PROCEEDINGS OF THE ROYAL SOCIETY A

Proc. R. Soc. A (2012) **468**, 1252–1279
doi:10.1098/rspa.2011.0616
Published online 5 January 2012

Geometric models of matter

By M. F. Atiyah[1], N. S. Manton[2] and B. J. Schroers[3,*]

[1]*School of Mathematics, University of Edinburgh, King's Buildings, Edinburgh EH9 3JZ, UK*
[2]*DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK*
[3]*Department of Mathematics, Heriot–Watt University, Riccarton, Edinburgh EH14 4AS, UK*

Inspired by soliton models, we propose a description of static particles in terms of Riemannian 4-manifolds with self-dual Weyl tensor. For electrically charged particles, the 4-manifolds are non-compact and asymptotically fibred by circles over physical 3-space. This is akin to the Kaluza–Klein description of electromagnetism, except that we exchange the roles of magnetic and electric fields, and only assume the bundle structure asymptotically, away from the core of the particle in question. We identify the Chern class of the circle bundle at infinity with minus the electric charge and, at least provisionally, the signature of the 4-manifold with the baryon number. Electrically neutral particles are described by compact 4-manifolds. We illustrate our approach by studying the Taub–Newman, Unti, Tamburino (Taub–NUT) manifold as a model for the electron, the Atiyah–Hitchin manifold as a model for the proton, $\mathbb{CP}^2$ with the Fubini–Study metric as a model for the neutron and S^4 with its standard metric as a model for the neutrino.

Keywords: self-dual 4-manifolds; Kaluza–Klein; soliton models for particles; proton; electron

1. Introduction

Geometry and the quantum mechanics of particles have an uneasy relationship, which is why general relativity is hard to incorporate into quantum field theory. String theory is an ambitious and remarkable attempt at unification with many successes, but the final theory has proved mysterious and elusive.

Einstein and Bohr fought a long battle on this front, and Bohr was generally deemed to have won, with the Copenhagen interpretation of quantum mechanics accepted. But Einstein's belief in the role of geometry made a partial come-back with the adoption of gauge theories as models of particle physics.

A more modest and limited role for geometry in nuclear physics was proposed by Skyrme (1961) with the solitonic model of baryons, i.e. proton, neutron and nuclei, now known as Skyrmions. These have been shown to be approximate models of the physical baryons occurring in gauge theories of quarks and gluons, and have been extensively studied (Brown & Rho 2010) with considerable success.

*Author for correspondence (b.j.schroers@hw.ac.uk).

Received 11 October 2011
Accepted 8 December 2011

This journal is © 2012 The Royal Society

In this paper, we explore a geometric model of particles that is inspired by Skyrme's idea, but with potential applications to both baryonic and leptonic particle physics. Our model differs from the Skyrme model in that it uses Riemannian geometry rather than field theory, and so it is closer in spirit to Einstein's ideas. Another key difference is that we absorb the Kaluza–Klein idea of an extra circle dimension to incorporate electromagnetism. However, we exchange the roles of electricity and magnetism relative to the standard Kaluza–Klein approach, so that the extra circle dimension is magnetic rather than electric, and it is the electric charge that is topologically quantized by the famous Dirac argument. We will also, initially, ignore time and dynamics, focusing on purely static models.

Our geometric models will therefore be four-dimensional Riemannian manifolds. We require the manifolds to be oriented and complete, but not usually compact: we use non-compact manifolds to model electrically charged particles, and compact manifolds for neutral particles. Both baryon number and electric charge will be encoded in the topology, with baryon number (at least provisionally) identified with the signature of the 4-manifold and electric charge with minus the Chern class of an asymptotic fibration by circles (the definition of signature for non-compact manifolds is reviewed in §8). In particular, the number of protons and the number of neutrons will therefore be determined topologically.

The manifolds will also have an 'asymptotic' structure that captures their relation to physical 3-space. The non-compact manifolds we consider have an asymptotic region that is fibred over physical 3-space, so no additional structure is required. For the compact (neutral) models, however, we fix a distinguished embedded surface X, where the 4-manifold M intersects physical 3-space. We call $M \setminus X$ the inside of M.

For single particles, the symmetry group of rotations should fix all of the above data. It should act isometrically on the 4-manifold and preserve the asymptotic structure. In the non-compact cases, this means that it should be a bundle map in the asymptotic region, covering the usual $SO(3)$-action on physical 3-space. In compact cases, it should preserve the distinguished surface X. In order to capture the fermionic nature of the particles considered in this paper, we also require spin structures on the non-compact manifolds and on the inside (in the sense defined above) of compact manifolds. Moreover, the lift of the rotation group action to the spin bundle (over the entire manifold in the non-compact case and over the inside in the compact case) should necessarily be an $SU(2)$-action, and this is what we mean by saying that our models are fermionic.

A key restriction on (the conformal classes of) our manifolds is that they are self-dual. Recall that the Riemann curvature is made up of the Ricci tensor plus the Weyl tensor W, which is conformally invariant. In dimension four, W is the sum of self-dual and anti-self-dual parts,

$$W = W^+ \oplus W^-. \tag{1.1}$$

A 4-manifold M is said to be self-dual if $W^- = 0$. These manifolds are precisely those that have twistor spaces in the sense of Penrose (1977). The latter are three-dimensional complex manifolds Z with a real S^2-fibration over M. The complex structure of Z (together with a real involution that is the antipodal map on each S^2) encodes the entire conformal structure of M. Even the Einstein metric can be captured by complex data on Z.

A simply connected self-dual 4-manifold, for which the Ricci tensor is also zero, is a hyperkähler manifold, whose structure group reduces to $SU(2)$. It is a complex Kähler manifold for an S^2-family of complex structures, but for any of these complex structures, and for the complex orientation, it would be anti-self-dual. As we want self-dual manifolds, we choose the opposite orientation.

While some of our particles, including the proton, will be modelled by hyperkähler manifolds, we do not want to be so restrictive. Instead, we will only require our 4-manifold models of particles to be self-dual and Einstein; so there can be a non-zero scalar curvature. Our model for the neutron is of this type, distinguishing it from the proton. The neutron will of course also have electric charge 0.

We should point out that reversing the orientation of a 4-manifold turns a self-dual manifold into an anti-self-dual one. This should be interpreted as giving the geometric model of an anti-particle.

Self-dual 4-manifolds are, in many ways, the four-dimensional analogue of Riemann surfaces, with H_2 replacing H_1 in homology. In particular, there are theorems (Donaldson & Friedman 1989) which assert that such manifolds admit connected sums although, unlike in the case of Riemann surfaces, there are restrictions on when this is possible. Such connected sums model composite objects like nuclei. Although we focus at present on static particles, we do envisage a deformation theory, using the moduli space of self-dual manifolds, which could underlie particle interactions.

Fortunately, a lot is now known about self-dual 4-manifolds with many metrics explicitly calculated. This makes it possible to put forward some definite models for the proton and neutron. Even though our ideas are inspired by Skyrme's theory of baryons, it turns out that geometric models of leptons, i.e. the electron and (electron-)neutrino, are even simpler, and we shall describe them too in the class of self-dual manifolds. Thus, somewhat surprisingly, our framework of self-dual manifolds allows us to describe baryons and leptons in a unified fashion.

The language and spirit of our model for particles is close to that of general relativity and suggests the possibility of a unification with gravity, but we do not address this issue here. In particular, we do not specify an action functional. Instead, we focus on how our model describes general features of particles such as their various quantum numbers. We are aware that a description of elementary particles as four-dimensional Riemannian manifolds is radically different from established treatments in terms of quantum field theory. What we aim to show in this paper is that such a geometric approach is possible, and that it has some surprising and attractive features, such as the possibility of describing the electron and the proton in one framework. While we do propose definite identifications of certain 4-manifolds with specific particles in §§3–5 of this paper, these should be seen as illustrations of the geometric approach, not necessarily as final proposals.

The paper is organized as follows. In §2, we outline the genesis of our geometric models of particles, starting with the Skyrme model of baryons. Electrically charged particles are necessarily described by non-compact 4-manifolds in our approach, and in §3 we explain how to model the electron and proton in terms of the Taub–Newman, Unti, Tamburino (Taub–NUT) and Atiyah–Hitchin manifolds, respectively. Neutral particles are described by compact 4-manifolds, and this is discussed in §§4 and 5. We propose $\mathbb{CP}^2$ as a model for the neutron and S^4 as a model for the neutrino. These are the simplest choices, but we also

discuss some more sophisticated versions. In §6, we describe how our particle models glue into empty space, and how the particles may interact with each other. Section 7 presents an outline of how our geometric models capture the spinorial nature of the particles they describe. In §8, we give the dictionary that translates topological properties of 4-manifolds into the electric charge and baryon number of particles, and discuss in some detail how these charges are related to fields and densities used in conventional Lagrangian models of particle physics. Section 9 contains our conclusion and some ideas for follow-up work. Conventions and calculations are collected in the appendix.

2. From Skyrmions to 4-manifolds

We begin by spelling out in detail how the Skyrme model suggests our 4-manifold model. The Skyrme model is based on a group-valued field from $\mathbb{R}^3$,

$$U : \mathbb{R}^3 \to G, \tag{2.1}$$

where the Lie group G is usually taken to be $SU(2)$, and $U(\boldsymbol{x}) \to 1$ as $|\boldsymbol{x}| \to \infty$. The degree of U as a map $S^3 \to SU(2)$ is identified with baryon number. The minima of the Skyrme energy, for each baryon number, are called Skyrmions.

Skyrmions are free to rotate both in physical space and through conjugation by elements of $SU(2)$. Quantizing this motion gives the Skyrmions spin and isospin. The proton and neutron are identified as the 'isospin up' and 'isospin down' states in an isospin doublet. The electric charge is related to a component of isospin when a $U(1)$ subgroup of the isospin symmetry is gauged.

Atiyah & Manton (1989) showed how to generate such Skyrme fields naturally by starting with an $SU(2)$ Yang–Mills gauge field on $\mathbb{R}^4$ and calculating the holonomy along the fourth direction. Suitable asymptotic behaviour on $\mathbb{R}^4$ guarantees a well-defined map U. Although this construction does not preserve the respective energy functionals, it does provide a good way of using instantons on $\mathbb{R}^4$ (i.e. self-dual gauge fields) to construct approximate minima of the Skyrme energy. It also identifies instanton number with the Skyrme degree. See also Sakai & Sugimoto (2005) and Sutcliffe (2010), where the difference between the Yang–Mills and Skyrme energy functionals is interpreted as due to an infinite tower of mesons.

Since the Yang–Mills energy functional in dimension four is conformally invariant, we could replace the decomposition

$$\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}^1 \tag{2.2}$$

by

$$S^4 \setminus S^2 = H^3 \times S^1, \tag{2.3}$$

where H^3 is hyperbolic 3-space. In fact, we can vary the curvature of H^3, provided we rescale the circle S^1 the opposite way, so that large circles correspond to almost flat H^3. We can now fix a gauge field on $S^4 \setminus S^2$ and take the holonomy round the circles. There are some technicalities (owing to base-points) that we shall ignore, but basically we expect to end up with a Skyrmion on H^3, an idea that has been explored by Atiyah & Sutcliffe (2005).

