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From Cohomology in Physics to q-connectivity in Social Science

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If we wish to be "scientific" in our approach to the Social Sciences then, in some way, we must want to make these disciplines more like the Physical Sciences. But here we meet the immediate snag that our intuition runs away from the thought that, as individual people, we can be adequately described as so many electrons and protons. This also leads to the fear that perhaps Physics is not formulated in such a manner as to show us the heart of its methods, as opposed to the fruits of its labours.

This paper is a review of a personal search over many years to find a formulation for physical science which would not do too much violence to accepted theories and yet show a way for its extension and generalization into fields of social science. Since the latter seems to require a language which mathematicians would call "combinatorial", being concerned with finite sets, it became a search for a mathematical language which would describe certain key properties of the familiar continuum but which would carry these over when that continuum should be replaced by a finite set of points.

The language which exhibits these properties is the twentieth-century development of algebraic topology (what used to be called combinatorial topology), and some of its basic concepts are referred to in the first half of the paper.

From this point of view an indication of the role of the cocycle in physics is first developed (although many of the intriguing details are given elsewhere), and this is replaced by the same idea but referred to a relation between finite sets. The notion of the simplicial complex is then developed as the vehicle for that sense of structure which is inherent in either the laws of physics or the behaviour of social systems. The result is that one finds, when applying it to some specific field of enquiry, that it is peculiarly powerful for the representation of social or human activities. It leads too to a view of data which is structural (in a multi-dimensional space) in a way which the statistical view is unable to penetrate. Hopefully the method will help to develop new techniques for understanding the data of relations.

1. The Illusion of the Continuum in Physics

When Descartes married the geometry of Euclid to the algebra of the reals he started a revolution in mathematical and scientific thinking which has

lasted for 400 years. More precisely, he postulated that the space of our physical experience (the so-called real space) can be identified with the three-dimensional geometry of Euclid. Furthermore he implied that the symbolism of elementary algebra provided a complete structural framework for that space. This meant that the points of real-space were to be set into a (1–1) "onto" correspondence with elements of the algebraic structure $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$; where $\mathbb{R}$ denotes the set of all real numbers and $\oplus$ denotes the algebraic direct sum. Since the backbone of this correspondence lay in the concept of the Pythagorean metric (the assertion of the truth of Pythagoras' theorem amounts to the completion of the rationals by adjoining the irrationals) it amounted to defining the real-space in the following topological-cum-algebraic way.

Real-space is a set of entities called *points* $\{P\}$ together with

(i) a *bijective map* $\rho : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\{P\}$, $\mathbb{R}$ denoting the continuum of the real numbers,

(ii) the pythagorean metric map $d : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, viz.,

$$d(x,y,z) = +\sqrt{(x^2+y^2+z^2)}.$$

This map ρ then provides an algebraic structure for $\{P\}$ via the induced isomorphism $\rho^* : \{P\} \cong \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$, whilst the metric d provides a topological structure for $\{P\}$ via the natural *metric topology*.

The complete collection of structures on the set $\{P\}$ is what we now mean by Euclidean three-space and which we denote by E^3. We shall try to avoid confusion by allowing this to be called real-space (because of the role of the real number system) but by coining the term *actual-space* for what we can observe (by whatever means).

Thus Descartes made the astonishing claim that

$$\text{actual-space is } E^3$$

and this idea has dominated physical theories to the present day, certainly to the beginning of the twentieth century. With the honourable exceptions of Leibniz, Hamilton, Cayley and Clifford, the eighteenth- and nineteenth-century mathematicians pursued this Cartesian assumption with an uncritical faith which finally killed the revolution and set up its own establishment of reaction; as one would expect. Finally, the facts of physical observations led to the fundamental rethinking of, on the one hand, Einstein relativity theory and, on the other hand, Quantum theory.

But the Pythagoreans knew that irrational numbers—as predicted by Pythagoras' theorem—are not observable quantities. The latter are rational

numbers (or the *ratio* of integers) only; $\sqrt{(2)}$ is not a number of the form p/q, p,q positive integers (without common factors). Why was this piece of ancient Greek knowledge the only piece which the scientists chose to ignore?

The argument that the physicist cannot measure, say $\sqrt{(2)}$, in the laboratory is countered, at the elementary level, by saying that it is possible to construct an isosceles right-angled triangle with equal sides of length 1. Then, by *Pythagoras' theorem*, the hypoteneuse *must be* of length $\sqrt{(2)}=\sqrt{(1^2+1^2)}$. But this clearly begs the question since Pythagoras' theorem can only be appealed to in actual-space if some successful measuring process has already established its validity. If the hypoteneuse is actually measured by the same devices as are used to measure the other two sides then the result will not be $\sqrt{(2)}$—in the sense that the mathematician understands that symbol.

Admittedly the measurements of AC by progressively sophisticated rulers might give the rational numbers

$$\frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \text{ etc., but the}$$

"etc"., is only a *finite set* of rationals.

As the ancient Greeks well-knew, AC in actual-space (in the physical laboratory) is a rational number. Hence Pythagoras' theorem cannot be regarded as law of (observable) nature.

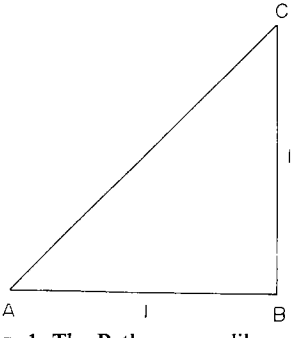


FIG. 1. The Pythagorean dilemma.

In mathematics (of a certain specified kind) the assertion of Pythagoras' theorem is permissible; it is part of a theoretical abstract structure. But in physics the assertion of *Pythagoras' Law* is not permissible.

But of course it is very tempting to "save the day" by pointing out that the act of measurement of the length AC at least gives us an approximate value and that, for *all practical purposes*, the degree of approximation is perfectly

adequate: we can "get as close as we need to" in measuring AC. So why not accept this approximation argument and use the reals $\mathbb{R}$ in our description of nature?

Although this might well do for measuring bits of string and sealing wax we know, to our cost, that the necessary pursuit of fine detail in physical observation has led us to abandon the reals for measurements at the atomic level. Thus no physicist would claim today that there is any point in contemplating an atomic energy level with value $h\sqrt{(2)}$, h being the rational number known as Planck's constant. There are many fundamental observations in which we must now assert the paramountcy of the rationals (actually, the integers) and the irrelevancy of the irrationals.

But, in addition to this, the idea that a rational number q is "close to" an irrational number ψ is itself dependent on the fundamental topological assumptions of the space $\{P\}$ (Hu, 1965). The Cartesian postulate has conditioned our minds to think in terms of the metric topology induced by the Pythagorean law—we might refer to it as the *Pythagorean topology* of the set $\{P\}$. Thus "near to" means "sharing a Pythagorean open set" in the metric topology on $\{P\}$. But if our actual-space consists of an observation of the points $\{P\}$ in it then if we assume the Pythagorean topology we must admit that between our rational q and irrational ψ there exists (for observation purposes) an *infinity of points* $\{P\}$, however "near" they are.

What if we observed points in a plane by instruments which allowed us to see only infinite strips (of varying width) parallel to the y-axis? Then the topology would no longer be Pythagorean (based as that is on the observation of circular discs) but would be an example of what is called a non-Hausdorff topology. Now points P_1,P_2 are "near" to each other if they share an observable strip. All points P_i on the same "vertical" line are equally near to each other; they are "coincident".

Thus "nearness" is not only a topological (mathematical) concept but is also deeply embedded in the process of observation. What kind of topology, of nearness, would we ascribe to real-space if we observed it by touch, or by probing it with feather dusters? And if we altered the topology in some such way would we find any significant role for the number system to play?

In fact our physics is a light-based physics; all our observations are subject to the light signal; we *see* the actual-space, literally. All our sophisticated instruments use the light signal for the final transmission of information to the observer. Thus the topology of actual-space must be determined by the relation between our sensory organs and the light signal. It has been shown, for example, by Zeeman (1961), that we only educate ourselves into the Pythagorean topology, and however convenient it might be for us all to be conditioned

in the same way, we must admit that the metric topology is not a fundamental *a priori* absolute of the actual-space.

