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# The logic of structures*

Pierre Ageron

*Groupe de Recherche Algorithmique & Logique, Département de Mathématiques, Université de Caen 14032 Caen Cedex, France*

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*Abstract*

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Accessible categories, first introduced by Grothendieck and Verdier, generalise in a natural way the domains used in denotational semantics. On the other hand, by Lair's theorem, accessible categories are exactly the categories of models of a (small) sketch. In this paper, we investigate some 'proof-theoretic' properties of various categories of accessible categories (cartesian closure, normal forms, links with linear logic) by relating them to 'model-theoretic' computations at the level of sketches.

## Preliminaries

The following conventions are observed throughout:
– We work in ZFC + the axiom of universes, and fix three Grothendieck universes $\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2$ (see [12]). Everything in $\mathcal{U}_0$ is said to be small. One exception: by a set, we always mean a small one (otherwise we call it a class).
– $\mathcal{S}et$ is the category of sets; $\mathcal{C}at$ is the class of small categories; CAT is the (not locally small) category of locally small categories.
– By a regular cardinal, we mean a (small) infinite one.
– By a diagram, a (co)cone, a (co)limit, we mean a small indexed one. We always assume that the indexation is a multiplicative graph in the sense of Ehresmann (this includes both cases of a graph and of a category).
– By a wide pullback, we mean a limit whose indexation has a final object (this includes finite and infinite pullbacks).
– By an ordered set, we mean a not-necessarily linearly one. Ordered sets are special cases of categories, so we have been careful that our terminology on ordered sets agrees with that on categories (complete, filtered, ...).

*Correspondence to*: P. Ageron, Groupe de Recherche Algorithmique & Logique, Département de Mathématiques, Université de Caen, 14032 Caen Cedex, France.

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– By a filtered ordered set or category, by a connected category, we mean a nonempty one.
– By a filtered (resp. connected, $\alpha$-small) (co)limit, we mean one whose indexation is filtered (resp. connected, of size $<\alpha$).

## 1. Elements of domain theory

It is well known that the so-called Curry–Howard isomorphism identifies mechanisable computations with intuitionistic proofs. Since D. Scott (1969), many authors have proposed semantics of computation, i.e. models of intuitionistic logic (cartesian closed categories) well suited to the intuitive notion of computable function. According to that tradition, all computable functions should be (at least) continuous (a requirement sometimes referred to as 'Scott's thesis'). A fundamental structure emerged from that stream of research, often called 'algebraic complete partial order'—but as this terminology is highly annoying for category theorists, we shall just call it a domain.

**Definition.** A *domain* is a (partially) ordered set $(X, \leq)$ satisfying the two following properties:

(1) every filtered subset has a lowest upper bound,

(2) $X$ contains a subset $X'$ whose elements are *finite* and such that every element $a$ of $X$ is the filtered l.u.b. of $\{x' \in X' \mid x' \leq a\}$.

Of course, we have to clarify what we mean by 'finite element'. The most usual notion is that of 'finite presentability', coming from universal algebra:

$p$ is *finitely presentable* if, whenever $p \leq \sup a_j$ (filtered l.u.b.), we already have $p \leq a_j$ for some $j$.

It is sometimes useful to view the elements of a domain $X$ as pieces of information on a certain datum: the essential property is that every piece of information can be expressed as a join of compatible finite pieces.

In this account, a computable function from $X$ to $Y$ is just an order-preserving map preserving filtered l.u.b.'s (note that it is completely determined by its restriction to the finitely presentable objects). These functions can also be seen as continuous maps for a suitable topology on $X$ and $Y$ (the so-called Scott's topology).

The semantical study of polymorphism (higher-order logical systems) led various workers in domain theory to looking for some generalized notion of domain affording a common structure at the level of one domain and at the level of all domains, and accordingly to replacing ordered sets by categories. For some reasons, the classes of categories they studied were very restrictive. But the natural translation of the definition of a domain gives the following definition:

**Definition.** Let $\mathbb{X}$ be a locally small category. Then $\mathbb{X}$ is said to be *finitely accessible* if

(1) $\mathbb{X}$ has filtered colimits,

(2) $\mathbb{X}$ contains a small full subcategory $\mathbb{X}'$ whose objects are finitely presentable and such that every object $A$ of $\mathbb{X}$ is canonically the filtered colimit of the canonical functor $X'/A \to \mathbb{X}$.

In fact (a variant of) this notion existed for a long time before Scott: it was introduced by Grothendieck and Verdier in [12], of course with completely different motivations.

**Remark.** Recall an object $P$ of $\mathbb{X}$ is called finitely presentable if the functor $\mathrm{Hom}(P, -)$ preserves filtered colimits. From the computer scientist's point of view, this might be a rather unsatisfactory notion of finiteness. On some occasions, it may appear too liberal, calling finite what (naively) is only compact or noetherian. In other cases, it may appear too restrictive: e.g. in Girard's category Coh-lin of coherence spaces and linear maps [8], every object can be approximated by finite spaces, but the latter are not finitely presentable, so that Coh-lin is not finitely accessible. (In fact it is not accessible at all—see Definition 3.1—because it can be shown that a complete self-dual accessible category must be a preorder.)

## 2. Elements of sketch theory

Sketches were introduced by Ehresmann in 1968 [6]. They are a geometric description of a given type of structures and morphisms. The most comprehensive exposition of sketch theory is [20]. See also [3, 11, 13, 22].

We recall here the main definitions and background results.

**Definitions.** $\mathbb{S} = (\underline{\mathbb{S}}, P, Q)$ is a *sketch* if $\underline{\mathbb{S}}$ is a small category, $P$ is a set of cones in $\underline{\mathbb{S}}$ (called the distinguished cones of $\mathbb{S}$) and $Q$ is a set of cocones in $\underline{\mathbb{S}}$ (called the distinguished cocones of $\mathbb{S}$).

We call this sketch *projective* if $Q = \emptyset$, *inductive* if $P = \emptyset$, *elementary* if it is both projective and inductive. Elementary sketches will be identified with small categories.

A *homomorphism* from the sketch $\mathbb{S}$ to the sketch $\mathbb{S}'$ is a functor $H : \underline{\mathbb{S}} \to \underline{\mathbb{S}}'$ that takes every distinguished cone of $\mathbb{S}$ to a distinguished cone of $\mathbb{S}'$ and every distinguished cocone of $\mathbb{S}$ to a distinguished cocone of $\mathbb{S}'$.

A *model* of the sketch $\mathbb{S}$ in the category $\mathbb{C}$ is a functor $M : \underline{\mathbb{S}} \to \mathbb{C}$ that turns every distinguished cone of $\mathbb{S}$ into a limit cone in $\mathbb{C}$ and every distinguished cocone of $\mathbb{S}$ into a colimit cocone in $\mathbb{C}$. $\mathrm{Mod}(\mathbb{S}, \mathbb{C})$ will denote the full

subcategory of $\mathbb{C}^{\underline{\mathbb{S}}}$ whose objects are the models of $\mathbb{S}$ in $\mathbb{C}$; $\mathrm{Mod}(\mathbb{S})$ will stand for $\mathrm{Mod}(\mathbb{S}, \mathrm{Set})$.

If $\alpha$ is a regular cardinal, let us say that $\mathbb{S}$ is an $\alpha$-sketch if all its distinguished cones are $\alpha$-small indexed. (The size of the distinguished cocones will play no role in this work.)

Sk (resp. $\alpha$-Sk) will denote the (locally small) category of all sketches (resp. $\alpha$-sketches) and homomorphisms between them. $\mathscr{Sk}$ (resp. $\alpha$-$\mathscr{Sk}$) will stand for its class of objects. The category Sk (resp. $\alpha$-Sk) is complete and cocomplete.

**Definition.** Given two sketches $\mathbb{S}$ and $\mathbb{T}$, $\mathbb{S} \otimes \mathbb{T}$ denotes the coarsest sketch structure on the product category $\underline{\mathbb{S}} \times \underline{\mathbb{T}}$ such that all canonical functors

$$(-, T) : \mathbb{S} \to \mathbb{S} \otimes \mathbb{T}, \quad T \in \mathrm{Ob}(\mathbb{T})$$

and

$$(S, -) : \mathbb{T} \to \mathbb{S} \otimes \mathbb{T}, \quad S \in \mathrm{Ob}(\mathbb{S})$$

become homomorphisms.

