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Algebra Automata II: The Categorical Framework for Dynamic Analysis[1]

Y. GIVE'ON AND M. A. ARBIB

Department of Electrical Engineering, Stanford University, Stanford, California 94305

We extend the treatment of algebra automata to automata employing algebras over arbitrary theories. To this end we re-present Eilenberg and Wright's approach to the notion of a theory and its algebras originated by Lawvere. Special attention is given here to the free theories which correspond, in ordinary automata theory, to the free monoids. For example, we prove that the free theories (like the free monoids) are characterized as the projective objects in the category of theories with surjective morphisms of theories only.

For a theory T, a T-automation is defined in a straightforward analogy with the notion of an automaton with an arbitrary input monoid. Our definition of a T-automaton becomes obvious once it is recognized that a T-algebra is the appropriate explication for a right-action of T on a set. Consequently, the construction of a minimal realization and of a minimal dynamics of any given response function f on a theory is analogous to the same construction with respect to a mapping f on a monoid. Also the basic division lemmata which underlie Krohn and Rhodes' composition theory of machines are true for response function on free theories.

3.[2] THEORIES

Our approach here follows the approach to ordinary automata theory taken by Give'on (1967) and the introduction of the study of algebras as associated with theories to generalized automata theory by Eilenberg and Wright (1967). Much of the first half of this paper is a re-presentation of the material of Eilenberg and Wright (1967). However, there are enough novelties (such as the notion of a carrier of a theory and of an

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[2] Sections 1 and 2 refer to those of Part I.

operative theory of an algebra) to justify a discussion which is essentially self-contained. The reader should consult Eilenberg and Wright (1967) for clarification of the more terse portions of this paper, and for a presentation of how the language approach to algebra automata, initiated by Mezei and Wright (1965), may be placed in a categorical framework. The discussion that follows demands knowledge of Part I. Notation and results of Part I will be used without comment.

We assume familiarity with only the most elementary and general notions of category theory (e.g., *category*, the definition of a category by means of *objects* and *morphisms*, *product* and *coproduct diagrams*, *functors* and *cofunctors*, the elementary properties of the *category S of sets* and of *concrete categories*). For exposition of these notions, the reader may consult MacLane (1965), Maclane and Birkhoff (1967), Freyd (1964), and Mitchell (1965). On the other hand, in some cases where we adopt a somewhat different approach from those presented in the literature, we will explicate some of these notions in the body of the paper itself (in small print).

Let us define the set-theoretic map

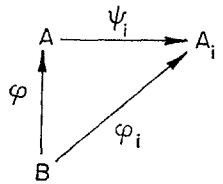
$$i_n : [1] \to [n] : 1 \mapsto i$$

for any $n \in N$ and $1 \leqq i \leqq n$, where $[p] = \{1, \cdots, p\}$ (so that $[0] = \emptyset$). This map determines (cf. 2.4 (ii)) for each set Q the projection

$$\hat{\imath}_n : Q^{[n]} \to Q^{[1]} : x \mapsto xi_n \; ; \tag{3.1}$$

i.e., $(q_1, \cdots, q_n) \mapsto q_i$.

Let us recall (from category theory) that a family of morphisms $\{\psi_i : A \to A_i, \text{for } i = 1, \cdots, n\}$ is a *product diagram* iff for each family of morphisms $\{\varphi_i : B \to A_i, \text{ for } i = 1, \cdots, n\}$ there exits a unique morphism $\varphi : B \to A$ such that



commutes for each $i = 1, \cdots, n$. (The dual notion of a *coproduct diagram* is obtained by reversing all the arrows in the above definition of a product diagram.)

2.4 (iii) simply says that the family $\{\hat{\imath}_n : Q^{[n]} \to Q^{[1]}, \text{ for } i = 1, \cdots, n\}$

is a product diagram in the completion $\langle \mathfrak{A} \rangle$ of a Σ-algebra $\mathfrak{A}$ with carrier Q. Note that $\langle \mathfrak{A} \rangle$ is obviously a subcategory of the category S of sets. Consequently, we may recast Theorem 2.4 in categorical form as a starting point for our categorical approach to algebra automata:

(3.2) THEOREM. *Let C be a subcategory of the category S of sets. Then C is the completion of a Σ-algebra $\mathfrak{A}$ with carrier Q, for some graded set Σ, iff the objects of C are the sets $Q^{[n]}$, $n \in N$, and the projections $\hat{\imath}_n : Q^{[n]} \to Q$ are morphisms of C and constitute, for each n, a product diagram in C.*

Let us consider, for any completion $\langle \mathfrak{A} \rangle$ of a Σ-algebra $\mathfrak{A}$ with carrier Q, the opposite category $\langle \mathfrak{A} \rangle^{\mathrm{op}}$ to $\langle \mathfrak{A} \rangle$, where each object $Q^{[n]}$ of $\langle \mathfrak{A} \rangle$ is relabeled by $[n]$. Thus, $\langle \mathfrak{A} \rangle^{\mathrm{op}}$ has a morphism $\varphi : [n] \to [p]$ just in case $\langle \mathfrak{A} \rangle$ contains the map $\varphi : Q^{[p]} \to Q^{[n]}$. The advantage of forming the opposite of $\langle \mathfrak{A} \rangle$ is that so long as $| Q | > 1$, the trivial maps $_{i}n$ in $\langle \mathfrak{A} \rangle$ may indeed be mirrored in $\langle \mathfrak{A} \rangle^{\mathrm{op}}$ by the maps $i_n : [1] \to [n]$ of S.

(3.3) Denoting by $\check{S}$ the full subcategory of S whose objects are all the sets $[n]$ for $n \in N$, we see that for any nontrivial Σ-algebra $\mathfrak{A}$ (i.e., the cardinality of the carrier of $\mathfrak{A}$ exceeds 1), $\langle \mathfrak{A} \rangle^{\mathrm{op}}$ is a theory, as defined by Eilenberg and Wright (1967):

(3.4) DEFINITION. A *theory* T is a category which is a morphic extension of $\check{S}$ (i.e., the objects of T are the objects of $\check{S}$ and $\check{S}$ is a subcategory of T) and in which the family

$$\{ i_n : [1] \to [n] : 1 \mapsto i, \quad \text{for} \quad i = 1, \cdots, n \}$$

is a coproduct diagram for each $n \in N$.

Thus $\check{S}$ is itself a theory.

(3.5) If a Σ-algebra $\mathfrak{A}$ is nontrivial, we denote by $T_{\mathfrak{A}}$ the theory obtained from the opposite category of $\langle \mathfrak{A} \rangle$, and call it the *operative theory* of $\mathfrak{A}$.

Thus a nontrivial Σ-algebra $\mathfrak{A}$ gives rise to a cofunctor $T_{\mathfrak{A}} \to \langle \mathfrak{A} \rangle$. In Section 4 we shall introduce the category of algebras associated with an arbitrary theory T as a category of certain cofunctors on T. Here we present some elementary properties of theories.

(3.6) Given a theory T and morphisms $\varphi_1, \cdots, \varphi_n : [1] \to [p]$ of T, let $\langle \varphi_1, \cdots, \varphi_n \rangle : [n] \to [p]$ denote the unique morphism $\varphi : [n] \to [p]$ determined by the coproduct diagram $\{ i_n : 1 \leqq i \leqq n \}$. Let $T([n], [p])$ denote the set of all morphisms $[n] \to [p]$ of T, and let T_p be short for $T([1], [p])$. Then the coproduct diagram $\{ i_n : 1 \leqq i \leqq n \}$ determines a bijection

$$T_p^{[n]} \to T([n], [p]) : (\varphi_1, \cdots, \varphi_n) \mapsto \langle \varphi_1, \cdots, \varphi_n \rangle \qquad (3.7)$$

for each n, $p \in N$. Henceforth we shall regard these bijections as identities and use $\langle \varphi_1, \cdots, \varphi_n \rangle$ both for the n-list of morphisms $\varphi_1, \cdots, \varphi_n$, and the morphism $[n] \to [p]$ determined by it.

