CHAPTER 2

Double categories and functors

1. Double categories

In this section, we introduce the definition of a double category, generalising that of both 2-categories and bicategories.

Definition 2.1. A *double category* $\mathbb{A}$ consists of the following data.

(DC1) A collection of *objects*.

(DC2) For each pair of objects A and B, a set of *tight morphisms*. A tight morphism from A to B is denoted by $A \to B$.

(DC3) For each pair of objects A and B, a collection of *loose morphisms*. A loose morphism from A to B is denoted by $A \nrightarrow B$.

(DC4) For each *frame* or *boundary* of tight and loose morphisms as follows, a set of *cells*. A cell with the frame below may be denoted by $p \to q$ or $f \nrightarrow g$.

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[q]{} & D \end{array}$$

(DC5) For each pair of tight morphisms $f\colon A \to B$ and $g\colon B \to C$, a *composite* tight morphism $g \circ f\colon A \to C$. For each pair of cells $\alpha\colon p \to q$ and $\beta\colon q \to r$, a composite cell $\beta \circ \alpha\colon p \to r$ whose frame is given as follows.

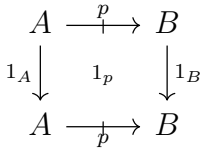
$$\begin{array}{ccc} A & \xrightarrow{p} & A' \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle f'} \\ B & \xrightarrow[q]{} & B' \\ {\scriptstyle g}\downarrow & \beta & \downarrow{\scriptstyle g'} \\ C & \xrightarrow[r]{} & C' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{p} & A' \\ {\scriptstyle g\circ f}\downarrow & \beta\circ\alpha & \downarrow{\scriptstyle g'\circ f'} \\ C & \xrightarrow[r]{} & C' \end{array}$$

(DC6) For each pair of loose morphisms $p\colon A \nrightarrow B$ and $q\colon B \nrightarrow C$, a *composite* loose morphism $p \odot q\colon A \nrightarrow C$. For each pair of cells $\alpha\colon f \nrightarrow g$ and $\beta\colon g \nrightarrow h$, a composite cell $\alpha \odot \beta\colon f \nrightarrow h$ whose frame is given as follows.

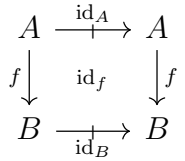
$$\begin{array}{ccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C \\ {\scriptstyle f}\downarrow & \alpha & {\scriptstyle g}\downarrow & \beta & \downarrow{\scriptstyle h} \\ A' & \xrightarrow[p']{} & B' & \xrightarrow[q']{} & C' \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{p\odot q} & C \\ {\scriptstyle f}\downarrow & \alpha\odot\beta & \downarrow{\scriptstyle h} \\ A' & \xrightarrow[p'\odot q']{} & C' \end{array}$$

(DC7) For each object A, an *identity* tight morphism $1_A\colon A \to A$. For each loose morphism $p\colon A \nrightarrow B$, an identity cell $1_p\colon p \to p$ whose frame is given as

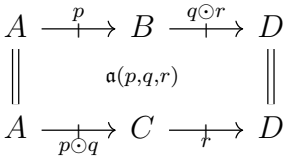
follows.



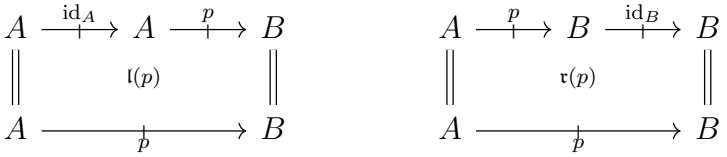
(DC8) For each object A, an *identity* loose morphism $\mathrm{id}_A \colon A \nrightarrow A$. For each tight morphism $f \colon A \to B$, an identity cell $\mathrm{id}_f \colon f \nrightarrow f$ whose frame is given as follows.



