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The intersection of algebra and coalgebra☆

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Abstract

Presheaf categories are well-known to be varieties of algebras and covarieties of coalgebras. We prove the converse: if a category is a variety as well as a covariety, then it is a presheaf category. Our main result is that all coalgebras on a set functor H form a presheaf category iff H is a reduction of a polynomial functor.
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1. Introduction

The aim of the paper is to prove the equation

$$\text{algebra} \cap \text{coalgebra} = \text{presheaves}$$

over many-sorted sets. In the more restrictive case of algebra and coalgebra over **Set** the equation is

$$\text{algebra} \cap \text{coalgebra} = \text{monoid actions}.$$

This shows that here, essentially, just sequential automata form the intersection of algebra and coalgebra. In fact, a sequential automaton can be viewed as an algebra, or as a coalgebra: the main ingredient, the next-state function

$$\delta\colon Q \times I \longrightarrow Q \quad (I = \text{the input set})$$

defines an algebra of the endofunctor $(-) \times I$ of **Set**, but by currying it

$$\hat{\delta}\colon Q \longrightarrow Q^I$$

one gets a coalgebra of the endofunctor $(-)^I$. Now suppose that M is a monoid, then the category of M-sets (i.e., sets with a monoid action of M) is a subcategory of the category of sequential automata with the input set $I = M^*$, and

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M-sets are easily seen to be both a variety and a covariety of sequential automata. Is this a unique such situation, or are there other interesting examples of coalgebras that are algebras? A surprisingly general example was discovered by James Worrell: he proved in [21] that for every (not necessarily finitary) signature Σ we can view Σ-coalgebras, i.e., coalgebras of the *polynomial endofunctor* of **Set**

$$H_\Sigma Q = \coprod_{\sigma\in\Sigma} Q^n \quad (n = \text{arity of } \sigma)$$

as a variety of algebras. In fact, the category **Coalg** H_Σ of all Σ-coalgebras is equivalent to a presheaf category $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ for some small category $\mathcal{A}$, see [21]. Now $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ is always a variety of unary algebras—but not always one-sorted! Thus, the slogan

all Σ-coalgebras form a variety of algebras

is, in general, only true if we move from one-sorted sets to many-sorted ones. Therefore, in the present paper we consider algebra and coalgebra over many-sorted sets (given by endofunctors of $\mathbf{Set}^S$, the category of S-sorted sets). Our results also hold for base categories of the form $\mathbf{Set}^{\mathcal{C}}$ where $\mathcal{C}$ is a small category, see 5.3.

We are going to describe the intersection of algebra and coalgebra, i.e., those categories which are at the same time varieties of F-algebras and covarieties of G-coalgebras for endofunctors F and G of many-sorted sets. We consider all these categories as *concrete categories*, i.e., pairs consisting of a category $\mathcal{V}$ and a faithful ("forgetful") functor $V\colon \mathcal{V} \longrightarrow \mathbf{Set}^S$. Given two concrete categories $V_i\colon \mathcal{V}_i \longrightarrow \mathbf{Set}^S$ for $i = 1, 2$ we call them *concretely equivalent* if there exists an equivalence functor $E\colon \mathcal{V}_1 \longrightarrow \mathcal{V}_2$ such that V_1 is naturally isomorphic to $V_2 \cdot E$; notation $\mathcal{V}_1 \simeq \mathcal{V}_2$ (see [15]).

We will strengthen the result of James Worrell in several directions:

(1) Considering presheaf categories $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ as concrete categories over **Set**, the category of Σ-coalgebras is *concretely* equivalent to a presheaf category. And we prove the converse, which is our main result: given an endofunctor H of **Set** such that **Coalg** H is concretely equivalent to a presheaf category, then H is a reduction of a polynomial functor—thus **Coalg** H is the category of Σ-coalgebras for some Σ. ("Reduction" means that the value at the empty set can be changed.)

(2) Considering $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ as a concrete category over $\mathbf{Set}^S$, where S is the set of objects of $\mathcal{A}$, we prove that the presheaf category is *always* concretely equivalent to a covariety of coalgebras. And conversely: every many-sorted variety concretely equivalent to a many-sorted covariety is a category of presheaves.

(3) In contrast to (2), only very special small categories $\mathcal{A}$ have the property that $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ is concretely equivalent to **Coalg** H over **Set**: $\mathcal{A}$ has to be equivalent to the Σ-tree category for some signature Σ. This category has all Σ-trees (see Example 2.7) as objects, and morphisms from t' to t are all nodes of t whose subtree (in t) is t'.

For all these results we work with concrete categories over $\mathbf{Set}^S$. The fact that Worrell's result about polynomial endofunctors of **Set** can be strengthened as in (1) above makes heavy use of concrete equivalence: we do not know the answer to the:

Open problem. For which endofunctors H of $\mathbf{Set}^S$ is the category **Coalg** H equivalent to a presheaf category?

The present paper is an expanded version of the paper [1] presented at the conference "Algebra and Coalgebra in Computer Science", CALCO 2005, in Swansea.

2. Varieties and covarieties

Remark 2.1. What is a many-sorted variety? Each of the following is a reasonable answer, depending on the generality one has in mind:

(a) An equationally presentable category of Σ-algebras, where Σ is a finitary, S-sorted signature, see [6] for the one-sorted case, and [7,13] for the many-sorted case.

(b) As above, but dropping "finitary". Thus, an *S-sorted signature* is a set Σ together with an *arity* of every operation symbol $\sigma \in \Sigma$ of the form

$$\sigma\colon (s_i)_{i<n} \longrightarrow s,$$

where s_i and s are sorts, and n is a cardinal number. The sort s is the *output* sort of σ; we denote by $\Sigma_s \subseteq \Sigma$ the set of all symbols of output sort s. The concept of an equationally presentable category of Σ-algebras is analogous to the finitary case (a).

(c) A full subcategory of **Alg** H, the category of H-algebras, for a *varietor* on $\mathbf{Set}^S$ (i.e., an endofunctor H of $\mathbf{Set}^S$ having free algebras), closed under products, subalgebras, and quotient algebras. See e.g. [17].

(d) A monadic category over $\mathbf{Set}^S$, see [14].

Fortunately, our results about algebra meeting coalgebra are independent of the generality one chooses: even if varieties are understood as monadic categories, the case (d), the meet of algebra and coalgebra is given by presheaf categories, a special case of (a). In fact, we can even move from varieties (equationally specified classes) to quasivarieties (implicationally specified classes) and obtain the same result: see Section 6.

Notation 2.2. For an endofunctor H of $\mathbf{Set}^S$ we denote by **Alg** H the category of all *H-algebras*, i.e., pairs (A, α) where A is an S-sorted set and $\alpha\colon HA \longrightarrow A$ an S-sorted function. Morphisms $f\colon (A, \alpha) \longrightarrow (B, \beta)$, called *homomorphisms*, are the S-sorted functions with $f \cdot \alpha = \beta \cdot Hf$.

For example,

$$\mathbf{Alg}\, H_\Sigma$$

is the usual category of Σ-algebras. Here $H_\Sigma\colon \mathbf{Set}^S \longrightarrow \mathbf{Set}^S$ is the *polynomial functor* defined on objects $X = (X_s)_{s\in S}$ by

$$(H_\Sigma X)_s = \coprod_{\sigma\in\Sigma_s} \prod_{i<n} X_{s_i} \quad \text{for all } s \in S,$$

where the arity of σ is $\sigma\colon (s_i)_{i<n} \longrightarrow s$ (and analogously on morphisms).