Now replace $H^3 \times S^1$ by any Riemannian 4-manifold M that is asymptotically fibred by circles over $\mathbb{R}^3$. This is the kind of Kaluza–Klein 4-manifold we are going to consider. An $SU(2)$ gauge field on M would then give a Skyrme field on 'the quotient of M by S^1'. Because we do not want to assume that there is a global circle fibration, this Skyrme field will only be defined asymptotically outside some 'core'. But an oriented 4-manifold has two natural $SU(2)$ bundles over it: the spin bundles S^+ and S^- (assuming that M is a spin-manifold, i.e. $w_2(M) = 0$). By picking one of these, say S^+, we get from the connection on S^+ a natural construction of an asymptotic Skyrme field on $\mathbb{R}^3$.

In the Skyrme model, the basic idea is that baryon number is identified with the degree of the map U in (2.1), or equivalently with the instanton number (or second Chern class) of the $SU(2)$ bundle over $\mathbb{R}^4$. This differs from the 4-manifold model we want to explore, where baryon number is, at least provisionally, identified with the signature of the 4-manifold. The signature is additive for connected sums of 4-manifolds (Atiyah & Singer 1968), and this captures the additivity of baryon number for composites of particles. The integrality of the signature is linked to it being an index of an elliptic operator. This means we are in the realm of K-theory rather than cohomology.

To sum up, in the Skyrme model, baryon number is cohomological, and electric charge arises at the quantum level. For our 4-manifold model, electric charge is cohomological, arising, as already explained, from the first Chern class of the asymptotic S^1-fibration, while baryon number should be seen as an index. Our model goes beyond the Skyrme model in aiming to understand topologically both the basic integer physical invariants, baryon number and electric charge. The two models are different but related in ways that we hope to explore at a later stage.

3. Models for the electron and proton

Models for the basic particles should exhibit a high degree of symmetry and we expect the rotation group $SO(3)$ of $\mathbb{R}^3$, or its double cover $SU(2)$, to act as isometries. For electrically charged particles, we take our geometric models to be non-compact hyperkähler manifolds. We also assume that the volume grows with the third power of the radius, to allow for an interpretation of the asymptotic region in terms of physical 3-space. As recently shown by Minerbe (2010), this forces the hyperkähler manifold to be asymptotically locally flat (ALF). We are therefore looking for rotationally symmetric and complete ALF hyperkähler manifolds. There are just two possibilities.

— The Taub–NUT manifold (Hawking 1977; Eguchi *et al.* 1980) depending on a positive parameter m (interpreted as mass in the gravitational context). For brevity, we denote it by TN.
— The Atiyah–Hitchin manifold, the (simply connected double cover of the) moduli space of centred $SU(2)$-monopoles of charge 2 (Atiyah & Hitchin 1988). For brevity, we denote it by AH.

Note that we could also single out TN and AH among non-compact, complete and rotationally symmetric hyperkähler manifolds by demanding that the $SU(2)$- (or $SO(3)$-) action rotates the complex structures (see our discussion following

equation (A 8)). This turns out to play a role in recovering the usual rotation action on physical 3-space in the asymptotic region of our geometric models, as discussed in §7.

Both TN and AH can be parametrized in terms of a radial coordinate r and angular coordinates on $SU(2)$ (for TN) or $SO(3)/\mathbb{Z}_2$ (for AH). Details are given in the appendix. In terms of the left-invariant 1-forms defined in equation (A 2), the metrics of both TN and AH can be written as

$$\mathrm{d}s^2 = f(r)^2\,\mathrm{d}r^2 + a(r)^2\eta_1^2 + b(r)^2\eta_2^2 + c(r)^2\eta_3^2, \tag{3.1}$$

with the functions f, a, b, c satisfying the self-duality equations

$$\frac{2bc}{f}\frac{\mathrm{d}a}{\mathrm{d}r} = (b-c)^2 - a^2, \quad + \text{cyclic}, \tag{3.2}$$

where + cyclic means we add the two further equations obtained by cyclic permutation of a, b, c. We adopt the convention

$$f(r) = -\frac{b(r)}{r}, \tag{3.3}$$

where (for reasons that will emerge later) the radial coordinate r has the range $[0,\infty)$ for TN and $[\pi,\infty)$ for AH. The self-duality equations (3.2) become

$$\left.\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}r} &= \frac{1}{2rc}(a^2-(b-c)^2),\\ \frac{\mathrm{d}b}{\mathrm{d}r} &= \frac{b}{2rca}(b^2-(c-a)^2)\\ \text{and}\qquad \frac{\mathrm{d}c}{\mathrm{d}r} &= \frac{1}{2ra}(c^2-(a-b)^2). \end{aligned}\right\} \tag{3.4}$$

This system has solutions in terms of elementary functions

$$a(r) = b(r) = r\sqrt{\epsilon + \frac{m}{r}} \quad \text{and} \quad c(r) = \frac{m}{\sqrt{\epsilon + m/r}}, \tag{3.5}$$

with parameters $\epsilon, m > 0$, associated with the TN manifold. The topology is that of $\mathbb{R}^4$, and as $\epsilon \to 0$, the metric tends to the flat metric. For $\epsilon > 0$, the manifold is asymptotic to an S^1 fibre-bundle over $\mathbb{R}^3$, with the length of the circle being $4\pi m/\sqrt{\epsilon}$. There is a $U(1)$ symmetry acting along the fibres, with just one fixed point at the origin, $r = 0$. The whole isometry group is $U(2)$. As $\epsilon \to 0$, the $U(1)$-action becomes the scalar action on $\mathbb{C}^2$. The complex orientation of $\mathbb{C}^2$ determines the orientation of TN as a self-dual manifold; this is opposite to the orientation given by any of the complex structures in the hyperkähler family (see the appendix for a discussion). At infinity, the $U(1)$-action gives the standard Hopf line bundle over $\mathbb{CP}^1$ with Chern class +1.

The TN metric with coefficient functions (3.5) has the following behaviour under scaling by non-vanishing real numbers α, β:

$$r \to \frac{\beta}{\alpha}r, \quad m \to \alpha\beta m, \quad \epsilon \to \alpha^2\epsilon, \quad \text{then } \mathrm{d}s^2 \to \beta^2\,\mathrm{d}s^2. \tag{3.6}$$

We use rescaling by α to set $\epsilon = 1$, and rescaling by β to set $m = 2$ from now on. This amounts to picking a unit of length for the radial coordinate r and to fixing an overall scale for the metric. Our choice is motivated by the asymptotic form of the AH metric to be discussed later. Note that, with this choice, the length of the asymptotic circle, in the length units chosen, is 8π.

The solution that gives rise to AH has the *asymptotic* form, for large r,

$$a(r) \sim b(r) \sim r\sqrt{1 - \frac{2}{r}} \quad \text{and} \quad c(r) \sim -\frac{2}{\sqrt{1 - 2/r}}. \tag{3.7}$$

These asymptotic expressions, a TN metric with $\epsilon = 1$, $m = -2$, also satisfy (3.4). However, $a(r)$ is not actually equal to $b(r)$, and r only extends down to π. For r near π, (3.7) is a poor approximation. Instead, one finds the leading terms

$$a(r) \sim 2(r - \pi), \quad b(r) \sim \pi + \tfrac{1}{2}(r - \pi) \quad \text{and} \quad c(r) \sim -\pi + \tfrac{1}{2}(r - \pi), \tag{3.8}$$

which we will need later in this paper.

The manifold AH is the complement of $\mathbb{RP}^2$ (the real projective plane) embedded in $\mathbb{CP}^2$, and the complex orientation of $\mathbb{CP}^2$ determines the orientation of AH as a self-dual manifold. As for TN, this is opposite to the orientation given by any of the complex structures in the hyperkähler family. AH has an $SO(3)$ symmetry with just one two-dimensional orbit at $r = \pi$, which is a minimal 2-sphere. We refer to this minimal 2-sphere, which is the totally imaginary conic in $\mathbb{CP}^2$ determined by $z_1^2 + z_2^2 + z_3^2 = 0$ in the homogeneous coordinates introduced in §4, as the core. Asymptotically, the manifold is fibred by circles. As further discussed later, neither the circles nor the base space of this asymptotic fibration are oriented because of a $\mathbb{Z}_2$-identification, given explicitly in equation (A 3).

The manifold TN is usually interpreted as the geometry of a Dirac monopole at the origin (Gross & Perry 1983; Pollard 1983; Sorkin 1983). For us, with electric and magnetic charges reversed, it has to be interpreted as an electrically charged particle. Because the signature of TN is zero (we discuss this further in §8), the particle is leptonic. We therefore interpret TN as a model for the electron. Down on $\mathbb{R}^3$, after factoring by $U(1)$, any 2-sphere surrounding the origin has an electric flux emerging from it due to the electron, which carries charge -1. This implies that there is a sign change in going from the Chern class to the electric charge.

The manifold AH has the opposite asymptotic behaviour with a sign change for m and an orientation change (see appendix) and so would lead us naturally to expect electric charge $+1$. Also, the topology at the core is different, with a 2-sphere instead of a point. As a result, AH has signature 1 (again discussed further in §8) and looks like the model we want for the proton (rather than the positron).

However, things are not quite that simple. The 'asymptotic boundary' of AH is not S^3 as for TN, but the boundary of a tubular neighbourhood of $\mathbb{RP}^2$ in $\mathbb{CP}^2$, which is S^3 divided by a cyclic group of order 4. Moreover, the base of this unoriented circle fibration is $\mathbb{RP}^2$, not S^2, and is non-orientable. This means that the 3-manifold that is the base of the asymptotic fibration is not $\mathbb{R}^3$, and has a fundamental group of order 2. It is not orientable.

This might seem to be a disaster, but we shall argue that, while unexpected, it is not as bad as it looks. The most convincing argument in favour of the model is

that the electric charge is well defined and equal to +1 as hoped. This argument goes as follows.

Our general prescription for the electric charge is that it is minus the first Chern class of the asymptotic fibration by circles, or equivalently, minus the self-intersection number X^2 of the surface X, which is the base of the asymptotic fibration. As we have observed, for AH, the surface X is the real projective plane $\mathbb{RP}^2$ in $\mathbb{CP}^2$. Because $\mathbb{RP}^2$ is non-orientable, it defines (locally) a homology class with twisted coefficients, but this still has a self-intersection number. Thus, we have to check that, for $\mathbb{RP}^2$ in $\mathbb{CP}^2$, the self-intersection number is -1.

This is most easily done by using the natural double covering of $\mathbb{CP}^2$ by $\mathbb{CP}^1 \times \mathbb{CP}^1$ (essentially given by the two roots of the quadratic whose three coefficients are the homogeneous coordinates of a point in $\mathbb{CP}^2$). The double covering is branched along the diagonal D (double roots), but is unbranched along the anti-diagonal D', which covers $\mathbb{RP}^2$. Now in $\mathbb{CP}^1 \times \mathbb{CP}^1$, the self-intersection numbers are

$$D^2 = 2 \quad \text{and} \quad (D')^2 = -2. \tag{3.9}$$

Dividing by 2 to get back to $\mathbb{CP}^2$, we see that the self-intersection number of $\mathbb{RP}^2$ in $\mathbb{CP}^2$ is indeed -1, as required.

