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Classical Mechanics in Galilean Space-Time

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Galilean space-time plays the same role in nonrelativistic physics that Minkowski space-time does in relativistic physics. In this paper, the fundamental concepts (velocity, momentum, kinetic energy, etc.) and principles (laws of motion and conservation laws) of classical physics are formulated in the language of Galilean space-time. Much of the development closely parallels the development of similar concepts and principles in the theory of special relativity.

1. INTRODUCTION

The term Galilean space-time has been used by Penrose,[1] Arnold,[2] and, presumably, others, to mean a four-dimensional flat continuum in whose geometry Galilean invariance is naturally incorporated in the same sense that Lorentz invariance is naturally incorporated in the geometry of Minkowski space-time. This built-in invariance acts as a filter to screen out meaningless statements; for example, one cannot say, in the language of Galilean space-time, that a particle is motionless, but only that its motion is uniform. Therefore, one is forced by the language of Galilean space-time to use only physically meaningful (i.e., Galilean-invariant) definitions and axioms in one's theories.

In this paper, we shall show how elementary classical-mechanical concepts have natural formulations in the language of Galilean space-time. The reader familiar with the theory of special relativity will not be able to avoid noticing the many parallels with that theory. A separate paper[3] will deal with the formulation of quantum mechanics within the framework of Galilean space-time; the needs of that paper dictate, in large measure, the choice of topics in this one.

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1.1. Notation and Conventions

The symbol $:=$ will be used to indicate equality by definition. The symbol $\mathbb{R}$ will be used to denote the set of real numbers. The space of linear mappings from a vector space U to a vector space V will be denoted $\mathrm{Lin}(\mathsf{U}, \mathsf{V})$ and the dual space of V will be denoted $\mathsf{V}^* := \mathrm{Lin}(\mathsf{V}, \mathbb{R})$. As is commonly done, we shall always identify the second dual V^{**} of a finite-dimensional vector space V with V itself, and we shall further identify V^* with V whenever V is equipped with an inner product.

Let U and V be finite-dimensional vector spaces and let $\mathbf{u} \in \mathsf{U}$ and $\mathbf{v} \in \mathsf{V}$ be given; then we define the *tensor product* $\mathbf{u} \otimes \mathbf{v} \in \mathrm{Lin}(\mathsf{V}^*, \mathsf{U})$ by

$$(\mathbf{u} \otimes \mathbf{v})\boldsymbol{\theta} := (\boldsymbol{\theta}\mathbf{v})\mathbf{u} \tag{1}$$

We shall abbreviate $\mathbf{v}^{\oplus 2} := \mathbf{v} \otimes \mathbf{v}$.

The *transpose* $L^T \in \mathrm{Lin}(\mathsf{V}^*, \mathsf{U}^*)$ of a linear mapping $L \in \mathrm{Lin}(\mathsf{U}, \mathsf{V})$ is defined by the rule

$$L^T\boldsymbol{\theta} := \boldsymbol{\theta}L \tag{2}$$

if $L = L^T$, we say that L is *symmetric*. The space of symmetric linear mappings from V to V^* is denoted

$$\mathrm{Sym}(\mathsf{V}, \mathsf{V}^*) := \{L \in \mathrm{Lin}(\mathsf{V}, \mathsf{V}^*) \mid L = L^T\} \tag{3}$$

When V is an inner-product space, so that $\mathsf{V} = \mathsf{V}^*$, we abbreviate

$$\mathrm{Sym}\,\mathsf{V} := \mathrm{Sym}(\mathsf{V}, \mathsf{V}) \tag{3a}$$

Let V be a vector space, let S be a subset of V, and let $t \in \mathbb{R}$ and $\mathbf{u} \in \mathsf{V}$ be given. We shall use such expressions as $\mathbf{u} + \mathsf{S} := \{\mathbf{u} + \mathbf{s} \mid \mathbf{s} \in \mathsf{S}\}$ and $t\mathsf{S} := \{t\mathbf{s} \mid \mathbf{s} \in \mathsf{S}\}$.

Let A and B be sets and let $f\colon \mathsf{A} \times \mathsf{B} \to \mathbb{R}$ be a function of two variables. For each $a \in \mathsf{A}$, we define the single-argument function $f(a, \cdot)\colon \mathsf{B} \to \mathbb{R}$ by $f(a, \cdot)(b) := f(a, b)$. In a corresponding manner, we define, for each $b \in \mathsf{B}$, a single-argument function $f(\cdot, b)\colon \mathsf{A} \to \mathbb{R}$.

2. FLAT SPACES[2]

A *flat space* E (also called an *affine space*) is a nonempty set, whose members are called *points*, which is endowed with structure by the prescription of:

[2] Most of the material in this section has been adapted from unpublished notes by Walter Noll.

1. a vector space denoted $\mathsf{E}-\mathsf{E}$ which is called the *translation space* of E and whose members are called either *translations* or *vectors*;

2. an operation of adding vectors to points which obeys Axioms (A1) and (A2) below. We write the sum of $\mathbf{x}\in\mathsf{E}$ and $\mathbf{v}\in\mathsf{E}-\mathsf{E}$ as $\mathbf{x}+\mathbf{v}$. The interpretation of this operation is suggested by Fig. 1.

(A1) The operation is an *action* of the additive group of $\mathsf{E}-\mathsf{E}$ on E, that is, for all $\mathbf{x}\in\mathsf{E}$ and all $\mathbf{u},\mathbf{v}\in\mathsf{E}-\mathsf{E}$

$$\mathbf{x}+\mathbf{O}=\mathbf{x} \qquad \text{and} \qquad \mathbf{x}+(\mathbf{u}+\mathbf{v})=(\mathbf{x}+\mathbf{u})+\mathbf{v}$$

(A2) For every pair $\mathbf{x},\mathbf{y}$ of points, there is a unique vector $\mathbf{v}$ which satisfies

$$\mathbf{x}+\mathbf{v}=\mathbf{y} \tag{4}$$

The *dimension* of a flat space is, by definition, the dimension of its translation space.

The essential difference between vector spaces and flat spaces is that the former have preferred points, namely their zeros, while the latter do not. (Thus the latter are more suitable as mathematical models of physical spaces and space-times.) Clearly, a vector space always has a natural structure as a flat space: it is its own translation space and the operation of adding vectors to points is simply vector addition.

For the remainder of this section, we let E be a flat space whose translation space will be denoted $\mathsf{T}:=\mathsf{E}-\mathsf{E}$.