Any small category $\mathbb{1}$ with only one object and one (identity) arrow is a unit for this tensor product.

Also note that $\mathbb{S} \otimes \mathbb{T}$ is an $\alpha$-sketch if $\mathbb{S}$ and $\mathbb{T}$ are.

It is easily verified that there are equivalences of categories $\mathrm{Mod}(\mathbb{S} \otimes \mathbb{T}) \simeq \mathrm{Mod}(\mathbb{S}, \mathrm{Mod}(\mathbb{T})) \simeq \mathrm{Mod}(\mathbb{T}, \mathrm{Mod}(\mathbb{S}))$ in the following cases:

(i) $\mathbb{S}$ and $\mathbb{T}$ are projective sketches,

(ii) $\mathbb{S}$ and $\mathbb{T}$ are inductive sketches,

(iii) $\mathbb{S}$ or $\mathbb{T}$ is an elementary sketch.

This equivalence is not true in the 'mixed' case, essentially because limits do not in general commute with colimits. Here is a simple counterexample: consider the projective sketch $\mathbb{S}$ such that $\underline{\mathbb{S}} = \mathbb{1}$ and with one distinguished cone with empty basis, and the inductive sketch $\mathbb{T}$ such that $\underline{\mathbb{T}} = \mathbb{1}$ and with one distinguished cocone with empty basis. Then $\mathrm{Mod}(\mathbb{S} \otimes \mathbb{T})$ is empty, while $\mathrm{Mod}(\mathbb{S}, \mathrm{Mod}(\mathbb{T}))$ has exactly one object. Note that $\mathrm{Mod}(\mathbb{T}, \mathrm{Mod}(\mathbb{S}))$ is a groupoid equivalent to $\mathrm{Mod}(\mathbb{S}, \mathrm{Mod}(\mathbb{T}))$.

**Definitions.** A (necessarily locally small) category $\mathbb{X}$ is said to be *sketchable* (resp. $\alpha$-sketchable) iff there exists a sketch (resp. an $\alpha$-sketch) $\mathbb{S}$ such that $\mathbb{X}$ is equivalent to $\mathrm{Mod}(\mathbb{S})$. Two sketches $\mathbb{S}$ and $\mathbb{S}'$ are said to be *equivalent* if the categories $\mathrm{Mod}(\mathbb{S})$ and $\mathrm{Mod}(\mathbb{S}')$ are equivalent.

Sketchable categories are exactly those equivalent to the category of models of an infinitary first-order theory with suitable relatively elementary morphisms as arrows; see [16, 22].

Any homomorphism between sketches (resp. $\alpha$-sketches) $H : \mathbb{S} \to \mathbb{S}'$ defines (by precomposition) a functor $\mathrm{Mod}(H) : \mathrm{Mod}(\mathbb{S}') \to \mathrm{Mod}(\mathbb{S})$. A functor equivalent to one that arises in that way is called sketchable (resp. $\alpha$-sketchable).

## 3. $\beta$-accessible categories and $\beta$-continuous functors

Let $\beta$ be a regular cardinal. A nonempty category is $\beta$-filtered if every $\beta$-small diagram in $\mathbb{I}$ is the basis of a cocone in $\mathbb{I}$. An object $P$ of a category is $\beta$-presentable iff the functor $\mathrm{Hom}(P, -)$ preserves $\beta$-filtered colimits. We recall from [18] or [23] the following definition:

**Definition 3.1.** Let $\beta$ be a regular cardinal and $\mathbb{X}$ a locally small category. Then $\mathbb{X}$ is said to be *$\beta$-accessible* iff:

(1) $\mathbb{X}$ has $\beta$-filtered colimits,

(2) $\mathbb{X}$ contains a small full subcategory $\mathbb{X}'$ whose objects are $\beta$-presentable and such that every object $A$ of $\mathbb{X}$ is canonically the $\beta$-filtered colimit of the canonical functor $\mathbb{X}'/A \to \mathbb{X}$.

Being $\aleph_0$-accessible clearly means to be finitely accessible, in the sense of Section 1. We say that $\mathbb{X}$ is *accessible* if there is a regular cardinal $\beta$ such that $\mathbb{X}$ is $\beta$-accessible. As we shall see, it is important *not* to restrict to the finitary case (or to any fixed level of accessibility).

**Theorem 3.2** (Lair). *Let $\mathbb{X}$ be a locally small category. Then the following are equivalent*:

(i) *$\mathbb{X}$ is accessible.*

(ii) *$\mathbb{X}$ is sketchable.* □

This fundamental result was originally stated and proofed in [18]. It also appears in [22]. Although neither implication is trivial, the most interesting one certainly is (ii)$\Rightarrow$(i). Let us recall the main lines of its proof. It makes use of Guitart–Lair's locally free diagram lemma [15] (which may be seen as a precise structural version of Löwenheim–Skolem's theorem):

**Lemma 3.3.** *Let $\mathbb{S}$ be an $\alpha$-sketch.*

(a) *Every object $M$ of* $\mathrm{Set}^{\mathbb{S}}$ *generates (at least) a* (small) *locally free diagram $L(M) = (L(M)_I)_{I \in \mathbb{I}}$ in* $\mathrm{Mod}(\mathbb{S})$.

(c) *There is a* (small) *regular cardinal $\beta \geq \alpha$ such that*

– *$\forall M$ ($M$ is $\alpha$-presentable)$\Rightarrow$(the $L(M)_I$ are $\beta$-presentable),*

– *the $\beta$-presentable objects in* $\mathrm{Mod}(\mathbb{S})$ *are exactly the $\beta$-small valued models of $\mathbb{S}$.* □

Recall from [14] that $(L(M)_I)_{I\in\mathbb{I}}$ is called a *locally free diagram* on $M$ iff there is a canonical natural bijection

$$\mathrm{Hom}(M, N) \cong \operatorname*{colim}_{I\in\mathbb{I}} \mathrm{Hom}(L(M)_I, N) .$$

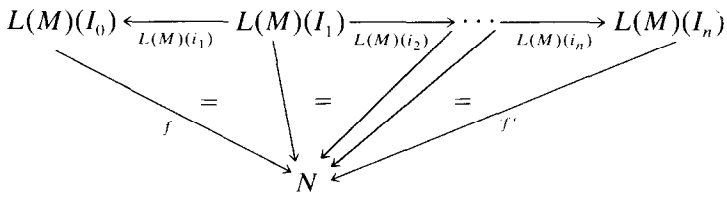
More explicitly, there exists $(z_I : M \to L(M)_I)$ such that:

(1) $\forall N \in \mathrm{Ob}(\mathrm{Mod}(\mathbb{S}))\ \ \forall z : M \to N\ \ \exists I \in \mathrm{Ob}(\mathbb{I})\ \ \exists f : L(M)_I \to N,\ \ z = f \cdot z_I$ (we say that $(I, f)$ is a factorisation for $z : M \to N$),

(2) if $(I, f)$ and $(I', f')$ are two factorisations for $z : M \to N$, then they are connected, i.e. there is a zigzag in $\mathbb{I}$

$$I = I_0 \xleftarrow[i_1]{} I_1 \xrightarrow[i_2]{} I_2 \xleftarrow[i_3]{} \cdots \xrightarrow[i_n]{} I_n = I'$$

and a commutative diagram in $\mathbb{X} = \mathrm{Mod}(\mathbb{S})$:




Now, with the notations of the lemma, it can be shown that the (obviously locally small) category $\mathbb{X} = \mathrm{Mod}(\mathbb{S})$ of models of an $\alpha$-sketch $\mathbb{S}$ is $\beta$-accessible. Indeed:

(1) The forgetful functor $\mathrm{Mod}(\mathbb{S}) \to \mathrm{Set}^{\mathbb{S}}$ creates $\alpha$-filtered colimits, hence a fortiori $\beta$-filtered colimits.

(2) Let $\mathbb{X}' = \mathbb{X}_\beta$ be the full subcategory of $\mathbb{X}$ whose objects are the $\beta$-presentable objects of $\mathbb{X}$. It is easily shown that $\mathbb{X}'$ is essentially small and (using the lemma) that every object $M$ of $\mathbb{X}$ is the $\beta$-filtered colimit of the canonical functor $\mathbb{X}'/M \to \mathbb{X}$.

