Structure in Mathematics and Logic: A Categorical Perspective

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> Since Hilbert and Dedekind, we have known very well that large parts of mathematics can develop logically and fruitfully from a small number of well-chosen axioms. That is to say, given the basis of a theory in an axiomatic form, we can develop the whole theory in a more comprehensible way than we could otherwise. This is what gave the general idea of the notion of mathematical structure. Let us say immediately that this notion has since been superseded by that of category and functor, which includes it in a more general and convenient form.
>
> Jean Dieudonné, *The Work of Nicholas Bourbaki.*

'Structure' is a fashionable word. But despite its rather lax usage in some disciplines it is a notion of apparent significance in mathematics, where it occurs often and with virtually invariant sense. Recently, several philosophers of mathematics have called attention to the notion of mathematical structure. For philosophical purposes, however, this notion still seems to require clarification. The purposes are those of what may be called 'philosophical structuralism', an approach to the ontology and epistemology of mathematics.[1]

At the same time, what may be called 'mathematical structuralism' has already met with considerable success through a century of work by mathematicians pursuing a structural approach to their subject. Indeed this success is reflected in the current prominence of the notion of structure in mathematics. My aim here is to suggest that a philosophically useful conception of mathematical structure can be found in this very mathematical practice, through closer attention to the methods developed by 'mathematical structuralists'. From Dedekind, through Noether, and to the work of Eilenberg and Mac Lane, the fact has clearly emerged that mathematical structure is determined by a system of objects *and their mappings*, rather than by any specific features of mathematical objects viewed in iso-

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[1] Cf. Resnik [1981], [1988]; Shapiro [1983], [1989]; Hellman [1989], [1990]; Parsons [1990]; Quine [1992].

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lation. To a great degree, the structural approach in modern mathematics is characterized by increased attention to (systems of) mappings, and the idea that mathematical objects are determined by their 'admissible transformations'.[2]

Surely it is this rise of the structural approach in modern mathematics that has sparked philosophical interest in structuralism. And yet the actual methods of mathematical structuralism seem to have been largely ignored; philosophical accounts often proceed instead either from model theory or from scratch. This neglect seems unfortunate; a view based instead on the methods of mathematical structuralism would at least be more consonant with current mathematical practice, and could benefit from this substantial tradition by drawing on a now well developed technical apparatus. The latter is, of course, the mathematical theory of categories: the abstract theory of mathematical objects and their mappings.

§1 below indicates how the techniques of category theory codify a structural approach to mathematics, providing a precise and yet flexible notion of mathematical structure. Category theory often serves as a tool to be applied in other branches of mathematics. §2 provides an example of such application to a familiar field, namely logic. While the subject matter may be familiar the structural approach is novel, and fruitful; it exhibits the close connection between logic and set theory, provides a structural theory of abstract sets, permits some surprising connections between logic and geometry to emerge, and generally treats logic in a fashion more uniform with the rest of modern mathematics.

It should be noted that my purpose is not to discuss categorical foundations of mathematics, or to present a comprehensive structuralist philosophy of mathematics based on category theory, but to elaborate a notion of mathematical structure from a categorical perspective, so that discussions of other issues may proceed directly. Although I think that some basic philosophical questions may fruitfully be addressed using this conception of mathematical structure, and that it serves the purposes of philosophical structuralism well, I have limited myself in §1 to only occasional indications in order to preserve the integrity and utility of the essay. The whole of §2 may then serve as a detailed example of the philosophical utility of this notion of structure.

1. Mathematical Structure

Mathematical objects are often said to have or admit various kinds of structure. A smooth manifold for example has point-set, topological, and differ-

[2] Cf. Mac Lane [1996] on mathematical structure; Stein [1988] on the development of the structural approach from Dedekind to Noether; Corry [1992] on structure in Bourbaki and the work of Eilenberg and Mac Lane; Mac Lane [1986] for a general, structural view of mathematics.

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entiable structures, and it may have more. Algebraic objects such as groups and rings are sets with additional structure. The unit circle S^1 in the plane is a group (under complex multiplication) which also bears many other structures; *e.g.*, it is a topological space and a smooth, one-dimensional manifold as well. Mappings between mathematical objects may preserve structure, or fail to. Continuous mappings are those preserving topological structure, regions of the complex plane have an additional conformal structure preserved by certain analytic mappings, and so on.

Model theory provides familiar definitions of the notions of a structure for a given first-order language and of a model of a first-order theory in that language. A more general explication applying to such non-elementary mathematical notions as 'topological space' is, say, that given by Bourbaki.[3] This model-theoretic conception of mathematical structure—let us call it the 'Bourbaki notion of structure'—is the result of a sharpening of the axiomatic method in modern mathematics, and it has proven to be a useful way of distinguishing and describing mathematical objects. Indeed, it has shaped our present-day conception of a 'mathematical object'.[4] It has also made it plain that at least some features of mathematical objects, some mathematical facts about them, depend *solely* on their structure. For example, once we have characterized the real numbers as a complete, ordered field, nothing but the common structure of complete, ordered fields is relevant to the mathematics of the real numbers. In particular, it does not matter that the individual real numbers must then be distinct sets differing with respect to their elements, although this is required by the very conception of structure that permitted the initial, structural characterization of the reals. And furthermore, as soon as we have *one* such characterization, we find that there are also others which would do as well. The Bourbaki description of mathematical objects as sets-with-structure leads to a useful, structural perspective, which the language and methods of model theory itself do not serve to describe particularly well. This aspect of the Bourbaki notion of mathematical structure has recently been discussed quite thoroughly; I only wish to call attention to it here, though a bit more will be said below.[5]

In mathematics, the Bourbaki notion of a possibly higher-order structure has proven far less useful than the influence of Bourbaki's *Elements* might suggest. This is not to say that mathematicians have not adopted a broadly

[3] Taking the definition of 'structure' in Bourbaki's *Elements*, vol. 1, §4, but only for the sake of concreteness.

[4] See Bourbaki [1950] on the axiomatic method, mathematical structure, and the objects of mathematics.

[5] See also Parsons [1990] for philosophical difficulties with the model-theoretical notion of structure, *e.g.*, problems of 'multiple reductions' and 'ontological commitments'. Hellman [1990] looks to modal logic for an alternative; Resnik [1988] would base a variety of structuralism on an entirely new notion: 'pattern'.

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structural view of mathematical objects, for they surely have. Few modern mathematicians are interested in any properties of the objects they study which do not respect a given, well defined notion of isomorphism. That is to say, the topologist does not care to distinguish among homeomorphic spaces by examining the set-theoretical structure of the points of a space, nor does the geometer wonder about the ordinal rank of a given smooth manifold. The object of modern mathematical study is rarely a specific set with given Bourbaki structure, but rather a mathematical object determined up to isomorphism, the various relations between such objects bearing similar structure, relations between different kinds of structure on such objects, and so on. While the Bourbaki notion of mathematical structure surely plays a role in such studies, the method of using mappings to isolate, describe, and compare different kinds of mathematical structure has emerged as a more effective tool.[6] Category theory thus arose, not as an alternative foundational scheme, but in response to a mathematical need for a language and methods well suited to problems involving different kinds of mathematical structure.

A category provides a way of characterizing and describing mathematical structure of a given kind, namely in terms of preservation thereof by mappings between mathematical objects bearing the structure in question. A category may be viewed as consisting of objects bearing a certain kind of structure together with mappings between such objects preserving that structure. For example, topological spaces and continuous mappings between them form a category, which we call **Top**. Similarly, **Groups** is the category consisting of groups and group homomorphisms, and **Sets** consists of sets and functions between them.[7] Now what do these examples, and obvious similar ones, have in common? We wish to axiomatize the notion of a 'kind of mathematical structure' in terms of that of a 'system of structure-preserving mappings'. For a mapping $f : A \to B$ to preserve some kind of structure, plainly both A and B must have it. If $g : B \to C$ also preserves it, then so does the composite mapping $g \circ f : A \to B \to C$. And every object has an identity mapping $1_A : A \to A$ which preserves all structure. The intuitive notion of a structure-preserving mapping may include much more, but presumably it includes at least this.

A *category* by definition thus consists of *objects* $A, B, C, \ldots$ and *morphisms* $f, g, h, \ldots$ such that: (i) every f has a unique domain A and a unique *codomain* B , written $f : A \to B$; (ii) given any $g : B \to C$ there is a unique *composite* $g \circ f : A \to C$, with composition being associative; (iii) each B has an identity $1_B : B \to B$ which is a unit for composition, *i.e.*,

[6] cf. Corry [1992].

[7] A function $f : A \to B$ between sets is a subset of the Cartesian product such that for each $a \in A$ there is a unique $b \in B$ with $\langle a, b \rangle \in f$; the range of f need only be contained in B, as with mappings in other branches of mathematics.

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$1_B \circ f = f$ and $g \circ 1_B = g$ for any f and g as stated.[8]

A category is *anything* satisfying these axioms. The objects need not have 'elements', nor need the morphisms be 'functions', although this is the case in some motivating examples. But also, for example, associated with any formal system of logic is a category, the objects of which are formulas and the morphisms of which are deductions from premises. We do not really care what non-categorical properties the objects and morphisms of a given category may have; that is to say, we view it 'abstractly' by restricting to the language of objects and morphisms, domains and codomains, composition, and identity morphisms.