4. The neutron

(*a*) *Complex projective plane*

Having put forward a definite proposal for the proton, we now have to face the neutron. Because the neutron has no electric charge, any non-compact model would need to have a trivial asymptotic circle fibration. The 4-manifold should have signature 1 and it should resemble the AH model of the proton in its $SO(3)$ orbit structure. However, the latter requirement rules out asymptotically trivial circle bundles over physical 3-space as the generic $SO(3)$ orbits would be two dimensional in that case. Therefore, we consider compact 4-manifolds. In fact, there is an obvious choice, which is just the complex projective plane $\mathbb{CP}^2$ with its Fubini–Study metric (and its natural complex orientation). This is a self-dual manifold of positive scalar curvature.

$\mathbb{CP}^2$ has even more symmetry than we need, as it is acted on by $SU(3)$. The rotation group $SO(3)$ sits inside as the subgroup that preserves the real structure given by complex conjugation. This preserves $\mathbb{RP}^2$ as in the case of the proton, so we fix this $\mathbb{RP}^2$ as the distinguished surface where the 4-manifold intersects physical 3-space, thus breaking the symmetry to $SO(3)$. The global $SU(3)$ symmetry might give us a link to quarks, but this remains to be explored.

The Fubini–Study metric is often written in coordinates that exhibit the invariance under $U(2) \subset SU(3)$. This brings out the parallels with TN (Gibbons & Pope 1978), but is not the symmetry we want in our neutron model. For the interpretation of $\mathbb{CP}^2$ as a neutron and for a comparison with the AH model of the proton, we need to write the Fubini–Study metric in coordinates adapted to the $SO(3)$-action, which is discussed in Bouchiat & Gibbons (1988) and Dancer & Strachan (1994). We write the results of Bouchiat & Gibbons (1988) in the conventions used in our discussion of the AH metric.

In terms of homogeneous coordinates, $\boldsymbol{z} \in \mathbb{C}^3 \setminus \{0\}$ (with the identification $\boldsymbol{z} \simeq \lambda \boldsymbol{z}$, $\lambda \in \mathbb{C}^*$), the Fubini–Study metric on $\mathbb{CP}^2$ is

$$\mathrm{d}s^2 = \frac{|\boldsymbol{z}^2| \, \mathrm{d}\boldsymbol{z}^\dagger \, \mathrm{d}\boldsymbol{z} - \boldsymbol{z}^\dagger \, \mathrm{d}\boldsymbol{z} \boldsymbol{z}^t \, \mathrm{d}\bar{\boldsymbol{z}}}{|\boldsymbol{z}|^4}. \tag{4.1}$$

For calculations, we can fix $|\boldsymbol{z}| = 1$ and parametrize

$$\boldsymbol{z} = \mathrm{e}^{\mathrm{i}\alpha} R \boldsymbol{z}_0, \tag{4.2}$$

where $R \in SO(3)$ can, in turn, be parametrized in terms of Euler angles as recalled in the appendix. The reference vector $\boldsymbol{z}_0$ (which depends on one parameter) should be a unit vector and we can assume, by adjusting the phase α if necessary, that its real and imaginary parts are orthogonal. For our purpose, it is convenient to single out the 1-axis and pick

$$\boldsymbol{z}_0 = \begin{pmatrix} 0 \\ a_2 \\ \mathrm{i}\, a_3 \end{pmatrix}, \quad a_2^2 + a_3^2 = 1. \tag{4.3}$$

Then, we find the following expression for the Fubini–Study metric in terms of the left-invariant forms (A 2) on $SO(3)$ (for details of an analogous calculation, we refer the reader to Bouchiat & Gibbons (1988)),

$$\mathrm{d}s^2 = \mathrm{d}a_2^2 + \mathrm{d}a_3^2 + (a_3^2 - a_2^2)^2 \eta_1^2 + a_3^2 \eta_2^2 + a_2^2 \eta_3^2. \tag{4.4}$$

Parametrizing

$$a_2 = \cos\left(\frac{\rho}{2} + \frac{\pi}{4}\right), \quad a_3 = \sin\left(\frac{\rho}{2} + \frac{\pi}{4}\right) \quad \text{and} \quad \rho \in \left[0, \frac{\pi}{2}\right], \tag{4.5}$$

we obtain, finally, the Fubini–Study metric in the form

$$\mathrm{d}s^2 = \frac{1}{4} \mathrm{d}\rho^2 + \sin^2 \rho \eta_1^2 + \sin^2\left(\frac{\rho}{2} + \frac{\pi}{4}\right) \eta_2^2 + \cos^2\left(\frac{\rho}{2} + \frac{\pi}{4}\right) \eta_3^2. \tag{4.6}$$

There is a simple interpretation of the geometry of $\mathbb{CP}^2$ and its orbit structure in terms of oriented ellipses up to scale (Atiyah & Manton 1993), which is useful for comparison with the AH metric. As already exploited above, we can adjust the phase in the homogeneous coordinate $\boldsymbol{z} = \boldsymbol{u} + \mathrm{i}\boldsymbol{v}$ (no longer fixed to satisfy $|\boldsymbol{z}| = 1$) so that the real vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal: if $\boldsymbol{z}^2 = 0$, this is automatic, and if $\boldsymbol{z}^2 \neq 0$, we multiply by a unit complex number to set $\mathrm{Im}(\boldsymbol{z}^2) = 2\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and $\mathrm{Re}(\boldsymbol{z}^2) = \boldsymbol{u}^2 - \boldsymbol{v}^2 < 0$ (we pick the negative sign to agree with the choice made in (4.5) above). We can interpret $\boldsymbol{v}$ and $\boldsymbol{u}$ as the major and minor axes of an ellipse. This ellipse is only determined up to scale (we can still rescale $\boldsymbol{z}$ by any positive real number), but it is oriented. The totally degenerate case $\boldsymbol{z} = 0$ is excluded by the definition of homogeneous coordinates, but circles ($\boldsymbol{z}^2 = 0$ or $|\boldsymbol{u}| = |\boldsymbol{v}|$) and lines ($|\boldsymbol{u}| = 0$) can occur.

In terms of our parametrization (4.2), the reference vector (4.3) and the definition of ρ in (4.5), we see that for $\rho = 0$, the ellipse is a circle and for $\rho = \pi/2$, it degenerates to a line. For generic values of ρ, the $SO(3)$-orbit is $SO(3)/\mathbb{Z}_2$, with the $\mathbb{Z}_2$ generated by the 180° rotation about the 1-axis, but for $\rho = 0$, the orbit

is a 2-sphere and for $\rho = \pi/2$, the orbit is $\mathbb{RP}^2$. This is the same as the orbit structure of AH compactified by an $\mathbb{RP}^2$ at infinity, although the metric is of course different.

We note that the Kähler form

$$\omega = \mathrm{i}\frac{|z^2|\,\mathrm{d}z^t \wedge \mathrm{d}\bar{z} - z^\dagger\,\mathrm{d}z \wedge z^t\,\mathrm{d}\bar{z}}{|z|^4} \tag{4.7}$$

takes the simple form

$$\omega = \cos\rho\,\eta_2 \wedge \eta_3 - \sin\rho\,\mathrm{d}\rho \wedge \eta_1, \tag{4.8}$$

which should be compared with the expression given in Gibbons & Pope (1978) in coordinates adapted to the $U(2)$ symmetry of $\mathbb{CP}^2$. The form ω is invariant under the 180° rotation about the 1-axis and hence well defined on the generic $SO(3)$-orbits. It is manifestly closed, but not exact: for $\rho > 0$, we can write $\omega = \mathrm{d}(\cos\rho\,\eta_1)$, but this expression is not valid on the exceptional $SO(3)$-orbit where $\rho = 0$, because η_1 is not well defined there.

The Kähler form ω is self-dual with respect to the complex orientation (with volume element $\mathrm{d}V = \omega^2 = -\sin(2\rho)\,\mathrm{d}\rho \wedge \eta_1 \wedge \eta_2 \wedge \eta_3$). Because it is closed, it is also harmonic. The existence of a non-exact harmonic, self-dual form on $\mathbb{CP}^2$ follows from the fact that the signature of $\mathbb{CP}^2$ is 1. In view of our interpretation of signature as baryonic charge, one might expect there to be a baryonic interpretation of ω. We return to this question when discussing the AH model of the proton further in §8.

(*b*) *Hitchin's one-parameter family of Einstein metrics*

If the $\mathbb{CP}^2$ model for the neutron turns out to be too naive, there is a more sophisticated variant that could be explored. This arises from the sequence of self-dual Einstein manifolds $M(k)$, for $k \geq 3$, studied by Hitchin (1996). The manifolds $M(k)$ for even $k \geq 4$ are all defined on the same space as our proton model, namely $\mathbb{CP}^2$ but with $\mathbb{RP}^2$ removed. $M(4)$ is $\mathbb{CP}^2$ with the Fubini–Study metric and all the metrics on $M(k)$, $k > 4$ and even, are incomplete on the open set $\mathbb{CP}^2 \backslash \mathbb{RP}^2$, but can be completed to metrics on $\mathbb{CP}^2$ with a conical singularity of angle $4\pi/(k-2)$ along $\mathbb{RP}^2$. For odd $k \geq 3$, the manifolds are defined on S^4, with $M(3)$ being S^4 with its standard metric. This time the metrics of $M(k)$, $k > 3$ and odd, are incomplete on the open set $S^4 \backslash \mathbb{RP}^2$, but can be completed to metrics on S^4 with a conical singularity of angle $2\pi/(k-2)$ along $\mathbb{RP}^2$.

The sequence of conical manifolds $M(k)$ for even $k \geq 4$ and starting with $\mathbb{CP}^2$ has decreasing scalar curvature and converges to AH as $k \to \infty$. It may turn out that some other value of k gives a better model for the neutron than $k = 4$. Note that for $k > 4$, the conical singularity breaks the symmetry down to $SO(3)$. Even for $k = 4$, we shall see later that other factors break the symmetry in this way.

Hitchin also pointed out (Hitchin 1995, 1996) that the family $M(k)$ can be extended to real parameter values. For any real $k > 0$, $M(k)$ is related to the moduli space of centred $SU(2)$ monopoles over hyperbolic space of curvature $-1/p^2$, where $p = (k-4)/4$. When k is not an integer, $M(k)$ is not an orbifold and the explicit methods of Hitchin (1996) then do not apply. Nonetheless, having

k as a real parameter gives useful room for manoeuvre in modelling the neutron and may provide contact with conventional nuclear models. In particular, $1/k$ may play a role as a small parameter that controls the breaking of isospin symmetry.

The forthcoming paper by Atiyah & LeBrun (in preparation) contains a signature formula for Riemannian manifolds with conical singularities, like the Hitchin manifolds $M(k)$. We summarize that result in §8.

5. The neutrino

Having put forward models of the electron, proton and neutron, it is then natural to look for a similar model of the neutrino. As it has no electric charge, it should, like the neutron, be modelled by a compact manifold. It should have symmetry similar to the $U(2)$ symmetry of the electron and should have positive curvature. It should have zero baryon number, that is, vanishing signature.