Let points $\mathbf{x},\mathbf{y}\in\mathsf{E}$ be given; we denote by $\mathbf{y}-\mathbf{x}$ the unique vector $\mathbf{v}\in\mathsf{T}$ which satisfies (4). For $\mathbf{x}\in\mathsf{E}$ and $\mathbf{v}\in\mathsf{T}$, we abbreviate $\mathbf{x}-\mathbf{v}:=\mathbf{x}+(-\mathbf{v})$. Generally speaking, the various operations of addition and subtraction may be safely composed. For example, we have $\mathbf{x}-\mathbf{x}=\mathbf{O}$, $\mathbf{x}+(\mathbf{y}-\mathbf{x})=\mathbf{y}$, and $(\mathbf{x}-\mathbf{u})-(\mathbf{y}+\mathbf{v})=(\mathbf{x}-\mathbf{y})+(\mathbf{u}-\mathbf{v})$ for all $\mathbf{x},\mathbf{y}\in\mathbf{E}$ and all $\mathbf{u},\mathbf{v}\in\mathbf{T}$.

A subset F of E with the property that the set

$$\mathsf{F}-\mathsf{F}:=\{\mathbf{x}-\mathbf{y}\mid\mathbf{x},\mathbf{y}\in\mathsf{E}\} \tag{5}$$

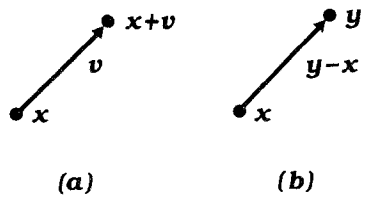


Fig. 1.

is a subspace of T is called a *flat*; such a subset F inherits from E a structure as a flat space whose translation space is F − F. For example, straight lines and planes are flats. We note that since every vector space is also a flat space, it is meaningful to speak of flats in vector spaces. Every subspace of a vector space is a flat, and a flat in a vector space is a subspace when it contains zero.

2.1. Gradient and Divergence

The familiar calculus on finite-dimensional vector spaces is easily extended to flat spaces. Suppose that E and F are finite-dimensional flat spaces and let $\mathbf{x} \in \mathsf{E}$, $\mathbf{v} \in \mathsf{T}$, and a mapping $\mathbf{f}$: $\mathsf{E} \to \mathsf{F}$ be given. We call the member

$$\nabla_{\mathbf{x}}^{\mathbf{v}}\mathbf{f} := \lim_{t \to 0} \frac{\mathbf{f}(\mathbf{x} + t\mathbf{v}) - \mathbf{f}(\mathbf{x})}{t} \tag{6}$$

of F − F the *directional gradient* of f at $\mathbf{x}$ in the direction $\mathbf{v}$ whenever it exists. Trivially, there is at most one linear mapping $L \in \mathrm{Lin}(T, \mathsf{F} - \mathsf{F})$ which satisfies

$$L\mathbf{v} = \nabla_{\mathbf{x}}^{\mathbf{v}}\mathbf{f} \tag{7}$$

for every $\mathbf{v} \in \mathsf{T}$. If such a mapping L exists, then it is denoted $\nabla_{\mathbf{x}}\mathbf{f}$ and is called the *gradient* at $\mathbf{x}$ of $\mathbf{f}$, and $\mathbf{f}$ is said to be *differentiable* at $\mathbf{x}$. If the translation space T of E is the set $\mathbb{R}$ of real numbers, then

$$\mathbf{f}'(\mathbf{x}) := \lim_{t \to 0} \frac{\mathbf{f}(\mathbf{x} + t) - \mathbf{f}(\mathbf{x})}{t} \in \mathsf{F} - \mathsf{F} \tag{8}$$

is, as usual, called the *derivative* at $\mathbf{x}$ of f whenever the right-hand side exists.

We shall use the term *field* to mean a mapping from E to some given vector space. Whenever it is convenient to do so, we shall denote the value of a field $\mathbf{v}$ at a point $\mathbf{x} \in \mathsf{E}$ by the symbol $\mathbf{v}_{\mathbf{x}}$ instead of the more common $\mathbf{v}(\mathbf{x})$.

Let a vector space W and a field

$$L\colon \mathsf{E} \to \mathrm{Lin}(\mathsf{T}^*, \mathsf{W})$$

of linear mappings b given. For each $\omega \in \mathsf{W}^*$, a vector field

$$\omega L\colon \mathsf{E} \to \mathsf{T}$$

is defined by putting

$$(\omega L)_{\mathbf{x}} := \omega L_{\mathbf{x}}$$

and then exploiting the identification of T with T**. We note that for each $\mathbf{x} \in \mathsf{E}$ at which L is differentiable, we have

$$\nabla_{\mathbf{x}}(\omega L) \in \operatorname{Lin} \mathsf{V}$$

it is meaningful to speak of the *trace* of such a linear operator. Now let $\mathbf{x} \in \mathsf{E}$ be given. If L is differentiable at $\mathbf{x}$, then there is a unique member of W, denoted $\operatorname{dib}_{\mathbf{x}} L$ and called the *divergence* at $\mathbf{x}$ of L, which satisfies the equality

$$\omega \operatorname{div}_{\mathbf{x}} L = \operatorname{tr} \nabla_{\mathbf{x}}(\omega L) \tag{9}$$

for every $\omega \in \mathsf{W}^*$. Since a vector field, say $\mathbf{v}$: $\mathsf{E} \to \mathsf{T}$, may be considered to take values in $\mathsf{T}^{**} = \operatorname{Lin}(\mathsf{T}^*, \mathbb{R})$, this definition includes as a special case the definition of the divergence of a vector field. By a direct application of this definition, one can show that the divergence of a vector field $\mathbf{v}$ at a point $\mathbf{x}$ is given by the formula

$$\operatorname{div}_{\mathbf{x}} \mathbf{v} = \operatorname{tr} \nabla_{\mathbf{x}} \mathbf{v} \tag{10}$$

We shall not hesitate to apply the formulas above to mappings and fields which are defined only on proper open subsets of E.

2.2. Euclidean Space

For our purposes, a *Euclidean space* is a flat space which is endowed with additional structure by the prescription of a genuine (i.e., positive-definite) inner product on its translation space.