Remark 2.3. What is a many-sorted covariety? Each of the following is a reasonable answer, depending on the generality one has in mind:

(a) A coequationaly presentable category of Σ-algebras, where Σ is a (not necessarily finitary) S-sorted signature. That is, a full subcategory of **Coalg** H_Σ presentable by subsets D of the cofree coalgebras $C_\Sigma(X)$ (i.e., sets D of colored trees, where a coloring is a map into X, see 2.7). A coalgebra satisfies D iff, under any coloring by colors from X, the tree expansions of all states are trees lying in D. See [18].

(b) A full subcategory of **Coalg** H, the category of H-coalgebras, for a *covarietor* on $\mathbf{Set}^S$ (i.e., an endofunctor H of $\mathbf{Set}^S$ having cofree coalgebras), closed under coproducts subcoalgebras, and quotient coalgebras. See [18].

(c) A comonadic category over $\mathbf{Set}^S$, see e.g. [3].

Fortunately, our results about algebra meeting coalgebra are independent of the generality one chooses: even if covarieties are understood as comonadic categories, the case (c), the meet of algebra and coalgebra is given by presheaf categories, a special case of (a). In fact, we can even move from covarieties (coequationally specified classes) to quasicovarieties (implicationally specified classes) and obtain the same results. See Section 6.

Notation 2.4. (1) For an endofunctor H of $\mathbf{Set}^S$ we denote by **Coalg** H the category of all *H-coalgebras*, i.e., pairs (A, α) consisting of an S-sorted set A and an S-sorted function $\alpha\colon A \longrightarrow HA$. Morphisms $f\colon (A, \alpha) \longrightarrow (B, \beta)$, called *homomorphisms*, are the S-sorted functions such that $\beta \cdot f = Hf \cdot \alpha$.

For example, a *Σ-coalgebra*, i.e., an object of **Coalg** H_Σ, consists of an S-sorted set A and an S-sorted function α assigning to every element $a \in A_s$ an n-tuple $(a_i)_{i<n}$ in $\prod_{i<n} A_{s_i}$ for some operation $\sigma\colon (s_i)_{i<n} \longrightarrow s$ of Σ_s. Notation:

$$\alpha(a) = \sigma(a_i)_{i<n}.$$

(2) Given an S-sorted set X (of colors), a *cofree coalgebra* on X is a coalgebra $C(X)$ with a structure map

$$\psi_X\colon C(X) \longrightarrow HC(X)$$

together with a universal coloring

$$C(X) \xrightarrow{\varepsilon_X} X.$$

The universal property means that for every coalgebra $\alpha: A \longrightarrow HA$ and every coloring $f: A \longrightarrow X$ there exists a unique coalgebra homomorphism $\bar{f}: A \longrightarrow C(X)$ such that $f = \varepsilon_X \cdot \bar{f}$.

Example 2.5. As observed by Jan Rutten [18], deterministic systems having binary input and halting states (not reacting to inputs) are coalgebras of the one-sorted polynomial endofunctor

$$H_\Sigma X = X \times X + 1.$$

A coalgebra consists of a set Q of states and a function $\alpha: Q \longrightarrow Q \times Q + 1$ assigning to a non-halting state the pair of next states, and to a halting state the element 1. A cofree coalgebra $C_\Sigma(k)$ on k colors is the coalgebra of all binary trees, possibly infinite, colored by k colors. In fact, given a coalgebra Q and a coloring $f: Q \longrightarrow k$ of its states, every state q in Q produces a k-colored tree $\bar{f}(q)$ which is the tree expansion of the system with q as initial state. This defines a function $\bar{f}: Q \longrightarrow C_\Sigma(k)$ which is the unique homomorphism extending f. Here are two concrete examples:

(i) Consider the subset

$$D = C(1) - \{t_0\}$$

of all trees distinct from the trivial, root-only, tree t_0. A coalgebra satisfies D iff it has no halting states. This covariety is also a variety: each such coalgebra is given by a function

$$\alpha: Q \longrightarrow Q \times Q$$

corresponding to a pair of endofunctions of Q, i.e., we have the corresponding algebra structure

$$\hat{\alpha}: Q + Q \longrightarrow Q.$$

Thus, the covariety presented by D is nothing else than the category

$$\mathbf{Alg}(\mathrm{Id} + \mathrm{Id})$$

of algebras of the functor $\mathrm{Id} + \mathrm{Id}$.

(ii) Next, consider the subset

$$D' \subseteq C(1)$$

of all finite trees. A coalgebra satisfies D' iff the system always halts in finitely many steps. Is the corresponding covariety a variety? No! It is easy to verify that the covariety presented by D' does not have products on the level of **Set**. (Example: if A is a coalgebra whose states are all halting and B is a coalgebra with no halting state, then $A \times B$ is empty.) Since varieties of one-sorted algebras have products on the level of **Set**, this finishes the argument.

Remark 2.6. (a) To say that H has cofree coalgebras means that the forgetful functor $U: \mathbf{Coalg}\, H \longrightarrow \mathbf{Set}^S$ has a right adjoint. Such functors H are called *covarietors* in [3].

(b) We can interpret $C(X)$ as the collection of all possible behaviors of systems (represented as coalgebras) whose states, colored by colors from X, are not observable, but their colors are. If a system has the state set Q and we color the states, using an arbitrary $f: Q \longrightarrow X$, we can assign to every state $a \in Q$ the corresponding behavior $\bar{f}(a)$ we observe.

Example 2.7 (*See [3]*). For a polynomial functor H_Σ we have the cofree coalgebra $C_\Sigma(X)$ of all X-colored Σ-trees. More detailed:

(1) By a *Σ-tree* τ is meant a tree[1] labelled in Σ in such a way that every node labelled by a symbol σ of arity $\sigma: (s_i)_{i<n} \longrightarrow s$ (such nodes are said to *have sort s*) has n children, and the ith child has sort s_i. The sort of the root is called the *sort of the tree* τ. We denote by T_Σ the S-sorted set of all Σ-trees. Then T_Σ is a terminal Σ-coalgebra whose coalgebra structure is the inverse of tree-tupling.

[1] Trees are understood to be ordered, with labelled nodes throughout the paper. And we consider trees always up to isomorphism (of labelled, ordered trees) only.

(2) Given an S-sorted set X of "colors", by an *X-colored Σ-tree* is meant a Σ-tree together with a coloring of its nodes: every node of sort s is colored by an element of X_s. The color of the root is called the *color of the tree*. We denote by $C_\Sigma(X)$ the S-sorted set of all X-colored trees. We have the universal map

$$\varepsilon_X : C_\Sigma(X) \longrightarrow X$$

assigning to every tree its color.

(3) The coalgebra structure ψ_X of the cofree coalgebra $C_\Sigma(X)$ has sorts

$$(\psi_X)_s : C_\Sigma(X)_s \longrightarrow \coprod_{\sigma\in\Sigma_s} \prod_{i<n} C_\Sigma(X)_{s_i}$$

which assign to every tree τ whose root is labelled by σ of arity $\sigma : (s_i)_{i<n} \longrightarrow s$ the n-tuple of its children in $\prod_{i<n} C_\Sigma(X)_{s_i}$.