(DC9) For each triple of loose morphisms $p \colon A \nrightarrow B$, $q \colon B \nrightarrow C$, and $r \colon C \nrightarrow D$, a cell $\mathfrak{a}(p, q, r)$ called the *associator*.



(DC10) For each loose morphism $p \colon A \nrightarrow B$, a pair of cells $\mathfrak{l}(f)$ and $\mathfrak{r}(f)$ called the *left unitor* and *right unitor*, respectively.

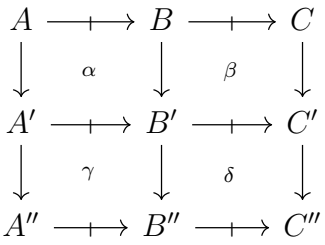


This data is required to satisfy the following axioms.

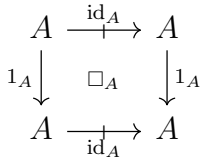
(DC11) The objects and tight morphisms form a category $\mathbb{A}_0$ with respect to the composition $g \circ f$ and identities 1_A.

(DC12) The loose morphisms and cells form a category $\mathbb{A}_1$ with respect to the composition $\beta \circ \alpha$ and identities 1_p.

(DC13) For each matrix of cells as follows, $(\gamma \odot \delta) \circ (\alpha \odot \beta) = (\gamma \circ \alpha) \odot (\delta \circ \beta)$.

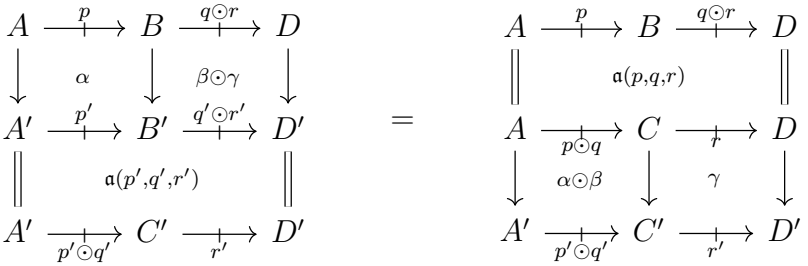


(DC14) For each object A, we have that $1_{\mathrm{id}_A} = \mathrm{id}_{1_A}$; we may denote this cell as follows.

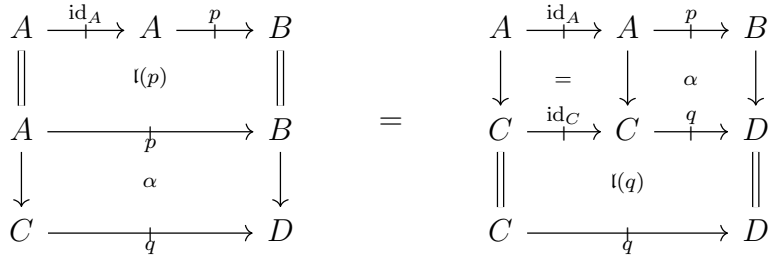


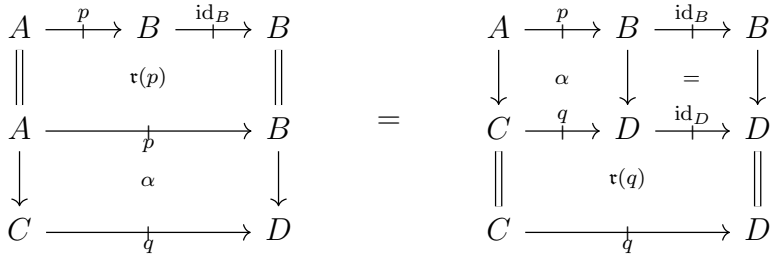
For each pair of loose morphisms $p\colon A \nrightarrow B$ and $q\colon B \nrightarrow C$ we have that $1_{p \odot q} = 1_p \odot 1_q$. For each pair of tight morphisms $f\colon A \to B$ and $g\colon B \to C$. we have that $\mathrm{id}_{g \circ f} = \mathrm{id}_g \circ \mathrm{id}_f$.