3. GALILEAN SPACE-TIME[3]

A *Galilean space-time* G is a four-dimensional flat space which is endowed with additional structure by the prescription of a nonconstant (i.e., nonzero) linear form $\tau \in (\mathsf{G} - \mathsf{G})^*$ called the *duration form* and a genuine inner product on the nullspace of τ. (The inner product of the vectors $\mathbf{u}$ and $\mathbf{v}$ will be denoted $\mathbf{u} \cdot \mathbf{v}$ and we shall abbreviate $\mathbf{v}^2 := \mathbf{v} - \mathbf{v}$). The physical interpretation of Galilean space-time is as follows. Points in G are events and vectors in $\mathsf{G} - \mathsf{G}$ may be considered to be translations in space and time. Given two events, say $\mathbf{x}, \mathbf{y} \in \mathsf{G}$, the real number $\tau(\mathbf{y} - \mathbf{x})$ is the *time-lapse* from $\mathbf{x}$ to $\mathbf{y}$. This time-lapse is greater than, less than, or equal to zero as $\mathbf{y}$

[3] Some of the terminology which is introduced in this section is patterned after terminology used by Noll in the context of neoclassical space time.[4]

occurs *before*, *after*, or *simultaneous* with $\mathbf{x}$. If the events $\mathbf{x}$ and $\mathbf{y}$ are simultaneous, then $(\mathbf{x}-\mathbf{y})^2$ is the square of the *distance* from $\mathbf{x}$ to $\mathbf{y}$. (If $\mathbf{x}$ and $\mathbf{y}$ are not simultaneous, then there is no defined distance between them.) The role of the flat structure of G *per se* is to be understood through the interpretation of a straight line l in G as an ordinary straight line in "space" if all events on the line are simultaneous and as a possible world-line (see Section 3.1) of a particle whose motion is uniform if the line contains a pair of nonsimultaneous events.

For the remainder of this paper, G will be a given Galilean space-time whose translation space will be denoted $\mathsf{V} := \mathsf{G} - \mathsf{G}$.

Let a vector $\mathbf{v} \in \mathsf{V}$ be given. We call the real number $\tau\mathbf{v}$ the *duration* of $\mathbf{v}$ and say that $\mathbf{v}$ is *future-directed*, *past-directed*, or *spatial* as its duration is greater than, less than, or equal to zero. We denote the set of spatial vectors by

$$\operatorname{Spa} \mathsf{G} := \{\mathbf{v} \in \mathsf{V} \mid \tau\mathbf{v} = 0\}; \tag{11}$$

this set, being the nullspace of the linear form τ, is obviously a three-dimensional subspace of V; it is, in fact, the space of vectors which one usually encounters in an elementary physics course. A vector of unit duration is said to be (for reasons which will become clearer later) *velocity-like*; the set of all such vectors is denoted

$$\operatorname{Vel} \mathsf{G} := \{\mathbf{v} \in \mathsf{V} \mid \tau\mathbf{v} = 1\}. \tag{12}$$

We note that Vel G is a three-dimensional flat in V and that the translation space of Vel G is Spa G.

The relation of simultaneity between events is obviously an equivalence relation. The equivalence classes of this relation are called *instants*. When we speak of the instant at which an event $\mathbf{x}$ occurs, we mean the equivalence class

$$\mathbf{x} + \operatorname{Spa} \mathsf{G} = \{\mathbf{y} \in \mathsf{G} \mid \tau(\mathbf{y}-\mathbf{x}) = 0\} \tag{13}$$

to which $\mathbf{x}$ belongs. It is easy to see that each instant, say I, is a flat in G whose translation space is the inner-product space

$$\mathsf{I} - \mathsf{I} = \operatorname{Spa} \mathsf{G} \tag{14}$$

thus, each instant is a three-dimensional Euclidean space. The set of all instants is denoted

$$\operatorname{Inst} \mathsf{G} := \{\mathbf{x} + \operatorname{Spa} \mathsf{G} \mid \mathbf{x} \in \mathsf{G}\} \tag{15}$$

This set aquires a natural structure as a one-dimensional flat space with tran-

slation space $\mathbb{R}$ when the sum of an instant I and a "time-lapse" $t \in \mathbb{R}$ is defined to be the instant

$$\begin{aligned} \mathsf{I} + t &:= \mathsf{I} + t \,\mathrm{Vel}\, \mathsf{G} \\ &= \{\mathbf{x} + \mathbf{v} \mid \mathbf{x} \in \mathsf{G}, \mathbf{v} \in \mathsf{V}, \text{ and } \tau\mathbf{v} = t\} \end{aligned} \tag{16}$$

When two instants, say I and J, are given, we call $\mathsf{J} - \mathsf{I} \in \mathbb{R}$ the *time-lapse* from I to J. (The expression $\mathsf{J} - \mathsf{I}$ is now ambiguous, but this ambiguity is easily resolved on a case-by-case basis either by context or by the use of such language as "the time-lapse $\mathsf{J} - \mathsf{I} \in \mathbb{R}$" as opposed to "the subset $\mathsf{J} - \mathsf{I}$ of V.")

3.1. Observers

By the term *straight world-line* we mean a straight line, say l, in G, which is not included in an instant. We interpret a straight world-line as the set of events experienced by an actual or possible object (observer or particle) whose motion (we use the term informally here) is uniform.[4] Let l be a straight world-line which we shall consider to be the world-line of an observer. Then the translation space $l - l$, which is a one-dimensional subspace of V, contains exactly one velocity-like vector (see (12) and Fig. 2), say $\mathbf{u} \in \mathrm{Vel}\, \mathsf{G}$; this vector is called the *velocity* of the observer. We note that this vector $\mathbf{u}$ is not the familiar spatial velocity relative to some fixed reference frame; instead, $\mathbf{u}$ represents velocity independently of reference

[4] Uniform motion is actually a primitive concept which is mathematically realized in the flat structure of Galilean space-time.

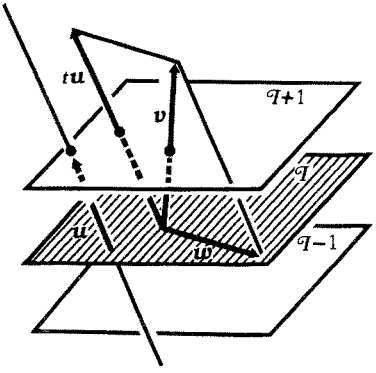


Fig. 2.

frame and corresponds to the so-called "4-velocity" in relativity theory. If $\mathbf{u}$ and $\mathbf{v}$ are the velocities of two observers, then the spatial vector $\mathbf{v} - \mathbf{u}$ is the (familiar) *relative velocity* of the second observer with respect to the first.