(4) For every Σ-coalgebra (A, α) and every coloring $f : A \longrightarrow X$ the unique homomorphism

$$\bar{f} : A \longrightarrow C_\Sigma(X) \quad \text{with } f = \varepsilon_X \cdot \bar{f}$$

can be described as follows. Given an element $a \in A_s$ then $\alpha(a)$ has the form

$$\alpha(a) = \sigma(a_i)_{i<n} \quad \text{for some } \sigma \in \Sigma_s$$

where the sorts s_i of a_i are such that σ has the arity $\sigma : (s_i)_{i<n} \longrightarrow s$. This gives us an unfolding tree τ of a: the root of τ is labelled by σ and colored by $f_s(a)$, the children a_i of the root are labelled (recursively) by an operation in Σ and colored by $f_{s_i}(a_i)$, etc. We put

$$\bar{f}_s(a) = \tau.$$

Definition 2.8 (*See [18]*). Let H be a covarietor. By a *coequation* is meant a subobject

$$d : D \hookrightarrow C(X)$$

of a cofree coalgebra. We say that a coalgebra A *satisfies* D provided that for every coloring $f : A \longrightarrow X$ the corresponding homomorphism $\bar{f} : A \longrightarrow C(X)$ factorizes through d. By a *covariety* of S-sorted coalgebras is meant a full subcategory of **Coalg** H, where H is a covarietor in $\mathbf{Set}^S$, which can be presented by coequations; or, equivalently, which is closed under coproducts, subcoalgebras, and quotient coalgebras.

Example 2.9. *M-sets as a variety.* Unary algebras with operations indexed by a set M are just algebras of the functor $X \longmapsto M \times X$. If $(M, *, e)$ is a monoid with unit e, recall that an *M-set* is a unary algebra $\alpha : M \times A \longrightarrow A$ such that

$$\alpha(e, x) = x \quad \text{for } x \in A$$

and

$$\alpha(u*v, x) = \alpha(u, \alpha(v, x)) \quad \text{for } x \in A \text{ and } u, v \in M.$$

This is, obviously, precisely the variety of unary algebras presented by the equations $e(x) = x$ and $w(x) = u(v(x))$ for all $w = u*v$ in M.

Example 2.10. *M-sets as a covariety.* Here we use the obvious currification for every unary algebra A:

$$\frac{M \times A \longrightarrow A}{A \longrightarrow A^M}$$

to obtain a coalgebra of the set functor $X \longmapsto X^M$. Observe that this functor is polynomial: consider one M-ary operation. The elements of a cofree coalgebra $C(X)$ are the trees colored in X such that the children of every node are indexed by M. Therefore, the nodes of such a tree are in a bijective correspondence with words in M^*: the empty word is the root, and the children of a word w are the words wm for all $m \in M$. Consequently, to present a member of

$C(X)$ means to present a coloring of the words in M^* by colors in X. Shortly: we can consider $C(X)$ as the set of all colorings:

$$C(X) = X^{M^*}.$$

An example of a coequation: the set

$$D \subseteq C(2)$$

of all colorings of M^* by two colors such that
(a) e has the same color as the empty word, and
(b) if $u*v = w$ in M then the one-letter word w has the same color as the two-letter word uv.
It is easy to verify that the coequation D presents precisely the M-sets in $\mathbf{Coalg}(-)^M$.

Remark 2.11. For every coequation $d\colon D \hookrightarrow C(X)$ denote by $\bar{d}\colon \overline{D} \hookrightarrow C(X)$ the largest subcoalgebra of $C(X)$ contained in D (i.e., the union of all subcoalgebras of $C(X)$ which factorize through d). Then D and $\overline{D}$ present the same coalgebras A. In other words, A satisfies D iff it satisfies $\overline{D}$, see [9].

3. Coalgebras as presheaves

Assumption 3.1. Unlike the rest of the paper, in the present section we work with one-sorted algebras and coalgebras only. Thus "concrete category" means here a category together with a faithful functor into **Set**. Examples include
(a) **Alg** H and **Coalg** H for endofunctors H of **Set**, and
(b) presheaf categories $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ (where $\mathcal{A}$ is any small category) endowed with the forgetful functor

$$V\colon \mathbf{Set}^{\mathcal{A}^{\mathbf{op}}} \longrightarrow \mathbf{Set}, \quad V(A) = \coprod_{s \in \mathbf{obj}\,\mathcal{A}} A(s).$$

Remark 3.2. For every one-sorted signature Σ each Σ-coalgebra (A, α) defines the following graph (with multiple, directed edges) on the set A: the edges from a node b into a node a with $\alpha(a) = \sigma(a_i)_{i<n}$ are precisely all the indices i with $b = a_i$.

For example, the graph of the terminal coalgebra T_Σ, see Example 2.7(1), has (uncolored) Σ-trees as nodes, and edges from t' to t are all children of t which are (isomorphic to) t'.

Notation 3.3. Let $\mathcal{T}_\Sigma$ be the free category on the graph given by the terminal Σ-coalgebra T_Σ; we can describe $\mathcal{T}_\Sigma$ as follows:
- objects are all Σ-trees;
- morphisms from t' to t are all nodes of t such that the corresponding subtree of t is (isomorphic to) t';
- identity morphisms are given by the roots of trees;
- composition is obvious, given by the transitivity of the relation "subtree".

In particular, every morphism is a composite of the *basic morphisms* $t' \longrightarrow t$ given by all children of t equal to t'.

Theorem 3.4. *For every one-sorted signature Σ the category of Σ-coalgebras is concretely equivalent, over* **Set**, *to the presheaf category of $\mathcal{T}_\Sigma$*:

$$\mathbf{Coalg}\, H_\Sigma \simeq \mathbf{Set}^{\mathcal{T}_\Sigma^{\mathbf{op}}}.$$

Remark. The equivalence of **Coalg** H_Σ to a presheaf category was proved by James Worrell [21]; he used wide-pullbacks preserving functors instead of polynomial functors, but for set functors these are the same. The proof below is due to Hans Porst and Christian Dzieron [16], except that the concreteness of the equivalence was not mentioned there.

Proof. For every H_Σ-coalgebra A denote by

$$h_A\colon A \longrightarrow T_\Sigma$$

the unique homomorphism (which takes every element to the tree unfolding). Define a functor $E\colon \mathbf{Coalg}\, H_\Sigma \longrightarrow \mathbf{Set}^{\mathcal{T}_\Sigma^{\mathbf{op}}}$ as follows. With every coalgebra A associate a presheaf $EA\colon \mathcal{T}_\Sigma^{\mathbf{op}} \longrightarrow \mathbf{Set}$ defined on objects $t \in T_\Sigma$ by

$$EA(t) = h_A^{-1}(t).$$

Thus, to every tree t we assign the set of all elements of A which unfold to t. For every basic morphism $j\colon t' \longrightarrow t$ (where $t = \sigma(t_i)_{i<n}$ and $t' = t_j$) the function $EA(j)\colon h_A^{-1}(t) \longrightarrow h_A^{-1}(t')$ assigns to every element a with $h_A(t) = a$ the element a_j, where $\alpha(a) = (a_i)_{i<n}$.

With every coalgebra homomorphism $f\colon (A, \alpha) \longrightarrow (B, \beta)$ we associate a natural transformation $Ef\colon EA \longrightarrow EB$ whose t-component is the restriction of f to $h_A^{-1}(t) \longrightarrow h_B^{-1}(t)$. Then E is an equivalence functor, see [16].

Recall that $U\colon \mathbf{Coalg}\, H_\Sigma \longrightarrow \mathbf{Set}$ denotes the forgetful functor; observe that the functor $V.E\colon \mathbf{Coalg}\, H_\Sigma \longrightarrow \mathbf{Set}$ takes a coalgebra A to the isomorphic copy of UA one gets from the bijection between UA and $\coprod h_A^{-1}(t)$. This isomorphism $UA \simeq VEA$ is clearly natural in A. □

Definition 3.5. A set functor H' is called a *reduction* of H if the domain restrictions of H and H' to all non-empty sets are naturally isomorphic.

