arXiv:2508.11611v1 [math.CT] 15 Aug 2025

EXPONENTIABLE VIRTUAL DOUBLE CATEGORIES AND PRESHEAVES FOR DOUBLE CATEGORIES

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Dedicated to Bob Paré, a profound influence on my understanding of category theory, in celebration of his 80th birthday.

Abstract. Given a pair of pseudo double categories $\mathbb{A}$ and $\mathbb{B}$, the lax functors from $\mathbb{A}$ to $\mathbb{B}$, along with their transformations, modules, and multimodulations, assemble into a virtual double category $\mathbb{L}\mathbf{ax}(\mathbb{A},\mathbb{B})$. We exhibit a universal property of this construction by observing that it arises naturally from the consideration of exponentiability for virtual double categories. In particular, we show that every pseudo double category is exponentiable as a virtual double category, whereby the virtual double category $\mathbb{L}\mathbf{ax}(\mathbb{A},\mathbb{B})$ of lax functors arises as the virtual double category $\mathbb{M}\mathbf{od}(\mathbb{B}^{\mathbb{A}})$ of monads and modules in the exponential $\mathbb{B}^{\mathbb{A}}$. We explore some consequences of this characterisation, demonstrating that it facilitates simple proofs of statements that heretofore required unwieldy computations. For instance, we deduce that the 2-category of pseudo double categories and lax functors is enriched in the 2-category of normal virtual double categories, and demonstrate that several aspects of the Yoneda theory of pseudo double categories – such as the correspondence between presheaves and discrete fibrations – are substantially simplified by this perspective.

1. Introduction

In the last twenty years, it has become increasingly clear that weak double categories play a fundamental role in category theory [GP99; GP04; Shu08]. However, due to its relative youth, several basic aspects of the theory of double categories have remained underdeveloped. One such aspect is the theory of *representability* for double categories. The notion of *presheaf* for double categories was first studied in detail by Paré [Par11], who showed that, for each double category $\mathbb{A}$, there exists a fully faithful functor $Y_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op\,t}}, \mathbb{S}\mathbf{pan})$, sending each object in $\mathbb{A}$ to a contravariant lax functor from $\mathbb{A}$ into the double category of sets and spans; the appropriateness of this definition is justified by a corresponding Yoneda lemma. The generalisation from categories to double categories is by no means routine: for a category $\mathbf{A}$, the presheaves on $\mathbf{A}$ assemble into a category $\mathbf{Set}^{\mathbf{A}^{\mathrm{op}}}$; on the other hand, $\mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op\,t}}, \mathbb{S}\mathbf{pan})$ is, in general, not a pseudo double category but merely a *virtual double category* – a structure which generalises a double category in the same way that a multicategory generalises a monoidal category. Thus, surprisingly, the theory of double categories appears to lead to the study of virtual double categories.

While the development of Paré [Par11] is powerful and insightful, the nature of the virtual double category of presheaves is left rather mysterious. For a category $\mathbf{A}$, the Yoneda embedding $y_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{Set}^{\mathbf{A}^{\mathrm{op}}}$ corresponds, via the cartesian closed structure of the 2-category $\mathbf{Cat}$, to the hom-functor $\mathbf{A}(-,-)\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$. This gives a particularly elementary perspective on the Yoneda embedding, every category being equipped with a hom-functor essentially by definition. We might therefore hope that something similar is true for the double categorical Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op\,t}}, \mathbb{S}\mathbf{pan})$: namely, that it corresponds, via some notion of 'transposition', to the lax hom-functor $\mathbb{A}(-,-)\colon \mathbb{A}^{\mathrm{op\,t}} \times \mathbb{A} \to \mathbb{S}\mathbf{pan}$. The present paper stemmed from the desire to understand to what extent this is true.

Exponentiability of virtual double categories. Given that, for $\mathbf{Cat}$, transposition is induced by cartesian closure, a first instinct might be to guess that the 2-category $\mathbf{VDblCat}$ of virtual double categories is cartesian closed and that, for each pair $\mathbb{A}$ and $\mathbb{B}$ of double categories, the

Date: 18 August 2025.

2020 *Mathematics Subject Classification.* 18D15,18D20,18D60,18M65,18N10.

The author was supported by a departmental postdoctoral grant from the Department of Software Science at Tallinn University of Technology.

virtual double category $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ of lax functors forms the exponential $\mathbb{B}^{\mathbb{A}}$. This is false on both counts, but nevertheless turns out to be a fruitful thought. To understand what is going on, we take inspiration from the theory of multicategories. Just as monoidal categories may be seen as double categories with a single object and single tight morphism, so too may multicategories be seen as virtual double categories with a single object and single tight morphism, and in many respects the theory of virtual double categories may thereby be seen to extend the theory of multicategories. The 2-category of multicategories is *not* cartesian closed, but it does admit some exponentials: Pisani [Pis14] established that a multicategory is exponentiable if and only if it is *promonoidal* in the sense of Day [Day70]. In particular, every monoidal category is promonoidal, hence exponentiable as a multicategory. Given monoidal categories $\mathbf{M}$ and $\mathbf{N}$, one might thus expect that their exponential $\mathbf{N}^{\mathbf{M}}$ in the category of multicategories to comprise the lax monoidal functors from $\mathbf{M}$ to $\mathbf{N}$. This is not true; however, we can obtain the lax monoidal functors from $\mathbf{M}$ to $\mathbf{N}$ as the *monoids* in $\mathbf{N}^{\mathbf{M}}$.

Guided by the one-object setting, we carry out a similar analysis for virtual double categories. As for multicategories, while the 2-category $\mathbf{VDblCat}$ is not cartesian closed, every double category is exponentiable and so, given double categories $\mathbb{A}$ and $\mathbb{B}$, we may form an exponential virtual double category $\mathbb{B}^{\mathbb{A}}$. The analogue of a monoid in a multicategory is a monad in a virtual double category[1], and for every virtual double category $\mathbb{X}$, there is a virtual double category $\mathbb{M}\mathbf{od}(\mathbb{X})$ of monads and modules in $\mathbb{X}$. Our main theorem establishes the expected analogue[2] of the multicategorical story.

Theorem (Theorem 6.2)**.** *Paré's virtual double category* $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ *is isomorphic to* $\mathbb{M}\mathbf{od}(\mathbb{B}^{\mathbb{A}})$.

In answer to our original question, Paré's Yoneda embedding may be decomposed into the following composite,

$$\mathbb{A} \longrightarrow \mathbb{M}\mathbf{od}(\mathbb{A}) \longrightarrow \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op\,t}}})$$

in which the first functor sends each object in $\mathbb{A}$ to the identity monad on that object, and in which the second functor is the image under $\mathbb{M}\mathbf{od}$ of the transposition $\mathbb{A} \hookrightarrow \mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op\,t}}}$ of the lax hom-functor $\mathbb{A}(-,-)\colon \mathbb{A}^{\mathrm{op\,t}} \times \mathbb{A} \to \mathbb{S}\mathbf{pan}$. Thus, the Yoneda embedding is obtained, as desired, by transposition of the hom-functor, 'up to $\mathbb{M}\mathbf{od}$'.

Enrichment and universality. While our given motivation for such a characterisation of $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ was entirely conceptual, it turns out to have a number of practical implications. For instance, it follows easily (Corollary 6.6) that the 2-category of double categories and lax functors is enriched in the 2-category of *normal* virtual double categories (which are virtual double categories that admit identity loose morphisms). Furthermore, the virtual double category $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ acquires a universal property from the universal properties of the $\mathbb{M}\mathbf{od}$ construction [CS10] and the exponential $\mathbb{B}^{\mathbb{A}}$, equipping the 2-category of virtual double categories with a partially-closed skew-monoidal structure (Theorem 6.10). As a consequence, it becomes essentially unnecessary to manipulate $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ directly; the benefits of this will be immediately apparent to readers familiar with previous work on virtual double categories of lax functors [Par11; Par13; Lam21].

Representability for double categories. Having answered our motivating question, we explore the extent to which this perspective simplifies and clarifies the theory of representability for double categories developed by Paré [Par11]. One of the most important aspects of category theory is the relationship between families and bundles. For categories, this relationship takes the form of an equivalence between the category of presheaves on a fixed category $\mathbf{A}$ and the category of discrete fibrations over $\mathbf{A}$. The situation for double categories is similar. Paré [Par11] observed that every lax functor $F\colon \mathbb{A}^{\mathrm{op\,t}} \to \mathbb{S}\mathbf{pan}$ admits the construction of a double category $\mathbb{E}\mathbf{l}(F)$ of elements, equipped with a projection functor $\pi_F\colon \mathbb{E}\mathbf{l}(F) \to \mathbb{A}$, which is a discrete fibration of double categories. Subsequently, Lambert [Lam21], established that, for $\mathbb{A}$ a *strict* double category, this assignment extends to an equivalence between the virtual double category $\mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op\,t}}, \mathbb{S}\mathbf{pan})$ and a virtual double category $\mathbb{D}\mathbf{Fib}(\mathbb{A})$ of discrete fibrations over $\mathbb{A}$. Unexpectedly, the assumption that $\mathbb{A}$ be strict is not simply a matter of convenience, but rather is crucial in Lambert's proof strategy; we

[1] In this paper, a *monad* in a virtual double category will always refer to a *loose monad*, rather than a *tight monad* (which can only be defined in a virtual double category with enough identity loose morphisms).

[2] In fact, the virtual double categorical story is better behaved than the multicategorical one, as the monoids in a multicategory merely form a category, rather than a multicategory, whereas the monads in a virtual double category form another virtual double category.

shall elaborate on this matter in Section 8. Later, Cruttwell, Lambert, Pronk and Szyld [CLPS22] extended the correspondence to permit $\mathbb{A}$ to be an arbitrary pseudo double category, but merely established an equivalence of categories rather than of virtual double categories.

To demonstrate the utility of our characterisation of $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$, we give in Section 8 a new proof of the correspondence between presheaves on double categories and discrete fibrations of double categories. Our strategy has several advantages over previous approaches. For instance, we establish an equivalence of virtual double categories, rather than merely of categories, with respect to an arbitrary pseudo double category $\mathbb{A}$. Our correspondence thus strictly generalises the previous correspondences. Furthermore, our proof strategy is entirely two-dimensional in nature and avoids manual calculation, by deriving the correspondence directly from that for categories, in contrast to [Lam21] and [CLPS22] respectively.

1.1. **Outline of paper.** We begin in Section 2 by recalling the basic definitions of double category theory (both pseudo and virtual) that we will use throughout the paper. In Section 3, we study exponentiability for virtual double categories and, in particular, establish that pseudo double categories are exponentiable virtual double categories (Theorem 3.9). In Section 4, we recall the definition of the virtual double category $\mathbb{M}\mathbf{od}(\mathbb{X})$ of monads and modules in a virtual double category $\mathbb{X}$, recall its universal property as the cofree normal virtual double category on $\mathbb{X}$ (Proposition 4.5), and study some of its properties, including its lax-idempotence (Section 4.2) and its interaction with the slice construction (Section 4.3). In Section 5, we recall the construction of the free normal virtual double category on $\mathbb{X}$ [FL25], and establish some properties useful for the following sections. In Section 6, we establish our main result, demonstrating that Paré's virtual double category $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ of lax functors between pseudo double categories arises as $\mathbb{M}\mathbf{od}(\mathbb{B}^{\mathbb{A}})$ (Theorem 6.2). The remainder of the paper is dedicated to exploring consequences of this result. For instance, we show that the 2-category of pseudo double categories and lax functors is enriched in the 2-category of (normal) virtual double categories (Corollary 6.6), and exhibit a universal property of $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ (Theorem 6.10). In Section 7, we study representability for presheaves of double categories, and show that our perspective simplifies aspects of the development of Paré [Par11]. Finally, in Section 8, we use the perspective to give a new and substantially simpler proof that contravariant $\mathbb{S}\mathbf{pan}$-valued lax functors from a pseudo double category $\mathbb{A}$ are equivalent to discrete fibrations over $\mathbb{A}$, generalising the main theorem of [Lam21] from strict double categories to pseudo double categories.

1.2. **Tight representability and loose representability.** Above, and in this paper as a whole, we focus on just one aspect of representability for double categories. However, we find it helpful to describe briefly how this paper fits into the larger picture, by clarifying the relationship between two approaches for representability for double categories that may be found in the literature.

A presheaf on a category is most helpfully considered as a special case of a more general concept: a *distributor* between categories, which provides a notion of *heteromorphism* between objects of different categories [Bén73]. Since there is a single notion of morphism between objects of a category, there is a single notion of distributor between categories. However, double categories have two kinds of morphism – the *tight morphisms* (which compose strictly), and the *loose morphisms* (which compose weakly) – and correspondingly there are two notions of distributor between double categories: *tight distributors* and *loose distributors*, respectively providing notions of *tight heteromorphisms* and *loose heteromorphisms* between objects of different double categories[3]. By specialising to (tight or loose) distributors with terminal domain or codomain, we obtain notions of *tight (co)presheaf* and *loose (co)presheaf*.

This reveals there are two notions of representability for double categories: *tight representability*, which is concerned with the representability of tight presheaves (and, more generally, tight distributors); and *loose representability*, which is concerned with the representability of loose presheaves (and, more generally, loose distributors). Paré's [Par11] presheaves for double categories – namely, lax functors $\mathbb{A}^{\mathrm{op\,t}} \to \mathbb{S}\mathbf{pan}$ – are the *tight* presheaves. Consequently, we focus in this paper purely on tight representability.

[3] Tight and loose distributors may be defined efficiently as (necessarily strict) functors of double categories into the free-standing tight morphism and into the free-standing loose morphism respectively. The reader is invited to verify that these two definitions recover the notions of (tight) presheaf of Paré [Par11] and (loose) presheaf of Fiore, Gambino and Kock [FGK12] respectively.

Loose representability has been considered in the literature by Fiore, Gambino and Kock [FGK12, §3], albeit only in the context of strict double categories. The precise connection between the two approaches deserves further clarification.

1.3. **Acknowledgements.** The author thanks Bryce Clarke for helpful discussions during the writing of the paper, and Bryce Clarke and Dylan McDermott for comments on the paper itself.

2. Pseudo double categories and virtual double categories

We begin by recalling the basic definitions of pseudo and virtual double categories, both for the benefit of the reader unfamiliar with these notions, and to establish our terminology and notation. The reader familiar with these concepts may prefer to skip directly to Section 3.

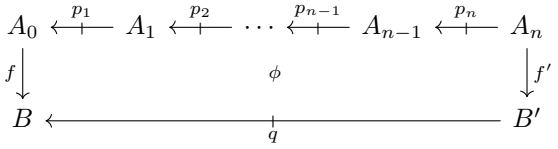
The central concept in this paper is that of a *virtual double category*, which is to a multicategory what a pseudo double category is to a monoidal category. Like a double category, a virtual double category is a structure with two kinds of morphism, which we call *tight morphisms* and *loose morphisms*. The objects and tight morphisms in a virtual double category assemble into a category, so that tight morphisms compose strictly associatively and unitally. However, it is not generally possible to compose the loose morphisms in a virtual double category. Instead, a virtual double category has *cells* that are framed at the top by a path of loose morphisms, representing a 'virtual composite'. If you are encountering the definition for the first time, this may appear a little intimidating. However, in a virtual double category, unlike in a pseudo double category, there is no coherence data, and it often turns out to be simpler to work with virtual double categories than with pseudo double categories for this reason.

Definition 2.1 ([Bur71, page 61; CS10, Definition 2.1])**.** A *virtual double category* $\mathbb{X}$ comprises the following data.

(1) A category $\underline{\mathbb{X}}$ of *objects* and *tight morphisms*. We denote a tight morphism f from an object A to an object B by an arrow $f\colon A \to B$; denote the composition of tight morphisms $f\colon A \to B$ and $g\colon B \to C$ both by $(f\,;g)\colon A \to C$ and by $gf\colon A \to C$; and denote the identity of an object A by $1_A\colon A \to A$, or simply by $=$ in pasting diagrams.

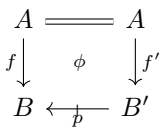
(2) For each pair of objects A and B, a class of *loose morphisms* from A to B. We denote a loose morphism p from A to B by an arrow with a vertical stroke $p\colon A \nrightarrow B$.

(3) For each chain of loose morphisms $p_1, \ldots, p_n$ $(n \geq 0)$ and compatible tight morphisms f, f' and loose morphism q (together forming a *frame*), a class of *cells* with the given frame.



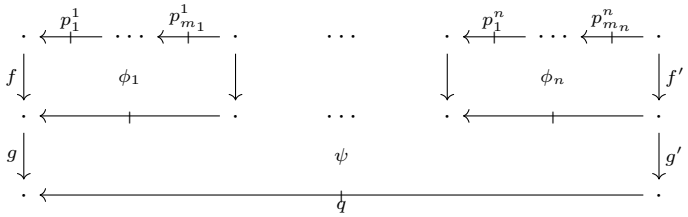
Observe that our pasting diagrams are written from right-to-left (matching nondiagrammatic composition order of loose morphisms). Such a cell is *globular* if f and f' are identities.

In pasting diagrammatic notion, we denote a nullary cell by a square of the following form[4].

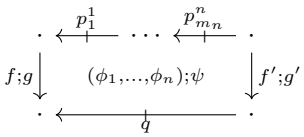


[4] Some authors, e.g. [CS10], prefer to render nullary cells as triangles. However, we find this to be both less uniform (for instance, such authors do not render binary cells as trapeziums) and less visually pleasing than rendering all cells as rectangles.

(4) For every configuration of cells of the following shape,



a cell,

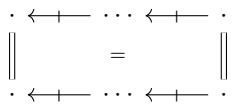


the *composite*.

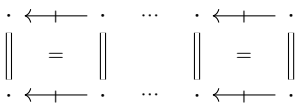
(5) For each loose morphism $p\colon A' \nrightarrow A$, a cell $1_p\colon p \Rightarrow p$, the *identity* of p, denoted simply as $=$ in pasting diagrams.

$$\begin{array}{ccc} A & \xleftarrow{p} & A' \\ \| & = & \| \\ A & \xleftarrow[p]{} & A' \end{array}$$

We shall sometimes denote by



the juxtaposition of identity cells



Composition of cells is required to be associative and unital in the evident manner [CS10, Definition 2.1]: associativity is implicit in pasting diagrams. ⌟

Example 2.2. Every category $\mathbf{A}$ forms a virtual double category $\overline{\mathbf{A}}$ with no loose morphisms or cells. ⌟

Example 2.3. Every multicategory $\mathbf{M}$ forms a virtual double category $\Sigma\mathbf{M}$ with a single object and single tight morphism, the loose morphisms of $\Sigma\mathbf{M}$ being the objects of $\mathbf{M}$, and the cells of $\Sigma\mathbf{M}$ being the multimorphisms of $\mathbf{M}$. ⌟

Example 2.4. The terminal virtual double category is denoted $\mathbb{1}$, and has a single object, single tight morphism, single loose morphism, and a single n-ary cell for each $n \in \mathbb{N}$. ⌟

Example 2.5. The virtual double category $\mathbb{S}\mathbf{pan}$ is defined as follows.

(1) The underlying category is $\mathbf{Set}$, the category of sets and functions.

(2) A loose morphism from A to B is a span $A \leftarrow X \rightarrow B$.

(3) A cell with the following frame

$$\begin{array}{ccccccccccccc} A_0 & \xleftarrow{p_1} & X_1 & \xrightarrow{q_1} & A_1 & \xleftarrow{p_2} & X_2 & \xrightarrow{q_2} & \cdots & \xleftarrow{p_n} & X_n & \xrightarrow{q_n} & A_n \\ {\scriptstyle f}\downarrow & & & & & & & & & & & & \downarrow{\scriptstyle f'} \\ B & & \xleftarrow[r]{} & & & & Y & & \xrightarrow[s]{} & & & & B' \end{array}$$

is a span morphism from the iterated pullback, as follows.

$$\begin{array}{ccccccccccc} A_0 & \xleftarrow{p_1} & X_1 & \xleftarrow{\pi_1} & X_1 \times_{A_1} X_2 \times_{A_2} \cdots \times_{A_n} X_n & \xrightarrow{\pi_n} & X_n & \xrightarrow{q_n} & A_n \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle f'} \\ B & \xleftarrow[r]{} & & & Y & & \xrightarrow[s]{} & & B' \end{array}$$

The identity on a span $A \leftarrow X \rightarrow B$ is given by 1_X, and the composition of cells is induced by the universal property of the pullback in the evident manner. ⌟

Example 2.6. More generally, for any category $\mathbf{E}$ with pullbacks, there is a virtual double category $\mathbb{S}\mathbf{pan}(\mathbf{E})$, whose underlying category is $\mathbf{E}$, in which a loose morphism is a span in $\mathbf{E}$, and whose cells and their composition is defined analogously to Example 2.5. In particular, $\mathbb{S}\mathbf{pan} := \mathbb{S}\mathbf{pan}(\mathbf{Set})$. ⌟

Example 2.7. The virtual double category $\mathbb{D}\mathbf{ist}$ is defined as follows.