In the laboratory (by which I mean to include the world of everyday experience) an observation of actual-space must be the observation of "real" things, or objects, or particles. We must adopt, as the foundation of physics, the Liebnizian view that *actual-space is the set of relationships between objects*; we cannot take as fundamental the Newtonian-Galileon view that *actual-space is absolute and real objects are observed in it*. The latter view is unrealistic as a description of the *physics* of space, however fascinating it might be as a point of philosophic discussion.

But if we are to abandon the Cartesian view what must we put in its place?

On the one hand we must clearly abandon the reals as a descriptive language and replace them by the rationals. Since the rationals are countable this amounts in many ways to replacing the reals by the positive integers.

On the other hand we must be able to so interpret the conventional theory (in E^3) that the successes are not lost. In other words is there any *combinatorial view* of natural laws in E^3 which expresses their essential truths (but which, by definition, does not involve the continuum).

2. Combinatorial Formulation of Physical Laws

It has been argued in Atkin (1971) that the natural laws of physics may be regarded as statements in a cohomology theory and that this theory is applicable equally well to finite sets of observation. In fact, the cohomology statement of a law of physics does not require the Cartesian assumption that "actual-space is E^3".

In an earlier paper (Atkin, 1965), I have explored the bilinear relation (the *homography*) as the fundamental law of macrophysics, (see also O'Hara & Ward, 1949). In particular the accepted laws of geometrical optics are illustrations of the homography

$$axy+bx+cy+d=0 \tag{1}$$

with a,b,c,d constants in R.

Since the homography preserves *cross-ratio* a set of four values of x, say, x_1,x_2,x_3,x_4, and the corresponding values y_1,y_2,y_3,y_4 of y are related by

$$(x_1x_2x_3x_4)=(y_1y_2y_3y_4)=k, \text{ a constant.} \tag{2}$$

If the set $\{x\}$ is the same as the set $\{y\}$ then it is meaningful to suppose that (1) may possess *double points*—at which $x=y$. Such double points are given by the roots of the quadratic

$$at^2+(b+c)+d=0. \tag{3}$$

When (1) possesses double points, say α and β, then all other pair of corresponding points x,x' are given by the statement

$$(x\ \alpha\ x'\ \beta)=k. \tag{4}$$

When the homography is symmetrical, so that if x corresponds to y then y corresponds to x, then it is called an *involution* and is characterized by $b=c$, or by

$$axy+b(x+y)=D=0. \tag{1a}$$

In this case (4) becomes

$$(x\ \alpha\ x'\ \beta)=-1 \tag{5}$$

so that *all pairs x,x' in the involution form an harmonic range with the double points α,β.*

Now every act of measurement consists of setting up a (1–1) correspondence between objects (or "points" in actual-space) and doing this by the aid of the light signal. This signal is a *sine qua non* of the act of measurement. Let us refer to the signal, or seeing-agent, as the *base-element* of the system which constitutes our act of observation, and denote it by ξ_o. If some other phenomenon which is observed in the system is denoted by ξ we can sum up the process of observation by saying that it is to be described by the symbol $S(\xi_o;\xi)$.

In particular we see that (cf. discussion in section 1) it is the base-element ξ_o which provides us with our basic topological structure of actual-space. This basic structure we would expect to identify with the special system $S(\xi_o;\xi_o)$—wherein *we observe the base-element via the base-element.* If we assume that $S(\xi_o;\xi_o)$ must be symmetrical then we can identify the involution range in this system and then corresponding pairs are given by (5).

This set of measures $S(\xi_o;\xi_o)$ will normally show a lack of ordering—because of the symmetry of the homography—and therefore will be time-independent. It is therefore plausible to refer to $S(\xi_o;\xi_o)$ as the set of *stationary measures*, and to associate such meaures with the involution (5).

If the measures of elements ξ_1,ξ_2 on $S(\xi_o)$ are co-ordinatized by mapping into some algebraic structure $\mathscr{A}$ (traditionally $\mathbb{R}$) then the *mappings must be defined by the experimental techniques.* The whole operation may be summarized in the diagram:

$$\begin{array}{ccc} \xi_1 & & \xi_2 \\ \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_2} \\ x_1 & \longleftrightarrow & x_2 \end{array} \qquad \begin{array}{l} \sigma_r \text{ defined by } S \\ \\ x_1,x_2\ \varepsilon\ \mathscr{A} \end{array}$$

Mappings m: $\mathscr{A}\to\mathscr{A}$ which are invariant under all σ_r (and therefore under all possible $S(\xi_o;\xi_1,\xi_2)$ will presumably express what we call the natural laws of physics.

If actual-space is observed via the light signal and instruments of elementary optics then x and y may denote co-ordinates of object and image respectively.

The various laws are then based on (for example)

(i) plane mirror $x+y=0$ (cf. (1a) with $a=d=0$),

(ii) concave/convex mirror $\frac{1}{x}+\frac{1}{y}=\frac{1}{f}$
which is $xy-f(x+y)=0$ (cf. (1a) with $d=0$).

(iii) Change of medium (introduction of refractive index μ) amounts to a transformation of co-ordinates $x\to\mu x$, etc.

The double points of the involution $x+y=0$ occur at $x=0$ and at $x=\infty$. The double points of $\frac{1}{x}+\frac{1}{y}=\frac{1}{f}$
occur at $x=0$ and $x=f/2$.

In a similar way we can see the formulation of special relativity as an example of (1)—and therefore as placed in a non-metrical context, since the theory of co-ordinatization which leads to (1) is quite independent of metrical notions. If particles are ascribed a co-ordinate called "velocity"—say, in an assigned direction—and if we contemplate two possible observers measuring the velocity of a single particle, then special relativity theory is based on two assumptions. One is that if observer S attributes a *unique* velocity v to the particle, observer S' attributes a *unique* velocity v' to the same particle. The other is that when S attributes a velocity V to S' then $(V\neq 0)$ $v=v'$ if and only if $v=v'=c$ (the velocity of the light-signal).

This is all contained in the postulate of an homography, viz.,

$$\alpha vv'+\beta v+\gamma v'+\delta=0 \tag{6}$$

with $\alpha,\beta,\gamma,\delta\,\varepsilon\mathbb{R}$.

Since we know the two pairs $(v=V,\ v'=0)$ and $(v=0,\ v'=-V)$ (6) reduces to

$$\alpha vv'+\gamma(v'-v)+\gamma V=0; \tag{7}$$

then with the double point $v=v'=c$ (7) becomes

$$Vvv'+c^2(v-v')-c^2V=0 \tag{8}$$

and this constitutes the macro-kinematics of special relativity.

It is worthy of note that the assumption that $c \to \infty$ reduces (8) to

$$v' - v + V = 0$$

which is the kinematics of Galileo–Newton.

Making (8) homogeneous by the substitutions $v = x/t$, $v' = x'/t'$, gives

$$Vxx' + c^2xt' - c^2x't - c^2vtt' = 0$$

which is

$$x'(Vx - c^2t) + t'(c^2x - c^2Vt) = 0. \tag{9}$$

Since each observer can choose his co-ordinates in a selfish and arbitrary manner we must suppose from (9) that there exists a separation constant k, independent of v, v' (but not necessarily of V, c) such that

$$k\left(ct - \frac{Vx}{c}\right) = ct'$$

and

$$k(x - Vt) = x'.$$

If we write $r = (x\ y\ z\ ct)$ these are

$$r' = L\,r \tag{10}$$

$$L = L(V) = \begin{pmatrix} k & 0 & 0 & -kv/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -kv/c & 0 & 0 & k \end{pmatrix}$$

Since we expect the roles of S, S' to be interchanged by $V \to -V$ we can take

$$L^{-1} \equiv L(-V)$$

so that $L(-V)\,L(V) = \mathrm{I}_4$. This gives

$$k^2(1 - V^2/c_2) = 1$$

so that (10) is the usual set of Lorentz–Einstein transformations. An important point of this derivation is that it is *independent of the notion of metric.*

3. The Cocycle Law Bridges Continuum Theory and Discrete Theory

The general homography over a field F, viz.

$$axy + bx + cy + d = 0$$

with $a, b, c, d, 0 \in F$, can be written in the homogeneous form

$$axy + bxs + cyt + dst = 0 \tag{11}$$

in which x,y have been replaced by x/t, y/s respectively. This can be characterized by a 2×2 matrix M

$$M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

since (11) is a scalar condition on a two-dimensional vector space V_2, viz.

$$XMY^T=0 \tag{12}$$

with $X=(x\ t)$, $Y=(y\ s)$. The condition for (11) to be non-degenerate is det $M=ad-bc\neq 0$. Hence (11) is invariant under the full linear group $LG(2)$.