It is essential to notice that we do *not* have ($\mathbb{X}$ is $\alpha$-sketchable) $\Rightarrow$ ($\mathbb{X}$ is $\alpha$-accessible)—however, the converse does hold. Here is a counterexample due to Makkai and Paré [22, 3.3.6]. Let $\mathbb{X}$ be the category whose objects are sequences $(X_i)_{i\in\mathbb{N}}$ of infinite sets, and whose arrows are sequences of bijections. Then $\mathbb{X}$ is $\aleph_0$-sketchable, but only $\aleph_1$-accessible (because $\mathbb{X}$ has no $\aleph_0$-presentable object!).

Let us also mention here (because it is generally not well known) that it is possible to give a semantical characterization of $\alpha$-sketchable categories, using Lair's notion of an $\alpha$-analysable category (see [20]).

We now turn to the categorical counterpart of Scott's continuous functions.

**Definitions 3.4.** Let $\beta$ be a regular cardinal and $F : \mathbb{X} \to \mathbb{Y}$ a functor. Then $F$ is said to be *$\beta$-continuous* iff $\mathbb{X}$ and $\mathbb{Y}$ are $\beta$-accessible categories and $F$ preserves

$\beta$-filtered colimits. If $\mathbb{X}$ and $\mathbb{Y}$ are $\beta$-accessible categories, the category of $\beta$-continuous functors from $\mathbb{X}$ to $\mathbb{Y}$ and natural transformations will be denoted by $\beta\text{-Cont}(\mathbb{X}, \mathbb{Y})$. The (not locally small) category of all $\beta$-accessible categories and $\beta$-continuous functors will be denoted by $\beta$-ACC.

The remainder of this section describes different economical representations of $\beta$-continuous functors.

Let $F : \mathbb{X} \to \mathbb{Y}$ be a $\beta$-continuous functor. It is more or less clear that $F$ is completely determined by its restriction to $\mathbb{X}_\beta$. More precisely we have the following well-known fact:

**Proposition 3.5.** *If $\mathbb{X}$ and $\mathbb{Y}$ are two $\beta$-accessible categories, then the categories $\beta\text{-Cont}(\mathbb{X}, \mathbb{Y})$ and $\mathbb{Y}^{\mathbb{X}_\beta}$ are equivalent.* □

One may ask if, when $\mathbb{X}$ and $\mathbb{Y}$ are $\beta$-accessible categories, $\beta\text{-Cont}(\mathbb{X}, \mathbb{Y})$ is again a $\beta$-accessible category. *This is not always the case.*

Let us try a sketch-theoretic proof to understand where the problem lies. Since $\mathbb{Y}$ is $\beta$-accessible, there is a $\beta$-sketch $\mathbb{T}$ such that $\mathbb{Y} \simeq \text{Mod}(\mathbb{T})$. Now if we view the (essentially) small category $\mathbb{X}_\beta$ as an elementary sketch, we get (using a result from Section 2):

$$\beta\text{-Cont}(\mathbb{X}, \mathbb{Y}) \simeq (\text{Mod}(\mathbb{T}))^{\mathbb{X}_\beta} \simeq \text{Mod}(\mathbb{T} \otimes \mathbb{X}_\beta) .$$

So $\beta\text{-Cont}(\mathbb{X}, \mathbb{Y})$ is $\beta$-sketchable, hence accessible—but we cannot conclude that it is $\beta$-accessible!

However, there are some important special cases where the rank of sketchability gives us a control on the level of accessibility. Section 4 of this paper is essentially devoted to the search for such situations.

Using Proposition 3.5, we now show how every $\beta$-continuous functor $F : \mathbb{X} \to \text{Set}$ can be expressed as a colimit of functors represented by $\beta$-presentable objects.

**Proposition 3.6.** *Let $F : \mathbb{X} \to \text{Set}$ be a $\beta$-continuous functor. Then*

$$F \cong \operatorname*{Colim}_{\substack{P \in \mathbb{X}_\beta \\ x \in F(P)}} \text{Hom}_{\mathbb{X}}(P, -) .$$

**Proof.** Let $F : \mathbb{X} \to \text{Set}$ be a $\beta$-continuous functor and $F'$ be its restriction to $\mathbb{X}_\beta$. Since the (opposite) Yoneda embedding $Y : \mathbb{X}_\beta \to (\text{Set}^{\mathbb{X}_\beta})^{\text{op}}$ is dense, we have:

$$F' \cong \text{Colim}\,(Y^{\text{op}}/F' \xrightarrow{\text{can}} \mathbb{X}_\beta^{\text{op}} \xrightarrow{Y^{\text{op}}} \text{Set}^{\mathbb{X}_\beta})$$

or equivalently

$$F' \cong \operatorname*{Colim}_{\substack{P \in \mathbb{X}_\beta \\ x \in F(P)}} \mathrm{Hom}_{\mathbb{X}_\beta}(P, -) .$$

Now the functor $\mathrm{Hom}_{\mathbb{X}_\beta}(P, -) : \mathbb{X}_\beta \to \mathrm{Set}$ is the restriction to $\mathbb{X}_\beta$ of the $\beta$-continuous functor $\mathrm{Hom}_{\mathbb{X}}(P, -) : \mathbb{X} \to \mathrm{Set}$ (because $\mathbb{X}_\beta$ is full and $P$ is $\beta$-presentable). So the equivalence of categories between $\beta$-Cont($\mathbb{X}$, Set) and $\mathrm{Set}^{\mathbb{X}_\beta}$ gives us the proposition. □

If the category $Y^{\mathrm{op}}/F'$ contains a discrete initial subcategory, this colimit can be computed as a coproduct. In that case we get some kind of 'power series expansion' of $F$ (see [9, 24]):

$$F(X) = \sum_P C_P X^P , \quad \text{with } C_P = \mathrm{card}\, F(P) \text{ and } X^P = \mathrm{Hom}_{\mathbb{X}}(P, X) .$$

Let $\mathbb{X}$ and $\mathbb{Y}$ be two $\beta$-accessible categories, $\mathbb{S}$ and $\mathbb{T}$ two $\beta$-sketches sketching respectively $\mathbb{X}$ and $\mathbb{Y}$. Then every homomorphism $H : \mathbb{T} \to \mathbb{S}$ gives rise to a $\beta$-continuous functor $F = \mathrm{Mod}(H) : \mathbb{X} \to \mathbb{Y}$. But one should not think that every $\beta$-continuous functor is obtained in that way. Here is an illuminating counterexample.

**Example 3.7.** Let $\mathbb{X} = \mathrm{Set}^A$ be the category of families of sets indexed by a fixed set $A$. Let $\mathbb{Y} = \mathrm{Set}^?$ be the category of families of sets indexed by a variable set (or Grothendieck construction associated to the functor $\mathrm{Set}^{\mathrm{op}} \to \mathrm{CAT}$, $A \mapsto \mathrm{Set}^A$). Let $F : \mathbb{X} \to \mathbb{Y}$ be the canonical inclusion functor. Then:

$\mathbb{X}$ is sketched by $\mathbb{S} = A$ (discrete elementary sketch);

$\mathbb{Y}$ is sketched by the elementary sketch $\mathbb{T}$ with two objects $U$, $I$ and an arrow $j : U \to I$ (imagine $I$ representing the variable indexation set and $U$ the disjoint union of the fibres of $j$).

There is clearly no homomorphism $H : \mathbb{T} \to \mathbb{S}$ sketching the functor $F$. (More generally, it is easy to argue that there can be no homomorphism between projective sketches sketching $F$, because $F$ does not preserve the final object.)

But $\mathbb{X}$ is (also) sketched by the (mixed) sketch $\mathbb{S}'$ with

three objects $U$, $I$, 1,

one arrow $J : U \to I$,

for every $a \in A$, one arrow $i_a : 1 \to I$,

the distinguished cone $( : 1 \to)_{\in \emptyset}$,

the distinguished cocone $(i_a : 1 \to I)_{a \in A}$,

and this time there exists an homomorphism $H : \mathbb{T} \to \mathbb{S}'$ (just the inclusion) sketching the functor $F$.

