Notes on mathematical writing

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Some points of style merely embody hoary aesthetics or clique membership; accordingly, their discussion is borne of orthodoxy or snobbery – qualities altogether at odds with the spirit, aims, and practise of mathematics. Other matters of style can aid authors in nobler pursuits – to be precise, intelligible, and interesting; I care only to help hit these targets.

Openings

I loathe admonitions to 'motivate' the study of a mathematical object. Surely the motivation will be the purest the species can muster – curiosity. That said, one has to choose which objects to contemplate. So you have to alert potential readers to the kinds of objects they'll find within your text; to this end you have: the AMS's Mathematical Subject Classification (MSC), a list of keywords, the Abstract, and the Introduction.

The classes of the MSC are woefully maladapted to current practise. Many subjects appear to be organised according to the trends of the early 1960s. Unhelpful.

A list of keywords, on the other hand, is well used by various archival tools but underused by writers themselves. List every new term you use and every notion that appears in an essential way in your main results.

The Abstract should describe in as few words as possible[1] the main contributions of your paper. If that is to prove a single theorem, state that theorem. No one will mind if your Abstract is only a sentence or two long.

[1] You can practise writing tersely by the trick of forbidding any use of the verb 'to be'.

But by far the most important way for you to invite or repel your reader is your Introduction. No one rejoices at an Introduction piled high with vague generalities. There is only one thing worse than a paper that begins with the phrase, 'Oftentimes in mathematics … ', but that at least can be treated with antibiotics.

Minimise the angular momentum of your readers' eyes with the following recipe for your Introduction.

1. Tell a story[2] your target audience knows – or should know – very well.[3] This needn't be something tremendously deep, though it's best if the objects under discussion are widely appreciated.
2. Identify certain aspects of the story that are mysterious, incomplete, heretofore inadequately explained, or not obviously generalisable.
3. State your contributions[4] as Theorems, labelled with letters A, B, …, as well as cross-references to the points in the paper where these results are proved.
4. If necessary, explain how 3 addresses the points raised in 2.
5. Repeat the steps 1–4 if needed until the main topics of the text are covered.[5]
6. Stop. Get on with the paper already.

[2] of a notion, an example, a technique, or a result

[3] In particular, you should know who that target audience is. For research articles, I recommend imagining someone who works in your area, but whose knowledge is about six months out of date.

[4] with slight oversimplifications as needed

[5] In particular, you should know what the main topics are.

Structure

Let us call every unique idea that could be cited[6] an *atom* – these atoms are the

[6] every Theorem, Proposition, Lemma, Conjecture, Definition, Construction, Example, Notation, Question, Problem, Remark

building blocks of your text. Any other writing in your text is, properly speaking, unnecessary. (I do not say undesirable.)

Give every atom a unique number; this number is always a subnumber of the section, which may itself be a subnumber of the chapter, which may itself be a subnumber of the part.[7] Make these numbers sequential – for God's sake. For every section S, there should be a monotonically increasing map from the atoms of section S to the page number.[8]

Organise your atoms into three rough classes: *significant atoms*, *technical atoms*, and *marginal atoms*.

- Labels for the significant atoms include *Theorems*, *Propositions*, *Conjectures*, *Examples*, and *Definitions*.
- Labels for the technical atoms include: *Lemmas*, *Questions*, *Problems* and *Notations*.
- Labels for the marginal atoms include *Remarks*, *Observations*, and 'unlabelled' numbers.

Give every labelled result[9] one of the following – in decreasing order of preference –

1. a proof,
2. a reference to a proof in a different text,
3. a deferment of the proof, or
4. a description of how the proof would go.

Try not to write the same proof more than once, and try to use earlier results, not their proofs. If you find yourself saying that a proof of a significant Theorem is 'similar to the argument of Theorem 5.7.15 except for the following alterations …' then you may not have the correct result in hand. Seek an interesting common generalisation of the two results; it may be easier to find than you think, and the search will often provide an insight.

Provide strategic remarks for sufficiently complex[10] proofs. Outline how the proof will proceed at the beginning; provide periodic progress reports. Explain why you're doing what you're doing.

If you have a constructive proof of the existence of an object with some property, first give an atom labelled Construction in which you build the object, and then state that your construction has the desired property as a Proposition or Theorem. Contrapositively, if the outermost quantifier of your Proposition or Theorem is existential, then your proof should be nonconstructive.

The Definition of any technical term must precede its usage. Italicise or boldify the word or phrase being defined.