We can therefore associate with the homography (11) the *exterior product space* (Flanders, 1963)

$$\Lambda V_2=\Lambda^0 V_2\oplus\Lambda^1 V_2\oplus\Lambda^2 V_2$$

in which $\Lambda^0 V_2=F$. If V_2 is spanned by $e_1=(1\ 0)$ and $e_2=(0\ 1)$ then $\Lambda^1 V_2=V$ and $\Lambda^2 V_2$ is one-dimensional with basis $e_1\Lambda e_2$. The elements of $\Lambda^2 V_2$ are *pseudoscalars*. Hence the symbol 0 in (11) may be viewed as $0\,\varepsilon\,\Lambda^0 V_2$ or as $0\,\varepsilon\,\Lambda^2 V_2$; zero is either a scalar or a pseudoscalar.

If we view $0\,\varepsilon\,\Lambda^0 V_2$ we can express the homography (11) as a one-cycle statement in an homology theory based on a boundary operator ∂ in

$$\Lambda^0 V_2 \xleftarrow{\partial_1} \Lambda^1 V_2 \xleftarrow{\partial_2} \Lambda^2 V_2$$

with $\partial_1\, o\, \partial_2=0$.

If we view $0\varepsilon\Lambda^2 V_2$ we can express the homography (11) as a one-cocycle statement in a cohomology theory based on a coboundary operator δ in

$$\Lambda^0 V_2 \xrightarrow{\delta_0} \Lambda^1 V_2 \xrightarrow{\delta_1} \Lambda^2 V_2$$

with $\delta_1\, o\, \delta_0=0$.

For the homology theory we define ∂_1 as $\partial_1^x\equiv XM=(ax+ct,bx+dt)$ so that (11) can be written as the statement "Y^T is a one-cycle", viz.

$$\partial_1^x Y^T=0$$

provided ∂_2 is defined so that $\partial_1\ o\ \partial_2=0$. This follows by defining $\partial_2=\partial_7^x$ where

$$\partial_2^x(\theta^1\Lambda\varphi^1)=\partial_1^x(\theta^1)\Lambda\varphi^1-\theta^1\Lambda\partial_1^x(\varphi^1)$$

where $\theta^1,\varphi^1 \varepsilon \Lambda^1 V_2$. Thus the homography (1) says: *given a vector* **X** *the corresponding* **Y** *is such that* Y^T *is a one-cycle in the homology on* ΛV_2 *defined by* ∂^x.

For the *cohomology theory* we first define the matrix $\overset{+}{M}$ via

$$\overset{+}{M}M=(\det M)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

so that $\overset{+}{M}$ "sees" the space ΛV_2 from "the other end", the matrix $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ being a representation of the basis $e_1 \Lambda e_2$.

This gives $\overset{+}{M}=\begin{pmatrix}-b & -d\\ a & c\end{pmatrix}$, and then we define $\delta_1=\delta_1^x$ via

$\delta_1^x=(MX^T)\Lambda$ so that the homography (11) can be written as

$$\delta_1^x Y\equiv(\overset{+}{M}X^T)\Lambda Y=0.$$

Defining $\delta_0^x=\delta_0$ where

$$\delta_0^x : k \to k(\overset{+}{M}X^T) \qquad k\varepsilon F$$

ensures $\delta_1 \, o \, \delta_0=0$ (the zero map).

Thus the homography (11) says: *given a vector* **X** *the corresponding vector* **Y** *is such that* **Y** *is a one-cocycle in the cohomology on* ΛV_2 *defined by* δ^x.

Furthermore, it is not difficult to show that, for example, "the homology is trivial" ≡ "the homography is non-degenerate". The same applies to the cohomology group since, if $(XM)Y^T=0$ then $(\overset{+}{M}X^T)\Lambda Y=0$.

The *natural law* contained in the homography (11) is a statement to the effect: *every observable macro-variable is expressible as a one-cocycle in a pattern of cohomology on an exterior product space* ΛV_2. Furthermore, the one-cocycle is a one-coboundary (the *cohomology is trivial*).

This pattern of cohomology can be traced in continuum micro-physics, where the space V_2 is replaced by a local space spanned by a set of differentials $\{dx^i\}$. That physics which is based on the formulation of differential equations can then be expressed via cocycles in a cohomology on an exterior product space $\Lambda\{dx^i\}$ under the exterior differential operator δ (De Rham, 1955). A discussion of Hamiltonian dynamics and the Maxwell equations of electromagnetism, from this point of view, will be found in various sources, e.g. Atkin (1971), Flanders (1963), also (via the theory of the Pfaff differential form) in Whittaker (1944). In Atkin (1971) the classical Hamilton–Jacobi

equation is shown to be the condition for a trivial cohomology in Hamiltonian dynamics.

The transition from classical theories of dynamics to quantum mechanics is discussed in detail in Atkin (1972) where it is shown that if the cocycle formulation of classical mechanics is modified by requiring integral values of the *action* then the Hamilton–Jacobi equation becomes cohomologously transformed into the Schrodinger wave-equation. In addition the role of special relativity theory is manifest through a modified cocycle statement so that the Dirac equations, Dirac (1934), can be understood as the conditions for two cocycles to be cohomologous.

The implications of this are that Quantum theory can be seen as a natural translation of classical (continuum) theory (in cocycle form) into one which *rejects the real-numbers as a value field for the basic observations.* Its complexities and difficulties are not so surprising when we realize that it is a theory trying to break free from a basis in the continuum, but not allowed to do so.

What is important to the thesis of this paper is that the cocycle law in its generality—that is to say *every observable is* expressible as a p-*cycle, for some integer* p—is independent of the continuum postulate, being equally meaningful in the context of countable sets (and *a fortiori* that of finite sets).

Sets for X or Y which are finite can exhibit the same homological structures as previously discussed. Thus if we return temporarily to the Euclidean plane of Descartes and consider the co-ordinates of points P which lie on a fixed straight line through the origin 0, we know that the familiar equation $y=mx$ (m a constant number) must apply. If we make the formal change $x \to x/t, y \to y/s$ this becomes

$$mxs - yt = 0.$$

The homography is therefore defined by

$$M = \begin{pmatrix} 0 & m \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \overset{+}{M} = \begin{pmatrix} -m & 0 \\ 0 & -1 \end{pmatrix}.$$

Now even if x,y,s,t may only take integral values (because of the physical observation conditions) the cocycle law $\delta_1^x Y = 0$, or the cycle law $\partial_1^x Y = 0$, has a well-defined meaning. In this case the actual-space (that due to the observational technique) will be a *mesh* of *discrete points*

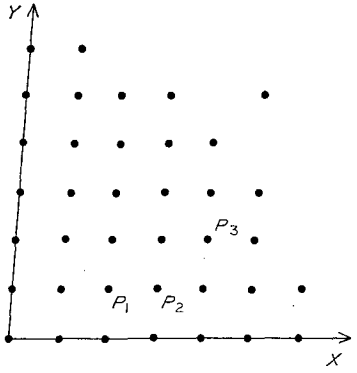


FIG. 2. A mesh of points in actual-space.

and this need not be considered as being embedded in the Euclidean plane—or in any other continuum geometry based on $\mathbb{R}\times\mathbb{R}$.