(DC15) The associator cells are *invertible*, as morphisms in $\mathbb{A}_1$, and *natural*, in the sense that for each triple of cells α, β, and γ, the following equation holds.

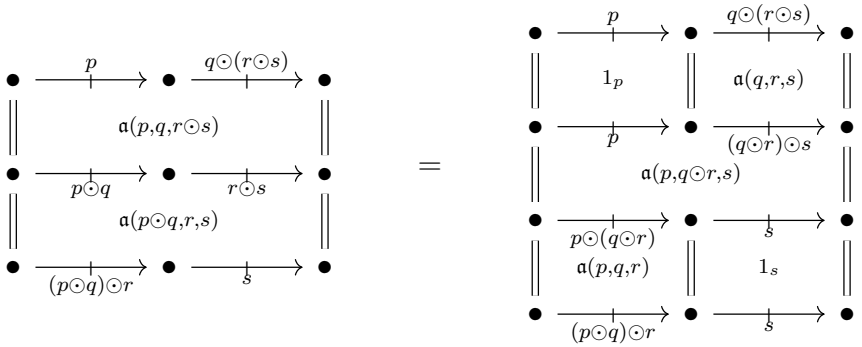


(DC16) The left and right unitor cells are *invertible*, as morphisms in $\mathbb{A}_1$, and *natural*, in the sense that for each cell α, the following equations holds.

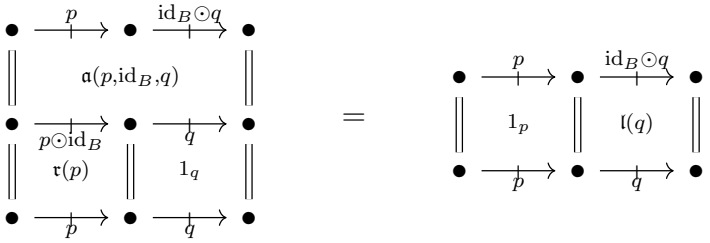




(DC17) *Coherence condition for the associator*: for each quadruple of loose morphisms $p\colon A \nrightarrow B$, $q\colon B \nrightarrow C$, $r\colon C \nrightarrow D$, and $s\colon D \nrightarrow E$, the following equation holds.



(DC18) *Coherence condition for the unitors*: for each pair of loose morphisms $p\colon A \nrightarrow B$ and $q\colon B \nrightarrow C$, the following equation holds.



This completes the definition. ⌟

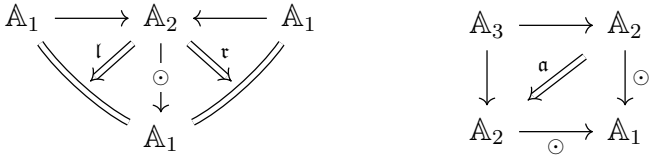
Definition 2.2. A double category is called *unitary* if its left and right unitor cells are identities. A double category is called *strict* if its associator, left unitor, and right unitor cells are identities. ⌟

In the definition of a double category $\mathbb{A}$, conditions (DC11) and (DC12) define categories $\mathbb{A}_0$ and $\mathbb{A}_1$, respectively. These categories are related by functors as depicted in the following diagram.

$$\mathbb{A}_0 \underset{\text{cod}}{\overset{\text{dom}}{\leftleftarrows}} \mathbb{A}_1 \xleftarrow{\odot} \mathbb{A}_1 \times_{\mathbb{A}_0} \mathbb{A}_1 \coloneqq \mathbb{A}_2 \qquad (1)$$