Generally, suppose we have somehow specified a particular kind of structure in terms of objects and morphisms, as in the examples above. Then that category characterizes that kind of mathematical structure, independently of the initial means of its specification. For example, the topology of a given space is determined by its continuous mappings to and from other spaces, regardless of whether it was initially specified in terms of open sets, limit points, a closure operator, or whatever. The category **Top** thus serves the purpose of characterizing the notion of 'topological structure'. In the case of set-based, algebraic objects such as groups, rings, and modules, the algebraic structure is given by certain operations and axioms; this is perhaps the paradigm of the Bourbaki notion. A morphism of such objects may be taken to be simply a set function respecting those operations. Thus a morphism of groups is a morphism of the underlying sets that respects the group multiplication and unit, *i.e.*, a group homomorphism. Now, the multiplication on a given group can be recovered from such homomorphisms, so the notion of a 'group structure' is also determined by the category of groups.[9] The group and topological structures on S^1 are then effectively distinguished by focusing on its group homomorphisms and continuous functions respectively, as is commonly done in practice. This kind of situation provides the analogy by which more general categories can usefully be viewed as categories of objects-with-structure and structure-preserving mappings. For example, the smooth structure on a differentiable manifold is in effect *given* by specifying which continuous mappings to and from other manifolds are smooth; this can be done in various different ways, any two of which are considered equivalent just if they give all the same smooth maps. The category whose objects are topological spaces and whose morphisms are continuous functions identified by homotopy provides another example of a kind of mathematical structure conveniently determined directly in terms of a category. Of course, there

[8] See Mac Lane [1971] for precise definitions of the basic concepts of category theory.

[9] Technical results concerning the recovery of syntactical 'structure' from a suitable category of models often fall under the heading of 'conceptual completeness' following Makkai and Reyes [1977]. Also see Pitts [1989] and the further references there.

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may also be Bourbaki structures of which the objects of a given category can be described as models, but the categorical method has certain advantages which have led to its wide-spread acceptance in mathematics. One such advantage can be indicated already; the categorical notion is 'syntax invariant', *i.e.*, it depends on no particular choice among the different possible Bourbaki descriptions of a given kind of mathematical structure.

Intuitively, two things with the same kind of structure may indeed have the same structure. The categorical notion of isomorphism may now serve as a *definition* of 'having the same structure of a given kind'. In any category, a morphism $f : A \to B$ is by definition an *isomorphism* iff there is a $g : B \to A$ with $g \circ f = 1_A$ and $f \circ g = 1_B$. We then write $A \simeq B$ and say that these objects are isomorphic. Note that isomorphism is always relative to a given kind of structure, since f and g are morphisms in a given category. An isomorphism of sets is also a bijection, an isomorphism of groups is also a bijective group homomorphism, but an isomorphism of topological spaces is not merely a continuous function which is bijective; homeomorphism requires more, namely that the inverse function also be continuous. The utility of categical theory in providing a uniform notion of structure is evident even at this simple level: given any category, one automatically knows the 'right' notion of having the same structure. Now, two objects should be said to bear the same structure just if they are structurally indistinguishable, *i.e.*, just if any 'structural property' enjoyed by one is also enjoyed by the other. This is ensured simply by taking the vague, intuitive notion of a structural property to be one that respects isomorphism in the category at issue. Thus, for example, a structural property of spaces is just one that respects homeomorphism, more usually called a topological property or invariant, such as having a particular fundamental group.

Another reason for the effectiveness of category theory in specifying and describing mathematical structure can now be put quite simply: any mathematical property or construction given solely in terms of structure-preserving mappings—*i.e.*, in a given category—will necessarily respect isomorphism in that category, and will thus be structural.[10] Since all categorical properties are thus structural, the only properties which a given object in a given category may have, *qua* object in that category, are structural ones. As sets-with-Bourbaki-structure, isomorphic groups, say, may differ radically with respect to their set-theoretical constitutions, but such differences are not generally mathematically relevant, at least from a structural point of view, nor are they even expressible in terms of the category of groups. Thus doing mathematics 'arrow-theoretically' automatically pro-

[10] 'Giving a property or construction in terms of mappings' requires some care with respect to mentioning *specific* objects or morphisms; for example the property of being the domain of some particular morphism f is plainly not structural, cf. McLarty [1993] for a precise statement of the relevant condition in a special case.

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vides a structural approach, and this has proven quite effective in attacking certain kinds of mathematical problems having to do with mathematical structure. For example, recall the utility in group theory of the basic homomorphism and isomorphism theorems, which are early examples of such methods. Earlier still, Dedekind's structural characterization of the natural numbers, and indeed of 'the infinite', also show how mappings can be used to determine structure. A more modern example is the notable effectiveness of homotopy and homology in topology; of course, this is the setting in which Eilenberg and Mac Lane first formulated the basic notions of category theory.[11]

A zealous structuralist holding that mathematics is in some sense purely structural could therefore make his case by somehow 'showing' that all of mathematics can be couched in category theory, much as the logicist Russell could have made his by showing that all of mathematics could be coded into some arguably logical system of logic. While the accuracy and utility of such restricted views of what constitutes mathematics should probably be doubted, it nonetheless seems significant how rich the language and methods of category theory have proven to be, perhaps confirming some more modest 'structuralist thesis'.[12] For example, the category of abstract sets described in §2 below provides a structural setting in which to conduct virtually any piece of mathematical reasoning that can be modeled in conventional, axiomatic set theory (provided it does not turn on some *particular* set-theoretical reconstruction). While such a setting is rarely the best one for displaying relevant structural relations, it does confirm that the methods of category theory will suffice for many present-day mathematical purposes which are intuitively structural.

Having now indicated in what sense a category determines a kind of mathematical structure, let no misunderstanding arise from the fact that the model-theoretic, Bourbaki notion of structure served as motivation and provides a source of examples. The claim being made here is not that *that* notion has been pointlessly reformulated in other terms, but that a categorical one different from it has proven more fruitful in the structural approach to mathematics, and that it also serves philosophical purposes better. Indeed, the utility of the categorical conception derives in part from how it differs. For example, given a Bourbaki specification of a kind of mathematical object, one may still ask 'what are the morphisms?' The answer to this not only determines what aspects of the specification are relevant; it also provides a second, semi-independent 'dimension' useful in determining different kinds of structure. Furthermore, many useful categories are not 'categories of Bourbaki-structures' in any sense.

The example of topological spaces will suffice to illustrate some of the

[11] Eilenberg and Mac Lane [1945].

[12] See Marquis [1993] for a related line of thought.

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differences between the two conceptions. Since spaces may be defined in several familiar ways, the objects of the category **Top** are described by various different Bourbaki structures, as already mentioned. In some cases, such as that of a base for a topology, some of the data used in the Bourbaki specification is thrown away in passing to **Top** . And no examination of any definition of topological spaces will reveal the definition of a continuous function, which does not preserve open sets, but takes them back. Indeed there are also other mathematically significant kinds of structure described by categories with spaces as objects, for example the homotopy category mentioned above. Finally, the category whose objects are the open sets of a particular space and whose morphisms are the inclusion maps between these provides a simple example of a kind of mathematical structure on objects which are not models of a Bourbaki structure in any conventional sense.

The general study of topological spaces involves consideration of all of these different categories, and their relations among one another. Topology also involves the relations between spaces and structures of other very different kinds, such as groups. A further, and very important, advantage of the categorical approach to mathematical structure is now evident: representing different kinds of structure as different categories provides a *uniform* notion of structure, making it easy to relate structures of different kinds. By 'relating structures of different kinds' is meant, of course, mapping one to the other; so category theory can be applied in particular to the theory of categories, which proves to be quite potent.[13] One thus studies a particular category, not just by contemplating its 'multiplication table' of morphisms, but by mapping it to and from other categories, for example by mapping **Top** into **Groups**. Indeed since the categorical perspective involves a shift of attention from objects alone to objects and morphisms, one of the central notions of category theory is that of a functor.

A *functor* is a morphism of categories, defined as a mapping from one category to another that preserves the expected category structure: taking objects to objects, morphisms to morphisms, and respecting domains, codomains, composition, and identity morphisms. Thus a functor $F : \mathbf{C} \to \mathbf{D}$ takes a commutative diagram as on the left below to another one as on the right:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g\circ f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[1_C]{} & C \end{array} \quad \text{in } \mathbf{C} \qquad\qquad \begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle F(g\circ f)}\downarrow & & \downarrow{\scriptstyle Fg} \\ FC & \xrightarrow[1_{FC}]{} & FC \end{array} \quad \text{in } \mathbf{D}.$$

[13] It also addresses the philosophical difficulty mentioned in §1 of Parsons [1990] of providing a structural account of structures.