4. The Simplicial Complex of Observations

If we begin with the postulate that all observation (whether in physical or in social science) is a matter of *identifying set-membership in a finite set*, then we can take the mesh of observed points (see above) as the result of observing actual-space in two dimensions. Such an observation must rest on a choice of base-element (usually the light signal) and a phenomenological interpretation of the intuitive "point" in $\{P\}$. At this fundamental level the set $\{P\}$ is without structure (questions of metric, linearity, betweenness, do not arise). But, anticipating the vocabulary and structure which we shall use later on, we shall describe this initial actual-space set of points as a *simplicial complex* $K(P)$ in which the only simplices are zero-simplices. Each point P is a zero-simplex σ_0 in the collection of simplices $K(P)$. It can be described technically, and intuitively, as totally disconnected.

If now we allow that the process of observation sets up a relation between the sets $\{P\}$ and $\{A\}$—where $\{A\}$ is the set of all the physically possible phenomena allowed by the technique—which is not merely diagonal, then the complex $K(P)$ can acquire a great deal more structure. As an example we have in mind the possibility that the observing technique might allow us to "see" the point-pair $\{P_iP_j\}$ or the point-triple $\{P_1P_2P_3\}$—which might correspond to two possible A's in the set $\{A\}$. Then the complete picture of the actual-space, for example, will be contained in some such relation λ between the sets $\{P\}$ and $\{A\}$ so that we write

$$\lambda \subset \{A\}\times\{P\}.$$

We regard each set $\{P\}$ or $\{A\}$ as finite; for generality rename them sets Y and X, with members X_i $(i=1, \ldots n)$ and Y_j $(j=1, \ldots m)$. Then observation defines a relation λ, in the sense that $\lambda \subset Y \times X$, and equally well in inverse relation λ^{-1}, in the sense that $\lambda^{-1} \subset X \times Y$. The relation, which is a binary relation, may be represented by a matrix array with entries 0,1; this is summarized in the notation

$$\begin{array}{c|c} \lambda & X \\ \hline Y & \Lambda=(\lambda_{ij}) \end{array} \tag{13}$$

where (λ_{ij}) denotes this *incidence matrix*. The incidence matrix of the inverse relation λ^{-1} is the *transpose* (τ_{ij}) with $\tau_{ij}=\lambda_{ji}$.

Associated with (1) are two simplicial complexes which we shall denote by $K_Y(X;\lambda)$ and $K_X(Y;\lambda^{-1})$, defined as follows.

(i) $K_Y(X;\lambda)$ is a collection of simplices $\{\sigma_p\}, p=0,1, \ldots N$;

(ii) each $\sigma_p \varepsilon K$ is a subset of $(p+1)$ distinct X_i provided there is at least one $Y_k \varepsilon Y$ such that $(X_i, Y_k) \varepsilon \lambda$ for each of the $(p+1)$ values of i;

(iii) the σ_o^i are identified with X_i, $i=1, \ldots n$;

(iv) every $(q+1)$ subset of a σ_p $(q \leqslant p)$ is called a q-face of σ_p and defines a $\sigma_p \varepsilon K$.

The finite number N in (i) is called the *dimension* of K and written dim K. It denotes the largest dimension of any $\sigma_p \varepsilon K$. The set of $\{X_i\}$ is also called the *vertex set* of $K_Y(X;\lambda)$. We notice too that each simplex $\sigma_p \varepsilon K$ corresponds to at least one $Y_k \varepsilon Y$ by (ii). When there is no danger of confusion we might speak loosely of the set Y denoting the simplices of $K_Y(X;\lambda)$—hence the notation.

In a similar way, if we let Y be the vertex set, λ^{-1} gives us the *conjugate complex* $K_X(Y;\lambda^{-1})$, in which the X_i denote simplices (in the above sense).

As a simple example let us look at the relation λ between sets X and Y defined by the matrix:

λ	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
Y_1	1	1	1	1	0	0	0	0	0
Y_2	0	0	0	0	0	1	0	0	1
Y_3	0	1	1	0	1	0	0	0	0
Y_4	0	0	1	1	1	0	0	0	0
Y_5	0	0	0	0	1	1	0	0	0
Y_6	0	0	0	0	1	0	1	1	0

Considering $K_Y(X;\lambda)$.

$<X_1X_2X_3X_4>$ constitutes a σ_3 since each vertex is related to Y_1

$<X_6X_9>$ constitutes a σ_1

$<X_2X_3X_5>$, $<X_3X_4X_5>$ and $<X_5X_7X_8>$ constitute three σ_2's

$<X_5X_6>$ constitutes a σ_1 and each $<X_i>$ constitutes a σ_0.

Thus we could say $K_Y(X;\lambda)=\{\sigma^i\} \cup \{\sigma_1{}^j\} \cup \{\sigma_2{}^k\} \cup \{\sigma_3\}$

with $i=1, \ldots 9; j=1,2; k=1,2,3$.

It is often helpful to think of a *geometrical realization* of the complex in which each σ_p is a closed convex polyhedron of $(p+1)$ vertices (in a suitable space E^H). Then a σ_3 becomes a tetrahedron (and all its faces), a σ_2 becomes a triangle (and all its faces) and a σ_1 becomes and edge (and its end-points). A geometric realization of (14) is given as follows:

$K_Y(X;\lambda)$

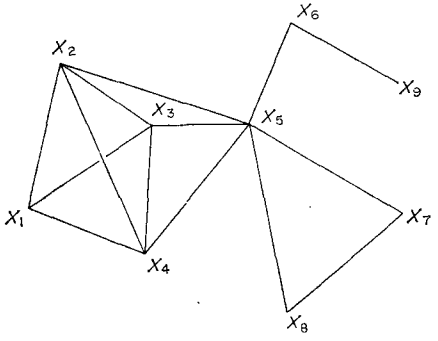


FIG. 3. The geometry of $K_Y(X;\lambda)$.

$K_X(Y;\lambda^{-1})$

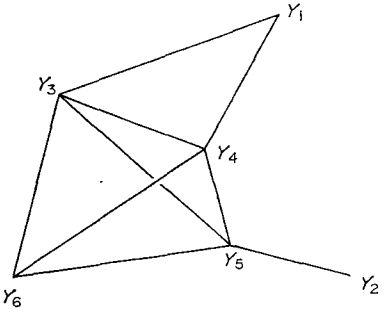


FIG. 4. The geometry of $K_X(Y;\lambda^{-1})$.

In each figure the tetrahedron is solid, so is each triangle except $X_2X_4X_5$ in $K_Y(X;\lambda)$—since $<X_2X_4X_5>$ is not a simplex in K.

Given such a complex K of simplices then there is a standard mathematical technique which leads to homology and cohomology theory on it. We give a brief sketch of the main ideas of this technique (see Hilton & Wylie, 1962). First, the vertex set, say X, is totally ordered; this identifies ± 1 with any selection of the X's (via even/odd permutations) and therefore induces an ordering (or *orientation*) of any simplex σ_p. Then the $\sigma_p{}^i$, for fixed p, are used to generate a free abelian group C_p; any $x \varepsilon C_p$ is the formal sum $\sum_i n_i \sigma_p{}^i$, with $n_i \varepsilon J$, the ring of integers. The members of C_p are called p-chains, and the boundary of a p-simplex σ_p, written $\partial\sigma_p$, is a $(p-1)$-chain defined (typically) by

$$\partial\sigma_p = \partial <X_1X_2 \ldots X_{p+1}> = \sum_i (-1)^i <X_1X_2 \ldots \hat{X}_i \ldots X_{p+1}>$$

where $\hat{X}_i$ means that X_i is omitted. This ∂ is a nilpotent homomorphism $C_p \to C_{p-1}$ ($\partial^2=0$) and $im\partial$ ($=B_p$) is a subgroup of C_{p-1} called the bounding cycles (or boundaries) in C_{p-1}. Any p-chain $Zp \varepsilon C_p$ such that $\partial Z_p = 0$ is called a p-cycle. Hence every boundary is a cycle and since $ker\ \partial (=Z_p)$ is a subgroup of C_p we see that B_p is a normal subgroup of Z_p. The factor group

$$H_p = Z_p/B_p \text{ or } ker\ \partial/im\ \partial$$

is called the pth. *homology group* (strictly the pth. integral homology group since C_p has coefficients in J). Since generally $H_p \cong G_p^o \oplus Tor_p$ where G_p^o is a free group and Tor_p is the torsion group, the rank of G_p^o is called the pth. *Betti number*, β_p. It is worthy of note that if the complex is arc-wise connected (as our examples above are) then $\beta_0 = 1$; β_0 equals the number of connected components of K.