(1) The underlying category is $\mathbf{Cat}$, the category of *small* categories and functors.

(2) A loose morphism from $\mathbf{A}$ to $\mathbf{B}$ is a distributor from $\mathbf{A}$ to $\mathbf{B}$, i.e. a functor $\mathbf{B}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$.

(3) A cell with the following frame

$$\begin{array}{ccccc} \mathbf{A}_0 & \xleftarrow{p_1} & \cdots & \xleftarrow{p_n} & \mathbf{A}_n \\ {\scriptstyle f}\downarrow & & \phi & & \downarrow{\scriptstyle f'} \\ \mathbf{B} & & \xleftarrow[q]{} & & \mathbf{B}' \end{array}$$

comprises a family of functions

$$\phi_{a_0,\ldots,a_n} \colon p_1(a_0, a_1) \times \cdots \times p_n(a_{n-1}, a_n) \to q(fa_0, f'a_n)$$

for each $a_0 \in |\mathbf{A}_0|, \ldots, a_n \in |\mathbf{A}_n|$, satisfying naturality conditions [AM24a, Definition 8.1].

The identity on a distributor is given by the identity function, and the composition of cells is induced by functoriality of the cartesian product. ⌟

Example 2.8. More generally, for any category $\mathbf{E}$ with pullbacks, there is a virtual double category $\mathbb{D}\mathbf{ist}(\mathbf{E})$, whose underlying category is $\mathbf{Cat}(\mathbf{E})$ (the category of categories and functors internal to $\mathbf{E}$), in which a loose morphism is an internal distributor, and whose cells are internal natural transformations. Since we will only refer to this example in passing, we will not spell it out, but will rather define $\mathbb{D}\mathbf{ist}(\mathbf{E})$ by means of a general construction on $\mathbb{S}\mathbf{pan}(\mathbf{E})$ in Example 4.2. In particular, $\mathbb{D}\mathbf{ist} := \mathbb{D}\mathbf{ist}(\mathbf{Set})$. ⌟

Definition 2.9 ([Bur71])**.** A *functor* of virtual double categories is a homomorphism, having an assignment on objects, tight morphisms, loose morphisms, and cells that (strictly) preserves identities and composites of tight morphisms and cells. Every functor of virtual double categories $F \colon \mathbb{X} \to \mathbb{X}'$ has an underlying functor $\underline{F} \colon \underline{\mathbb{X}} \to \underline{\mathbb{X}'}$ of categories. ⌟

At first, Definition 2.9 may appear too strict, as we are asking for all structure to be preserved strictly. However, as we shall see in Theorem 2.15, such functors correspond to *lax functors* between (non-virtual) double categories. This illustrates the general principle we mentioned in the beginning of the section: that virtual double categories provide a coherence-free approach to double category theory.

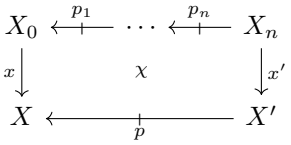
Definition 2.10 ([DPP06, page 489])**.** Let $F, G \colon \mathbb{X} \to \mathbb{Y}$ be functors of virtual double categories. A *transformation* Φ from F to G comprises the following data.

(1) A tight morphism $\Phi_X \colon FX \to GX$ in $\mathbb{Y}$ for each object of $\mathbb{X}$.

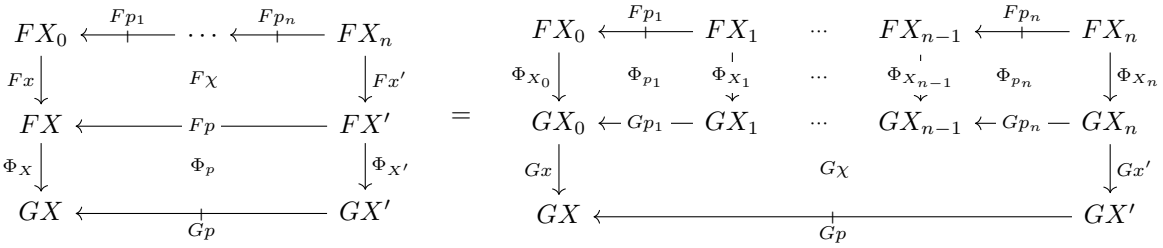
(2) A cell Φ_p in $\mathbb{Y}$ for each loose morphism $p \colon X' \nrightarrow X$ of $\mathbb{X}$.

$$\begin{array}{ccc} FX & \xleftarrow{Fp} & FX' \\ {\scriptstyle \Phi_X}\downarrow & \Phi_p & \downarrow{\scriptstyle \Phi_{X'}} \\ GX & \xleftarrow[Gp]{} & GX' \end{array}$$

The family of tight morphisms is required to form a natural transformation $\underline{F} \Rightarrow \underline{G}$, and the family of cells is required to be natural in the sense that, for each cell χ in $\mathbb{X}$ as follows,



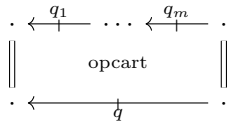
the following equation holds in $\mathbb{Y}$.



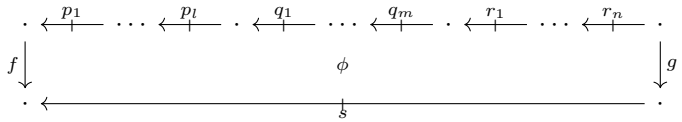
Proposition 2.11 ([DPP06][5])**.** *Virtual double categories, functors, and transformations form a 2-category* **VDblCat**.

While virtual double categories do not admit composites of loose morphisms in general, we may characterise the existence of particular composites by a universal property, in the same way that we may characterise the existence of certain tensor products in a multicategory.

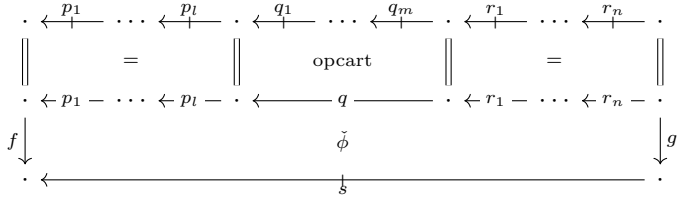
Definition 2.12 ([DPP06, Definition 2.7; CS10, Definition 5.1])**.** A cell



in a virtual double category is *opcartesian* if any cell

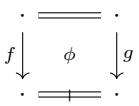


factors uniquely therethrough:



In this case, we call q the *(loose) composite* $q_1 \odot \cdots \odot q_m$ of $q_1, \ldots, q_m$. When $m = 0$, we call $q\colon A \nrightarrow A$ the *loose identity* and denote it by $A(1,1)$, or simply by $=\!\!=$ in pasting diagrams.

We denote a nullary loose morphism with loose identity codomain by $\phi\colon f \Rightarrow g$.



A virtual double category is *representable* when it admits all loose composites (including loose identities). ⌟

[5] Note that Dawson, Paré and Pronk [DPP06] call virtual double categories *lax double categories*, though, as discussed in [Par11, §4], this name is better reserved for a notion of double category in which loose morphisms may be composed, but only associatively and unitally up to noninvertible cells (called a *slack double category* in [DPP06]).

Loose composition is associative and unital up to coherent isomorphism, e.g. we have $(p \odot q) \odot r \cong p \odot q \odot r \cong p \odot (q \odot r)$ when each of these composites exist.

Definition 2.13 ([DPP06, Definition 2.7; CS10, Remark 5.8])**.** If the universal property of Definition 2.12 holds only for $l = 0$ and $r = 0$, we say that the cell is *weakly opcartesian*, and that q is a *weak loose composite* or *weak loose identity*. A virtual double category is *weakly representable* when it admits all weak loose composites. ⌟

Remark 2.14 (Terminological warning)**.** The term *representability* is used in two senses in this paper. When speaking of pseudo double categories, it is in the sense of *representable presheaves*; whereas, when speaking of virtual double categories, it is in the sense of Definition 2.12, i.e. the existence of loose composites. ⌟

Pseudo double categories and their morphisms are typically defined explicitly in terms of pseudocategories in the 2-category **Cat** [GP99, §7.1; ABK24, Example 5.13]. For our purposes, it will be more useful to view pseudo double categories as certain kinds of virtual double categories, which is justified by the following result of Dawson, Paré and Pronk [DPP06].

Theorem 2.15. *Denote by* $\mathbf{DblCat}_l$ *the 2-category of pseudo double categories, lax functors, and transformations [GP99, §7.1 – 7.3]. There is a fully faithful 2-functor* $\mathbf{DblCat}_l \hookrightarrow \mathbf{VDblCat}$ *whose image comprises the representable virtual double categories.*

Proof. The 2-category $\mathbf{DblCat}_l$ is the Kleisli 2-category for a lax-idempotent 2-comonad on $\mathbf{DblCat}_s$ [DPP06, Proposition 1.19], whereas the 2-category **VDblCat** is the 2-category of strict coalgebras [DPP06, Theorem 2.13]. The image of the fully faithful inclusion of the Kleisli 2-category into the 2-category of coalgebras comprises the representable virtual double categories [DPP06, Proposition 2.8]. □

Remark 2.16. Similarly, *colax double categories* may be identified with the virtual double categories that are weakly representable [DPP06, §2.9; CS10, Example 9.7]. ⌟

In this paper, we define a *pseudo double category* to be a representable virtual double category and a *lax functor* to be a functor of virtual double categories between representable virtual double categories.

Example 2.17. The virtual double category $\Sigma\mathbf{M}$ arising from a multicategory $\mathbf{M}$ is representable if and only if $\mathbf{M}$ is representable in the sense of Hermida [Her00], i.e. induced by a monoidal category. ⌟

Example 2.18 ([GP99, §3.2])**.** $\mathbb{S}\mathbf{pan}$ is a pseudo double category. The loose identity associated to a set A is the identity span $A = A = A$, and the composite of two spans is given by the pullback of the inner cospan. More generally, for any category $\mathbf{E}$ with pullbacks, $\mathbb{S}\mathbf{pan}(\mathbf{E})$ is a pseudo double category. ⌟

Example 2.19 ([GP99, §3.1])**.** $\mathbb{D}\mathbf{ist}$ is a pseudo double category. The loose identity associated to a small category $\mathbf{A}$ is the hom distributor $\mathbf{A}(-,-)\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$, and the composite of two distributors $p\colon \mathbf{B} \nrightarrow \mathbf{A}$ and $q\colon \mathbf{C} \nrightarrow \mathbf{B}$ is given by the following coend.

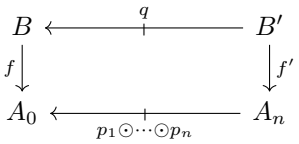
$$p \odot q := \int^{b \in |\mathbf{B}|} p(-,b) \odot q(b,-)$$

More generally, for any category $\mathbf{E}$ with pullbacks and reflexive coequalisers that are stable under pullback, $\mathbb{D}\mathbf{ist}(\mathbf{E})$ is a pseudo double category (e.g. see [Joh77, Proposition 2.43]). ⌟

Example 2.20. For any pseudo double category $\mathbb{A}$, we may define a pseudo double category $\mathbb{A}^{\mathrm{op\,t}}$, the *tight opposite* of $\mathbb{A}$, whose underlying category is $(\underline{\mathbb{A}})^{\mathrm{op}}$, whose loose morphisms are the same as those of $\mathbb{A}$, and in which a cell with the following frame in $\mathbb{A}^{\mathrm{op\,t}}$

$$\begin{array}{ccccc} A_0 & \xleftarrow{p_1} & \cdots & \xleftarrow{p_n} & A_n \\ {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle f'} \\ B & & \xleftarrow[q]{} & & B' \end{array}$$

is given by a cell with the following frame in $\mathbb{A}$.



⌟

To study exponentiability of virtual double categories in the next section, we observe that the 2-category **VDblCat** admits binary products, which are given componentwise in the usual manner. Rather than spell this out, we appeal to the following stronger observation.

Proposition 2.21 ([DPP06, Remark 2.14])**.** **VDblCat** *is a complete and cocomplete 2-category.*

3. Exponentiable virtual double categories

Our investigation begins with the study of exponentiability in the 2-category **VDblCat** of virtual double categories. As we shall see, **VDblCat** is not cartesian closed (Corollary 3.18), but there are nevertheless several interesting classes of exponentiable objects. To analyse exponentials of virtual double categories, it will be useful to first spend a little time studying the relationship between virtual double categories and graphs internal to categories.

Definition 3.1. Denote by $\mathbf{G} := \{G_1 \rightrightarrows G_0\}$ the free-standing parallel pair.

(1) $\mathbf{Grph} := [\mathbf{G}, \mathbf{Set}]$ is the category of graphs.

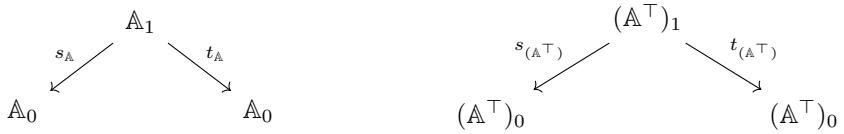
(2) $\mathbf{Grph}(\mathbf{Cat}) := [\mathbf{G}, \mathbf{Cat}]$ is the 2-category of graphs internal to categories. ⌟

There are two perspectives on graphs internal to categories, which we make explicit to avoid potential confusion about our notational conventions.

Proposition 3.2. *The 2-category* $\mathbf{Grph}(\mathbf{Cat})$ *is isomorphic to the 2-category* $\mathbf{Cat}(\mathbf{Grph})$ *of categories internal to* **Grph**.

Proof. Follows from the fact that limits in functor categories are computed componentwise. Abstractly, this is an instance of the symmetry of internalisation for two-dimensional limit sketches [ABK24, Theorem 7.5]. □

Explicitly, the objects of $\mathbf{Grph}(\mathbf{Cat})$ comprise graphs of of categories (below left), where $\mathbb{A}_0$ is a category of *objects* and *tight morphisms*, and $\mathbb{A}_1$ is a category of *loose morphisms* and *unary cells*. On the other hand, objects of $\mathbf{Cat}(\mathbf{Grph})$ comprise internal categories (below right), where $(\mathbb{A}^\top)_0$ is a graph of objects and loose morphisms, and $(\mathbb{A}^\top)_1$ is a graph of tight morphisms and unary cells. The isomorphism between the two 2-categories may therefore be seen as a kind of transposition operation, analogous to that for strict double categories [GP99, §1.2].



In this paper, we will focus on the perspective of $\mathbf{Grph}(\mathbf{Cat})$; our notation is chosen to reflect this, as well as to match the typical convention for pseudo double categories viewed as pseudocategories.

Every virtual double category $\mathbb{X}$ has an underlying graph $\mathbb{X}_-$ internal to categories, given by forgetting the non-unary cells. Conversely, we may view a graph $\mathbb{A}$ internal to categories as a virtual double category $\mathbb{A}^-$ with no non-unary cells. The precise relationship between categories, graphs internal to categories, and virtual double categories is summarised by the following pair of adjunctions.

Proposition 3.3. *There are coreflective 2-adjunctions as follows.*

$$\mathbf{Cat} \underset{(-)_0}{\overset{\mathbf{0}\rightrightarrows(-)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Grph}(\mathbf{Cat}) \underset{(-)_-}{\overset{(-)^-}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{VDblCat} \qquad = \qquad \mathbf{Cat} \underset{\underline{(-)}}{\overset{\overline{(-)}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{VDblCat}$$

Proof. The existence of the leftmost 2-adjunction follows from initiality of the empty category $\mathbf{0}$. The rightmost 2-adjunction $\mathbf{VDblCat}(\mathbb{A}^{-}, \mathbb{X}) \cong \mathbf{Grph}(\mathbf{Cat})(\mathbb{A}, \mathbb{X}_{-})$ is elementary. In both cases, the right adjoints are retractions of the left adjoints. □

Remark 3.4. Neither $\mathbf{0} \rightrightarrows (-)\colon \mathbf{Cat} \to \mathbf{Grph}(\mathbf{Cat})$ nor $\overline{(-)}\colon \mathbf{Cat} \to \mathbf{VDblCat}$ admits a left adjoint, since neither preserves the terminal object. ⌟

It follows from Proposition 3.3 that the forgetful functor $\underline{(-)}\colon \mathbf{VDblCat} \to \mathbf{Cat}$ is given by $\mathbf{VDblCat}(\overline{\mathbf{1}}, -)$, where $\mathbf{1}$ is the terminal category. In what follows, we may refer to the underlying category of a virtual double category $\mathbb{X}$ as either $\underline{\mathbb{X}}$ or $\mathbb{X}_0$ (thereby leaving $(-)_$ implicit), depending on the intuition we wish to convey.

Definition 3.5. A virtual double category is *unary* if each of its cells is unary, i.e. it is in the image of $(-)^{-}\colon \mathbf{Grph}(\mathbf{Cat}) \hookrightarrow \mathbf{VDblCat}$. ⌟

While exponentiability of general virtual double categories is subtle, *unary* virtual double categories are much better behaved.

Proposition 3.6. *The 2-category* $\mathbf{Grph}(\mathbf{Cat})$ *is cartesian closed. Explicitly, given unary virtual double categories* $\mathbb{A}$ *and* $\mathbb{X}$*, the exponential* $\mathbb{X}^{\mathbb{A}}$ *in* $\mathbf{Grph}(\mathbf{Cat})$ *is given by the following unary virtual double category.*

- *The underlying category is the functor category* $[\mathbb{A}_0, \mathbb{X}_0]$*.*
- *A loose morphism from* F *to* F' *is a graph morphism.*

$$\begin{array}{ccccc} \mathbb{A}_0 & \xleftarrow{s} & \mathbb{A}_1 & \xrightarrow{t} & \mathbb{A}_0 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \Xi} & & \downarrow{\scriptstyle F'} \\ \mathbb{X}_0 & \xleftarrow[s]{} & \mathbb{X}_1 & \xrightarrow[t]{} & \mathbb{X}_0 \end{array}$$

  Explicitly, this comprises the following data.

  (1) For each loose morphism $p\colon X \nrightarrow Y$ *in* $\mathbb{A}$*, a loose morphism* $\Xi p\colon FX \nrightarrow GY$*.*

  (2) For each cell in $\mathbb{A}$*,*

$$\begin{array}{ccc} Y & \xleftarrow[p]{} & X \\ {\scriptstyle y}\downarrow & \alpha & \downarrow{\scriptstyle x} \\ Y' & \xleftarrow[p']{} & X' \end{array}$$

  a cell in $\mathbb{X}$*.*

$$\begin{array}{ccc} GY & \xleftarrow{\Xi p} & FX \\ {\scriptstyle Gy}\downarrow & \Xi\alpha & \downarrow{\scriptstyle Fx} \\ GY' & \xleftarrow[\Xi p']{} & FX' \end{array}$$

  These are required to satisfy the following laws.

  (3) $\Xi(1_p) = 1_{\Xi(p)}$*.*

  (4) $\Xi(\alpha\,;\alpha') = \Xi(\alpha)\,;\Xi(\alpha')$*.*

- *A cell with the following frame*

$$\begin{array}{ccc} G & \xleftarrow{\Xi} & F \\ {\scriptstyle \Gamma}\downarrow & \xi & \downarrow{\scriptstyle \Phi} \\ G' & \xleftarrow[\Xi']{} & F' \end{array}$$

  is a natural transformation $\xi\colon \Xi \Rightarrow \Xi'$ *that commutes with* Φ *and* Γ*.*

Proof. $\mathbf{Grph}(\mathbf{Cat})$ is a presheaf 2-category, hence is cartesian closed. The explicit description of the exponential is given by evaluating the exponentials in the presheaf 2-category using the Yoneda lemma. □

Our strategy for studying exponentiability of general virtual double categories stems from the observation that exponentials of virtual double categories are closely related to exponentials of unary virtual double categories.

Definition 3.7. A 2-adjunction $L \dashv R$ between cartesian monoidal 2-categories satisfies the *reciprocity condition* if the canonical 1-cell $\langle \varepsilon_X \circ L\pi_1, L\pi_2 \rangle \colon L(RX \times Y) \to X \times LY$ is invertible for all objects X and Y. ⌟

It is a straightforward exercise to verify that a 2-adjunction satisfies the reciprocity condition if and only if the right adjoint preserves exponentials (cf. [Law70, page 6]).