Let $\mathbf{u} \in \text{Vol G}$ and $\mathbf{v} \in \text{V}$ be given; we shall consider $\mathbf{u}$ to be the velocity of an observer. Then there are unique $t \in \mathbb{R}$ and $\mathbf{w} \in \text{Spa G}$ which satisfy (see Fig. 2)

$$\mathbf{v} = t\mathbf{u} + \mathbf{w} \tag{17}$$

it is clear that $t = \tau\mathbf{v}$ is the duration of $\mathbf{v}$; we call $\mathbf{w} = \mathbf{v} - t\mathbf{u}$ the *spatial part of* $\mathbf{v}$ *with respect to* $\mathbf{u}$ (or "with respect to the observer whose velocity is $\mathbf{u}$"). The linear operator which assigns to each vector its spatial part with respect to $\mathbf{u}$ is denoted $\prod_{\mathbf{u}} \in \text{Lin}(\text{V}, \text{Spa G})$ and is called the *spatial projection operator* with respect to $\mathbf{u}$. Clearly,

$$\prod_{\mathbf{u}} \mathbf{v} = \mathbf{v} - (\tau\mathbf{v})\mathbf{u} \tag{18}$$

for all $\mathbf{v} \in \text{V}$.

4. PARTICLE MECHANICS

4.1. Kinematics

By a *world-line* we mean a connected subset, say l, of G, with the property that l contains exactly[5] one event of each instant; we interpret l as the set of events experienced by a particle (or observer). By a *motion* we mean a continuous mapping, say

$$\mathbf{s}: \text{Inst G} \to \text{G} \tag{19}$$

which assigns to each instant I an event $\mathbf{s}(\text{I})$; we consider it understood that some particle is under discussion and interpret $\mathbf{s}(\text{I})$ as the event experienced by that particle at the instant I. It is easy to see that each motion's range is a world-line and that each world-line is the range of a unique motion; this establishes a natural one-to-one correspondence between world-lines and motions. If the motion $\mathbf{s}$ of a particle is differentiable at the instant I, we call its derivative

$$\mathbf{v} := \mathbf{s}'(\text{I}) \tag{20}$$

[5] It is sometimes useful to generalize this concept by replacing "exactly" with "at most." There is an obvious parallel generalization of the concept of a motion.

the *velocity* of the particle at that instant. If **s** is twice differentiable at the instant l, then its second derivative

$$\mathbf{a} := \mathbf{s}''(\mathrm{l}) \tag{21}$$

is called the *acceleration* of the particle (see Fig. 3). Clearly,

$$\tau\mathbf{v} = \lim_{t\to 0} \frac{\tau(\mathbf{s}(\mathrm{l}+t) - \mathbf{s}(\mathrm{l}))}{t}$$

$$= \lim_{t\to 0} \left(\frac{t}{t}\right) = 1$$

so **v** is, indeed, a velocity-like vector. We interpret this velocity **v** in the same sense that we have already interpreted velocities associated with straight world-lines. We note that if **u** is the velocity of an observer, then the spatial vector

$$\prod_{\mathbf{u}} \mathbf{v} = \mathbf{v} - \mathbf{u} \tag{22}$$

is the (familiar) relative velocity of the particle with respect to that observer. We also see that

$$\tau\mathbf{a} = \lim_{t\to 0} \frac{\tau(\mathbf{s}'(\mathrm{l}+t) - \mathbf{s}'(\mathrm{l}))}{t}$$

$$= \lim_{t\to 0} \frac{1-1}{t} = 0$$

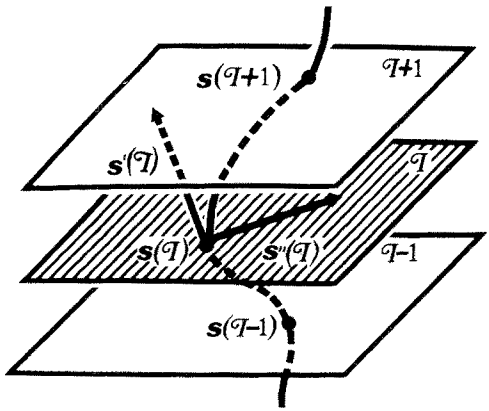


Fig. 3.

so **a** is a spatial vector; indeed, **a** is the familiar acceleration vector of the particle.

4.2. Dynamics

We again consider a particle with motion **s** being observed by an observer with velocity **u**; we now denote by $m \in \mathbb{R}$ the mass (assumed to be greater than zero) of the particle. We let an instant I be given and we suppose that the velocity $\mathbf{v} := \mathbf{s}'(\mathsf{I})$ exists. Then the vector

$$\mathbf{p} := m\mathbf{v} \tag{23}$$

is called the *momentum* of the particle at the instant I. The duration of **p** is equal to m and the spatial vector $\prod_{\mathbf{u}} \mathbf{p}$ is the familiar *relative momentum* of the particle with respect to the observer.

A *force field* $\mathbf{F}$: G → Spa G is a spatial vector field; the value of the field at $\mathbf{x} \in \mathsf{G}$ is a spatial vector denoted $\mathbf{F}_{\mathbf{x}}$. We say that a particle feels a *force* $F_{\mathbf{x}}$ when it experiences event **x** and we assume that for each $\mathsf{I} \in \operatorname{Inst} \mathsf{G}$, the motion **s** of a particle of mass m obeys the classical equation of motion

$$m\mathbf{a} = \mathbf{F}_{\mathbf{x}} \tag{24}$$

where we have abbreviated

$$\mathbf{a} := \mathbf{s}''(\mathsf{I}) \qquad \text{and} \qquad \mathbf{x} := \mathbf{s}(\mathsf{I}) \tag{25}$$

4.3. Phase Space-Time

Let a force field $\mathbf{F}$: G → Spa G be given. We assume that $\mathbf{F}$ is sufficiently well-behaved that we may, in principle, find the entire motion **s** of a particle of mass m by integrating the law of motion (24), provided only that we know the position and momentum of the particle at one instant. The set

$$\mathsf{G} \times m \operatorname{Vel} \mathsf{G} = \{(\mathbf{x}, \mathbf{p}) \mid \mathbf{x} \in \mathsf{G}, \mathbf{p} \in \mathsf{V}, \text{ and } \boldsymbol{\tau}\mathbf{p} = m\} \tag{26}$$

of *phases* (i.e., position–momentum pairs) is called *phase space-time* (for a single particle of mass m). This phase space-time is a seven-dimensional flat space whose translation space is the vector space[6]

$$\mathsf{V} \times \operatorname{Spa} \mathsf{G} = \{(\mathbf{v}, \mathbf{w} \in \mathsf{V} \text{ and } \boldsymbol{\tau}\mathbf{w} = 0\} \tag{27}$$

[6] The *cartesian product* $\mathsf{U} \times \mathsf{V}$ of two vector space is sometimes written $\mathsf{U} \oplus \mathsf{V}$ and called their *direct sum*.