In the example of $K_Y(X;\lambda)$ above there is no non-bounding cycle. Hence we get

$$H \cong J;\ H_1 \cong 0;\ H_2 \cong 0.$$

Similarly in $K_X(Y;\lambda^{-1})$ there are no non-bounding one-cycles, so that

$$H \cong J;\ H_1 \cong 0;\ H_2 \cong 0.$$

If we consider the dual modules C^p to C_p, of all homomorphisms from C_p to (say) J, we obtain a cohomology theory. Denote by c^p any member of C^p so

that $c^p: C_p \to J$ (or $C_p \in \mathscr{J}^{C_p}$) and let (c_p, c^p) be the value of c^p on c_p. Then δ defines an adjoint coboundary operator $\delta: C^p \to C^{p+1}$ such that

$$(\partial c_p, c^{p-1}) \overset{Def}{} (c_p, \delta c^{p-1}).$$

The *cohomology groups* H^p are given by

$$H^p = ker\ \delta / im\ \delta = Z^p / B^p.$$

With that very brief outline let us return now to the postulated mesh of discrete points in our observed actual-space. If the observational technique allows (say) some point-pairs (not necessarily all point-pairs) to be observed then (e.g.) the simplicial complex of observations $K_A(P;\lambda)$ will contain the one-simplex $\sigma_2 = <P_1P_2>$. This allows us to entertain the notion of *connectedness* in our actual-space, strictly arc-wise connectedness. If $K_A(P;\lambda)$ contains (say) a $\sigma_2 = <P_1P_2P_3>$ then we may allow the introduction of concepts which are two-dimensional-based (cf. concept of Euclidean area). But also, if it is physics which we insist on studying, the existence of at least one σ_1 allows the notion of *geometrical* (spatial) *displacement*, *velocity*, possibly *acceleration*—because each of these requires a point-pair, or a σ_1. Let us issue ourselves with an important caveat: a *point-pair*, or σ_1, $<P_1P_2>$ is *qualitatively distinct from the pair of points* $<P_1> \cup <P_2>$, and this is why we must allow for it by way of its being a characteristic of the observational technique, of the system $S(\xi_o;\xi_r)$. The role of the base-element ξ_o in this can obviously be crucial.

By way of example suppose our observer stands at a point O on a military parade ground before 32 cannon (loaded with blanks) and that a point P in real-space is to be "the sight of the muzzle of a firing cannon". Then he sees (presumably) a space of 32 points $\{P\}$ and suppose that he feels able to see neighbouring point-pairs, that is to say, $<P_1P_2>$ is a simplex in $K_A(P;\lambda)$ and only if P_1, P_2 are neighbouring cannon. This might constitute the whole of K—since he might not be able to decide whether or not three cannon explode simultaneously. Thus $K_A(P;\lambda)$ consists of 32 zero-simplices and 31 one-simplices. This is an observational system $S(\xi_o;P)$ where ξ_o is the light signal.

What happens to $K_A(P;\lambda)$ if now our observer closes his eyes and *listens* to the explosions; when ξ_o is then the sound-signal? To make the point more obvious suppose the chief gunner had all along arranged for the even-numbered cannon to trigger noiseless electric lights rather than explosives. Then our observer "sees" $K_A(P;\lambda)$ as a complex with only 16 zero-simplices, 0 one-simplices (at the worst).

In the actual-space complex based on our mesh the *speed* of a body which is associated with P_1, P_2 at times t_1, t_2 is an example of a mapping $v: <P_1P_2> \to \mathscr{Q}$ (the rationals). Such a mapping we shall call a *co-simplex*, not a cochain

because it is not a homomorphism: $C_1 \to \mathcal{Q}$—in general, $v\{<P_1P_2>+<P_2P_3>\} \neq v<P_1P_2>+v<P_2P_3>$. Whereas displacement often behaves like a cochain when, for example, $d\{<P_1P_2>+<P_2P_3>\}=d<P_1P_2>+d<P_2P_3>$—constituting the familiar *parallelogram law*. By linking the numerical speed to the notion of displacement we can obtain a similar cochain-type *velocity*-vector. In a similar way we commonly associate other physical concepts with appropriate simplices in $K_A(P;\lambda)$ and an important class of such concepts are those genuine cochains commonly represented by integrals over real-space. The cochain c^p has the formal properties of an integral in the sense that the value of c^p on c_p, viz. (c_p,c^p) can be thought of as a suitable integral $\int_{c_p} c^p$. But whereas an integral is well-defined, and makes intuitive sense, on a general chain c_p (which might consist of disconnected simplices $\sigma_p{}^i$), *the example of vector-displacement shows that some quantities must be normally tied to certain kinds of connected chains* c_p. Thus we can hardly apply the parallelogram law to the sum $<P_1P_2>+<P_3P_4>$ when $P_3 \neq P_2$.

This suggests that *the significance of the complex* $K_A(P;\lambda)$, *in its most general occurrence, lies in a detailed view of its inner connections*—and not only in its homological structure. Accordingly we look for this inner connectedness via the concept of q-*connectivity*, defined in the next section. Such a notion can be regarded as giving us (more of) a *local* view of the complex; homology gives us a *global* view.

5. The Definition of q-Connectivity

Given two simplices σ_p,σ_r of a complex K, we shall say that they are joined by a *chain of connection* if there exists a finite sequence of simplices

$$\sigma_{\alpha_1}, \sigma_{\alpha_2}, \ldots \sigma_{\alpha_h}$$

such that

(i) σ_{α_1} is a face of σ_p;

(ii) σ_{α_h} is a face of σ_r;

(iii) σ_{α_i} and $\sigma_{\alpha_{i+1}}$ have a common face (say σ_{β_i}) for $i=1, \ldots (h-1)$.

We shall say that this chain of connection is a q-*connectivity* if q is the least of the integers

$$\{a_1,\beta_1,\beta_2, \ldots \beta_{h-1}, a_h\}.$$

As a special case we see that a simplex σ_p must be regarded as p-connected to itself, since a chain of connection exists consisting of σ_p only. Thus σ_p is p-connected but not $(p+1)$-connected.

It is not difficult to prove that if σ_p and σ_r are q-connected then they are also $(q-1)$—, .., 1—, 0-connected in K. The zero-order Betti number β_o denotes the numbers of 0-connected (i.e. arc wise connected) components in K. The process of identifying those pieces of K which are q-connected for $q>0$ can now be seen as a new generalization of β_o, away from the usual homology theory.

If we define a relation γ_q as meaning "is q-connected with" then γ_q is an *equivalence relation* on the simplices of K. The classes of γ_q are now the pieces of K which are separately q-connected. If the cardinality of the classes of γ_q is Q_q we see that, although $Q_o=\beta_o$, in general $Q_q \neq \beta_q$, $q \neq 0$.

Considering (14) of section 4 we have the analysis:

For $K_Y(X;\lambda)$.

$Q_q=0$ for $q>3$;

$Q_3=1$; there exists one $\sigma_3=<X_1X_2X_3X_4>$ only

$Q_2=4$; components are σ_3, $<X_2X_3X_5>$, $<X_3X_4X_5>$ and $<X_5X_7X_8>$

$Q_1=4$; components are σ_3 $<X_2X_3X_5>$ $<X_3X_4X_5>$, $<X_5X_7X_8>$, $<X_5X_6>$ and $<X_6X_9>$

$Q_o=1$; since K is 0-connected.

For $K_X(Y;\lambda^{-1})$

$Q_q=0$ for $q>3$

$Q_3=1$; there is one $\sigma_3=<Y_3Y_4Y_5Y_6>$

$Q_2=2$; components are σ_3 and $<Y_1Y_3Y_4>$

$Q_1=2$; components are σ_3, $<Y_1Y_3Y_4>$ and $<Y_5Y_2>$

$Q_0=1$; since K is 0-connected.

