINITION. An *elementary topos*, or simply a *topos*, is a category which has a terminal object, pullbacks, exponentials, and a subobject classifier.

Clearly $\mathbf{S}$ is a topos; moreover, it can be shown that $\mathbf{S}^P$, $\mathbf{Shv}(X)$, $V^{(B)}$, $\mathbf{S}_H$ and $\mathbf{S}^M$ are toposes as well (see [4] or [5]).

It is an interesting fact that for a category to be a topos it is enough that it have a terminal object, products, a subobject classifier Ω, pullbacks of all diagrams of the form

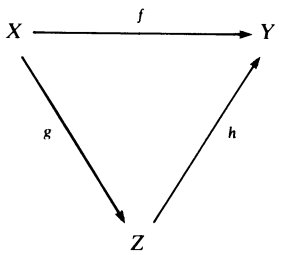


and all exponentials of the form Ω^X.

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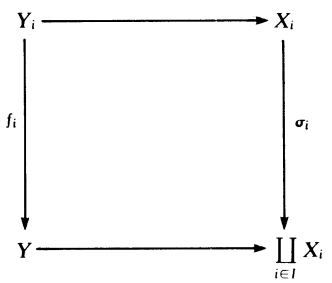
The topos axioms are very rich in consequences, in that many properties of the category **S** can be shown to obtain in arbitrary toposes. For example, the following assertions are easily seen to hold in **S** and can be shown also to hold in an arbitrary topos **E** (for proofs, see [3] or [5]).

1. **E** has an initial object 0 and any arrow with domain 0 is monic.
2. In **E**, an arrow is an isomorphism iff it is both epic and monic.
3. Any arrow $X \xrightarrow{f} Y$ in **E** admits a unique factoring (up to isomorphism)



with g epic and h monic. (The arrow h is called the *image* of f and the object Z the *image* of X.)

4. In **E**, pullbacks of epics are epic.
5. **E** admits the construction of finite products and coproducts.
6. Coproducts in **E** are (i) *disjoint* and (ii) *pullback-stable.* This means: (i) if $\{X_i: i \in I\}$ is any family of objects and $\{X_i \xrightarrow{\sigma_i} \amalg_{i\in I} X_i\}_{i\in I}$ is a coproduct diagram, then for any $i \neq j$ in I the pullback of X_i against X_j is 0. (ii) Under the same conditions as (i), for any arrow $Y \xrightarrow{f} \amalg_{i\in I} X_i$, if for each $i \in I$



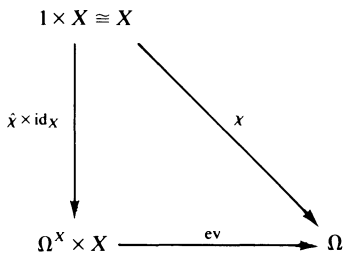
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is a pullback, then $\{Y_i \xrightarrow{f_i} Y\}_{i \in I}$ is a coproduct diagram. ("A pullback of a coproduct is a coproduct".)

At this point we stipulate that any topos we consider be non-degenerate, i.e. $0 \not\cong 1$. This is easily seen to be equivalent to asserting that there is no arrow $1 \to 0$.

In a topos $\mathbf{E}$ we might with some justice regard arrows $x\colon 1 \to X$ as the "elements" of an object X; they evidently do correspond to elements when $\mathbf{E} = \mathbf{S}$. However, there are usually not "enough" of these elements in an arbitrary topos because of the presence of non-zero objects with no elements.[2] It is thus helpful to think of arrows $u\colon U \to X$ from an *arbitrary* domain U as "generalized" elements of X.

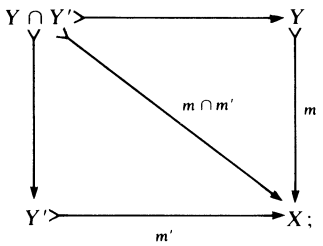
Arrows $\chi\colon X \to \Omega$ in a topos $\mathbf{E}$ may be regarded, by analogy with $\mathbf{S}$, as *predicates* defined on X. The fact that predicates are present in $\mathbf{E}$ enables some of the usual notions of set theory and logic to be formalized in $\mathbf{E}$. For example, corresponding to each object X of $\mathbf{E}$ there is a power set object $PX = \Omega^X$. Now, by the definition of the subobject classifier, there is a one-one correspondence between subobjects on an object X and predicates $\chi\colon X \to \Omega$ defined on X; since, as is easily verified, $X \cong 1 \times X$ these correspond, in turn, via the diagram



to arrows $\phi = \hat{\chi}\colon 1 \to PX$, i.e. to *elements* of X. And similarly, predicates $\psi\colon W \times X \to \Omega$ of two variables correspond to arrows $\phi\colon W \to PX = \Omega^X$, i.e. to *generalized* elements of PX. We may regard this correspondence as a kind of *comprehension principle*, by analogy with the fact that in $\mathbf{S}$, ϕ would send each element $w \in W$ to the set $\{x\colon \psi(w, x)\}$. This is an *internal* version of the usual comprehension principle because the predicate ψ here is actually an entity (an arrow) *in* the system, not a formula in its associated formal language.

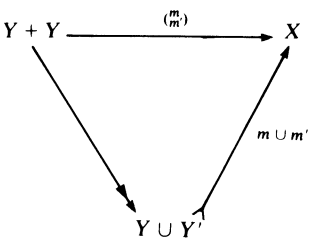
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Consider now the set Sub(X) of all subobjects of an object X in a topos **E**. For the time being we shall identify each element $[m]$ of Sub(X) with its representative monic $m\colon Y \rightarrowtail X$. This set Sub($X$) can be partially ordered by the obvious inclusion of subobjects: if $Y \overset{m}{\rightarrowtail} X$ and $Y' \overset{m'}{\rightarrowtail} X$ are subobjects of X, put $m \subseteq m$ (and $Y \subseteq Y'$) if there is a monic $Y \overset{n}{\rightarrowtail} Y'$ such that $m = nm'$. This partial ordering turns Sub(X) into a *lattice*: the infimum $m \cap m'$ of m and m' is the intersection constructed by pullback:



we write $Y \cap Y'$ for the domain of $m \cap m'$.

The supremum $m \cup m'$ is the *union* of m and m', i.e. the image of the map $\binom{m}{m'}\colon Y + Y' \to X$ in X formed by taking the epic-monic factorization:



we write $Y \cup Y'$ for the domain of $m \cup m'$.

In fact the lattice Sub(X) is a *Heyting algebra*, that is, a lattice L which, regarded as a category (see Example 5 of Section 1), admits exponentials. Since the infimum $\wedge$ in a lattice L is easily seen to be the *product* in the associated category, the exponential assigns to each pair of elements b, c of L an element c^b, also written $b \Rightarrow c$,

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such that, for all $a \in L$,

$$a \wedge b \leq c \quad \text{iff} \quad a \leq (b \Rightarrow c).$$

(Observe that, since $x \leq y$ in L iff $\hom(x, y) \neq \emptyset$, this statement is exactly the bijection (*) of Section 1 defining exponentials, written in lattice language.) In any Heyting algebra L with least element 0, the negation $\neg a$ of an element a may be defined as:

$$\neg a = a \Rightarrow 0;$$

i.e. $\neg a$ is the largest element x of L such that $a \wedge x = 0$.

Remark. It is not hard to show that a complete lattice has the structure of a Heyting algebra iff it satisfies the distributive law:

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} x \wedge y_i.$$

This explains why in Example 11 of Section 1 we defined complete Heyting algebras in the way we did.

For the generic example of a (complete) Heyting algebra – that of a topology on a set X – the operation $\Rightarrow$ is defined by $U \Rightarrow V$ = interior of $V \cup (X - U)$.

The operations in the Heyting algebra Sub(1) of all subobjects of 1 in a topos **E** are particularly easy to define. First, we note (cf. 2.2) that if U is an object of E and the unique arrow $U \to 1$ is *monic* then for any object X of **E** there is *at most* one arrow $X \to U$ (and conversely). It follows that the elements of Sub(1) may be regarded as (isomorphism classes) of domains of monics into 1. Then if $U, V \in \text{Sub}(1)$, i.e. if $U \rightarrowtail 1$, $V \rightarrowtail 1$, we have

$$\begin{aligned} U \cap V &= U \times V \\ U \cup V &= \text{image of } U + V \text{ in } 1 \\ U \Rightarrow V &= V^U \\ \neg U &= 0^U. \end{aligned}$$

This Heyting algebra Sub(1) of all subobjects of 1 in a topos **E** is called the algebra of (global) *truth values* of **E** because it is in natural one-one correspondence with the set $E(1, \Omega)$ of elements of Ω. It plays a particularly important role in determining the structure of **E**. For example, when **E** is **Shv**(X), Sub(1) is the Heyting algebra of open subsets of X; in this case **E** can be recaptured as the category of sheaves over its algebra of truth values. Suitable generalizations of

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the notion of sheaf and algebra of truth values enables this assertion to be carried over to arbitrary toposes. Thus a topos may be thought of as a generalized "category of sheaves" as well as a generalized "category of sets"; it is this fact which places topos theory at the junction of algebraic geometry and set theory.