Thanks to the equation

$$\langle \varphi_1, \cdots, \varphi_m \rangle \langle \psi_1, \cdots, \psi_n \rangle = \langle \langle \varphi_1, \cdots, \varphi_m \rangle \psi_1, \cdots, \langle \varphi_1, \cdots, \varphi_m \rangle \psi_n \rangle, \tag{3.8}$$

the composition rule in a theory T is reduced to the family of maps

$$T_p^{[n]} \times T_n \to T_p : (\langle \varphi_1, \cdots, \varphi_m \rangle, \psi) \mapsto \langle \varphi_1, \cdots, \varphi_m \rangle \psi \tag{3.9}$$

one for each n, $p \in N$.

(3.10) DEFINITION. For any theory T we call the graded set

$$\underline{T} = (T_0, T_1, \cdots, T_p = T([1], [p]), \cdots)$$

the (*graded*) *carrier* of T.

(3.11) FACT. Let $\underline{T} = (T_0, T_1, \cdots,)$ be any graded set provided with a family of maps (3.9) such that:

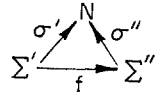
(i) $\check{S}_p \subseteq T_p$ for each $p \in N$; and

(ii) for any $\langle \varphi_1, \cdots, \varphi_n \rangle \in T_p^{[n]}$ and any $1 \leqq i \leqq n$ we have

$$\langle \varphi_1, \cdots, \varphi_n \rangle i_n = \varphi_i .$$

Then the category T defined by $T([n], [p]) = T_p^{[n]}$ and whose composition rule is defined by (3.8) is a theory with carrier $\underline{T}$.

(3.12) Let S^w be the *category of graded sets* with objects the graded sets (Σ, σ) and with morphisms $f: (\Sigma', \sigma') \to (\Sigma'', \sigma'')$, one for each map $f: \Sigma' \to \Sigma''$ for which



commutes. We often refer to the S^w-morphisms as *graded maps*.

(3.13) DEFINITION. The *category of theories Th*, is the category whose objects are theories and whose morphisms are functors which leave fixed the subcategory $\check{S}$.

(3.14) FACT. The restriction $\underline{h}: \underline{T}' \to \underline{T}''$ of a *Th*-morphism h to carriers is a graded map which preserves $\check{S}$-morphisms and satisfies the expected condition that

$$\underline{h}(\langle \varphi_1, \cdots, \varphi_n \rangle \psi) = \langle h(\varphi_1), \cdots, h(\varphi_n) \rangle h(\psi)$$

for all $\langle \varphi_1, \cdots, \varphi_n \rangle \in T_p'^{[n]}$ and $\psi \in T_m'$. The graded map $\underline{h}$ is called the *underlying graded map* of h. A graded map $k: \underline{T}' \to \underline{T}''$ is an underlying graded map of a Th-morphism $h: T' \to T''$ (and then

$$h(\langle \varphi_1, \cdots, \varphi_n \rangle) = \langle k(\varphi_1, \cdots, k(\varphi_n)\rangle,)$$

iff k preserves the $\check{S}$-morphisms and satisfies the above expected condition.

(3.15) FACT. The assignment

$$T \longmapsto \underline{T} \quad \text{and} \quad h \longmapsto \underline{h}$$

determines a functor $\mathfrak{U}: Th \to S^\omega$ called the *forgetful functor* on Th. It has the following properties:

(i) it is *locally injective*, i.e., for any two Th-morphisms $h_1, h_2: T' \to T''$

$$\underline{h}_1 = \underline{h}_2 \quad \text{iff} \quad h_1 = h_2,$$

but not injective (for there exist different theories T' and T'' with $\underline{T}' = \underline{T}''$);

(ii) it *relates the basic types of morphisms*, i.e., a Th-morphism is injective, surjective, or bijective iff its underlying graded map is.

Consequently, Th is a *regular* category, i.e., each Th-morphism h is the (unique, up to an appropriate isomorphism) composition $h = je$ of a surjective Th-morphism e followed by an injective Th-morphism j.

(3.16) Theories can be viewed as "sets with additional structure" in two manners. The obvious one regards a theory T as a set of *all* the morphisms of T together with the composition rule of T with the property of being a morphic extension of $\check{S}$ with $[n]$ remaining a coproduct of n copies of [1] as stated in 3.4. According to this point of view T is first of all a category. On the other hand, the forgetful functor $\mathfrak{U}: Th \to S^\omega$ is defined so that one may consider a theory T as the graded set $\underline{T}$ enjoying the structure specified by the maps (3.9) as in 3.11. Consequently, Th can be regarded as a *concrete category* in two complementary fashions. While we feel that the "natural" carrier of a theory T is the graded set $\underline{T}$, it is still useful to retain the attitude towards a theory as a category. Definitions 3.4 and 3.13 support this point of view.

(3.16) In any concrete category C (cf. MacLane and Birkhoff, 1967) a *C-congruence* on an object A is any *kernel equivalence* of a morphism $h: A \to B$ i.e., the equivalence relation E_h defined by

$$\alpha E_h \alpha' \quad \text{iff} \quad h(\alpha) = h(\alpha')$$

(cf. MacLane and Birkhoff, 1967). Thus an *S^w-congruence* on a graded set (Σ, σ) is an equivalence E Σ for which

$$fEf' \Rightarrow \sigma(f')$$

holds for all $f, f' \in \Sigma$ (i.e., an S^w-congruence on (Σ, σ) is any *refinement* of the kernel equivalence of σ).

(3.17) A *Th-congruence* on a theory T is the kernel equivalence of a *Th*-morphism with domain T. The characterization of the *Th*-congruences due to the view of a theory T as the graded set $\underset{\sim}{T}$ with additional structure is immediate and it is as follows. Since *Th*-morphisms are determined by S^ω-morphisms which satisfy additional requirements (cf. 3.14), it follows that *Th*-congruences on a theory T are determined by the S^ω-congruences on $\underset{\sim}{T}$ which satisfy conditions comparable to 3.14. So, for an arbitrary S^ω-congruence E on the graded carrier $\underset{\sim}{T}$ of a theory T, denote by E^* the equivalence defined on the morphisms of T by: if $\varphi, \varphi' \in T_p^{[n]}$, then

$$\varphi E^* \varphi' \quad \text{iff for all } i_n\text{: } \varphi i_n E \varphi' i_n .$$

Clearly, E^* is a *Th*-congruence iff:

(i) E is the identity on $\check{S}$;

(ii) if $\langle \varphi_1, \cdots, \varphi_n \rangle, \langle \varphi_1, \cdots, \varphi_n{}' \rangle \in T_p^{[n]}$ and $\psi, \psi' \in T_p$ are such that $\psi E \psi'$ and $\varphi_i E \varphi_i{}'$ for all $i = 1, \cdots, n$; then

$$\langle \varphi_1, \cdots, \varphi_n \rangle \psi E \langle \varphi_1{}', \cdots, \varphi_n{}' \rangle \psi'.$$

Furthermore, if E is a *Th*-congruence on T and $\underset{\sim}{E}$ is the restriction of E to $\underset{\sim}{T}$, then $(\underset{\sim}{E})^* = E$.

(3.18) For any set X and an equivalence E on X we denote by $[x]_E$ the E-equivalence class of $x \in X$, by X/E the set of all E-equivalence classes, and by $X \to X/E$ the *canonical* E-surjection $x \longmapsto [x]_E$ whose kernel equivalence is obviously E. In any concrete category C, if E is a C-congruence on an object A, then A/E enjoys the structure of an object of C such that the surjection $A \to A/E$ determines a morphism of C.

Thus, we have

(3.19) FACT. (i) If E is a *Th*-congruence on a theory T, the collection of the sets $[n]$ as objects with morphisms $[\varphi]_E\text{:}[n] \to [p]$, for $\varphi\text{:}[n] \to [p]$ in T, determine a theory T/E, the *quotient* theory of T modulo E, and the map

$$T \to T/E\text{:}\varphi \longmapsto [\varphi]_E$$

is a surjective Th-morphism whose kernel equivalence is E. The composition rule of T/E is given, of course, by

$$[\varphi]_E[\psi]_E = [\varphi\psi]_E .$$

(ii) If $\{E_i : i \in I\}$ is any family of Th-congruences on a theory T indexed by a set I, then (a) the intersection $\bigcap\{E_i : i \in I\}$ is also a Th-congruence on T; (b) the equivalence closure of the union $\bigcup\{E_i : i \in I\}$ is a Th-congruence on T so long as this closure does not make equivalent distinct $\check{S}$-morphisms.

We end our discussion of theories by exposing the natural directed system (cf. Mitchell, 1965, p. 47) structure of the carrier of any given theory T.