Thus, although we know that $K_Y(X;\lambda)$ and $K_X(Y;\lambda^{-1})$ are homologically isomorphic, Dowker (1952), it is clear that a similar isomorphism cannot be assumed under a q-connectivity analysis (or Q-analysis).

Finally, this idea of q-connection may be alternatively described in terms of algebraic category theory (Mitchell, 1965), as follows.

(a) The simplices $\{\sigma_p\}$ of K may be viewed as *objects* in an algebraic category Γ_k.

(b) The morphisms of Γ_k are pairs (σ_p,σ_r) which are q-connected, for some q.

(c) The q-connectivity defines a *filtration* on Γ_k in the sense that

$$\Gamma_k = \bigcup_q \Gamma_k^q$$

and $\Gamma_k{}^q$ denotes that sub-category composed of pairs (σ_p,σ_r) which are q-connected. Then we can write

$$\Gamma_k=\Gamma_k^o\supset\Gamma_k{}^1\supset\ldots\supset\Gamma_k{}^n$$

where n=dim K.

6. Physical Science and Social Science via some $K_A(\mathbf{P};\lambda)$

Even social scientists seem to be convinced that the people and institutions of a society manifest themselves in terms of relations. It is perhaps not so clear as to whether they always have in mind the mathematical expression of these relations. But if not, then we can only urge them to do so. By setting up a series of complexes $K_A(P;\lambda)$ and studying both the local and global connectivity patterns we would expect a big pay-off in the shape of sharper definitions and unambiguous theories.

Suppose we take a simple-minded view of the social scene and notice that, for example, a given n people $P_1P_2,\ldots P^n$ share, in varying degrees, a certain set of m attributes (or activities) $A_1,A_2,\ldots A_m$. This may be described by a relation $\lambda\subset A\times P$ and a consequent complex of observations $K_A(P;\lambda)$—and by implication $K_P(A;\lambda^{-1})$.

If A_1 is the attribute of " being a student " then the finite set of people who share this attribute form a simplex A_1 in $K_A(P;\lambda)$. Some of these people might happily share the attribute A_2 of " being a diligent worker " in which case the simplex A_2 will thereby be q-connected (for some q) to A_1.

If P_1 is a person who is entirely defined (in this context) by the attributes A_3 (the activity of parenthood), A_4 (salaried worker), A_5 (local politician) and A_6 (enjoying classical music), then P_1 is a σ_3 (a tetrahedron) in $K_P(A;\lambda^{-1})$. If P_2 is entirely defined by A_1,A_3,A_6 then he is a σ_2 (a triangle) in the same complex. A geometrical representation of this piece of $K_P(A;\lambda^{-1})$ is given in Fig. 5.

Now P_1 and P_2 (represented by their simplices $<A_3A_4A_5A_6>$ and $A_3A_6A_1>$) are q-connected at the level $q=1$; they share a one-simplex $<A_3A_6>$. The *face* (in mathematics and in common speech) which P_1 "turns to" P_2 is $<A_3A_6>$. In this context it is the only face which P_1 and P_2 can share. Any dialogue between P_1 and P_2 must take place "on" (or "through") $<A_3A_6>$. Hence they converse with each other by discussing matters of parenthood and/or the delights of classical music.

The implications of this structure are clear and profound. The chains of connection in both conjugate complexes have immediate significance for the social milieu. *They define a kind of structural social cement*; if they do not exist then neither does the society. On the one hand $K_P(A;\lambda^{-1})$ gives us a

view of the society which is *person-oriented.* On the other, $K_A(P;\lambda)$, gives us a view which is A-oriented (whatever the set A might comprise). Thus we can obtain a specifically political, cultural, educational, industrial, etc., view of society by suitable choices of A.

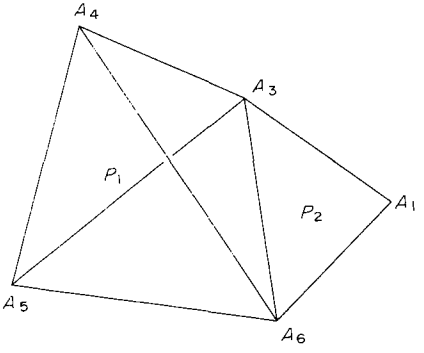


FIG. 5. P_1 and P_2 speak through $(A_3\ A_6)$.

As another example let us take a standard organization in which there is an elaborate "decision-making process" in the shape of n committees $C_1 \ldots C_n$, and suppose there are m "areas of policy" $A_1 \ldots A_m$ in which the committees function. Hence committee C_1 might consider A_1 (limited financial decisions in the area of plant maintenance), A_2 (appointment of junior secretaries), A_3 (social amenities for after-hours), etc. When C_1 makes (in addition) recommendations to a "higher" committee then it is sharing certain areas of policy. Now the connectivity analysis of $K_A(C;\lambda)$ gives a policy-area-oriented view of the system, whilst a Q-analysis of $K_C(A;\lambda^{-1})$ gives a committee-oriented view of the system. If there is one supra-committee to which all other committees must send recommendations on all matters then this one committee will be a σ_m, $m=\dim K_C(A;\lambda^{-1})$, and every other committee will be a face of this σ_m. In this special case $K_C(A;\lambda^{-1})=\sigma_m$, and incidentally,

$$Q_m=1,\ Q_r=1 \text{ for } 0 \leqslant r < m.$$

Chains of connection in this $\mathrm{K_C(A;\lambda^{-1})}$ *must act literally as the channels of communication for discussion and decision.*

These examples show up the interesting parallel between observations in a social science context and those in physical science (see also Atkin & Bray, 1972). In the latter we have seen that the cycle/cocycle seems to express the idea of a natural law, independently of assumptions of continuity or metric

under $\mathbb{R}$. In the former we can see that complexes $K_A(P;\lambda)$, etc., seem to be a natural language to posit the problems, and notions of q-connectivity express intuitive relations which are the counterpart of the physicists backcloth of real-space. Whereas the physicist's system of observation $S(\xi_o;\xi)$ is a rigid system which springs from properties of the inanimate elements ξ_o,ξ, etc., the social system is one which (perhaps) could be denoted by $S(P_o;P)$. The P_o denotes the observer himself, who is a base-element among all other elements (people) P. In this case *the complexes which a social scientist must consider need not be rigid frameworks but can grow and change.* Perhaps biologists might like to ask whether this is the crucial difference between the animate and the inanimate. The idea is contained in the following scheme of things.

Physical science	*Social science*
Select a base-element ξ_o (usually light signal).	Select an observer P_o.
Set up observing system $S(\xi_o;\xi_r)$.	Set up observing system $S(P_o;\xi_r)$.
Observe finite sets of some $\xi_k \equiv A$ and $\xi_h \equiv P$, and set up a base-complex for reference $K_A(P;\lambda)$ (usually real-space).	Observe finite sets of some $\xi_k \equiv A$ and $\xi_h \equiv P$, and set up a base-complex $K_A(P;\lambda)$.
q-chains in $K_A(P)$ and $K_P(A)$ describe connectivity structure of the base-complex.	q-chains in $K_A(P)$ and $K_P(A)$ describe connectivity structure of the base-complex.
All other ξ_r ($r \neq h,k$) are represented by cochains $c^p: K \to J$.	All other ξ_r are represented by co-simplices $\sigma^p: K \to J'$.
Natural laws are $\delta z^p = 0$.	Natural laws not known.
Global and local structure of base-complex is rigid.	Structure of base-complex is fluid.

7. The Idea of a Q-space Associated to a Complex

We notice that in a complex K the numbers Q_q are not independent of each other. This is because of the fact that connectivity defines a filtration, not a gradation. Thus since $Q_n = 1$ (at least), when dim $K = n$, it must follow that $Q_{n-1} \geqslant 1, \ldots Q_o \geqslant 1$. If $Q_o = 1$ the complex K is in one piece, and the numbers

$$Q_n - 1, Q_{n-1} - 1, \ldots Q_1 - 1, Q_o - 1 \tag{15}$$

are independent of each other (with respect to the definition of connectivity). The set of integers (15) is therefore a characteristic of the complex K itself, vis-a-vis its structure.