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Such a functor F thus gives a 'picture' of $\mathbf{C}$ in $\mathbf{D}$. The above says just that $F(1_C) = 1_{FC}$ and $F(g \circ f) = F(g) \circ F(f)$, so functors also take isomorphisms to isomorphisms. A single functor $F : \mathbf{C} \to \mathbf{D}$ then gives rise to lots of 'structural properties' on $\mathbf{C}$; most simply, for each $\mathbf{D}$-object D the property $FC \simeq D$ is plainly structural in C (for example, take F to be the fundamental group functor). Indeed any structural property of $\mathbf{D}$-objects can be thus 'composed' with F to give one of $\mathbf{C}$-objects. Note that a functor depends only on the 'multiplication table' of the morphisms in the domain category, and not on any non-categorical properties of the objects and morphisms. This 'explains' the utility of functors in studying a given category, in the same way that a particular category is useful in describing a particular kind of structure on a given object. A familiar functor is the 'forgetful' one U : **Groups** $\to$ **Sets** that takes a group to the set of its elements and a group homomorphism to the underlying function. Also, any particular group is itself a category with just one object, with each element as a morphism, the unit as the identity morphism, and with the group multiplication as composition of morphisms; a functor between such categories is then exactly a group homomorphism. And a functor from a group G to the category of vector spaces (say, over the Reals) and linear transformations is exactly a linear representation of G. Functors are everywhere in mathematics. Recognizing a particular construction as a functor permits application of the general theory of functors to that particular case, just as recognizing a group action somewhere can provide a wealth of group-theoretic information.

Of course, we need not stop here; having defined functors one may ask what *their* morphisms are. Given categories $\mathbf{C}$ and $\mathbf{D}$, we wish to define the functor category $\mathbf{D}^{\mathbf{C}}$ with functors from $\mathbf{C}$ to $\mathbf{D}$ as *objects*. What should a morphism of functors be? Given two functors F and G from $\mathbf{C}$ to $\mathbf{D}$, a morphism $\alpha : F \to G$, called a *natural transformation*, consists of one morphism $\alpha_C : FC \to GC$ in $\mathbf{D}$ for each C in $\mathbf{C}$ such that for any $j : C \to D$ in $\mathbf{C}$ one has $\alpha_D \circ Fj = Gj \circ \alpha_C$ (a commutative square). On the category $\mathbf{V}$ of real vector spaces and linear transformations between them, there is, *e.g.*, a natural transformation α from the (evident) identity functor on $\mathbf{V}$ to the functor that takes each such vector space to its double dual and each linear transformation to the double dual transformation; the component $\alpha_V : V \to V^{**}$ of α at a vector space V is the linear transformation taking each vector x in V to the linear transformation $x^{\wedge} : V^{*} \to R$ given by evaluation at x. Thus α is a morphism in the functor category $\mathbf{V}^{\mathbf{V}}$, and it is well known that it is an isomorphism if $\mathbf{V}$ is restricted to finite-dimensional vector spaces. There is no such natural transformation to the single dual, although each such finite-dimensional vector space is also isomorphic to its single dual. This is just what is meant by saying that such a vector space is 'naturally' isomorphic to its double dual, but not to its dual.

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One of the most fruitful notions to come to light under the categorical perspective is that of a pair of adjoint functors or 'adjunction', which, among other things, generalizes the notion of an isomorphism of categories. A functor $F : \mathbf{C} \to \mathbf{D}$ is an isomorphism of categories if there is a functor coming back $U : \mathbf{D} \to \mathbf{C}$ with $1_{\mathbf{C}} = U \circ F$ and $F \circ U = 1_{\mathbf{D}}$. A pair of functors going both ways are said to be *adjoint* if instead there are natural transformations $\eta : 1_{\mathbf{C}} \to U \circ F$ and $\epsilon : F \circ U \to 1_{\mathbf{D}}$ satisfying a further, special condition which, frankly, looks a bit complicated at first sight.

Consider first the 'free group' functor $F : \mathbf{Sets} \to \mathbf{Groups}$, which is adjoint to the forgetful functor U. If S is a set, UG the set of elements of some group G, and $f : S \to UG$ any function, then there is a unique group homomorphism $h : FS \to G$ making the triangle below commute

$$\begin{array}{ccc} S & \xrightarrow{f} & UG \\ {\scriptstyle \eta_S}\downarrow & \nearrow_{Uh} & \\ UFS & & \end{array} \qquad\qquad \begin{array}{c} G \\ \nearrow_{h} \\ FS \end{array}$$

where η_S is the map that takes an element of S to itself as a 'word' in the free group FS on S. Thus we have a bijection between group homomorphisms $h : FS \to G$ and set functions $f : S \to UG$, which we display schematically thus:

$$\frac{FS \to G \quad \text{in } \mathbf{Groups}}{S \to UG \quad \text{in } \mathbf{Sets}.}$$

Now in the general case mentioned above, we take any objects C in $\mathbf{C}$ and D in $\mathbf{D}$ and require that morphisms $FC \to D$ correspond bijectively to morphisms $C \to UD$, with the correspondence mediated by $\eta : 1_{\mathbf{C}} \to U \circ F$ as in the example. We then call F the *left-adjoint*, U the *right-adjoint*, and η the *unit*, and we write the adjunction thus:

$$\frac{FC \to D \quad \text{in } \mathbf{D}}{C \to UD \quad \text{in } \mathbf{C}.}$$

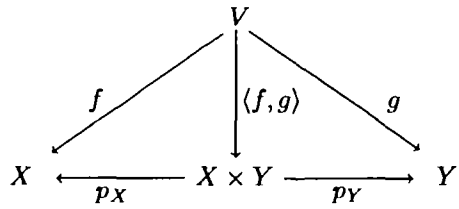
In this schema, putting FC for D throughout and 1_{FC} for the $\mathbf{D}$-morphism on top gives a $\mathbf{C}$-morphism $C \to UFC$ on the bottom which is just the unit η_C at C. In the example, the insertion of generators into the (underlying set of the) free group is the 'universal morphism from S to a group', since any f as in the diagram factors uniquely through it by (the underlying function of) a group homomorphism. In the general case, each η_C is similarly a 'universal morphism from C to (objects in the image of) U'. This condition on F, G, and η suffices to determine the natural transformation $\varepsilon : F \circ U \to 1_{\mathbf{D}}$, called the *counit* of the adjunction; the component $\varepsilon_D : FUD \to D$ at D in $\mathbf{D}$ is given in the schema above as the morphism on top, after putting UD for C throughout and taking the identity morphism on UD on the bottom. The definition of adjoints could just as well be stated in terms of

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the counit rather than the unit, and each ε_D has a similar universal mapping property in **D**. The counit of the free/forgetful adjunction for example is the group homomorphism that presents any group as a factor group of the free group on its elements. It should be noted that if a given functor has an adjoint, then it is uniquely determined up to a unique natural isomorphism, as can be seen without difficulty. So, for example, the free group FS on a set S is characterized up to isomorphism by the universal mapping property stated above.

Adjoint functors occur throughout mathematics and often express significant relations between different kinds of structure. The 'Galois correspondence' in algebra expresses an adjunction, and the Stone-Cech compactification in topology is characterized up to isomorphism by an adjoint functor.[14] Duality theorems such as the Gelfand duality in functional analysis and Stone duality in its many forms are also examples of adjunctions.[15] In logic, as we shall see in §2 below, the propositional and (typed) lambda calculi follow from a few basic adjunctions; even more striking is the fact that the quantifiers $\exists$ and $\forall$ are also adjoint functors, and all of their basic properties follow from adjointness.

Adjunctions are sometimes stated in terms of the associated universal mapping properties of the unit or counit; the reader may be familiar with a characterization of polynomial rings or tensor products by such a universal mapping property. A basic and important example is the categorical definition of a product of two objects in a category. A *product* of X and Y in **C** consists of an object $X \times Y$ and morphisms $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ such that, given any object V and morphisms $f : V \to X$, $g : V \to Y$, there is a unique morphism $\langle f, g \rangle : V \to X \times Y$ such that $f = p_X \circ \langle f, g \rangle$ and $g = p_Y \circ \langle f, g \rangle$ all as in the commutative diagram



The universal mapping property of the product states that the pair of morphisms p_X, p_Y to X and Y with common domain are 'universal' among such pairs, in that any other such pair (here f and g) factor uniquely through them (here by $\langle f, g \rangle$). If such a product exists in **C**, then it is unique up to isomorphism. If every pair of objects in **C** has a product, one says that **C** 'has products'. As with all characterizations by universal

[14] See Mac Lane [1971] for these and many other examples.

[15] See *ibid.* and also Johnstone [1982].

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mapping properties, there is an equivalent characterization of products in terms of adjoint functors. Specifically, let $\mathbf{C}^2$ be the category with pairs of objects of $\mathbf{C}$ as objects and pairs of morphisms of $\mathbf{C}$ as morphisms, with the expected identity morphisms, domains, and codomains. Then the product is a functor $\times : \mathbf{C}^2 \to \mathbf{C}$, defined on morphisms in the evident way, and the universal mapping property stated above can be recognized as the statement that the functor $\times$ is right adjoint to the functor $\Delta : \mathbf{C} \to \mathbf{C}^2$ that simply 'doubles' each object and morphism of $\mathbf{C}$.