(3.19) The obvious insertions $j:[p] \to [p']:i \mapsto i$ for $p \leqq p'$ determine injections

$$j^*: T_p \to T_{p'}: \varphi \mapsto j\varphi$$

which impose the structure of a directed system on the carrier of T. In certain contexts it is natural to regard these injections as insertions. Thus, if we consider the operative theory $T_{\mathfrak{A}}$ of a Σ-algebra $\mathfrak{A}$ with carrier Q, this corresponds to regarding a function $\varphi: Q^{[p]} \to Q^{[n]}$ as a function $\hat{j}\varphi: Q^{[p']} \to Q^{[n]}$ for any $p' \geqq p$, simply by adding "dummy variables" $q_{p+1}, \cdots, q_{p'}$. This observation underlies our statement prior to 2.1 that every $\Sigma_{[p]}$-DOAG is a $\Sigma_{[p']}$-DOAG whenever $p' \geqq p$.

4. THE ALGEBRAS OF A THEORY

After 3.5 we noted that each nontrivial Σ-algebra $\mathfrak{A}$ gives rise to a cofunctor on $T_{\mathfrak{A}}$, the operative theory of $\mathfrak{A}$, with values in the category S of sets. The "image" of this cofunctor is the category $\langle\mathfrak{A}\rangle$ which is the completion of $\mathfrak{A}$. The motivation for defining the notion of a theory is that with each theory T, the algebras whose "structure is indicated and described by T" are defined by means of certain cofunctors on T. The idea is that the morphisms of T are to be thought of as (not necessarily unique) labels for functions of an algebra completion. The relations that hold between the morphisms of T become, therefore, laws satisfied by all the algebras associated with T.

(4.1) DEFINITION. Let Q be an arbitrary set. By an *algebra* (*with carrier* Q) we mean the completion of any Σ-algebra with carrier Q for some graded set Σ. If T is a theory, then by a *T-algebra* $\mathfrak{A}$ (*with carrier* Q) we mean any cofunctor

$$\mathfrak{A}: T \to S$$

such that for every $\check{S}$-morphism $\varphi:[n] \to [p]$ in T, $\mathfrak{A}(\varphi)$ is the map

$$\hat{\varphi}:Q^{[p]} \to Q^{[n]}:(q_1, \cdots, q_p) \mapsto (q_{\varphi(1)}, \cdots, q_{\varphi(n)}),$$

and, therefore, $\mathfrak{A}([n]) = Q^{[n]}$ for each object $[n]$ of T.

Thus, for each nontrivial Σ-algebra $\mathfrak{A}$ the co-isomorphism $T_{\mathfrak{A}} \to \langle\mathfrak{A}\rangle \subseteq S$ is, in fact, a $T_{\mathfrak{A}}$-algebra when considered as an injective cofunctor from $T_{\mathfrak{A}}$ to S. Also, it is clear that for any T-algebra $\mathfrak{A}:T \to S$ with carrier Q, $\mathfrak{A}(T)$, the image category of $\mathfrak{A}$, is an algebra with carrier Q, called the *image algebra* of $\mathfrak{A}$ and is denoted also by $\langle\mathfrak{A}\rangle$.

By definition, each Σ-algebra gives rise to a certain T-algebra for some theory T. It is natural to inquire whether there exists a theory, $\mathfrak{F}(\Sigma)$, such that for any Σ-algebra $\mathfrak{A}$, $\langle\mathfrak{A}\rangle$ is the image algebra of some $\mathfrak{F}(\Sigma)$-algebra. The constructions employed in Part I imply the existence of a uniform theory $\mathfrak{F}(\Sigma)$ for all Σ-algebras. We defer the details to Section 5.

(4.2) FACT. Given a nontrivial T-algebra $\mathfrak{A}:T \to S$ (i.e., the cardinality of whose carrier exceeds 1) the kernel equivalence $E_{\mathfrak{A}}$ of $\mathfrak{A}$ is a Th-congruence on T. The quotient theory $T_{\mathfrak{A}} = T/E_{\mathfrak{A}}$ is called the *operative theory* of $\mathfrak{A}$ and is co-isomorphic to the image algebra of $\mathfrak{A}$. In fact, $T_{\mathfrak{A}}$ is the minimal theory for the image algebra of $\mathfrak{A}$; i.e., for any T'-algebra $\mathfrak{A}'$ with image equal to that of $\mathfrak{A}$ there exists a unique surjective Th-morphism $T' \to T_{\mathfrak{A}}$ which factors $\mathfrak{A}'$. Note also that for any T-algebra $\mathfrak{A}:T \to S$ and any [surjective] Th-morphism $e:T' \to T$, $\mathfrak{A}e:T' \to S$ is a T'-algebra [with image equal to that of $\mathfrak{A}$].

We now rephrase 4.1 in a way which makes more evident the relevance of the concept of a T-algebra to generalized automata theory. For any two entities (e.g. sets) X and Y, $X \supseteq\!\!\to Y$ denotes a map of a part of X into Y; i.e., a *partial assignment* on X with values in Y. For any set Q we denote by Q^* the graded set with $Q_p{}^* = Q^{[p]}$.

(4.3) FACT. Let T be a theory and Q a set. Then, a T-algebra $\mathfrak{A}:T \to S$ with carrier Q is completely specified by any partial assignment

$$\mathfrak{A}:Q^* \times T \supseteq\!\!\to Q^*:(x, \varphi) \mapsto x\cdot\varphi \tag{4.4}$$

(where $x\cdot\varphi$ denotes the value of $\mathfrak{A}(\varphi)$ on x), which satisfies the conditions:

(i) $x\cdot\varphi$ is defined iff, for some n, $p \in N$: $x \in Q^{[p]}$ and $\varphi \in T_p^{[n]}$; and then $x\cdot\varphi \in Q^{[n]}$;

(ii) for any $x \in Q^*$, if φ is an $\check{S}$-morphism in T for which $x\cdot\varphi$ is defined, then $x\cdot\varphi = x\varphi$;

(iii) for any $x \in Q^*$ and morphisms φ and ψ in T such that $x \cdot \varphi$ and $(x \cdot \varphi) \cdot \psi$ are defined, then $(x \cdot \varphi) \cdot \psi = x \cdot (\varphi\psi)$. Note also that for a T-algebra $\mathfrak{A}$ as in 4.4, the congruence $E_{\mathfrak{A}}$ is essentially defined by "$x \cdot \varphi = x \cdot \varphi'$ for all $x \in Q^*$."

(4.5) NOTATION. For any two graded sets Σ, Σ', $[\Sigma, \Sigma']$ denotes the union of the product sets $\Sigma_p \times E_p'$ for all $p \in N$. For any T-algebra

$$\mathfrak{A}: Q^* \times T \supseteq \rightarrow Q^*$$

we call

$$\underset{\sim}{\mathfrak{A}}: [Q^*, \underset{\sim}{T}] \rightarrow Q: (x, \varphi) \mapsto x \cdot \varphi$$

the *underlying action* of $\mathfrak{A}$.

(4.6) FACT. For an arbitrary theory T and any set Q, an *arbitrary* map

$$\alpha: [Q^*, \underset{\sim}{T}] \rightarrow Q: (x, \varphi) \mapsto x \cdot \varphi$$

is the underlying action of a T-algebra $\mathfrak{A}_\alpha$ (and then

$$\mathfrak{A}_\alpha: Q^* \times T \supseteq \rightarrow Q^*: (x, \varphi) \mapsto x \cdot \varphi$$

is given by

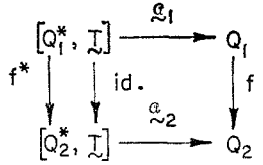
$$x \cdot \langle \varphi_1, \cdots, \varphi_n \rangle = (x \cdot \varphi_1, \cdots, x \cdot \varphi_n),)$$

iff α satisfies the requirements:

(i) $x \cdot i_n = x i_n$ for all $x \in Q^*$ and $i_n \in \check{S}_n$;

(ii) $x \cdot (\langle \varphi_1, \cdots, \varphi_n \rangle \psi) = (x \cdot \varphi_1, \cdots, x \cdot \varphi_n) \cdot \psi$ for all $\langle \varphi_1, \cdots, \varphi_n \rangle \in T_p^{[n]}$ and $\psi \in T_p$.