If $Q_o=k$, where $k>1$, then the complex K is in k pieces. In this case we can regard K as the union of complexes $K^{(i)}$, $i=1 \ldots n$ and so obtain k sets of numbers like (1), in each of which $Q_o=1$. We shall therefore restrict the discussion to the case when K possesses only one component at the 0-level ($Q_o=1$).

We tidy this up by associating to the (whole family of) complexes a vector space over a field F (or more delicately, a module over a ring J) which we shall call the *Q-space*. This must be of sufficient dimension to accommodate any of the finite dimensional complexes which interest us. Then, given a complex K with dim $K=n$, we introduce a *first structure vector* $\mathbf{Q}(\mathrm{K})$ defined by

$$\mathbf{Q}(\mathrm{K})=\mathbf{Q}=\{Q_n, Q_{n-1}, \ldots Q_1, Q_o\}. \tag{16}$$

In this Q-space we identify the *unit point* with vector

$$\mathbf{U}=\{1, 1, \ldots 1\}$$

and, when there is no fear of ambiguity, we shall use the same letter to denote the projection of $\mathbf{U}$ on the n-dimensional subspace which contains $\mathbf{Q}(K)$.

We pick out the significance of the set (15) by defining the *obstruction vector* of K as $\hat{\mathbf{Q}}(\mathrm{K})$ where

$$\hat{\mathbf{Q}}(\mathrm{K})=\mathbf{Q}(\mathrm{K})-\mathbf{U}(\mathrm{K})$$

Thus $\hat{\mathbf{Q}}$ possesses the components in the set (15), and the relation between the first structure vector and the obstruction vector of K is shown in the diagram.

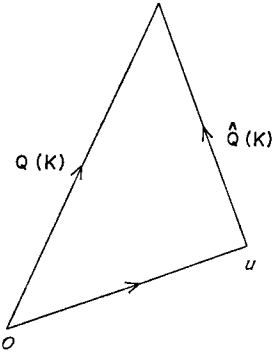


FIG. 6. Obstruction vector in Q-space.

So far we have not assumed that the Q-space possesses any metrical properties. If any such were to be introduced then we would expect to have

good reasons to do so. In fact, it will be shown in a later paper that in the social scientific context of urban structure there is an important part to be played by the introduction of what is called an *ultrametric* on the Q-space. But now we shall suppose that, for whatever reasons, the Q-space is equipped with a norm and a scalar product, and that these are denoted in the usual way by $|\mathbf{Q}|$ and $(\mathbf{Q},\mathbf{Q}')$, with values in F or J.

When the Q-space is a vector space over a field F we shall identify the quantity h, where

$$|\mathbf{Q}| \cdot |\mathbf{Q}'| \cdot h = (\mathbf{Q},\mathbf{Q}') \tag{17}$$

and in the special case when $\mathbf{Q}(K)$ and $\mathbf{Q}'(K')$ refer to conjugate complexes ($K_Y(X;\lambda)$ and $K_X(Y;\lambda^{-1})$) we shall call h the *first structure coefficient* of the relation λ (or of the two complexes defined by λ).

In the case when the norm is Euclidean we see that

$$|\mathbf{Q}| = \{Q_o^2 + Q_1^2 + \ldots + Q_n^2\}^{\frac{1}{2}}$$

and

$$(\mathbf{Q},\mathbf{Q}') = Q_o Q_o' + Q_1 Q_1' + \ldots + Q_n Q_n'$$

where $n = \max\{\dim K, \dim K'\}$. The value of h is then the value of $\cos\theta$, θ being the angle between the vectors $\mathbf{Q}$ and $\mathbf{Q}'$, provided of course that $\dim K = \dim K'$.

We illustrate the sums in the Euclidean case by referring to the numerical examples in section 4. These give the following:

on $K_Y(X;\lambda)$ $\quad \mathbf{Q}=\{1, 4, 4, 1\}$ $\quad (n=3)$

and $\quad \hat{\mathbf{Q}}=\{0, 3, 3, 0\}$

on $K_X(Y;\lambda^{-1})$ $\quad \mathbf{Q}=\{1, 2, 2, 1\}$

and $\quad \hat{\mathbf{Q}}=\{0, 1, 1, 0\}.$

The first structure coefficient for this λ is given by

$$\sqrt{34} \cdot \sqrt{10} \cdot h = 1+8+8+1 = 18$$

or

$$h = 0{\cdot}98.$$

We notice too that the norms of the obstruction vectors are respectively $\sqrt{18}$ and $\sqrt{2}$.

But we emphasize that the concept of structure vector and of obstruction vector is quite independent of metrical notions. The numerical sums above which have used a Euclidean metric (and $F=\mathbb{R}$) are intended to illustrate the comparison between those structural notions and the statistical analysis of the

data in section 4. Thus, referring to the tabulated data for the relation λ on p. 151 we can obtain a graph in E^2 in which the co-ordinates of (X_i, Y_j) are (i,j). This is shown in the diagram.

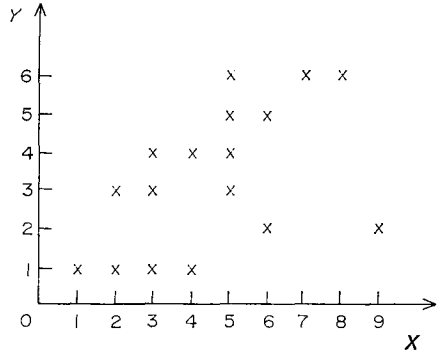


FIG. 7. Regression analysis loses the structure.

If we try to "make sense" out of the relation λ by finding the regression lines (of Y on X and of X on Y) then we are attempting to replace the simplicial complexes by new ones in which the only simplices are of 0-order, and each K is the union of points only. Thus, the regression line must destroy the connectivity pattern we have discussed. The linear correlation coefficient r is a measure of the separation of the two "conjugate" regression lines ($r=\cos\theta$, θ being the angle between the two lines). In this particular case we obtain the value $r=0{\cdot}49$ corresponding to the regression lines

$$Y=1{\cdot}46+0{\cdot}41\ X \quad \text{and} \quad X=2{\cdot}63+0{\cdot}58\ Y$$

We compare this with the first coefficient of structure $h=0{\cdot}98$: thus there is a relatively high "matching" of the structures of $K_Y(X)$ and $K_X(Y)$—as seen from the analysis of the q-connectivity of λ—and this cannot be found from a consideration of the linear regression lines.

8. The Obstructionist Nature of the Obstruction Vector $\hat{\mathbf{Q}}$

If a complex K consists of an n-simplex only, then it is easy to see that $\hat{\mathbf{Q}}(\mathrm{K})$ is the zero vector. Thus, in some sense we would expect to regard such a complex as possessing zero obstruction to "something"—presumably a zero obstruction to a possible change or flow. We shall attempt to make this clearer by a specific example.

Suppose that the set X is a set of social amenities provided in a certain town T, and that Y is the set of residential areas defined conveniently by specifying collections of streets and houses in T. X will contain things like public parks, libraries, schools, meeting halls, sports stadia, etc. There will be a relation λ between the sets X and Y and consequent complexes $K(X)$ and $K(Y)$. For example, X_i is related via λ to Y_j if the residential area Y_j possesses immediate access to the amenity X_i. In the complex $K(X)$ the residential areas Y_j correspond to simplices of various dimensions, and when an individual person goes to the town T to buy a house he consciously or unconsciously takes cognizance of the dimensions of the Y_j (see also Atkin, Johnson & Mancini, 1971).

Now this complex $K(X)$ can be regarded as the *backcloth* against which the human activities of the town are carried on; to be sure of this we need only ensure that the set X is sufficiently exhaustive. (In practice we can arrange for our Q-analysis to be performed by a computer—work which has been carried out so far offers no difficulties with sets X and Y possessing cardinalities up to a few hundred.) In this sense it functions as the base complex referred to in section 6. In this context additional mathematical functions may be defined on the complex in such a way as to represent many of the human activities which are found in the residential areas. These functions will be co-simplices, as previously discussed, and for future reference we shall call such a set of co-simplices, which are specific to some human activity, a *pattern*, π, on $K(X)$.