Lemma 3.8. *The 2-adjunctions of Proposition 3.3 satisfy the reciprocity condition. Consequently, the 2-functors* $(-)_0 \colon \mathbf{VDblCat} \to \mathbf{Grph}(\mathbf{Cat})$ *and* $(-)_0 \colon \mathbf{Grph}(\mathbf{Cat}) \to \mathbf{Cat}$*, and their composite* $\underline{(-)} \colon \mathbf{VDblCat} \to \mathbf{Cat}$*, preserve exponentials.*

Proof. The reciprocity condition for the first adjunction states that the canonical functor $(\mathbb{X}_- \times \mathbb{A})^- \to \mathbb{X} \times \mathbb{A}^-$ is invertible for each virtual double category $\mathbb{X}$ and unary virtual double category $\mathbb{A}$. Since the n-ary cells in a product are given by pairs of n-ary cells in each component, and $\mathbb{A}^-$ has only unary cells, this is clear. The reciprocity condition for the second adjunction states that the canonical functor $(\mathbf{0} \rightrightarrows (\mathbb{X}_0 \times \mathbf{A})) \to \mathbb{X} \times (\mathbf{0} \rightrightarrows \mathbf{A})$ is invertible for each unary virtual double category $\mathbb{X}$ and category $\mathbf{A}$. This is clear for essentially the same reason, since $\mathbf{0} \rightrightarrows \mathbf{A}$ has no cells whatsoever. □

In particular, observe that, upon taking $\mathbf{A} = \mathbf{1}$ to be the terminal category in the reciprocity condition for the 2-adjunction $\overline{(-)} \dashv \underline{(-)} \colon \mathbf{VDblCat} \to \mathbf{Cat}$, we have that $\mathbb{X} \times \overline{\mathbf{1}} \cong \overline{(\underline{\mathbb{X}})}$ for every virtual double category $\mathbb{X}$, i.e. taking the product of any virtual double category $\mathbb{X}$ with the virtual double category $\overline{\mathbf{1}}$ having a single tight morphism and no loose morphisms forgets the loose morphisms in $\mathbb{X}$.

We are now ready to prove the main theorem of this section. As we have mentioned (and will substantiate in Corollary 3.18), **VDblCat** is not cartesian closed. However, there is an important class of exponentiable objects: namely, the pseudo double categories.

Theorem 3.9. *Representable virtual double categories are exponentiable.*

Proof. Let $\mathbb{A}$ be a pseudo double category and let $\mathbb{X}$ be a virtual double category. We shall define a virtual double category $\mathbb{X}^{\mathbb{A}}$, guided by Lemma 3.8, which tells us that the unary virtual double category $(\mathbb{X}^{\mathbb{A}})_-$ underlying $\mathbb{X}^{\mathbb{A}}$ must be $(\mathbb{X}_-)^{(\mathbb{A}_-)}$. Thus, it is only necessary to exercise creativity in coming up with the appropriate definition of the multiary cells and composition; the remaining data is determined.

(1) The underlying category is given by $\underline{\mathbb{X}^{\mathbb{A}}}$.

(2) A loose morphism from F to F' is given by a graph morphism as follows (see Proposition 3.6).

$$\begin{array}{ccccc} \mathbb{A}_0 & \xleftarrow{s} & \mathbb{A}_1 & \xrightarrow{t} & \mathbb{A}_0 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \Xi} & & \downarrow{\scriptstyle F'} \\ \mathbb{X}_0 & \xleftarrow[s]{} & \mathbb{X}_1 & \xrightarrow[t]{} & \mathbb{X}_0 \end{array}$$

(3) A cell with the following frame

$$\begin{array}{ccccc} F_0 & \xleftarrow{\Xi_1} & \cdots & \xleftarrow{\Xi_n} & F_n \\ {\scriptstyle \Phi}\downarrow & & \xi & & \downarrow{\scriptstyle \Phi'} \\ F & & \xleftarrow[\Xi]{} & & F' \end{array}$$

comprises, for each chain of loose morphisms $p_1, \ldots, p_n$ in $\mathbb{A}$, a cell in $\mathbb{X}$,

$$\begin{array}{ccccc} F_0A_0 & \xleftarrow{(\Xi_1)p_1} & \cdots & \xleftarrow{(\Xi_n)p_n} & F_nA_n \\ {\scriptstyle \Phi A_0}\downarrow & & \xi(p_1,\ldots,p_n) & & \downarrow{\scriptstyle \Phi' A_n} \\ FA_0 & & \xleftarrow[\Xi(p_1 \odot \cdots \odot p_n)]{} & & F'A_n \end{array}$$

which is natural in the sense that, for each family of cells ϖ_i in $\mathbb{A}$ $(0 \leq i < n)$,

$$\begin{array}{ccc} A'_i & \xleftarrow{p'_i} & A'_{i+1} \\ {\scriptstyle a_i}\downarrow & \varpi_{i+1} & \downarrow{\scriptstyle a_{i+1}} \\ A_i & \xleftarrow[p_i]{} & A_{i+1} \end{array}$$

the following two cells in $\mathbb{X}$ are equal.

$$\begin{array}{ccccccccc} F_0A'_0 & \xleftarrow{\Xi_1(p'_1)} & F_1A'_1 & \longleftarrow & \cdots & \longleftarrow & F_{n-1}A'_{n-1} & \xleftarrow{\Xi_n(p'_n)} & F_nA'_n \\ {\scriptstyle F_0a_0}\downarrow & \Xi_1(\varpi_1) & \downarrow{\scriptstyle F_1a_1} & & \cdots & & \downarrow{\scriptstyle F_{n-1}a_{n-1}} & \Xi_n(\varpi_n) & \downarrow{\scriptstyle F_na_n} \\ F_0A_0 & \xleftarrow{\Xi_1(p_1)} & F_1A_1 & \longleftarrow & \cdots & \longleftarrow & F_{n-1}A_{n-1} & \xleftarrow{\Xi_n(p_n)} & F_nA_n \\ {\scriptstyle \Phi A_0}\downarrow & & & & \xi(p_1,\dots,p_n) & & & & \downarrow{\scriptstyle \Phi' A_n} \\ FA_0 & & & \xleftarrow[\Xi(p_1\odot\cdots\odot p_n)]{} & & & & & F'A_n \end{array}$$

$$\begin{array}{ccccccccc} F_0A'_0 & \xleftarrow{\Xi_1(p'_1)} & F_1A'_1 & \longleftarrow & \cdots & \longleftarrow & F_{n-1}A'_{n-1} & \xleftarrow{\Xi_n(p'_n)} & F_nA'_n \\ {\scriptstyle \Phi A'_0}\downarrow & & & & \xi(p'_1,\dots,p'_n) & & & & \downarrow{\scriptstyle \Phi A'_n} \\ FA'_0 & & & \xleftarrow{\Xi(p'_1\odot\cdots\odot p'_n)} & & & & & F'A'_n \\ {\scriptstyle Fa_0}\downarrow & & & & \Xi(\varpi_1\odot\cdots\odot\varpi_n) & & & & \downarrow{\scriptstyle Fa_n} \\ FA_0 & & & \xleftarrow[\Xi(p_1\odot\cdots\odot p_n)]{} & & & & & F'A_n \end{array}$$

(4) The identity cell on $\Xi\colon F' \nrightarrow F$ is given by sending each loose morphism p in $\mathbb{A}$ to the identity cell $1_{\Xi(p)}$ in $\mathbb{X}$.

(5) The composite of cells

$$\begin{array}{ccccccccc} F_0^1 & \xleftarrow{\Xi_1^1} & \cdots & \xleftarrow{\Xi_{n_1}^1} & F_{n_1}^1 & \longleftarrow & \cdots & \longleftarrow & F_0^m & \xleftarrow{\Xi_1^m} & \cdots & \xleftarrow{\Xi_{n_m}^m} & F_{n_m}^m \\ {\scriptstyle \Phi_0}\downarrow & \xi^1 & & & \downarrow{\scriptstyle \Phi_1} & & \cdots & & \downarrow{\scriptstyle \Phi_{m-1}} & \xi^m & & & \downarrow{\scriptstyle \Phi_m} \\ F_0 & \xleftarrow{\Xi_1} & & & F_1 & \longleftarrow & \cdots & \longleftarrow & F_{m-1} & \xleftarrow{\Xi_m} & & & F_m \\ {\scriptstyle \Phi}\downarrow & & & & & & \xi & & & & & & \downarrow{\scriptstyle \Phi'} \\ F & & & & & \xleftarrow[\Xi]{} & & & & & & & F' \end{array}$$

is given by the cell sending each chain $p_1^1, \dots, p_{m_n}^m$ of loose morphisms in $\mathbb{A}$ to the following cell in $\mathbb{X}$.

$$\begin{array}{ccccccccc} F_0^1A_0^1 & \xleftarrow{\Xi_1^1(p_1^1)} & \cdots & \xleftarrow{\Xi_{n_1}^1(p_{n_1}^1)} & F_{n_1}^1A_{n_1}^1 & \longleftarrow & \cdots & \longleftarrow & F_0^mA_0^m & \xleftarrow{\Xi_1^m(p_1^m)} & \cdots & \xleftarrow{\Xi_{n_m}^m(p_{n_m}^m)} & F_{n_m}^mA_{n_m}^m \\ {\scriptstyle \Phi_0A_0^1}\downarrow & \xi^1(p_1^1,\dots,p_{n_1}^1) & & & \downarrow{\scriptstyle \Phi_1A_{n_1}^1} & & \cdots & & \downarrow{\scriptstyle \Phi_{m-1}A_0^m} & \xi^m(p_1^m,\dots,p_{n_m}^m) & & & \downarrow{\scriptstyle \Phi_mA_{n_m}^m} \\ F_0A_0^1 & \xleftarrow{\Xi_1(p_1^1\odot\cdots\odot p_{n_1}^1)} & & & F_1A_{n_1}^1 & \longleftarrow & \cdots & \longleftarrow & F_{m-1}A_0^m & \xleftarrow{\Xi_m(p_1^m\odot\cdots\odot p_{n_m}^m)} & & & F_mA_{n_m}^m \\ {\scriptstyle \Phi A_0^1}\downarrow & & & & & & \xi(p_1^1\odot\cdots\odot p_{n_1}^1,\dots,p_1^m\odot\cdots\odot p_{n_m}^m) & & & & & & \downarrow{\scriptstyle \Phi' A_{n_m}^m} \\ FA_0^1 & & & & & \xleftarrow{\Xi((p_1^1\odot\cdots\odot p_{n_1}^1)\odot\cdots\odot(p_1^m\odot\cdots\odot p_{n_m}^m))} & & & & & & & F'A_{n_m}^m \\ \| & & & & & & \Xi(\cong) & & & & & & \| \\ FA_0^1 & & & & & \xleftarrow[\Xi(p_1^1\odot\cdots\odot p_{n_1}^1\odot\cdots\odot p_1^m\odot\cdots\odot p_{n_m}^m)]{} & & & & & & & F'A_{n_m}^m \end{array}$$

That composition is unital and associative follows from unitality and associativity of composition in $\mathbb{A}$ and $\mathbb{X}$, together with naturality of the cells.

We must show that, given a virtual double category $\mathbb{W}$, there is an isomorphism of categories $\mathbf{VDblCat}(\mathbb{W}\times\mathbb{A},\mathbb{X}) \cong \mathbf{VDblCat}(\mathbb{W},\mathbb{X}^{\mathbb{A}})$, 2-natural in $\mathbb{W}$ and $\mathbb{X}$. First, note that this isomorphism

certainly holds on the underlying categories of objects and tight morphisms, i.e. $\mathbf{Cat}(\underline{\mathbb{W}} \times \underline{\mathbb{A}}, \underline{\mathbb{X}}) \cong \mathbf{Cat}(\underline{\mathbb{W}}, \underline{\mathbb{X}}^{\underline{\mathbb{A}}})$, so it remains to check assignments on the loose morphisms and cells.

(1) A functor $F\colon \mathbb{W} \times \mathbb{A} \to \mathbb{X}$ comprises an assignment sending pairs of loose morphisms, $p\colon W' \nrightarrow W$ in $\mathbb{W}$ and $q\colon A' \nrightarrow A$ in $\mathbb{A}$, to a loose morphism $F(p,q)\colon F(W',A') \nrightarrow F(W,A)$ in $\mathbb{X}$; and an assignment sending pairs of a cell in $\mathbb{W}$ and a cell in $\mathbb{A}$,

$$\begin{array}{ccccc} W_1 & \xleftarrow{p_1} & \cdots & \xleftarrow{p_n} & W_n \\ {\scriptstyle w}\downarrow & & \omega & & \downarrow{\scriptstyle w'} \\ W & & \xleftarrow[p]{} & & W' \end{array} \qquad \begin{array}{ccccc} A_1 & \xleftarrow{q_1} & \cdots & \xleftarrow{q_n} & A_n \\ {\scriptstyle a}\downarrow & & \alpha & & \downarrow{\scriptstyle a'} \\ A & & \xleftarrow[q]{} & & A' \end{array}$$

to a cell in $\mathbb{X}$ as follows.

$$\begin{array}{ccccc} F(W_1,A_1) & \xleftarrow{F(p_1,q_1)} & \cdots & \xleftarrow{F(p_n,q_n)} & F(W_n,A_n) \\ {\scriptstyle F(w,a)}\downarrow & & F(\omega,\alpha) & & \downarrow{\scriptstyle F(w',a')} \\ F(W,A) & & \xleftarrow[F(p,q)]{} & & F(W',A') \end{array}$$

(2) A functor $G\colon \mathbb{W} \to \mathbb{X}^{\mathbb{A}}$ comprises an assignment on loose morphisms sending a loose morphism $p\colon W' \nrightarrow W$ in $\mathbb{W}$ to a morphism of spans,

$$\begin{array}{ccccc} \mathbb{A}_0 & \xleftarrow{s} & \mathbb{A}_1 & \xrightarrow{t} & \mathbb{A}_0 \\ {\scriptstyle G(W')}\downarrow & & {\scriptstyle G(p)}\downarrow & & \downarrow{\scriptstyle G(W)} \\ \mathbb{X}_0 & \xleftarrow[s]{} & \mathbb{X}_1 & \xrightarrow[t]{} & \mathbb{X}_0 \end{array}$$

which sends each loose morphism $q\colon A' \nrightarrow A$ in $\mathbb{A}$ to a loose morphism $G(p)(q)\colon G(W')(A') \nrightarrow G(W)(A)$ in $\mathbb{X}$, and sends each unary cell in $\mathbb{A}$

$$\begin{array}{ccc} A_0 & \xleftarrow{q} & A_1 \\ {\scriptstyle a}\downarrow & \alpha & \downarrow{\scriptstyle a'} \\ A & \xleftarrow[q']{} & A' \end{array}$$

to a unary cell in $\mathbb{X}$ as follows.

$$\begin{array}{ccc} G(W)(A_0) & \xleftarrow{G(p)(q)} & G(W')(A_1) \\ {\scriptstyle G(W)(a)}\downarrow & G(p)(\alpha) & \downarrow{\scriptstyle G(W')(a')} \\ G(W)(A) & \xleftarrow[G(p)(q')]{} & G(W')(A') \end{array}$$

It also comprises an assignment of cells ω in $\mathbb{W}$ as above to a cell in $\mathbb{X}^{\mathbb{A}}$ with the following frame,

$$\begin{array}{ccccc} G(W_1)(-) & \xleftarrow{G(p_1)(-)} & \cdots & \xleftarrow{G(p_n)(-)} & G(W_n)(-) \\ {\scriptstyle G(w)(-)}\downarrow & & G(\omega)(-) & & \downarrow{\scriptstyle G(w')(-)} \\ G(W)(-) & & \xleftarrow[G(p)(-)]{} & & G(W')(-) \end{array}$$

hence, for each chain of loose morphisms $q_1, \ldots, q_n$ in $\mathbb{A}$, a cell in $\mathbb{X}$ as follows.

$$\begin{array}{ccccc} G(W_1)(A_1) & \xleftarrow{G(p_1)(q_1)} & \cdots & \xleftarrow{G(p_n)(q_n)} & G(W_n)(A_n) \\ {\scriptstyle G(w)(A_1)}\downarrow & & G(\omega)(q_1,\ldots,q_n) & & \downarrow{\scriptstyle G(w')(A_n)} \\ G(W)(A_1) & & \xleftarrow[G(p)(q_1 \odot \cdots \odot q_n)]{} & & G(W')(A_n) \end{array}$$

It is clear that the action of (1) and (2) above on loose morphisms and unary cells is the same. To establish the one-dimensional universal property of the exponential, it remains only to verify that their action on arbitrary cells is the same.

Observe that, since $\mathbb{A}$ is representable, every cell α in $\mathbb{A}$ factors uniquely through an opcartesian cell as follows.

$$\begin{array}{ccccc}
A_1 & \xleftarrow{q_1} & \cdots & \xleftarrow{q_n} & A_n \\
\| & & \text{opcart} & & \| \\
A_1 & \longleftarrow & q_1 \odot \cdots \odot q_n & \longrightarrow & A_n \\
{\scriptstyle a}\downarrow & & \check{\alpha} & & \downarrow{\scriptstyle a'} \\
A & & \xleftarrow[q]{} & & A'
\end{array}$$

Consequently, given ω and α as above, we have by (2) a cell in $\mathbb{X}$ as follows.

$$\begin{array}{ccccc}
G(W_1)(A_1) & \xleftarrow{G(p_1)(q_1)} & \cdots & \xleftarrow{G(p_n)(q_n)} & G(W_n)(A_n) \\
{\scriptstyle G(w)(A_1)}\downarrow & & G(\omega)(p_1,\ldots,p_n) & & \downarrow{\scriptstyle G(w')(A_n)} \\
G(W)(A_1) & \longleftarrow & G(p)(q_1 \odot \cdots \odot q_n) & \longrightarrow & G(W')(A_n) \\
{\scriptstyle G(W)(a)}\downarrow & & G(p)(\check{\alpha}) & & \downarrow{\scriptstyle G(W')(a')} \\
G(W)(A) & & \xleftarrow[G(p)(q)]{} & & G(W')(A)
\end{array}$$

Conversely, for each chain of loose morphisms $q_1, \ldots, q_n$ in $\mathbb{A}$, we have by (1) a cell in $\mathbb{X}$ as follows.

$$\begin{array}{ccccc}
G(W_1, A_1) & \xleftarrow{G(p_1,q_1)} & \cdots & \xleftarrow{G(p_n,q_n)} & G(W_n, A_n) \\
{\scriptstyle G(w,A_1)}\downarrow & & G(\omega,\text{opcart}) & & \downarrow{\scriptstyle G(w',A_n)} \\
G(W, A_1) & & \xleftarrow[G(p,q_1 \odot \cdots \odot q_n)]{} & & G(W')(A_n)
\end{array}$$

That this correspondence is bijective follows from opcartesianness.

Finally, we must show that the two-dimensional universal property is satisfied. This is simple, since only unary cells are involved in the data of a transformation (Definition 2.10).

(1) A transformation between lax functors $F\colon \mathbb{W} \times \mathbb{A} \to \mathbb{X}$ comprises, for each $p\colon W' \nrightarrow W$ in $\mathbb{W}$ and $q\colon A' \nrightarrow A$ in $\mathbb{A}$, a cell in $\mathbb{X}$ as follows.

$$\begin{array}{ccc}
F(X, A) & \xleftarrow{F(p,q)} & F(X', A') \\
{\scriptstyle \phi_{X,A}}\downarrow & \phi_{p,q} & \downarrow{\scriptstyle \phi_{X',A'}} \\
F'(X, A) & \xleftarrow[F'(p,q)]{} & F'(X', A')
\end{array}$$

(2) A transformation between lax functors $G\colon W \to \mathbb{X}^{\mathbb{A}}$ comprises, for each $p\colon W' \nrightarrow W$ in $\mathbb{W}$, a cell in $\mathbb{X}^{\mathbb{A}}$ as follows,

$$\begin{array}{ccc}
G(X)(-) & \xleftarrow{G(p)(-)} & G(X')(-) \\
{\scriptstyle \gamma_X}\downarrow & \gamma_p & \downarrow{\scriptstyle \gamma_{X'}} \\
G'(X)(-) & \xleftarrow[G'(p)(-)]{} & G'(X')(-)
\end{array}$$

hence, for each loose morphism $q\colon A' \nrightarrow A$ in $\mathbb{A}$, a cell in $\mathbb{X}$ as follows.

$$\begin{array}{ccc}
G(X)(A) & \xleftarrow{G(p)(q)} & G(X')(A') \\
{\scriptstyle (\gamma_X)_A}\downarrow & (\gamma_p)_q & \downarrow{\scriptstyle (\gamma_{X'})_{A'}} \\
G'(X)(A) & \xleftarrow[G'(p)(q)]{} & G'(X')(A')
\end{array}$$

These data, subject to the naturality conditions of Definition 2.10 and (3), are elementarily equivalent. The verification that these assignments are 2-natural is routine. □

Remark 3.10. The reader familiar with Pisani's work on exponentiable multicategories [Pis14] will observe similarities in our analysis of exponentiability for virtual double categories. However, we note that there is a subtlety that arises for virtual double categories that is not present in the

theory of multicategories. First, recall from [Pis14, §2.1] that there is a 2-adjunction as follows, where the left adjoint interprets a category as a multicategory in which every multimorphism is unary.

$$\mathbf{Cat} \underset{(-)_*}{\overset{(-)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Multicat}$$

This adjunction turns out to play both the role of the adjunction $(-)^* \dashv (-)_* \colon \mathbf{VDblCat} \to \mathbf{Grph}(\mathbf{Cat})$, as well as the role of the adjunction $\overline{(-)} \dashv \underline{(-)} \colon \mathbf{VDblCat} \to \mathbf{Cat}$ in the theory of virtual double categories. This is because the 2-category $\mathbf{Cat}$ appears in the theory of multicategories in two ways.