Just as $\mathbb{CP}^2$ is the most obvious model for the neutron, the standard 4-sphere, S^4, is the most obvious model for the neutrino. Again this has more symmetry than we need, $SO(5)$ instead of $SO(3)$. Just as a distinguished $\mathbb{RP}^2$ in $\mathbb{CP}^2$ picks out the smaller $SO(3)$ symmetry, so a distinguished S^2 in S^4 is needed to cut down the symmetry of the neutrino.

To exhibit the symmetry, we parametrize S^4 in terms of vectors $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{y} \in \mathbb{R}^2$ satisfying the constraint

$$\boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{y} \cdot \boldsymbol{y} = 1. \tag{5.1}$$

The metric is then

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} + \mathrm{d}\boldsymbol{y} \cdot \mathrm{d}\boldsymbol{y}. \tag{5.2}$$

The group $SO(3) \times SO(2)$ acts in the obvious way on the pair of vectors $(\boldsymbol{x}, \boldsymbol{y}) \in S^4$ and preserves the metric. In order to compare with other metrics discussed in this paper, we parametrize

$$\left.\begin{aligned} \boldsymbol{x} &= \sin\rho(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ \text{and} \qquad \boldsymbol{y} &= \cos\rho(\cos\chi, \sin\chi), \end{aligned}\right\} \tag{5.3}$$

with $\rho \in [0, \pi/2]$, $\chi \in [0, 2\pi)$ and the usual ranges for the polar coordinates θ, ϕ on S^2, and find the expression

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \sin^2\rho(\eta_1^2 + \eta_2^2) + \cos^2\rho\,\mathrm{d}\chi^2. \tag{5.4}$$

Here, we used that $\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 = \eta_1^2 + \eta_2^2$ in terms of the left-invariant 1-forms defined in equation (A 2). The generic $SO(3) \times SO(2)$ orbit is $S^2 \times S^1$, but this collapses to S^1 when $\rho = 0$ and to the distinguished S^2 when $\rho = \pi/2$.

Note that S^4 is conformally flat, so that the Weyl tensor vanishes and is trivially self-dual. S^4 has no middle-dimensional homology, so the signature is zero and hence our model neutrino has zero baryon number, as required. Because S^4 also has an orientation-reversing isometry, our model seems to suggest that the neutrino coincides with the anti-neutrino. For this and other reasons (see later), our choice of S^4 is very tentative and provisional. As with the $\mathbb{CP}^2$ model for the neutron, it should be regarded at present as a prototype.

To address the symmetry breaking issue, and several others, we will, in the next section, discuss how our various models fit into conventional 3-space.

6. Attaching the models to space

So far our models are abstract objects, 4-manifolds on their own, which are supposed to model four basic particles of nature. How are we to view them in the real world?

Let us begin with the easiest case, that of the electron. Thought of originally as the Dirac monopole, the idea is well known. We consider Kaluza–Klein space as a Riemannian 4-manifold with a circle action. Away from matter, this space is assumed to be a circle bundle over $\mathbb{R}^3$. Outside a given region in $\mathbb{R}^3$ that is electrically neutral, the bundle is assumed to be (topologically) just the product space. If a region is electrically charged, the circle bundle over the boundary is supposed to have a Chern class equal to minus the charge. If we start from the vacuum, then inserting one electron amounts to attaching a truncated version of TN to the boundary. This truncation turns the idealized model into a more realistic model of a particle. If other particles are present, TN will be an approximation to the precise metric. This approximation is some measure of the force exerted on the electron. In a dynamical theory, forces should emerge from the equations, a task for the future.

Next, let us move on to the proton. This is found to be similar, using a truncated version of AH. However, as pointed out earlier, the circle fibration is not oriented now so that the asymptotic 3-space is not $\mathbb{R}^3$ (unless our region contains equal numbers of protons and anti-protons).

The model for the neutron is compact, so that there is no way to attach it to a boundary. Instead, we propose that our model neutron (a copy of $\mathbb{CP}^2$) intersects our 4-space in a surface. This surface should project to a surface in 3-space that is the 'boundary' of the neutron as seen by an observer. Because we want to keep the neutron similar to the proton (except for the charge), it seems reasonable to take this surface to be a copy of $\mathbb{RP}^2$. But as the charge is now zero, the circle bundle over the surface should be trivial. This could arise as follows. Pick a point in $\mathbb{R}^3$ and blow it up to give an $\mathbb{RP}^2$, so that we are modifying 3-space in this neighbourhood. Keep the circle bundle trivial, so making the charge zero. Lift this $\mathbb{RP}^2$ into the total space of the circle bundle and let the $\mathbb{CP}^2$ neutron model intersect 4-space in this $\mathbb{RP}^2$, which is the distinguished $\mathbb{RP}^2$ in $\mathbb{CP}^2$ (hence breaking the larger symmetry of $\mathbb{CP}^2$). In this construction, the metric on $\mathbb{CP}^2$ need not be changed. The only change that is needed is the change in metric on the background 4-space obtained from the blowing-up process in 3-space.

Finally, we come to the neutrino. From the other examples, it is clear what is required. This time we just take a 3-ball in $\mathbb{R}^3$ with a boundary 2-sphere. The circle bundle over it is trivial and we lift the sphere to the total 4-space. We now require the S^4 neutrino model to intersect our 4-space in the chosen 2-sphere. The choice of the 2-sphere in S^4 again breaks the symmetry, down to $SO(3) \times SO(2)$. The 2-sphere need not be a great (geodesic) sphere, and this provides a parameter to play with.

Note that, in this case, we could reinterpret the picture as the surgery that kills off the circle and leaves a 2-sphere. This means that for the electron, the proton and the neutrino, we can still think of our 'space' as a 4-manifold. But this does not seem to work for the neutron, where we have to settle for a 4-space with intersecting components like a complex algebraic surface with double curves.

The four models that we have proposed for the four basic particles should be geometrically related in some way to account for the process of beta decay in which a neutron breaks up into a proton, an electron and an anti-neutrino. The opposite asymptotic behaviour of AH and TN is a good start, but the difference in the asymptotic fundamental groups presents a problem. This suggests that the model of the neutrino should somehow bridge the gap and it argues against the simplicity of the 4-sphere. We hope to pursue this question.

7. Spin 1/2

In all our models, we have a natural action of the symmetry group of rotations ($SO(3)$ or $SU(2)$) preserving the metric and the 'asymptotics', the details of which differ according to the cases. For the neutral, compact models, we interpret 'asymptotic' to mean the behaviour near the distinguished surface where the 4-manifold intersects 3-space, which is either an $\mathbb{RP}^2$ or an S^2. For the electrically charged, non-compact models, we have an asymptotic fibration by circles over physical 3-space; rotations preserve this fibration and induce an $SO(3)$-action on the base. The hyperkähler structures on TN (Gibbons & Ruback 1988) and AH (Olivier 1988) can be used to construct Cartesian coordinates on physical 3-space with the physically required transformation properties under spatial rotations. Here, we make essential use of the fact that the complex structures on TN and AH transform as vectors under rotations, as explained at the beginning of §3.

For a model to represent a particle of spin 1/2, we must include the data necessary to lift the rotation group action to an $SU(2)$ action, and to construct its two-dimensional representation. To achieve this for non-compact (electrically charged) models, we require a spin structure on the 4-manifold, while for compact (neutral) models, we only require a spin structure on the inside, obtained by removing a distinguished surface X from the 4-manifold. In this short section, we explain why, in each of the models considered in this paper, the lift of the rotation group action to the spin bundle is an $SU(2)$-action. We do not attempt to construct naturally associated spin 1/2 representations here, but comment on how this may be done.

For the TN model of the electron, there is nothing to do because the rotation group action on TN is an $SU(2)$-action. For the neutron model, we view the required data as the compactification of the proton model AH, not just topologically, but also with the action of the rotation group. In particular, the $\mathbb{RP}^2$ at infinity is part of the data. We now simply require the extra data of a spin structure on the inside $\mathbb{CP}^2\backslash\mathbb{RP}^2$, i.e. on AH.

It might be thought that as the spin structure on AH is unique, there is nothing gained by the additional data, but this is to ignore the interaction with the symmetries. We must now require the rotation group to lift to the spin bundle, and this may require us to pass to $SU(2)$, in which case we label the model as fermionic. Otherwise, we call it bosonic. To see that, in principle, either case could occur, consider the two manifolds $Y_1 = SO(3) \times \mathbb{R}$ and $Y_2 = \mathbb{R}^3 \times \mathbb{R}$. We take the left-translation action of $SO(3)$ on Y_1 and the standard action on Y_2. For Y_1, the tangent bundle is trivial and we can choose the trivial spin

bundle (though, because Y_1 is not simply connected, there is another choice). The $SO(3)$-action extends without going to $SU(2)$, so Y_1 is bosonic in our terminology. For Y_2, the fixed point at the origin of $\mathbb{R}^3$ means that we can only lift to the spin bundle after passing to $SU(2)$, so Y_2 is fermionic.

For AH, we have to show that, with its action of $SO(3)$, it is fermionic. There are several ways to do this. Perhaps the simplest (in line with the example Y_1 above) is to note that the action is not free and that the isotropy group of a point on the core is $SO(2)$. To lift even this subgroup to the spin bundle over the fixed point requires us to go to the double cover.

A comparison between our models for the electron and proton is illuminating. As we pointed out, $SU(2)$ acts on the electron but only $SO(3)$ acts on the proton. Thus, the fermionic natures of both our models differ. In one case, it is inherent in the symmetry, while in the other, it is geometric or topological.

Our tentative model for the neutrino is just the round 4-sphere, with a distinguished 2-sphere at infinity given by the decomposition $\mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R}^2$ and the corresponding action of $SO(3)$. The inside, got by removing S^2, has an infinite cyclic fundamental group, so there are infinitely many spin structures, but only one extends to the whole of S^4. If we pick this, then it is easy to see that the lift of $SO(3)$ to the spin structure requires us to pass to $SU(2)$ (for example, we can use an $SO(2)$ fixing a point at infinity and argue as with the proton). Thus, our model of the neutrino is fermionic.

Our discussion so far shows that our geometric approach furnishes fermionic models, but it does not establish that they necessarily give spin 1/2. This requires constructing the two-dimensional representation of $SU(2)$ and relating it to the asymptotic region. We expect that the required two-dimensional representations can be constructed in terms of eigenspaces of the Dirac operator on our model manifolds. In the case of $\mathbb{CP}^2$, this may involve a Spin^c-structure or restricting attention to its inside; in the non-compact cases, one should couple the Dirac operator to a $U(1)$-bundle with the curvature proportional to one of the harmonic 2-forms discussed in §8.

8. Charges, energies and fluxes

(*a*) *General remarks*

So far, we have focused on topological and geometrical features of our models and explained how they describe general properties of particles—like baryon number, electric charge and location in space. We want to keep an open mind about how our geometric models make quantitative contact with the physics of elementary particles. In particular, we do not assume that this should necessarily happen in the standard framework of Lagrangian field theory, where dynamics, conservation laws and even the quantum theory are all derived from an action functional.