Examples of patterns, in this context, will be (i) population density, (ii) land values, (iii) rateable values, (iv) density of garage space, (v) capital investment in amenities, (vi) commuter traffic density, and many others.

Any change in a given pattern amounts to a flow of the cosimplex values throughout the complex K, and this flow can be thought of as actually taking place along the chains of connectivity in K. Since the pattern π is a function of the vertices of the backcloth $K(X)$ there will be a relation between the cosimplices which make up π and the set X. This relationship defines a connectivity structure for the cosimplices which constitute the pattern. Thus the pattern π requires a certain connectivity structure vis-a-vis the vertex set X of the backcloth; by "requires" we mean that in the presence of such connectivity the pattern π freely adjusts its values; there is no obstruction to a change in the pattern π. Hence it follows that if the connectivity between the simplices Y_j in $K_Y(X)$ matches that required by the pattern π then the complex K will be effectively offering zero obstruction to changes in the values of the cosimplices contained in π.

In a simple case we might suppose that π is characterized by a t-connectivity. That is to say, that each cosimplex in π is determined by $p+1$ of the

vertices in X and that this p is always greater than or equal to the fixed value of t. This is not to say that it is always the same set of $p+1$ vertices which determine different co-simplices. In this case we could denote the pattern by the symbol π_t and deduce that on every simplex of $K(X)$ of order greater than $(t-1)$ there is to be zero obstruction to changes in π_t.

This notion clearly fits into the definition of the obstruction vector, in the case that K is a single t-simplex.

If there are two simplices in K, each of which is of order t, but which are only connected at a level $q<t$ then there is an obstruction to changes in π. In the extreme case, when K is in two quite separate pieces ($Q_o=2$), then any flow of a pattern from one piece to the other is impossible; this is because every pattern π must be a π_t for some $t\geqslant 0$.

It can be seen therefore that if a complex K changes in such a way as to increase any component of the obstruction vector $\hat{\mathbf{Q}}(\mathrm{K})$ then this must result in a tendency for associated patterns to be more rigid. Conversely when K changes so as to become one single (large dimension) simplex then all obstruction to pattern flow drops to zero. The same applies to each component of K, for a given q.

By way of example, the town of Southend-on-Sea in Essex, U.K., publishes (as do many other towns) its "town map". This town map shows 19 residential areas within the borough boundaries and a total of 30 civic and social amenities—which range from parks, schools (various) to the sea-side amusements and the waterworks. Although the data is somewhat minimal and rather crude, nevertheless it is sufficient to illustrate the ideas.

The Q-analysis of the relation λ gave the following results.

(i) The outstanding residential area was a simplex of order 14.

(ii) The next two areas appeared at $q=10$, but at that stage none of the three areas were connected; thus $Q_{10}=3$.

(iii) Other areas appeared in the analysis at lower q-values, but the first sign of a lengthy chain of connection only occurred at $q=5$; at $q=6$ we have $Q_6=9$ whilst at $q=5$ we have $Q_5=3$.

(iv) Overall the structure vector $\mathbf{Q}(\mathrm{K})$ became

$$\mathbf{Q}(K)=\{1, 1, 1, 1, 3, 5, 8, 11, 9, 3, 4, 1, 1, 1, 1\}$$

(v) The obstruction vector was consequently

$$\hat{\mathbf{Q}}(\mathrm{K})=\{0, 0, 0, 0, 2, 4, 7, 10, 8, 2, 3, 0, 0, 0, 0\}$$

(vi) This demonstrated that any pattern π_t for which $t>4$ would suffer obstruction to change, with a decided increase in this at $t=6$. We could immediately say that, for example, if the population density pattern is

rigid then it must correspond to a π_t with $t>4$. Generally it was borne out in fact that the obstruction vector for the town, based on this patricular data, corresponded to well-established rigidities in a number of the basic patterns one would normally consider.

Thus we can see that the interaction between human activities and the spatial distribution in a typical town structure is something which must be played out against a backcloth of a base complex with all its connectivities. The consequences of this for planning and urban development are clear for all to see.

A fascinating philosophical consequence of this point of view is that we can no longer assume that this particular human drama can be adequately fitted into the physicist's four-dimensional world of conventional space-time. The dimensionality of this human world of interaction is determined by the complexes $K(X)$ and $K(Y)$ which are used to describe the backcloth.

9. Conclusion

When a physicist collects up his measuring rods and light signals and goes out into the desert to observe the geometry of three-space, he does not in fact observe the Cartesian E^3. On the contrary, he obtains a specifically Leibnitzian view of the space, and this is the view of that relation which is set up between himself, his light sources, and his measuring rods. Although he might like to conclude that the finite mesh of "points" he obtains constitute a complex K without any holes, this eludes him. The presence of a boulder means that there is a hole in the complex; he can build chains of connection around this hole, but he cannot go through it; it is opaque to his light signal. This is the point of view we have stressed in this paper, viz., the physicist "sees" only the cycles of a complex K as the "objects" of his world. These cycles (or holes) conversely set up a relation between themselves which recognizes that their boundaries consist of points (via light+rods) which can be connected by suitable q-chains in the mesh-complex known as actual-space. The idea that space is three-dimensional now means that all our observations, in this style, result in the chains of connection being at least two-chains. We would now say that $Q_2=1$. The fact that we think of a solid body as enduring in time amounts to saying that the space-time mesh is a complex with $Q_3=1$ (in this we are taking the word "body" to mean something homeomorphic to a tetrahedron).

If we take a basic set of observed objects as defining the mesh complex (in this Leibnitzian sense) then all other bodies will be supposed to be defined (and observed) against this backcloth. The dynamics of these other bodies will

be described as patterns π on the base complex. For example the velocity of a particle will be given by a cosimplex on the base complex K; as the particle moves it literally moves down the chains of connection (three-chains) of K and this requires a successive change in the pattern π. Thus the dynamics of particles, on this view of physics, is governed by the obstruction vector of the base complex. If the particle meets a brick wall then it has come to the end of its three-chain, and since it requires a three-connection for zero obstruction (free passage) the two-connection (which would take it around the wall) is inadequate; it therefore moves in some manner as to stay in the three-connected component of K—in which it started. The phenomenon of possessing velocity is characterized by being associated with a π_3 pattern on the actual-space complex K.

The same language can be used, as we have seen, to describe the structure and dynamics of social systems. Here the base complex which we can choose is not one inherited from our forefathers and therefore is not one on which we are all likely to readily agree. However we can experiment with various complexes in the different fields of social science—which suggests that there is unlikely to be a drab uniformity in approach.

What is clear however, above all, is that this sort of analysis is only possible because we live in the computer-age. Setting up incidence matrices with many hundreds of rows and columns and performing the logical and search operations which are required for a complete Q-analysis is now well within the powers of modern computers. The significance of this, apart from the obvious one of being able to do it, lies in the freedom it gives us in the realm of ideas. What we are in fact doing is using the computer to do something which we find difficult to do, viz. to think in multi-dimensional space. What the computer does not know is why it is thinking in multi-dimensional space. In that, perhaps, we still have the edge?

This situation seems to point in the direction which computers want to go (it may use the word "want" in this context). For surely the old idea that all we want is for the computer to do "clerking" jobs one million times as fast as the human clerks, is a poor use of the computer potential. It is also, in our terms, a rather large obstruction vector to that flow of intellectual ideas which should stimulate research. Until the cupboards which contain all the old problems of research (but which required millions of man-hours to solve by old methods) have been cleared and emptied by the clerking expertise of the computers, we cannot hope to make the machines our slaves—as opposed to being slaves ourselves.

Nor is it obvious that the sort of analysis which is required for this theory also requires excessively large computers. The work which we are doing at the

University of Essex on Urban Structure, and which involves apparently large matrix arrays, has all been done (or equally well could have been done) on a small machine. In this respect the co-operation and expertise of Mr Brian Gaines of the Department of Electrical Engineering has been much appreciated.

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