Let $\mathbf{s}$ be the motion of a particle of mass m and assume that $\mathbf{s}$ is differentiable. The mapping

$$(\mathbf{s}, m\mathbf{s}'): \text{Inst G} \to \text{G} \times m \text{ Vel G} \tag{28}$$

whose value at an instant I is the phase $(\mathbf{s}(\mathsf{I}), m\mathbf{s}'(\mathsf{I}))$ associated with the particle at that instant, is called the *phase-space-time motion* of the particle. The range of this mapping (28) is called the *phase-space-time world-line* of the particle. There is an obvious natural one-to-one correspondence between phase-space-time world-lines and phase-space-time motions.

Recall that $\mathbf{F}$ is given and well behaved. A phase-space-time world-line of a particle of mass m is said to be *legitmate* if the motion, say $\mathbf{s}$, which induces it obeys the law of motion (24). Let a phase $(\mathbf{x}, \mathbf{p})$ be given. Since the field $\mathbf{F}$ is well behaved, there is exactly one legitimate phase-space-time world-line, say l, through $(\mathbf{x}, \mathbf{p})$; let us denote by $\mathbf{s}$ the motion associated with l. Then l is tangent at $(\mathbf{x}, \mathbf{p})$ to the derivative

$$(\mathbf{s}''(\mathsf{I}), m\mathbf{s}''(\mathsf{I})) = (\mathbf{p}/m, \mathbf{F}_{\mathbf{x}}) \tag{29}$$

at the instant $\mathsf{I} := \mathbf{x} + \text{Spa G}$ of the phase-space-time motion $(\mathbf{s}, m\mathbf{s}')$.

Let $\mathsf{I} \in \text{Inst G}$ and $m > 0$ be given. Then the set

$$\mathsf{I} \times m \text{ Vel G} = \{(\mathbf{x}, \mathbf{p}) \mid \mathbf{x} \in \mathsf{I}, \mathbf{p} \in \mathsf{V}, \text{ and } \tau\mathbf{p} = m\} \tag{30}$$

is called *phase space* at the instant I (for a single particle of mass m). It is clear that this phase space is a six-dimensional Euclidean space with translation space $(\text{Spa G})^2$.

5. A RAREFIED GAS

5.1. Phase-Space Densities

We shall now study the evolution of a swarm of *noninteracting* particles, each of mass m, under the influence of a force field $\mathbf{F}$. At each instant I, the swarm (more accurately, the property of the swarm which we shall study) is completely described by a nowhere-negative function[7]

$$f_{\mathsf{I}}: \mathsf{I} \times m \text{ Vel G} \to \mathbb{R} \tag{31}$$

on the phase space at the instant I. We interpret f_{I} as follows: if X and P are

[7] We shall implicitly assume that the function f_{I} is quite well-behaved.

measurable subsets of the "positition space" I and "momentum space" m Vel G, respectively, then the integral

$$\int_X \int_P f_I(\mathbf{x}, \mathbf{p})\, d\mathbf{x}\, d\mathbf{p} \tag{32}$$

is (approximately) the number of particles in the region X whose momenta lie in the region P. We note that, as a second possible interpretation, the function f_I, if normalized, could be a probability density describing our less-than-complete knowledge of the position and momentum of a single particle. It is also possible to fuse these two interpretations into a third by interpreting f_I as the sum of several probability densities, each of which represents our state of knowledge of a single particle. The reason that all three interpretations are consistent with the same dynamics is that we have assumed that the various particles (if indeed, more than one particle is under consideration) do not interact. We will, henceforth, be intentionally vague concerning the choice of interpretation and shall call f_I simply the *density function* (or, if confusion is likely, the *phase-space density function*) of the swarm at the instant I. The entire history of the swarm will be described by a single function $f: \mathrm{G} \times m\,\mathrm{Vel}\,\mathrm{G} \to \mathbb{R}$ on phase space-time called the *phase-space-time density function* of the swarm. It is related to the single-instant density functions by the formula

$$f(\mathbf{x}, \mathbf{p}) = f_I(\mathbf{x}, \mathbf{p}) \tag{33}$$

which holds for every instant I, all $\mathbf{x} \in \mathrm{I}$, and all $\mathbf{p} \in m\,\mathrm{Vel}\,\mathrm{G}$.

5.2. Dynamical Equation

Let a force field $\mathbf{F}: \mathrm{G} \to \mathrm{Spa}\,\mathrm{G}$ be given. Then the motion of a swarm of particles (each of mass m) is governed by the requirement that the density function f of the swarm is constant along each legitimate world-line in phase space-time. This hypothesis is standard in analytical mechanics and is motivated by the presumption that each individual particle in the swarm should follow a legirimate world-line in phase space-time. When appropriate smoothness is assumed, this is equivalent to requiring that the partial differential equation

$$\nabla_{(\mathbf{x},\mathbf{p})}^{(\mathbf{p}/m,\,\mathbf{F}_\mathbf{x})} f = 0 \tag{34}$$

holds for all $(\mathbf{x}, \mathbf{p})$ in phase space-time. We shall call (34) the *dynamical equation* for a swarm of particles; it is also known as the *continuity equation.*

It is sometimes convenient to separate partial derivatives and rewrite (34) as

$$\nabla_{\mathbf{x}}^{\mathbf{p}} f(\cdot, \mathbf{p}) = -m\mathbf{F}_{\mathbf{x}} \cdot \nabla_{\mathbf{p}} f(\mathbf{x}, \cdot) \tag{35}$$

5.3. Gross Fields[8]

Let $m \in \mathbb{R}$ and f: $\mathsf{G} \times m\,\mathrm{Vel}\,\mathsf{G} \to \mathbb{R}$ be given as before. We shall assume that f is sufficiently well-behaved that the definitions below make sense. We define the following *gross fields*:

$$\textit{(number) density} \qquad n\colon \mathsf{G} \to \mathbb{R}, \quad n_{\mathbf{x}} := \int f(\mathbf{x}, \mathbf{p})\, d\mathbf{p} \tag{36}$$

$$\textit{(number) flux vector} \qquad \mathbf{j}\colon \mathsf{G} \to \mathsf{V}, \quad \mathbf{j}_{\mathbf{x}} := m^{-1} \int f(\mathbf{x}, \mathbf{p})\, \mathbf{p}\, d\mathbf{p} \tag{37}$$

these are volume integrals over the three-dimensional "momentum space" $m\,\mathrm{Vel}\,\mathsf{G}$. It is clear that $n = \tau\mathbf{j}$: on the subset $\mathsf{G}_+ := \{\mathbf{x} \in \mathsf{G} \mid n_{\mathbf{x}} > 0\} = \{\mathbf{x} \in \mathsf{G} \mid n_{\mathbf{x}} \neq 0\}$ of G where n does not vanish, we define the gross field

$$\textit{(material) velocity} \quad \mathbf{v}\colon \mathsf{G}_+ \to \mathsf{V}, \quad \mathbf{v}_{\mathbf{x}} := (\mathbf{n}_{\mathbf{x}})^{-1} \mathbf{j}_{\mathbf{x}} \tag{38}$$

Wherever the field $\mathbf{v}$ exists (namely on G_+), it provides a natural frame of reference (i.e., observer), the *material frame*, which of course varies from point to point.