Remark 3.6. In case H' is a reduction of H, $\mathbf{Coalg}\, H$ is, obviously, concretely equivalent to $\mathbf{Coalg}\, H'$. Thus we have an immediate:

Corollary 3.7. *For every reduction H of a polynomial functor the category* $\mathbf{Coalg}\, H$ *is concretely equivalent to a presheaf category.*

Example 3.8. (i) For the constant functor C_1 of value 1 we have the reduction $\widehat{C}_1$ with $\widehat{C}_1\emptyset = \emptyset$.

(ii) Let Σ be the signature with a nullary symbol, then H_Σ has C_1 (the constant functor of value 1) as a summand. Consequently, reductions H'_Σ of H_Σ can have *any* value at $\emptyset$: define $H'_\Sigma\emptyset = R$ (an arbitrary set) and for the empty map $f\colon \emptyset \longrightarrow X$ let $H'_\Sigma f\colon M \longrightarrow H_\Sigma X$ factor through the summand $1 = C_1X \hookrightarrow H_\Sigma X$ above.

(iii) If Σ has no nullary symbols, then H_Σ has no reductions $H \neq H_\Sigma$: any element of $H\emptyset$ leads to a natural transformation from $\widehat{C}_1$ to H_Σ. But since clearly $\hom(\widehat{C}_1, H_\Sigma) = \emptyset$, we get $H\emptyset = \emptyset$. This means $H = H_\Sigma$.

Theorem 3.9. *For a set functor H the category* $\mathbf{Coalg}\, H$ *is concretely equivalent to a presheaf category iff H is a reduction of a polynomial functor.*

Proof. Sufficiency is Corollary 3.7. For the necessity, let $\mathcal{A}$ be a small category such that $\mathbf{Coalg}\, H$ is concretely equivalent to $V\colon \mathbf{Set}^{\mathcal{A}^{\mathbf{op}}} \longrightarrow \mathbf{Set}$ of 3.1(b). Observe that V is a coproduct of hom-functors: in fact, the functor $A \longmapsto A(s)$ of evaluation at s is, by the Yoneda lemma, representable by the object $\mathcal{A}(-, s)$ of $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$. Since hom-functors preserve limits, and coproducts commute in $\mathbf{Set}$ with connected limits, we conclude that

V preserves connected limits.

Consequently, so does the forgetful functor $U\colon \mathbf{Coalg}\, H \longrightarrow \mathbf{Set}$ (since $U \approx V \cdot E$ for some equivalence functor $E\colon \mathbf{Coalg}\, H \longrightarrow \mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$).

We will prove next that H preserves nonempty wide pullbacks, i.e., limits of cocones $p_i\colon P_i \longrightarrow P$ $(i \in I)$ in $\mathbf{Set}$, provided that the limit cone $q_i\colon Q \longrightarrow Q_i$ $(i \in I)$ has nonempty domain, $Q \neq \emptyset$. In fact, given elements

$$x_i \in HP_i \ (i \in I) \quad \text{and} \quad x \in HP$$

with $Hp_i(x_i) = x$, it is our task to prove that there is a unique $y \in HQ$ with $x_i = Hq_i(y)$ for all $i \in I$. We can endow each P_i with the constant function of value x_i and obtain an H-coalgebra; analogously with P. This turns each p_i into a coalgebra homomorphism. Now $\mathbf{Coalg}\, H$ is complete (because $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ is), thus, it has a wide pullback of that cocone of homomorphisms. Since U preserves this wide pullback, we have a coalgebra structure $\rho\colon Q \longrightarrow HQ$ turning each q_i into a homomorphism (and forming the domain of a wide pullback in $\mathbf{Coalg}\, H$). For every element $z \in Q$ the element $y = \rho(z)$ of HQ has the desired property: for each $i \in I$ we have

$$Hq_i(y) = (Hq_i \cdot \rho)(z) = (\mathrm{const}x_i \cdot q_i)(z) = x_i.$$

And y is unique: suppose y' also fulfils $Hq_i(y') = x_i$, then the constant function of value y' turns each q_i into a coalgebra homomorphism, from which it easily follows that ρ is equal to this constant function; thus, $y = y'$.

We are ready to prove that H is a reduction of a polynomial functor, using results of Věra Trnková on the structure of set functors. Express H as a coproduct of functors H_i preserving terminal objects, $H_i 1 \approx 1$, which is possible by [19], see I.11. It is clear that since H preserves nonempty wide pullbacks, so does each H_i. It is sufficient to prove that, then, H_i is a reduction of a representable functor—it follows that H is a reduction of a coproduct of representable functors. This concludes the proof: polynomial functors are precisely the coproducts of representable functors.

Since H_i preserves terminal objects and nonempty wide pullbacks, it preserves nonempty products. Then the reduction $\widehat{H}_i$ of H_i with $\widehat{H}_i\emptyset = \emptyset$ preserves products. Věra Trnková calls a set functor separating if it preserves disjointness of pairs of subobjects. Every separating functor preserves finite intersections, see Corollary 2.1 in [19]. Consequently, if $\widehat{H}_i$ is separating, then it preserves limits, thus, it is representable. If $\widehat{H}_i$ is not separating, then it is naturally isomorphic to $\widehat{C}_1$, see 3.8, by [20], IV.4. This is a reduction of C_1 (representable by $\emptyset$). Thus, $\widehat{H}_i$ is a reduction of a representable functor, which implies the same for H_i. □

4. Presheaves as coalgebras

Notation 4.1. Throughout this section $\mathcal{A}$ denotes a small category, and we consider S-sorted sets for

$$S = \mathbf{obj}\,\mathcal{A}.$$

Thus, unlike 3.1(b), the presheaf category is now endowed with the forgetful functor

$$U\colon \mathbf{Set}^{\mathcal{A}^{\mathbf{op}}} \longrightarrow \mathbf{Set}^S, \quad A \longmapsto (A(s))_{s\in S}.$$

Remark 4.2. The presheaf category is a variety of S-sorted unary algebras: take $\Sigma = \mathbf{mor}\mathcal{A}$ as the signature, and given a morphism $u \in \mathcal{A}(t, s)$ let $u\colon s \longrightarrow t$ be the arity of u. Then the following equations present $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ in $\mathbf{Alg}\,\Sigma$:

$$\mathrm{id}_s(x) = x \quad \text{and} \quad u(v(x)) = w(x),$$

where s is any sort, x is a variable of sort s, and $w \in \mathcal{A}(t, s)$ is a morphism which factorizes as $w = v \cdot u$ in $\mathcal{A}$.

We are now going to show that $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ is also a covariety of coalgebras. This also follows from Proposition B.2.3.16 in [11].

Observation 4.3. Generalizing Example 2.10 to presheaves

$$A\colon \mathcal{A}^{\mathbf{op}} \longrightarrow \mathbf{Set}$$

observe that A is completely specified by (a) the underlying S-sorted set $UA = (A_s)_{s\in S}$ and (b) the functions

$$\mathcal{A}(t, s) \times A_s \longrightarrow A_t \quad \text{for } (t, s) \in S \times S$$

which assign to every pair $u\colon t \longrightarrow s$ in $\mathcal{A}$ and $x \in A_s$ the value $Au(x)$ in A_t. We use, as in 2.10, currification:

$$\frac{\mathcal{A}(t, s) \times A_s \longrightarrow A_t}{A_s \longrightarrow A_t^{\mathcal{A}(t,s)}}$$

and, for a given sort $s \in S$, we collect all these functions together by using product:

$$A_s \longrightarrow \prod_{t\in S} A_t^{\mathcal{A}(t,s)}.$$

In this sense, every presheaf is a coalgebra of the endofunctor $H^{\mathcal{A}}$ of $\mathbf{Set}^S$ given on objects A by

$$(H^{\mathcal{A}}A)_s = \prod_{t\in S} A_t^{\mathcal{A}(t,s)} \quad \text{for all } s \in S. \tag{1}$$

This is a polynomial functor (see 2.2) of the signature Σ which has for every sort s precisely one operation $\sigma_s \in \Sigma_s$ of arity

$$\sigma_s\colon (t\ t\ \ldots\ t\ t'\ t'\ldots\ t'\ldots) \longrightarrow s$$

in which every sort $t \in S$ is repeated precisely $\mathcal{A}(t, s)$-times.