The purpose of this section is to illustrate that our geometric models for particles nevertheless contain natural candidates for the kind of quantities that arise in Lagrangian models, like energy density and electric fields. We concentrate on the non-compact models and show that electric charge can be represented by a harmonic 2-form, thus making contact with the usual description of electric

Table 1. Geometric properties of 4-manifolds and their physical interpretation.

	$-X^2$ (electric charge)	τ (baryon number)	χ	$\|R\|^2/(8\pi^2)$
TN (electron)	−1	0	1	1
AH (proton)	1	1	2	2
$\mathbb{CP}^2$ (neutron)	0	1	3	3
S^4 (neutrino)	0	0	2	2

flux. One important feature of the densities and fields considered in this section is that they are defined on the 4-manifold so that they can only be interpreted as conventional *spatial* densities and fields in the asymptotic region of the 4-manifold that projects down to physical 3-space.

We begin with a summary of the topological quantities and their physical interpretation for each of the 4-manifolds considered thus far. Recall that, for non-compact manifolds, the electric charge is minus the self-intersection number X^2 of the manifold X representing infinity in their compactification. For compact manifolds, the electric charge is 0. The compactification is $\mathbb{CP}^2$ for both TN and AH, but $X = \mathbb{CP}^1$ for TN (a line in $\mathbb{CP}^2$) with self-intersection number $X^2 = 1$, while $X = \mathbb{RP}^2$ for AH with self-intersection number $X^2 = -1$.

Recall also that we have identified baryon number with the signature τ of the 4-manifold. For a non-compact oriented manifold, the signature is defined as the signature of the image of the compactly supported cohomology in the full cohomology (Atiyah *et al.* 1975). The topology of TN is that of $\mathbb{R}^4$, so the signature vanishes. The signature of AH is 1. This follows from the fact that its two-dimensional homology is generated by the core 2-sphere, and that the self-intersection number of this 2-sphere is positive (in fact equal to +4). The same argument applies to the sequence of Hitchin manifolds $M(k)$ for $k \geq 4$ and even, reviewed in §4.

In table 1, we list the electric charge and baryon number as well as the Euler characteristic χ and the squared L^2-norm $\|R\|^2$ of the Riemann curvature for the four 4-manifolds mainly discussed in this paper. The Euler characteristic is homotopy invariant, and so can be computed for TN and AH by noting that the former retracts to a point and the latter to a 2-sphere.

(*b*) *Signature, Euler characteristic and L^2-norm of the Riemann curvature*

For compact Riemannian 4-manifolds, there are standard formulae for the Euler characteristic and signature in terms of integrals over the 4-manifold involving the Riemann curvature. In the non-compact cases, these bulk contributions need to be supplemented by boundary integrals and (for the signature) a subtle spectral contribution (η-invariant); see Eguchi *et al.* (1980) and Atiyah & LeBrun (in preparation) for a summary. For manifolds with conical singularities like the sequence of Hitchin manifolds $M(k)$, a signature formula was recently found (Atiyah & LeBrun in preparation), which we also review below. We first discuss the bulk contributions.

Writing R for the Riemann tensor, we define the squared L^2-norm of R (for compact and non-compact manifolds) as

$$\|R\|^2 = \int_M \sum_{a<b} R_{ab} \wedge *R_{ab}, \tag{8.1}$$

where $*R_{ab}$ is the Hodge dual of R_{ab}. The form that integrates to the first Pontrjagin class on a compact manifold is

$$p_1 = \frac{1}{4\pi^2} \sum_{a<b} R_{ab} \wedge R_{ab}, \tag{8.2}$$

and the form that integrates to the signature τ in the compact case is

$$S = \frac{1}{3} p_1, \tag{8.3}$$

so that

$$\tau(M) = \int_M S = \frac{1}{3} \int_M p_1 = \frac{1}{12\pi^2} \int_M \sum_{a<b} R_{ab} \wedge R_{ab}. \tag{8.4}$$

The form that integrates to the Euler characteristic χ in the compact case is

$$e = \frac{1}{16\pi^2} \sum_{a<b} \epsilon^{abcd} R_{ab} \wedge R_{cd}. \tag{8.5}$$

Note that, with our conventions and for compact Einstein manifolds (LeBrun 2004),

$$\chi(M) = \int_M e = \frac{1}{8\pi^2} \|R\|^2, \tag{8.6}$$

which determines $\|R\|^2$ for $\mathbb{CP}^2$ and S^4 in terms of their topology.

The Riemann curvature on a 4-manifold may also be thought of as a mapping of 2-forms. Exploiting the fact that the space Λ^2 of 2-forms on an oriented 4-manifold decomposes into $\pm$-eigenspaces $\Lambda^{\pm}$ of the Hodge star operator $*$, we get a corresponding decomposition of the Riemann curvature into irreducible pieces (Eguchi *et al.* 1980),

$$R = \begin{pmatrix} W^+ + \dfrac{s}{12} & B \\ *B & W^- + \dfrac{s}{12} \end{pmatrix}. \tag{8.7}$$

Here, $W^{\pm}$ are the self-dual and anti-self-dual parts of the Weyl tensor, s is the scalar curvature and B amounts to the trace-free part of the Ricci curvature. Then, the signature of a compact manifold can also be expressed as (LeBrun 2004)

$$\tau(M) = \frac{1}{12\pi^2} (\|W^+\|^2 - \|W^-\|^2). \tag{8.8}$$

For a self-dual manifold, $W^- = 0$; so, up to the factor $12\pi^2$, τ is given by the L^2-norm of the Weyl tensor W^+, and is non-negative.

Because the metrics on both TN and AH are hyperkähler, B and s vanish, so the full Riemann curvature is self-dual. Therefore, the bulk contribution to

both the signature and the Euler characteristic can be expressed in terms of the L^2-norm of the Riemann curvature, which equals the L^2-norm of W^+. In the appendix, we show that

$$\|R\|^2_{\mathrm{TN}} = 8\pi^2 \quad \text{and} \quad \|R\|^2_{\mathrm{AH}} = 16\pi^2, \tag{8.9}$$

so that the bulk contributions to the Euler characteristic are both in agreement with the topological results listed above,

$$\int_{\mathrm{TN}} e = \frac{8\pi^2}{8\pi^2} = 1 \quad \text{and} \quad \int_{\mathrm{AH}} e = \frac{16\pi^2}{8\pi^2} = 2. \tag{8.10}$$

The bulk contributions to the signatures, on the other hand, turn out to be fractional,

$$\int_{\mathrm{TN}} S = \frac{8\pi^2}{12\pi^2} = \frac{2}{3} \quad \text{and} \quad \int_{\mathrm{AH}} S = \frac{16\pi^2}{12\pi^2} = \frac{4}{3}. \tag{8.11}$$

As shown in Atiyah & LeBrun (in preparation), the fall-off of the spin connection and curvature implies that the boundary integrals do not contribute in the limit to either the Euler characteristic or the signature. However, the fractional values of the bulk integrals for the signature show that there must be a non-zero contribution from the η-invariant.

In Atiyah & LeBrun (in preparation), a signature formula is derived for Riemannian 4-manifolds with conical singular metrics. The Hitchin manifolds $M(k)$ are of this type, and $M(\infty)$ can be identified with AH. The signature of a 4-manifold M whose metric has a conical singularity of angle $2\pi/\kappa$ along a surface X is

$$\tau(M) = \frac{1}{3}\int_M p_1 - \frac{1}{3}\left(1 - \frac{1}{\kappa^2}\right) X^2, \tag{8.12}$$

where p_1 is the form (8.2) that integrates to the Pontrjagin class, and X^2 is the self-intersection number of X in M.

The squared L^2-norm (8.1) of the Riemann curvature is a geometric analogue of the Yang–Mills functional used for defining the energy in gauge theories. It is a natural candidate for measuring the energy or mass in our geometric models of matter. For certain configurations, the value of the integral (8.1) is related to the Euler characteristic and the signature of the underlying 4-manifold. This, too, is reminiscent of the Yang–Mills functional in four dimensions whose value on self-dual configurations (instantons) is proportional to the second Chern number. However, in the geometric setting, the details are more complicated (see LeBrun 2004).

We have already seen that for hyperkähler metrics, the integral (8.1) is proportional to the bulk contribution for both the Euler characteristic and the signature. We now study the integrand of (8.1) for the TN and AH manifolds to shed light on its physical interpretation. In the notation of the appendix, the integrand of (8.1) in both the TN and AH case can be written as

$$\mathrm{d}F \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 = \frac{r}{ab^2c}\frac{\mathrm{d}F}{\mathrm{d}r}\,\mathrm{d}V, \tag{8.13}$$

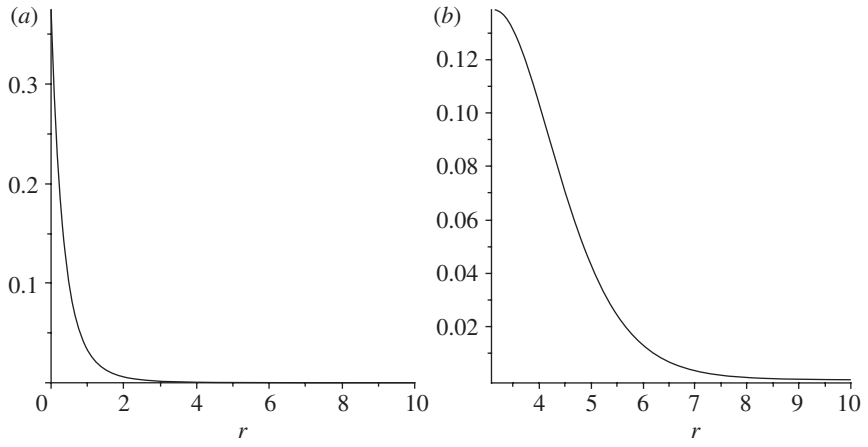


Figure 1. The density (8.14) for (*a*) TN and (*b*) AH.

where $\mathrm{d}V$ is the volume element. If the integral (8.1) is interpreted as an energy, then the combination

$$\frac{r}{ab^2c}\frac{\mathrm{d}F}{\mathrm{d}r} \tag{8.14}$$

should be viewed as an energy density. We plot this density for TN and AH in figure 1. It is finite at, respectively, the origin and the core, and for large r, falls off like $1/r^6$. In the AH case, we compute the functions a, b, c and hence F numerically; in the TN case, we have the explicit formula

$$\frac{r}{ab^2c}\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{24}{(r+2)^6}. \tag{8.15}$$

The qualitative features of the plots in figure 1 are consistent with the interpretation of (8.14) as an energy density for the electron (TN) and the proton (AH). They are peaked at, respectively, the origin and the core, and the peak is much broader in the AH model than in the TN model (we will return to the interpretation of length scales in the model in §9). However, the decay according to $1/r^6$ in both cases is difficult to interpret in conventional terms: the electric Coulomb energy density should fall off like $1/r^4$, while any baryonic energy contribution should vanish in the case of the electron. We therefore look at other candidates for energy densities in the next section.