We give Definitions for two rather different things – *Nonce Words* and *Notions*. Nonce Words are terms of convenience – a way to package up some jumble of conditions for quick use in a complicated argument; when you introduce a Nonce Word, you don't really expect the term to be used again outside your work.[11] A Notion, by contrast, is a compact expression of a discrete mathematical idea.

[7] E.g., '4.2.19'.

[8] Certain flavours of LaTeX don't ensure this by default.

[9] Theorem, Proposition, or Lemma

[10] more than 1 page or so

[11] You see Nonce Words all the time when someone is doing an induction argument and says, 'let's say that a so-and-so is *good* if …'.

When one gives a Definition – let's say of the word *widget* – then one can tell whether this is a Nonce Word or a Notion by applying the following criteria.

- When you write a Theorem about widgets, are the examples to which you'd like that Theorem to apply all widgets?
- Does the collection of all widgets have pleasant formal properties?

If the answer to each of these questions is yes, it is more likely that you have a Notion, and not a Nonce Word.

Notions are always more desirable than Nonce Words; they are the Definitions upon which time looks kindly.[12]

If you find that you're using a lot of Nonce Words, there are two possibilities:

- Your subject has become baroque.[13]
- You have not found the 'right' Definitions. They no doubt 'work' in the sense that they permit you to prove the correct theorems, but they are not expressions of the real ideas that are running your proofs.

Now a certain amount of Noncemanship is inevitable, but you should aim to develop your subject in a way that gravitates toward Notions. You'll find that this impetus pairs nicely with the suggestion not to write the same proof more than once.

Follow every significant Definition with at least one nontrivial example and, ideally, some non-examples as well. The reader should be able to 'run' proofs of results on the examples and non-examples, to see where the hypotheses are used.

A Conjecture is an assertion that meets all the following criteria.

- It is precise and unambiguous.[14]
- The author strongly suspects that it is the case.
- The author considers the assertion interesting or difficult.
- The author has seriously attempted to prove it.
- Nevertheless, the author does not know how to prove it.

Anything that satisfies the final condition but not all of the others is a Question or a Problem. Do not fear formulating plenty of Questions and Problems.

Notation

Any piece of mathematical notation must:

- be used prominently in most of the literature in your area,
- have been used in no more than 3 pages prior to the current usage,
- have been defined or recalled in no more than 3 pages prior to the current usage,
- be recalled in no more than 3 lines after the usage, or
- appear in a glossary of notation.

[12] When Manin said that 'All the other vehicles of mathematical rigour are secondary [to Definitions], even that of rigorous proof', he wasn't talking about Nonce Words.

[13] Looking at you, general topology.

[14] In particular, it does not depend on a notion the author does not know how to define.

No mathematical notation may begin a clause.

Never use notation that suggests something you don't mean (e.g., **Top** for the ∞-category of spaces).

An instance of mathematical notation is unnecessary if and only if its use lengthens a sentence that would be otherwise unambiguous. Use no unnecessary mathematical notation.[15]

If a result or idea can be expressed as a short sentence, then it should be.

ONLY INTRODUCE FIXED NOTATION[16] at the beginning of every division[17] in which it appears. Make it into a technical atom labelled Notation in the sense above.

Make explicit all non-fixed notations and all assumptions in every significant and technical atom explicit, even if you've been talking about the same things for awhile. It should always be possible to read, understand, and believe your writing *locally*.

[15] E.g., one should shorten a sentence like 'If X is a compact convex subset of $\mathbf{R}^n$ and if f is a continuous map $X \longrightarrow X$, then there exists $x \in X$ such that $f(x) = x$.' to a sentence more like 'Every continuous self-map of a compact convex subset of euclideans space admits a fixed point.'

[16] e.g., 'in this section, let p be a fixed prime'

[17] document, part, chapter, section, etc.

Grammar~

Above all, it should be trivial for your readers to rewrite every mathematical sentence you write with one of the two quantifiers $\forall$ and $\exists$ attached to each variable. If you ensure this, you'll have already done more than many authors.

It is a matter of some subtle linguistics whether a particular use of the word 'any' refers to a universal quantifier or an existential one. This is confusing to nonnative English speakers. It's better to use consistently 'every', 'each', and 'all' for the universal quantifiers and 'there exists' and 'for some' for the existential ones.

THE USE OF THE WORDS 'which' and 'that' generally depends upon the removability of the clause they introduce. When the clause can be removed from a sentence without risking a change to its truth value[18], use the word 'which', and set the clause off from the rest of the sentence with commas. When the clause cannot be removed from a sentence without risking a change to its truth value[19], use the word 'that', and do not precede it with a comma.[20] The following sentences illustrate the shade of meaning here:

The vector space V, which is of finite dimension, has a natural basis.