Remark 4.4. Let us describe cofree coalgebras

$$C(X)$$

for the above polynomial functor $H^{\mathcal{A}}$, see Example 2.7. An element τ of $C(X)_s$ is a tree whose root is colored in X_s and labelled in Σ_s—but since Σ_s has just one element, we can forget the latter label. Thus we only need to remember that the children of the root of sort t correspond to $\mathcal{A}(t, s)$, for any $t \in S$. And we need to know the color, from X_t, of each child of sort t. Analogously, the children of a child of sort t correspond to $\mathcal{A}(t', t)$ for any $t' \in S$ and are colored in $X_{t'}$. Etc. Thus, such a tree τ is completely described by its sort s and by a coloring of all finite sequences of composable morphisms

$$t_0 \xrightarrow{u_0} t_1 \xrightarrow{u_1} t_2 \longrightarrow \cdots \xrightarrow{u_{n-1}} t_n = s \tag{2}$$

of $\mathcal{A}$ by colors from X_{t_0}.

Denote by

$$\mathcal{A}\langle t, s\rangle$$

the set of all sequences (2) in $\mathcal{A}$ with $t = t_0$. This includes, for $t = s$, the empty sequence $e_s \in \mathcal{A}\langle s, s\rangle$. Then all colorings of sequences (2) form the set $X_t^{\mathcal{A}\langle t,s\rangle}$, and we can consider $C(X)$ as the coalgebra of all such colorings of sequences:

$$C(X)_s = \prod_{t\in S} X_t^{\mathcal{A}\langle t,s\rangle} \quad \text{for all } s \in S$$

(compare with $C(X) = X^{M^*}$ in 2.10). Here $(\varepsilon_X)_s\colon C(X)_s \longrightarrow X_s$ takes each coloring to the color of the empty sequence:

$$(\varepsilon_X)_s\colon \varphi \longmapsto \varphi_s(e_s) \quad \text{for all } \varphi = (\varphi_t)_{t\in S}.$$

And the coalgebra structure

$$(\psi_X)_s\colon C(X)_s \longrightarrow (H^{\mathcal{A}}C(X))_s = \prod_{p\in S}\prod_{t\in S} X_t^{\mathcal{A}\langle t,p\rangle\times\mathcal{A}(p,s)}$$

is determined by the obvious concatenation maps $\mathcal{A}\langle t, p\rangle \times \mathcal{A}(p, s) \longrightarrow \mathcal{A}\langle t, s\rangle$. Thus $(\psi_X)_s$ assigns to a coloring $\varphi = (\varphi_t)_{t\in S}$ of sequences the tuple (indexed by all morphisms $u\colon p \longrightarrow s$ of $\mathcal{A}$) of colorings $\varphi_s(\ldots p \xrightarrow{u} s)$ obtained by concatenating u at the end of the sequence and then using φ_s.

Notation 4.5. The S-sorted set whose sorts are $\{0, 1\}$ is denoted by **bool**.

Proposition 4.6. *The presheaf category* $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ *is a covariety of* $H^{\mathcal{A}}$*-coalgebras presented by the coequation*

$$D \subseteq C(\mathbf{bool})$$

which consists of precisely those colorings $(\varphi_t)_{t\in S}$ *such that*

(i) *the empty sequence and* $s \xrightarrow{\mathrm{id}} s$ *have the same color, i.e.,*

$$\varphi_s(\mathrm{id}) = \varphi(e_s)$$

and

(ii) $\varphi_t(t \xrightarrow{u_0} t_1 \xrightarrow{u_1} s) = \varphi_t(t \xrightarrow{u_1 \cdot u_0} s)$ *for every composable pair* u_0, u_1 *of morphisms of* $\mathcal{A}$.

Proof. Let A be a presheaf. For every map $f: UA \longrightarrow \mathbf{bool}$ we prove that the homomorphism $\bar{f}: A \longrightarrow C(\mathbf{bool})$ factorizes through D. In fact, that homomorphism assigns to every element $a \in A_s$ the following coloring $\varphi = (\varphi_t)_{t \in S}$ of sequences:

$$t \xrightarrow{u_0} t_1 \xrightarrow{u_1} \cdots\cdots \xrightarrow{u_{n-1}} s \quad \text{is colored by} \quad f_t(Av(a)),$$

where $v: s \longrightarrow t$ is the composite $u_0 \cdot u_1 \cdots\cdots u_{n-1}$ in $\mathcal{A}^{\mathbf{op}}$. This coloring φ fulfils (i) because A preserves identity morphisms, and (ii) because A preserves composition. Therefore, $\bar{f}_s(a) \in D$, which proves that A satisfies D.

Conversely, let A be a coalgebra of $H^{\mathcal{A}}$ satisfying D. The underlying S-sorted set $UA = (A_s)_{s \in S}$ gives the object-function of a presheaf. The coalgebra structure

$$\alpha_s: A_s \longrightarrow \prod_{t \in T} A_t^{\mathcal{A}(t,s)} \quad (s \in S)$$

defines, for every morphism $u: t \longrightarrow s$ of $\mathcal{A}$, a function $A(u): A_s \longrightarrow A_t$ assigning to every element $a \in A_s$ the u-component of $\alpha_s(a)$. This would give the morphism-function of a presheaf, in case we verify the preservation of identity morphisms and composition. In fact:

(i) $A(\mathrm{id}_s)(a) = a$ follows from the property that every coloring $f: UA \longrightarrow \mathbf{bool}$ assigns the same color to a and to $A(\mathrm{id}_s)(a)$. This property is a consequence of (i) in the definition of D and the fact that $\bar{f}_s(a) \in D_s$ because $\bar{f}_s(a)$ colors $s \xrightarrow{\mathrm{id}} s$ with the color of $A(\mathrm{id}_s)(a)$.

(ii) $A(u_0 \cdot u_1)(a) = Au_0(Au_1(a))$ follows, for every composable pair $t_0 \xrightarrow{u_0} t_1$ and $t_1 \xrightarrow{u_1} s$ in $\mathcal{A}$, from the property that every coloring $f: UA \longrightarrow \mathbf{bool}$ colors both sequences with the same color. This property is a consequence of (ii) in the definition of D and the fact that $\bar{f}_s(a) \in D_s$ because $\bar{f}_s(a)$ colors

$$t_0 \xrightarrow{u_0} u_1 \xrightarrow{u_1} s \quad \text{and} \quad t_0 \xrightarrow{u_1 \cdot u_0} s$$

with the above two colors. □

5. Algebra ∩ coalgebra

Remark 5.1. We saw in Section 2 that the category

$$M\text{-}\mathbf{Set} \quad (M \text{ a monoid})$$

is equal both to a variety of one-sorted algebras and to a covariety of one-sorted coalgebras. Well, "equal" is a bit strong: it only is a variety and a covariety up to a concrete equivalence over **Set**. In the present section we prove that there are no other such categories, that is, for categories concrete over **Set** we prove that

$$\text{algebra} \cap \text{coalgebra} = M\text{-sets}. \tag{3}$$

Is this not in a contradiction to the result of Section 3 that **Coalg** H_Σ is both a covariety and a presheaf category? No contradiction! If $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ is viewed as a concrete category over **Set**, see 3.1(b), then, unless $\mathcal{A}$ is equivalent to a monoid, the presheaf category is never a variety. (Proof: assuming that V of 3.1(b) is naturally isomorphic to a forgetful functor of a variety, then V preserves products. Consider the product $1 = 1 \times 1$ of the terminal presheaf 1 with itself: if V preserves this product, then **obj** $\mathcal{A}$ is a product of two copies of itself. Thus, **obj** $\mathcal{A}$ has at most one element.)