The functors $\text{dom}\colon \mathbb{A}_1 \to \mathbb{A}_0$ and $\text{cod}\colon \mathbb{A}_1 \to \mathbb{A}_0$ assign a loose morphism $A \nrightarrow B$ to its domain A and codomain B, respectively; similarly for cells. The functor $\text{id}\colon \mathbb{A}_0 \to \mathbb{A}_1$ assigns an object A to its identity loose morphism id_A, and a tight morphism $f\colon A \to B$ to its identity cell id_f defined in (DC8). The category $\mathbb{A}_2$ is constructed as the pullback of the functors dom and cod; its objects are composable pairs of loose morphisms. The functor $\odot\colon \mathbb{A}_2 \to \mathbb{A}_1$ assigns each composable pair of loose morphisms $p\colon A \nrightarrow B$ and $q\colon B \nrightarrow C$ to their composite $p \odot q\colon A \nrightarrow C$ as defined in (DC6); similarly for cells. In summary, the diagram (1) accounts for the data (DC1) to (DC8) of the double category.

The associator, left unitor, and right unitor cells may be encoded as suitable invertible natural transformations as follows (where some of the functor labels have been omitted).



(2)

This accounts for the remaining data (DC9) and (DC10) of a double category, as well as the conditions (DC15) and (DC16).

In summary, a double category may be encoded via the diagrams (1) and (2) subject to the conditions (DC13), (DC14), (DC17), and (DC18). Formally, this yields an equivalence between double categories and *pseudo category objects*. A detailed examination of this perspective is postponed until a later chapter.

Proposition 2.3. *A double category is equivalent to a pseudo category object in the* 2*-category* $\mathbb{C}\text{AT}$ *of (locally small) categories.*

Corollary 2.4. *A strict double category is equivalent to a category object (or internal category) in the* 1*-category* $\mathbb{C}\mathrm{AT}$ *of (locally small) categories.*

Remark 2.5 (Notation)**.** To ease notation when drawing diagrams, the identity tight morphisms and identity loose morphisms may be drawn, respectively, as follows.

$$\begin{array}{c} A \\ \| \\ A \end{array} \qquad\qquad A \mathrel{=\!\!=\!\!=} A$$

We will also frequently write gf and $\beta\alpha$ for the composites of tight morphisms $g \circ f$ and cells $\beta \circ \alpha$, respectively. The identity cells 1_p on a loose morphism p and id_f on a tight morphism f are often denoted by "=". Cells which arise as a composite of associator, left unitor, and right unitor cells are also often denoted by "≅". ⌟

Definition 2.6. A cell is called *globular* if both of the tight morphisms in its frame are identities.

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ \| & \alpha & \| \\ A & \xrightarrow[q]{} & B \end{array}$$

A globular cell $\alpha\colon p \to q$ which is invertible (that is, there is a cell $\beta\colon q \to p$ such that $\beta\alpha = 1_p$ and $\alpha\beta = 1_q$) is called an *isocell.* ⌟

Lemma 2.7. *Each double category* $\mathbb{A}$ *has an underlying bicategory* $\mathcal{L}\mathrm{oose}(\mathbb{A})$ *whose* 0*-cells are the objects of* $\mathbb{A}$*, whose* 1*-cells are the loose morphisms of* $\mathbb{A}$*, and whose* 2*-cells are the globular cells of* $\mathbb{A}$*.*

$$A \;\overset{p}{\underset{q}{\Downarrow \alpha}}\; B \qquad := \qquad \begin{array}{ccc} A & \xrightarrow{p} & B \\ \| & \alpha & \| \\ A & \xrightarrow[q]{} & B \end{array}$$

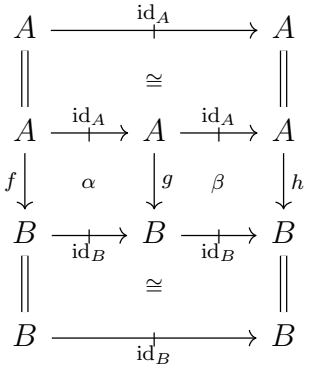
Composition in the bicategory is inherited directly from composition in the double category.