(4.7) DEFINITION. *The category T^b of T-algebras* has for objects all T-algebras and for morphisms $f: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ (where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two T-algebras with carriers Q_1 and Q_2 respectively), the mappings $f: Q_1 \rightarrow Q_2$ which render commutative the diagram



where

$$f^*: Q_1^* \rightarrow Q_2^*: x \mapsto fx.$$

(4.8) FACT. $f: Q_1 \to Q_2$ determines a $T^\flat$-morphism $f: \mathfrak{A}_1 \to \mathfrak{A}_2$ iff $f^*: {Q_1}^* \to {Q_2}^*$, as a graded map, determines a natural transformation $\mathfrak{A}_1 \to \mathfrak{A}_2$ of the cofunctors $\mathfrak{A}_1: T \to S$ and $\mathfrak{A}_2: T \to S$.

(4.9) Clearly, for each theory T the category $T^\flat$ is a concrete category. Hence, we assume as known the notions of a $T^\flat$-congruence on a T-algebra, of a *quotient* T-algebra $\mathfrak{A}/E$ of $\mathfrak{A}$ modulo E, and of the canonical surjective $T^\flat$-morphism $\mathfrak{A} \to \mathfrak{A}/E$ whose kernel equivalence is E. Note that for any family $\{E_i: i \in I\}$ of $T^\flat$-congruences on a T-algebra $\mathfrak{A}$ (indexed by I), both the intersection and the equivalence closure of the union of $\{E_i: i \in I\}$ are $T^\flat$-congruences on $\mathfrak{A}$. For a detailed study of the categories of a form $T^\flat$ (as "varieties of universal algebras") the reader is referred to Cohn (1965).

(4.10) For each theory T, the *forgetful functor* $\mathfrak{U}: T^\flat \to S$ is the functor which assigns to each T-algebra its carrier, and to each Th-morphism $f: \mathfrak{A}_1 \to \mathfrak{A}_2$ its *underlying* mapping $f: Q_1 \to Q_2$. This functor is *locally injective* (i.e., given any two T-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with carrier Q_1 and Q_2 then the restriction of $\mathfrak{U}$

$$T^\flat(\mathfrak{A}_1, \mathfrak{A}_2) \to S(Q_1, Q_2)$$

is injective) but not always injective. (It is the identity functor of S for $T = \check{S}$, since $\check{S}^\flat = S$, but in general, it is not injective; e.g., there are distinct monoids with identical carriers.) $T^\flat$-morphisms are said to be *surjective*, *injective*, or *bijective* depending whether their underlying mappings are. While the bijective and the injective $T^\flat$-morphisms are the invertible and the monic morphisms (respectively) in $T^\flat$, in general it is only true that the surjective $T^\flat$-morphisms are epic in $T^\flat$. For example, the insertion of the cyclic free monoid into the additive monoid of integers is epic in $T^\flat$ for the suitable theory T of monoids, it is not surjective. The reader who is interested in the problem of when epic morphisms in $T^\flat$ are surjective is referred to Linton (1965). We will demonstrate now the existence of a functor $F_T: S \to T^\flat$ *adjoint* to $\mathfrak{U}: T^\flat \to S$ (cf. MacLane and Birkoff, 1967) which assigns to each set V the *free* T-algebra $F_T(V)$ on V.

(4.11) For any theory T and any set V consider the equivalence $\equiv$ defined on the set $[V^*, T]$ (cf. 4.5) as the equivalence closure of the relation $(x, \varphi) \sim (x', \varphi')$ which holds iff $x \in V^{[p]}$, $x' \in V^{[p']}$, $\varphi \in T_p$ and $\varphi' \in T_{p'}$ for some $p, p' \in N$, and there exists an $\check{S}$-morphism $\tau: [p] \to [p']$ such that $x = x'\tau$ and $\varphi' = \tau\varphi$. Put differently, $(x, \varphi) \equiv (x', \varphi')$ iff in *any* T-algebra with carrier V we have $x \cdot \varphi = x' \cdot \varphi'$.

Let $V \cdot T$ be the set $[V^*, T]$ reduced by the equivalence $\equiv$. The T-algebra $F_T(V)$ is defined as the partial assignment

$$F_T(V) : (V \cdot T)^* \times T \supseteq \rightarrow V \cdot T$$

determined by

$$(((x_1, \varphi_1), \cdots, (x_n, \varphi_n)), \psi) \longmapsto (x_1 x_2 \cdots x_n, (\varphi_1 \times \cdots \times \varphi_n)\psi)$$

for $x_1 \in V^{[p_1]}, \cdots, x_n \in V^{[p_n]}$, $\varphi_1 \in T_{p_1}, \cdots, \varphi_n \in T_{p_n}$, $\psi \in T_n$, and where $\varphi_1 \times \cdots \times \varphi_n \in T_{p_1}^{[n]} + \cdots + p_n$ is defined as $\langle \tau_1 \varphi_1, \cdots, \tau_n \varphi_n \rangle$ where for $j = 1, \cdots, n$:

$$\tau_j : [p_j] \rightarrow [p_1 + \cdots + p_n] : i \longmapsto (\sum_{k<j} p_k) + i.$$

The reader may check that $F_T(V)$ is well defined and it is in fact a T-algebra with carrier $V \cdot T$ and that for the injection

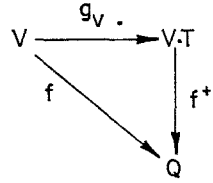
$$g_V : V \rightarrow V \cdot T : v \longmapsto [(v, 1_1)]_{\equiv}$$

we have:

(4.12) PROPOSITION. *Let V be an arbitrary set and $\mathfrak{A}$ an arbitrary T-algebra with carrier Q. Then for any function $f : V \rightarrow Q$ there exists a unique T^b-morphism*

$$f^\dagger : F_T(V) \rightarrow \mathfrak{A}$$

for which



commutes.

(4.13) Note that in case $V = [n] = \{1, 2, \cdots, n\}$ for some $n \in N$, then

$$V \cdot T \rightarrow T_n : [(i, \varphi)]_{\equiv} \longmapsto i_n \varphi$$

is a well-defined bijection and under this bijection the underlying action of $F_T([n])$ is simply given by the composition rule of T itself. Clearly, the insertion $g_{[n]}$ associated with $F_T([n])$ is

$$g_{[n]} : [n] \rightarrow T_n : i \longmapsto i_n .$$

We observe therefore, that a theory T can be regarded as a *disjoint*

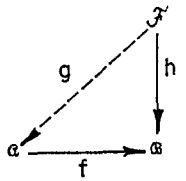
union of the free T-algebras

$$F_T([0]), F_T([1]), \cdots$$

even though T itself is not a T-algebra unless we modify our notion of T-algebra defined in 4.1 to replace Q by a *graded set* with structure comparable to that of a carrier of a theory.

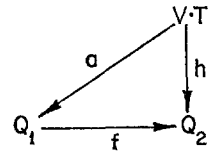
We now give a classical corollary of Proposition 4.12:

(4.14) COROLLARY. *Let $\mathfrak{F}$ be any free T-algebra. Then, for any $T^\flat$-morphism $h:\mathfrak{F} \to \mathfrak{B}$ and any surjective $T^\flat$-morphism f: $\mathfrak{A} \to \mathfrak{B}$ there exists at least one $T^\flat$-morphism $g:\mathfrak{F} \dashrightarrow \mathfrak{A}$ for which*



commutes.

Proof. Let $\mathfrak{F} = F_T(V)$ and let the carriers of $\mathfrak{A}$ and $\mathfrak{B}$ be Q_1 and Q_2 respectively. Then, there exists a function $a: V \cdot T \to Q_1$ for which



commutes in S. Consider the $T^\flat$-morphism

$$g = (ag_V)^\dagger : \mathfrak{F} \to \mathfrak{A}$$

whose existence is guaranteed by 4.12. It follows from 4.12 that $fg = f(ag_V)^\dagger = h$ in $T^\flat$. Q.E.D.

(4.15) *Remark.* Due to 4.14 we say that the free T-algebras are $\mathfrak{U}$-*projective*. Since the epic morphisms in $T^\flat$ are not necessarily surjective $T^\flat$-morphisms, we cannot identify the $\mathfrak{U}$-projective objects in $T^\flat$ with the *projective* objects in $T^\flat$ without additional arguments. Though it follows that every projective T-algebra must be $\mathfrak{U}$-projective, the example of the category of monoids shows that $\mathfrak{U}$-projective T-algebras need not be projective (see Give'on, 1967). The example of the category of R-modules where $R = Z \oplus Z$ shows that $\mathfrak{U}$-projective T-algebras need not be free (see MacLane, 1963).