- From the perspective of generalised multicategories [CS10], it is the 2-category underlying the virtual double category on which the monad parameterising the notion of multicategory is defined (namely, the *strict monoidal category* monad on $\mathbb{D}\mathbf{ist}$).
- From the perspective of 2-category theory, it is the base of enrichment.

For virtual double categories, these 2-categories no longer coincide: the former is the 2-category $\mathbf{Grph}(\mathbf{Cat}) \cong \mathbf{Cat}(\mathbf{Grph})$, whereas the latter is again $\mathbf{Cat}$. For this reason, when one wishes to extend a given result from multicategories to virtual double categories, it typically takes a little thought to identify the role that $\mathbf{Cat}$ is playing in that instance. ⌟

Representability is merely a sufficient condition for exponentiability, not a necessary condition. To demonstrate this, we exhibit another class of exponentiable virtual double categories, not contained in the class of representable virtual double categories. We first observe that categories, in addition to being coreflective in virtual double categories by Proposition 3.3, are also reflective therein [FL25, §6].

Proposition 3.11. *The forgetful 2-functor* $\underline{(-)} \colon \mathbf{VDblCat} \to \mathbf{Cat}$ *admits a fully faithful right adjoint* $\mathbb{C}\mathbf{h} \colon \mathbf{Cat} \to \mathbf{VDblCat}$ *sending each category to the representable virtual double category having the same underlying category, a unique loose morphism between every pair of objects, and a unique cell filling every frame.*

Proof. Trivial. □

Since $\mathbb{C}\mathbf{h}(\mathbf{A})$ is representable for each category $\mathbf{A}$, it is exponentiable by Theorem 3.9. In addition to the representable virtual double categories, every discrete virtual double category on a category is exponentiable[6].

Proposition 3.12. *Let* $\mathbf{A}$ *be a category.* $\overline{\mathbf{A}}$ *is exponentiable, but is representable if and only if* $\mathbf{A} = \mathbf{0}$*. Concretely,* $\mathbb{Y}^{\overline{\mathbf{A}}} \cong \mathbb{C}\mathbf{h}(\underline{\mathbb{Y}}^{\mathbf{A}})$ *for any virtual double category* $\mathbb{Y}$*.*

Proof. We have the following chain of isomorphims, natural in virtual double categories $\mathbb{X}$ and $\mathbb{Y}$.

$$\begin{aligned}
\mathbf{VDblCat}(\mathbb{X} \times \overline{\mathbf{A}}, \mathbb{Y}) &\cong \mathbf{VDblCat}(\overline{\underline{\mathbb{X}} \times \mathbf{A}}, \mathbb{Y}) && \text{(Lemma 3.8)} \\
&\cong \mathbf{Cat}(\underline{\mathbb{X}} \times \mathbf{A}, \underline{\mathbb{Y}}) && \text{(Proposition 3.3)} \\
&\cong \mathbf{Cat}(\underline{\mathbb{X}}, \underline{\mathbb{Y}}^{\mathbf{A}}) && (\mathbf{Cat} \text{ is cartesian closed}) \\
&\cong \mathbf{VDblCat}(\mathbb{X}, \mathbb{C}\mathbf{h}(\underline{\mathbb{Y}}^{\mathbf{A}})) && \text{(Proposition 3.11)}
\end{aligned}$$

Finally, note that $\overline{\mathbf{A}}$ has no loose morphisms, so cannot be representable unless it has no objects (in which case it is trivially representable). □

In particular, we have $\mathbb{X}^{\overline{\mathbf{1}}} \cong \mathbb{C}\mathbf{h}(\underline{\mathbb{X}})$ for every virtual double category $\mathbb{X}$, and so the 2-adjunction $(-) \times \overline{\mathbf{1}} \dashv (-)^{\overline{\mathbf{1}}}$ recovers the 2-adjunction $\underline{(-)} \dashv \mathbb{C}\mathbf{h}$ of Proposition 3.11.

[6] This has also been observed by Fujii and Lack [FL25, Remark 6.1].

3.1. **An exponentiability conjecture.** It would certainly be of interest to characterise the exponentiable virtual double categories generally. We shall not do so here, as our motivating applications require only the exponentiability of pseudo double categories. However, we shall draw an analogy to the theory of multicategories to make a conjecture. The exponentiable multicategories may be characterised as the promonoidal categories [Pis14, Proposition 2.8] (i.e. the pseudocomonoids in the monoidal bicategory **Dist**), which may be viewed as multicategories that are generated from their nullary and binary multimorphisms. This motivates the following definition.

Definition 3.13. A *pro-double category* is a pseudocategory in the bicategory **Dist** of small categories, distributors, and natural transformations for which the source and target projections are representable distributors.[7] Explicitly, this comprises categories $\mathbb{A}_0$ and $\mathbb{A}_1$, functors $s, t\colon \mathbb{A}_1 \to \mathbb{A}_0$, distributors $\mathsf{l}\colon {\mathbb{A}_1}^{\mathrm{op}} \times \mathbb{A}_0 \to \mathbf{Set}$ and $\odot\colon {\mathbb{A}_1}^{\mathrm{op}} \times \mathbb{A}_2 \to \mathbf{Set}$ together with natural transformations expressing compatibility, and associativity and unitality constraints. ⌟

In particular, every probicategory in the sense of Day [Day73] (i.e. a **Dist**-enriched bicategory) gives rise to a pro-double category whose underlying category of objects and tight morphisms is discrete (this is a bicategorical analogue of the classical correspondence between enriched and internal categories [Ver92, §2.2]). Pro-double categories may be viewed as virtual double categories that are generated from their nullary and binary cells.

In practice, being promonoidal is not a particularly convenient property with which to work. It is therefore useful to identify an intrinsic property of a multicategory that corresponds to its being induced by a promonoidal category. Following Román [Rom24, Definition 2.3.3 & Proposition 2.3.10], we shall refer to the appropriate property as *malleability*: conceptually, it says that the composition operation of a multicategory is invertible, in that every multimorphism may be decomposed in a suitable manner. The following definition captures the analoguous property for virtual double categories.

Definition 3.14. Let T be the monad on $\mathbb{S}\mathbf{pan}(\mathbf{Grph})$ whose algebras are categories, inducing a monad $\mathbb{M}\mathbf{od}(T)$ on $\mathbb{D}\mathbf{ist}(\mathbf{Grph})$ [CS10, Examples 4.7 & 8.13]. A virtual double category is *malleable* if, viewed as a $\mathbb{M}\mathbf{od}(T)$-*monoid* in the sense of Cruttwell and Shulman [CS10, Definition 4.3], its multiplication cell is cartesian. ⌟

Finally, recall that the promonoidal categories may be characterised as the cocontinuously monoidal presheaf categories [Day70], which, in light of Pisani's work [Pis14, Proposition 2.12], suggests that the cartesian multicategory **Set** plays a distinguished role with respect to exponentiability, analogous to the Sierpiński locale in the category of locales (cf. [Tow06, Theorem 2]). These observations motivate the following conjecture.

Conjecture 3.15. The following are equivalent for a virtual double category $\mathbb{X}$.

(1) $\mathbb{X}$ is exponentiable.

(2) $\mathbb{X}$ is malleable.

(3) $\mathbb{X}$ is a pro-double category.

(4) The exponential $\mathbb{S}\mathbf{pan}^{\mathbb{X}}$ exists.

3.2. **Relation to multicategories.** **Multicat** is a full sub-2-category of **VDblCat**: its image comprises the virtual double categories whose underlying category is terminal (i.e. the virtual double categories having a single object and single tight morphism as in Example 2.3). Following from the work of Pisani [Pis14], every representable multicategory (i.e. monoidal category) is exponentiable, since monoidal categories are in particular promonoidal. In fact, the exponentiability of monoidal categories is a special case of a more general phenomenon: that multicategories may be powered by exponentiable virtual double categories.

Proposition 3.16. **Multicat** *is powered over* $\mathbf{VDblCat}_{\mathrm{exp}}$ *(the full sub-2-category of* **VDblCat** *spanned by exponentiable virtual double categories) and the 2-functor* $\Sigma\colon \mathbf{Multicat} \hookrightarrow \mathbf{VDblCat}$

[7] Abstractly, the inclusion of bicategories $\mathbf{Cat} \to \mathbf{Dist}$ transfers finite product sketch structure, and we are defining a pro-double category to be a pseudocategory in this sketch structure on **Dist**, in the sense of [ABK24] (modulo the distinction between sketches on 2-categories and sketches on bicategories, which is easily resolved).

strictly preserves these powers. Explicitly, for a multicategory $\mathbf{M}$ *and pseudo double category* $\mathbb{A}$*, the power* $\mathbb{A} \pitchfork \mathbf{M} := (\Sigma\mathbf{M})^{\mathbb{A}}$ *has objects the functors* $\mathbb{A}_1 \to \mathbf{M}_1$ *and as multimorphisms* $F_1, \dots, F_n \to G$ *families of multimorphisms*

$$\{F_1(p_1), \dots, F_n(p_n) \to G(p_1 \odot \cdots \odot p_n)\}_{p_1,\dots,p_n \in \mathbb{A}_n}$$

natural in $p_1, \dots, p_n$*. Consequently,* Σ *preserves exponentials by monoidal categories.*

Proof. Let $\mathbf{M}$ be a multicategory and let $\mathbb{A}$ be an exponentiable virtual double category. Since the underlying category of $\Sigma\mathbf{M}$ is terminal, so is that of $(\Sigma\mathbf{M})^{\mathbb{A}}$ by Lemma 3.8: we denote the corresponding multicategory $\mathbb{A} \pitchfork \mathbf{M}$. We therefore obtain a functor $\pitchfork\colon \mathbf{VDblCat}_{\mathrm{exp}}{}^{\mathrm{op}} \times \mathbf{Multicat} \to \mathbf{Multicat}$. That this exhibits $\mathbf{Multicat}$ as powered follows from the fact that exponentiation exhibits $\mathbf{VDblCat}$ as powered over $\mathbf{VDblCat}_{\mathrm{exp}}$, and $\mathbf{Multicat}$ is a full subcategory of $\mathbf{VDblCat}$. Unwrapping the description of the exponential in Theorem 3.9 gives the stated definition of $\mathbb{A} \pitchfork \mathbf{M}$ when $\mathbb{A}$ is representable.

For $\mathbf{A}$ a monoidal category, viewed as a representable multicategory, $\Sigma\mathbf{A}$ is also representable, and $\mathbf{M}^{\Sigma\mathbf{A}}$ coincides with $\Sigma(\mathbf{M}^{\mathbf{A}})$ by inspection of the definition of exponentials of multicategories [Pis14, §2.9]. □

Given a promonoidal category $\mathbf{A}$ and a cocomplete monoidal category $\mathbf{M}$, the exponential multicategory $\mathbf{M}^{\mathbf{A}}$ is representable, and the corresponding monoidal structure coincides with Day's [Day70] convolution monoidal structure [Pis14, Proposition 2.12]. Consequently, the operation of powering a multicategory by a virtual double category may be viewed as a substantial generalisation of convolution for monoidal categories.

We expect Σ to preserve exponentials generally; this would follow straightforwardly from a concrete description of exponentials following Conjecture 3.15. However, we can show that Σ reflects exponentiability.

Lemma 3.17. Σ *reflects exponentiability: if* $\mathbf{A}$ *is a multicategory for which* $\Sigma\mathbf{A}$ *is exponentiable as a virtual double category, then* $\mathbf{A}$ *is exponentiable as a multicategory.*

Proof. Suppose the virtual double category $\Sigma\mathbf{A}$ is exponentiable. We have:

$$\begin{aligned}
\mathbf{Multicat}(\mathbf{M} \times \mathbf{A}, \mathbf{N}) &\cong \mathbf{VDblCat}(\Sigma(\mathbf{M} \times \mathbf{A}), \Sigma\mathbf{N}) && (\Sigma \text{ is fully faithful}) \\
&\cong \mathbf{VDblCat}(\Sigma\mathbf{M} \times \Sigma\mathbf{A}, \Sigma\mathbf{N}) && (\Sigma \text{ preserves products}) \\
&\cong \mathbf{VDblCat}(\Sigma\mathbf{M}, \Sigma\mathbf{N}^{\Sigma\mathbf{A}}) && (\Sigma\mathbf{A} \text{ is exponentiable}) \\
&\cong \mathbf{VDblCat}(\Sigma\mathbf{M}, \Sigma(\Sigma\mathbf{A} \pitchfork \mathbf{N})) && (\Sigma \text{ preserves powers}) \\
&\cong \mathbf{Multicat}(\mathbf{M}, \Sigma\mathbf{A} \pitchfork \mathbf{N}) && (\Sigma \text{ is fully faithful})
\end{aligned}$$

which exhibits $\Sigma\mathbf{A} \pitchfork \mathbf{N}$ as the exponential $\mathbf{N}^{\mathbf{A}}$. □

As a consequence, the failure of cartesian closure of multicategories implies the failure of cartesian closure of virtual double categories.

Corollary 3.18. $\mathbf{VDblCat}$ *is not cartesian closed.*

Proof. If every virtual double category were exponentiable, then every multicategory would also be exponentiable by Lemma 3.17. However, the only exponentiable multicategories are the promonoidal ones, by [Pis14, Proposition 2.8]. □

It is worth noting that (a special case of) the power structure of Proposition 3.16 has been previously studied in the literature. In [BMZ23], Behr, Melliès and Zeilberger show that, given a small pseudo double category $\mathbb{A}$, the category of set-valued functors from the category $\mathbb{A}_1$, of loose morphisms and cells in $\mathbb{A}$, admits a colax monoidal structure that recovers the usual convolution monoidal structure of Day [Day70] when $\mathbb{A}$ is the delooping of a monoidal category.

Proposition 3.19. *For a pseudo double category* $\mathbb{A}$*, viewing* $\mathbf{Set}$ *as a cartesian multicategory, the power* $\mathbb{A} \pitchfork \mathbf{Set}$ *is weakly representable, and the corresponding colax monoidal structure on the category* $\mathbf{Set}^{\mathbb{A}_1}$ *is the convolution structure of Behr, Melliès and Zeilberger [BMZ23, Theorem 3.5].*

Proof. A multimorphism $F_1, \dots, F_n \to G$ in the convolution structure of Behr, Melliès and Zeilberger is given by a natural transformation

$$\{\int^{(p_1,\dots,p_n)\in\mathbb{A}_n} \mathbb{A}_1(p_1 \odot \cdots \odot p_n, r) \times F_1(p_1) \times \cdots \times F_n(p_n) \to G(r)\}_{r\in\mathbb{A}_1}$$

which, by the universal property of the coend, is equivalent to a natural family

$$\{\mathbb{A}_1(p_1 \odot \cdots \odot p_n, r) \times F_1(p_1) \times \cdots \times F_n(p_n) \to G(r)\}_{(p_1,\dots,p_n)\in\mathbb{A}_n, r\in\mathbb{A}_1}$$

which, by naturality, is equivalent to

$$\{F_1(p_1) \times \cdots \times F_n(p_n) \to G(p_1 \odot \cdots \odot p_n)\}_{(p_1,\dots,p_n)\in\mathbb{A}_n, r\in\mathbb{A}_1}$$

which are precisely the multimorphisms of $\mathbb{A} \pitchfork \mathbf{Set}$. □

Remark 3.20. We shall not study the representability of exponentials of virtual double categories here, as it is tangential to our interests. However, the observation that the specific exponentials appearing in Proposition 3.19 are weakly representable suggest it would be worthwhile to investigate the general phenomenon. We conjecture that, for every pseudo double category $\mathbb{A}$ and locally cocomplete and weakly representable virtual double category $\mathbb{X}$, the exponential $\mathbb{X}^{\mathbb{A}}$ is again weakly representable. If $\mathbb{A}$ furthermore satisfies a suitable factorisation property, such as the *AFP condition* of Paré [Par13, Definition 3.1.1] or the *n-cylindrical decomposition property* of Behr, Melliès and Zeilberger [BMZ23, Definition 5.1] (both of which are satisfied, for example, by pseudo double categories with companions and conjoints), and $\mathbb{X}$ admits non-nullary composites, we expect that $\mathbb{X}^{\mathbb{A}}$ also admits non-nullary composites. As observed in [BMZ23, Example 4.5], loose identities rarely exist in exponentials, even assuming $\mathbb{X}$ admits them. While the absence of loose identities may seem undesirable, we will observe in Remark 6.4 that this is less restrictive than might at first be imagined. ⌟

4. The virtual double category of loose monads

As outlined in the introduction, we will show in Section 6 that exponentiation of virtual double categories is closely connected to Paré's virtual double category of lax functors. To explain the connection, we must recall a second construction on virtual double categories.

Definition 4.1 ([Lei04, Definition 5.3.1])**.** Let $\mathbb{X}$ be a virtual double category. The virtual double category $\mathbb{M}\mathbf{od}(\mathbb{X})$ is defined as follows.

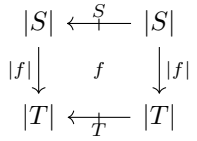
(1) An object is a *loose monad* in $\mathbb{X}$, comprising an object $|T| \in \mathbb{X}$, a loose morphism $T\colon |T| \nrightarrow |T|$, and cells



subject to the following equations.

$$(\circ_T, 1_T)\,;\circ_T = (1_T, \circ_T)\,;\circ_T \qquad (\mathsf{I}_T, 1_T)\,;\circ_T = 1_T \qquad (1_T, \mathsf{I}_T)\,;\circ_T = 1_T$$

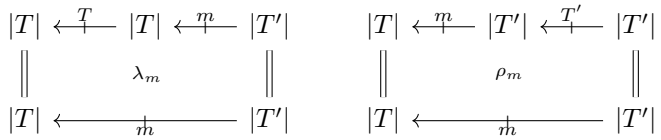
(2) A tight morphism from $(|S|, S, \circ_S, \mathsf{I}_S)$ to $(|T|, T, \circ_T, \mathsf{I}_T)$ is a *loose monad morphism*, comprising a tight morphism $|f|\colon |S| \to |T|$ and a cell



subject to the following equations.

$$\circ_S\,; f = (f, f)\,;\circ_T \qquad\qquad \mathsf{I}_S\,; f = 1_{|f|}\,;\mathsf{I}_T$$

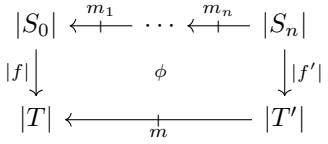
(3) A loose morphism is a *loose monad module*, comprising a loose morphism $m\colon |T|' \nrightarrow |T|$ and cells



subject to the following equations.

$$(\circ_T, 1_m)\,;\lambda_m = (1_T, \lambda_m)\,;\lambda_m \qquad (\mathsf{l}_T, 1_m)\,;\lambda_m = 1_m$$
$$(1_m, \circ_{T'})\,;\rho_m = (\rho_m, 1_{T'})\,;\rho_m \qquad (1_m, \mathsf{l}_{T'})\,;\rho_m = 1_m$$
$$(\lambda_m, 1_{T'})\,;\rho_m = (1_T, \rho_m)\,;\lambda_m$$

(4) A cell is *loose monad transformation*, comprising a cell in $\mathbb{X}$



subject to the following equations[8].

$$(f, \phi)\,;\lambda_m = (\phi, f')\,;\rho_m \qquad (n = 0)$$
$$(\lambda_{m_1}, 1_{m_2}, \ldots, 1_{m_n})\,;\phi = (f, \phi)\,;\lambda_m \qquad (n \geq 1)$$
$$(1_{m_1}, \ldots, 1_{m_{n-1}}, \rho_{m_n})\,;\phi = (\phi, f')\,;\rho_m \qquad (n \geq 1)$$
$$(1_{m_1}, \ldots, \lambda_{m_{i+1}}, \ldots, 1_{m_n})\,;\phi = (1_{m_1}, \ldots, \rho_{m_i}, \ldots, 1_{m_n})\,;\phi \qquad (1 \leq i < n)$$

Identities and composition is given componentwise. ⌟

Since the only kinds of monads we consider in this paper are loose monads, we will typically refer to loose monads simply as *monads*. Many category-like structures arise by applying the $\mathbb{M}\mathbf{od}$ construction to a simpler virtual double category. The prototypical example, stemming from observations of Bénabou [Bén67] and Burroni [Bur71], is the virtual double category of categories and distributors.

Example 4.2. $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}) \cong \mathbb{D}\mathbf{ist}$. More generally, for any category $\mathbf{E}$ with pullbacks, $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathbf{E})) \cong \mathbb{D}\mathbf{ist}(\mathbf{E})$. ⌟

Many more instances of virtual double categories arising from the $\mathbb{M}\mathbf{od}$ construction may be found in the work of Cruttwell and Shulman [CS10]; we shall see examples of a different flavour shortly.