This example suggests that in order to represent at the syntactical level all $\beta$-continuous functors from Mod($\mathbb{S}$) to Mod($\mathbb{T}$), we have to consider the homomorphisms from $\mathbb{T}$ to any $\beta$-sketch $\mathbb{S}'$ equivalent to $\mathbb{S}$. More precisely:

**Definition 3.8.** Let $\mathbb{S}$ and $\mathbb{T}$ be two $\beta$-sketches. A *relational morphism* from $\mathbb{T}$ to $\mathbb{S}$ is a pair $R = (H, e)$, where $H : \mathbb{T} \to \mathbb{S}'$ is a homomorphism between $\beta$-sketches and $e$ an equivalence between the categories $\mathrm{Mod}(\mathbb{S}')$ and $\mathrm{Mod}(\mathbb{S})$.

A relational morphism $R$ from $\mathbb{T}$ to $\mathbb{S}$ clearly gives rise to a functor $\mathrm{Mod}(R) : \mathrm{Mod}(\mathbb{S}) \to \mathrm{Mod}(\mathbb{T})$.

**Proposition 3.9.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be two $\beta$-accessible categories, $\mathbb{S}$ and $\mathbb{T}$ two $\beta$-sketches respectively $\mathbb{X}$ and $\mathbb{Y}$, and $F : \mathbb{X} \to \mathbb{Y}$ a functor. The following are equivalent*:

(i) *$F$ is $\beta$-continuous*,

(ii) *there is a relational morphism $R$ from $\mathbb{T}$ to $\mathbb{S}$ such that $F$ is equivalent to* $\mathrm{Mod}(R)$.

**Proof.** Essentially, [22], 4.4.2(i), (ii) and 3.3.5(iii). □

The following theorem generalizes Lemma 3.3:

**Theorem 3.10.** *Let $F : \mathbb{X} \to \mathbb{Y}$ be a $\beta$-continuous functor. Then every object $B$ of $\mathbb{Y}$ generates a (small) locally free diagram $(A_I)_{I \in \mathbb{I}}$ in $\mathbb{X}$.*

*More explicitly, there is a family $(z_I : B \to F(A_I))_{I \in \mathrm{Ob}(\mathbb{I})}$ of arrows in $\mathbb{X}$ such that*:

(1) $\forall A \in \mathrm{Ob}(\mathbb{X})\ \forall z : B \to F(A)\ \exists I \in \mathrm{Ob}(\mathbb{I})\ \exists f : A_I \to A,\ z = F(f).z_I$,

(2) *two such factorisations $(I, f)$ and $(I', f')$ are connected.* □

Very natural examples of locally free diagrams arise in Galois theory (see [19]), in algebraic topology (see [13]). All normal form theorems and various concepts of 'trace' in proof theory and denotational semantics are also instances of that notion, usually in cases where some unicity and finiteness properties can be expected (stable functors, normal functors). See, for example, [9, 10].

## 4. Cartesian closed categories of accessible categories

**Definitions 4.1.** A *variety* (of sketches) is any class of sketches. An $\alpha$-variety is any class of $\alpha$-sketches. An $\alpha$-variety $\mathscr{S}$ is *conservative* iff for every $\mathbb{S} \in \mathscr{S}$ the category $\mathrm{Mod}(\mathbb{S})$ is $\alpha$-accessible. If $\mathscr{S}$ is a conservative $\alpha$-variety, $\mathrm{MOD}(\mathscr{S})$ is the full subcategory of $\alpha$-ACC whose objects are the categories sketchable by a sketch in $\mathscr{S}$ (so the arrows are the $\alpha$-continuous functors between these categories). Two conservative $\alpha$-varieties $\mathscr{S}$ and $\mathscr{S}'$ are equivalent iff $\mathrm{MOD}(\mathscr{S}) = \mathrm{MOD}(\mathscr{S}')$.

**Definition 4.2.** An $\alpha$-variety of sketches $\mathscr{S}$ is *linear* iff

(a) $\emptyset \in \mathscr{S}$,

(b) $\forall \mathbb{S}_1 \in \mathscr{S}\ \forall \mathbb{S}_2 \in \mathscr{S},\ \mathbb{S}_1 + \mathbb{S}_2 \in \mathscr{S}$,
(c) $\forall \mathbb{S} \in \mathscr{S}\ \forall \mathbb{C} \subset \mathscr{C}at,\ \mathbb{S} \otimes \mathbb{C} \subset \mathscr{S}$.

**Theorem 4.3.** *Let $\mathscr{S}$ be a linear conservative $\alpha$-variety of sketches. Then the category* MOD($\mathscr{S}$) *is cartesian closed.*

**Proof.** First by (a) the category $\mathbb{1} = \mathrm{Mod}(\emptyset)$ belongs to MOD($\mathscr{S}$), and it is a final object because the unique functor from any $\beta$-accessible $\mathbb{X}$ to $\mathbb{1}$ is obviously $\beta$-continuous. Given $\mathbb{X}_1 \simeq \mathrm{Mod}(\mathbb{S}_1)$ and $\mathbb{X}_2 \simeq \mathrm{Mod}(\mathbb{S}_2)$, with $\mathbb{S}_1, \mathbb{S}_2 \in \mathscr{S}$, their product in CAT is

$$\begin{array}{ccc} \mathrm{Mod}(\mathbb{S}_1 + \mathbb{S}_2) & \xrightarrow[P_1]{} & \mathrm{Mod}(\mathbb{S}_1) \\ {\scriptstyle P_2}\downarrow & & \\ \mathrm{Mod}(\mathbb{S}_2) & & \end{array}$$

where $P_i$ is the functor induced by the coprojection $\mathbb{S}_i \to \mathbb{S}_1 + \mathbb{S}_2$.

Let us show that the same cone is still a product in MOD($\mathscr{S}$): by (b), $\mathrm{Mod}(\mathbb{S}_1 + \mathbb{S}_2)$ belongs to MOD($\mathscr{S}$); $P_1$ and $P_2$ are $\alpha$-sketchable, hence $\alpha$-continuous; if $\mathbb{X}$ to MOD($\mathscr{S}$) and $Q_1 : \mathbb{X} \to \mathbb{X}_1$, $Q_2 : \mathbb{X} \to \mathbb{X}_2$ are $\alpha$-continuous functors, then the unique factorisation (in CAT) $H : \mathbb{X} \to \mathrm{Mod}(\mathbb{S}_1 + \mathbb{S}_2)$ is still $\alpha$-continuous, since for any $\alpha$-filtered colimit in $\mathbb{X}$ we have

$$\begin{aligned} H(\mathrm{colim}\ X_J) &\cong (P_1(H(\mathrm{colim}\ X_J)), P_2(H(\mathrm{colim}\ X_J)) \\ &\cong (Q_1(\mathrm{colim}\ X_J), Q_2(\mathrm{colim}\ X_J)) \\ &\cong (\mathrm{colim}\ Q_1(X_J), \mathrm{colim}\ Q_2(X_J)) \\ &\cong \mathrm{colim}\ (Q_1(X_J), Q_2(X_J)) \\ &\cong \mathrm{colim}\ H(X_J)\,. \end{aligned}$$

Last, given $\mathbb{X} \simeq \mathrm{Mod}(\mathbb{S})$ and $\mathbb{Y} \simeq \mathrm{Mod}(\mathbb{T})$, $\mathbb{S}, \mathbb{T} \in \mathscr{S}$, we know by the discussion following Proposition 3.5 that $\alpha$-Cont($\mathbb{X}, \mathbb{Y}$) is equivalent to $\mathrm{Mod}(\mathbb{T} \otimes \mathbb{X}_\alpha)$, hence (by (c)) belongs to MOD($\mathscr{S}$). It remains to check that

$$\mathrm{Hom}_{\mathrm{MOD}(\mathscr{S})}(\mathbb{Z} \times \mathbb{X}, \mathbb{Y}) \simeq \mathrm{Hom}_{\mathrm{MOD}(\mathscr{S})}(\mathbb{Z}, \alpha\text{-Cont}(\mathbb{X}, \mathbb{Y}))$$

which holds because $\alpha$-filtered colimits are computed in $\alpha$-Cont($\mathbb{X}, \mathbb{Y}$) as in $\mathbb{Y}^{\mathbb{X}_\alpha}$, that is, pointwise (the 'points' being here the objects of $\mathbb{X}_\alpha$). □

**4.4. Elementary examples and counterexamples.** $\mathscr{S} = \{\emptyset\}$ is a linear conservative $\alpha$-variety, for any $\alpha$. So is the class of elementary sketches (i.e. $\mathscr{C}at$). But note that the variety of discrete elementary sketches (i.e. $\mathscr{S}et$) is not linear: this

explains why Girard's quantitative domains [9] did not lead to cartesian closed categories.