Thus in any topos $\mathbf{E}$ the set $\mathrm{Sub}(X)$ of subobjects of X is a Heyting algebra; since the correspondence $m \mapsto \chi_m$ establishes a bijection between $\mathrm{Sub}(X)$ and $\mathbf{E}(X, \Omega)$, we may transfer the Heyting algebra structure of the former to the latter and so transform it into a Heyting algebra with operations $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\Rightarrow}$, $\tilde{\neg}$. It is now possible to transmit the Heyting algebra structure of $\mathbf{E}(X, \Omega)$ to Ω itself, yielding arrows ("logical operations"):

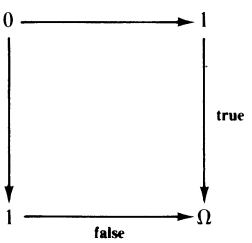
$$\Omega \times \Omega \xrightarrow{\tilde{\wedge}} \Omega, \qquad \Omega \times \Omega \xrightarrow{\tilde{\vee}} \Omega, \qquad \Omega \times \Omega \xrightarrow{\tilde{\Rightarrow}} \Omega, \qquad \Omega \xrightarrow{\tilde{\neg}} \Omega$$

in such a way that a given operation in Ω induces the corresponding operation in $\mathbf{E}(X, \Omega)$, as in

$$X \xrightarrow{\langle \phi, \psi \rangle} \Omega \times \Omega \xrightarrow{\tilde{\wedge}} \Omega = X \xrightarrow{\phi \tilde{\wedge} \psi} \Omega$$

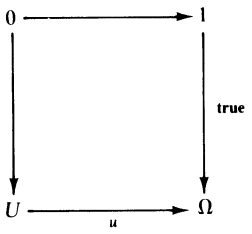
for any $\phi, \psi\colon X \to \Omega$. Here, by taking $X = \Omega \times \Omega$, $\phi = \pi_1$, $\psi = \pi_2$, we see that $\wedge$ must be $\pi_1 \wedge \pi_2$ and from this one easily shows that $\wedge$ is the characteristic arrow of $\langle \mathbf{true}, \mathbf{true} \rangle$.

Similarly, one shows that $\neg\colon \Omega \to \Omega$ is the negation $\tilde{\neg}\mathrm{id}_\Omega$ of the element id_Ω of $\mathbf{E}(\Omega, \Omega)$. Now id_Ω classifies the element $\mathbf{true}\colon 1 \to \Omega$ of $\mathrm{Sub}(\Omega)$ and so $\neg$ classifies the negation of this subobject, i.e. the largest subobject of Ω disjoint from $\mathbf{true}$. What is this subobject? We claim that it is the characteristic arrow $\mathbf{false}\colon 1 \to \Omega$ of the monic $0 \rightarrowtail 1$. For certainly $\mathbf{false} \cap \mathbf{true} = 0$ since

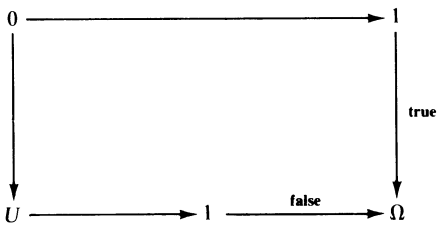


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is a pullback. Moreover, let $U \overset{u}{\rightarrowtail} \Omega$ be a subobject of Ω disjoint from **true**. This means that

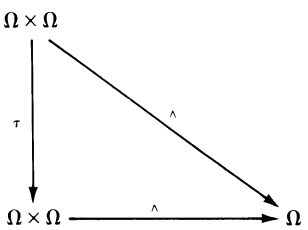


is a pullback, and therefore so is



Hence $0 \rightarrowtail U$ is classified both by $U \to 1 \xrightarrow{\textbf{false}} \Omega$ and by $U \overset{u}{\to} \Omega$, so these latter are equal. Therefore $U \rightarrowtail 1$ and $u \subseteq \textbf{false}$. This proves the claim.

A Heyting algebra may be defined by equations valid for these operations $\wedge$, $\vee$, $\Rightarrow$, $\neg$ and each such equation may be expressed as a suitable commutative diagram in E. For example, the commutative law for the infimum operation may be expressed by the commutative diagram



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where τ is the canonical isomorphism asserting that the categorical product is commutative. In view of the commutativity of these diagrams, Ω is called a Heyting algebra *object* in **E**. Since the algebraic laws holding in Heyting algebras correspond to the laws of intuitionistic propositional logic, we may thus say that intuitionistic logic is "internalized" in **E**.

We recall that a Heyting algebra is called a *Boolean algebra* if $x \vee \neg x = 1$ for any x. This is equivalent to $\neg\neg$ being the identity in H. (So, if a Heyting algebra corresponds to intuitionistic logic, a Boolean algebra corresponds to *classical* logic in which the law of excluded middle holds.) Now while it is clear that, in **S**, the truth value object $2 = \{0, 1\}$ is a Boolean algebra, this is no longer the case for an arbitrary topos. For example, in $\mathbf{Shv}(X)$, the corresponding Heyting algebra Ω is the Heyting algebra of open sets in X which is not Boolean in general because double negation is the interior of the closure. We can, however, give a number of simple necessary and sufficient conditions for Ω to be a Boolean algebra. Calling a topos *Boolean* if its Ω is a Boolean algebra, we have

2.1 THEOREM. The following conditions in a topos **E** are equivalent:

(i) **E** is Boolean;

(ii) $\neg\neg = \mathrm{id}_\Omega$;

(iii) any subobject in **E** has a *complement*, i.e. given $Y \overset{y}{\rightarrowtail} X$ there is $Z \overset{z}{\rightarrowtail} X$ such that $Y + Z \xrightarrow{(\overset{y}{z})} X$ is an isomorphism;

(iv) $1 + 1 \xrightarrow{(\overset{\text{true}}{\text{false}})} \Omega$ is an isomorphism.

Proof. (i) $\Rightarrow$ (ii) is immediate from the corresponding result in **S**.

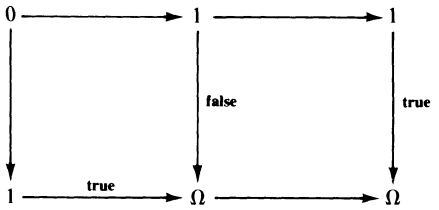
(i) $\Rightarrow$ (iii). Let $X \xrightarrow{\chi} \Omega$ be the characteristic arrow of $Y \overset{y}{\rightarrowtail} X$, and let Z be the subobject of X classified by $\neg\chi$. From the definition of a Boolean algebra we have $Y \cap Z = 0$, $Y \cup Z = X$ as subobjects of X; but this is equivalent to saying that $Y + Z \cong X$.

(iii) $\Rightarrow$ (iv). Assuming (iii), the subobject $1 \overset{\text{true}}{\rightarrowtail} \Omega$ has a complement which must be $1 \overset{\text{false}}{\rightarrowtail} \Omega$ since the latter is the largest subobject of Ω whose intersection with **true** is 0.

(iv) $\Rightarrow$ (ii). First note that we have, by definition of $\neg$,

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false. $\neg$ = **true**. Also, the two constituent squares in the rectangle:



are pullbacks, and so therefore is the whole rectangle. Therefore **true**. $\neg$ classifies $0 \rightarrowtail 1$, and so **true**. $\neg$ = **false**. It follows that

$$\begin{pmatrix}\textbf{true}\\ \textbf{false}\end{pmatrix}.\neg.\neg = \begin{pmatrix}\textbf{true}.\neg.\neg\\ \textbf{false}.\neg.\neg\end{pmatrix} = \begin{pmatrix}\textbf{true}\\ \textbf{false}\end{pmatrix}$$

So if $\begin{pmatrix}\textbf{true}\\ \textbf{false}\end{pmatrix}$ is an isomorphism, $\neg\neg = \mathrm{id}_\Omega$. ■

EXAMPLES.