4.1. **Normality.** The $\mathbb{M}\mathbf{od}$ construction admits a convenient universal property: it is the cofree *normal* virtual double category construction. This property will be central to our study of lax functors in Section 6.

Definition 4.3. A *normal* virtual double category is a virtual double category equipped with a choice of loose identities. A *normal functor* strictly preserves the chosen loose identities. Denote by $\mathbf{NVDblCat}$ the 2-category of normal virtual double categories, normal functors, and transformations. There is a forgetful 2-functor $(-)'\colon \mathbf{NVDblCat} \to \mathbf{VDblCat}$. ⌟

Our definition of normal functor may seem overly strict. However, any functor that preserves loose identities up to isomorphism is isomorphic to a functor that preserves loose identities strictly. Thus there is no loss in generality in considering the stricter notion, which allows us to work with stricter universal properties.

Just as every monoid acts on itself, every loose monad is a module for itself: this exhibits $\mathbb{M}\mathbf{od}(\mathbb{X})$ as a normal virtual double category.

[8] Note that [Lei04, Definition 5.3.1] is incomplete, as it omits the coherence condition for nullary cells (as observed in [AM24b, Footnote 3]).

Lemma 4.4 ([CS10, Proposition 5.5])**.** *Let $\mathbb{X}$ be a virtual double category. The virtual double category $\mathbb{M}\mathbf{od}(\mathbb{X})$ is normal: the loose identity on a monad T in $\mathbb{X}$ is T itself.*

In fact, $\mathbb{M}\mathbf{od}$ is universal in this respect.

Proposition 4.5 ([CS10, Proposition 5.14])**.** *There is a 2-adjunction as follows.*

$$\mathbf{NVDblCat} \underset{\mathbb{M}\mathbf{od}}{\overset{(-)'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{VDblCat}$$

Note that Cruttwell and Shulman [CS10] only establish the existence of a *pseudo*adjunction, since their normal functors preserve loose identities only up to isomorphism. Since we will have need to analyse the unit and counit of this 2-adjunction further in what follows, we find it helpful to provide a self-contained proof of Proposition 4.5.

Proof. First, it is clear that $\mathbb{M}\mathbf{od}$ is functorial, since its definition is diagrammatic in nature. We define the unit

$$\eta_{\mathbb{X}} \colon \mathbb{X} \to \mathbb{M}\mathbf{od}(\mathbb{X}') \tag{1}$$

at a normal virtual double category $\mathbb{X}$ to be the functor sending an object A to the loose monad $A(1,1)$; a tight morphism $f \colon A \to B$ to the monad morphism $(f, 1_f)$; a loose morphism p to the monad module p; and a cell φ to the monad transformation φ. The functor $\eta_{\mathbb{X}}$ is normal, since it sends a loose identity $A(1,1) \colon A \nrightarrow A$ in $\mathbb{X}$ to the monad module $A(1,1) \colon A(1,1) \nrightarrow A(1,1)$, which is the identity on the loose monad $A(1,1)$. It is clear that η is 2-natural with respect to strictly normal functors.

We define the counit

$$\varepsilon_{\mathbb{Y}} \colon \mathbb{M}\mathbf{od}(\mathbb{Y})' \to \mathbb{Y} \tag{2}$$

at a virtual double category $\mathbb{Y}$ to be the forgetful functor sending a loose monad T with carrier B to the object B; a monad morphism (f, ϕ) to the tight morphism f; a monad module m to the loose morphism m; and a monad transformation φ to the cell φ. ε is trivially 2-natural.

The left triangle identity says that the carrier of $A(1,1)$ is A. The right triangle identity says that if we view a loose monad T as a monad module $T \colon T \nrightarrow T$, which forms a loose monad in $\mathbb{M}\mathbf{od}(\mathbb{Y})'$, its underlying loose monad is precisely T.

$$\begin{array}{ccc} \mathbb{X}' & \xrightarrow{\eta_{\mathbb{X}}'} & \mathbb{M}\mathbf{od}(\mathbb{X}')' \\ & \searrow & \downarrow{\scriptstyle \varepsilon_{\mathbb{X}'}} \\ & & \mathbb{X}' \end{array} \qquad \begin{array}{ccc} \mathbb{M}\mathbf{od}(\mathbb{Y}) & \xrightarrow{\eta_{\mathbf{Mnd}(\mathbb{Y})}} & \mathbb{M}\mathbf{od}(\mathbb{M}\mathbf{od}(\mathbb{Y})') \\ & \searrow & \downarrow{\scriptstyle \mathbb{M}\mathbf{od}(\varepsilon_{\mathbb{Y}})} \\ & & \mathbb{M}\mathbf{od}(\mathbb{Y}) \end{array}$$

□

4.2. **Lax-idempotence.** The counit of the 2-adjunction in Proposition 4.5 gives, for each virtual double category $\mathbb{Y}$, a forgetful functor $\varepsilon_{\mathbb{Y}} \colon \mathbb{M}\mathbf{od}(\mathbb{Y})' \to \mathbb{Y}$, while the unit gives, for each *normal* virtual double category $\mathbb{X}$, a normal functor $\eta_{\mathbb{X}} \colon \mathbb{X} \to \mathbb{M}\mathbf{od}(\mathbb{X}')$. We might hope that these are related by adjointness, and indeed this is true[9].

Definition 4.6 ([CN24, Theorem 3.15])**.** A 2-adjunction $L \dashv R$ with unit η and counit ε is *lax-idempotent* if there is an adjunction $L\eta \dashv \varepsilon L \colon LRL \Rightarrow L$ with invertible unit. ⌟

Theorem 4.7. *The 2-adjunction $(-)' \dashv \mathbb{M}\mathbf{od}$ is lax-idempotent.*

Proof. We must show that, for each normal virtual double category $\mathbb{X}$, we have a reflective adjunction as follows.

$$\mathbb{X}' \underset{\varepsilon_{\mathbb{X}'}}{\overset{\eta_{\mathbb{X}}'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbb{M}\mathbf{od}(\mathbb{X}')'$$

The unit of the adjunction is the identity, the carrier of each identity loose monad $A(1,1)$ on an object A simply being A. The counit $\eta_{\mathbb{X}}' \circ \varepsilon_{\mathbb{X}'} \Rightarrow 1$ is the transformation specified by, for each loose

[9] This has been independently observed by Fujii and Lack [FL25, Remark 7.8].

monad S on A, the loose monad morphism $(1_A, \mathsf{l}_S)\colon A(1,1) \to S$; and, for each module $p\colon S \nrightarrow T$, the identity cell on p.

$$\begin{array}{ccc} A(1,1) & \xrightarrow{p} & B(1,1) \\ {\scriptstyle (1_A,\mathsf{l}_S)}\downarrow & 1_p & \downarrow{\scriptstyle (1_B,\mathsf{l}_T)} \\ S & \xrightarrow[p]{} & T \end{array}$$

The left triangle identity expresses that, for each object A, the unit of the loose monad $A(1,1)$ is the identity. The right triangle identity expresses that, for each loose monad S on A, the first component of $(1_A, \mathsf{l}_S)$ is the identity. □

Remark 4.8. Since the 2-adjunction is lax-idempotent, so too is the induced 2-monad $\mathbb{M}\mathbf{od}(-')$ on $\mathbf{NVDblCat}$.[10] This permits a simple characterisation of the pseudoalgebras for the 2-monad, which we shall show in future work are related to the *exact virtual double categories* of [AM24b, §5] (cf. [AM24b, Remark 5.4]). ⌟

We record a useful consequence of lax-idempotence.

Definition 4.9. A functor of virtual double categories is *fully faithful* if each of the actions on tight morphisms and on cells is invertible. ⌟

Corollary 4.10. *For each normal virtual double category $\mathbb{X}$, the unit component $\eta_{\mathbb{X}}\colon \mathbb{X} \to \mathbb{M}\mathbf{od}(\mathbb{X}')$ (1) is fully faithful.*

Proof. Immediate from Theorem 4.7, since it is the left adjoint of a coreflective adjunction. □

Lemma 4.11. *If a functor $F\colon \mathbb{X} \to \mathbb{Y}$ between virtual double categories is fully faithful, then so is $\mathbb{M}\mathbf{od}(F)\colon \mathbb{M}\mathbf{od}(\mathbb{X}) \to \mathbb{M}\mathbf{od}(\mathbb{Y})$.*

Proof. The unit preservation condition for a loose monad morphism between identity loose monads implies that it is determined by the tight morphism component. Loose monad transformations are trivially determined by their underlying cells. □

Proposition 4.12. *If a functor $F\colon \mathbb{X}' \to \mathbb{Y}$ from a normal virtual double category to a virtual double category is fully faithful, then so is its transpose $\mathbb{M}\mathbf{od}(F) \circ \eta_{\mathbb{X}}\colon \mathbb{X} \to \mathbb{M}\mathbf{od}(\mathbb{X}') \to \mathbb{M}\mathbf{od}(\mathbb{Y})$.*

Proof. By Corollary 4.10 and Lemma 4.11, the transpose is the composite of two fully faithful functors. □

4.3. **Slicing over a monad.** We now have almost all the properties of the $\mathbb{M}\mathbf{od}$ construction that we need for what follows. One further observation will be relevant: when we analyse the relationship between presheaves and discrete fibrations for double categories in Section 8, we shall need to understand how the $\mathbb{M}\mathbf{od}$ construction interacts with slicing. The slice of a pseudo double category over an object is well known [GP99, §1.7]; more generally, we may slice over a loose monad in a pseudo double category [GP99, §2.5]. Since the 2-category $\mathbf{VDblCat}$ is complete (Proposition 2.21), we may similarly slice over loose monads in a virtual double category; observe that a functor of virtual double categories $\mathbb{1} \to \mathbb{X}$ expresses the data of a loose monad in $\mathbb{X}$.

Definition 4.13. Let $\mathbb{X}$ be a virtual double category and let T be a loose monad in $\mathbb{X}$. The *slice virtual double category* $\mathbb{X}/T$ is the slice object in the 2-category of virtual double categories, i.e. the following comma object in $\mathbf{VDblCat}$.

$$\begin{array}{ccc} \mathbb{X}/T & \xrightarrow{\langle\rangle} & \mathbb{1} \\ {\scriptstyle \pi}\downarrow & \swarrow & \downarrow{\scriptstyle T} \\ \mathbb{X} & = & \mathbb{X} \end{array}$$

For any object $X \in \mathbb{X}$ admitting a loose identity, denote by $\mathbb{X}/X$ the slice $\mathbb{X}/X(1,1)$. ⌟

[10] Geoff Cruttwell and Michael Shulman have independently observed that the 2-monad $\mathbb{M}\mathbf{od}(-')$ is lax-idempotent.

When $\mathbb{X}$ is normal (so that all loose identities exist), the slice over an object is the slice in the 2-category of normal virtual double categories, observing that a normal functor $\mathbb{1} \to \mathbb{X}$ expresses the data of an object in $\mathbb{X}$, and that the forgetful functor $(-)' \colon \mathbf{NVDblCat} \to \mathbf{VDblCat}$ preserves limits (as will follow, for instance, from Proposition 5.2).

We shall be interested in slicing virtual double categories of the form $\mathbb{M}\mathbf{od}(\mathbb{X})$, for which the following lemma will be useful.

Lemma 4.14. *Let $f \colon A \to B$ and $a \colon 1 \to A$ be 1-cells in a 2-category with a terminal object, and suppose that the slice object B/fa exists. If f is representably fully faithful, then the comma object A/a exists if and only if the following pullback exists, in which case they are isomorphic.*

$$\begin{array}{ccc} A/a & \longrightarrow & B/fa \\ {\scriptstyle \pi_a}\downarrow & pb & \downarrow{\scriptstyle \pi_{fa}} \\ A & \xrightarrow[f]{} & B \end{array}$$

Proof. A cone for the pullback comprises a 1-cell $a' \colon X \to A$ and a cell β. Since f is representably fully faithful, there exists a cell α such that $f\alpha = \beta$, exhibiting a cone for the slice object.

$$\begin{array}{ccc} X & \xrightarrow{\langle\rangle} & 1 \\ {\scriptstyle fa'}\downarrow & \swarrow{\scriptstyle \beta} & \downarrow{\scriptstyle fa} \\ B & = & B \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\langle\rangle} & 1 \\ {\scriptstyle a'}\downarrow & \swarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle a} \\ A & = & A \end{array}$$

□

The following describes the structure of the slice virtual double category in alternative terms, which shows our definition matches that of Kawase [Kaw25, Definition 4.30].

Proposition 4.15. *Let $\mathbb{X}$ be a normal virtual double category and let T be a loose monad in $\mathbb{X}$. The following diagram forms a pullback in* **VDblCat**.

$$\begin{array}{ccc} \mathbb{X}/T & \longrightarrow & \mathbb{M}\mathbf{od}(\mathbb{X})/T \\ {\scriptstyle \pi}\downarrow & pb & \downarrow{\scriptstyle \pi} \\ \mathbb{X} & \longrightarrow & \mathbb{M}\mathbf{od}(\mathbb{X}) \end{array}$$

Proof. Immediate from Lemma 4.14, since $\mathbb{X} \to \mathbb{M}\mathbf{od}(\mathbb{X})$ is representably fully faithful by Corollary 4.10. □

The following strengthens [Kaw25, Corollary 4.34(ii)] (which additionally assumes that $\mathbb{X}$ is normal and admits restrictions along isomorphisms in the sense of [CS10, Definition 7.1]). It should be compared to the classical result concerning slices of presheaf categories: since $\mathbb{M}\mathbf{od}$ is lax-idempotent, it behaves in a suitable sense like a free cocompletion of virtual double categories (namely under *collapses* of monads and their modules, cf. [AM24b, Definition 5.2]).

Proposition 4.16. *Let $\mathbb{X}$ be a virtual double category and let T be a loose monad in $\mathbb{X}$. There is an isomorphism of virtual double categories as follows.*

$$\begin{array}{ccc} \mathbb{M}\mathbf{od}(\mathbb{X}/T) & \xrightarrow{\cong} & \mathbb{M}\mathbf{od}(\mathbb{X})/T \\ & {\scriptstyle \mathbb{M}\mathbf{od}(\pi)}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & \mathbb{M}\mathbf{od}(\mathbb{X}) & \end{array}$$

Proof. $\mathbb{M}\mathbf{od}$ is a right 2-adjoint by Proposition 4.5, and so preserves slice objects, observing that $\mathbb{M}\mathbf{od}(T) \colon \mathbb{M}\mathbf{od}(\mathbb{1}) \cong \mathbb{1} \to \mathbb{M}\mathbf{od}(\mathbb{X})$ picks out the identity module on the monad T, which is T itself. □

5. The free normal virtual double category

We continue our short detour from exponentiability to further explore the relationship between normal virtual double categories and virtual double categories. In particular, we observe that, in addition to admitting a right adjoint, $(-)' \colon \mathbf{NVDblCat} \to \mathbf{VDblCat}$ also admits a left adjoint. Viewing normality as a representability condition for virtual double categories, which are a kind of

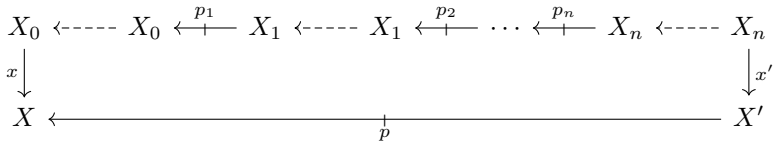
generalised multicategory [Bur71], the existence of the left adjoint $\mathbb{F}\colon \mathbf{VDblCat} \to \mathbf{NVDblCat}$ is suggested by the theory of representability for generalised multicategories [Her01] (though there is a subtlety, in that normality is only a *partial* representability condition). This construction is implicit in the work of Dawson, Paré and Pronk [DPP06] and was made explicit in the work of Fujii and Lack [FL25].

Definition 5.1 ([FL25, §7.2])**.** Let $\mathbb{X}$ be a virtual double category. The virtual double category $\mathbb{F}(\mathbb{X})$ is defined as follows.

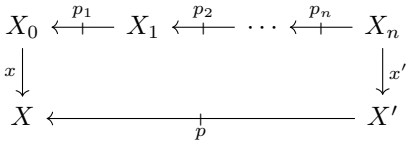
(1) The underlying category is $\underline{\mathbb{X}}$.

(2) The loose morphisms from X to Y comprise the loose morphisms $X \nrightarrow Y$ in $\mathbb{X}$ and, when $X = Y$, an endomorphism $X(1,1)\colon X \nrightarrow X$.

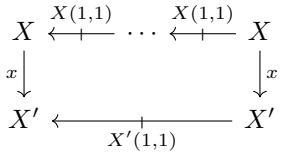
(3) A cell with the following frame (where $p_1, \ldots, p_n, p$ are loose morphisms in $\mathbb{X}$, and a dashed arrow denotes a finite chain of endomorphisms $X_i(1,1)$),



is given by a cell in $\mathbb{X}$ with the following frame (i.e. given by removing the endomorphisms $X_i(1,1)$ in the domain).



Furthermore, for each tight morphism $x\colon X \to X'$, there is a unique cell with the following frame.



Identities and composites are inherited from $\mathbb{X}$. ⌟

Proposition 5.2. *There is a lax-idempotent and monadic 2-adjunction as follows.*

$$\mathbf{NVDblCat} \underset{(-)'}{\overset{\mathbb{F}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{VDblCat}$$

Proof. That $\mathbb{F}$ defines a left 2-adjoint is stated in [FL25, §7.2]. For lax-idempotence, observe that, since $\mathbb{F} \dashv \mathbb{M}\mathbf{od}$ is lax-idempotent by Theorem 4.7, it induces a colax-idempotent 2-comonad. A 2-monad that is left-adjoint to a colax-idempotent 2-comonad is lax-idempotent. The $\mathbb{F}'$-algebras are consequently characterised in terms of existence of a left-adjoint retraction to the unit $\mathbb{X} \to \mathbb{F}(\mathbb{X})'$, which corresponds precisely to the normality condition: the left adjoint provides the interpretation of the formal loose identities in $\mathbb{F}(\mathbb{X})$ as loose morphisms in $\mathbb{X}$, as well as the interpretation of cells involving them; and the unit of the adjunction and triangle identities specify the nullary opcartesian cells in $\mathbb{X}$. □

Consequently, we have a 2-adjoint triple $\mathbb{F} \dashv (-)' \dashv \mathbb{M}\mathbf{od}$, inducing an adjoint 2-monad–2-comonad pair $\mathbb{F}' \dashv \mathbb{M}\mathbf{od}'$. This situation is useful, as we may take advantage of the calculus of adjoint triples, properties of one adjunction being deducible from properties of the other [EM65].

Corollary 5.3. $(-)' \dashv \mathbb{M}\mathbf{od}$ *is 2-comonadic.*

Proof. Follows from 2-monadicity of $\mathbb{F} \dashv (-)'$ by the **Cat**-enriched analogue of [EM65, Proposition 3.3]. □

Remark 5.4. $\mathbb{F}$ does not preserve the terminal object $\mathbb{1}$, so does not admit a further left adjoint. $\mathbb{M}\mathbf{od}$ does not admit a further right adjoint. Suppose that it did, so that we have a natural isomorphism as follows.

$$\mathbf{NVDblCat}(\mathbb{M}\mathbf{od}(\mathbb{X}), \mathbb{Y}) \cong \mathbf{VDblCat}(\mathbb{X}, R\mathbb{Y})$$

Taking $\mathbb{X} = \overline{\mathbf{1}}$, and denoting by $\mathbb{0} = \overline{\mathbf{0}}$ the empty virtual double category, we have

$$\mathbf{1} \cong \mathbf{NVDblCat}(\mathbb{0}, \mathbb{Y}) \cong \mathbf{NVDblCat}(\mathbb{M}\mathbf{od}(\overline{\mathbf{1}}), \mathbb{Y}) \cong \mathbf{VDblCat}(\overline{\mathbf{1}}, R\mathbb{Y}) \cong \underline{R\mathbb{Y}}$$

which implies that $R\mathbb{Y}$ has a unique object for all $\mathbb{Y}$. However, taking $\mathbb{Y} = \mathbb{0}$, we have

$$\mathbf{0} \cong \mathbf{NVDblCat}(\mathbb{M}\mathbf{od}(\mathbb{X}), \mathbb{0}) \cong \mathbf{VDblCat}(\mathbb{X}, R\mathbb{0})$$

which implies that $R\mathbb{0} = \mathbb{0}$, which is a contradiction. ⌟

Remark 5.5. The 2-functor $\mathbb{C}\mathbf{h}\colon \mathbf{Cat} \to \mathbf{VDblCat}$ is similar in nature to the 2-functor $\mathbb{F}(\overline{(-)})'\colon \mathbf{Cat} \to \mathbf{VDblCat}$, the difference being that the former adjoins a unique loose morphism $!_{a,a'}\colon a \nrightarrow a'$ between every pair of objects a, a', whereas the latter does so only for $a = a'$. Consequently, for a category $\mathbf{A}$, $\mathbb{C}\mathbf{h}(\mathbf{A}) \cong \mathbb{F}(\overline{\mathbf{A}})'$ if and only if $\mathbf{A}$ has at most one object. In particular, $\mathbb{F}(\overline{\mathbf{1}})' \cong \mathbb{1}$, since $\mathbb{C}\mathbf{h}$ is a right adjoint (Proposition 3.11) and thus preserves the terminal object. ⌟

As we observed in Proposition 2.21, $\mathbf{VDblCat}$ is known to be complete and cocomplete as a 2-category and hence admit powers and copowers by categories. However, the methodology used to establish this fact relies upon the construction of limits in 2-categories of coalgebras and colimits in 2-categories of algebras [DPP06], and therefore does not produce concrete descriptions of either. In contrast, the adjoint triple between virtual double categories and normal virtual double categories gives a concrete description of both powers and copowers.