The variety $\alpha = \mathscr{Sk}$ of all $\alpha$-sketches is obviously linear, but is not conservative, as we have seen in Section 3. If we restrict ourselves to those $\alpha$-sketches $\mathbb{S}$ such that $\mathrm{Mod}(\mathbb{S})$ is $\alpha$-accessible, we of course get a conservative $\alpha$-variety (the greatest one), but it is no longer linear!

A straightforward application of Zorn's lemma (for classes) shows that the following:

**Proposition 4.5.** *Every linear conservative $\alpha$-variety is contained in a maximal one.* □

Of course, it would be quite interesting to determine some or all—up to equivalence—of these maximal linear conservative $\alpha$-varieties, and to compare them to the (four-elements) 'maximal spectrum' of cartesian closed categories of domains, as determined by Jung in [17] (using previous results of Smyth).

Here we shall just describe three (model-theoretically and domain-theoretically) significant linear conservative $\alpha$-varieties.

Our first example is related to Gabriel and Ulmer's work [7]. The following two results are well known:

**Lemma 4.6.** *Let $\mathbb{S}$ be a projective $\alpha$-sketch.*

(a) *Every object $M$ of $\mathrm{Set}^{\underline{\mathbb{S}}}$ generates a free object $L(M)$ in $\mathrm{Mod}(\mathbb{S})$.*

(b) *The model $L(M)$ is unique up to isomorphism.*

(c) *If $M$ is $\alpha$-presentable, then $L(M)$ is $\alpha$-presentable.* □

**Theorem 4.7.** *Let $\mathbb{X}$ be a category. The following are equivalent:*

(1) *$\mathbb{X}$ is sketchable by a projective $\alpha$-sketch,*

(2) *$\mathbb{X}$ is $\alpha$-accessible and has all limits,*

(3) *$\mathbb{X}$ is $\alpha$-accessible and has all colimits,*

(4) *$\mathbb{X}$ is $\alpha$-accessible and $\mathbb{X}_\alpha$ has $\alpha$-small colimits,*

(5) *$\mathbb{X}$ is locally $\alpha$-presentable (in the sense of [7]).*

**Proof** (outline). Let us quickly explain how assuming (1) we can ensure $\alpha$-accessibility (and not just accessibility). Let $\mathbb{S}$ be a projective $\alpha$-sketch and $Y : \underline{\mathbb{S}} \to (\mathrm{Set}^{\underline{\mathbb{S}}})^{\mathrm{op}}$ be the Yoneda embedding. Let $\mathbb{X}''$ be the full subcategory of $\mathbb{X}$ whose objects are the $L(Y(S))$, $S \in \mathrm{Ob}(\underline{\mathbb{S}})$. Then $\mathbb{X}''$ is small and its objects are (by the lemma) $\alpha$-presentable objects of $\mathbb{X}$. Let $\mathbb{X}'$ be the closure of $\mathbb{X}''$ for $\alpha$-colimits. Then $\mathbb{X}'$ has the properties ensuring $\alpha$-accessibility for $\mathbb{X}$.

On the other hand, if $\mathbb{X}$ is locally $\alpha$-presentable, we get a projective $\alpha$-sketch by distinguishing in (the skeleton of) $\mathbb{X}_\alpha^{\mathrm{op}}$ all cones that arise as dual of an $\alpha$-colimit in $\mathbb{X}_\alpha$. This canonical sketch will be denoted by $/\mathbb{X}_\alpha^{\mathrm{op}}/$: it is easily seen that $\mathrm{Mod}(/\mathbb{X}_\alpha^{\mathrm{op}}/) \simeq \mathbb{X}$. □

From Theorems 4.7 and 4.3 we get the following corollary:

**Corollary 4.8.** *The class of projective $\alpha$-sketches is a linear conservative $\alpha$-variety. The category $\alpha$-*LOCP *whose objects are locally $\alpha$-presentable categories and whose arrows are $\alpha$-continuous functors is cartesian closed.* □

Our second example is related to Diers's work [5].

**Definition 4.9.** A sketch $\mathbb{S}$ is called *local* if every distinguished cocone of $\mathbb{S}$ is discretely indexed.

Of course, every projective sketch is a particular local sketch.

In [14] the following was shown:

**Lemma 4.10.** *Let $\mathbb{S}$ be a local $\alpha$-sketch.*

(a) *Every object $M$ of* $\mathrm{Set}^{\mathbb{S}}$ *generates a locally free* family $L(M) = (L(M)_x)_{x \in X}$ *in* $\mathrm{Mod}(\mathbb{S})$.

(b) *The set $X$ is unique up to bijection and the models $L(M)_x$ are unique up to isomorphism.*

(c) *If $M$ is $\alpha$-presentable, then the $L(M)_x$ are $\alpha$-presentable.* □

Using this lemma we can adapt the proof of Theorem 4.7 and state the following characterizations:

**Theorem 4.11.** *Let $\mathbb{X}$ be a category. The following are equivalent:*

(1) $\mathbb{X}$ *is sketchable by a local $\alpha$-sketch,*

(2) $\mathbb{X}$ *is $\alpha$-accessible and has all connected limits,*

(3) $\mathbb{X}$ *is $\alpha$-accessible and has all multicolimits,*

(4) $\mathbb{X}$ *is $\alpha$-accessible and $\mathbb{X}_\alpha$ has $\alpha$-small multicolimits,*

(5) $\mathbb{X}$ *is locally $\alpha$-multipresentable (in the sense of* [5]). □

**Corollary 4.12.** *The class of local $\alpha$-sketches is a linear conservative $\alpha$-variety. The category $\alpha$-*LOCMUP *whose objects are locally $\alpha$-multipresentable categories and whose arrows are $\alpha$-continuous functors is cartesian closed.* □

Let us turn to our third example, inspired by the work of Lamarche [21]. This one will be treated in some detail, since the corresponding variety of sketches is introduced here for the first time.

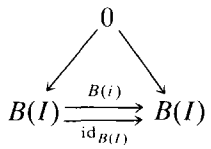
**Definition 4.13.** Let $\mathbb{I}$ be a small category whose only arrows are automorphisms and $D : \mathbb{I} \to \mathbb{C}$ be a *faithful* diagram of shape $\mathbb{I}$ in a category $\mathbb{C}$. Then we say that $D$ is a *disjoint sum of subgroups* in $\mathbb{C}$.

**Definition 4.14.** We shall call a sketch $\mathbb{S}$ *galoisian* iff:

(1) $\mathbb{S}$ has a distinguished cocone with empty basis, whose vertex is denoted by 0;

(2) for every distinguished cocone $q = (q_I : B(I) \to A)_{I \in \mathbb{I}}$ of $\mathbb{S}$

- $B$ is a disjoint sum of subgroups in $\underline{\mathbb{S}}$,
- if $i$ is an arrow of $\mathbb{I}$ which is not an identity, the following cone is distinguished:




**4.15. Comments and notations.** Note that condition (1) is compatible with condition (2). Note also that every local sketch is equivalent to a particular galoisian sketch.

If $\mathbb{S}$ is a galoisian sketch, $M$ a model of $\mathbb{S}$ and $q = (q_I : B(I) \to A)_{I \in \mathbb{I}}$ a distinguished cocone of $\mathbb{S}$, then for every $I \in \mathrm{Ob}(\mathbb{I})$, the set $X_I = M(B(I))$ carries a left action of the group $\mathrm{Aut}(I)$, defined by $i \odot x = M(B(i))(x)$.