1. For T_0-spaces, X, $\mathbf{Shv}(X)$ is Boolean iff X is discrete.

2. Let M be a monoid. Then the topos $\mathbf{S}^M$ of M-sets is Boolean iff M is a group. For if M is a group, then the set-theoretical complement of any sub-M-set Y of an M-set X is itself a sub-M-set and therefore the complement (in $\mathbf{S}^M$) of Y. Conversely, if M is not a group, then the set N of non-invertible elements of M (regarded as an M-set with the natural multiplication on the left by members of M), is a sub-M-set which doesn't have a complement because any sub-M-set of M must – as is easily verified – intersect N.

Before proceeding further it will be helpful to prove a simple lemma.

2.2 LEMMA. Let $\mathbf{E}$ be a topos and let U be an object in $\mathbf{E}$. Then the following are equivalent:

(i) U is a subobject of 1, i.e. the unique arrow $U \rightarrow 1$ is monic;

(ii) for any X, there is at most one arrow $X \rightarrow U$;

(iii) any arrow with domain U is monic.

Proof. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) is obvious, so it remains to prove (i) $\Rightarrow$ (ii). Suppose (i) holds and $f, g\colon X \rightarrow U$. Then $fh = gh$, where $U \xrightarrow{h} 1$ is monic, so $f = g$. Hence (ii). ■

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We now formulate some conditions on a topos which will turn out to be important in our characterization of **S**.

A topos **E** is said to be:

• *extensional* if whenever $f, g: X \to Y$ with $f \neq g$ there is $x: 1 \to X$ such that $xf \neq xg$; the arrow x is said to *distinguish* f and g. (This is a version of the axiom of extensionality for arrows.)

• 2-*valued* if **true** and **false** are the only arrows $1 \to \Omega$; equivalently, if 0 and 1 are the only subobjects – up to isomorphism – of 1.

Clearly **S** is both extensional and 2-valued.

Now we can prove

2.3 LEMMA. Any extensional topos is 2-valued and Boolean.

Proof. Suppose **E** is extensional, and let $X \ncong 0$. Then the monics $0 \rightarrowtail X$ and $X \overset{\mathrm{id}_X}{\rightarrowtail} X$ are non-equivalent and so therefore are their characteristic arrows to Ω. Since **E** is extensional, there must be an arrow $1 \to X$ distinguishing them, hence certainly an arrow $1 \to X$.

Now let $U \rightarrowtail 1$. If $U \ncong 0$, then by the above there exists an arrow $1 \to U$ and $1 \to U \to 1 = \mathrm{id}_1$ forces $U \to 1$ to be epic. Hence $U \cong 1$ and so **E** is 2-valued.

To show that **E** is Boolean, we prove $\neg\neg = \mathrm{id}_\Omega$. In the proof of 2.1 we established that

$$\mathbf{true}.\neg.\neg = \mathbf{true} = \mathbf{true}.\mathrm{id}_\Omega$$
$$\mathbf{false}.\neg.\neg = \mathbf{false} = \mathbf{false}.\mathrm{id}_\Omega.$$

But since **E** is 2-valued, **true** and **false** are the only arrows $1 \to \Omega$ so, since **E** is extensional, we must have $\neg\neg = \mathrm{id}_\Omega$. Therefore **E** is Boolean. ■

EXAMPLE. We have seen that the topos $\mathbf{S}^M$ of M-sets (for a monoid M) is only Boolean if M is a group. But it is *always* 2-valued. For the terminal object in $\mathbf{S}^M$ is the one-point set 1 with trivial M-action and evidently this has only the 2 subobjects 0, 1. Thus when M is not a group, the algebra Sub(1) of global truth values is Boolean (since it has only two elements) while the subobject classifier Ω is not internally a Boolean algebra.

We can also formulate a version of the *axiom of choice* in a topos (in fact in an arbitrary category).

AC: for each epic $X \overset{f}{\to} Y$ there is a (necessarily monic) $Y \overset{g}{\rightarrowtail} X$ such that $gf = \mathrm{id}_Y$.

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It is a remarkable fact, due to Diaconescu, that *any topos in which* **AC** *holds is Boolean.* We shall not give a formal proof of this result (for such a proof, see [5]), but rather a "heuristic" proof in **S** which can be translated into an arbitrary topos. The idea is to use the axiom of choice to construct a complement for a subset of an arbitrary set, *without using the fact that logic in* **S** *is Boolean*, i.e. that the law of excluded middle holds.

So assume **AC**, let X be a set and let $Y \subseteq X$. We want to construct a complement Y' for Y in X.

First, we take the disjoint union (coproduct) $X + X$ of two disjoint copies of X. We may think of the points of the first copy of X as being coloured white and those of the second as being coloured black. So each point of X has been "split" into a "white" copy and a "black" copy.

Next, we identify each copy of a point of Y in the first (white) copy of X with its mate in the second (black) copy of X; the points arising in this way (which constitute a copy of Y) we agree to colour grey, say. In this way we obtain a set Z consisting of white, black and grey points, and an epic map $X + X \twoheadrightarrow Z$.

Now we use the axiom of choice to assign to each point z of Z a point z' of $X + X$ in such a way that z' is mapped to z under the function $X + X \twoheadrightarrow Z$ above. Then this process, P, say, transforms each point x of $X + X$ into a point x' – possibly the same – of $X + X$.

Finally, we define

X_1 = set of points of X whose black copy remains black under P
X_2 = set of points of X whose black copy becomes white under P
X_3 = set of points of X whose white copy remains white under P
X_4 = set of points of X whose white copy becomes black under P.

Clearly $X = X_1 + X_2 = X_3 + X_4$; so

$$X = (X_1 \cap X_3) + (X_1 \cap X_4) + (X_2 \cap X_3) + (X_2 \cap X_4).$$

Also, it is easy to see that $X_2 \cap X_4 = \emptyset$ and

$$Y = (X_1 \cap X_3) + (X_1 \cap X_4)$$

So Y has complement $X_1 \cap X_3$.

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3. COPOWERED TOPOSES AND THE CHARACTERIZATION OF S.

In this section we characterize those toposes which are equivalent to **S** in a natural sense. In order to do this we must introduce a condition on a topos which is not first-order.

A topos **E** is said to be *copowered* if it has arbitrary copowers of its objects, i.e. if for each object X and each set I the I-fold copower $I.X$ of X exists in **E**. It is easy to see that for this to be the case it is sufficient that for each set I the I-fold copower $I.1$ of the terminal object 1 exist in **E**, for then $I.X$ may be constructed as $X \times (I.1)$. For each set I we write $\hat{I}$ for $I.1$; $\hat{I}$ may be thought of as the object of **E** *representing* the set I. (Clearly, when $\mathbf{E} = \mathbf{S}$, $\hat{I} \cong I$ for each set I.)

We also observe that in a copowered topos we can form arbitrary coproducts of subobjects of any given object X. For let $\{U_i \overset{u_i}{\rightarrowtail} X\}_{i \in I}$ be a family of subobjects of X; let $X \overset{g_i}{\rightarrow} \Omega$ be the characteristic arrow of u_i and let $U \rightarrowtail I.X$ be the subobject classified by $[g_i]_{i \in I}: I.X \rightarrow \Omega$. Then one can show that $U \cong \amalg_{i \in I} U_i$. (It is easy to verify that this procedure works in **S**!)

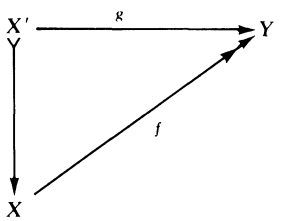
It follows that, in a copowered topos **E**, the lattice $\mathrm{Sub}(X)$ of subobjects of any object **E** is complete (in fact, a complete Heyting algebra.) That is, we can form, for any family $\{U_i \overset{u_i}{\rightarrowtail} X\}_{i \in I}$ of subobjects of X, the supremum or union $\bigcup_{i \in I} U_i$. For given such a family we can form the coproduct $\amalg_{i \in I} U_i$, and then $\bigcup_{i \in I} U_i \rightarrowtail X$ is the image of the arrow $[u_i]_{i \in I}: \amalg_{i \in I} U_i \rightarrow X$.