(4.16) FACT. For any (surjective) Th-morphism $H': T \to T'$, denote by $H_p': T_p \to T_p'$ the restriction of H' to T_p, and by $E_{H_p'}$ the kernel equivalence of H_p'. Then, $E_{H_p'}$ is a $T^\flat$-congruence on the T-algebra $F_T([p])$. Denote by $\mathfrak{A}_{H_p'}$ the T-algebra $F_T([p])/E_{H_p'}$. Then for any (surjective) Th-morphism $H: T' \to T''$, $H_p: T_p' \to T_p''$ determines a (surjective) $T^\flat$-morphism

$$H_p: \mathfrak{A}_{H_p'} \to \mathfrak{A}_{H_p''}$$

where $H'' = HH'$. Note that for a surjective Th-morphism $H': T \to T'$ the carrier of $\mathfrak{A}_{Hp'}$ can be taken to be T_p' and then the underlying action of $\mathfrak{A}_{Hp'}$ is given by

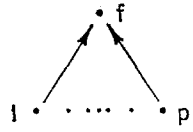
$$\langle \varphi_1, \cdots, \varphi_n \rangle \cdot \psi = \langle \varphi_1, \cdots, \varphi_n \rangle H(\psi)$$

for all $\langle \varphi_1, \cdots, \varphi_n \rangle \in (T')_p^{[n]}$ and $\psi \in T_p$.

5. FREE THEORIES

We demonstrate now the existence of a functor $\mathfrak{F}: S^\omega \to Th$ adjoint to the forgetful functor $\mathfrak{U}: Th \to S^\omega$ on the category Th of theories (cf. 3.15). For any graded set Σ, the theory $\mathfrak{F}(\Sigma)$, which is called the *free theory on* Σ, is constructed directly from the $\Sigma_{[p]}$-trees discussed in Section 2, and it is characterized in Th in much the same way as a free monoid is characterized in the category of monoids.

(5.1) In order to construct the free theory $\mathfrak{F}(\Sigma)$ on Σ from the $\Sigma_{[p]}$-trees, we have to identify $\Sigma_{[p]}$-trees that are isomorphic from the graph-theoretic point of view, and to associate with each (representative of a graph-theoretic isomorphism type of a) $\Sigma_{[p]}$-tree an external label n with $n \geqq p$, such that, with this label, each Σ_p-tree determines a unique morphism $[1] \to [n]$ in the carrier of $\mathfrak{F}(\Sigma)$. Thus, henceforth, by a $\Sigma_{[p]}$-tree we mean a pair $(\mathfrak{I}, p)$ where $\mathfrak{I}$ is an isomorphism-type of a $\Sigma_{[p]}$-tree, usually represented by a specific $\Sigma_{[p]}$-tree. Accordingly, we identify the $\check{S}$-morphism $i_p: [1] \to [p]: 1 \mapsto i$ with the $\Sigma_{[p]}$-tree represented by the single-node tree labeled by i. For any $f \in \Sigma_p$ we identify f with the $\Sigma_{[p]}$-tree represented by the tree



which becomes the single-node tree labeled by f in case $p = 0$. Conse-

quently, the carrier of $\mathfrak{F}(\Sigma)$ can be described inductively by:

(i) for any $p \in N$: $\check{S}_p \subseteq \mathfrak{F}(\Sigma)_p$ and $\Sigma_p \subseteq \mathfrak{F}(\Sigma)_p$;

(ii) if $f \in \Sigma_k$ and $(\varphi_1, \cdots, \varphi_k) \in (\mathfrak{F}(\Sigma)_p)^{[k]}$, then $(\varphi_1, \cdots, \varphi_k)f$ is the Σ_p-tree obtained from the $\Sigma_{[k]}$-tree f, as in (2.1), by replacing each initial node of f labeled i by an appropriate copy of p_i. Note that the $\Sigma_{[p]}$-tree $(\varphi_1, \cdots, \varphi_k)f$ can be denoted uniquely by the formal term "$(\varphi_1, \cdots, \varphi_k)f$".

In order to complete the definition of $\mathfrak{F}(\Sigma)$, we take $\mathfrak{F}(\Sigma)([n], [p]) = (\mathfrak{F}(\Sigma)_p)^{[n]}$, and define the maps

$$(\mathfrak{F}(\Sigma)_p)^{[n]} \times \mathfrak{F}(\Sigma)_n \to \mathfrak{F}(\Sigma)_p : ((\varphi_1, \cdots, \varphi_n), \psi) \mapsto (\varphi_1, \cdots, \varphi_n)\psi$$

where $(\varphi_1, \cdots, \varphi_n)\psi$ is obtained from ψ, as in (2.1), by replacing each initial node of ψ labeled i by an appropriate copy of φ_i. By 3.11 it follows that $\mathfrak{F}(\Sigma)$ is a theory.

Note that the inductive definition of the carrier of $\mathfrak{F}(\Sigma)$ indicates that it can be defined as a graded set of *formal terms* where each formal term is either an element of Σ, a morphism in the carrier of $\check{S}$, or of the form $(\varphi_1, \cdots, \varphi_k))f$ where $\varphi_1, \cdots, \varphi_k$ are appropriate formal terms and f is an appropriate element of Σ.

(5.2) LEMMA. *Let Σ be a graded set and Q a set. Then for each mapping*

$$\alpha : [Q^*, \Sigma] \to Q$$

there exists a unique $\mathfrak{F}(\Sigma)$-algebra

$$\mathfrak{A}_\alpha : Q^* \times \mathfrak{F}(\Sigma) \supseteq \to Q^*$$

whose underlying action agrees with α (i.e., *for each $x \in Q^{[p]}$ and $f \in \Sigma_p$: $\alpha(x, f) = x \cdot f$*).

Proof. Let $\mathfrak{A}$ be the Σ-algebra determined by α, and recall 2.2. For each $x \in Q^{[p]}$ and $\psi \in \mathfrak{F}(\Sigma)_p$ define

$$x \cdot \psi = (x)\varphi_\psi^{\mathfrak{A},[\nu]},$$

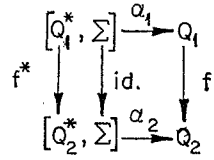
where ν is the terminal node of the $\Sigma_{[p]}$-tree ψ. By 4.6 it is easily verified that this defines an $\mathfrak{F}(\Sigma)$-algebra $\mathfrak{A}_\alpha$ whose underlying action agrees with α. The uniqueness of $\mathfrak{A}_\alpha$ also follows easily from 2.2 and 4.6. Q.E.D.

(5.3) COROLLARY. *An algebra is the completion of a Σ-algebra iff it is the image algebra of an $\mathfrak{F}(\Sigma)$-algebra.*

(5.4) COROLLARY. *Let Σ be a graded set, Q_1 and Q_2 sets and let*

$$\alpha_i : [Q_i^*, \Sigma] \to Q_i \,;\, i = 1, 2;$$

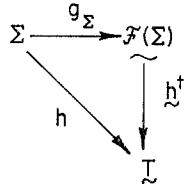
be any two maps. Then, a function $f: Q_1 \to Q_2$ *determines an* $\mathfrak{F}(\Sigma)^\flat$*- morphism* $f: \mathcal{A}_{\alpha_1} \to \mathcal{A}_{\alpha_2}$ *iff*



commutes.

The justification for calling $\mathfrak{F}(\Sigma)$ the *free* theory on Σ is based on the following characteristic property of $\mathfrak{F}(\Sigma)$.

(5.5) PROPOSITION. *For any theory* T *and any graded map* $h: \Sigma \to \underset{\sim}{T}$*, there exists a unique morphism* $h^\dagger: \mathfrak{F}(\Sigma) \to T$ *of theories for which*



commutes (in S^ω*) where*

$$g_\Sigma: \Sigma \to \underset{\sim}{\mathfrak{F}(\Sigma)}$$

is the graded map which identifies each $f \in \Sigma_p$ *with the* $\Sigma_{[p]}$*-tree denoted by* f (cf. 5.1).