Denote by $\mathbb{S}^-$ the sketch obtained from $\mathbb{S}$ by deleting all distinguished cocones as well as those distinguished cones whose presence is required by condition (2). Since $\mathbb{S}^-$ is a projective sketch, we have at our disposal the canonical model $Y : \mathbb{S}^- \to (\mathrm{Mod}(\mathbb{S}^-))^{\mathrm{op}}$ such that $\mathrm{Hom}_{\mathrm{Mod}(\mathbb{S}^-)}(Y(-), M) \cong M(-)$, for every model $M$. If $q = (q_I : B(I) \to A)_{I \in \mathbb{I}}$ is a distinguished cocone of $\mathbb{S}$, then its image by $Y$ is a cone $f = (f_I : C \to D(I))_{I \in \mathbb{I}}$ in $\mathrm{Mod}(\mathbb{S}^-)$. Now if $M$ is a model of $\mathbb{S}^-$, we define the relation $M <!!\ f$ by:

- $\forall c : C \to M\ \exists! I \in \mathrm{Ob}(\mathbb{I})\ \exists h : D(I) \to M,\ c = h \cdot f_I$,
- $\forall I \in \mathrm{Ob}(\mathbb{I})\ \forall h, h' : D(I) \to M,\ (h \cdot f_I = h' \cdot f_I) \Rightarrow (\exists! i : I \to I,\ h' = h \cdot D(i))$.

**Proposition 4.16.** *Let $\mathbb{S}$ be a galoisian sketch and $M$ be a model of $\mathbb{S}^-$. Then the following are equivalent:*

(a) *$M$ is a model of $\mathbb{S}$,*

(b) *for every distinguished cocone $q = (q_I : B(I) \to A)_{I \in \mathbb{I}}$ of $\mathbb{S}$, $M(q)$ is a colimit and the left actions of the groups $\mathrm{Aut}(I)$ on the sets $M(B(I))$ have trivial point stabilisers,*

(c) *for every distinguished cocone $q$ of $\mathbb{S}$, $M <!!\ Y(q)$.*

**Proof.** (b) $\Leftrightarrow$ (a) The assertion

$$\forall x \in X_I\ ,\quad \mathrm{Stab}_{\mathrm{Aut}(I)}(x) = \{\mathrm{id}_I\}$$

means exactly that

$$\forall i \in \mathrm{Aut(I)}\backslash\{\mathrm{id}_I\} \quad \{x \in X_I \mid i \odot x = x\} = \emptyset$$

or equivalently

$$\forall i \in \mathrm{Aut}(I)\backslash\{\mathrm{id}_I\} \quad \emptyset \to X_I \underset{\mathrm{id}_{X_I}}{\overset{M(B(i))}{\rightrightarrows}} X_I \quad \text{is a kernel in Set},$$

that is to say

$$\forall i \in \mathrm{Aut}(I)\backslash\{\mathrm{id}_I\} \quad M(0 \to B(I) \underset{\mathrm{id}_{B(I)}}{\overset{B(i)}{\rightrightarrows}} B(I)) \quad \text{is a kernel}.$$

(c) $\Leftrightarrow$ (b) Define $M <\!\!|\!| f$ just as $M <\!\!! f$ except that unicity of $i$ is not required. It is easily checked (see [20]) that $M <\!\!|\!| Y(q)$ means that $M(q)$ is a colimit cocone. Now the unicity assumption means that the right action of $\mathrm{Aut}(I)$ on $\mathrm{Hom}(D(I), M)$ defined by $h^* i = h \cdot D(i)$ has trivial point stabilisers. But $\mathrm{Hom}(D(I), M) = \mathrm{Hom}(Y(B(I), M) \cong M(B(I))$ and similarly $-^* i = \mathrm{Hom}(D(i), M) = \mathrm{Hom}(Y(B(i)), M) \cong M(B(i)) = i \odot -$, so this also amounts to saying that the left action of $\mathrm{Aut}(I)$ on $M(B(I))$ has trivial point stabilisers. □

**Proposition 4.17.** *Let* $\mathbb{S}$ *be a galoisian sketch and* $U : \mathrm{Mod}(\mathbb{S}) \to \mathrm{Mod}(\mathbb{S}^-)$ *be canonical forgetful functor. Then* $\mathrm{Mod}(\mathbb{S})$ *has and* $U$ *creates and preserves wide pullbacks.*

**Proof.** Let $f = (f_I : C \to D(I))_{I \in \mathbb{I}}$ be any cone in $\mathrm{Mod}(\mathbb{S}^-)$. We assume that $p = (p_I : L \to G(J))_{J \in \mathbb{J}}$ is a wide pullback in $\mathrm{Mod}(\mathbb{S}^-)$ and that $G(J) <\!\!! f$ for each $J \in \mathrm{Ob}(\mathbb{J})$. We are going to prove that $L <\!\!! f$. This will give us our result just by applying the characterization 4.16(c) of models.

Let us prove the first assertion in the definition of $L <\!\!! f$. Consider any arrow $c : C \to L$ in $\mathrm{Mod}(\mathbb{S}^-)$, and fix an object $J$ of $\mathbb{J}$. Define $c_J = p_J \cdot c$. Since $G(J) <\!\!! f$, we have:

$$\exists! I(J) \in \mathrm{Ob}(\mathbb{I}) \ \exists h_J : D(I(J)) \to G(J), \quad c_J = h_J \cdot f_{I(J)}.$$

Now if $j : J \to J'$ is any arrow in $\mathbb{J}$, we compute $h_{J'} \cdot f_{I(J')} = c_{J'} = p_{J'} \cdot c = G(j) \cdot p_J \cdot c = G(j) \cdot c_J = G(j) \cdot h_J \cdot f_{I(J)}$, so by the definition of $G(J) <\!\!! f$, we conclude $I(J) = I(J')$ and $\exists! i_{JJ'}$, $h_{J'} = G(j) \cdot h_J \cdot D(i_{JJ'})$. Recall that $\mathbb{J}$ has a final object $Z$: this implies (by connectedness) that all $I(J)$'s have a common value $I$, and that we get a cone $(h_J \cdot D(i_{JZ}) : D(I) \to G(J))_{J \in \mathbb{J}}$ (because $i_{ZZ} = \mathrm{id}_{D(I)}$). But $p$ is a limit cone: there is a unique $h : D(I) \to L$ such that $p_J \cdot h = h_J \cdot D(i_{JZ})$, for every $J$. Do we have $c = h \cdot f_I$? Yes, because for every $J$, $p_J \cdot (h \cdot f_I) = (p_J \cdot h) \cdot f_I = h_J \cdot D(i_{JZ}) \cdot f_I = h_J \cdot f_I = c_J = p_J \cdot c$. So this $h$ solves our problem (clearly there can be no factorisation of $c$ through any $I' \neq I$).

We now turn to the second assertion. Given $I \in \mathrm{Ob}(\mathbb{I})$, suppose $h \cdot f_I = h' \cdot f_I$. Then for every $J \in \mathrm{Ob}(\mathbb{J})$ we have $(p_J \cdot h) \cdot f_I = (p_J \cdot h') \cdot f_I$. From $G(J) <\!\!! f$,

we infer $\exists! i_J, p_J \cdot h' = p_J \cdot h \cdot D(i_J)$. If $j : J \to Z$ is the unique arrow from $J$ to the final object, we compute $p_Z \cdot h' = G(j) \cdot p_J \cdot h' = G(j) \cdot p_J \cdot h \cdot D(i_J) = p_Z \cdot h \cdot D(i_J)$, but also $p_Z \cdot h' = p_Z \cdot h \cdot D(i_Z)$: this gives us $D(I_J) = D(i_Z)$, hence all $i_J$ has a common value $i$. For every $J \in \mathrm{Ob}(\mathbb{J})$, we have $p_J \cdot h' = p_J \cdot h \cdot D(i)$, so $h' = h \cdot D(i)$. We have not proved yet that $i$ is unique, but this is clear, since $h \cdot D(i_1) = h \cdot D(i_2)$ implies (for any $J$) that $p_J \cdot c' \cdot D(i_1) = p_J \cdot c' \cdot D(i_2)$, whence $i_1 = i_2$. □

The key lemma (analogous to Lemmas 4.6 and 4.10) is the following:

**Lemma 4.18.** *Let $\mathbb{S}$ be a galoisian $\alpha$-sketch.*

(a) *Every object $M$ of $\mathrm{Set}^{\mathbb{S}}$ generates a locally free diagram $L(M) = (L(M)_I)_{I \in \mathbb{I}}$ in* Mod($\mathbb{S}$) *with the following properties*:

- *$L(M)$ is a disjoint sum of subgroups,*
- *two factorisations of the same arrow through $L(M)$ are* uniquely *connected.*

(b) *$\mathbb{I}$ is unique up to bijection and group isomorphisms and the models $L(M)_I$ are unique up to isomorphism.*

(c) *If $M$ is $\alpha$-presentable, then the $L(M)_I$ are $\alpha$-presentable.*

**Proof.** (a) We already know that every object $M$ of $\mathrm{Set}^{\mathbb{S}}$ generates some locally free diagram relatively to $U$ (Lemma 3.3), and that the canonical forgetful functor $\mathrm{Mod}(\mathbb{S}) \to \mathrm{Set}^{\mathbb{S}}$ preserves wide pullbacks (clear from Proposition 4.17, since $\mathbb{S}^-$ is a projective sketch); this makes it possible to apply Taylor's factorisation theorem [25]: every $M$ has a factorisation through a candidate. Now it is easily checked that the full subcategory of $M/U$ whose objects are the candidates over $M$ is a locally free diagram and satisfies the required properties.