Proposition 5.6. *The 2-category* $\mathbf{VDblCat}$ *is powered and copowered over* $\mathbf{Cat}$*. For a category* $\mathbf{A}$ *and virtual double categories* $\mathbb{X}$ *and* $\mathbb{Y}$*:*

$$\mathbf{A} \cdot \mathbb{X} \cong \mathbb{F}(\overline{\mathbf{A}})' \times \mathbb{X} \qquad\qquad \mathbf{A} \pitchfork \mathbb{Y} \cong \mathbb{Y}^{\mathbb{F}(\overline{\mathbf{A}})'}$$

Proof. First, $\mathbb{F}(\overline{\mathbf{A}})'$ is representable, its only loose morphisms being loose identities, hence is exponentiable by Theorem 3.9. We have $\mathbf{VDblCat}(\mathbb{X}, \mathbb{Y}^{\mathbb{F}(\overline{\mathbf{A}})'}) \cong \mathbf{VDblCat}(\mathbb{F}(\overline{\mathbf{A}})' \times \mathbb{X}, \mathbb{Y})$. We must show that $\mathbf{VDblCat}(\mathbb{F}(\overline{\mathbf{A}})' \times \mathbb{X}, \mathbb{Y})$ is naturally isomorphic to $\mathbf{Cat}(\mathbf{A}, \mathbf{VDblCat}(\mathbb{X}, \mathbb{Y}))$. An object of the latter comprises, for each object $a \in |\mathbf{A}|$, a functor $F_a\colon \mathbb{X} \to \mathbb{Y}$ and, for each morphism $\alpha\colon a \to a'$, a transformation $F_\alpha\colon F_a \Rightarrow F_{a'}$, functorial in $a \in |\mathbf{A}|$.

We obtain such an assignment from a functor $\mathbb{F}(\overline{\mathbf{A}})' \times \mathbb{X} \to \mathbb{Y}$ by evaluation. For each object $a \in |\mathbf{A}|$, there is a unique (necessarily normal) functor $!_a\colon \mathbb{1} \to \mathbb{F}(\overline{\mathbf{A}})'$ sending the unique object of $\mathbb{1}$ to a. Similarly, for each morphism $\alpha\colon a \to a'$, there is a unique transformation $!_\alpha\colon !_a \Rightarrow !_{a'}$ whose component at the unique object of $\mathbb{1}$ is α. This induces a family of functors $\mathbf{VDblCat}(!_a \times \mathbb{X}, \mathbb{Y})\colon \mathbf{VDblCat}(\mathbb{F}(\overline{\mathbf{A}})' \times \mathbb{X}, \mathbb{Y}) \to \mathbf{VDblCat}(\mathbb{1} \times \mathbb{X}, \mathbb{Y}) \cong \mathbf{VDblCat}(\mathbb{X}, \mathbb{Y})$ natural in $a \in |\mathbf{A}|$.

Conversely, given $F_{(-)}\colon \mathbf{A} \to \mathbf{VDblCat}(\mathbb{X}, \mathbb{Y})$, we define a functor $F_{(-_1)}(-_2)\colon \mathbb{F}(\overline{\mathbf{A}})' \times \mathbb{X} \to \mathbb{Y}$. For the assignment on loose morphisms, observe that loose morphisms in $\mathbb{F}(\overline{\mathbf{A}})'$ are in bijection with objects of $\mathbf{A}$. Therefore, loose morphisms in $\mathbb{F}(\overline{\mathbf{A}})' \times \mathbb{X}$ are indexed by an *object* of $\mathbf{A}$ and a *loose morphism* of $\mathbb{X}$, thus making this assignment well defined.

It remains to verify that these assignments are inverse to one another and natural in each variable, which is a straightforward calculation. □

It follows from Proposition 5.6 that the embedding $\mathbb{F}(\overline{(-)})'\colon \mathbf{Cat} \to \mathbf{VDblCat}$ plays an analogous role for virtual double categories that the cocartesian (a.k.a. sequential) multicategory construction plays for multicategories [Pis13, Proposition 2.1; Pis14, Corollary 3.4]. For instance, the following is an analogue of the fact that the cocartesian multicategory construction is left-adjoint to the category of monoids construction [Pis14, Corollary 3.10].

Proposition 5.7. *The 2-functor* $\mathbb{F}(\overline{(-)})'\colon \mathbf{Cat} \to \mathbf{VDblCat}$ *is fully faithful and left-adjoint to* $\underline{\mathbb{M}\mathbf{od}(-)'}\colon \mathbf{VDblCat} \to \mathbf{Cat}$*.*

Proof. First, full faithfulness follows because the action on loose morphisms is uniquely determined. Adjointness is given by composition of the 2-adjunctions of Propositions 3.3, 4.5 and 5.2.

$$\mathbf{Cat} \underset{\underline{(-)}}{\overset{\overline{(-)}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{VDblCat} \underset{\mathbb{M}\mathbf{od}(-)'}{\overset{\mathbb{F}(-)'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{VDblCat}$$

□

Remark 5.8. Since **VDblCat** is copowered over **Cat**, we obtain, for each fixed virtual double category $\mathbb{X}$, a 2-adjunction $(-) \cdot \mathbb{X} \dashv \mathbf{VDblCat}(\mathbb{X}, -) \colon \mathbf{VDblCat} \to \mathbf{Cat}$.

(1) When $\mathbb{X} = \overline{\mathbf{1}}$, this recovers the free–forgetful 2-adjunction $\overline{(-)} \dashv \underline{(-)}$ of Proposition 3.3.

(2) When $\mathbb{X} = \mathbb{1}$, this recovers the 2-adjunction $\mathbb{F}(\overline{(-)})' \dashv \underline{\mathbb{M}\mathbf{od}(-)'}$ of Proposition 5.7. ⌟

Therefore, **Cat** is embedded coreflectively in **VDblCat** in two ways, identifying the categories as both the 'loosely empty' virtual double categories (having no loose morphisms) and the 'loosely discrete' virtual double categories (having only loose identities); and, by Proposition 3.11, is embedded reflectively by identifying the categories as the 'loose codiscrete' virtual double categories. Note that $\underline{\mathbb{F}(\overline{(-)})'} \colon \mathbf{Cat} \to \mathbf{Cat}$ is isomorphic to the identity, so that $\underline{(-)} \colon \mathbf{VDblCat} \to \mathbf{Cat}$ provides a retraction of both coreflections.

6. Lax functors as monads

We are now in position to explain the relationship between exponentiation for virtual double categories and the virtual double category of lax functors introduced by Paré [Par11]. As a warm up, we observe that the category of monads and monad morphisms in a virtual double category $\mathbb{X}$ is given by the category of functors from the terminal virtual double category $\mathbb{1}$ into $\mathbb{X}$.

Lemma 6.1. $\mathbf{VDblCat}(\mathbb{1}, \mathbb{X}) \cong \underline{\mathbb{M}\mathbf{od}(\mathbb{X})'}$.

Proof. Immediate from the fact that $\mathbb{1} \cong \mathbb{F}(\overline{\mathbf{1}})'$ (Remark 5.5) together with Proposition 5.7. □

The following observation, which extends the above from an isomorphism of categories to an isomorphism of virtual double categories, provides our original motivation for considering exponentiation of virtual double categories.

Theorem 6.2. *Let $\mathbb{A}$ be a pseudo double category and let $\mathbb{X}$ be a virtual double category. The virtual double category $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X})$ of [Par11, Theorem 4.3]* [11] *is isomorphic to $\mathbb{M}\mathbf{od}(\mathbb{X}^{\mathbb{A}})$.*

Proof. First, we observe that the underlying categories are isomorphic. Using Lemma 6.1 for the final isomorphism, we have the following.

$$\underline{\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X})} \cong \mathbf{VDblCat}(\mathbb{A}, \mathbb{X}) \cong \mathbf{VDblCat}(\mathbb{1}, \mathbb{X}^{\mathbb{A}}) \cong \underline{\mathbb{M}\mathbf{od}(\mathbb{X}^{\mathbb{A}})'}$$

Next, we must check the data of the loose morphisms and cells. A loose morphism in $\mathbb{M}\mathbf{od}(\mathbb{X}^{\mathbb{A}})$ is given by a module between loose monads. Explicitly, it is given by a loose morphism in $\mathbb{X}^{\mathbb{A}}$, i.e. a graph morphism, equipped with two cells in $\mathbb{X}^{\mathbb{A}}$ corresponding to the left and right actions. Comparing with the data and axioms for a module in [Par11, Definition 3.2], (M1 & M2) express the data of a graph morphism; (M3) expresses the data of the left and right actions; (M4) expresses functoriality of the graph morphism (see Proposition 3.6); (M5) expresses naturality of the left and right actions; (M6) expresses the compatibility of the right action with the multiplication, the compatibility of the left action with the right action, and the compatibility of the left action with the multiplication; (M7) expresses the compatibility of the right action with the unit, and the compatibility of the left action with the unit.

A cell in $\mathbb{M}\mathbf{od}(\mathbb{X}^{\mathbb{A}})$ is given by a cell in $\mathbb{X}^{\mathbb{A}}$ satisfying several compatibility conditions. Comparing with the data and axioms for a multimodulation in [Par11, Definition 4.2], (mm1) expresses the data of a cell; (mm2) expresses naturality; ($\mathrm{mm3_l}$) expresses to the compatibility condition for the left action with itself; ($\mathrm{mm3_r}$) expresses to the compatibility condition for the right action with itself; ($\mathrm{mm3_i}$) expresses to the compatibility condition for the left action with the right action; ($\mathrm{mm3_0}$) expresses the nullary compatibility condition.

[11] Paré only defined $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X})$ under the additional assumption that $\mathbb{X}$ is representable, but the definition extends in the evident way.

Paré [Par11] does not define composition of cells in $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X})$, therein deferring them to [Par13], where they are also not defined explicitly. However, it is clear that they must coincide with those of $\mathbb{M}\mathbf{od}(\mathbb{X}^{\mathbb{A}})$. □

Beyond being purely of conceptual interest, this observation has many practical consequences. For one, it provides a simpler construction of $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ that makes concrete computations substantially easier (e.g. Section 6.1). For another, it provides a universal property of $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ that means that concrete computations are often rendered unnecessary (e.g. Section 6.2). In the remainder of the paper, we will explore some of these consequences.

First, normality of the virtual double category of lax functors follows as immediate corollary of Theorem 6.2.

Corollary 6.3 ([Par11, Theorem 4.3; Par13, Theorem 5.1.10])**.** $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X})$ *is normal.*

Proof. Immediate from Theorem 6.2 and Lemma 4.4. □

Remark 6.4. If $\mathbb{X}$ admits non-nullary composites, restrictions, and local reflexive coequalisers, then $\mathbb{M}\mathbf{od}(\mathbb{X})$ is representable: this is stated in [CS10, Theorem A.1] under the further assumptions that $\mathbb{X}$ admits arbitrary composites and local coequalisers, but inspection of the proof reveals the stronger assumptions to be unnecessary. It is straightforward to show that, if $\mathbb{X}$ admits restrictions and local reflexive coequalisers, then so too does each exponential $\mathbb{X}^{\mathbb{A}}$. Consequently, supposing the conjecture in Remark 3.20 holds, it would follow that, given a pseudo double category $\mathbb{A}$ satisfying a suitable factorisation condition and a locally cocomplete pseudo double category $\mathbb{X}$ with restrictions, the virtual double category $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X})$ is representable (and is furthermore locally cocomplete with restrictions), which would recover [Par13, Theorem 4.0.1]. Our expectation is that this proof strategy would result in a substantial simplication of the proof ibid., given the relative simplicity of $\mathbb{X}^{\mathbb{A}}$ compared to $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X})$. ⌟

In the following subsections, we will explore some of the more interesting consequences of Theorem 6.2.

6.1. **Enrichment of double categories.** The reader of [Par11] may be led to wonder whether the assignment $(\mathbb{A}, \mathbb{B}) \mapsto \mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ of pairs of pseudo double categories to the normal virtual double category of lax functors between them defines an enrichment of pseudo double categories in normal virtual double categories. While verifying this by hand is not a difficult exercise, it does involve some tedious calculations. On the other hand, it is a straightforward corollary of Theorem 6.2. We require a preparatory lemma.

Lemma 6.5. *Let $(\mathscr{D}, \otimes, I)$ be a symmetric monoidal bicategory and let $\mathscr{C} \hookrightarrow \mathscr{D}$ be a full subbicategory. Suppose that, for each $A, B \in \mathscr{C}$, the internal hom $[A, B]$ exists in $\mathscr{D}$. Then $\mathscr{C}$ is canonically equipped with a $\mathscr{D}$-enriched structure.*

Proof. The enriched structure on $\mathscr{C}$ is defined in the same way as the canonical enriched structure on a closed monoidal bicategory [Lac95, Proposition 3.9.6], with $\mathscr{C}(A, B) := [A, B]$. Full faithfulness ensures that the underlying bicategory of the enriched bicategory coincides with $\mathscr{C}$, since $\mathscr{D}(I, [A, B]) \simeq \mathscr{D}(A, B) \simeq \mathscr{C}(A, B)$. □

Corollary 6.6. *The 2-category* $\mathbf{DblCat}_l$ *of pseudo double categories and lax functors admits the following enrichments.*

(1) In the cartesian monoidal 2-category $\mathbf{VDblCat}$ *via* $\mathbf{DblCat}_l(\mathbb{A}, \mathbb{B}) := \mathbb{B}^{\mathbb{A}}$.

(2) In the cartesian monoidal 2-category $\mathbf{NVDblCat}$ *via* $\mathbf{DblCat}_l(\mathbb{A}, \mathbb{B}) := \mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$.

Proof. First, the inclusion $\mathbf{DblCat}_l \to \mathbf{VDblCat}$ is a fully faithful 2-functor (Theorem 2.15). (1) thus follows from Lemma 6.5 using that representable virtual double categories are exponentiable (Theorem 3.9). For (2), we use change of base along $\mathbb{M}\mathbf{od}$, using that $\mathbb{M}\mathbf{od}$ is right-adjoint and so preserves products, along with the characterisation of $\mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})$ in Theorem 6.2. It is clear that the underlying 2-category is precisely $\mathbf{DblCat}_l$, since $\mathbf{NVDblCat}(\mathbb{1}, \mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{B})) \cong \mathbf{VDblCat}(\mathbb{A}, \mathbb{B})$. □

Remark 6.7. Bicategories enriched in the cartesian monoidal 2-category $\mathbf{NVDblCat}$ are generalisations of the locally cubical bicategories of [GG09, Definition 11], which are bicategories enriched

in the cartesian monoidal 2-category $\mathbf{DblCat}_p$ of pseudo double categories and pseudo functors. Note that each of the 2-functors in the following commutative square preserves products and so, by change of base for enriched bicategories [GS16, §13.2], every locally cubical bicategory induces bicategories enriched in $\mathbf{DblCat}_l$, $\mathbf{NVDblCat}$, and $\mathbf{VDblCat}$. However, only change of base along the vertical functors below preserves underlying 2-categories (since normal functors from $\mathbb{1}$ pick out objects, whereas non-normal functors from $\mathbb{1}$ pick out loose monads).

$$\begin{array}{ccc} \mathbf{DblCat}_p & \longrightarrow & \mathbf{DblCat}_l \\ \downarrow & & \downarrow \\ \mathbf{NVDblCat} & \longrightarrow & \mathbf{VDblCat} \end{array} \qquad \lrcorner$$

6.2. **Skew-closed structure.** The universal properties of exponentiation and of $\mathbb{M}\mathbf{od}$ supply $\mathbb{L}\mathbf{ax}$ with a universal property of its own.

Lemma 6.8. *For each normal virtual double category $\mathbb{W}$, exponentiable virtual double category $\mathbb{A}$, and virtual double category $\mathbb{X}$, we have a 2-natural isomorphism of categories:*

$$\mathbf{VDblCat}(\mathbb{W}' \times \mathbb{A}, \mathbb{X}) \cong \mathbf{NVDblCat}(\mathbb{W}, \mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X}))$$

Consequently, for each virtual double category $\mathbb{W}$, exponentiable virtual double category $\mathbb{A}$, and virtual double category $\mathbb{X}$, we have a 2-natural isomorphism of categories:

$$\mathbf{VDblCat}((\mathbb{F}\mathbb{W})' \times \mathbb{A}, \mathbb{X}) \cong \mathbf{VDblCat}(\mathbb{W}, \mathbb{L}\mathbf{ax}(\mathbb{A}, \mathbb{X})')$$

Proof. For the first isomorphism, we use Theorem 6.2 together with the following 2-adjunction.

$$\mathbf{NVDblCat} \underset{\mathbb{M}\mathbf{od}}{\overset{(-)'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{VDblCat} \underset{(-)^{\mathbb{A}}}{\overset{(-)\times\mathbb{A}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{VDblCat}$$

For the second isomorphism, we precompose the adjunction $\mathbb{F} \dashv (-)'$. □

Remark 6.9. In particular, for a category $\mathbf{A}$ and virtual double categories $\mathbb{X}$ and $\mathbb{Y}$, we have the following chain of natural isomorphisms,

$$\begin{aligned} \mathbf{VDblCat}(\mathbb{F}(\overline{\mathbf{A}})' \times \mathbb{X}, \mathbb{Y}) &\cong \mathbf{VDblCat}(\overline{\mathbf{A}}, \mathbb{L}\mathbf{ax}(\mathbb{X}, \mathbb{Y})') && \text{(Lemma 6.8)} \\ &\cong \mathbf{Cat}(\mathbf{A}, \underline{\mathbb{L}\mathbf{ax}(\mathbb{X}, \mathbb{Y})'}) && \text{(Proposition 3.3)} \\ &\cong \mathbf{Cat}(\mathbf{A}, \mathbf{VDblCat}(\mathbb{X}, \mathbb{Y})) && \text{(Theorem 6.2)} \end{aligned}$$

recovering the description of the copower $\mathbf{A} \cdot \mathbb{X}$ in Proposition 5.6. However, note that this does not give an alternative proof of Proposition 5.6, as it only establishes the universal property for *exponentiable* $\mathbb{X}$. ⌟

Lemma 6.8 indicates that $\mathbb{L}\mathbf{ax}(-,-)$ forms something resembling a closed monoidal structure on the 2-category of virtual double categories. Since not every virtual double category is exponentiable, such a monoidal structure will be at best partially closed, but we may still hope that the monoidal structure itself is totally defined. This is, in fact, very close to being the case: while the closed structure expressed by $\mathbb{L}\mathbf{ax}(-,-)$ does not correspond to a monoidal structure, it does correspond to a *left-skew monoidal* structure in the sense of Szlachányi [Szl12].

Theorem 6.10. *The assignment $(\mathbb{W}, \mathbb{X}) \mapsto (\mathbb{F}\mathbb{W})' \times \mathbb{X}$ equips $\mathbf{VDblCat}$ with the structure of a right-normal left-skew monoidal 2-category $(\mathbf{VDblCat}, \circledcirc, \mathbb{1})$. Furthermore, for each exponentiable virtual double category $\mathbb{A}$, we have $(-) \circledcirc \mathbb{A} \dashv \mathbb{L}\mathbf{ax}(\mathbb{A}, -)'$.*

Proof. Every 2-monad admits a unique colax monoidal structure with respect to cartesian monoidal structure, and so the given assignment defines a left-skew monoidal structure by the two-dimensional analogue of [LS12, Example 3.8]. Right-normality follows because $(\mathbb{F}\mathbb{W})' \times \mathbb{1} \cong (\mathbb{F}\mathbb{W})'$. That this skew-monoidal structure is right-closed with respect to exponentiable virtual double categories follows from Lemma 6.8. □

Corollary 6.11. *The 2-category $\mathbf{DblCat}_l$ is equipped with right-normal left-skew monoidal structure $(\circledcirc, \mathbb{1})$.*

Proof. It suffices to check that $\circledcirc$ restricts to pseudo double categories. This is trivial, as $\mathbb{F}$ preserves representability, as does the cartesian product. □

7. Presheaves for double categories

In this section, we examine some implications of Theorem 6.2 for the virtual double category $\mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}\,\mathrm{t}}, \mathbb{S}\mathbf{pan})$ of presheaves on a double category $\mathbb{A}$, which was studied extensively by Paré [Par11]. We begin by spelling out the definition of the exponential $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$, which admits some simplification over the general case.