The product is a fine example of a categorical specification of an intuitively structural notion. **Sets** has products, given by Cartesian products defined, say, in terms of the Kuratowski ordered pair. As with products in all categories, this Cartesian product of two sets is characterized up to isomorphism by the universal mapping property above. Indeed, any other definition of the ordered pair—and there are infinitely many—also determines a product, and each is as good as the next from a categorical perspective, for they are necessarily all isomorphic. Furthermore, one and the same categorical definition describes also products of topological spaces, groups, vector bundles on a smooth manifold, or whatever. The definition above provides a uniform, structural characterization of a product of two objects in terms of their relations to other objects and morphisms in a category, in contrast to 'material' set-theoretic definitions which depend on specific and often irrelevant features of the objects involved, introducing unwanted additional structure. Indeed it is just this material aspect of conventional set theory that gives rise to such pseudo-problems as whether the number 1 is 'really' the set $\{\emptyset\}$, or whether the real numbers are 'really' cuts in the rationals.[16]

Often, when a construction is recognized as an adjoint to some simple or familiar functor, the resulting structural characterization seems to capture its essential properties (recall that adjoints are unique up to isomorphism). This is so, for example, for the above definitions of products and free groups as for the following definition of an exponential. Given sets X, Y, Z, let Z^Y be the set of functions $g : Y \to Z$. Then there is a familiar bijection

$$(*) \qquad \frac{F : X \times Y \to Z}{f : X \to Z^Y}$$

given by $f(x)(y) = F(x, y)$ between functions F in two variables and those f in one and taking values in Z^Y. This bijection is easily seen to be natural in both X and Z, and so $(*)$ states an adjunction: for fixed Y, the functor 'product with Y' is left adjoint to the functor 'exponential by Y'. The counit $\epsilon : Z^Y \times Y \to Z$ at any set Z is just the usual evaluation function

[16] The categorical definition of a product, due to Mac Lane [1950], is one of the first examples of category theory being used to give a purely structural characterization of an important basic mathematical notion.

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$\varepsilon(g, y) = g(y)$; as always, it has a corresponding universal mapping property which serves to define exponentials in any category. In any category $\mathbf{C}$ with products, the functor $- \times Y : \mathbf{C} \to \mathbf{C}$ for fixed Y in $\mathbf{C}$ has a right adjoint iff every Z in $\mathbf{C}$ has an exponential Z^Y by Y. The adjunction $(*)$ then holds just as in **Sets**, and for each Z there is a morphism $\varepsilon : Z^Y \times Y \to Z$ universal from products with Y just like the evaluation function. A further example is provided by functor categories $\mathbf{D}^{\mathbf{C}}$, which are exponentials in **Cat**, the category of categories and functors; this also confirms that 'natural transformation' was the 'right' definition of a morphism of functors.

The universal mapping properties arising from adjunctions are examples of structural definitions, *i.e.*, characterizations up to isomorphism. Any two objects satisfying such a definition are thus structurally indistinguishable. Here is another example of such a structural definition.

In any category $\mathbf{C}$, an *initial object* is one 0 from which each object C has exactly one morphism $o_C : 0 \to C$. A *terminal object* is an object 1 to which each object C has exactly one morphism $!_C : C \to 1$. Any two initial (terminal) objects are plainly isomorphic; these are just simple universal mapping properties with equivalent formulations in terms of adjoints. In **Sets** the null set is initial, and any singleton is terminal. Now define a 'thingamajig' in **Sets** to be a set A together with functions $a : 1 \to A$ and $s : A \to A$. A morphism $F : (A, a, s) \to (A', a', s')$ of thingamajigs shall be a function $f : A \to A'$ such that $f \circ a = a'$ and $f \circ s = s' \circ f$, *i.e.*, a commutative triangle and a commutative square. These morphisms compose in an obvious way; so we have a category of thingamajigs in **Sets**. Now, an initial object in this category is exactly 'the' natural numbers, with zero and successor specified. Indeed, the universal mapping property 'initial object in the category of thingamajigs' suffices to prove Peano's postulates, and nothing which is not a consequence thereof.[17] Note that a thingamajig is just a set with a 'starting point' a and an operation 'take x to sx'; characterizing the natural numbers as an *initial* such thing amounts to saying just that they uniquely index any inductively defined sequence. This example nicely illustrates the utility of a uniform notion of structure; not only does Dedekind's categoricity theorem follow immediately from the fact that anything defined by adjoints (or a universal mapping property) is unique up to isomorphism, one also sees that the structural character of the natural numbers is no oddity, but a general feature of all mathematical objects so determined.

Another notion which, perhaps surprisingly, can be treated structurally is elementhood, and more generally membership in 'subobjects', *i.e.*, subsets, subgroups, and such. In **Top**, the one-point space 1 is terminal, and the points of a given space X correspond exactly to continuous maps

[17] The characterization is due to Lawvere, cf. Lawvere [1964]. See McLarty [1993] for a precise sense in which the natural numbers so determined have no further properties.

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$p : 1 \to X$. In any category $\mathbf{C}$ with a terminal object 1, a morphism $p : 1 \to C$ is called a point of the object C. In **Sets** the points of a set correspond exactly to its elements. To the extent that an object in any category is determined by its morphisms, sets are determined by their points, for if two functions $f, g : A \to B$ have $fa = ga$ for all points $a : 1 \to A$, then f and g must be identical. Now, this is a special property of sets, and it need not hold in an arbitrary category with a terminal object. For example, each group has just one point: the trivial homomorphism from the one-element group. Thus groups have more structure than is revealed by their points.

Whether or not the objects of a category have enough points to distinguish morphisms, it always suffices to consider instead arbitrary morphisms $x : X \to C$, for arbitrary X, as 'generalized elements' of C; for in any category two morphisms $f, g : C \to D$ are identical iff $fx = gx$ for all such x ('if' by taking x to be 1_C). A generalized element $x : X \to C$ can be viewed as a variable element of the object C, and a point as a constant element, as long as one recalls that 'element' is itself being used in a somewhat perverse sense.

Now this structural notion of elementhood generalizes to membership in subobjects, providing an alternative to the set-theoretical conception of membership as primitive. Here is how. In any category $\mathbf{C}$, a morphism $m : U \to C$ is a *monomorphism* iff $mu = mu'$ implies $u = u'$ for all $u, u' : X \to U$. Monomorphisms into an object C represent subobjects of C as follows: call two monos $m : U \to C$ and $n : V \to C$ equivalent if m factors through n as in the commutative triangle

$$\begin{array}{ccc} U & \xrightarrow{k} & V \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle n} & \\ & C & \end{array}$$

for some (necessarily unique) $k : U \to V$, and also n factors similarly through m. The domains of equivalent monos are then isomorphic objects in $\mathbf{C}$. For any object C in $\mathbf{C}$ there is then a category $\mathbf{Sub(C)}$ with these equivalence classes of monos into C as objects, and with a unique morphism $|m| \to |n|$ between such objects iff the mono m factors through the mono n. We may as well write $|m| \leq |n|$ for $|m| \to |n|$ since $\mathbf{Sub(C)}$ is partially ordered by its morphisms, *i.e.*, there is at most one morphism between any two objects. The objects of $\mathbf{Sub(C)}$ are then called *subobjects* of C. Subsets, subgroups, submodules, *etc.* correspond exactly to such subobjects in their respective categories, and so we confuse the objects of $\mathbf{Sub(C)}$ with the domains of their representing monos whenever possible, writing $|m| \to C$ for the subobject determined by a mono $m : U \to C$. We then say that an element $x : X \to C$ is 'in the subobject' $|m| \to C$, and write $x \in_C |m|$, if x factors through m, which it then does uniquely since m is monic. Note that the truth of $x \in_C |m|$ does not depend on the choice of

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representative $m : U \rightarrow C$ of $|m| \rightarrow C$.

One way of comprehending subobjects in many categories is by taking solutions to equations with variables. The resulting 'equational subvarieties' are described categorically by equalizers. For a parallel pair of morphisms $f, g : C \rightarrow D$ in a category $\mathbf{C}$, an equalizer of f and g is a morphism $e : E \rightarrow C$ with $fe = ge$ and universal with this property; *i.e.*, given any $x : X \rightarrow C$ with $fx = gx$, x factors uniquely through e as in the diagram:

$$\begin{array}{ccccc} E & \xrightarrow{e} & C & \overset{f}{\underset{g}{\rightrightarrows}} & D. \\ {\scriptstyle u}\uparrow & \nearrow_{x} & & & \\ X & & & & \end{array}$$

In **Groups**, for example, take g in the diagram to be the trivial homomorphism, then the kernel of f (with its inclusion) is an equalizer. If there is such an equalizer e in a category $\mathbf{C}$, then it must be monic, and for any $x : X \rightarrow C$ we then have $x \in_C |e|$ iff $fx = gx$. So any equalizer of f and g determines the 'subobject of all x with $fx = gx$', which we may denote accordingly as $\{x \in C \mid fx = gx\} \rightarrow C$.