(*c*) *Electric and baryonic fluxes*

We now construct fields on TN and AH that carry the electric flux measured by the self-intersection numbers X^2 tabulated in table 1. We do this by systematically studying rotationally symmetric harmonic 2-forms on both TN and AH. The question of computing harmonic 2-forms on the TN and AH manifolds arose in physics in the context of testing S-duality; see Sen (1994) for AH and Lee *et al.* (1996) and Gauntlett & Lowe (1996) for TN. Harmonic forms also play a role as curvatures of line bundles over TN and AH; see Manton & Schroers (1993)

for a discussion of this for AH. Here, we also require a harmonic form on the branched cover of AH that has not previously been considered in the literature.

On TN, one finds (Brill 1964; Pope 1978; Gauntlett & Lowe 1996; Lee *et al.* 1996) that the only square-integrable harmonic and rotationally symmetric 2-form is, up to an overall arbitrary constant,

$$\omega_3^+ = \left(\frac{r}{r+2}\eta_1 \wedge \eta_2 + \frac{2}{(r+2)^2}\,\mathrm{d}r \wedge \eta_3\right) = \mathrm{d}\left(\frac{r}{r+2}\eta_3\right). \tag{8.16}$$

The form

$$\omega_{\mathbb{CP}^1} = -\frac{1}{4\pi}\omega_3^+ \tag{8.17}$$

is Poincaré dual to the $\mathbb{CP}^1$ at infinity because

$$\int_{\mathbb{CP}^1} \omega_{\mathbb{CP}^1} = 1, \tag{8.18}$$

which is the self-intersection number of the line $\mathbb{CP}^1$ in $\mathbb{CP}^2$.

As the 2-form $-\omega_{\mathbb{CP}^1}$ is harmonic and because its total flux through infinity equals the electric charge, we interpret it as the electric field of the electron. Although we have adopted a viewpoint dual to standard electromagnetism, with a purely spatial 2-form being interpreted as electric rather than magnetic flux, it is interesting that the self-dual 2-form (8.16) also contains a term that allows for a conventional electric interpretation: when we contract $\omega_{\mathbb{CP}^1}$ with the vector field ∂_ψ along the fibres (using $\eta_3(\partial_\psi) = 1$), we obtain a purely radial field that, asymptotically, falls off like $1/r^2$.

It follows from the self-duality of the electric field that its squared L^2-norm equals the self-intersection number (i.e. minus the electric charge),

$$\int_{\mathrm{TN}} \omega_{\mathbb{CP}^1} \wedge \omega_{\mathbb{CP}^1} = 1. \tag{8.19}$$

Ignoring overall factors and working with ω_3^+, we write

$$\|\omega_3^+\|^2 = \int_{\mathrm{TN}} \omega_3^+ \wedge \omega_3^+ = \int \rho_3^+\,\mathrm{d}V, \tag{8.20}$$

with the TN volume element $\mathrm{d}V$ defined in equation (A 9). We then interpret the integrand

$$\rho_3^+(r) = \frac{2}{(r+2)^4} \tag{8.21}$$

as an electric energy density. For comparison with the AH case provided later, we plot the profile function $r/(r+2)$ appearing in (8.16) and the energy density ρ_3^+ in figure 2.

On AH, there are only two independent $SO(3)$-invariant harmonic forms that respect the identification (A 3) (Gibbons & Ruback 1988; Manton & Schroers 1993; Sen 1994). They have the structure

$$\Omega_1^\pm = G_1^\pm \eta_2 \wedge \eta_3 + \mathrm{d}G_1^\pm \wedge \eta_1, \tag{8.22}$$

for functions $G_1^\pm$ of r, which satisfy certain ordinary differential equations. One shows that only G_1^+ is finite at the core and decays at infinity. In fact, the solution

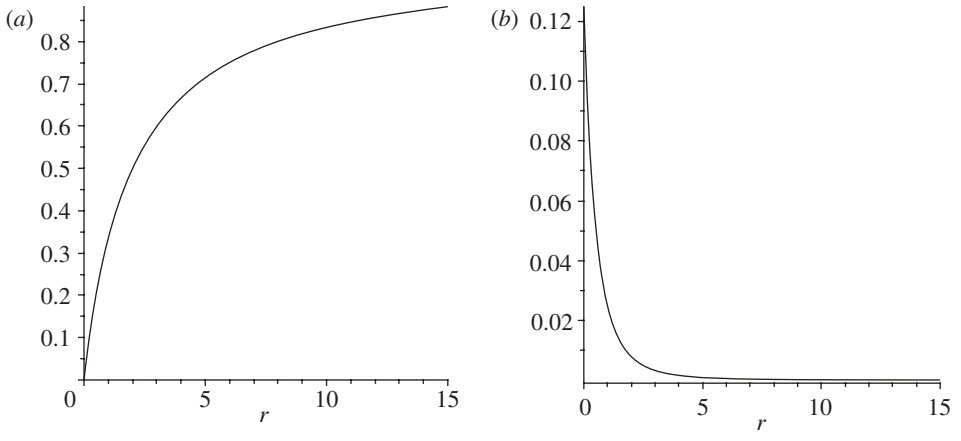


Figure 2. (*a*) The 'electric' profile function $r/(r+2)$ and (*b*) the energy density ρ_3^+.

decays exponentially fast at infinity, with the leading term proportional to $\mathrm{e}^{-r/2}$. We normalize $G_1^+(\pi)=1$, so that near the core

$$G_1^+(r)\sim 1-\frac{1}{\pi^2}(r-\pi)^2,\quad \text{for } (r-\pi) \text{ small.} \tag{8.23}$$

The 2-form Ω_1^+ has no interpretation in terms of electric flux. However,

$$\Omega_{\mathrm{core}}=-\frac{1}{\pi}\Omega_1^+ \tag{8.24}$$

is dual to the core S^2 in AH because

$$\int_{\mathrm{core}}\Omega_{\mathrm{core}}=4, \tag{8.25}$$

which is the self-intersection number of the core (a conic) in $\mathbb{CP}^2$. In the branched covering of AH by $\mathbb{CP}^1\times\mathbb{CP}^1$, the core is the diagonal D described at the end of §3. The squared L^2-norm on AH is again the self-intersection number, i.e. $\int_{\mathrm{AH}}\Omega_{\mathrm{core}}\wedge\Omega_{\mathrm{core}}=4$.

The harmonic form (4.8) on $\mathbb{CP}^2$ is related to the signature of $\mathbb{CP}^2$ through the fact that the signature of a compact 4-manifold equals the difference of the dimensions of the spaces of self-dual and anti-self-dual harmonic representatives of the second de Rham cohomology. It seems likely that the (up to scale) unique bounded and rotationally symmetric self-dual harmonic form Ω_{core} on AH is related to the self-dual harmonic form (4.8) on $\mathbb{CP}^2$ via the sequence of Hitchin manifolds reviewed in §4.

As the existence of Ω_{core} on AH is linked to the signature of AH being 1, and because signature represents baryon number in our approach, it may be possible to interpret the detailed structure of Ω_{core} in baryonic terms. The exponential fall-off exhibited by Ω_{core} is reminiscent of the proton's pion field in the Yukawa description of the nuclear force. In figure 3, we plot the numerically computed

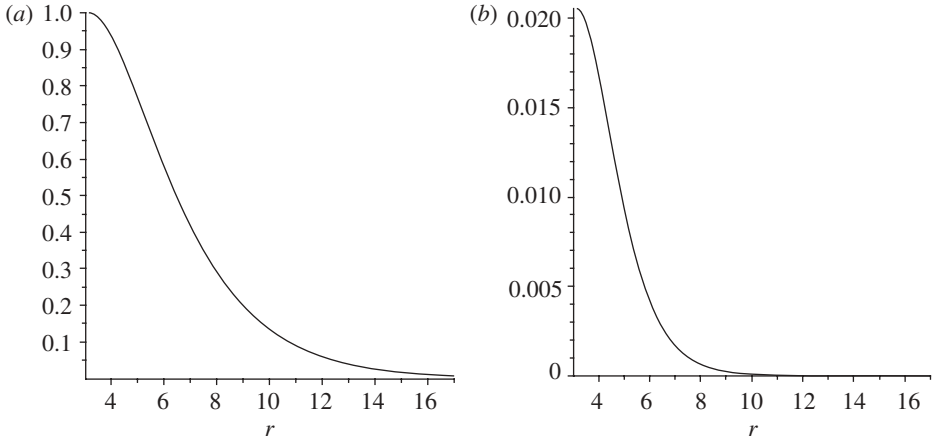


Figure 3. (*a*) The 'baryonic' profile function G_1^+ and (*b*) the energy density ρ_1^+.

profile function G_1^+ and the associated energy density, defined in analogy to the TN energy density (8.20),

$$\rho_1^+ = 2\frac{(G_1^+)^2}{b^2c^2}. \tag{8.26}$$

This is finite at the core and decays exponentially according to e^{-r} for large r.

To find a harmonic 2-form on AH that can play the role of the proton's electric field, we need to go to a branched cover of AH, denoted AH′. The metric on the branched cover is not smooth at the core, but this will not affect the following calculations near infinity. Dropping the requirement that forms are invariant under the identification (A 3), we find that the closure condition on self- or anti-self-dual rotationally symmetric 2-forms has only one further solution that is finite at the core and that remains bounded for large r. This is the 2-form

$$\Omega_3^- = G_3^-\eta_1\wedge\eta_2 + \mathrm{d}G_3^-\wedge\eta_3, \tag{8.27}$$

with a radial function G_3^- satisfying the differential equation

$$\frac{\mathrm{d}G_3^-}{\mathrm{d}r} = \frac{cf}{ab}G_3^-. \tag{8.28}$$

The solution vanishes at the core as

$$G_3^-(r) \sim C\sqrt{r-\pi}, \quad \text{for } (r-\pi) \text{ small}, \tag{8.29}$$

for some constant C. The large-r behaviour is

$$G_3^-(r) \sim \tilde{C}\frac{r-2}{r}, \tag{8.30}$$

where $\tilde{C}$ is another constant. This leads to an anti-self-dual form on AH′, which is square-integrable (as we shall show later) and which has not been considered previously. It is our candidate for the electric field of the proton.