Proof. For a given $h: \Sigma \to \underset{\sim}{T}$ define $h': \underset{\sim}{\mathfrak{F}(\Sigma)} \to \underset{\sim}{T}$ by induction on the structure of $\underset{\sim}{\mathfrak{F}(\Sigma)}$ as in 5.1:

(i) $h'(\varphi) = \varphi$ for each $\check{S}$-morphism in $\underset{\sim}{\mathfrak{F}(\Sigma)}$ and $h'(f) = h(f)$ for each element f in Σ;

(ii) $h'((\varphi_1, \cdots, \varphi_k)f) = \langle h'(\varphi_i), \cdots, h'(\varphi_k)\rangle\, h(f)$.

It follows now by induction and by 3.14 and 3.15 that h' is the underlying graded map of a unique Th-morphism $h^\dagger: \mathfrak{F}(\Sigma) \to T$ for which $\underset{\sim}{h^\dagger} g_\Sigma = h$. The uniqueness of h' also follows by induction on the structure of the carrier of $\mathfrak{F}(\Sigma)$.

(5.6) Several conclusions can be drawn (in a classical manner) from 5.5. First, it follows that the assignment $\Sigma \mapsto \mathfrak{F}(\Sigma)$ determines a functor

$$\mathfrak{F}: S^\omega \to Th;$$

for each graded map $f:\Sigma' \to \Sigma''$, $\mathfrak{F}(f):\mathfrak{F}(\Sigma') \to \mathfrak{F}(\Sigma'')$ is defined by

$$\mathfrak{F}(f) = (g_{\Sigma''}f)^\dagger. \tag{5.7}$$

Proposition 5.5 implies that $\mathfrak{F}$ is in fact a functor and that it is adjoint (cf. MacLane and Birkhoff, 1967) to the forgetful functor $\mathfrak{U}:Th \to S^\omega$. Furthermore, the proof of 4.14 as a corollary of 4.12 can be applied here to derive that every free theory is $\mathfrak{U}$-projective. On the other hand, by using techniques similar to those applied to the category of monoids (e.g., as in Give'on, 1967), we will prove now that every $\mathfrak{U}$-projective theory must be free.

(5.8) Let T be a theory and $g:\Sigma \to \underset{\sim}{T}$ an injective graded map (say, Σ is a graded subset of the carrier of T). Consider the Th-morphism

$$g^\dagger:\mathfrak{F}(\Sigma) \to T.$$

The image of $g^\dagger$ is said to be the *subtheory of T generated* by Σ and is denoted by $T\langle\Sigma\rangle$. If $g^\dagger$ is surjective, then $g:\Sigma \to \underset{\sim}{T}$ is said to be a *generating set* of the theory T. Clearly, if Σ is a graded subset of $\underset{\sim}{T}$, then the carrier of $T\langle\Sigma\rangle$ consists of the $\check{S}$-morphisms that belong to $\underset{\sim}{T}$, the morphisms that belong to Σ, and the "products" $(\varphi_1, \cdots, \varphi_n)\psi$ of morphisms $\varphi_1, \cdots, \varphi_n$ which belong to $T\langle\Sigma\rangle$ and of ψ which belongs to Σ.

Consider the morphisms in the carrier of the free theory $\mathfrak{F}(\Sigma')$. Let Σ'' be any graded subset of $\underset{\sim}{\mathfrak{F}(\Sigma')}$. A morphism φ in $\underset{\sim}{\mathfrak{F}(\Sigma')}$ is said to be *basic to* Σ'' in $\mathfrak{F}(\Sigma)$ iff it belongs to Σ'' and whenever

$$\varphi = (\varphi_1, \cdots, \varphi_n)f$$

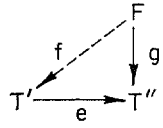
(where f belongs to Σ' and $\varphi_1, \cdots, \varphi_n$ to $\mathfrak{F}(\Sigma')$) we always have that all the $\varphi_1, \cdots, \varphi_n$ are $\check{S}$-morphisms. Thus a morphism in Σ'' is basic to Σ'' iff it is not a "nontrivial product" of other morphisms in Σ''. By induction on the structure of the morphisms in the carrier of a free theory, the reader can easily prove

(5.9) LEMMA. *If T is a subtheory of a free theory F, then the graded set Σ of all the morphisms in $\underset{\sim}{T}$ which are basic to $\underset{\sim}{T}$ in F (or more precisely, the insertion $\Sigma \to \underset{\sim}{T}$) is a generating set of T which is included in any generating set of T.*

We prove now:

(5.10) THEOREM. *A theory F is free iff it is $\mathfrak{U}$-projective. That is, F is free iff for any Th-morphism $g:F \to T''$ and any Th-morphism $e:T' \to T''$ such that $\mathfrak{U}(e):\underset{\sim}{T'} \to \underset{\sim}{T''}$ is surjective* (i.e., *e is surjective*), *there exists at*

least one Th-morphism $f: F \to T'$ *for which*



commutes.

Proof. We noted above that the proof of 4.14 as a corollary of 4.12 can be applied here to derive that every free theory is $\mathfrak{U}$-projective. To prove the converse, note that every $\mathfrak{U}$-projective theory F must be isomorphis to a subtheory of a free theory. For any theory T is a *Th*-morphic image of a free theory: $\underset{\sim}{T} \to T$ is a generating set of T. Hence we have a surjective *Th*-morphism $\mathfrak{F}(\underset{\sim}{F}) \to F$, and since F is $\mathfrak{U}$-projective, there exists at least one *Th*-morphism $F \to \mathfrak{F}(F)$ such that $F \to \mathfrak{F}(\underset{\sim}{F}) \to F$ is the identity on F, and $F \to \mathfrak{F}(\underset{\sim}{F})$ must be injective.

By Lemma 5.9 we know that if F is $\mathfrak{U}$-projective then there exists a generating set $b: \Sigma \to \underset{\sim}{F}$ of F with the property that

$$(\mathfrak{U}(b^\dagger))(\psi) = b(f) \quad \text{iff} \quad \psi = g_\Sigma(f) \tag{5.11}$$

for any ψ in $\mathfrak{F}(\Sigma)$ and f in Σ (for note that $(\mathfrak{U}(b^\dagger))(\psi)$ is a "nontrivial product" iff $\psi = g_\Sigma(f)$). Since $b^\dagger: \mathfrak{F}(\Sigma) \to F$ is surjective, and F is $\mathfrak{U}$-projective, there exists a *Th*-morphism $j: F \to \mathfrak{F}(\Sigma)$ such that

$$b^\dagger j = i_F \tag{5.12}$$

and therefore

$$\mathfrak{U}(b^\dagger)\mathfrak{U}(j)b = b.$$

Now, for any f in Σ we have

$$\mathfrak{U}(b^\dagger)[\mathfrak{U}(j)b(x)] = b(x),$$

which, by 5.11, implies

$$\mathfrak{U}(j)b = g_\Sigma .$$

The last equality implies

$$g_\Sigma = \mathfrak{U}(j)b = \mathfrak{U}(j)(\mathfrak{U}(b^\dagger)g_\Sigma) = \mathfrak{U}(jb^\dagger)g_\Sigma ,$$

and therefore, by 5.5, we have

$$jb^\dagger = i_{\mathfrak{F}(\Sigma)} .$$

Hence, by 5.12, F is isomorphic to $\mathfrak{F}(\Sigma)$. Q.E.D.

6. ELEMENTARY PROPERTIES OF AUTOMATA OVER THEORIES

In Part I we studied Σ-automata as Σ-algebras with output. Since we may identify Σ-algebras with $\mathfrak{F}(\Sigma)$-algebras, we realize that we can introduce the notion of a T-automaton, for an arbitrary theory T, as a T-algebra with output.

(6.1) DEFINITION. For a theory T, a *T-automaton* is a quadruple

$$M = \langle Q, \mathfrak{A}, Y, \lambda \rangle$$

where:

(i) Q and Y are sets (the *state* and *output sets* of M respectively);

(ii) $\mathfrak{A}: Q^* \times T \supseteq \to Q^*$ is a T-algebra with carrier Q (the *algebra* of M); and

(iii) $\lambda: Q \to Y$ is a function (the *output function* of M). The theory T is said to be the *input theory* of M, and in case $\mathfrak{A}$ is nontrivial, $T_{\mathfrak{A}}$ is said to be the *operative theory* of M.