Note that this structural notion of membership is always 'local', in that it is a relation between morphisms into and subobjects of a given object C. It specializes to something like the usual notion in the category of sets, as will become clear in §2. Here we note the important difference that in the category of sets, elements cannot themselves have elements. Using points, one recovers some of the 'internal structure' of sets as a special case of 'external'—*i.e.*, categorical—structure, but this never goes deeper than one $\in$, so to speak. And this is exactly the notion of elementhood appropriate to the categorical conception of 'set structure'; for the structure of sets characterized by the category **Sets** includes just what is preserved by set functions, and so *not* internal $\in$-relations.

2. Logical Structure

Our point of view has been that a category determines a particular kind of structure, which each of its objects has. Now, the objects of any category, and indeed all mathematical objects, may be said to bear additional 'logical structure'; it is this we now wish to specify. Given the discussion of §1, the way to specify a kind of structure is clear enough; we will therefore determine a species of category, each consisting of (i) objects with some arbitrary structure, *i.e.*, with morphisms between them, and (ii) 'everything that can be constructed from these by logical means'. Such a category will be called a 'topos'. We will arrive at the definition of a topos through a mock analysis of 'logical structure'; of course, this is purely rhetorical.[18]

The notion of a topos is quite different from the categories considered up to now; we are not dealing with a particular category but rather a kind

[18] See McLarty [1990] for a careful historical account.

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of category, to be characterized directly in structural terms rather than by specifying objects and morphisms. This is a notion of a different order, reflecting the difference between logical structure in general and a particular kind of mathematical structure.

An object X of a topos will be viewed as a type, or sort, or species, or generalized set, or class of things—the X's . Thus an object X is the 'object of x's' in the same way that a product $X \times Y$ in any category is the object of pairs $\langle x, y \rangle$ where x is an X and y is a Y . We may refer generally to an object of a topos as a 'type'. The basic means of logical construction in a topos will be comprehension of subobjects by 'properties', a notion we now specify. The following tenets are taken as fundamental:

i) *Properties are local.* A property is always a property of x's of some type X, thus every property has a fixed domain of significance.
ii) *Properties are variable propositions.* If ϕ is a property with domain of significance X, and $a : 1 \to X$ is a constant element of type X, then $\phi(a)$ is a proposition.

Of course, this is just Russell's notion of a 'propositional function'.[19] The term conveys the functional aspect of the notion we seek better than do 'property' or 'predicate'. So in a topos a property with domain of significance X will be called a *propositional function on* X. Now every 'function' must of course also have a codomain; so a topos will include an object of propositions P, the points $p : 1 \to P$ of which (if any) are propositions and the generalized elements $\phi : X \to P$ of which are variable propositions and hence propositional functions. Thus if the proposition p factors as $p = \phi a : 1 \to X \to P$, then p results from evaluating the propositional function ϕ at the point a of X.

Next, we say what makes P an object of propositions by saying, not what it is, but how it works. This we of course do in terms of its morphisms. If there are to be any propositions $p : 1 \to P$ at all in a topos, then there should also be the true proposition '$p = p$': $1 \to P$. We therefore assume that there is at least the one proposition $true : 1 \to P$, and that there is a terminal object for this purpose. Then for any object X, the composite $true \circ !_X : X \to 1 \to P$ is a constant, true-valued propositional function on X, which we abbreviate to $true_X$. Propositional functions will serve to specify subobjects, as follows. Given a propositional function $\phi : X \to P$,

[19] Cf. *Principles of Mathematics*, §22: 'We may explain... [propositional function] as follows: ϕx is a propositional function if, for every value of x, ϕx is a proposition, determinate when x is given.' Also *Principia Mathematica*, p. 14: 'Let ϕx be a statement containing a variable x and such that it becomes a proposition when x is given any fixed determined meaning. Then ϕx is called a "propositional function"...' and p. 161: '... there will be arguments x with which ϕx is meaningless, *i.e.*, with which as arguments ϕ does not have any value. The arguments with which ϕx has values form what we will call the "range of significance" of ϕx. A "type" is defined as the range of significance of some [propositional] function'.

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we want the subobject of x's for which ϕ is true. This we know how to get: as an equalizer of ϕ and $true_X$. Then in particular for any point a of X we will have $a \in_X \{x \in X \mid \phi x = true_X\}$ iff $\phi a = true$. We will call (the subobject determined by) this equalizer the *extension* of the propositional function ϕ, and will denote it simply $\{x \in X \mid \phi\} \rightarrow X$. The extension is viewed as the propositional function 'objectified'.[20]

We have not said what propositional functions there are to be aside from the various $true_X$ for objects X. Nor will we, except to require that any given subobject $\sigma \rightarrow X$ be the extension of a unique propositional function on X. This suits the purpose of describing only the additional logical structure on objects which may also bear some other structure. The uniqueness requirement will make propositional functions 'extensional', to the same end. Despite having specified none, there will still be many non-trivial propositional functions in a topos, for a topos will also be required to have products and exponentials, and as adjoints these bring universal morphisms which will give rise to significant propositional functions. In logical terms, products provide for relations as propositional functions on the product, and exponentials provide for propositional functions of propositional functions—*i.e.*, higher-order logic—as propositional functions on P^X, the object of propositional functions on X.

This, then, will be sufficient to characterize the notion of logical structure: a *topos* is defined to be a category with a terminal object, products, exponentials, and an object of propositions. The latter is characterized by a 'comprehension axiom': there is a morphism $true : 1 \rightarrow P$ such that for each $\phi : X \rightarrow P$ there is an equalizer of ϕ and $true_X$, and each mono $m : M \rightarrow X$ is such an equalizer for a unique ϕ. More simply put: every propositional function has a unique extension, and these are the only subobjects.[21]

We next sketch the development of the familiar logical calculus in a topos, emphasizing the appearance of adjoints as a significant aspect of the structural approach to logic. Recall that this also serves as a sample application of the methods presented in §1; so the reader who has just made their acquaintance may find this a bit foreign and wish to skim forward to the discussion.

[20] *Principles*, §84: 'When we consider the x's such that ϕx, where ϕx is a propositional function, we are introducing a notion of which, in the calculus of propositions, only a very shadowy use is made—I mean the notion of *truth*. We are considering, among all the propositions of the type ϕx, those that are true: the corresponding values of x give the class defined by the function ϕx. It must be held, I think, that every propositional function which is not null defines a class, which is denoted by "x's such that ϕx".'

[21] See Mac Lane and Moerdijk [1992], IV.1 for equivalent standard definitions.

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Atomic Propositions

Given objects X, Y in a topos (or any category) $\mathbf{T}$, write $\mathbf{T}[X, Y]$ for the collection of morphisms $X \to Y$ in $\mathbf{T}$, so that, in a topos, $\mathbf{T}[X \times Y, P]$ is relations on X and Y, and the product/exponential adjunction provides an isomorphism $\mathbf{T}[X \times Y, P] \simeq \mathbf{T}[X, P^Y]$.

By the comprehension axiom, $\mathbf{T}[X, P]$ (propositional functions on X) inherits a category structure from $\mathbf{Sub}(X)$, with a unique morphism $\phi \to \psi$ between propositional functions iff $\{x \in X \mid \phi\} \leq \{x \in X \mid \psi\}$ in $\mathbf{Sub}(X)$. Then $\{x \in X \mid -\}$ is a functor

$$\{x \in X \mid -\} : \mathbf{T}[X, P] \to \mathbf{Sub}(X),$$

and in fact an isomorphism of categories (this is exactly the content of the comprehension axiom).

Denote the inverse of $\{x \in X \mid -\}$ by

$$x \in_X - : \mathbf{Sub}(X) \to \mathbf{T}[X, P].$$

That is, if $\sigma \to X$ is a subobject of X, then $x \in_X \sigma : X \to P$ is the unique propositional function on X of which σ is the extension. This 'internalizes' the membership relation defined in §1 in a way to be seen presently. For an element $z : Z \to X$ of X we then write $(x \in_X \sigma) \circ z$ as $z \in_X \sigma$, treating x as a free variable. We may drop the type indicator X in $\{x \in X \mid -\}$ and $\in_X$ when it is clear from the context.

The unit and counit of the $x \in -/\{x \mid -\}$ adjunction[22] are just the familiar identities:

$$\sigma = \{x \mid x \in \sigma\}, \qquad x \in \{x \mid \phi\} = \phi.$$

The injectivity and surjectivity of the isomorphism

$$x \in - : \mathbf{Sub}(X) \simeq \mathbf{T}[X, P]$$

correspond to the usual extensionality and comprehension principles respectively: '$x \in \sigma = x \in \tau$ implies $\sigma = \tau$' says just that $x \in -$ is injective, and 'for each ϕ there is a σ with $x \in \sigma = \phi$' says just that $x \in -$ is surjective.