Theorem 5.2 (*algebra* ∩ *coalgebra* = *presheaves*). *For every concrete category* $\mathcal{V}$ *on* S*-sorted sets the following conditions are equivalent*:

(i) $\mathcal{V}$ *is concretely equivalent both to a variety of* S*-sorted algebras and to a covariety of* S*-sorted coalgebras*;

(ii) $\mathcal{V}$ *is concretely equivalent to* $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ *for a small category* $\mathcal{A}$ *with* $S =$ **obj** $\mathcal{A}$.

Remark. (a) The statement (and the proof) of the above theorem does not depend on the generality of the concept of variety we use: this can mean an equationally defined class of Σ-algebras, or, more generally, any monadic category over

$\mathbf{Set}^S$. The same goes for covariety: this can mean a coequationally defined class of Σ-coalgebras, or, more generally, a comonadic category over $\mathbf{Set}^S$, see Section 2.

(b) In the one-sorted case we get Eq. (3).

Proof. Let $V: \mathcal{V} \longrightarrow \mathbf{Set}^S$ be a concrete category concretely equivalent to a variety and concretely equivalent to a covariety. It is sufficient to prove that the monad $\mathbb{T} = (T, \eta, \mu)$ induced by V is isomorphic to the monad induced, for some category $\mathcal{A}$ with $S = \mathbf{obj}\,\mathcal{A}$, by the forgetful functor $U: \mathbf{Set}^{\mathcal{A}^{\mathbf{op}}} \longrightarrow \mathbf{Set}^S$ of Notation 4.1. In fact, the functor V is monadic, i.e., the category $\mathcal{V}$ is concretely equivalent to the category of $\mathbb{T}$-algebras. But also U is monadic, so that $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ is also concretely equivalent to the category of $\mathbb{T}$-algebras. We denote by $P_s: \mathbf{Set}^S \longrightarrow \mathbf{Set}$ be the s-th projection, which, being representable, is a right adjoint.

Since V, due to the coalgebra part, preserves colimits, its composite T with a left adjoint of V also preserves colimits. By Special Adjoint Functor Theorem, see [14], T has a right adjoint $R: \mathbf{Set}^S \longrightarrow \mathbf{Set}^S$. Each component of R,

$$P_s \cdot R: \mathbf{Set}^S \longrightarrow \mathbf{Set} \quad (s \in S)$$

is, then, also a right adjoint—thus, it is representable by an S-sorted set. Let $M(s, t)$ denote, for $t \in S$, the components of that representing set. Then R is naturally isomorphic to the functor

$$X \longmapsto \left(\prod_{t \in S} X_t^{M(s,t)} \right)_{s \in S}$$

whose left adjoint, T, is given (by currification) as

$$X \longmapsto \left(\coprod_{t \in S} M(s, t) \times X_t \right)_{s \in S}.$$

Thus, without loss of generality we can assume that T is defined by

$$(TX)_s = \coprod_{t \in S} M(s, t) \times X_t.$$

More succinctly:

$$P_s \cdot T = \coprod_{t \in S} M(s, t) \bullet P_t \quad \text{for all } s \in S$$

(where $M\bullet$ denotes copowers indexed by M).

The unit of the monad $\mathbb{T}$

$$\eta: \mathrm{Id} \longrightarrow T$$

yields natural transformations

$$P_s\eta: P_s \longrightarrow \coprod_{t \in S} M(s, t) \bullet P_t.$$

Since P_s is representable by the S-sorted set whose components are empty except the sth one which is 1, we see that, by the Yoneda lemma, $P_s\eta$ is just a choice of an element

$$e_s \in M(s, s).$$

Then η_s has components

$$(\eta_s)_X: X_s \longrightarrow \coprod_{t \in S} M(s, t) \times X_t$$

which are the coproduct injections corresponding to e_s.

Analogously, the monad multiplication

$$\mu: T \cdot T \longrightarrow T$$

yields natural transformations μ_s from the functor

$$P_s \cdot T \cdot T = \coprod_{t \in S} M(s,t) \bullet P_t T = \coprod_{t,t' \in S} (M(s,t) \times M(t,t')) \bullet P_{t'}$$

into $P_s \cdot T$. By using the Yoneda lemma on the (u, v)-component of μ_s for any $(u, v) \in M(s,t) \times M(t,t')$:

$$P_{t'} \longrightarrow P_s \cdot T \cdot T \xrightarrow{\mu_s} P_s \cdot T = \coprod_{t \in T} M(s,t) \bullet P_t,$$

we see that μ chooses for u, v an element in $M(s,t')$, which we denote by $v \cdot u$. Then μ_s has components

$$(\mu_s)_X \colon \coprod_{t,t' \in S} [M(s,t) \times M(t,t')] \bullet P_{t'} X \longrightarrow \coprod_{t' \in T} M(s,t') \bullet P_{t'} X$$

given, for every (u, v)-copy of $P_{t'}X$, by the coproduct injection corresponding to $v \cdot u \in M(s,t')$.

Let us prove that we obtain a category $\mathcal{A}$ with the set of objects S and hom-sets $M(s,t)$ where e_s is the identity morphism and $(u, v) \longmapsto v \cdot u$ is the composition. In fact, the unit law of monads

$$\mu \cdot T\eta = \mu \cdot \eta T = \mathrm{id}$$

tells us precisely that e_s is a unit of the composition. And the associative law of monads

$$\mu \cdot T\mu = \mu \cdot \mu T$$

tells us precisely that the above composition is associative. Thus, we obtained a category $\mathcal{A}$.

It remains to verify that the forgetful functor $U \colon \mathbf{Set}^{\mathcal{A}^{\mathbf{op}}} \longrightarrow \mathbf{Set}^S$ induces the above monad $\mathbb{T}$. In fact, for every S-sorted set X we have an obvious presheaf $X^* \colon \mathcal{A}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ with

$$UX^* = TX :$$

put $X^*(s) = \coprod_{t \in S} \mathcal{A}(s,t) \times X_t$ on objects, and for a morphism $w \colon s' \longrightarrow s$ of $\mathcal{A}$ define $X^*(w) \colon X^*(s) \longrightarrow X^*(s')$ at a pair $(u, a) \in \mathcal{A}(s,t) \times X_t$ to be the pair $(u \cdot w, a) \in \mathcal{A}(s',t) \times X_t$, shortly: $X^*(w)(u, a) = (u \cdot w, a)$. The above natural transformation $\eta_X \colon X \longrightarrow TX = UX^*$ is clearly a universal arrow for U. Since U is monadic on $\mathbf{Set}^S$, this proves that U induces the monad $\mathbb{T}$. □