Lemma 2.8. *Each double category* $\mathbb{A}$ *has an underlying* 2*-category* $\mathcal{T}\mathrm{ight}(\mathbb{A})$ *whose* 0*-cells are the objects of* $\mathbb{A}$*, whose* 1*-cells are the tight morphisms of* $\mathbb{A}$*, and whose* 2*-cells are given as follows.*

$$A \;\overset{f}{\underset{g}{\Downarrow \alpha}}\; B \qquad := \qquad \begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ B & \xrightarrow[\mathrm{id}_B]{} & B \end{array}$$

The composition of morphisms in the hom-category $\mathcal{T}\mathrm{ight}(\mathbb{A})(A, B)$ *is given as follows, and is well-defined by the invertibility and naturality of the unitors together with the*

coherence conditions.



1.1. Dualities for double categories.

Definition 2.9. Let $\mathbb{A}$ be a double category. The *loose opposite* $\mathbb{A}^{\mathrm{op}}$ is a double category consisting of the following data.

(a) The objects are the objects of $\mathbb{A}$.

(b) The tight morphisms $A \to B$ are the tight morphisms $A \to B$ in $\mathbb{A}$.

(c) The loose morphisms $A \nrightarrow B$ are the loose morphisms $B \nrightarrow A$ in $\mathbb{A}$

(d) The cells $f \nrightarrow g$ are the cells $g \nrightarrow f$ in $\mathbb{A}$. ⌟

Definition 2.10. Let $\mathbb{A}$ be a double category. The *tight opposite* $\mathbb{D}^{\mathrm{co}}$ is a double category consisting of the following data.

(a) The objects are the objects of $\mathbb{A}$.

(b) The tight morphisms $A \to B$ are the tight morphisms $B \to A$ in $\mathbb{A}$.

(c) The loose morphisms $A \nrightarrow B$ are the loose morphisms $A \nrightarrow B$ in $\mathbb{A}$.

(d) The cells $p \to q$ are the cells $q \to p$ in $\mathbb{A}$. ⌟

Remark 2.11 (Transpose)**.** If $\mathbb{A}$ is a *strict* double category, it is common to consider a further duality called the *transpose* of $\mathbb{A}$ which exchanges the tight and loose morphisms. This notion may not always been well-defined due to size constraints, and even in the instances where it is well-defined, it is often difficult to find a compelling reason to consider it. ⌟

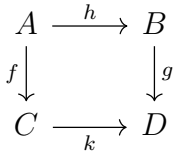
1.2. Basic examples.

Example 2.12. Let $\mathcal{C}$ be a category. There is a strict double category $\mathbb{S}\mathbf{q}(\mathcal{C})$, called the *double category of squares*, which consists of the following data.

(a) The objects are the objects of $\mathcal{C}$.

(b) The tight morphisms and loose morphisms are the morphisms of $\mathcal{C}$.

(c) The cells are commutative squares in $\mathcal{C}$. That is, a cell with the following frame exists if and only if $kf = gh$ in $\mathcal{C}$.



Composition and identities in $\mathbb{S}\mathbf{q}(\mathcal{C})$ are inherited from those of $\mathcal{C}$. ⌟

Example 2.13. Let $\mathcal{K}$ be a 2-category. There is a strict double category $\mathbb{Q}(\mathcal{K})$, called the *double category of quintets*, which consists of the following data.

(a) The objects are the 0-cells of $\mathcal{K}$.

(b) The tight morphisms and loose morphisms are the 1-cells of $\mathcal{K}$.

(c) The cells with the following frame are 2-cells $\alpha\colon kf \Rightarrow gh$ in $\mathcal{K}$.

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ C & \xrightarrow[k]{} & D \end{array}$$

Composition and identities in $\mathbb{Q}(\mathcal{C})$ are inherited from those of $\mathcal{K}$. In particular, composition of cells in $\mathbb{Q}(\mathcal{C})$ is well-defined by pasting of 2-cells in $\mathcal{K}$.