The presence of 1, products, and exponentials in a topos provides a way of 'internalizing' $x \in -$, $\{x \mid -\}$, $\mathbf{T}[X, P]$, and $\mathbf{Sub}(X)$. We have for each object X, $X \simeq 1 \times X$, so by the product/exponential adjunction, points $1 \to P^X$ correspond uniquely to propositional functions $\phi : X \to P$ and hence to subobjects $\{x \mid \phi\} \to X$. Generally, the point corresponding

[22] Recall that isomorphism is a special case of adjunction; so the topos axioms consist of four adjunctions. Cf. Lawvere [1968] for a more general view of comprehension as an adjoint functor.

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to a morphism $f : X \to Z$ under such 'exponential transposition' may be denoted $\lambda_x.f : 1 \to Z^X$, and similarly for the transposition from $X \times Y \to Z$ to $X \to Z^Y$. The counit $\varepsilon : X \times P^X \to P$ of this adjunction at P factors each $\phi : X \to P$ uniquely as $\phi = \varepsilon \circ \langle 1_X, \lambda_x.\phi \circ !_X \rangle$ (the reader may wish to draw the diagram). Now, in general, $\varepsilon : X \times P^X \to P$ is 'evaluation'. But in this case, as a propositional function on a product, it can also be viewed as a relation; indeed it is *membership*. For any constant element $a : 1 \to X$ we have

$$\varepsilon \circ \langle a, \lambda_x.\phi \rangle = \phi a = (x \in \{x \mid \phi\}) \circ a = a \in \{x \mid \phi\}$$

and similarly for generalized elements. Thus we can identify $\mathbf{Sub}(X)$ with $\mathbf{T}[1, P^X]$, and the subobject $\{x \mid \phi\} \to X$ with the exponential transpose $\lambda_x.\phi : 1 \to P^X$ of $\phi : X \to P$. Given $\sigma : 1 \to P^X$ we then get $x \in \sigma : X \to P$ just by transposing back. In this way, the $x \in -/\{x \mid -\}$ isomorphism is nothing but 'lambda conversion' at P, *i.e.*, exponential transposition.

The membership relation $\in$ thus 'is' the counit of the product/exponential adjunction at P, and other universal morphisms also give rise to familiar logical notions. Indeed, 'identity' arises from products. For any object X, the unit of the adjunction that defines products is the 'diagonal morphism' $\Delta_X = \langle 1_X, 1_X \rangle : X \to X \times X$. It is always monic; so it is the extension of a unique propositional function, denoted $\delta_X : X \times X \to P$, which is thus a relation on X. For any $y, z : V \to X$ we then have $\delta_X \circ \langle y, z \rangle = true_V$ iff $y = z$. So $\delta_X : X \times X \to P$ is the *identity relation* on X. We also write $y =_X z$ for $\delta_X \circ \langle y, z \rangle$. The exponential transpose $\{-\}_X : X \to P^X$ of δ_X has

$$y \in \{z\}_X = \varepsilon \circ \langle y, \{z\}_X \rangle = \delta_X \circ \langle y, z \rangle = y =_X z,$$

so $\{-\}_X$ is the *singleton morphism* of X.

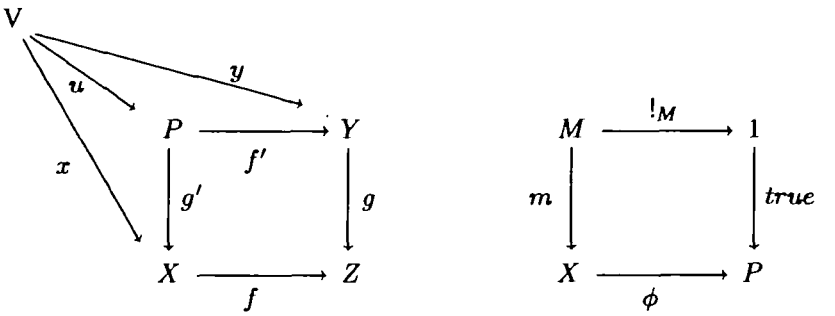
These basic 'set-theoretical' notions thus arise solely from the general adjunctions defining a topos; from our present point of view, they belong to the logic of objects with arbitrary structure, and not just the theory of sets. Their structural characterization not only permits application in a wider variety of settings, it has allowed us to recognize them as closely related to other important constructions, with which they perhaps seemed to have little in common.

Propositional Calculus

We next require some definitions, a lemma, and a theorem.

In any category, given a corner of morphisms f, g as on the left below, a *pullback* of f and g consists of a pair f', g' with common domain making a commutative square, and universal with this property; *i.e.*, given any pair x, y with $fx = gy$ there is a unique u with $g'u = x$ and $f'u = y$, all as in the commutative diagram on the left:

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Now in a topos, the right-hand diagram above is a pullback iff m is an equalizer for ϕ and $true_X$. For, in either case, m is universal with $\phi m = true_M = true_X m$. So by the comprehension axiom each mono $m : M \to X$ in a topos is a pullback of *true* by a unique propositional function on X, *i.e.*, m fits into a pullback square as on the right above for exactly one ϕ. It can then be seen that a topos in fact has a pullback for *every* corner of morphisms as on the left above, and so has all 'finite limits'.[23] That was the lemma.

The finite limit notions have (categorical) duals, whose definitions are obtained by reversing all the arrows in the respective limit definitions. Thus 'initial object' is the dual of 'terminal object', a *coproduct* for a pair of objects consists of an object receiving a pair of morphisms from those objects and universal with this property, and similarly for coequalizers. In **Sets**, disjoint unions provide coproducts. Quotient objects such as factor groups and general quotients by equivalence relations arise from coequalizers. The notions initial object, coproduct, coequalizer, and pushout ('co-pullback') are called finite colimits. We will not prove the surprising fact that a topos must also have all finite colimits. That is the theorem.[24]

Now, the propositional calculus is just the 'algebra' of the category of propositions $\mathbf{T}[1, P] \simeq \mathbf{Sub}(1)$. For any propositions p, q we may interpret $p \leq q$ as 'p implies q'; for given $p \leq q$, if $p = true$ then $q = true$, since *true* is terminal in $\mathbf{T}[1, P]$. The category of propositions also has an initial object $false : 1 \to P$; it is the constant proposition arising from the mono $0 \to 1$ from the initial object in the topos (any morphism from 0 is monic). Thus for any proposition p,

$$false \leq p, \qquad p \leq true. \tag{1}$$

Given a proposition $p : 1 \to P$, write $p^* \to 1$ for the subobject of 1 that is its extension. For propositions p, q the unique morphism $p^* \times q^* \to 1$

[23] Terminal object, product, pullback, and equalizer have a common generalization in the notion of a finite limit, which we need not define.

[24] Mikkelson [1976], cf. Mac Lane and Moerdijk [1992], IV.5.

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is also monic; so (by comprehension) there is a unique proposition $p \wedge q$ such that $(p \wedge q)^* = p^* \times q^*$. Indeed $\wedge$ is itself a propositional function $\wedge : P \times P \to P$, namely that of the mono $\langle true, true \rangle : 1 \to P \times P$, for which $p \wedge q = \wedge \circ \langle p, q \rangle$. For propositions p, q, r we then have

$$(2) \qquad r \leq (p \wedge q) \quad \text{iff} \quad r \leq p \text{ and } r \leq q$$

by the definition of the product. In particular $p \wedge q = true$ iff $p = true$ and $q = true$, so $\wedge$ is indeed *conjunction*, and the logical product of two propositions is just their categorical product.

Just as $p^* \times q^*$ gives rise to $p \wedge q$, so the unique morphism $(q^*)^{p^*} \to 1$ from the exponential is monic, and thus determines a proposition which we denote $p \Rightarrow q$. By the product/exponential adjunction

$$(3) \qquad (p \wedge q) \leq r \quad \text{iff} \quad p \leq (q \Rightarrow r)$$

for any propositions p, q, r. In particular, $p \leq q$ iff $p \Rightarrow q = true$; so the logical exponential is the *conditional* 'if-then'. Now $\Rightarrow$ is also a propositional function on $P \times P$, defined so that $(p \Rightarrow q)^*$ is the equalizer of p and $p \wedge q$, *i.e.*, $(p \Rightarrow q) = (p =_1 p \wedge q)$. Using $\Rightarrow$ one then defines *negation* by $\neg p = (p \Rightarrow false)$, so that $\neg p = true$ iff $p = false$. Plainly also $\neg : P \to P$.

The operations $- \wedge q$ and $q \Rightarrow -$ are easily seen to be functors on the category of propositions; by (3) they are left and right adjoint respectively. There is at most one morphism between any two objects in this category, and so the bijection condition for adjunctions takes the form $Fx \leq y$ iff $x \leq Uy$, and naturality is automatic. (1) and (2) are similarly just the usual adjunctions for initial object, terminal object, and product, applied in the category of propositions. Much of the propositional calculus follows already from these adjunctions. For example, a basic property of right adjoints is that they preserve products, so $p \Rightarrow (q \wedge r) = (p \Rightarrow q) \wedge (p \Rightarrow r)$. The counit of (3) is $p \wedge (p \Rightarrow q) \leq q$, *i.e.*, *modus ponens*. And putting $false$ for q and applying the rule above for $\neg$ gives $\neg(p \wedge \neg p) = true$, *i.e.*, the 'law of contradiction'.