These material reference frames give us the means to decompose fields associated with the material system into spatial and temporal parts in a natural way.

Now let S be a subset of G (in our applications, S will be either G or G_+), let W be a vector space, and let $\mathbf{w}$: $\mathsf{S} \times m\,\mathrm{Vel}\,\mathsf{G} \to \mathsf{W}$ be a function which is sufficiently well-behaved that the following fields may be defined:

w-density $\quad \mathbf{w}^{\mathrm{den}}\colon \mathsf{S} \to \mathsf{W}$

$$\mathbf{w}_{\mathbf{x}}^{\mathrm{den}} := \int f(\mathbf{x}, \mathbf{p})\, \mathbf{w}(\mathbf{x}, \mathbf{p})\, d\mathbf{p} \tag{39}$$

spatial w-flux vector $\quad \mathbf{w}^{\mathrm{spa}}\colon \mathsf{G}_+ \cap \mathsf{S} \to \mathrm{Lin}(\mathsf{V}^*, \mathsf{W})$

$$\mathbf{w}_{\mathbf{x}}^{\mathrm{spa}} := m^{-1} \int f(\mathbf{x}, \mathbf{p})\, \mathbf{w}(\mathbf{x}, \mathbf{p}) \otimes (\mathbf{p} - m\mathbf{v}_{\mathbf{x}})\, d\mathbf{p} \tag{40}$$

[8] For a comparable treatment of gross fields and transport and balance laws in the framework of absolute space-time, see Truesdell and Muncaster.[5] Of course their treatment of the latter is more complicated (and more general) because they deal with colliding particles.

w-flux vector $\mathbf{w}^{\text{flux}}: \mathsf{S} \to \text{Lin}(\mathsf{V}^*, \mathsf{W})$

$$\mathbf{w}^{\text{flux}}_{\mathbf{x}} := m^{-1} \int f(\mathbf{x}, \mathbf{p}) \mathbf{w}(\mathbf{x}, \mathbf{p}) \otimes \mathbf{p} \, d\mathbf{p} \tag{41}$$

it is clear that

$$\mathbf{w}^{\text{flux}} = \mathbf{w}^{\text{den}} \otimes \mathbf{v} + \mathbf{w}^{\text{spa}} \tag{41a}$$

The three most important examples of such functions $\mathbf{w}$ are those which generate the densities and flux-vectors of number, momentum, and (internal kinetic) energy. These functions and the symbols assigned to the densities and fluxes are as follows:

	$\mathbf{w}(\mathbf{x}, \mathbf{p})$	$\mathbf{w}^{\text{den}}$	$\mathbf{w}^{\text{spa}}$	$\mathbf{w}^{\text{flux}}$
number	1	n	$\mathbf{0}$	$\mathbf{j}$
momentum	$\mathbf{p}$	$m\mathbf{j}$	M	T
energy	$(2m)^{-1}(\mathbf{p} - m\mathbf{v}_{\mathbf{x}})^2$	$mn\varepsilon$	$\mathbf{q}^{\text{spa}}$	$\mathbf{q}$

(42)

The number density and flux have already been introduced. It is easy to show that $M_{\mathbf{x}} = m^{-1} \int f(\mathbf{x}, \mathbf{p})(\mathbf{p} - m\mathbf{v}_{\mathbf{x}})^{\otimes 2} \, d\mathbf{p}$ so M is the familiar *(Maxwell) pressure tensor.* The *momentum flux tensor* T is a nonrelativistic analog of the stress-energy tensor of relativity theory. The use of the superscript "spa" in the symbol for the spatial energy flux is inconsistent with its generic use in the symbol $\mathbf{w}^{\text{spa}}$, but there is little chance of confusion.

The significance of the density $\mathbf{w}^{\text{den}}$ is that for each (measurable) region X of an instant, the volume-integral $\int_{\mathsf{X}} \mathbf{w}^{\text{den}}$ is the amount of the physical quantity associated with $\mathbf{w}$ to be found in that region. Similarly, for each well-behaved oriented surface at an instant, the surface integral $\int_{\mathsf{S}} \mathbf{w}^{\text{spa}}$ is the rate at which the physical quantity represented by $\mathbf{w}$ is flowing through the surface. It is shown in Appendix B that there is a natural way of integrating fields of the type $\mathbf{w}^{\text{flux}}$ over three-dimensional hyper-surfaces in space-time. For each well-behaved oriented hyper-surface H in G, the hyper-surface-integral $\int_{\mathsf{H}} \mathbf{w}^{\text{flux}}$ is the total flux of the quantity associated with $\mathbf{w}$ through the hyper-surface. In the case where H is a subset of an instant, this hyper-surface integral is equal to the ordinary volume-integral $\int_{\mathsf{H}} \mathbf{w}^{\text{den}}$.

5.4. Spatial Gradients and Diverges and Material Time-Derivatives

Let $g: \mathsf{G} \to \mathsf{E}$ be a field which takes values in a flat space E with translation space $\mathsf{T} := \mathsf{E} - \mathsf{E}$. (We shall not hesitate to apply our definitions to fields defined on smaller domains.) We define (whenever possible) the

(familiar) *spatial gradient* field grad $\mathbf{g}$: $G \to \mathrm{Lin}(\mathrm{Spa}\, G, T)$ by prescribing, for each instant I and each $\mathbf{x} \in I$,

$$\mathrm{grad}_{\mathbf{x}}\, \mathbf{g} := \nabla_{\mathbf{x}}(\mathbf{g}\,|_I) \tag{43}$$