Now we can prove the important

3.1 THEOREM. Let **E** be an extensional copowered topos. Then (assuming **AC** holds in **S**) **AC** holds in **E**.

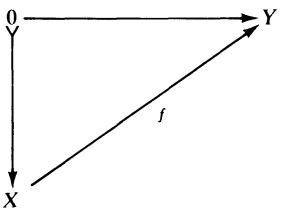
Proof. The argument here is almost a verbatim translation to **E** of the standard derivation of **AC** from Zorn's lemma in **S**.

Let $X \overset{f}{\rightarrow} Y$ be an epic in **E**. Define a *partial inverse* of f to be a pair (X', g) with $X' \rightarrowtail X$ and $X' \overset{g}{\rightarrow} Y$ such that the diagram

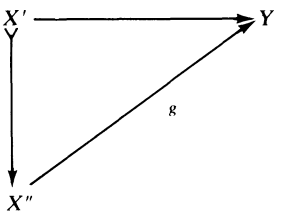


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commutes. Let P be the set of all partial inverses of f; P is non-empty since we always have the trivial partial inverse

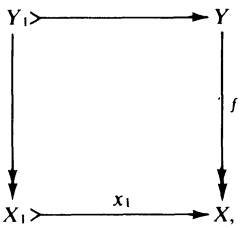


We partially order P by agreeing that $(X', g) \leq (X'', g')$ if $X' \subseteq X''$ as subobjects of X and the diagram



commutes. Then P satisfies the conditions of Zorn's lemma, since we can form arbitrary unions of subobjects of a given object in **E**. So P has a maximal element (X_0, g) and it suffices to show that the monic $X_0 \overset{x}{\rightarrowtail} X$ is an isomorphism.

Suppose not; then since **E** is Boolean (2.3) the complement $X_1 \overset{x_1}{\rightarrowtail} X$ of $X_0 \overset{x}{\rightarrowtail} X$ is non-zero and hence so is Y_1 in the pullback



where $Y_1 \twoheadrightarrow X_1$ is epic as the pullback of the epic f.

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Since **E** is extensional, there is an arrow $1 \xrightarrow{a} Y_1$. Now the composition

$$b = 1 \xrightarrow{a} Y_1 \overset{c}{\rightarrowtail} Y \overset{f}{\twoheadrightarrow} X$$

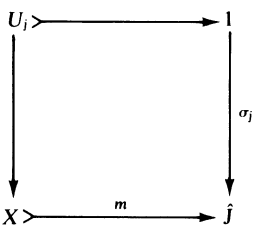
is monic, and clearly $b \subseteq x$ as subobjects of X. So $b \cap x_0 \subseteq x_1 \cap x_0 = 0 \rightarrowtail X$. But this implies that the union of the subobject $1 \overset{b}{\rightarrowtail} X$ with $X_0 \rightarrowtail X$ is a disjoint one and so we can properly extend our partial inverse (X_0, g) to the partial inverse $(X_0 + 1, h)$ by putting $h = \binom{g}{c}$. This contradicts the maximality of (X_0, g). So x_0 must be an isomorphism, and **AC** follows. ■

3.2 COROLLARY. Let **E** be an extensional copowered topos. Then for each object X of **E** there is a set I such that $X \cong \hat{I}$ in **E**.

Proof. We first show that for each object X there is a set J and an epic $\hat{J} \twoheadrightarrow X$. Let J be a set indexing all the arrows $1 \to X$ and for each $j \in J$ let f_j be the j^{th} such arrow. Put $f = [f_j]_{j \in J} : \hat{J} \to X$.

We claim that f is epic. For suppose $g, h : X \to Y$ are distinct. Then there is, since **E** is extensional, an arrow $f_{j_0} : 1 \to X$ such that $f_{j_0} g \neq f_{j_0} h$. It follows immediately that $fg \neq fh$, so f is epic.

Since **AC** holds in **E** (by 3.1), there is a monic $X \overset{m}{\rightarrowtail} \hat{J}$ (such that $mf = \mathrm{id}_X$). For each $j \in J$ we form the pullback



Since **E** is 2-valued (by 2.3), $U_j \cong 0$ or $U_j \cong 1$; put $I = \{j \in J : U_j \cong 1\}$. Then, by the pullback-stability of coproducts in **E**, we have $X \cong \coprod_{j \in J} U_j$, and clearly

$$\coprod_{j \in J} U_j \cong \coprod_{i \in I} U_i \cong \hat{I}. \quad \blacksquare$$

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It is now the right time to introduce the notion of a structure-preserving map or *functor* between categories.

Let $\mathbf{E}$ and $\mathbf{E}'$ be categories. A *functor* $F\colon \mathbf{E} \to \mathbf{E}'$ is a function F which assigns to each object X of E an object $F(X)$ of $\mathbf{E}$ and to each arrow $X \xrightarrow{f} Y$ of $\mathbf{E}$ an arrow $F(X) \xrightarrow{F(f)} F(Y)$ of $\mathbf{E}'$ in such a way that

- if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{E}$, then $F(fg) = F(f)F(g)$;
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any $X \in \mathbf{E}$.

A functor is said to be an *isomorphism* if it is bijective on objects and arrows. Two categories are *isomorphic* if there is an isomorphism between them.

In practice the notion of isomorphism of categories is replaced by the weaker notion of *equivalence*. A functor $F\colon \mathbf{E} \to \mathbf{E}'$ is said to be *faithful* if it is injective on arrows, and *full* if it maps each $\hom(X, Y)$ onto $\hom(F(X), F(Y))$. F is an *equivalence* if it is full and faithful and bijective on *isomorphism classes* of objects, i.e. if for each object Y of $\mathbf{E}'$ there is an object X of $\mathbf{E}$ such that $Y \cong F(X)$. Two categories $\mathbf{E}$ and $\mathbf{E}'$ are said to be *equivalent*, written $\mathbf{E} \simeq \mathbf{E}'$, if there is an equivalence between them. To assert that two categories are equivalent is essentially to assert that they have exactly the same category-theoretic properties.

We can now characterize the category of sets up to equivalence.

3.3 THEOREM. Let $\mathbf{E}$ be a category. Then $\mathbf{E} \simeq \mathbf{S}$ iff $\mathbf{E}$ is an extensional copowered topos.

Proof. One way round is clear, since $\mathbf{S}$ is an extensional copowered topos and equivalence preserves these properties.

Conversely, let $\mathbf{E}$ be an extensional copowered topos. We define a functor $F\colon \mathbf{S} \to \mathbf{E}$ by putting, for each set I,

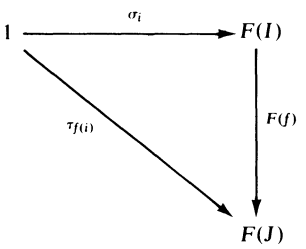
$$F(I) = \hat{I} \ (= I.1)$$

If $f\colon I \to J$ in $\mathbf{S}$, we define

$$F(f)\colon F(I) \to F(J)$$

to be the unique arrow in $\mathbf{E}$ making the diagrams

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commute for each $i \in I$, where σ_i, τ_j are the canonical injections. It is easy to check that F is a functor, and by 3.2 each object in **E** is isomorphic to one in the range of F. Thus, to show that F is an equivalence it suffices to show that it is full and faithful.

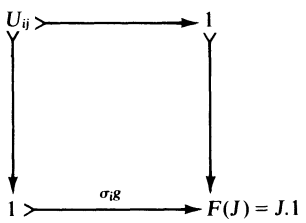
To verify the fidelity of F, observe that

$$\begin{aligned} F(f) = F(g) &\Leftrightarrow \tau_{f(i)} = \tau_{g(i)}, \text{ all } i \quad (*) \\ &\Leftrightarrow f(i) = g(i), \text{ all } i \quad (**) \\ &\Leftrightarrow f = g. \end{aligned}$$

(To see that (*) and (**) are equivalent, it suffices to show that, for $j \neq j' \in J$, $\tau_j \neq \tau_{j'}$. But this follows easily from the disjointness of coproducts in **E**.)