Let $\mathbb{T}\mathbf{i}(\mathcal{K})$ denote the restriction of $\mathbb{Q}(\mathcal{K})$ to the strict double category whose loose morphisms are all identities. Let $\mathbb{L}\mathbf{o}(\mathcal{K})$ denote the restriction of $\mathbb{Q}(\mathcal{K})$ to the strict double category whose tight morphisms are all identities.

If $\mathcal{K}$ happens to be *locally discrete* — that is, it is a 1-category — then the double category of quintets is the same as the double category of squares. ⌟

Example 2.14. Let $\mathcal{B}$ be a bicategory. There is a double category $\mathbb{L}\mathbf{o}(\mathcal{B})$ consisting of the following data.

(a) Objects are the 0-cells of $\mathcal{B}$.

(b) Tight morphisms are just the identities.

(c) Loose morphisms are the 1-cells of $\mathcal{B}$.

(d) Cells with the following frame are 2-cells $\alpha\colon f \Rightarrow g$ in $\mathcal{B}$.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & \alpha & \| \\ A & \xrightarrow[g]{} & B \end{array} \quad := \quad A \;\overset{p}{\underset{q}{\Downarrow \alpha}}\; B$$

The identities and composition are inherited from the bicategory, as are the associator and unitor cells.

If $\mathcal{B}$ happens to be a 2-category — that is, the associator and unitor cells are identities — then the double category $\mathbb{L}\mathbf{o}(\mathcal{B})$ agrees with $\mathbb{L}\mathbf{o}(-)$ defined in Example 2.13; thus we use the same notation. On the other hand, it is not possible to construct analogous double categories $\mathbb{Q}(-)$ or $\mathbb{T}\mathbf{i}(-)$ for a bicategory. ⌟

Example 2.15. Let $(\mathcal{V}, I, \otimes)$ be a monoidal category. There is a double category $\mathbb{L}\mathbf{o}(\mathcal{V}, I, \otimes)$ consisting of the following data.

(a) A single object $*$.

(b) The identity tight morphism $1_*\colon * \to *$.

(c) Loose morphisms are the objects of $\mathcal{V}$.

(d) Cells with the following frame are morphisms $f\colon A \to B$ in $\mathcal{V}$.

$$\begin{array}{ccc} * & \xrightarrow{A} & * \\ \| & f & \| \\ * & \xrightarrow[B]{} & * \end{array}$$

Composition of loose morphisms is given by the monoidal product; similarly for composition of cells in the loose direction. Composition of cells in the tight direction is just ordinary composition in $\mathcal{V}$. The associator and unitor cells are inherited from the monoidal category. ⌟

Example 2.16. Let $(\mathcal{V}, I, \otimes)$ be a distributive monoidal category whose initial object is denoted 0. There is a double category $\mathbb{M}\mathbf{at}(\mathcal{V})$ consisting of the following data.

(a) Objects are sets.

(b) Tight morphisms are functions.

(c) Loose morphisms $A \nrightarrow B$ are functors $A \times B \to \mathcal{V}$, where A and B are considered as discrete categories. This is called a $\mathcal{V}$-*matrix*.

(d) Cells with the following frame are natural transformations α as depicted below, where the function $f \times g$ is considered as a functor between discrete categories.

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ C & \xrightarrow[N]{} & D \end{array} \quad \coloneqq \quad \begin{array}{ccc} A \times B & \xrightarrow{M} & \mathbf{Set} \\ {\scriptstyle f\times g}\downarrow & \alpha\Downarrow & \| \\ C \times D & \xrightarrow[N]{} & \mathbf{Set} \end{array}$$

Since $A \times B$ is discrete, this amounts to a family of morphisms of $\mathcal{V}$ given by $\{\alpha(a,b)\colon M(a,b) \to N(fa, gb)\}_{a\in A,\, b\in B}$.