the field $g\,|_I \to E$ is just the restriction to I of the field $\mathbf{g}$. Let W be a vector space. It is clear that a spatially differentiable field L: $G \to \mathrm{Lin}(\mathrm{Spa}\, G, W)$ will have a *spatial divergence* field associated in the obvious way with its spatial gradient. However, we need not introduce a distinctive symbol for such a divergence because, as we shall see, the spatial divergence of L is equal to the (space-time) divergence of a field which may be identified with L. Indeed, for each $\boldsymbol{\lambda} \in V^*$, there is a unique $\mathbf{s} \in \mathrm{Spa}G$ such that $\boldsymbol{\lambda}\mathbf{u} = \mathbf{s} \cdot \mathbf{u}$ for all $\mathbf{u} \in \mathrm{Spa}\, V$. The associated mapping from V^* to Spa G is linear and its nullspace is $\mathbb{R}\boldsymbol{\tau}$. It induces a linear mapping from $\mathrm{Lin}(\mathrm{Spa}\, G, W)$ to $\mathrm{Lin}(V^*, W)$. (The range of this last mapping is the subspace $\{L \in \mathrm{Lin}(V^*, W) \mid L\boldsymbol{\tau} = \mathbf{0}\}$). Thus, a field L with values in $\mathrm{Lin}(\mathrm{Spa}\, G, W)$ may be identified with a field which takes values in $\mathrm{Lin}(V^*, W)$. It is easy to show that the spatial divergence of the former field is equal to the (space-time) divergence of the latter. It is worth noting that, conversely, if a field L: $G \to \mathrm{Lin}(V^*, W)$ has the property that $L\boldsymbol{\tau} = \mathbf{0}$, then its divergence may be interpreted as a spatial divergence; in such a case $\mathrm{div}_{\mathbf{x}} L$ depends only upon the values of L at the instant $\mathbf{x} + \mathrm{Spa}\, G$. Every spatial flux is an example of such a field.

We again let $\mathbf{g}$: $G \to E$ be a field which takes values in the flat space E, but we now suppose that a material system (presumably described by a phase-space-time density f) with a velocity field $\mathbf{v}$ has been specified. As mentioned earlier, the velocity field provides a preferred frame of reference at each event in G_+. We can compute time-derivatives with respect to such frames, and we define the *material time-derivative* $\dot{\mathbf{g}}$: $G_+ \to T$ by the prescription

$$\dot{\mathbf{g}}_{\mathbf{x}} := \nabla_{\mathbf{x}}^{\mathbf{v}_{\mathbf{x}}}\, \mathbf{g} \tag{44}$$

The decomposition (41a) of a flux into spatial and temporal parts induces the following decomposition of its divergence:

$$\mathrm{div}\, \mathbf{w}^{\mathrm{flux}} = \dot{\mathbf{w}}^{\mathrm{den}} + \mathrm{div}\, \mathbf{w}^{\mathrm{spa}} + (\mathrm{div}\, \mathbf{v})\mathbf{w}^{\mathrm{den}} \tag{45}$$

5.5. Transport and Balance Laws

In Appendix A, we show that if a phase-space-time density f obeys the dynamical equation (34), then the flux of a quantity $\mathbf{w}$ associated with it obeys the following transport law:

$$\operatorname{div}_{\mathbf{x}} \mathbf{w}^{\text{flux}} = m^{-1} \int f(\mathbf{x}, \mathbf{p})\, \nabla_{\mathbf{x}}^{\mathbf{p}} \mathbf{w}(\cdot, \mathbf{p})\, d\mathbf{p} + \int f(\mathbf{x}, \cdot)\, \nabla_{\mathbf{p}} \mathbf{w}(\mathbf{x}, \cdot)\, d\mathbf{p} \mathbf{F}_{\mathbf{x}} \quad (46)$$

When this law is applied to the three important cases [see (42)], we obtain the following *balance laws*:

$$\textit{number}: \qquad \operatorname{div} \mathbf{j} = 0 \quad (47)$$

$$\textit{momentum}: \quad \operatorname{div} T = n\mathbf{F} \quad (48)$$

$$\textit{energy}: \qquad \operatorname{div} \mathbf{q} = -\operatorname{tr}(M \operatorname{grad} \mathbf{v}) \quad (49)$$

The interpretations of the transport and balance laws as they stand depend (as do the interpretations of balance laws in relativity theory) on a version of Stokes's theorem which implies that the hyper-volume-integral of $\operatorname{div} \mathbf{w}^{\text{flux}}$ over a space-time region S is equal to the hyper-surface-integral of $\mathbf{w}^{\text{flux}}$ over the boundary of S (see Appendix B). Alternatively, the identity (45) can be used to rewrite the laws in more familiar forms. In any case, we note that since M is symmetric, the tensor $\operatorname{grad} \mathbf{v}$ in (49) may be (and often is) replaced by its symmetric part $\frac{1}{2}(\operatorname{grad} \mathbf{v} + (\operatorname{grad} \mathbf{v})^T)$.

APPENDIX A. TRANSPORT LAW

We let the following be given:

1. a scalar $m > 0$, representing the mass of a particle;
2. a continuously differentiable nowhere-negative function f: $\mathsf{G} \times m\,\mathrm{Vel}\,\mathsf{G} \to \mathbb{R}$ which represents, at each instant, number density on phase space;
3. a vector space W and a continuously differentiable function $\mathbf{w}$: $\mathsf{G} \times m\,\mathrm{Vel}\,\mathsf{G} \to \mathbb{R}$, the latter representing a physical quantity in which we are interested;
4. a continuously differentiable vector field F: $\mathsf{G} \to \mathrm{Spa}\,\mathsf{G}$, which represents an external force field.

We implicitly assume that f and $\mathbf{w}$ satisfy certain growth conditions so that all the integrals which appear below do, indeed, exist.

In proving the theorem below, we shall make use of the following easily-derived formula for the divergence of the tensor product of a field $\mathbf{u}$: $\mathsf{E} \to \mathsf{U}$ with a field $\mathbf{t}$: $\mathsf{E} \to \mathsf{T}$ (here E is a flat space with translation space T, and U is an arbitrary vector space):

$$\operatorname{div}(\mathbf{u} \otimes \mathbf{t}) = \nabla^{\mathbf{t}} \mathbf{u} + (\operatorname{div} \mathbf{u}) \otimes \mathbf{t} \quad (A.1)$$

Theorem. *If f satisfies the dynamical equation* (34) *then* $\mathbf{w}^{\text{flux}}$ (*see* 41) *satisfies the transport law* (46).