The passage from $\mathfrak{F}(\Sigma)$-automata to T-automata, for an arbitrary theory T, is not only formally justified. Since every theory is a quotient theory of a free theory, the class of all T-automata can be considered as a subclass of those Σ-automata which satisfy certain laws which amount to a Th-congruence E for which $\mathfrak{F}(\Sigma)/E = T$. This is analogous not only to the algebraic point of view but also to the current view in ordinary automata theory: Abelian automata are automata with Abelian input monoids and permutation automata have groups as their input monoids.

The response function of a Σ-automaton was defined based on the point of view that the Σ-algebra of a Σ-automaton, tacitly specifies the "initial states" of the given Σ-automaton (by means of its nullary operators). Following this point of view we define:

(6.2) DEFINITION. By a *response function* on a theory T (with values in a set Y) we mean any function $f: T_0 \to Y$. The *response function* of a T-automaton $M = \langle Q, \mathfrak{A}, Y, \lambda \rangle$ is simply the map

$$T_0 \xrightarrow{f_M} Y = T_0 \xrightarrow{\mathfrak{A}_0} Q \xrightarrow{\lambda} Y \tag{6.3}$$

where $\mathfrak{A}_0$ is the restriction of $\mathfrak{A}$ to $Q^{[0]} \times T_0$ identified with T_0 (i.e., $f_M(\varphi) = \lambda(\Lambda \cdot \varphi)$ where $Q^{[0]} = \{\Lambda\}$). We say that M *realizes* f iff $f = f_M$.

Looking for an automaton which realizes f is simply looking for factorizations of f like that of (6.3). Note also that

$$\mathfrak{A}_0: T_0 \to Q: \varphi \mapsto \Lambda \cdot \varphi$$

(as in 6.3), is the underlying function of the unique morphism

$$F_T([0]) \to \mathfrak{A}$$

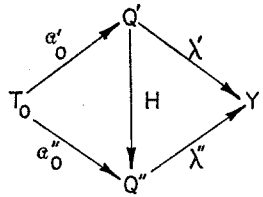
of T-algebras.

In fact, the realizations of a response function $f: T_0 \to Y$ form a category:

(6.4) The *category of* (T-*automaton*) realizations of f has for objects all T-automata which realize f, and for morphisms, say

$$H: \langle Q', \mathfrak{A}', Y, \lambda' \rangle \to \langle Q'', \mathfrak{A}'', Y, \lambda'' \rangle$$

those $T^\flat$-morphisms $H: \mathfrak{A}' \to \mathfrak{A}''$ for which



(6.5)

commutes.

(6.6) A realization $M = \langle Q, \mathfrak{A}, Y, \lambda \rangle$ of f is said to be surjective iff $\mathfrak{A}_0: T_0 \to Q$ is surjective. Since $\mathfrak{A}_0$ is the underlying algebra of the unique $T^\flat$-morphism $F_T([0]) \to \mathfrak{A}$, every T-automaton M has a (unique) subautomaton which is a surjective realization of f_M. We denote by $f/T^\flat$ the full subcategory of the surjective realizations of f. The objects of $f/T^\flat$ can be identified as the factorizations of f into a surjective $T^\flat$-morphism followed by a map, and they are completely determined by the $T^\flat$-congruences on $F_T([0])$ which refine the kernel equivalence of f on T_0, the carrier of $F_T([0])$. Consequently, $f/T^\flat$ is a *partial* order category (i.e., for any two objects r_1 and r_2 of $f/T^\flat$ there exists at most one morphism $H: r_1 \to r_2$, determined, in fact, by a surjective $T^\flat$-morphism) and to look for a minimal realization $M(f)$ of f is simply to look for a terminal object in $f/T^\flat$.

(6.7) PROPOSITION. *For any response function $f: T_0 \to Y$ on the theory T, the category $f/T^\flat$ has an initial object* (*called the* ***free realization of*** f) *and a terminal object $M(f)$* (*called the* ***minimal realization of*** f).

Proof. The initial object of $f/T^\flat$ is obviously the free algebra $F_T([0])$ equipped with $f: T_0 \to Y$ as an output function. We shall shortly give an automaton-theoretic construction of the terminal object of $f/T^\flat$. Here we just note that the simplest proof for the existence of $M(f)$ is based

on the fact that $f/T^\flat$ can be regarded as a directed system and has a direct limit determined by the equivalence closure of all the kernal equivalences of the surjective $T^\flat$-morphisms $\mathfrak{a}_0$ for all the T-algebras $\mathfrak{a}$ appearing in $f/T^\flat$ (cf. 4.19). Q.E.D.

(6.8) Automata theory suggests a direct construction of the minimal realization $M(f) = \langle Q_f, \mathfrak{a}_f, Y, \lambda_f \rangle$ of $f: T_0 \to y$ mimicking that in 2.13.

We define the equivalence relation E_f on T_0 by

$$\varphi' E_f \varphi'' \quad \text{iff} \quad f(\varphi'\varphi) = f(\varphi''\varphi) \quad \text{for all} \quad \varphi \in T_1 \tag{6.9}$$

and set $Q_f = T_0/E_f$. Letting $[\varphi]$ denote the equivalence class of φ modulo E_f, we then define the underlying action of $\mathfrak{a}_f$ by

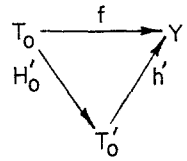
$$([\varphi_1], \cdots, [\varphi_n]) \cdot \psi = [\langle \varphi_1, \cdots, \varphi_n \rangle \psi]. \tag{6.10}$$

This is well defined by (6.9) and by the equality

$$\langle \varphi_1, \cdots, \varphi_n \rangle = \varphi_n \langle j\varphi_1, \cdots, j\varphi_{n-1}, 1_1 \rangle$$

where $j: [0] \to [1]$ is the only $\check{S}$-morphism $[0] \to [1]$. Finally, we define λ_F by $\lambda_f([\varphi]) = f(\varphi)$. The reader may readily verify that $M(f)$ so defined is indeed a terminal object of $f/T^\flat$; i.e., a minimal realization of f.

(6.11) We associate with a given response function $f: T_0 \to Y$ on T another category based on the realization that T_0 is the first component of the carier of T. Let f/Th, the *category* of Th-factorizations of f, have for objects pairs (H', h') where $H': T \to T'$ is a surjective Th-morphism and $h': T_0' \to Y$ is a response function such that



commutes, where H_0' (as in 4.16), is the restriction of $H': T \to T'$ to T_0. The morphisms of f/Th are the obvious surjective Th-morphisms:

$$H: (H', h') \to (H'', h'')$$

is a morphism in f/Th iff $HH' = H''$ (and then $h''H_0 = h'$).

(6.12) Fact. The assignment which maps each object (H, h) of f/Th to the realization of f with algebra $\mathfrak{a}_{H_0}$ (cf. 4.16) and output function h, and each morphism H in f/Th to H_0 is an injective functor

$$\mathfrak{J}: f/Th \to f/T^\flat$$

and, therefore, f/Th can be regarded as a subcategory of $f/T^{\flat}$. Since the free realization of f can be regarded as belonging also to f/Th (in the form of (i_T, f) where i_T is the identity functor on T), the category f/Th has an initial object.

We now prove

(6.13) PROPOITION. *If $f: T_0 \to Y$ is a nonconstant response function on the theory T, then f/Th has a terminal object (called the minimal dynamics of f). If we denote by*

$$H^f: T \to T_{\mathfrak{a}_f}$$

the obvious surjective Th-morphism from T onto the operative theory of the algebra $\mathfrak{a}_f$ of the minimal realization of f, then there exists a unique response function h^f on $T_{\mathfrak{a}_f}$ such that (H^f, h^f) is a terminal object in f/Th.

Proof. Since $M(f)$ realizes f, $\mathfrak{a}_f$ cannot be trivial when f is a nonconstant response function. The kernel equivalence E^f of $H^f: T \to T_{\mathfrak{a}_f}$ is characterized by:

$$\varphi' E^f \varphi'' \quad \text{iff for all} \quad x \in Q_f{}^*: x \cdot \varphi' = x \cdot \varphi'' \tag{6.14}$$

(whenever defined) for all $\varphi', \varphi'' \in T([n], [p]), n, p \in N$. It follows that for $\varphi', \varphi'' \in T_0$, $\varphi' E^f \varphi''$ implies $f(\varphi') = f(\varphi'')$, and therefore there exists a unique response function h^f on $T_{\mathfrak{a}_f}$ for which (H^f, h^f) is an object of f/Th. All we have to prove is that for any object (H, h) of f/Th, the kernel equivalence of H implies E^f. By 3.17, we can restrict our argument to the carrier of T.