Definition 7.1. For a virtual double category $\mathbb{X}$, denote by $\mathbb{X}_n$ the category whose objects are chains of loose morphisms of length n, and whose morphisms comprise n cells in $\mathbb{X}$ with compatible source and target.

$$\begin{array}{ccccccc} X_0 & \xleftarrow{p_1} & X_1 & \xleftarrow{p_2} & \cdots & \xleftarrow{p_n} & X_n \\ {\scriptstyle x_0'}\downarrow & \chi_1 & \downarrow{\scriptstyle x_1'} & \chi_2 & \cdots & \chi_n & \downarrow{\scriptstyle x_n'} \\ X_0' & \xleftarrow[p_1']{} & X_1' & \xleftarrow[p_2']{} & \cdots & \xleftarrow[p_n']{} & X_n' \end{array}$$

Example 7.2. Observe that the category $\mathbb{S}\mathbf{pan}_1$ of spans and span morphisms is isomorphic to $\mathbf{Set}^{\{\cdot \leftarrow \cdot \rightarrow \cdot\}}$, the category of $\mathbf{Set}$-valued functors from the free-standing span, and that the inclusion $1 \to \{\cdot \leftarrow \cdot \rightarrow \cdot\}$ that picks out the middle object of the span induces a functor $m \colon \mathbf{Set}^{\{\cdot \leftarrow \cdot \rightarrow \cdot\}} \to \mathbf{Set}$.

The underlying category of $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$ is the presheaf category $\mathbf{Set}^{(\mathbb{A}_0)^{\mathrm{op}}}$. A loose morphism from F to F' is given by a functor and a pair of natural transformations as follows.

$$\begin{array}{ccccc} \mathbb{A}_0{}^{\mathrm{op}} & \xleftarrow{s^{\mathrm{op}}} & \mathbb{A}_1{}^{\mathrm{op}} & \xrightarrow{t^{\mathrm{op}}} & \mathbb{A}_0{}^{\mathrm{op}} \\ {\scriptstyle F}\downarrow & \Xi_s \Longleftarrow & \downarrow{\scriptstyle \Xi} & \Longrightarrow \Xi_t & \downarrow{\scriptstyle F'} \\ \mathbf{Set} & = & \mathbf{Set} & = & \mathbf{Set} \end{array}$$

A cell with the following frame (left) is given by a natural transformation (right).

$$\begin{array}{ccccc} F_0 & \xleftarrow{\Xi_1} & \cdots & \xleftarrow{\Xi_n} & F_n \\ {\scriptstyle \Phi}\downarrow & & \xi & & \downarrow{\scriptstyle \Phi'} \\ F & & \xleftarrow[\Xi]{} & & F' \end{array} \qquad \begin{array}{ccccc} \mathbb{A}_n{}^{\mathrm{op}} & & \xrightarrow{\odot^{\mathrm{op}}} & & \mathbb{A}_1{}^{\mathrm{op}} \\ {\scriptstyle \langle \Xi_1, \ldots, \Xi_n \rangle}\downarrow & & \xi \Longrightarrow & & \downarrow{\scriptstyle \Xi} \\ \mathbb{S}\mathbf{pan}_n & \xrightarrow[\odot]{} & \mathbb{S}\mathbf{pan}_1 & \xrightarrow[m]{} & \mathbf{Set} \end{array}$$

such that the following equations hold.

$$\begin{array}{ccccc} \mathbb{A}_n{}^{\mathrm{op}} & \xrightarrow{\odot^{\mathrm{op}}} & \mathbb{A}_1{}^{\mathrm{op}} & \xrightarrow{s^{\mathrm{op}}} & \mathbb{A}_0{}^{\mathrm{op}} \\ {\scriptstyle \langle \Xi_1, \ldots, \Xi_n \rangle}\downarrow & \xi \Longrightarrow & \downarrow{\scriptstyle \Xi} & \Xi_s \Longrightarrow & \downarrow{\scriptstyle F'} \\ \mathbb{S}\mathbf{pan}_n & \xrightarrow[m \circ \odot]{} & \mathbf{Set} & = & \mathbf{Set} \end{array} = \begin{array}{ccccc} \mathbb{A}_n{}^{\mathrm{op}} & \xrightarrow{\odot^{\mathrm{op}}} & \mathbb{A}_1{}^{\mathrm{op}} & \xrightarrow{s^{\mathrm{op}}} & \mathbb{A}_0{}^{\mathrm{op}} \\ {\scriptstyle \langle \Xi_1, \ldots, \Xi_n \rangle}\downarrow & \pi_n \Longrightarrow & \downarrow{\scriptstyle \Xi_n} & (\Xi_n)_s \Longrightarrow & \downarrow{\scriptstyle F_n} \overset{\Phi'}{\Longrightarrow} \downarrow{\scriptstyle F'} \\ \mathbb{S}\mathbf{pan}_n & \xrightarrow[m \circ \odot]{} & \mathbf{Set} & = & \mathbf{Set} \end{array}$$

$$\begin{array}{ccccc} \mathbb{A}_n{}^{\mathrm{op}} & \xrightarrow{\odot^{\mathrm{op}}} & \mathbb{A}_1{}^{\mathrm{op}} & \xrightarrow{t^{\mathrm{op}}} & \mathbb{A}_0{}^{\mathrm{op}} \\ {\scriptstyle \langle \Xi_1, \ldots, \Xi_n \rangle}\downarrow & \xi \Longrightarrow & \downarrow{\scriptstyle \Xi} & \Xi_t \Longrightarrow & \downarrow{\scriptstyle F} \\ \mathbb{S}\mathbf{pan}_n & \xrightarrow[m \circ \odot]{} & \mathbf{Set} & = & \mathbf{Set} \end{array} = \begin{array}{ccccc} \mathbb{A}_n{}^{\mathrm{op}} & \xrightarrow{\odot^{\mathrm{op}}} & \mathbb{A}_1{}^{\mathrm{op}} & \xrightarrow{t^{\mathrm{op}}} & \mathbb{A}_0{}^{\mathrm{op}} \\ {\scriptstyle \langle \Xi_1, \ldots, \Xi_n \rangle}\downarrow & \pi_1 \Longrightarrow & \downarrow{\scriptstyle \Xi_1} & (\Xi_1)_t \Longrightarrow & \downarrow{\scriptstyle F_0} \overset{\Phi}{\Longrightarrow} \downarrow{\scriptstyle F} \\ \mathbb{S}\mathbf{pan}_n & \xrightarrow[m \circ \odot]{} & \mathbf{Set} & = & \mathbf{Set} \end{array}$$

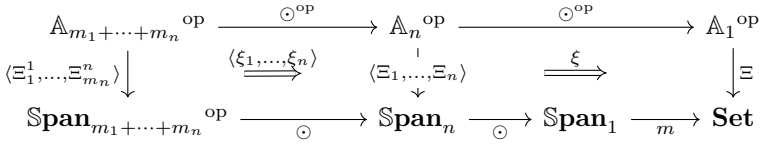
Above, we have denoted by $\odot \circ \langle \Xi_1, \ldots, \Xi_n \rangle$ the functor sending a chain of loose morphisms $p_1, \ldots, p_n$ in $\mathbb{A}$ to the limit (i.e. iterated pullback) of the following chain of spans.

$$\begin{array}{ccccccccc} & \Xi_1(p_1) & & & \Xi_2(p_2) & & & \cdots & \\ {\scriptstyle \Xi_t(p_1)}\swarrow & & \searrow{\scriptstyle \Xi_s(p_1)} \quad \swarrow{\scriptstyle \Xi_t(p_2)} & & & \searrow{\scriptstyle \Xi_s(p_2)} \quad \swarrow{\scriptstyle \Xi_t(p_3)} & & & \searrow{\scriptstyle \Xi_s(p_n)} \\ F_0(X_0) & & & F_1(X_1) & & & F_2(X_2) & & F_n(X_n) \end{array}$$

The identity cell on Ξ is given by the identity natural transformation on $\Xi \colon (\mathbb{A}_0)^{\mathrm{op}} \to \mathbf{Set}$.

$$\begin{array}{ccccc} \mathbb{A}_1{}^{\mathrm{op}} & & = & & \mathbb{A}_1{}^{\mathrm{op}} \\ {\scriptstyle \Xi}\downarrow & & = & & \downarrow{\scriptstyle \Xi} \\ \mathbb{S}\mathbf{pan}_1 & = & \mathbb{S}\mathbf{pan}_1 & \xrightarrow[m]{} & \mathbf{Set} \end{array}$$

Composition is given as follows.



Example 7.3. Let $\mathbf{E}$ be a small category, viewed as a loosely discrete strict double category $\mathbb{E} := \mathbb{F}(\overline{\mathbf{E}})'$, so that $\mathbb{E}_0 = \mathbb{E}_1 = \mathbb{E}_n$ $(n \geq 0)$. By direct computation, $\mathbb{S}\mathbf{pan}^{\mathbb{E}}$ (which, by Proposition 5.6, is the power of $\mathbf{E} \pitchfork \mathbb{S}\mathbf{pan}$) is representable and is equivalent to the pseudo double category $\mathbb{S}\mathbf{pan}([\mathbf{E}, \mathbf{Set}])$. By Example 4.2 and Theorem 6.2, we have

$$\mathbb{D}\mathbf{ist}([\mathbf{E}, \mathbf{Set}]) \cong \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}([\mathbf{E}, \mathbf{Set}])) \cong \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}^{\mathbb{E}}) \cong \mathbb{L}\mathbf{ax}(\mathbb{E}, \mathbb{S}\mathbf{pan})$$

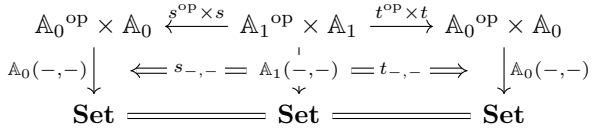
whose underlying isomorphism of categories, by Proposition 5.7, recovers the symmetry between categories internal to presheaf categories and $\mathbf{Cat}$-valued presheaves.

$$\mathbf{Cat}([\mathbf{E}, \mathbf{Set}]) \cong \mathbf{VDblCat}(\mathbb{E}, \mathbb{S}\mathbf{pan}) \cong [\mathbf{E}, \mathbf{Cat}]$$

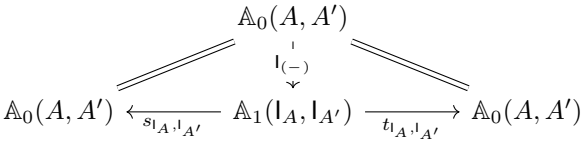
Just as to every category $\mathbf{A}$, there is an associated hom-functor $\mathbf{A}(-,-)\colon \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathbf{Set}$ sending every pair of objects to the set of morphisms between them, to every pseudo double category $\mathbb{A}$, there is an associated lax hom-functor $\mathbb{A}(-,-)\colon \mathbb{A}^{\mathrm{op\,t}} \times \mathbb{A} \to \mathbb{S}\mathbf{pan}$ sending every pair of objects to the set of tight morphisms between them, and every pair of loose morphisms to the set of cells between them.

Proposition 7.4 ([Par11, §2.1])**.** *For every pseudo double category $\mathbb{A}$, there is a lax hom-functor $\mathbb{A}(-,-)\colon \mathbb{A}^{\mathrm{op\,t}} \times \mathbb{A} \to \mathbb{S}\mathbf{pan}$.*

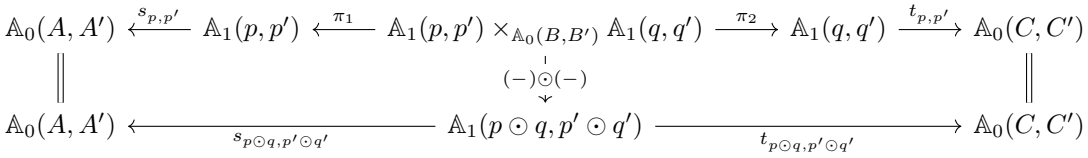
Proof. The action on objects and tight morphisms is given by $\mathbb{A}_0(-,-)$ and the action on loose morphisms and cells is given by $\mathbb{A}_1(-,-)$ together with the source and target projections, as in the following diagram. Below, $s_{-,-}$ and $t_{-,-}$ are postcomposition by s and t respectively.



For each pair of objects A and A', the unitor is given by the following functor.



For each pair of composable pairs of loose morphisms $A \overset{p}{\nleftarrow} B \overset{q}{\nleftarrow} C$ and $A' \overset{p'}{\nleftarrow} B' \overset{q'}{\nleftarrow} C'$ in $\mathbb{A}$, the compositor is given by the following functor.



That this forms a lax functor follows from coherence for the pseudo double category $\mathbb{A}$. □

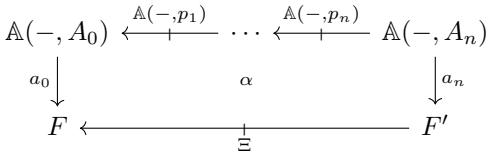
By transposing the hom-functor, we obtain a functor of virtual double categories as follows.

$$\mathbb{A}(-_2, -_1)\colon \mathbb{A} \to \mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op\,t}}} \tag{3}$$

Since the underlying category of $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op\,t}}}$ is the category of presheaves on the category $\underline{\mathbb{A}}$ underlying $\mathbb{A}$, we automatically obtain a Yoneda lemma with respect to the objects and tight morphisms of $\mathbb{A}$, given simply by the ordinary Yoneda lemma with respect to $\underline{\mathbb{A}}$. However, we also have a Yoneda lemma with respect to the loose morphisms and cells of $\mathbb{A}$; the idea is contained in [Par11,

Theorem 4.4], but the following proposition is simpler than the theorem ibid., as it concerns cells in $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$ rather than cells in $\mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}\,\mathrm{t}}, \mathbb{S}\mathbf{pan})$.

Proposition 7.5 (Yoneda lemma)**.** *Cells in* $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$ *of the following form*



are in natural bijection with elements of $\Xi(p_1 \odot \cdots \odot p_n)$.

Proof. By the Yoneda lemma for the category $\underline{\mathbb{A}}$, the tight morphisms framing α correspond to elements $a_0 \in F(A_0)$ and $a_n \in F'(A_n)$. By definition (Example 7.2), the cell α corresponds to a natural transformation,

$$\mathbb{A}(-, p_1) \times_{\mathbb{A}(-,A_1)} \mathbb{A}(-, p_2) \times_{\mathbb{A}(-,A_2)} \cdots \times_{\mathbb{A}(-,A_{n-1})} \mathbb{A}(-, p_n) \to \mathbb{A}(-, p_1 \odot \cdots \odot p_n)$$

which, upon taking the component at $p_1, \ldots, p_n$ and applying the corresponding function at $1_{p_1}, \ldots, 1_{p_n}$, produces an element of $q(p_1 \odot \cdots \odot p_n)$. Naturality implies that this is the only possible assignment. □

Corollary 7.6. *Every chain of representables* $\mathbb{A}(-, p_1), \ldots, \mathbb{A}(-, p_n)$ *in* $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$ *admits a weak composite* $\mathbb{A}(-, p_1 \odot \cdots \odot p_n)$.

Proof. Taking $q = \mathbb{A}(-, p_1 \odot \cdots \odot p_n)$ in Proposition 7.5, we get a globular cell of the appropriate shape, which is weakly opcartesian by another application of Proposition 7.5. □

This gives us all the pieces needed to recover the Yoneda embedding of Paré [Par11, Theorem 4.8]: as desired, it is obtained by transposing the hom-functor, and then applying the loose monads and modules construction.

Proposition 7.7. *For each pseudo double category* $\mathbb{A}$*, there is a fully faithful functor* $\mathbb{A}(-_2, -_1) \colon \mathbb{A} \to \mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$ *of virtual double categories, hence also a fully faithful normal functor* $Y_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}\,\mathrm{t}}, \mathbf{Set})$.

Proof. As described above (3), the functor $\mathbb{A}(-_2, -_1)$ is obtained by transposing the lax-hom functor of Proposition 7.4. Full faithfulness on tight morphisms follows from full faithfulness of the Yoneda embedding $\underline{\mathbb{A}} \hookrightarrow \mathbf{Set}^{\mathbb{A}^{\mathrm{op}}}$, while full faithfulness on cells follows from Proposition 7.5. Therefore, by Proposition 4.5 and Theorem 6.2, we obtain a normal functor of virtual double categories $\mathbb{A} \to \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}) \cong \mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}}, \mathbb{S}\mathbf{pan})$, which is fully faithful by Proposition 4.12. □

Explicitly, the embedding $Y_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}\,\mathrm{t}}, \mathbb{S}\mathbf{pan})$ sends an object $A \in \mathbb{A}$ to the image under $\mathbb{A}(-_2, -_1) \colon \mathbb{A} \to \mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$ of the identity loose monad $A(1, 1)$ on A. The Yoneda lemmas for $\mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}\,\mathrm{t}}, \mathbb{S}\mathbf{pan})$ [Par11, Theorems 2.3, 3.18 & 4.4] thereby follow readily from the Yoneda lemmas for $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$. We leave this as an instructive exercise for the reader. (For a hint, note that the Yoneda lemma for lax functors invokes both the ordinary Yoneda lemma and Proposition 7.5; whereas the Yoneda lemma for multimodulations makes use of the property that cells in $\mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}\,\mathrm{t}}, \mathbb{S}\mathbf{pan})$ are precisely cells in $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$ satisfying some additional properties.) We suggest this indicates that $\mathbb{A}(-_2, -_1) \colon \mathbb{A} \to \mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$ may be the more fundamental of the two 'Yoneda embeddings' for double categories.

Remark 7.8. In [FM24], Fröhlich and Moser study a variant of the Yoneda embedding with respect to *normal* lax functors $\mathbb{A}^{\mathrm{op}\,\mathrm{t}} \to \mathbb{D}\mathbf{ist}$. However, under the universal property of the $\mathbb{M}\mathbf{od}$ construction recalled in Proposition 4.5, their theory is entirely equivalent to Paré's [Par11]. For instance their normal lax hom-functor $\mathbb{A}^{\mathrm{op}\,\mathrm{t}} \times \mathbb{A} \to \mathbb{D}\mathbf{ist}$ [FM24, Construction 3.4.1] is obtained from Proposition 7.4 by transposition and their Yoneda lemma [FM24, Theorem 4.2] follows from Paré's likewise. ⌟

8. The elements construction

The data of the exponential $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$ in Example 7.2 is expressed in terms of presheaves (of categories) and natural transformations therebetween, and we are inevitably led to try transferring this data across the correspondence between presheaves and discrete fibrations. This motivates the following definition.

Definition 8.1. Let $\mathbb{A}$ be a pseudo double category. We define a virtual double category $\mathbb{L}\mathbf{vwAmIsof}(\mathbb{A})$ (for *levelwise amnestic isofibrations*) as follows.