Proof. Let $\mathbf{x} \in G$ be given and suppose that f does satisfy (34) at $\mathbf{x}$. Then, in view of (41) and applying (A.1) twice, first with $\mathbf{u} = \mathbf{w}(\cdot, \mathbf{p})$ and $\mathbf{t} = f(\cdot, \mathbf{p})\mathbf{p}$, then with $\mathbf{u} = f(\cdot, \mathbf{p})$ and $\mathbf{t} = \mathbf{p}$, we have

$$\begin{aligned} \operatorname{div}_{\mathbf{x}} \mathbf{w}^{\text{flux}} &= m^{-1} \int \operatorname{div}_{\mathbf{x}}(\mathbf{w}(\cdot, \mathbf{p}) \otimes f(\cdot, \mathbf{p}))\, d\mathbf{p} \\ &= m^{-1} \int f(\mathbf{x}, \mathbf{p})\, \nabla_{\mathbf{x}}^{\mathbf{p}} \mathbf{w}(\cdot, \mathbf{p})\, d\mathbf{p} \qquad \text{(A.2)} \\ &\quad + m^{-1} \int (\nabla_{\mathbf{x}}^{\mathbf{p}} \mathbf{f}(\cdot, \mathbf{p})) \otimes \mathbf{w}(\mathbf{x}, \mathbf{p})\, d\mathbf{p} \end{aligned}$$

The second term on the right-hand side may be rewritten by making use of the dynamical equation (35), applying the definition (1) of tensor product, and then integrating by parts as follows:

$$\begin{aligned} m^{-1} \int (\nabla_{\mathbf{x}}^{\mathbf{p}} \mathbf{f}(\cdot, \mathbf{p}) \otimes \mathbf{w}(\mathbf{x}, \mathbf{p})\, d\mathbf{p} &= -\int (\mathbf{F}_{\mathbf{x}} \cdot \nabla_{\mathbf{p}} \mathbf{f}(\mathbf{x}, \cdot))\, \mathbf{w}(\mathbf{x}, \mathbf{p})\, d\mathbf{p} \\ &= -\int w(\mathbf{x}, \mathbf{p}) \otimes \nabla_{\mathbf{p}} \mathbf{f}(\mathbf{x}, \cdot)\, d\mathbf{p} \mathbf{F}_{\mathbf{x}} \\ &= -\int f(\mathbf{x}, \cdot)\, \nabla_{\mathbf{p}} \mathbf{w}(\mathbf{x}, \cdot)\, d\mathbf{p} \mathbf{F}_{\mathbf{x}} \end{aligned}$$

APPENDIX B. INTEGRATION ON HYPERSURFACES

Throughout this appendix, W is a given finite-dimensional vector space. Let M be a finite-dimensional manifold. We denote the tangent space of M at the point $\mathbf{x} \in \mathsf{M}$ by $\operatorname{Tan}_{\mathbf{x}} \mathsf{M}$. Now let a field W on M be given whose value at each $\mathbf{x} \in \mathsf{M}$ is a member of the space $\operatorname{Alt}_4((\operatorname{Tan}_{\mathbf{x}} \mathsf{M})^4, \mathsf{W})$ of alternating quadrilinear mappings from $(\operatorname{Tan}_{\mathbf{x}} \mathsf{M})^4$ to W.[9] According to the standard theory of integration on manifolds, such a field may, when certain integrability conditions are met, be integrated to obtain a member of W which is denoted $\int_{\mathsf{M}} W$.

We shall shortly show how certain integrals mentioned in Section 5 may be defined in terms of integrals of the sort just described. First, however, we

[9] Discussions of alternating multilinear linear forms and their role in the theory of integration on manifolds may be found in both the mathematical[6] and physical[7] literature.

choose, once and for all, an orientation for Spa G and state without proof that there is exactly one member, say Ω, of $\mathrm{Alt}_4(V^4, \mathbb{R})$ with the property that

$$\Omega(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{v}) = \tau\mathbf{v} \tag{B.1}$$

for every oriented orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of Spa G and every vector $\mathbf{v} \in V$. (Actually Ω is characterized by the weaker requirement that (B.1) hold for some such $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and some nonspatial $\mathbf{v} \in V$.)

Now let an open subset $\mathcal{O}$ of G and a field $\mathbf{w}$: $G \to W$ be given. We denote by $\mathbf{w}\Omega$ the field on $\mathcal{O}$ defined by $(\mathbf{w}\Omega)_x := \mathbf{w}_x\Omega$; since $\mathcal{O}$ may be considered to be a manifold with the property that $\mathrm{Tan}_x\mathcal{O} = V$ for every $x \in \mathcal{O}$, the field $\mathbf{w}\Omega$ is of the right type to be integrated according to the standard theory discussed earlier. We define the *hyper-volume integral* of $\mathbf{w}$ over $\mathcal{O}$ by

$$\int_{\mathcal{O}} \mathbf{w} := \int_{\mathcal{O}} \mathbf{w}\Omega \tag{B.2}$$

whenever the right-hand side exists.

We now consider hyper-surface integrals. We first note that Ω induces an alternating trilinear mapping $\hat{\Omega} \in \mathrm{Alt}_3(V^3, V^*)$ via the following prescription: for each triple $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in V^3$, the member $\hat{\Omega}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ of V^* is defined by the rule

$$(\hat{\Omega}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3))\mathbf{u} := \Omega(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{u}) \tag{B.3}$$

Now let a hypersurface (i.e., embedded three-dimensional submanifold) H in G and a field L: $G \to \mathrm{Lin}(V^*, W)$ be given. (We note the following important special case: in view of the identification of V with $V^* = \mathrm{Lin}(V^*, \mathbb{R})$, the field L may be a vector field.) We define the field L on H by the following prescription: for each $\mathbf{x} \in H$, the alternating trilinear mapping $\hat{L}_x \in \mathrm{Alt}_3((\mathrm{Tan}_x H)^3, W)$ is defined by the rule

$$\hat{L}_x(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := L_x\hat{\Omega}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \tag{B.4}$$

(we have here identified the tangent spaces of H with subspaces of V in the usual fashion, so $\hat{L}_x$ is the appropriate restriction of $L_x\hat{\Omega}$). Finally, we define the hyper-surface integral of L over H by

$$\int_H L := \int_H \hat{L}$$

whenever the right-hand side exists.

Let O be a bounded open subset of G whose boundary ∂O is a hypersurface, and let a field L: $G \to \mathrm{Lin}(V^*, W)$ be given. When O and L are sufficiently well behaved, Stokes's theorem for a manifold can be translated into the following *divergence theorem*:

$$\int_O \operatorname{div} L = \int_{\partial O} L$$

REFERENCES

1. R. Penrose, *in Battelle Rencontres: 1967 Lectures in Mathematics and Physics*, C. DeWitt and J. A. Wheeler, eds. (Benjamin, New York, 1968).
2. V. I. Arnold, *Mathematical Methods of Classical Physics* (Springer-Verlag, New York, 1978).
3. R. E. Artz, *Found. Phys.*, Quantum mechanics in Galilean space-time (to appear).
4. W. Noll, *Arch. Rational Mech. Anal.* **52**, (1973).
5. C. Truesdell and R. G. Muncaster, *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas* (Academic Press, New York, 1979).
6. R. Marsden, *Foundations of Mechanics* (Benjamin, Reading, Massachusetts, 1967).
7. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).