We explained at the end of §3 that AH′ compactifies to $\mathbb{CP}^1\times\mathbb{CP}^1$ as a manifold. The surface $X=\mathbb{RP}^2$ that compactifies AH to $\mathbb{CP}^2$ becomes the anti-diagonal D' in $\mathbb{CP}^1\times\mathbb{CP}^1$. As we saw at the end of §3, the self-intersection number

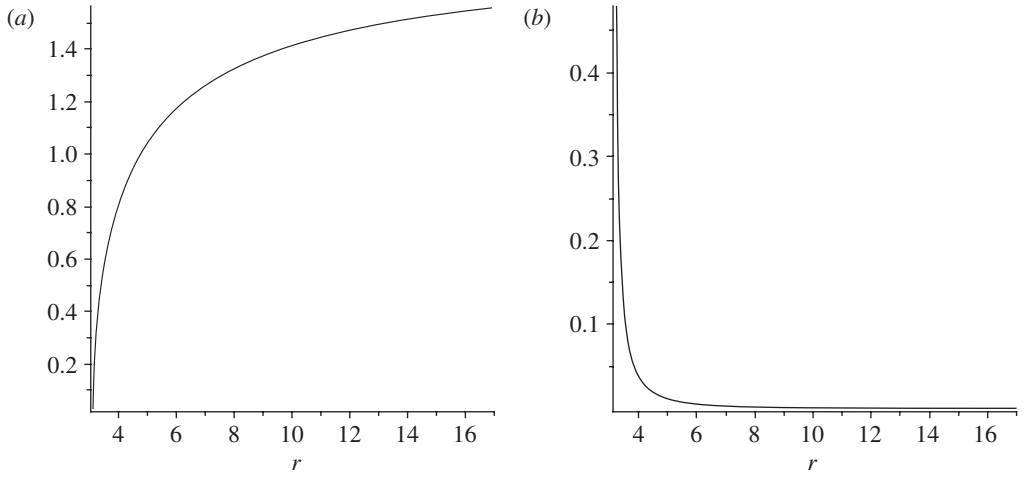


Figure 4. (*a*) The 'electric' profile function G_3^- and (*b*) the energy density ρ_3^-.

of D' is -2. Hence, choosing $\tilde{C}=1$, a harmonic representative of the Poincaré dual class to this D' is

$$\Omega_{D'}=\frac{1}{2\pi}\Omega_3^-. \tag{8.31}$$

This satisfies $\int_{D'}\Omega_{D'}=-2$, and its squared L^2-norm is 2 (minus the self-intersection number, because $\Omega_{D'}$ is anti-self-dual).

As the 2-form $-\Omega_{D'}$ on AH′ is harmonic and because its total flux through infinity, suitably interpreted, equals the electric charge, we think of $-\Omega_{D'}$ as representing the electric field of the proton. The comments made after equation (8.18) about the possibility of recovering a conventional electric field by contracting the electric 2-form on TN with the vector field along asymptotic fibres apply, suitably modified, to AH′.

In figure 4, we plot the numerically computed profile function G_3^- for $C=1$ (as defined in (8.29)) and the associated energy density

$$\rho_3^-=2\frac{(G_3^-)^2}{a^2b^2}. \tag{8.32}$$

This function has a $(r-\pi)^{-1}$ singularity at $r=\pi$, and falls off like r^{-4} for large r as in the TN case.

9. Conclusion and outlook

In introducing and illustrating geometric models of matter in this paper, we have concentrated on general, global properties of particles such as baryon number and electric charge. A pleasing aspect of our geometric approach is its prediction of precisely two stable electrically charged particles, one leptonic and one baryonic, with opposite electric charges.

An important theme throughout this paper is the implementation of rotational symmetry in our models. The rotation action preserves the metric and, in the compact cases, the distinguished surface where the 4-manifold intersects physical

3-space. We have shown that our models are fermionic in the sense that the lift of the rotation action to spin bundles (over the 'inside' of the 4-manifold in the compact cases) is necessarily an $SU(2)$-action, but we have left the explicit construction of spin 1/2 states for future work.

Non-vanishing electric charge and baryon number give rise to harmonic 2-forms on our model manifolds. These have allowed us to make contact with the conventional description of charged particles in terms of associated fields and fluxes.

There are various geometrically natural candidates for measuring energy or mass in our models, but we have not committed ourselves to any particular energy functional in this paper. In the absence of an energy measure, we are not able to make contact with the experimental data about particle masses or forces between particles.

We end our summary with some observations about relative scales predicted by our models for charged particles. The AH model relates three scale parameters: the size of the core (radius $\approx \pi$ in our geometric units), the size of the asymptotic circles (radius 2 in geometric units) and the scale 1 implicit in the exponential corrections to the asymptotic form of the coefficient functions (3.7) for the AH metric, which are proportional to e^{-r} (see Gibbons & Manton (1986) and Schroers (1991) for a discussion of these corrections in the context of magnetic monopoles). Interpreting the core radius as the proton radius and the scale in the exponential decay as the Compton wavelength λ_π of the pion, we find that our model correctly assigns the same order of magnitude to those two quantities. The details are not quite right (experimentally, the proton radius is just over half of λ_π), but this is not surprising, given our very rudimentary understanding of how our model relates to experiment. In any case, these considerations suggest that we should pick a length unit of about 1 fermi or 10^{-15} m in the AH model.

Because the asymptotic fibration by circles arises for all electrically charged particles, we expect the size of the asymptotic circle to have a purely electric interpretation, and we also assume that it is the same for AH and TN (this was implicit in the way we fixed scale and units for TN). One natural guess, at least for TN, is to relate the size of the asymptotic circle to the classical electron radius. Our models would then equate the orders of magnitude of the classical electron radius with the Compton wavelength of the pion, which is phenomenologically right. It is an attractive feature of TN as a model for the electron that it has a length scale, but no core structure.

Many avenues remain to be explored. In our geometric approach, fusion and fission of nuclei as well as decay processes involving both baryons and leptons (like the beta decay of the neutron) should have a description in terms of gluing and deformation of self-dual 4-manifolds. In order to study masses and binding energies of particles, we need to pick an energy measure. This could involve the norms of curvatures and harmonic forms discussed here, but may take a less conventional form. Our ideas about spin 1/2 need to be fleshed out. The Dirac operator on our model manifolds is likely to play a role in combining energy and spin considerations, and Seiberg–Witten theory on the model manifolds seems to be relevant too. In order to move beyond static considerations, time needs to be introduced. Unstable particles such as pions need to be included, presumably as fluctuations of the geometry, and one would like to see how quarks fit into our picture.

The list of open issues may seem daunting but, in each case, the geometric framework introduced here suggests natural lines of attack. We hope to pursue them in future work.

We thank Nigel Hitchin and Claude LeBrun for assistance and José Figueroa-O'Farrill for working with us during the early stages of the project.

Appendix A. Geometry and sign conventions

(*a*) *Orientation of self-dual manifolds*

We want to analyse various sign conventions in relation to self-dual manifolds. Referring to our discussion of the signature and Euler characteristic in §8, we note that $\mathbb{CP}^2$ with the Fubini–Study metric is self-dual for its complex orientation (Eguchi *et al.* 1980), agreeing with the positive signature $+1$.

We recall the argument that a hyperkähler manifold is self-dual for the orientation opposite to the complex orientation given by one of the family of complex Kähler structures. It is purely local, and just depends on the fact that the bundle of self-dual 2-forms for these complex orientations is flat. More generally, on any Kähler manifold, the bundle of self-dual 2-forms is the direct sum of the canonical line bundle K, its dual and a trivial bundle generated by the Kähler form.

Applying this argument to the Taub–NUT manifold TN, we recall that this has the isometry group $U(2)$ and that its topology (and symmetry) is that of $\mathbb{C}^2$ with the central $U(1)$ giving the scalar action. The other $U(1)$ actions, inside $SU(2)$, define the complex hyperkähler structures, which have the opposite orientation. Hence, TN is self-dual for the orientation that becomes that of $\mathbb{C}^2$ in the limit, discussed after equation (3.5), when the parameter ϵ goes to 0.

Next, we look at the Atiyah–Hitchin manifold AH, which is an open subspace of $\mathbb{CP}^2$ got by removing $\mathbb{RP}^2$. Its self-dual orientation, opposite to the complex orientations given by its hyperkähler metric, is just the orientation given by its complex structure as an open subset of $\mathbb{CP}^2$. This can be checked directly, but it is best seen by using the results of Hitchin (1996), reviewed in §4, which give a whole sequence of self-dual manifolds on the same space, starting from the Fubini–Study metric of $\mathbb{CP}^2$ and converging to AH.

(*b*) *Metric properties of TN and AH*

In this paper, we need an explicit parametrization of $SU(2)$ at various points. We define generators $t_a = (\mathrm{i}/2)\tau_a$, $a = 1, 2, 3$, where τ_a are the Pauli matrices (so $[t_a, t_b] = -\epsilon_{abc} t_c$) and parametrize $g \in SU(2)$ as

$$g(\phi, \theta, \psi) = \mathrm{e}^{\phi t_3} \mathrm{e}^{\theta t_2} \mathrm{e}^{\psi t_3}, \tag{A 1}$$

with ranges $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$. By defining left-invariant 1-forms η_a on $SU(2)$ via $g^{-1}\,\mathrm{d}g = \eta_1 t_1 + \eta_2 t_2 + \eta_3 t_3$, one finds

$$\left.\begin{aligned} \eta_1 &= -\sin\psi\,\mathrm{d}\theta + \cos\psi \sin\theta\,\mathrm{d}\phi, \\ \eta_2 &= \cos\psi\,\mathrm{d}\theta + \sin\psi \sin\theta\,\mathrm{d}\phi, \\ \eta_3 &= \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi \end{aligned}\right\} \tag{A 2}$$

and $\mathrm{d}\eta_i = \frac{1}{2}\epsilon_{ijk}\eta_j \wedge \eta_k$. They descend to $SO(3)$ by simply restricting ψ to $[0, 2\pi)$.

Both TN and AH can be parametrized in terms of Euler angles and a radial coordinate. For TN, the angular ranges are $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$. For AH, they are $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$ and $\psi \in [0, 2\pi)$ with the additional $\mathbb{Z}_2$ identification

$$(\theta, \phi, \psi) \simeq (\pi - \theta, \phi + \pi, -\psi), \tag{A 3}$$

which, in the asymptotic region, is the simultaneous reversal of spatial and fibre direction. Note that η_1 is invariant under equation (A 3), but η_2 and η_3 change sign. Computing angular volumes by integrating $\eta_1 \wedge \eta_2 \wedge \eta_3$, we find

$$\mathrm{Vol}(SU(2)) = -16\pi^2, \quad \mathrm{Vol}(SO(3)) = -8\pi^2 \quad \text{and} \quad \mathrm{Vol}(SO(3)/\mathbb{Z}_2) = -4\pi^2. \tag{A 4}$$

For both the TN and AH metrics (3.1), we introduce the tetrad

$$\theta^1 = a\eta_1, \quad \theta^2 = b\eta_2, \quad \theta^3 = c\eta_3 \quad \text{and} \quad \theta^4 = f\,\mathrm{d}r. \tag{A 5}$$

The self-duality conditions for the Riemann tensor

$$R_{12} = R_{34}, \quad R_{23} = R_{14} \quad \text{and} \quad R_{31} = R_{24} \tag{A 6}$$

computed from (3.1) with respect to the volume element

$$\mathrm{d}V = a\eta_1 \wedge b\eta_2 \wedge c\eta_3 \wedge f\,\mathrm{d}r = -fabc\,\mathrm{d}r \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 \tag{A 7}$$

are then equivalent to the set of ordinary differential equations

$$\frac{2bc}{f}\frac{\mathrm{d}a}{\mathrm{d}r} = (b - c)^2 - a^2 + 2\lambda bc, \quad + \text{cyclic}, \tag{A 8}$$

where '+ cyclic' means we add the two further equations obtained by cyclic permutation of a, b, c, and λ is a parameter that has to be either 0 or 1. In all cases, the resulting metrics are hyperkähler, but in order to obtain metrics whose hyperkähler structures are rotated by the $SU(2)$ action, we need to set $\lambda = 0$. This is the case for both TN and AH, so $\lambda = 0$ in equation (A 8).