(e) Given loose morphisms $M\colon A \nrightarrow B$ and $N\colon B \nrightarrow C$, their composite $M \odot N\colon A \nrightarrow C$ is given by the following functor.

$$A \times C \longrightarrow \mathbf{Set}, \qquad (a,c) \longmapsto \sum_{b\in B} M(a,b) \otimes N(b,c)$$

(f) Given a set A, the identity $\mathcal{V}$-matrix $A \nrightarrow A$ is given by the following functor.

$$\Delta_A\colon A \times A \longrightarrow \mathbf{Set}, \qquad \Delta_A(a,a') = \begin{cases} I & \text{if } a = a' \\ 0 & \text{if } a \neq a' \end{cases}$$

The left and right unitors are constructed from those of $\mathcal{V}$, together with the universal property of coproducts. The associator is constructed from that of $\mathcal{V}$, together with distributivity of coproducts in $\mathcal{V}$. ⌟

Example 2.17. Let $\mathbf{2} \coloneqq \{\bot \to \top\}$ denote the *interval category*, with two objects $\bot$ and $\top$, and a single non-identity morphism $\bot \to \top$ between them. This category has products, denoted $\wedge$ with terminal object $\top$, and coproducts, denoted $\vee$ with initial object $\bot$. With these operations, $\mathbf{2}$ becomes a distributive monoidal category.

The define the *double category of relations* to be $\mathbb{R}\mathbf{el} \coloneqq \mathbb{M}\mathbf{at}(\mathbf{2})$, which consists of the following data.

(a) Objects are sets.

(b) Tight morphisms are functions.

(c) Loose morphisms $A \nrightarrow B$ are functors $A \times B \to \mathbf{2}$. This is precisely a *relation* between sets, where the fibre over $\top$ determines a subset of $R_\top \rightarrowtail A \times B$.

(d) Cells with the following frame are natural transformations α as depicted below, where the function $f \times g$ is considered as a functor between discrete categories.

$$\begin{array}{ccc} A & \xrightarrow[R]{} & B \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ C & \xrightarrow[S]{} & D \end{array} \quad \coloneqq \quad \begin{array}{ccc} A \times B & \xrightarrow{R} & \mathbf{2} \\ {\scriptstyle f\times g}\downarrow & \alpha\Downarrow & \| \\ C \times D & \xrightarrow[S]{} & \mathbf{2} \end{array}$$

This is equivalent to a commutative square of functions as depicted below.

$$\begin{array}{ccc} R_\top & \rightarrowtail & A \times B \\ {\scriptstyle \alpha_\top}\downarrow & & \downarrow{\scriptstyle f\times g} \\ S_\top & \rightarrowtail & C \times D \end{array}$$

Note that since $S_\top \rightarrowtail C \times D$ is a monomorphism, there is *at most* one cell in $\mathbb{R}\mathbf{el}$ for each possible frame.

(e) Given relations $R\colon A \nrightarrow B$ and $S\colon B \nrightarrow C$, their composite $R \odot S\colon A \nrightarrow C$ is given by the following functor/function.

$$A \times C \to \mathbf{2}, \qquad (a, c) \longmapsto \bigvee_{b \in B} R(a, b) \wedge S(b, c)$$

The fibre over $\top$ is given by the following set.

$$\{(a, c) \mid \text{there exists } b \in B \text{ such that } R(a, b) = \top \text{ and } S(b, c) = \top\} \subseteq A \times C$$

(f) Given a set A, the identity relation $A \nrightarrow A$ is given by the following functor/function.

$$\Delta_A\colon A \times A \longrightarrow \mathbf{2}, \qquad \Delta_A(a, a') = \begin{cases} \top & \text{if } a = a' \\ \bot & \text{if } a \neq a' \end{cases}$$

The underlying bicategory $\mathcal{L}\mathrm{oose}(\mathbb{R}\mathbf{el})$ is the usual bicategory of sets and relations. ⌟

Example 2.18. There is a double category $\mathbb{D}\mathbf{ist}$ consisting of the following data.