- The underlying category is the full subcategory of the slice category $\mathbf{Cat}/\mathbb{A}_0$ spanned by the amnestic isofibrations[12].
- A loose morphism from $D\colon \mathbf{D} \to \mathbb{A}_0$ to $D'\colon \mathbf{D}' \to \mathbb{A}_0$ is given by an amnestic isofibration $E\colon \mathbf{E} \to \mathbb{A}_1$ and a span rendering the following diagram commutative.

$$\begin{array}{ccccc} \mathbf{D} & \xleftarrow{E_s} & \mathbf{E} & \xrightarrow{E_t} & \mathbf{D}' \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle E} & & \downarrow{\scriptstyle D'} \\ \mathbb{A}_0 & \xleftarrow[s]{} & \mathbb{A}_1 & \xrightarrow[t]{} & \mathbb{A}_0 \end{array}$$

- A cell with the following frame

$$\begin{array}{ccccc} D_0 & \xleftarrow{E_1} & \cdots & \xleftarrow{E_n} & D_n \\ {\scriptstyle H}\downarrow & & \zeta & & \downarrow{\scriptstyle H'} \\ D & & \xleftarrow[E]{} & & D' \end{array} \qquad \begin{array}{ccccc} \mathbf{D}_{i-1} & \xleftarrow{(E_i)_t} & \mathbf{E}_i & \xrightarrow{(E_i)_s} & \mathbf{D}_i \\ {\scriptstyle D_{i-1}}\downarrow & & \downarrow{\scriptstyle E_i} & & \downarrow{\scriptstyle D_i} \\ \mathbb{A}_0 & \xleftarrow[t]{} & \mathbb{A}_1 & \xrightarrow[s]{} & \mathbb{A}_0 \end{array}$$

  is given by a functor ζ making the following diagrams commute.

$$\begin{array}{ccc} \mathbf{E}_{1,\ldots,n} & \xrightarrow{\zeta} & \mathbf{E} \\ {\scriptstyle E_{1,\ldots,n}}\downarrow & & \downarrow{\scriptstyle E} \\ \mathbb{A}_n & \xrightarrow[\odot]{} & \mathbb{A}_1 \end{array} \qquad \begin{array}{ccccccccc} \mathbf{D}_0 & \xleftarrow{(E_1)_t} & \mathbf{E}_1 & \xleftarrow{\pi_1} & \mathbf{E}_{1,\ldots,n} & \xrightarrow{\pi_n} & \mathbf{E}_n & \xrightarrow{(E_n)_s} & \mathbf{E}_n \\ {\scriptstyle H}\downarrow & & & & \downarrow{\scriptstyle \zeta} & & & & \downarrow{\scriptstyle H'} \\ \mathbf{D} & & & \xleftarrow[t]{} & \mathbf{E} & \xrightarrow[s]{} & & & \mathbf{D}' \end{array}$$

  Above, we have denoted by $\mathbf{E}_{1,\ldots,n}$ the limit (i.e. iterated pullback) of the following chain of spans, and by $E_{1,\ldots,n}$ the canonical functor induced by the functors $E_1, \ldots, E_n$.

$$\begin{array}{ccccccccccc} & \mathbf{E}_1 & & & \mathbf{E}_2 & & & \cdots & & & \\ {\scriptstyle (E_1)_t}\swarrow & & \searrow{\scriptstyle (E_1)_s} & {\scriptstyle (E_2)_t}\swarrow & & \searrow{\scriptstyle (E_2)_s} & {\scriptstyle (E_3)_t}\swarrow & & \searrow{\scriptstyle (E_n)_s} & & \\ \mathbf{D}_0 & & \mathbf{D}_1 & & \mathbf{D}_2 & & & & \mathbf{D}_n & & \end{array}$$

- The identity cell on $E\colon \mathbf{E} \to \mathbb{A}_1$ is given by the identity functor on $\mathbf{E}$.
- For composition, observe that the diagram on the left commutes, but the diagram on the right commutes only up to isomorphism, using the structural transformations in $\mathbb{A}$.

$$\begin{array}{ccccc} \mathbf{E}_{1,\ldots,m^n_n} & \xrightarrow{\zeta_1,\ldots,\zeta_n} & \mathbf{E}_{1,\ldots,n} & \xrightarrow{\zeta} & \mathbf{E} \\ {\scriptstyle E_{1,\ldots,m^n_n}}\downarrow & & \downarrow{\scriptstyle E_{1,\ldots,n}} & & \downarrow{\scriptstyle E} \\ \mathbb{A}_{\sum_{1\le i\le n} m_i} & \xrightarrow[\odot]{} & \mathbb{A}_n & \xrightarrow[\odot]{} & \mathbb{A}_1 \end{array} \qquad \begin{array}{ccccc} \mathbf{E}_{1,\ldots,m^n_n} & \xrightarrow{\zeta_1,\ldots,\zeta_n} & \mathbf{E}_{1,\ldots,n} & \xrightarrow{\zeta} & \mathbf{E} \\ {\scriptstyle E_{1,\ldots,m^n_n}}\downarrow & & \cong & & \downarrow{\scriptstyle E} \\ \mathbb{A}_{\sum_{1\le i\le n} m_i} & & \xrightarrow[\odot]{} & & \mathbb{A}_1 \end{array}$$

  Since $E\colon \mathbf{E} \to \mathbb{A}_1$ is an amnestic isofibration, the invertible natural transformation on the right above lifts, defining a functor $\mathbf{E}_{1,\ldots,m^n_n} \to \mathbf{E}$ as below, which we take to be the composite.

$$\begin{array}{ccccc} \mathbf{E}_{1,\ldots,m^n_n} & \xrightarrow{\zeta_1,\ldots,\zeta_n} & \mathbf{E}_{1,\ldots,n} & \xrightarrow{\zeta} & \mathbf{E} \\ {\scriptstyle E_{1,\ldots,m^n_n}}\downarrow & & \cong & & \downarrow{\scriptstyle E} \\ \mathbb{A}_{\sum_{1\le i\le n} m_i} & & \xrightarrow[\odot]{} & & \mathbb{A}_1 \end{array}$$

[12] An amnestic isofibration is analogous to a discrete (op)fibration, but in which we only ask for unique lifts of isomorphisms.

That composition of cells is indeed associative and unital follows from uniqueness of liftings for the amnestic isofibrations.

Define $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ (for *levelwise discrete fibrations*) to be the full sub-virtual double category of $\mathbb{L}\mathbf{vwAmIsof}(\mathbb{A})$ whose objects and loose morphisms are the discrete fibrations over $\mathbb{A}_0$ and $\mathbb{A}_1$ respectively. ⌟

Remark 8.2. We introduced $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ by way of $\mathbb{L}\mathbf{vwAmIsof}(\mathbb{A})$ above to emphasise which properties of discrete fibrations are important in defining the virtual double category structure. However, in what follows, we will focus on $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$, since it is this construction that is relevant to the theory of Section 7. Let us simply remark that several of the properties below (e.g. Lemmas 8.4 and 8.7) also hold at the greater level of generality of amnestic isofibrations. ⌟

This definition is justified by the following, which establishes that $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ is indeed the fibrational analogue of $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}}$.

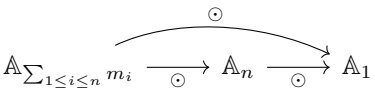
Proposition 8.3. *For each pseudo double category $\mathbb{A}$, there is an equivalence of virtual double categories $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op}\,\mathrm{t}}} \simeq \mathbb{L}\mathbf{vwDFib}(\mathbb{A})$.*

Proof. The structure of $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ is obtained by transfer of structure across the correspondence between presheaves (and natural transformations therebetween), and discrete fibrations (and commutative squares therebetween), using that precomposition for presheaves corresponds to pullback for discrete fibrations. □

As is suggested by Theorem 6.2, it is instructive to consider loose monads in $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$. These admit a particularly simple description in the special case in which $\mathbb{A}$ is strict, by virtue of the fact that strict double categories are precisely monads in $\mathbb{S}\mathbf{pan}(\mathbf{Cat})$ (in contrast, pseudo double categories are merely pseudomonads, the consideration of which requires three-dimensional structure as in [CLPS22]). To explain this, we will need a couple of preparatory lemmas.

Lemma 8.4. *For each strict double category $\mathbb{A}$, the virtual double category $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ is isomorphic to the full sub-virtual double category of the slice $\mathbb{S}\mathbf{pan}(\mathbf{Cat})/\mathbb{A}$ (Definition 4.13) spanned by the span morphisms whose components are discrete fibrations.*

Proof. When $\mathbb{A}$ is strict, the following diagram commutes.



Consequently, the lift appearing in the definition of composition in $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ is trivial, so that composition is precisely composition of spans via pullback. □

Remark 8.5. Let $\mathbf{A}$ be a strict monoidal category. By Lemma 8.4, the full sub-virtual double category of $\mathbb{S}\mathbf{pan}(\mathbf{Cat})/(\Sigma\mathbf{A})$ spanned by the span morphisms whose components are discrete fibrations is isomorphic to $\mathbb{L}\mathbf{vwDFib}(\Sigma\mathbf{A})$ and hence, by Proposition 8.3, is equivalent to $\mathbb{S}\mathbf{pan}^{\Sigma(\mathbf{A}^{\mathrm{op}})}$. We may restrict this equivalence on both sides to the one-object sub-virtual double categories on the terminal objects (i.e. the identity functor on the terminal category on one hand, and the terminal presheaf on the other), obtaining an equivalence of monoidal categories. By inspection of Example 7.2, the full sub-virtual double category of $\mathbb{S}\mathbf{pan}^{\Sigma(\mathbf{A}^{\mathrm{op}})}$ on the terminal presheaf is precisely the delooping of the exponential multicategory $\mathbf{Set}^{\mathbf{A}^{\mathrm{op}}}$. As discussed in Section 3.2, this is precisely the convolution monoidal structure on $\mathbf{A}$. We thereby recover [EHL18, Theorem 3.13]. The authors of [EHL18] indicated that it would be desirable to extend their work to the setting in which $\mathbf{A}$ is not assumed strict [EHL18, §4], for which they anticipated working in a three-dimensional setting. Our Lemma 8.4 suggests an alternative approach would be to work with (a full sub-virtual double category of) $\mathbb{L}\mathbf{vwDFib}(\Sigma\mathbf{A})$ instead of $\mathbb{S}\mathbf{pan}(\mathbf{Cat})/(\Sigma\mathbf{A})$. ⌟

Definition 8.6. $\mathbf{DFib}$ is the full subcategory of the arrow category $\mathbf{Cat}^{\to}$ spanned by the discrete fibrations. ⌟

Lemma 8.7. *For each strict double category $\mathbb{A}$, the apex of the pullback*[13]

$$\begin{array}{ccc} \mathbb{D}\mathbf{ist}(\mathbf{DFib})_{\mathbb{A}} & \longrightarrow & \mathbb{D}\mathbf{ist}(\mathbf{DFib}) \\ \downarrow & pb & \downarrow{\scriptstyle \mathbb{D}\mathbf{ist}(\mathrm{cod})} \\ \mathbb{1} & \xrightarrow[\mathbb{A}]{} & \mathbb{D}\mathbf{ist}(\mathbf{Cat}) \end{array}$$

is isomorphic to the full sub-virtual double category of $\mathbb{D}\mathbf{ist}(\mathbf{Cat})/\mathbb{A}$ (Definition 4.13) spanned by the span morphisms whose components are discrete fibrations.

Proof. Consider the following diagram. Since $\mathbf{Cat}^{\to}$ is a power by the arrow category, hence a comma object, and $\mathbb{D}\mathbf{ist}$ preserves 2-limits (since $\mathbb{D}\mathbf{ist} := \mathbb{M}\mathbf{od} \circ \mathbb{S}\mathbf{pan}$, and both $\mathbb{S}\mathbf{pan}$ and $\mathbb{M}\mathbf{od}$ preserve 2-limits), the right-hand square is also a comma diagram. By the pasting lemma for comma objects, the outer rectangle is a comma object, which is precisely the universal property of the slice $\mathbb{S}\mathbf{pan}(\mathbf{Cat})/\mathbb{A}$.

$$\begin{array}{ccccc} \mathbb{D}\mathbf{ist}(\mathbf{Cat}^{\to})_{\mathbb{A}} & \longrightarrow & \mathbb{D}\mathbf{ist}(\mathbf{Cat}^{\to}) & \xrightarrow{\mathbb{D}\mathbf{ist}(\mathrm{dom})} & \mathbb{D}\mathbf{ist}(\mathbf{Cat}) \\ \downarrow & pb & {\scriptstyle \mathbb{D}\mathbf{ist}(\mathrm{cod})}\downarrow & \swarrow & \| \\ \mathbb{1} & \xrightarrow[\mathbb{A}]{} & \mathbb{D}\mathbf{ist}(\mathbf{Cat}) & = & \mathbb{D}\mathbf{ist}(\mathbf{Cat}) \end{array}$$

That this restricts to discrete fibrations as stated follows easily. □

Discrete fibrations of strict and pseudo double categories have been appeared before in the literature in several guises (perhaps the earliest allusion is [Par11, Remark 3.10], though a definition is not given there). The following relates our virtual double category $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ with each of the notions appearing in the literature.

Theorem 8.8. *Let $\mathbb{A}$ be a pseudo double category.*

(1) For $\mathbb{A}$ strict, the virtual double category $\mathbb{D}\mathbf{Fib}(\mathbb{A})$ of [Lam21, Definition 3.24] is equivalent to $\mathbb{M}\mathbf{od}(\mathbb{L}\mathbf{vwDFib}(\mathbb{A}))$.

(2) For $\mathbb{A}$ strictly normal, the category $\mathbf{DDblFib}(\mathbb{A})$ of [CLPS22, Definition 3.43 & (3.20)] is equivalent to the underlying category of $\mathbb{M}\mathbf{od}(\mathbb{L}\mathbf{vwDFib}(\mathbb{A}))$.

(3) For $\mathbb{A}$ strict, the 2-category $\mathrm{D}\mathcal{F}\mathrm{ib}(\mathbb{A})$ of [FM24, Notation 5.2] is equivalent to the underlying 2-category of $\mathbb{M}\mathbf{od}(\mathbb{L}\mathbf{vwDFib}(\mathbb{A}))$.

Proof. (1) The virtual double category of discrete fibrations over a strict double category $\mathbb{A}$ is defined by Lambert as $\mathbb{D}\mathbf{ist}(\mathbf{DFib})_{\mathbb{A}}$, which, by Lemma 8.7, is equivalent to the full sub-virtual double category of $\mathbb{D}\mathbf{ist}(\mathbf{Cat})/\mathbb{A}$ spanned by the span morphisms whose components are discrete fibrations. By Lemma 8.4, we have that $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ is isomorphic to the full sub-virtual double category of $\mathbb{S}\mathbf{pan}(\mathbf{Cat})/\mathbb{A}$ spanned by the span morphisms whose components are discrete fibrations. Finally, we have $\mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathbf{Cat})/\mathbb{A}) \cong \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}(\mathbf{Cat}))/\mathbb{A} \cong \mathbb{D}\mathbf{ist}(\mathbf{Cat})/\mathbb{A}$ by Proposition 4.16, from which the equivalence follows by restricting to sub-virtual double categories.

(2) Explicitly, an object of $\mathbf{DDblFib}(\mathbb{A})$ is a strict functor of double categories $E\colon \mathbb{E} \to \mathbb{A}$ such that $E_0\colon \mathbb{E}_0 \to \mathbb{A}_0$ and $E_1\colon \mathbb{E}_1 \to \mathbb{A}_1$ are discrete fibrations; and a morphism is a commutative triangle of strict functors of double categories. Now, it is clear that the unit and multiplication for a loose monad in $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$, being defined via lifting the loose identities and loose composition in $\mathbb{A}$, define a pseudo double category structure on $\mathbb{A}$, and that $E\colon \mathbb{E} \to \mathbb{A}_1$ thereby defines a strict functor. That the components are discrete fibrations is by definition. Similarly, a loose monad morphism in $\mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ is a strict functor, the data of the tight morphism determining the action on $\mathbb{E}_0$ and the data of the cell determining the action on $\mathbb{E}_1$.

(3) The 2-category of discrete fibrations over a strict double category $\mathbb{A}$ is defined by Fröhlich and Moser as the underlying 2-category of the full sub-virtual double category of $\mathbb{S}\mathbf{pan}(\mathbf{Cat})/\mathbb{A}$

[13] The reader should be warned that Lambert [Lam21, Example 3.23] uses the notation $\mathbb{D}\mathbf{ist}(\mathbf{DFib})/\mathbb{A}$ in place of $\mathbb{D}\mathbf{ist}(\mathbf{DFib})_{\mathbb{A}}$ (though it is not a slice construction).

spanned by the spans whose components are discrete fibrations. Therefore, the equivalence follows as in (1). □

A virtual double category of discrete fibrations over a *pseudo* double category has not previously been defined in the literature. Theorem 8.8 justifies us in defining $\mathbb{D}\mathbf{Fib}(\mathbb{A})$ to be the virtual double category $\mathbb{M}\mathbf{od}(\mathbb{L}\mathbf{vwDFib}(\mathbb{A}))$. Having set up our definitions, we immediately obtain a correspondence between presheaves on, and discrete fibrations over, a pseudo double category $\mathbb{A}$.

Corollary 8.9. *For each pseudo double category $\mathbb{A}$, there is an equivalence of virtual double categories $\mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}}, \mathbb{S}\mathbf{pan}) \simeq \mathbb{D}\mathbf{Fib}(\mathbb{A})$.*

Proof. From Proposition 8.3, we have $\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op\,t}}} \simeq \mathbb{L}\mathbf{vwDFib}(\mathbb{A})$, whereupon applying $\mathbb{M}\mathbf{od}$ gives $\mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}}, \mathbb{S}\mathbf{pan}) \cong \mathbb{M}\mathbf{od}(\mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op\,t}}}) \simeq \mathbb{M}\mathbf{od}(\mathbb{L}\mathbf{vwDFib}(\mathbb{A})) \cong \mathbb{D}\mathbf{Fib}(\mathbb{A})$ by Theorems 6.2 and 8.8. □

When $\mathbb{A}$ is strict, Corollary 8.9 recovers [Lam21, Theorem 4.14]; the underlying equivalence of 2-categories, together with Proposition 4.5, recovers [FM24, Theorem 6.13]. When $\mathbb{A}$ is a strictly normal pseudo double category, the underlying equivalence of categories recovers [CLPS22, Theorem 3.45]. When $\mathbb{A}$ is a pseudo double category, the full faithfulness of the functor $\mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}}, \mathbb{S}\mathbf{pan}) \to \mathbf{DFib}(\mathbb{A})$ recovers [Par11, Theorems 3.8 & 3.11 & Proposition 3.9]. Finally, when $\mathbb{A}$ is a bicategory, Corollary 8.9 recovers an unpublished correspondence due to Cockett–Niefield–Wood [Nie12].

Remark 8.10. We note that the correspondence proven by Cruttwell, Lambert, Pronk and Szyld permits the base $\mathbb{A}$ to vary. Our view is that it is the correspondence with the fixed base that is primary; we explain briefly how one might go about obtaining the correspondence for a varying base from the correspondence for a fixed base. There are 2-functors $\mathbb{L}\mathbf{ax}(-^{\mathrm{op}}, \mathbb{S}\mathbf{pan}) \colon \mathbf{DblCat}_l \to \mathbf{NVDblCat}$ and $\mathbb{D}\mathbf{Fib}(-) \colon \mathbf{DblCat}_l \to \mathbf{NVDblCat}$, which are 2-naturally equivalent. Our expectation is that such functors should admit an elements construction (necessarily of a three-dimensional nature). Thereafter, applying the elements construction to the equivalence $\mathbb{L}\mathbf{ax}(-^{\mathrm{op}}, \mathbb{S}\mathbf{pan}) \simeq \mathbb{D}\mathbf{Fib}(-)$ would produce the desired correspondence for a varying base. ⌟

We conclude with some simple but useful consequences of the correspondence between presheaves and discrete fibrations.

Proposition 8.11. *The embedding $\mathbb{A}(-_2, -_1) \colon \mathbb{A} \to \mathbb{S}\mathbf{pan}^{\mathbb{A}^{\mathrm{op\,t}}}$ corresponds under Proposition 8.3 to the functor $\mathbb{A} \to \mathbb{L}\mathbf{vwDFib}(\mathbb{A})$ sending a loose morphism $p \colon A \nrightarrow B$ to the following span morphism.*

$$\begin{array}{ccccc} \mathbb{A}_0/A & \xleftarrow{s/p} & \mathbb{A}_1/p & \xrightarrow{t/p} & \mathbb{A}_0/B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \pi_p} & & \downarrow{\scriptstyle \pi_B} \\ \mathbb{A}_0 & \xleftarrow[s]{} & \mathbb{A}_1 & \xrightarrow[t]{} & \mathbb{A}_0 \end{array}$$

Consequently, the embedding $Y_{\mathbb{A}} \colon \mathbb{A} \to \mathbb{L}\mathbf{ax}(\mathbb{A}^{\mathrm{op}}, \mathbb{S}\mathbf{pan})$ corresponds under Corollary 8.9 to the functor $\mathbb{A} \to \mathbb{D}\mathbf{Fib}(\mathbb{A})$ each object $A \in \mathbb{A}$ to the projection functor from the slice double category $\pi_A \colon \mathbb{A}/A \to \mathbb{A}$.

Proof. The equivalence of Proposition 8.3 is levelwise, and so the results follow from the corresponding result for categories by unwrapping the definition of the embeddings in Proposition 7.7. □

Corollary 8.12. *A discrete fibration $\mathbb{E} \to \mathbb{A}$ of pseudo double categories corresponds under the equivalence of Corollary 8.9 to a representable presheaf if and only if its $\mathbb{E}_0$ admits a terminal object whose identity loose morphism is terminal in $\mathbb{E}_1$.*

Proof. Following Proposition 8.11, a discrete fibration $\mathbb{E} \to \mathbb{A}$ is representable if and only if each component discrete fibration $\mathbb{E}_0 \to \mathbb{A}_0$ and $\mathbb{E}_1 \to \mathbb{A}_1$ corresponds to a representable presheaf – i.e. $\mathbb{E}_0$ and $\mathbb{E}_1$ admit terminal objects – and furthermore, the representing object for $\mathbb{E}_1 \to \mathbb{A}_1$ is the loose identity on the representing object for $\mathbb{E}_0 \to \mathbb{A}_0$. Thus, the characterisation for discrete fibrations of double categories follows directly from the characterisation for discrete fibrations of categories, applied componentwise. □

For instance, in combination with Proposition 7.5, Proposition 8.11 extends [FM24, Theorem 7.4] from strict double categories to pseudo double categories, while Corollary 8.12 extends [FM24, Theorems 7.7 & 7.9] likewise.

Remark 8.13. For simplicity, we have chosen to focus here on the elements correspondence for tight presheaves and discrete fibrations of double categories, but note that precisely the same techniques apply to obtain a correspondence between tight distributors and two-sided discrete fibrations of pseudo double categories. ⌟

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