To verify the fullness of F, let $g\colon F(I) \to F(J)$ and for each $i \in I$, $j \in J$ let

(***)



be a pullback. Then $U_{ij} \cong 0$ or 1 since **E** is 2-valued (2.3), and by the pullback stability of coproducts in **E**,

$$\coprod_{j \in J} U_{ij} \cong 1.$$

It follows easily from this that for each $i \in I$ there is *exactly one* $j \in J$

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for which $U_{ij} \cong 1$. Define $f\colon I \to J$ by

$$f(i) = \text{unique } j \in J \text{ for which } U_{ij} \cong 1.$$

We claim that $F(f) = g$. But for this to be the case it suffices that

$$\sigma_i g = \sigma_i F(f) = \tau_{f(i)}, \quad \text{all } i \in I,$$

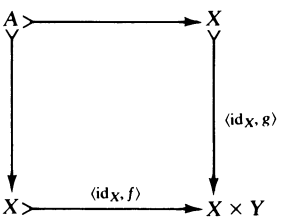
and this clearly follows from (***). Thus F is full and the theorem is proved. ■

S can also be characterized in terms of the satisfaction of a version of the axiom of choice, instead of an extensionality principle.

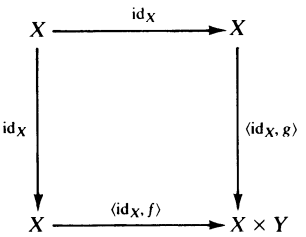
Let us say that a category **E** satisfies the *strong axiom of choice* (**SAC**) if for each arrow $X \xrightarrow{f} Y$ with $X \not\cong 0$ there is $g\colon Y \to X$ such that $fgf = f$. Then we can prove

3.4 THEOREM. Let **E** be a category. Then $\mathbf{E} \simeq \mathbf{S}$ iff **E** is a copowered topos satisfying **SAC**.

Proof. By 3.3, it is enough to show that any topos satisfying **SAC** is extensional. So suppose **E** is a topos satisfying **SAC** and let $f, g\colon X \to Y$ be distinct arrows of **E**. Form the pullback



(observing that $\langle \mathrm{id}_X, f\rangle$ and $\langle \mathrm{id}_X, g\rangle$ are monic!). Then $A \rightarrowtail X$ is not an isomorphism, for if it were then the diagram

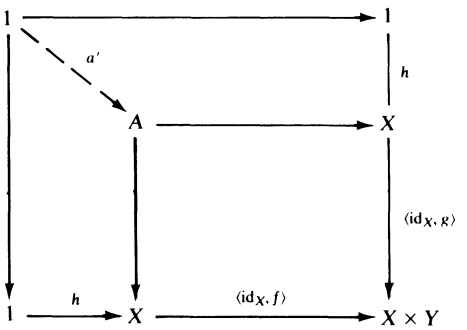


would commute, so $\langle \mathrm{id}_X, f\rangle = \langle \mathrm{id}_X, g\rangle$ and $f = g$.

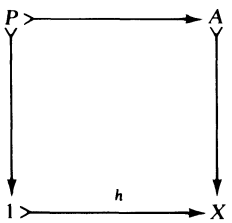
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Since **E** satisfies **SAC**, it is easy to see that it satisfies **AC**, and so is Boolean. Therefore the complement $A' \rightarrowtail X$ of $A \rightarrowtail X$ in X is non-zero and by **SAC** there is an arrow $1 \xrightarrow{a} A'$. Let h be the composition $1 \xrightarrow{a} A' \rightarrow X$; we show that $hf \neq hg$.

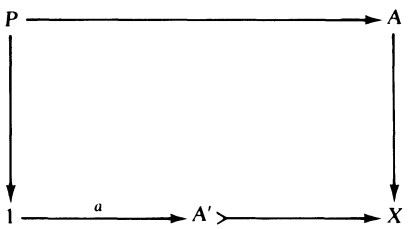
For suppose $hf = hg$. Then $\langle h, hg \rangle = \langle h, hf \rangle$ and so the diagram



commutes. But the inner diagram is a pullback and so there is $1 \xrightarrow{a'} A$ making the resulting left-hand trapezoid commute. It follows that there must be an arrow $1 \rightarrow P$, where



is a pullback. But clearly $P \cong 0$, since the diagram



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commutes, and the pullback of A against A' in X is 0. Since $0 \not\cong 1$, we cannot have an arrow $1 \to 0$, so we get a contradiction. Hence $hf \neq hg$ and $\mathbf{E}$ is extensional. The result follows. ■

One can obtain similar characterizations of the categories equivalent to toposes of the form $V^{(B)}$ and S_H (and also $\mathbf{S}^P$ and $\mathbf{Shv}(X)$). Let us call a topos *weakly extensional* if it satisfies the following weakened version of the extensionality principle: given two distinct arrows f, g: $X \to Y$, there is a subobject $U \rightarrowtail 1$ and an arrow $U \xrightarrow{x} X$ such that $xf \neq xg$. Then an argument similar to the one in 3.3 shows:

$\mathbf{E} \simeq V^{(B)}$ for some complete Boolean algebra B

$\Leftrightarrow \mathbf{E}$ is a copowered weakly extensional Boolean topos

$\Leftrightarrow \mathbf{E}$ is a copowered topos satisfying **AC**.

More complicated reasoning (see [4] or [5]) proves:

$\mathbf{E} \simeq \mathbf{S}_H$ for some cHa H

$\Leftrightarrow \mathbf{E}$ is a copowered weakly extensional topos.

4. TOPOLOGIES AND SHEAVES IN TOPOSES

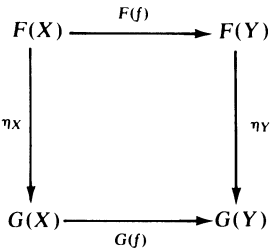
In Section 1 we defined the concept of a *sheaf*; it is clear from the definition that the notion of a sheaf on a topological space depends only on the lattice of open subsets of the space, and in no way on its "points". This suggests the possibility of extending this notion to categories more general than lattices of open sets. We shall see not only that this possibility can be realized for an arbitrary small[3] category $\mathbf{C}$, but that the resulting theory is an important and illuminating generalization of the classical case.

In order to formulate the notion of sheaf on a small category $\mathbf{C}$, we need two concepts to begin with: that of *presheaf* on $\mathbf{C}$ and that of *covering family* in $\mathbf{C}$.

Now the notion of presheaf on $\mathbf{C}$ is a straightforward generalizationof the classical idea. First, we define the *opposite* category $\mathbf{C}^{\mathrm{op}}$ to have the same objects as $\mathbf{C}$, but arrows going the "opposite" way to those of $\mathbf{C}$, i.e. we have an arrow $X \xrightarrow{f} Y$ in $\mathbf{C}^{\mathrm{op}}$ just when we have an arrow $Y \xrightarrow{f} X$ in $\mathbf{C}$. Then a *presheaf* F on $\mathbf{C}$ is just a functor F:

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$\mathbf{C}^{op} \to \mathbf{S}$. We turn the collection of presheaves on $\mathbf{C}$ into a category $\mathbf{S}^{\mathbf{C}^{op}}$ by defining an *arrow* between presheaves F and G to be a *natural transformation* between F and G, i.e. a family of maps η_X: $F(X) \to G(X)$ (one for each object X in C) such that, for each arrow f: $X \to Y$ in $\mathbf{C}^{op}$ (i.e. each arrow f: $Y \to X$ in $\mathbf{C}$), the diagram



commutes. Composition of arrows and the identity arrows are defined in the obvious way. $\mathbf{S}^{\mathbf{C}^{op}}$ is called the *category of presheaves* on $\mathbf{C}$; it can be shown that it is always a topos (see [3], [4], or [5]).

Important examples of presheaves are the so-called *representable presheaves* h_X for objects X of $\mathbf{C}$. Here h_X is defined by

$$h_X(Y) = \hom_{\mathbf{C}}(Y, X)$$

and if $Y \xrightarrow{f} Z$ is an arrow in $\mathbf{C}$, then

$$h_X(f)\colon h_X(Z) \to h_X(Y)$$

is given by $h_X(f)(g) = fg$ for $g \in h_X(Z)$.

Given two presheaves F and G on $\mathbf{C}$, F is called a *subpresheaf* of G if $F(X) \subseteq G(X)$ for all objects X of $\mathbf{C}$, and for any arrow $X \xrightarrow{f} Y$ in $\mathbf{C}$, we have $F(f) = G(f) \restriction F(Y)$.