Remark 5.3. The result of 5.2 trivially generalizes to algebra and coalgebra over $\mathbf{Set}^{\mathcal{C}}$ for small categories $\mathcal{C}$. In fact, these categories are always monadic, i.e.,

$$\mathbf{Set}^{\mathcal{C}} \simeq \mathbf{Set}^{\mathbb{C}},$$

where $\mathbb{C}$ is a monad on **Set** preserving colimits. Given a variety $V \colon \mathcal{V} \longrightarrow \mathbf{Set}^{\mathcal{C}}$ presented by a monad $\mathbb{T}$ on $\mathbf{Set}^{\mathcal{C}}$, it follows that there is a Beck's distributive law between $\mathbb{T}$ and $\mathbb{C}$, see [5], such that $\mathcal{V}$ is concretely isomorphic to $\mathbf{Set}^{\mathbb{C} \otimes \mathbb{T}}$, the category of algebras of the corresponding tensor-product monad $\mathbb{C} \otimes \mathbb{T}$. Now if $\mathcal{V}$ is also a covariety, then $\mathbb{T}$ preserves colimits, thus, $\mathbb{C} \otimes \mathbb{T}$ preserves colimits—and then $\mathcal{V} \simeq \mathbf{Set}^{\mathbb{C} \times \mathbb{T}}$ is a presheaf category, as we have just proved.

6. Quasivarieties and quasicovarieties

Remark 6.1. In the present section the equation

$$\text{algebra} \cap \text{coalgebra} = \text{presheaves}$$

is extended to quasivarieties, i.e., implicational classes of algebras, and quasicovarieties, i.e., implicational classes of coalgebras.

Whereas the most general concept of variety considered in the preceding section was $(\mathbf{Set}^S)^{\mathbb{T}}$, the category of algebras of a monad $\mathbb{T}$ on $\mathbf{Set}^S$, the most general concept of quasivariety we consider now is a full subcategory of $(\mathbf{Set}^S)^{\mathbb{T}}$ presented by implications

$$\bigwedge_{i \in I} E_i \Longrightarrow E \quad \text{where } E \text{ and } E_i \text{ are equations.}$$

Or, equivalently, a subcategory presented by injectivity w.r.t. regular epimorphisms—this equivalence was already observed by Banaschewski and Herrlich [4]. Thus, a class $\mathcal{C}$ of $\mathbb{T}$-algebras is a *quasivariety* iff there is a collection of regular epimorphisms $e_i\colon B_i \longrightarrow B_i'$ $(i \in I)$ such that a $\mathbb{T}$-algebra A lies in $\mathcal{C}$ iff it is injective w.r.t. each e_i (i.e., every homomorphism $B_i \longrightarrow A$ with $i \in I$ factors through e_i). These are precisely the classes of $\mathbb{T}$-algebras closed under products and subalgebras, see [4]. Dually, we use regular monomorphisms in categories of coalgebras. For endofunctors of $\mathbf{Set}^S$ these are precisely the embeddings of subcoalgebras, as proved in [10, Theorem 3.4] (formulated for endofunctors of **Set**, but the proof is valid in $\mathbf{Set}^S$). This leads us to the following:

Definition 6.2 (*See [8]*). (1) By an *implication* for H-coalgebras is meant an expression

$$B \Longrightarrow B'$$

where B is a coalgebra and B' is a subcoalgebra of B. A coalgebra A *satisfies* the implication if it is projective w.r.t. the embedding $m\colon B' \hookrightarrow B$; that is, every homomorphism $A \longrightarrow B$ factors through m.

(2) A full subcategory of $\mathbf{Coalg}\, H$ is a *quasicovariety* if it can be presented by a collection of implications.

Example 6.3. Consider graphs as coalgebras of the power-set functor.

(1) The implication

$$1 \Longrightarrow \emptyset,$$

where 1 is the single loop and $\emptyset$ is the empty graph, presents all graphs which have a leaf. This follows from the observation that a nonempty graph A has a coalgebra homomorphism into 1 iff it has no leaves.

(2) We can present the quasicovariety

$$\mathcal{V} = \text{all graphs such that every vertex can be reached from a loop by a path}$$

by the following (large) collection of implications:

$$A \Longrightarrow A',$$

where A is a graph and A' is the subcoalgebra of A on all vertices reachable from a loop. In fact, if B is a graph with each vertex reachable from a loop, and if $h\colon B \longrightarrow A$ is a homomorphism, then h takes loops of B to loops of A, consequently, $h[B] \subseteq A'$. Conversely, if B satisfies all the implications above, then it lies in our class: just put $A = B$ and use the identity homomorphism.

Remark 6.4. In case of covarities of accessible functors Jan Rutten proved in [18] that a *single* coequation is enough for a presentation. This sharply contrasts to Example 6.3(2), where a proper class of implications is used. In fact, no small set of implications can present that class:

Proposition 6.5. *The quasicovariety of all graphs such that every node is reachable from a loop cannot be presented by a* (*small*) *set of implications.*

Proof. We will show that if $A_i \Longrightarrow A_i'$ $(i \in I)$ is a presentation of the above quasicovariety, then for every cardinal k there exists $i \in I$ such that card $A_i \geq k$. Assuming the contrary, we have cardinal $k >$ card A_i for all $i \in I$; we derive a contradiction. For every implication $A_i \Longrightarrow A_i'$ in the given presentation let $\hat{A}_i$ denote the subcoalgebra of A_i on all vertices reachable from loops in A_i. Then $\hat{A}_i$ lies in our class, thus, the inclusion homomorphism $\hat{A}_i \hookrightarrow A_i$ factorizes through A_i' (because $\hat{A}_i$ satisfies the implication $A_i \Longrightarrow A_i'$). Consequently:

$$\hat{A}_i \subseteq A_i' \quad \text{for every } i \in I.$$

Let B be a clique of cardinality k. Since B has no loops, it does not lie in our quasicovariety. Nonetheless, we prove that B satisfies each $A_i \Longrightarrow A_i'$, which is the desired contradiction. Given a homomorphism $f\colon B \longrightarrow A_i$, since card $B >$ card A_i there exist distinct vertices $x \neq y$ in B with $f(x) = f(y) = a$. Since $x \longrightarrow y$ is an edge and B is

a clique, we have a loop on a in A_i. For every vertex $z \neq x$ of B the fact that $x \longrightarrow z$ is an edge implies that $f(z)$ is reachable (in one step) from the loop a. Consequently,

$$f[B] \subseteq \hat{A}_i \subseteq A_i'.$$

This proves that B satisfies $A_i \Longrightarrow A_i'$. □

Theorem 6.6 (*Characterization of quasicovarieties*). *Let H be an endofunctor of $\mathbf{Set}^S$ preserving monomorphisms. A class of coalgebras is a covariety iff it is closed under coproducts and quotients in* $\mathbf{Coalg}\, H$.

Remark. For endofunctors of **Set** the assumption that H preserve monomorphisms can be the left out. In fact, let H' be the reduction of H with $H'\emptyset = \emptyset$ (and $H'X = HX$ for all $X \neq \emptyset$). Then H' preserves monomorphisms, and $\mathbf{Coalg}\, H'$ is isomorphic to $\mathbf{Coalg}\, H$.