(a) Objects are categories.

(b) Tight morphisms are functors.

(c) Loose morphisms $\mathcal{A} \nrightarrow \mathcal{B}$ are functors $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathbf{Set}$. These are called *distributors*; the terms *profunctor* and *bimodule* are also common.

(d) Cells with the following boundary are natural transformations α as shown.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow[P]{} & \mathcal{B} \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ \mathcal{C} & \xrightarrow[Q]{} & \mathcal{D} \end{array} \quad \coloneqq \quad \begin{array}{ccc} \mathcal{A}^{\mathrm{op}} \times \mathcal{B} & \xrightarrow{P} & \mathbf{Set} \\ {\scriptstyle f\times g}\downarrow & \alpha\Downarrow & \| \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{D} & \xrightarrow[Q]{} & \mathbf{Set} \end{array}$$

(e) Given distributors $P\colon \mathcal{A} \nrightarrow \mathcal{B}$ and $Q\colon \mathcal{B} \nrightarrow \mathcal{C}$, their composite $P \odot Q\colon \mathcal{A} \nrightarrow \mathcal{C}$ is given by the following functor computed by a *coend*.

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \longrightarrow \mathbf{Set}, \qquad (a, b) \mapsto \int^{b \in \mathcal{B}} P(a, b) \times Q(b, c)$$

(f) Given a category $\mathcal{A}$, the identity distributor is given by the hom-set functor $\mathcal{A}(-,-)\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathbf{Set}$.

The associator and unitors are induced by the universal property of the coend. The underlying bicategory $\mathcal{L}\mathrm{oose}(\mathbb{D}\mathbf{ist})$ is the usual bicategory of distributors, while the underlying 2-category is $\mathcal{T}\mathrm{ight}(\mathbb{D}\mathbf{ist}) \cong \mathbf{Cat}$. ⌟

Example 2.19. Let $\mathcal{C}$ be a category with pullbacks. There is a double category $\mathbb{S}\mathbf{pan}(\mathcal{C})$ consisting of the following data.

(a) Objects are the objects of $\mathcal{C}$.

(b) Tight morphisms are the morphisms of $\mathcal{C}$.

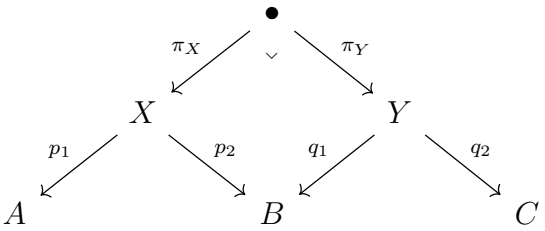
(c) Loose morphisms $A \nrightarrow B$ are spans of morphisms in $\mathcal{C}$, that is, diagrams of the following shape.

$$A \leftarrow X \rightarrow B$$

(d) A cell with the following frame are morphisms $\alpha\colon X \to Y$ in $\mathcal{C}$ making the diagram commute.

$$\begin{array}{ccc} A & \overset{p}{\nrightarrow} & B \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ C & \underset{q}{\nrightarrow} & D \end{array} \quad := \quad \begin{array}{ccccc} A & \xleftarrow{p_1} & X & \xrightarrow{p_2} & B \\ {\scriptstyle f}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xleftarrow[q_1]{} & Y & \xrightarrow[q_2]{} & D \end{array}$$

(e) Given spans $p\colon A \nrightarrow B$ and $q\colon B \nrightarrow C$ their composite span $p \odot q\colon A \nrightarrow C$ is given by pullback as follows.



(f) Given an object A, the identity span is given the span of identity morphisms.

$$A \xleftarrow{1_A} A \xrightarrow{1_A} A$$

The associator and unitors are constructed via the universal property of the pullback. ⌟