Next, we turn to the notion of *covering family* in $\mathbf{C}$. Let us first re-examine this concept in the case where $\mathbf{C}$ is the lattice $\mathcal{O}(X)$ of open sets (ordered by inclusion) in a topological space X. Given $U \in \mathcal{O}(X)$, we define a *sieve* on U to be a family R of subsets of U such that $R \subseteq \mathcal{O}(X)$, and $W \in \mathcal{O}(X)$, $W \subseteq V \in R \Rightarrow W \in R$. Thus a sieve on U is just a filter over U in the partially ordered set $(\mathcal{O}(X), \supseteq)$. Notice that each family $Q \subseteq \mathcal{O}(X)$ of subsets of U generates the sieve $\bar{Q}$ on U given by

$$\bar{Q} = \{V \in \mathcal{O}(X)\colon V \subseteq \cup Q\}$$

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and that Q covers U iff $\bar{Q}$ does: for this reason in analyzing the idea of a covering family we may confine our attention to sieves. For each $U \in \mathcal{O}(X)$ we define $J(U)$ to be the set of all sieves R on U such that $\bigcup R = U$; the members of $J(U)$ are called the *covering sieves* on U. It is now clear that the $J(U)$ satisfy the following conditions:

(C1) $\{V \in \mathcal{O}(X)\colon V \subseteq U\} \in J(U)$

(i.e. the *maximal* sieve is a covering sieve);

(C2) if $R \in J(U)$ and $V \in \mathcal{O}(X)$, then

$V^*R = \{W\colon W \in R \text{ and } W \subseteq V\} \in J(V)$ (i.e. the restriction of a covering sieve to a smaller open set is a covering sieve);

(C3) if $R \in J(U)$ and S is a sieve on U such that for each $V \in R$ we have $V^*S \in J(V)$, then $S \in J(U)$ (i.e., if the restriction of a sieve S to each member of a covering sieve R is a covering sieve, then S is a covering sieve).

These three conditions were taken by Grothendieck as being *characteristic* of the motion of covering sieve for sheaf-theoretic purposes because, as we shall see, they generalize easily to an arbitrary small category $\mathbf{C}$ in such a way as to enable the notion of sheaf to be naturally carried over to $\mathbf{C}$.

So let $\mathbf{C}$ be any small category. If U as an object of $\mathbf{C}$, we define (by analogy with the original definition of sieve) a *sieve* on U to be a family R of arrows with codomain U such that, if $(V \xrightarrow{\alpha} U) \in R$ then $(W \xrightarrow{\beta\alpha} U) \in R$ for any $W \xrightarrow{\beta} V$. Now we can introduce the notion of *covering sieve* in $\mathbf{C}$ via the idea of *Grothendieck topology*, as follows.

A *Grothendieck topology*, or simply a *topology*, in $\mathbf{C}$ is a function J which assigns to each object U of $\mathbf{C}$ a family $J(U)$ of sieves on U called *covering sieves* of the topology (or J-covering sieves) in such a way that the following conditions are satisfied:

(G1) for any U, the maximal sieve $\{\alpha\colon \text{codomain}(\alpha) = U\} \in J(U)$;

(G2) if $R \in J(U)$ and $V \xrightarrow{f} U$ is any arrow, then the sieve $f^*(R) = \{W \xrightarrow{\alpha} V\colon \alpha f \in R\} \in J(V)$;

(G3) if $R \in J(U)$ and S is a sieve on U such that for each $V \xrightarrow{f} U$ in R we have $f^*(S) \in J(V)$, then $S \in J(U)$.

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Clearly, (G1)–(G3) are immediate generalizations of (C1)–(C3) to the category **C**.

A small category equipped with a Grothendieck topology is called a *site*. If X is a topological space and J is defined on the lattice $\mathcal{O}(X)$ of open sets of X by

$$J(U) = \text{set of all sieves } R \text{ on } U \text{ such that } \cup R = U;$$

then $(\mathcal{O}(X), J)$ is a site – the site *canonically* associated with X. In view of this we may follow Grothendieck in thinking of *general* sites as "generalized topological spaces" or "spaces without points".

Now let us see how to formulate the notion of *sheaf* for a Grothendieck topology. Again, we turn for inspiration to the classical case in which we are concerned with sheaves over an ordinary topological space X. Given an open set U in X and a sieve R on U we may identify R with the presheaf on X defined by

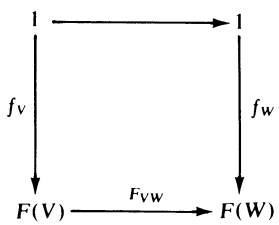
$$R(V) = \begin{cases} 1 \text{ if } V \in R \\ 0 \text{ if } V \notin R. \end{cases}$$

Also the representable presheaf h_U may be regarded as the function on $\mathcal{O}(X)$ defined by

$$h_U(V) = \begin{cases} 1 \text{ if } V \subseteq U \\ 0 \text{ if } V \not\subseteq U. \end{cases}$$

Then clearly $R(V) \subseteq h_U(V)$ for each $V \in \mathcal{O}(X)$, so R may be regarded as a ***subpresheaf*** of h_U.

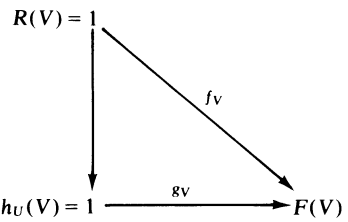
Given a presheaf F on X, an arrow $R \xrightarrow{f} F$ in $\mathbf{S}^{\mathcal{O}(X)^{\mathrm{op}}}$ determines – and is determined by – a function f_V: $\{0\} = 1 \to F(V)$ for each $V \in R$ such that, for $W \subseteq V$, the diagram



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commutes. In other words, such an arrow is uniquely determined by the choice of an element $\sigma_V = f_V(0) \in F(V)$ for each $V \in R$ in such a way that, for $W \subseteq V$, $\sigma_V \restriction_F W = F_{VW}(\sigma_V) = \sigma_W$. Because R is a sieve, this amounts precisely to specifying an *F-compatible* family of elements $\sigma_V \in F(V)$ for $V \in R$.

To say that F is a *sheaf* on X is thus to say that, for any $U \in \mathcal{O}(X)$, any covering sieve R on U (i.e. any subpresheaf of h_U) and any arrow $R \xrightarrow{f} F$ in $S^{\mathcal{O}(X)^{op}}$, there is a *unique* element $\sigma \in F(U)$ such that $\sigma \restriction_F V = \sigma_V = f_V(0)$ for each $V \in R$. But now each element $\sigma \in F(U)$ is uniquely associated with the arrow $h_U \xrightarrow{g} F$ in $S^{\mathcal{O}(X)^{op}}$ defined by $g_V(0) = \sigma \restriction_F V$ for $V \subseteq U$, and the assertion that $\sigma \restriction_F V = \sigma_V$ for all $V \in R$ amounts to saying that the diagrams



commute, i.e. that g is an *extension* of f to h_U.

Putting all these things together, we conclude that

(4.1) a presheaf F on X is a sheaf iff for each $U \in \mathcal{O}(X)$ and each covering sieve R on U, any arrow $R \to F$ in $S^{\mathcal{O}(X)^{op}}$ has a unique extension to an arrow $h_U \to F$.

Now let $\mathbf{C}$ be a small category. We have seen that in $\mathcal{O}(X)$ covering sieves may be identified with subpresheaves of representable presheaves. Similarly, any sieve R on an object U of $\mathbf{C}$ may be identified with a subpresheaf of the representable presheaf h_U, namely the presheaf

$$V \mapsto \{\alpha \in R\text{: domain } (\alpha) = V\}.$$

Conversely, any subpresheaf R of h_U determines a sieve on U, namely $R(U)$. *So sieves in* $\mathbf{C}$ *may be identified with subpresheaves of representable presheaves on* $\mathbf{C}$. This identification, together with the idea of a Grothendieck topology suggests the following generalization of (4.1) to an arbitrary site $(\mathbf{C}, J)$, namely

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4.2 DEFINITION. Let $(\mathbf{C}, J)$ be a site, F a presheaf on $\mathbf{C}$. F is said to be a *sheaf for the topology* J, or J-*sheaf*, if, for any object U of $\mathbf{C}$ and any $R \in J(U)$, each arrow $R \xrightarrow{f} F$ in $\mathbf{S}^{\mathbf{C}^{op}}$ has a unique extension to an arrow $h_U \xrightarrow{g} F$ in $\mathbf{S}^{\mathbf{C}^{op}}$.

The category $\mathbf{Shv}(\mathbf{C}, J)$ of J-sheaves has as objects all J-sheaves on $\mathbf{C}$ and as arrows all arrows arising between them in $\mathbf{S}^{\mathbf{C}^{op}}$. It can be shown that $\mathbf{Shv}(\mathbf{C}, J)$ is always a topos (see [5]). A category equivalent to one of this form is called a *Grothendieck topos*; thus a Grothendieck topos is a "generalized category of sheaves" over a "generalized topological space". This is the origin of the term *topos*!