The propositional calculus includes finally *disjunction* $\vee : P \times P \to P$, defined so that for any propositions p, q, r,

$$(4) \qquad (p \vee q) \leq r \quad \text{iff} \quad p \leq r \text{ and } q \leq r.$$

(4) is dual to (2), making $p \vee q$ the coproduct of p and q in the category of propositions. In particular, then $(p \vee q) = false$ iff $p = false$ and $q = false$, but in general this disjunction need not satisfy the classical semantics for 'or'.

Since the propositional connectives $\wedge$, $\vee$, $\Rightarrow$, $\neg$ are themselves propositional functions on $P \times P$, respectively P, the propositional calculus lifts

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to any category of propositional functions $\mathbf{T}[X, P]$ by composition. *E.g.*, the conjunction $\phi \wedge \psi$ of two propositional functions on an object X is simply $\wedge \circ \langle \phi, \psi \rangle$, and the same rules and tautologies carry over with general propositional functions in place of propositions.

The entire propositional calculus now results from the adjunctions (1)–(4); the algebra so determined is in fact—remarkably—exactly intuitionistic propositional logic, and classical propositional logic results just in case $\neg \circ \neg = 1_P : P \to P$.

Quantification

One of the fruits of the structural approach to logic is the recognition of the quantifiers as adjoint functors. This discovery not only elegantly solves the long-standing problem of how to treat quantification algebraically, it also relates quantification to the many other adjoint situations throughout mathematics.

Let $\mathbf{T}$ be a topos and X an object in $\mathbf{T}$. Given a propositional function $\phi : X \to P$ on X, $\exists_x.\phi$ and $\forall_x.\phi$ will be certain propositions $1 \to P$, with X the 'domain of quantification'.

For any proposition $p : 1 \to P$ let $x^*.p$ denote the constant, p-valued propositional function on X, given by

$$x^*.p = p \circ !_X : X \to 1 \to P.$$

If $p \leq q$, then $x^*.p \leq x^*.q$; so x^* is a functor $x^* : \mathbf{T}[1, P] \to \mathbf{T}[X, P]$ from propositions to propositional functions on X. In fact, x^* also preserves all of the propositional connectives $\vee$, $\Rightarrow$, *etc.*; so x^* is a homomorphism of propositional logic. From a logical point of view, x^* is quite trivial. For any proposition p and any $a : 1 \to X$ one has $(x^*.p)a = p$. Thus x^* in effect just adds a dummy variable x over X to a proposition; it is the inclusion of propositional logic into quantificational logic over X. Trivial as it may be, the functor x^* is important, for it has both left and right adjoints, and these are the quantifiers $\exists_x$ and $\forall_x$ respectively.

The morphism

$$\forall_X : P^X \longrightarrow P$$

may be defined as the propositional function of the mono $\lambda_x.true_X : 1 \to P^X$. Recall that points of P^X correspond to propositional functions on X under exponential transposition. Composing with $\forall_X$ thus takes (transposed) propositional functions on X to propositions. Explicitly, for a propositional function $\phi : X \to P$ on X, take the corresponding point $\lambda_x.\phi : 1 \to P^X$ and compose with $\forall_X : P^X \to P$ to get a proposition $\forall_X \circ \lambda_x.\phi : 1 \to P$. We then abbreviate $\forall_X \circ \lambda_x.\phi$ to $\forall_x.\phi$.

Now this mapping

$$\forall_x : \mathbf{T}[X, P] \simeq \mathbf{T}[1, P^X] \longrightarrow \mathbf{T}[1, P]$$

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taking $\phi : X \to P$ to $\forall_x.\phi = \forall_X \circ \lambda_x.\phi : 1 \to P$ is easily seen to be a functor, which is to say that if ϕ and ψ are propositional functions on X, and $\phi \leq \psi$, then $\forall_x.\phi \leq \forall_x.\psi$ (corresponding to a familiar logical fact). Furthermore, one shows that for any proposition p we then have:

$$x^*.p \leq \phi \quad \text{iff} \quad p \leq \forall_x.\phi\,.$$

Of course, this says exactly that $\forall_x$ is right adjoint to the 'dummy variable' functor x^*. Note in particular that $\forall_x.\phi = \mathit{true}$ iff $\phi = \mathit{true}_X$, since $x^*.\mathit{true} = \mathit{true}_X$. And the counit of the adjunction yields the familiar $(\forall_x.\phi \Rightarrow \phi) = \mathit{true}$.

The definition of the left adjoint $\exists_x$ requires a bit more work. Suffice it to say, for each object X, there is also a morphism

$$\exists_X : P^X \longrightarrow P$$

such that $\exists_x.\phi = \exists_X \circ \lambda_x.\phi$ defines a functor $\exists_x : \mathbf{T}[X, P] \to \mathbf{T}[1, P]$ for which

$$\exists_x.\phi \leq q \quad \text{iff} \quad \phi \leq x^*.q$$

for any proposition q and propositional function ϕ on X. Thus $\exists_x$ is left adjoint to x^*. The unit of this adjunction is $\phi \leq \exists_x.\phi$, giving the familiar $(\phi \Rightarrow \exists_x.\phi) = \mathit{true}$.

Note that $\exists_X$ and $\forall_X$ are themselves propositional functions on P^X, and thus 'second-order' propositional functions, as they should be.[25]

The general case of a propositional function in several variables

$$\psi : X \times Y_1 \times \ldots \times Y_n \longrightarrow P$$

reduces to the case $n = 1$ by putting $Y = Y_1 \times \ldots \times Y_n$. There is then a morphism $\lambda_x.\psi : Y \to P^X$, and the quantifiers may be defined as before by

$$\forall_x.\psi = \forall_X \circ \lambda_x.\psi : Y \longrightarrow P^X \longrightarrow P,$$

and similarly $\exists_x.\psi = \exists_X \circ \lambda_x.\psi$. Thus $\forall_x.\psi$ and $\exists_x.\psi$ are propositional functions on Y. As in the special case, these operations are functors

$$\exists_x, \forall_x : \mathbf{T}[X \times Y, P] \longrightarrow \mathbf{T}[Y, P].$$

Now let $\phi : Y \to P$ be a propositional function on Y. One adds a dummy variable over X to ϕ to get a propositional function on $X \times Y$ by precomposing ϕ with the projection $\pi : X \times Y \to Y$. Thus we put

$$\pi^*.\phi = \phi \circ \pi : X \times Y \to Y \to P.$$

[25] Using the rather subtle notion of an internal category, one can also recognize $\exists_X$ and $\forall_X$ as coproduct and product functors respectively.

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Given any pair $\langle a, b\rangle : 1 \to X \times Y$, we then have $\pi^*.\phi(a,b) = \phi \circ \pi(a,b) = \phi(b)$. This dummy variable operation

$$\pi^* : \mathbf{T}[Y, P] \longrightarrow \mathbf{T}[X \times Y, P]$$

is also a functor, and of course $\pi^* = x^*$ in case $Y \simeq 1$.

The general quantifiers $\exists_x, \forall_x : \mathbf{T}[X \times Y, P] \to \mathbf{T}[Y, P]$ are then left and right adjoint to the dummy variable functor π^*, just as before. The adjunctions above then take the general form:

$$\exists_x.\psi \leq \phi \quad \text{iff} \quad \psi \leq \phi \tag{5}$$

$$\phi \leq \psi \quad \text{iff} \quad \phi \leq \forall_x.\psi \tag{6}$$

for any ψ on $X \times Y$ and ϕ on Y. Here we have omitted the dummy variable over X on ϕ in the Northeast and Southwest positions. This corresponds exactly to the usual condition that the quantified variable x over X not occur free in ϕ. Indeed, these two equivalences are just the usual Gentzen-rules for the quantifiers, while the unit of (5) and the counit of (6) give just the usual quantifier axioms:

$$(\psi \Rightarrow \exists_x.\psi) = \mathit{true}\,, \quad (\forall_x.\psi \Rightarrow \psi) = \mathit{true}.$$

This is worth stressing: *All of the essential properties of quantification are summarized by the statement: the quantifiers $\exists_x$ and $\forall_x$ are left and right adjoint to adding a dummy variable.* By uniqueness of adjoints, this fact depends in no way on the particular construction sketched above, nor indeed on the specifics of the topos-theoretic approach to logic.[26]

Discussion

The notion of a topos was arrived at above to specify a particular kind of structure, *viz.* logical structure. So it may well be asked how the logic just sketched compares to the traditional conception, and whether the notion of a topos 'adequately characterizes' that of logical structure, in some sense. Recall, however, that a topos should not describe the logical structure of anything in particular, but rather the general logic of objects which may also bear some further, unspecified structure. And in fact there are many different topoi, *i.e.*, many non-equivalent categories satisfying the topos axioms, and each has its own 'internal logic', *i.e.*, different propositions are $= \mathit{true}$ in each.

[26] The treatment of quantifiers as adjoints is due to F. W. Lawvere, as is much of the categorical analysis of logic; cf. Lawvere [1969]. There also results a lovely geometric interpretation of quantification, for which see also Mac Lane and Moerdijk [1992], I.9.