(b) This was proved in [19].

(c) Let $X = \mathrm{colim}_{J \in \mathbb{J}} X(J)$ be an $\alpha$-filtered colimit in $\mathbb{X} = \mathrm{Mod}(\mathbb{S})$, let $(q_J)$ be the corresponding family of coprojections. We have to prove that the canonical map

$$\mathscr{A} : \mathrm{colim}_J \mathrm{Hom}(L(M)(I), X(J)) \to \mathrm{Hom}(L(M)(I), X)$$

is a bijection.

First let us prove that $\mathscr{A}$ is surjective. Given $x : L(M)(I) \to X$, define $z = x \cdot z_I : M \to X$. Since $M$ is $\alpha$-presentable, there exist $J \in \mathrm{Ob}(\mathbb{J})$ and $y : M \to X(J)$ such that $q_J \cdot y = z$. We have a factorisation of $y$ through the locally free diagram $L(M)$:

$$\exists \tilde{I} \in \mathrm{Ob}(\mathbb{I})\ \exists \tilde{y} : L(M)(\tilde{I}) \to X(J)\,, \quad \tilde{y} \cdot z_{\tilde{I}} = y\,.$$

But now $q_J \cdot \tilde{y} \cdot z_{\tilde{I}} = q_J \cdot y = z$, so we have found two factorisations of $z$ through $L(M)$: $z = x \cdot z_I$ and $z = (q_J \cdot \tilde{y}) \cdot z_{\tilde{I}}$. From the geometry of $L(M)$, we conclude that $\tilde{I} = I$ and that there is an $i : I \to I$ in $\mathbb{I}$ such that $q_J \cdot \tilde{y} = x \cdot g$ (where

$g = L(M)(i)$). Whence the existence of a decomposition of $x$ through the $X(J)$'s: $x = q_J \cdot (\tilde{y} \cdot g^{-1})$.

To establish that $\mathscr{A}$ is injective, it will be enough (since $\mathbb{J}$ is $\alpha$-filtered) to show that any two different decompositions $x = q_J \cdot t_J = q_{J'} \cdot t_{J'}$ (with $t_J : L(M)(I) \to X(J)$ and $t_{J'} : L(M)(I) \to X(J')$) of the same $x$ have a common refinement.

Assume those two decompositions. Then $q_J \cdot (t_J \cdot z_I) = q_{J'} \cdot (t_{J'} \cdot z_I)$. Using the $\alpha$-presentability of $M$, there are $j : J \to J''$, $j' : J' \to J''$ in $\mathbb{J}$ and $y : M \to X(J'')$ in $\mathbb{X}$ such that $X(j) \cdot (t_J \cdot z_I) = X(j') \cdot (t_{J'} \cdot z_I) = y$.

But now we have two factorisations of $y$ through $L(M)$: $y = (X(j) \cdot t_J) \cdot z_I$ and $y = (X(j') \cdot t_{J'}) \cdot z_I$.

From the geometry of $L(M)$, we conclude that $\tilde{I} = I$ and that there is $i : I \to I$ such that $X(j') \cdot t_{J'} = X(j) \cdot t_J \cdot g$ (where $g = L(M)(i)$). Next we notice that

$$x = q_{J'} \cdot t_{J'} = q_{J''} \cdot X(j') \cdot t_{J'} = q_{J''} \cdot X(j) \cdot t_J \cdot g = q_J \cdot t_J \cdot g = x \cdot g .$$

Define $z = x \cdot z_I$; again, we have two factorisations of $z$ through $L(M)$: $z = x \cdot z_I$ and $z = (x \cdot g) \cdot z_I$. Using the assumption of unicity of the connection, we assert:

$$\exists!\, i' : I \to I , \quad x \cdot g = x \cdot L(M)(i') .$$

As both $i' = i$ and $i' = \mathrm{id}_I$ will work, we conclude $i = \mathrm{id}_I$. Finally, we have obtained $j$, $j'$ with $X(j') \cdot t_{J'} = X(j) \cdot t_J$, the common refinement we were looking for. □

Using this lemma we can adapt the proof of Theorem 4.7 and state the following characterizations:

**Theorem 4.19.** *Let $\mathbb{X}$ be a category. The following are equivalent*:

(1) *$\mathbb{X}$ is sketchable by a galoisian $\alpha$-sketch*,

(2) *$\mathbb{X}$ is $\alpha$-accessible and has all wide pullbacks*,

(3) *$\mathbb{X}$ is $\alpha$-accessible and has all polycolimits*,

(4) *$\mathbb{X}$ is $\alpha$-accessible and $\mathbb{X}_\alpha$ has $\alpha$-small polycolimits*,

(5) *$\mathbb{X}$ is locally $\alpha$-polypresentable (in the sense of* [21]). □

**Corollary 4.20.** *The class of galoisian $\alpha$-sketches is a linear conservative $\alpha$-variety. The category $\alpha$-LOCPOP whose objects are locally polypresentable categories and whose arrows are $\alpha$-continuous functors is cartesian closed.* □

**Remarks.** The cartesian closedness of $\alpha$-LOCP and $\alpha$-LOCMUP were already proved in [1], but a misunderstanding of Lamarche's notion of a polycolimit made me state in that paper that the sketch method did not work for $\alpha$-LOCPOP. By very different methods, Taylor proved that the category whose objects are locally $\alpha$-polypresentable categories and whose arrows are *stable* $\alpha$-continuous functors is cartesian closed [26].

## 5. Some linear aspects

In this section, we describe some 'linear' features of the theory of locally $\alpha$-presentable categories (the word 'linear' is meant here in the sense of Girard's *linear logic*: [8, 23]). The results do not extend easily to other kinds of $\alpha$-accessible categories.

**Definitions 5.1.** Given two locally small categories $\mathbb{X}$ and $\mathbb{Y}$, a functor from $\mathbb{X}$ to $\mathbb{Y}$ will be called *linear* iff it preserves all colimits. $\mathrm{Lin}(\mathbb{X}, \mathbb{Y})$ denotes the category of linear functors from $\mathbb{X}$ to $\mathbb{Y}$ and natural transformations.

First we give a version of Girard's connective 'of course'. If $\mathbb{X}$ is a locally $\alpha$-presentable category, define the following:

**Definition 5.2.** $!_\alpha\mathbb{X} = \mathrm{Set}^{\mathbb{X}_\alpha^{\mathrm{op}}}$.

We have at our disposal the Yoneda embedding $\mathcal{Y} : \mathbb{X}_\alpha \to !_\alpha\mathbb{X}$. As in [10], we can make every $\alpha$-continuous functor linear, provided we change the source category:

**Theorem 5.3.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be two locally $\alpha$-presentable categories. Then there is an equivalence of categories*

$$\alpha\text{-}\mathrm{Cont}(\mathbb{X}, \mathbb{Y}) \underset{\text{Delin}}{\overset{\text{Lin}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathrm{Lin}(!_\alpha\mathbb{X}, \mathbb{Y})$$

*such that*
– *for every linear functor $G : !_\alpha\mathbb{X} \to \mathbb{Y}$, $\mathrm{Delin}(G)$ is the (unique) $\alpha$-continuous functor from $\mathbb{X}$ to $\mathbb{Y}$ whose restriction to $\mathbb{X}_\alpha$ is $G \circ \mathcal{Y}$;*
– *for every $\alpha$-continuous functor $F : \mathbb{X} \to \mathbb{Y}$, $\mathrm{Lin}(F)$ is the left Kan extension along $\mathcal{Y}$ of the restriction of $F$ to $\mathbb{X}_\alpha$.* □

Let us say (as in [7]) that a category is locally $\alpha$-copresentable if its opposite category is locally $\alpha$-presentable. (It is proved in [7] that the categories which are both locally presentable *and* locally copresentable are exactly those equivalent to a complete ordered set, i.e. such that every subset has a greatest lower bound—hence a least upper bound.)