Let $\varphi', \varphi'' \in T_p$ for some $p \in N$ such that $H(\varphi') = H(\varphi'')$. Then, for all $\varphi \in T_1$ and $\varphi_1, \cdots, \varphi_p \in T_0$, we clearly have

$$H(\langle \varphi_1, \cdots, \varphi_p \rangle \varphi' \varphi) = H(\langle \varphi_1, \cdots, \varphi_p \rangle \varphi'' \varphi). \tag{6.15}$$

Since (H, h) is an object of f/Th (i.e., $hH_0 = f$) and since both $\langle \varphi_1, \cdots, \varphi_p \rangle \varphi' \varphi$ and $\langle \varphi_1, \cdots, \varphi_p \rangle \varphi'' \varphi$ belong to T_0, 6.15 implies

$$f(\langle \varphi_1, \cdots, \varphi_p \rangle \varphi' \varphi) = f(\langle \varphi_1, \cdots, \varphi_p \rangle \varphi'' \varphi),$$

and, therefore, by 6.9,

$$\langle \varphi_1, \cdots, \varphi_p \rangle \varphi' E_f \langle \varphi_1, \cdots, \varphi_p \rangle \varphi''$$

which implies, by 6.10 and 6.14, that $\varphi' E^f \varphi''$. Q.E.D.

(6.16) The construction of a minimal realization $M(f)$ and a minimal dynamics (H^f, h^f) of a given response function $f: T_0 \to Y$ on a theory T is clearly analogous to the construction of a minimal realization $M(f)$ and a minimal dynamics (H^f, h^f) of a given function $f: W \to Y$ on a monoid W. The minimal realization of $f: W \to Y$ yields the minimal state space sufficient for the realization of f, while the minimal dynamics of

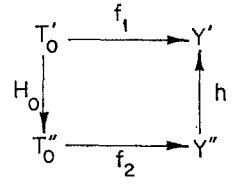
$f: W \to Y$ yields the minimal monoid automaton sufficient for the realization of $f: W \to Y$. The well-known phenomenon that the minimal monoid automaton can be defined from the opposite monoid of the transformation monoid of the minimal realization of $f: W \to Y$ is rephrased and generalized here in Proposition 6.13. The minimal dynamics of a response function $f: T_0 \to Y$ is centered around the opposite category of the image algebra of the algebra $\mathfrak{A}_f$ of the minimal realization of f.

(6.17) Let $\mathfrak{Z}$ be any class of theories. We say that $\mathfrak{Z}$ *admits quotients* iff $\mathfrak{Z}$ is closed under surjective *Th*-morphisms (i.e., if T' belongs to $\mathfrak{Z}$ and there exists a surjective *Th*-morphism $T' \to T''$ then T'' also belongs to $\mathfrak{Z}$). By an automaton *over* $\mathfrak{Z}$ we mean a T-automaton (for some theory T) whose *operative theory* is in $\mathfrak{Z}$. We clearly can infer from Proposition 6.13.

(6.18) COROLLARY. *Let $\mathfrak{Z}$ be any class of theories which admits quotients and let f be any nonconstant response function. Then f is realized by some automaton over $\mathfrak{Z}$ iff $T_{\mathfrak{A}_f}$ belongs to $\mathfrak{Z}$.*

(6.19) We extend now the automaton-theoretic study of response functions on theories and introduce the notion of *division* of response functions and related concepts in analogy with the Krohn–Rhodes' notion of division of machines (cf. Arbib, 1968). The results that follow our definitions are not surprising. Henceforth we assume that all the response functions under consideration are *nonconstant*.

(6.20) DEFINITION. Let T', T'' be theories and $f_1: T_0' \to Y'$ and $f_2: T_0'' \to Y''$ be response functions. We say that f_1 *divides* f_2 (written $f_1 \mid f_2$) iff there exist a *Th*-morphism $H: T' \to T''$ and a function $h: Y'' \to Y'$ such that

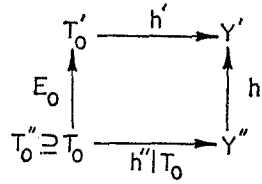


commutes. The response functions f_1 and f_2 are said to be *equivalent* iff $f_1 \mid f_2$ and $f_2 \mid f_1$.

(6.21) DEFINITION. By a *theory with output* we simply mean a pair (T, f) of a theory T and a response function f on T. If f is a response function on T, then by the *normal form* (T^f, h^f) of f we mean the theory with output determined by the minimal dynamics of f (i.e., $T^f = T_{\mathfrak{A}_f}$; cf. 6.13). The theory T^f is called the *normal theory* of f.

(6.22) DEFINITION. Let T' and T'' be theories. We say that T' *divides* T'' iff T' is the morphic image of a subtheory of T''. Let (T', h') and

(T'', h'') be theories with output. We say that (T', h') *divides* (T'', h'') iff there exist a subtheory T of T'', a surjective *Th*-morphism $E: T \to T'$ (and therefore T' divides T'') and a function $h: Y'' \to Y'$ such that



(6.23)

commutes (where $h'' \mid T_0$ is the restriction of h'' to T_0).

Analogous to the diagram-chasing in ordinary automata theory (Give'on, 1967) we have:

(6.24) THEOREM. *For arbitrary response functions f_1 and f_2 :*

(i) *$f_1 \mid f_2$ always implies that (T^{f_1}, h^{f_1}) divides (T^{f_2}, h^{f_2});*

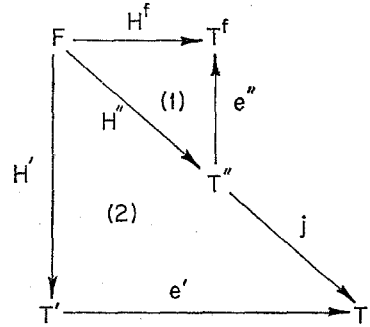
(ii) *if f_1 is a response function on a free theory and (T^{f_1}, h^{f_1}) divides (T^{f_2}, h^{f_2}) then $f_1 \mid f_2$.*

We now prove:

(6.25) THEOREM. *Let $f: F_0 \to Y$ be a response function on a free theory F, and let $\epsilon_T: T_0' \to T_0$ be the response function on a theory T' induced by a surjective Th-morphism $e': T' \to T$ by restriction to T_0'. Then f divides ϵ_T iff T^f, the normal theory of f, divides T.*

Proof. Noting that T is the normal theory of ϵ_T, by 7.17 it follows that if $f \mid \epsilon_T$ then T^f divides T.

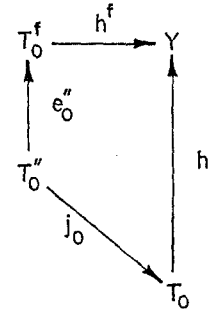
Conversely, assume that T'' is a subtheory of T with a surjective *Th*-morphism $e'': T'' \to T^f$, and let $j: T'' \to T$ denote the insertion *Th*-morphism for $T'' \subseteq T$. Then by 5.10 there exist, first a *Th*-morphism $H'': F \to T''$ for which part (1) in 6.26 commutes, and then a *Th*-morphism $H': F \to T'$ for which part (2) of 6.26 commutes.



(6.26)

On the other hand, since $j_0: T_0'' \to T_0$ is injective, (and since every non-

empty set is an *injective object* in the category of sets) there exists at least one function $h: T_0 \to Y$ for which



(6.27)

commutes.

Since clearly,

$$he_0'H_0' = h(j_0H_0'') = (h^f e_0'')H_0'' = h^f H^f = f,$$

we have $f \mid \epsilon_T$. Q.E.D.

The reader may readily verify the:

(6.28) FACT. For any response function $f: T_0 \to Y$ on *any* theory T, f and $h^f: T_0^f \to Y$ are equivalent response functions.

With these basic results on the minimization, normal form and divisibility of response functions on theories, we have paved the way for the development, in Part III, of the appropriate generalization of the Krohn–Rhodes composition theory for finite-state sequential machines.

(6.29) *Remark.* The divisibility results (i.e., 6.24 through 6.28) as well as the rest of our results can be extended to cover constant response functions by extending the category *Th* in adding to it a terminal object as the normal theory of the constant response functions.

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