The only solutions of equation (A 8) with $\lambda = 0$ that give rise to complete manifolds whose generic $SU(2)$- or $SO(3)$-orbit is three dimensional are the TN and AH metrics, whose coefficient functions we discussed in §3. It is important that the coefficient function c in (3.1) is negative for all r in the AH manifold. It implies in particular that the canonical volume element/orientation $\mathrm{d}V$ in equation (A 7), with $f = -b/r$,

$$\mathrm{d}V = -fabc\,\mathrm{d}r \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 = -\frac{ab^2c}{r}\sin\theta\,\mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi \wedge \mathrm{d}\psi \tag{A 9}$$

has opposite signs for TN and AH, in the following sense. Assuming the same orientation $\eta_1 \wedge \eta_2 \wedge \eta_3$ of the $SU(2)$ and $SO(3)$ orbits in TN and AH, respectively, the radial direction is oppositely oriented in the two cases: the natural radial line element $-fabc\,\mathrm{d}r$ has the same orientation as that of $\mathrm{d}r$ for TN, but the opposite orientation for AH. This will be important when evaluating various integrals. It means that, when using the coordinates r, θ, ϕ, ψ, we can use the conventional orientations for these coordinates when computing for AH, but should integrate in the negative r-direction when calculating on TN.

(*c*) L^2-*norms of the Riemann curvature*

The squared L^2-norm of the Riemann tensor is

$$\|R\|_M^2 = \int_M 2(R_{12} \wedge R_{12} + R_{23} \wedge R_{23} + R_{31} \wedge R_{31}), \tag{A 10}$$

where M is TN or AH, and we have used the self-duality of the curvature. The integrand can be simplified (see also Sethi *et al.* (1995), where this calculation is carried out for the AH metric) and becomes

$$2(R_{12} \wedge R_{12} + R_{23} \wedge R_{23} + R_{31} \wedge R_{31}) = \mathrm{d}F \wedge \eta_1 \wedge \eta_2 \wedge \eta_3, \tag{A 11}$$

where

$$F = 2(\mu_1 + \mu_2 + \mu_3 - 1)^2 - 8\mu_1\mu_2\mu_2, \tag{A 12}$$

$$\mu_1 = \frac{b^2 + c^2 - a^2}{2bc}, \quad \mu_2 = \frac{c^2 + a^2 - b^2}{2ca} \quad \text{and} \quad \mu_3 = \frac{a^2 + b^2 - c^2}{2ab}. \tag{A 13}$$

We denote the function F in the TN and AH cases by F_{TN} and F_{AH}.

For TN, we can compute explicitly and find

$$F_{\mathrm{TN}}(r) = -8\frac{2r+1}{(r+2)^4}. \tag{A 14}$$

Integrating equation (A 11) and using equation (A 4) as well as our sign convention for the volume form, we deduce

$$\|R\|_{\mathrm{TN}}^2 = (-16\pi^2)(F_{\mathrm{TN}}(0) - F_{\mathrm{TN}}(\infty)) = 8\pi^2. \tag{A 15}$$

For AH, we need

$$\mu_1(\infty) = \mu_2(\infty) = \frac{c}{2a}(\infty) = 0, \quad \mu_3(\infty) = 1 - \frac{c^2}{2a^2}(\infty) = 1, \tag{A 16}$$

as well as

$$\mu_1(\pi) = -1, \quad \mu_2(\pi) = \frac{1}{2} \quad \text{and} \quad \mu_3(\pi) = \frac{1}{2}, \tag{A 17}$$

where we took careful limits using equations (3.7) and (3.8). Therefore,

$$F_{\mathrm{AH}}(\infty) = 0 \quad \text{and} \quad F_{\mathrm{AH}}(\pi) = 4, \tag{A 18}$$

so

$$\|R\|_{\mathrm{AH}}^2 = (-4\pi^2)(F_{\mathrm{AH}}(\infty) - F_{\mathrm{AH}}(\pi)) = 16\pi^2. \tag{A 19}$$

References

Atiyah, M. & Hitchin, N. 1988 *The geometry and dynamics of magnetic monopoles*. Princeton, NJ: Princeton University Press.

Atiyah, M. F. & LeBrun, C. In preparation. The signature of 4-manifolds with conical singular metric.

Atiyah, M. F. & Manton, N. S. 1989 Skyrmions from instantons. *Phys. Lett. B* **222**, 438–442. (doi:10.1016/0370-2693(89)90340-7)

Atiyah, M. F. & Manton, N. S. 1993 Geometry and kinematics of two Skyrmions. *Commun. Math. Phys.* **153**, 391–422. (doi:10.1007/BF02096649)

Atiyah, M. & Singer, I. M. 1968 The index of elliptic operators: III. *Ann. Math.* **87**, 546–604. (doi:10.2307/1970717)
Atiyah, M. & Sutcliffe, P. 2005 Skyrmions, instantons, mass and curvature. *Phys. Lett. B* **605**, 106–114. (doi:10.1016/j.physletb.2004.11.015)
Atiyah, M. F., Patodi, V. K. & Singer, I. M. 1975 Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Camb. Phil. Soc.* **77**, 43–69. (doi:10.1017/S0305004100049410)
Bouchiat, C. & Gibbons, G. W. 1988 Non-integrable quantum phase in the evolution of a spin-1 system: a physical consequence of the non-trivial topology of the quantum state-space. *J. Phys. France* **49**, 187–199. (doi:10.1051/jphys:01988004902018700)
Brill, D. R. 1964 Electromagnetic fields in a homogeneous, nonisotropic universe. *Phys. Rev.* **133**, B845–B848. (doi:10.1103/PhysRev.133.B845)
Brown, G. E. & Rho, M. 2010 *The multifaceted skyrmion*. Singapore: World Scientific.
Dancer, A. S. & Strachan, I. A. B. 1994 Kähler–Einstein metrics with $SU(2)$ action. *Math. Proc. Camb. Phil. Soc.* **115**, 513–525. (doi:10.1017/S0305004100072273)
Donaldson, S. & Friedman, R. 1989 Connected sums of self-dual manifolds and deformations of singular spaces. *Nonlinearity* **2**, 197–239. (doi:10.1088/0951-7715/2/2/002)
Eguchi, T., Gilkey, P. B. & Hanson, A. J. 1980 Gravitation, gauge theories and differential geometry. *Phys. Rep.* **66**, 213–393. (doi:10.1016/0370-1573(80)90130-1)
Gauntlett, J. P. & Lowe, D. A. 1996 Dyons and S-duality in $N = 4$ supersymmetric gauge theory. *Nucl. Phys. B* **472**, 194–206. (doi:10.1016/0550-3213(96)00218-0)
Gibbons, G. W. & Manton, N. S. 1986 Classical and quantum dynamics of BPS monopoles. *Nucl. Phys. B* **274**, 183–224. (doi:10.1016/0550-3213(86)90624-3)
Gibbons, G. W. & Pope, C. N. 1978 $\mathbb{CP}^2$ as a gravitational instanton. *Commun. Math. Phys.* **61**, 239–248. (doi:10.1007/BF01940766)
Gibbons, G. W. & Ruback, P. J. 1988 The hidden symmetries of multicenter metrics. *Commun. Math. Phys.* **115**, 267–300. (doi:10.1007/BF01466773)
Gross, D. J. & Perry, M. J. 1983 Magnetic monopoles in Kaluza-Klein theories. *Nucl. Phys. B* **226**, 29–48. (doi:10.1016/0550-3213(83)90462-5)
Hawking, S. W. 1977 Gravitational instantons. *Phys. Lett. A* **60**, 81–83. (doi:10.1016/0375-9601(77)90386-3)
Hitchin, N. J. 1995 Twistor spaces, Einstein metrics and isomonodromic deformations. *J. Diff. Geom.* **42**, 30–112.
Hitchin, N. J. 1996 A new family of Einstein metrics. In *Manifolds and Geometry, Symp. Math., Pisa, September 1993*, vol. XXXVI, pp. 190–222. Cambridge, UK: Cambridge University Press.
LeBrun, C. 2004 Curvature functionals, optimal metrics, and the differential topology of 4-manifolds. In *Different faces of geometry* (eds S. Donaldson, Ya. Eliashberg & M. Gromov). New York, NY: Kluwer Academic/Plenum.
Lee, K.-M., Weinberg, E. J. & Yi, P. 1996 Electromagnetic duality and $SU(3)$ monopoles. *Phys. Lett. B* **376**, 97–102. (doi:10.1016/0370-2693(96)00286-9)
Manton, N. S. & Schroers, B. J. 1993 Bundles over moduli spaces and the quantisation of BPS monopoles. *Ann. Phys.* **225**, 290–338. (doi:10.1006/aphy.1993.1060)
Minerbe, V. 2010 On the asymptotic geometry of gravitational instantons. *Ann. Scient. Éc. Norm. Sup.* **43**, 883–924.
Olivier, D. 1988 Complex coordinates and Kähler potential for the Atiyah–Hitchin metric. *Gen. Rel. Grav.* **23**, 267–300.
Penrose, R. 1977 The twistor programme. *Rep. Math. Phys.* **12**, 65–76. (doi:10.1016/0034-4877(77)90047-7)
Pollard, D. 1983 Antigravity and classical solutions of five-dimensional Kaluza–Klein theory. *J. Phys. A* **16**, 565–574. (doi:10.1088/0305-4470/16/3/015)
Pope, C. N. 1978 Axial-vector anomalies and the index theorem in charged Schwarzschild and Taub-NUT spaces. *Nucl. Phys. B* **141**, 432–444. (doi:10.1016/0550-3213(78)90038-X)
Sakai, T. & Sugimoto, S. 2005 Low energy hadron physics in holographic QCD. *Prog. Theor. Phys.* **113**, 843–882. (doi:10.1143/PTP.113.843)
Schroers, B. J. 1991 Quantum scattering of BPS monopoles at low energy. *Nucl. Phys. B* **367**, 177–214. (doi:10.1016/0550-3213(91)90047-2)

Sen, A. 1994 Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and SL(2, Z) invariance in string theory. *Phys. Lett. B* **329**, 217–221. (doi:10.1016/0370-2693(94)90763-3)

Sethi, S., Stern, M. & Zaslow, E. 1995 Monopole and dyon bound states in $N = 2$ supersymmetric Yang–Mills theories. *Nucl. Phys. B* **457**, 484–510. (doi:10.1016/0550-3213(95)00517-X)

Skyrme, T. H. R. 1961 A non-linear field theory. *Proc. R. Soc. Lond. A* **260**, 127–138. (doi:10.1098/rspa.1961.0018)

Sorkin, R. D. 1983 Kaluza–Klein monopoles. *Phys. Rev. Lett.* **51**, 87–90. (doi:10.1103/PhysRevLett.51.87)

Sutcliffe, P. 2010 Skyrmions, instantons and holography. *J. High. Energy Phys.* **1008**, 1–25. (doi:10.1007/JHEP08(2010)019)