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The general logic common to all topoi can be codified in a traditional, deductive system of higher-order logic, essentially by replacing $\leq$ with a deducibility relation (with $\vdash p$ being $true \vdash p$ and hence equivalent to $p = true$), collecting up the adjunctions (1)–(6) as rules of inference, and stating the defining adjunctions as logical axioms. The result is an elegant formulation of higher-order logic, *i.e.*, simple type theory, except of course that the logic of topoi is in general intuitionistic.[27]

Now a noteworthy theorem asserts that various topoi correspond in a natural way to different theories (in the technical sense) in higher-order logic, when the latter is formulated as just mentioned. In fact, every such theory generates a topos in which its theorems are true propositions, and conversely the internal logic of every topos determines such a theory which, in turn, generates a topos equivalent to the original one. As might be guessed, this correspondence too is by a pair of adjoint functors. This theorem provides strong confirmation that the definition of a topos does indeed 'adequately characterize' the notion of logical structure.[28]

The theorem just mentioned also points out the fact that structural characteristics of the objects in a topos are reflected in the internal logic of the topos. For example, in topoi the law of excluded middle in the form $\forall_p(p \vee \neg p) = true : 1 \to P$ is equivalent to the statement, for every monomorphism $m : M \to X$ there is another $m' : M' \to X$ such that $M + M' \simeq X$ (+ for coproduct). Furthermore, in topoi excluded middle has been shown to follow from the axiom of choice.[29] From a structural point of view, choice says that every epimorphism[30] $e : E \to X$ in the topos 'splits'; *i.e.*, that there is an $s : X \to E$ with $e \circ s = 1_X$. This in turn is equivalent to the statement that every object in the topos is 'projective', a condition familiar from homological algebra as a way of saying that something has a particularly simple structure.

The interplay between the logic and structure of objects in a topos permits, for example, simple logical and structural characterizations of the category of sets. Indeed, **Sets** is a topos, and it is distinguished by the fact that its objects bear a particularly 'pure' logical structure. In addition to all epis splitting there, **Sets** is bivalent; *i.e.*, there are exactly the two propositions $true, false : 1 \to P \simeq 2$. The logic of **Sets** is then pure, classical logic, for which the following 'semantics' hold for any propositions

[27] Cf. Lambek and Scott [1986] and Fourman [1977], also for what follows.

[28] This representation theorem thus relates the topos approach to higher-order logic to that *via* deductive systems in much the way that Cayley's theorem relates abstract group theory to the study of transformation groups. One consequence is that higher-order logic is deductively complete with respect to models in topoi, in contrast to its Gödel incompleteness with respect to models in the category of sets.

[29] Diaconescu [1975].

[30] $f : X \to Y$ is by definition an *epimorphism* iff for any $\alpha, \beta : Y \to Z$, $\alpha \circ f = \beta \circ f$ implies $\alpha = \beta$. In **Sets**, the epis are exactly the surjections.

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p, q, and propositional function $\psi : X \times Z \to P$:

$$\begin{aligned}
\neg p = \mathit{true} &\text{ iff } p \neq \mathit{true},\\
p \wedge q = \mathit{true} &\text{ iff } p = \mathit{true} \text{ and } q = \mathit{true},\\
p \vee q = \mathit{true} &\text{ iff } p = \mathit{true} \text{ or } q = \mathit{true},\\
p \Rightarrow q = \mathit{true} &\text{ iff if } p = \mathit{true} \text{ then } q = \mathit{true},\\
\exists_x.\psi(x,z) = \mathit{true}_Z &\text{ iff for some } u : Z \to X, \psi(u,z) = \mathit{true}_Z,\\
\forall_x.\psi(x,z) = \mathit{true}_Z &\text{ iff for all } u : Z \to X, \psi(u,z) = \mathit{true}_Z.
\end{aligned}$$

Conversely, any topos in which the above 'semantics' hold is bivalent and has choice, *i.e.*, all epis split.[31]

We thus have equivalent logical and structural characterizations of the category of sets (ignoring axioms of infinity for present purposes). Adding bivalence and choice to the topos axioms, the local $\in_X$ relation then permits a logical-style set theory which is adequate for most mathematical purposes involving sets (outside set theory itself), and is in fact equivalent in a strong sense to a certain modification of conventional Zermelo-Fraenkel set theory.[32] These axioms thus provide a structural characterization of the category of sets through an elementary axiomatization of set theory in terms of functions between sets, with composition as a primitive notion, in contrast to conventional elementary set theory based on the primitive relation $\in$.[33] This of course is just how we would choose to characterize the kind of structure 'set', given the discussion of §1. Note that this approach does not produce distinguished sets $1 = \{\emptyset\}, P = \{\emptyset, \{\emptyset\}\}$, *etc.*, nor distinguished products or power sets, for these sets are determined only up to isomorphism, and isomorphic sets are structurally indistinguishable. The structural approach thus yields only sets consisting of *lauter Einsen*, themselves mere points with no internal structure.

Conclusion

On reflection, it is not so surprising that set theory, and logic generally, should admit such a structural treatment. The Frege-Russell conception of logic was based on a 'functional' analysis of predication, while the structural approach simply employs a further analysis of 'functionality' via the notion of a category. Under that analysis, set theory's $\in$ is related to predication, and the notion of a set to that of a property, in a perspicuous way.

[31] Routine proof using the theorem of Diaconescu cited. In particular, the clause for $\exists_x.\psi(x,z)$ makes epis split; for if $e : X \to Z$ is any epi, then the 'graph' $\langle 1_X, e\rangle : X \to X \times Z$ is a mono, with a propositional function $\psi(x,z)$ such that $\exists_x.\psi(x,z) = \mathit{true}_Z$. Applying that clause gives a morphism $u : Z \to X$ which splits e.

[32] See Johnstone [1977], 9.3 for a precise statement and proof of an 'equiconsistency theorem' via a correspondence between models.

[33] An equivalent categorical set theory was first presented in Lawvere [1964]. While we have not stressed it, all of the axioms involved (category, topos, *etc.*) are indeed elementary, *i.e.*, they can be directly stated in conventional first-order logic.

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Perhaps unexpected, however, is the reappearance of 'logical types'; the local character of membership in a topos—and of quantification, lambda conversion, and the other logical operations—results in a conception closer to the early, type-theoretical approach than to axiomatic set theory.

Whether construed as 'pure', classical, higher-order, logic or 'structural' set theory, the axioms for a bivalent topos with choice (perhaps including the natural numbers as determined in §1) provide a structural axiomatization of the category of sets and so a structural 'foundation for mathematics', to the extent that one takes set theory as such. I do not think too much should be made of this. For one thing, the structural perspective is at odds with the idea that all mathematical objects exist in a single, comprehensive universe of sets. The methods of topoi make it more natural to study the 'local' logic of various kinds of structure than to force all kinds of structure into a 'global' logic, say of the category of sets. Moreover, the very idea of 'foundations of mathematics' becomes less significant from a categorical perspective than, say, organizing and unifying the language and methods of mathematics. Such unification often points up related structural phenomena in disparate fields, suggesting fruitful directions for further research.

A striking example is the invention of topos theory itself, which has brought to light some unexpected connections between logic and geometry. Our approach to topoi was motivated by logic, but this is not their only interpretation; in fact, it is not even the principal one! Topoi first arose in the Grothendieck school of algebraic geometry, where they appeared as generalized topological spaces. Their logical aspects were first studied by F. W. Lawvere and M. Tierney, who then used topoi to give a 'geometric' proof of the independence of the continuum hypothesis.[34] The fruitfulness of the uniform, structural approach is also nicely illustrated in the case of logic by the profound analysis of quantifiers as adjoints, which subsumes what is perhaps the most significant discovery of modern logic under a general notion that permeates mathematics.

The structural perspective on mathematics codified by categorical methods might be summarized in the slogan: The subject matter of pure mathematics is invariant form, not a universe of mathematical objects consisting of logical atoms. This trivialization points to what may ultimately be an insight into the nature of mathematics. The tension between mathematical form and substance can be recognized already in the dispute between Dedekind and Frege over the nature of the natural numbers, the former determining them structurally, and the latter insisting that they be logical objects.[35] My aim here was not to make the case for philosophical structuralism, but to suggest that it be pursued using a technical apparatus other than that developed by logical atomists since Frege, one with a

[34] Lawvere [1970], Tierney [1972].

[35] See Tait [forthcoming] for a careful discussion of this issue.

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mathematical heritage sufficiently substantial, and mathematical applications sufficiently uniform, to render significant a view of mathematics based on the notion of 'structure'.[36]

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ABSTRACT. A precise notion of 'mathematical structure' other than that given by model theory may prove fruitful in the philosophy of mathematics. It is shown how the language and methods of category theory provide such a notion, having developed out of a structural approach in modern mathematical practice. As an example, it is then shown how the categorical notion of a topos provides a characterization of 'logical structure', and an alternative to the Fregean approach to logic which is continuous with the modern structural approach in mathematics.

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