In C_p, every element that is not the identity generates.

[18] i.e., when the clause is *non-restrictive*

[19] i.e., when the clause is *restrictive*

[20] In British English, one also sees 'which' – still sans comma – in restrictive clauses, but in American English generally does not.

ALL TRUE MATHEMATICAL SENTENCES are necessarily true, and they are currently the case. Therefore use the present tense and the indicative mood for any such sentence. In a short proof by contradiction, use the subjunctive mood.[21] In a longer proof by contradiction, begin with a clear declaration of the assumption and remain in the indicative mood.[22]

[21] E.g., 'If there were only finitely many primes $p_1, \ldots, p_n$, then the number $p_1 + \cdots + p_n + 1$ would admit no prime factorsation.'

[22] E.g., 'Assume, for the purpose of contradiction, that ν is a Hodge class that is not algebraic…'

THE PASSIVE VOICE is nearly not as objectionable as some people pretend. Nevertheless, you shouldn't use it in more than 3 clauses in a row, because it becomes confusing and dull.

DO NOT USE pronouns whose antecedents are unclear. This is what I'm doing here.

Lexicon

The words 'obvious', 'evident', and 'clear' mean that the discussion up to this point ensures that any attempt to justify the claim further is liable to confuse the reader more than saying nothing; to use one of these words is to make an assertion about the expected mental state of the reader.

The words 'trivial' or 'tautological', by contrast, mean that the relevant collection of conditions has become empty; to use one of these words is to make an assertion about a mathematical object.

WORD CHOICE is sometimes subtle, and one often sees the following rules violated in mathematical writing:

- Don't use 'alternately' when you mean 'alternatively'.
- Don't use the word 'given' unless there is an identifiable giver.
- Use 'different from', not 'different than' or 'different to'.
- Don't use the phrase 'associated to'.
- Don't use the phrase 'P due to Q' unless P and Q are noun phrases. And even then, ask yourself: *could this sentence be clearer*?[23]

[23] And then answer: *yes*.

Formality

There are those who wish to bar contractions in 'formal' writing, but I've never seen a cogent explanation for this bias.[24] Used correctly, contractions are clear and natural. On this matter, I say, render unto copyeditors what is copyeditors'; if they want to expand every contraction to reflect the lingo of Cold War robots, let them do so.

[24] The Chicago Manual of Style offers this: 'Most types of writing benefit from the use of contractions. If used thoughtfully, contractions in prose sound natural and relaxed and make reading more enjoyable.'

COLOURFUL, STYLISH, OR AMUSING language is no sin, irrespective of the surly pronouncements of bland referees. Do not discard a sense of fun out of a misplaced sense of professionalism. That said, however, there are two rules about this.

- Clichés are always tedious. If you're going to be colourful, be interesting and colourful. I like the Orwell Injunction: don't use any figure of speech you are used to reading.
- Never let your linguistic cleverness or artfulness exceed your mathematical cleverness or artfulness.

Typography

Move a formula or other symbolic expression out of the midst of a paragraph[25] and onto a line of its own[26] if it contains any of the following:

[25] 'inline'

[26] 'displaymath'

- a symbol with a double-subscript, a double-superscript, or a subscript and a superscript;
- a 'big' operator[27] with a limit that is more than three characters in length;

[27] e.g. $\sum, \bigcup$

- more than one of these 'big' operators;
- more than one arrow;
- more than 10 characters.

EXHIBIT any list of conditions, cases, instances, etc. with either more than 3 items or more than 3 words per item as a list[28]. These items may be part of a single – correctly punctuated and grammatically flawless – sentence, or each item may be an individual sentence. Choose, and be consistent.

[28] 'itemize' or 'enumerate'

Strategy

To enact all the advice here, write each paper or chapter in the following order.

1. List the expected sections.
2. Populate the sections with the necessary definitions.
3. Formulate the significant results.
4. Provide proofs of increasing detail of the results you've listed, highlighting the difficult bits.
5. Formulate Lemmas that will sharpen your proofs.
6. Distribute examples throughout.
7. Repeat steps 1–6 until the logical flow is perfectly clean and legible.[29]
8. Introduce interstitial comments that may clarify the thread of an argument, compare or contrast with something in the literature, or warn the reader of pitfalls.

[29] Do not be afraid to discard a significant portion of what you have written, but do be sure you keep the discarded writing in a file somewhere.

SOME SUGGEST saving the writing of the introduction till the end, but this isn't important. The main point is that you should write it when you have the energy and the inspiration to do so.

ANYTHING WORTH WRITING is worth rewriting. If what's in front of you bears much relation to your first draft, you are not done revising.