Next, we succumb to the temptation of following Lawvere and Tierney in generalizing the notion of sheaf to the case where $\mathbf{S}^{\mathbf{C}^{op}}$ is replaced by an arbitrary topos $\mathbf{E}$.

We remark first that it is not hard to show that the subobject classifier Ω in $\mathbf{S}^{\mathbf{C}^{op}}$ is given by

$$\Omega(U) = \text{set of all sieves on } U$$

and, for $V \xrightarrow{f} U$,

$$\Omega(f)(R) = f^*(R) = \{W \xrightarrow{\alpha} V \colon \alpha f \in R\}$$

for $R \in \Omega(U)$. (Cf. the description of Ω in $\mathbf{S}^P$ given in § 1.) The arrow $\mathbf{1} \xrightarrow{\mathbf{true}} \Omega$ in $\mathbf{S}^{\mathbf{C}^{op}}$ is defined by

$$\mathbf{true}_U \colon 0 \mapsto \{\alpha \colon \text{codomain}(\alpha) = U\}.$$

It follows immediately from (G2) that a Grothendieck topology J in $\mathbf{C}$ is essentially just a subobject (subpresheaf) of Ω in $\mathbf{S}^{\mathbf{C}^{op}}$. So we can consider the characteristic arrow $j \colon \Omega \to \Omega$ of J. It can then without much difficulty be shown that in $\mathbf{S}^{\mathbf{C}^{op}}$ a subobject $J \rightarrowtail \Omega$ satisfies conditions (G1) and (G3) (i.e. is a Grothendieck topology) iff its characteristic arrow $j \colon \Omega \to \Omega$ satisfies the conditions:

(LT1) $\mathbf{true}.j = j$
(LT2) $j.j = j$
(LT3) $\wedge.j = (j \times j).\wedge$

Replacing $\mathbf{S}^{\mathbf{C}^{op}}$ by an arbitrary topos $\mathbf{E}$, we define a *Lawvere-Tierney topology* on $\mathbf{E}$ to be an arrow $j \colon \Omega \to \Omega$ satisfying conditions (LT1)–(LT3) above. This is the appropriate generalization of the notion of Grothendieck topology to arbitrary toposes; from the above we see

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that there is a natural bijection between Grothendieck topologies on **C** and Lawvere-Tierney topologies in $\mathbf{S}^{\mathbf{C}^{\mathrm{op}}}$.

Given a Lawvere-Tierney topology j (which we shall abbreviate simply to *topology*) in a topos **E**, how shall we define the notion of *sheaf* for j? To do this we introduce the idea of a *uniform closure operation* on **E**.

First, we define a *closure operation* on an object X of **E** to be a map which assigns to each subobject Y of X of a subobject $\bar{Y}$ of X in such a way that, for all subobjects Y, Y' of X, we have

$$Y \subseteq Y' \Rightarrow \bar{Y} \subseteq \overline{Y'}$$
$$\bar{\bar{Y}} = Y$$
$$Y \subseteq \bar{Y}.$$

A uniform closure operation on **E** is now defined to be a specification of a closure operation on each object X of **E** in such a way that closure commutes with pullback along arrows of **E**, i.e. given $Z \xrightarrow{f} X$ and a subobject Y of X, we have $f^*(\bar{Y}) \cong \overline{f^*(Y)}$ as subobjects of Z, where $f^*(Y)$ is the pullback of $Y \rightarrowtail X$ along f.

It is easy to see that each topology j gives rise to a unique uniform closure operation on **E**; namely, if a subobject $Y \rightarrowtail X$ has characteristic arrow χ, then $\bar{Y}$ is the subobject classified by $\chi.j$. Conversely, each universal closure operation on **E** yields a topology j on **E** as follows: apply the closure operation to the subobject $1 \xrightarrow{\text{true}} \Omega$, obtaining a subobject $J \rightarrowtail \Omega$; j is the characteristic arrow of J. So there is a bijection between topologies in **E** and universal closure operations on **E**.

One must not confuse the notion of uniform closure operation on a topos with the more familiar notion of Kuratowski closure operation defined on the lattice of all subsets of topological space. To highlight the difference between the two notions, recall that Kuratowski closure commutes with finite unions but not in general with intersections, while it follows from (LT3) that uniform closure commutes with intersections, though not normally with unions. In fact, the "generic" example of a (uniform) closure operation commuting with intersections may be extracted from the situation in which J is the *canonical* Grothendieck topology defined on the lattice $\mathcal{O}(X)$ of open sets in a topological space X. Here $J(U)$ is the set of all covering sieves on an open set U and the corresponding closure operation is

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defined on the set $\Omega(U)$ of all sieves on U by

(4.3) $\quad \bar{R} = \{V \in \mathcal{O}(X)\colon V \subseteq UR\}.$

It is easy to verify directly that this operation satisfies: $R \subseteq \bar{R}$, $\bar{\bar{R}} = R$, $\overline{R \cap S} = \bar{R} \cap \bar{S}$. It is a point of genuine historical interest that the characteristic properties of closure operations derived from the notion of *covering* were discovered almost exactly half a century after the isolation of the characteristic properties of closure operations derived from the notion of *neighbourhood.*

Given a topology j in a topos $\mathbf{E}$ with associated uniform closure operation $^-$, we say that a subobject $Y \rightarrowtail X$ is *j-dense* if $\bar{Y} = X$ and *j-closed* if $\bar{Y} = Y$. In the case where $\mathbf{E}$ is $\mathbf{Shv}(X)$ and J is the canonical Grothendieck topology on X introduced above, the closure $\bar{R}$ of a sieve $R \rightarrowtail h_U$ is given by (4.3), so R is closed iff UR is the collection of all subsets of UR; and R is dense in h_U iff $UR = U$.

Now let us return to the case of a site $(\mathbf{C}, J)$. Let j be the corresponding topology in $\mathbf{S}^{\mathbf{C}^{\mathrm{op}}}$. It is then not hard to show that a subpresheaf $R \rightarrowtail h_U$ is j-dense iff it is an element of $J(U)$, i.e. a J-covering sieve on U. It follows that 4.2 may be reformulated as:

(4.4) A presheaf $F \in \mathbf{S}^{\mathbf{C}^{\mathrm{op}}}$ is a J-sheaf iff for any object U of $\mathbf{C}$, any j-dense monic $R \overset{\sigma}{\rightarrowtail} h_Y$ and any arrow $R \overset{f}{\rightarrow} F$ in $\mathbf{S}^{\mathbf{C}^{\mathrm{op}}}$, there is a unique $h_U \overset{g}{\rightarrow} F$ such that $f = \sigma g$.

Now it can be shown, using the fact that every object X of $\mathbf{S}^{\mathbf{C}^{\mathrm{op}}}$ is a "colimit" (or "inductive limit") of objects of the form h_U (see [5]) that condition (4.4) continues to hold when h_U is replaced by an *arbitrary* object X in $\mathbf{S}^{\mathbf{C}^{\mathrm{op}}}$. So the condition for F to be a J-sheaf now reads:

•For any object X of $\mathbf{C}$, any j-dense monic $R \overset{\sigma}{\rightarrowtail} F$ and any arrow $R \overset{f}{\rightarrow} F$ there is a unique $X \overset{g}{\rightarrow} F$ such that $f = \sigma g$.

It is this form of the condition that we adopt in formulating the notion of sheaf in an arbitrary topos. Thus, if j is a topology in $\mathbf{E}$, we make the

4.5 DEFINITION. An object F of E is said to be a *j-sheaf* if, given any j-dense monic $R \overset{\sigma}{\rightarrowtail} X$ and any $R \overset{f}{\rightarrow} F$, there is a unique $X \overset{g}{\rightarrow} F$ such that $\sigma g = f$.

We write $\mathbf{Sh}_j(\mathbf{E})$ for the category whose objects are all j-sheaves

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and whose arrows are all arrows between them in **E**. It can be shown that $\mathbf{Sh}_j(\mathbf{E})$ is always a topos (see [3] or [5]).