**Theorem 5.4.** *Let $\mathbb{X}$ be a locally $\alpha$-presentable category and $\mathbb{Y}$ be a locally $\alpha$-copresentable category. Then the category $\mathrm{Lin}(\mathbb{X}, \mathbb{Y})$ is locally $\alpha$-copresentable.*

*More precisely, if $\mathbb{S}$ and $\mathbb{T}$ are two projective $\alpha$-sketches such that $X \simeq \mathrm{Mod}(\mathbb{S})$ and $\mathbb{Y} \simeq \mathrm{Mod}(\mathbb{T})$, then $\mathrm{Lin}(\mathbb{X}, \mathbb{Y})^{\mathrm{op}} \simeq \mathrm{Mod}(\mathbb{S} \otimes \mathbb{T})$.*

**Proof.** Consider the category Diag($\mathbb{X}$) of all diagrams in $\mathbb{X}$. If $\mathbb{X}$ is locally $\alpha$-presentable, it is known that Diag($\mathbb{X}$) is also locally $\alpha$-presentable (construct a sketch!—or see [22]) and that the $\alpha$-presentable objects of Diag($\mathbb{X}$) are just the $\alpha$-small diagrams of $\alpha$-presentable objects of $\mathbb{X}$. It follows that every object $D$ of Diag($\mathbb{X}$) is an $\alpha$-filtered colimit of such diagrams. We now claim that a functor $F : \mathbb{X} \to \mathbb{Y}$ is linear if (and only if) $F$ is $\alpha$-continuous (i.e. preserves $\alpha$-filtered colimits) and preserves $\alpha$-small colimits of $\alpha$-presentable objects. Indeed, let $D = \operatorname{colim}_J D_J$ be an arbitrary diagram in $\mathbb{X}$, written as an $\alpha$-filtered colimit of $\alpha$-presentable diagrams. Then

$$
\begin{aligned}
& F(\operatorname{colim} D) \\
& \quad \cong F(\operatorname{colim}(\operatorname{colim}_J D_J)) \\
& \quad \cong F(\operatorname{colim}_J(\operatorname{colim} D_J)) \quad (1) \\
& \quad \cong \operatorname{colim}_J F(\operatorname{colim} D_J) \quad (2) \\
& \quad \cong \operatorname{colim}_J \operatorname{colim}(F \circ D_J) \quad (2) \\
& \quad \cong \operatorname{colim} \operatorname{colim}_J (F \circ D_J) \\
& \quad \cong \operatorname{colim}(F \circ \operatorname{colim}_J D_J) \quad (3) \\
& \quad \cong \operatorname{colim}(F \circ D) .
\end{aligned}
$$

(1): because the functor colim : Diag($\mathbb{X}$) $\to \mathbb{X}$ is linear (it is a left adjoint).
(2): by hypothesis on $F$.
(3): because the functor Diag($F$) is $\alpha$-continuous.

In other words, in the equivalence $\alpha$-Cont($\mathbb{X}, \mathbb{Y}$) $\simeq \mathbb{Y}^{\mathbb{X}_\alpha}$ (which clearly extends to this case because $\mathbb{Y}$ has $\alpha$-filtered colimits), the linear functors correspond to those functors from $\mathbb{X}_\alpha$ to $\mathbb{Y}$ that preserve colimits of $\alpha$-presentable objects of $\mathbb{X}$ (these colimits are known to be themselves $\alpha$-presentable). This means that we have the following equivalence of categories:

$$
\operatorname{Lin}(\mathbb{X}, \mathbb{Y}) \simeq \operatorname{Mod}(/\mathbb{X}_\alpha^{\mathrm{op}}/^{\mathrm{op}}, \mathbb{Y}) \simeq \operatorname{Mod}(/\mathbb{X}_\alpha^{\mathrm{op}}/, \mathbb{Y}^{\mathrm{op}})^{\mathrm{op}} ,
$$

where $/\mathbb{X}_\alpha^{\mathrm{op}}/$ is the projective $\alpha$-sketch described in the proof of Theorem 4.7. Suppose now that $\mathbb{Y}^{\mathrm{op}} \simeq \operatorname{Mod}(\mathbb{T}, \mathrm{Set})$, with $\mathbb{T}$ a projective $\alpha$-sketch, then

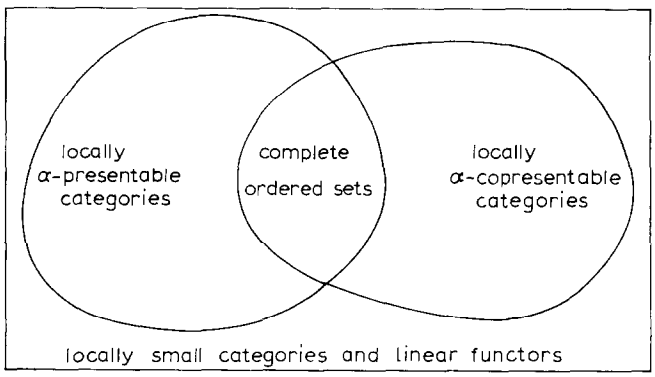
$$
\begin{aligned}
\operatorname{Lin}(\mathbb{X}, \mathbb{Y})^{\mathrm{op}} &\simeq \operatorname{Mod}(/\mathbb{X}_\alpha^{\mathrm{op}}/, \operatorname{Mod}(\mathbb{T})) \simeq \operatorname{Mod}(\mathbb{T}, \operatorname{Mod}(/\mathbb{X}_\alpha^{\mathrm{op}}/)) \\
&\simeq \operatorname{Mod}(\mathbb{T}, \mathbb{X}) \simeq \operatorname{Mod}(\mathbb{T}, \operatorname{Mod}(\mathbb{S})) \simeq \operatorname{Mod}(\mathbb{S} \otimes \mathbb{T})
\end{aligned}
$$

and $\mathbb{S} \otimes \mathbb{T}$ is again a projective $\alpha$-sketch. □

Locally $\alpha$-presentable categories do not seem to give rise to any *-autonomous

structure (or model of linear logic, see [23]). But we do find a slightly weaker structure:

**Corollary 5.5.** *The following picture*




*with the pairing* $\mathbb{X} - \circ \mathbb{Y} = \mathrm{Lin}(\mathbb{X}, \mathbb{Y})$ *and the dualizing object* $\perp = \mathrm{Set}^{\mathrm{op}}$ *describes a pre-*-autonomous situation in the sense of* [2]. □

**Conclusion.** The reader familiar with coherence spaces semantics [10] may have noticed that our computations with accessible categories at the level of sketches are somewhat analogous to the computations with coherence spaces at the level of webs. This suggests in return to view the web of a coherence space as some kind of logical theory, and indeed, there is some evidence for that.

**Note added in March 1991.** It is well known that there can be no denotational semantics for classical logic, because boolean bicartesian closed categories are degenerate. In spite of that, Girard proposed some weeks ago a denotational semantics for classical logic! (*A new constructive logic: classical logic*, manuscript, 44 pp.) Although I am not sure whether his constructions deserve to be called classical logic, they are very interesting. Using his *coherence spaces*, he describes nothing but a particular *pre-*-autonomous situation*, from which he builds up the 'classical' connectives. (Obviously, the 'and' is not a categorical product.) The categorical analysis of what he does and the links with other examples of pre-*-autonomous situations (like Corollary 5.5 above) will be studied in a forthcoming paper.

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