In the classical case where **E** is the category $\mathbf{S}^{\mathcal{O}(X)^{\mathrm{op}}}$ of presheaves on a topological space X, there is a natural functor (the "associated sheaf" or "sheafification" functor) $L\colon \mathbf{S}^{\mathcal{O}(X)^{\mathrm{op}}} \to \mathbf{Shv}(X)$ which assigns to each presheaf F on X a sheaf LF on X in such a way that the following "universal" condition is satisfied: there is an arrow $i_F\colon F \to LF$ such that, for each arrow $f\colon F \to G$ to a sheaf G, there is a unique arrow $g\colon LF \to G$ such that $f = i_F g$. (Thus LF is the "most economical" way of turning the presheaf F into a sheaf.) This result can be extended to the general case (see [3] or [5]): given a topology j in a topos **E**, there is a functor $L\colon \mathbf{E} \to \mathbf{Sh}_j(\mathbf{E})$ and for each object F of **E** an arrow $i_F\colon F \to LF$ such that, for each j-sheaf G and each arrow $F \xrightarrow{g} G$ in **E**, there is a unique $g\colon LF \to G$ such that $f = i_F g$. (In category jargon, L is left adjoint to the inclusion functor $\mathbf{Sh}_j(\mathbf{E}) \to \mathbf{E}$.)

One of the most interesting examples of a topology in a topos is the so-called *double negation topology* $\neg\neg\colon \Omega \to \Omega$. It is easily verified that this is a topology: in MacLane's [8] memorable phrase, "[here] the topologist Brouwer meets the intuitionist Brouwer". As we shall see, this topology is implicitly present in Cohen-style independence proofs in set theory.

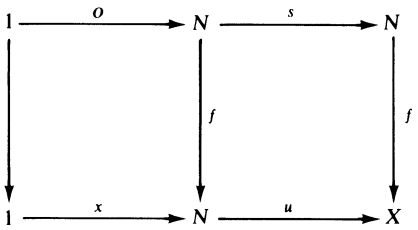
Let $\langle P, \leq \rangle$ be a partially ordered set of forcing conditions in the usual Cohen sense (cf. [1]). We form the topos $\mathbf{S}^P$ of sets varying over P; here the subobject classifier is not in general a Boolean algebra and the "internal logic" is intuitionistic. Now we form the topos $\mathbf{E} = \mathbf{Sh}_{\neg\neg}(\mathbf{S}^P)$ of "double negation" sheaves (i.e. $\neg\neg$-sheaves) in $\mathbf{S}^P$. It can then be verified that **E** is Boolean and is in fact categorically *equivalent* to the Boolean extension $V^{(B)}$, where B is the Boolean completion of P. Thus the process of constructing Boolean valued models à la Scott-Solovay amounts to taking double negation sheaves in an intuitionistic model consisting of sets varying over some set P of forcing conditions.

More generally, this construction can be carried out when **S** is replaced by any *set-like* topos **E**, i.e. one satisfying **AC** and the two-valuedness axiom (see Section 2). In this way set-theoretical independence proofs using Cohen's method can be translated into the topos framework. For example, let us see how the independence proof for the *continuum hypothesis* appears in this setting.

First, we need to introduce the notion of a *natural number object* in

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a topos. A *natural number object* in a topos **E** is an object N together with arrows $1 \xrightarrow{o} N \xrightarrow{s} N$ such that, for any diagram $1 \xrightarrow{x} X \xrightarrow{u} X$ in **E** there is a unique $N \xrightarrow{f} X$ such that the diagram



commutes. This definition (due to Lawvere) tells us that arrows with domain N can be defined by recursion, just as in **S**. In any topos with a natural number object we can introduce the concept of finite cardinal and finite object, again just as in **S**.

If N is a natural number object in **E**, we may think of its power object Ω^N as representing the *continuum* in **E**. So we may say that the continuum hypothesis *fails* in **E** if there is an object I in **E** and monics

$$N \rightarrowtail I \rightarrowtail \Omega^N$$

but *no* epics $N \twoheadrightarrow I$ or $I \twoheadrightarrow \Omega^N$. (If this is the case, we shall also say that I violates the continuum hypothesis in **E**.)

Now let **E** be a set-like topos with a natural number object N. Let I be an object of **E** such that there is a monic $\Omega^N \rightarrowtail I$ but no epics $\Omega^N \twoheadrightarrow I$. (For example, take $I = \Omega^{\Omega^N}$.) Following Cohen, we let P be the "partially ordered object" in **E** of partial arrows $I \times N \to \Omega$ with finite domains ordered by the relation $f \leq g \Leftrightarrow f$ extends g. (Think of the elements of P as finite sets of "forcing conditions", each of which says that a particular element of N is or is not in a particular member of I.) Now form the topos $\mathbf{E}^P$ of "**E**-objects varying over P" by mimicking the construction of $\mathbf{S}^P$ in **E**, and from this obtain the topos $\mathbf{D} = \mathbf{Sh}_{\neg\neg}(\mathbf{E}^P)$ of double negation sheaves in $\mathbf{E}^P$. It can be shown that **AC** holds in **D** (so **D** is Boolean), it has a natural number object N, and the object I of **E** has a natural image $\tilde{I}$ in **D** which violates the continuum hypothesis in **D**.

In the classical case we can go on to "factor" a Boolean valued

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model $V^{(B)}$ by an ultrafilter U in B to obtain an ordinary model. A version of this procedure can also be carried out in the case at hand, as follows. Since **D** is Boolean, its algebra Sub(1) of global truth values is a Boolean algebra in **E**; by the axiom of choice in **E** there is a subobject F of B which is an ultrafilter in B. Now let Σ be the class of arrows $X \xrightarrow{f} Y$ in **D** for which there is $U \rightarrowtail 1$ such that[4] $U \in F$ and $f \times \mathrm{id}_U\colon X \times U \to Y \times U$ is an isomorphism. (In $V^{(B)}$, this is equivalent to saying that the Boolean-value of the statement "f is a bijection" is a member of F.) A standard category-theoretic construction now yields the category $\mathbf{D}[\Sigma^{-1}] = \mathbf{E}'$ of "fractions" in which all the arrows in Σ are turned into isomorphisms. (In the case where **D** is $V^{(B)}$ one can show that $\mathbf{E}'$ is (equivalent to) the quotient $V^{(B)}/F$.) Then $\mathbf{E}'$ is a set-like topos in which the continuum hypothesis is violated. (For a rigorous account of these matters, see [5].)

Thus, starting with any set-like topos with a natural number object, we can construct a set-like topos in which the continuum hypothesis is violated: this is the topos-theoretical version of Cohen's independence proof for the continuum hypothesis.

Finally, since we have seen that topos theory and set theory have many points in common, it is worth mentioning the precise *logical* connection between the two theories. In fact it can be shown that the theory of *extensional* toposes is *logically equivalent* to a certain version Z_0 of Zermelo set theory without the axiom of infinity in which the separation scheme is restricted to formulas with *bounded* quantifiers. This logical equivalence is established by showing that each model **M** of Z_0 gives rise to an extensional topos **E** and vice-versa. It is quite easy to go from **M** to **E**, for we need merely take **E** to be the category of sets and functions in **M** (i.e., the category **S** constructed in **M**). The converse, however, is considerably more difficult and is essentially achieved by characterizing the notion of *transitive* set directly as objects in **E**. (See [5].)

Further reading. An excellent introduction to the subject for non-specialists is provided by Goldblatt in [4]. For those with a good background in category theory (and mathematics generally), [5] is indispensible. Category theory is given a thorough development in the classic [6]. One of the first articles on the subject is [3]; it is still very useful. Finally, [7] is a lucid expository article, and [8] a masterly review.

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NOTES

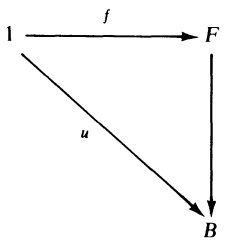
* This paper is based on a course of lectures delivered while I was visiting the Mathematics Department of the University of Singapore. I would like to thank the University for their generous support.

[1] If f is a function and $A \subseteq \text{domain}(f)$, we write $f \restriction A$ for the restriction of f to A.

[2] For example, in the topos $\mathbf{S}^M$ of M-sets, arrows $1 \to X$ correspond to the $x \in X$ such that $\alpha.x = x$ for all $\alpha \in M$. Thus, if M is a group with more than one member, M itself (with the obvious M-action), is a non-zero object in $\mathbf{S}^M$ with no arrows from 1 to it.

[3] A category is *small* if its collections of arrows and objects are both sets.

[4] To be precise, U corresponds to an "element" $1 \xrightarrow{u} B$ in $\mathbf{E}$, and "$U \in F$" means that there is an arrow $1 \xrightarrow{f} F$ in $\mathbf{E}$ such that the diagram



commutes.

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