Proof. We apply Proposition 1 of [4] which, in the dual form, states that whenever $\mathcal{K}$ is a cocomplete category which has (epi, regular mono)-factorizations of morphisms and is wellpowered w.r.t regular monomorphisms, then the following are equivalent for full subcategories $\mathcal{V}$ of $\mathcal{K}$:
(i) $\mathcal{V}$ can be presented by projectivity w.r.t. regular monomorphisms
and
(ii) $\mathcal{V}$ is closed under coproducts and quotients in $\mathcal{K}$.
Now $\mathcal{K} = \mathbf{Coalg}\, H$ is always cocomplete, see [18]. By [9], Theorem 3.4, regular monomorphisms are precisely the embeddings of subcoalgebras. And since H preserves monomorphisms, the (epi, regular mono)-factorizations lift from $\mathbf{Set}^S$ to $\mathbf{Coalg}\, H$, see [3]. Consequently, $\mathbf{Coalg}\, H$ is wellpowered w.r.t. regular monomorphisms and has (epi, regular mono)-factorizations. □

Theorem 6.7. *For every concrete category $\mathcal{V}$ on S-sorted sets the following conditions are equivalent*:
(i) *$\mathcal{V}$ is concretely equivalent both to a quasivariety of S-sorted algebras and to a quasicovariety of S-sorted coalgebras*;
(ii) *$\mathcal{V}$ is concretely equivalent to $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ for a small category $\mathcal{A}$ with $S = \mathbf{obj}\,\mathcal{A}$.*

Proof. We only need to prove (i) $\longrightarrow$ (ii). Let $V\colon \mathcal{V} \longrightarrow \mathbf{Set}^S$ be a quasivariety concretely isomorphic to a quasicovariety. We use the fact, established in the proof of Theorem 5.2, that the monad $\mathbb{T}$ induced by the forgetful functor V is the monad of $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ for some small category $\mathcal{A}$ with $S = \mathbf{obj}\,\mathcal{A}$. Now $\mathcal{V}$ is concretely equivalent to a full subcategory of $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ closed under
(a) products and subobjects (since $\mathcal{V}$ is a quasivariety)
and
(b) colimits (since $\mathcal{V}$ is concretely equivalent to a quasicovariety, thus, V preserves colimits).

It only remains to prove that every full subcategory $\mathcal{W}$ of $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ closed under products, subobjects and colimits is concretely equivalent to $\mathbf{Set}^{\mathcal{B}^{\mathbf{op}}}$ for some category $\mathcal{B}$ with $S = \mathbf{obj}\,\mathcal{B}$.

(I) $\mathcal{W}$ is a reflective subcategory of $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$. This follows from (a) since $\mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}$ is complete and cowellpowered, see [2, Theorem 16.8].

(II) Let $\sim$ be the congruence of $\mathcal{A}$ generated by $\mathcal{W}$, i.e., for parallel morphisms f, g of $\mathcal{A}$ we put

$$f \sim g \text{ iff } Mf = Mg \quad \text{for all } M \in \mathcal{W}. \tag{4}$$

We denote by

$$Q\colon \mathcal{A} \longrightarrow \mathcal{B}$$

the corresponding quotient category, that is,

$$\mathbf{obj}\,\mathcal{B} = \mathbf{obj}\,\mathcal{A} \quad (= S)$$

and the morphisms of $\mathcal{B}$ are the congruence classes Qf of morphisms f of $\mathcal{A}$. The functor

$$E\colon \mathbf{Set}^{\mathcal{B}^{\mathbf{op}}} \longrightarrow \mathbf{Set}^{\mathcal{A}^{\mathbf{op}}}, \quad M \longmapsto M\cdot Q^{\mathbf{op}}$$

is, obviously, a full concrete embedding preserving colimits.

We will prove that $\mathcal{W}$ is concretely equivalent to $\mathbf{Set}^{\mathcal{B}^{\mathbf{op}}}$ by verifying that $\mathcal{W}$ is the image of E:

$$\mathcal{W} = E\left[\mathbf{Set}^{\mathcal{B}^{\mathbf{op}}}\right].$$

In other words, given a functor $M\colon \mathcal{A}^{\mathbf{op}} \longrightarrow \mathbf{Set}$ we prove that

$$M \text{ lies in } \mathcal{W} \text{ iff } M \text{ factorizes through } Q^{\mathbf{op}}. \tag{5}$$

We first verify a special property of reflections of presentable presheaves in $\mathcal{W}$.

(III) For every object A of $\mathcal{A}$ let M_A be a reflection of $\mathcal{A}[-, A]$ in $\mathcal{W}$. That is, we have $M_A \in \mathcal{W}$ and there exists an element $a \in M_A A$ universal in the following sense:

> for every $M \in \mathcal{W}$ and every element $x \in MA$ there exists a unique natural transformation $\tau\colon M_A \longrightarrow M$ with $x = \tau_A(a)$.

It follows that given morphisms $f, g\colon A \longrightarrow B$ in $\mathcal{A}$ then

$$f \sim g \text{ iff } M_A f(a) = M_A g(a). \tag{6}$$

In fact, if $f \sim g$ then $M_A f = M_A g$, see (4). Conversely, suppose $M_A f(a) = M_A g(a)$, then for every $M \in \mathcal{W}$ we are to verify that $Mf = Mg$, i.e.,

$$Mf(x) = Mg(x) \quad \text{for all } x \in MA. \tag{7}$$

Choose $\tau\colon M_A \longrightarrow M$ as above and use the naturality:

$$Mf(x) = Mf\cdot\tau_A(a) = \tau_B\cdot M_A f(a)$$

and analogously for $Mg(x)$. This proves (7).

(IV) Let us prove (5). If $M \in \mathcal{W}$ then (4) implies that whenever $f \sim g$ then $Mf = Mg$—this is equivalent to saying that M factorizes through $Q^{\mathbf{op}}$.

Conversely, let M factorize through $Q^{\mathbf{op}}$:

$$M = \overline{M}\cdot Q^{\mathbf{op}} \quad \text{for some } \overline{M}\colon \mathcal{B}^{\mathbf{op}} \longrightarrow \mathbf{Set}.$$

We can suppose, without loss of generality, that $\overline{M}$ is representable. The general case is then obtained as follows: express $\overline{M}$ as a colimit of representable functors, $\overline{M} = \operatorname{colim}_{i\in I}\mathcal{B}[-, A_i]$, then $M = \overline{M}\cdot Q^{\mathbf{op}} = \operatorname{colim}_{i\in I}\mathcal{B}[-, A_i]\cdot Q^{\mathbf{op}}$ is a colimit of functors which, due to the special case (to be proved below) all lie in $\mathcal{W}$; since $\mathcal{W}$ is closed under colimits, we conclude $M \in \mathcal{W}$.

Now in case $\overline{M} = \mathcal{B}[-, A]$ for some object $A \in \mathcal{A}$, we can define a natural transformation from $M = \mathcal{B}[Q^{\mathbf{op}}-, A]$ to the above reflection $M_A \in \mathcal{W}$,

$$\mu\colon M \longrightarrow M_A,$$

by components μ_B, $B \in \mathbf{obj}\,\mathcal{A}$, as follows:

$$\mu_B([f]) = M_A f(a) \quad \text{for all } [f] \in \mathcal{B}[B, A].$$

It follows from (6) that μ_B is a monomorphism for every B, thus, M is a subfunctor of M_A. Since $\mathcal{W}$ is closed under subobjects, this proves $M \in \mathcal{W}$. □

7. Conclusions and open questions

We proved that if algebra is understood as the study of varieties and coalgebra as the study of covarieties, then

algebra ∩ coalgebra = presheaves.

The result turned out to be "robust" w.r.t. what precisely a variety or a covariety is, and it also generalizes to quasivarieties and quasicovarieties. This holds for algebra and coalgebra over many-sorted sets.

All our results depend on viewing categories of algebras and coalgebras as concrete categories. For the corresponding question concerning abstract categories we do not know the answer at present.

A surprising fact proved by Linton and Paré [12] is that every covariety is equivalent to the dual of a variety. We do not know the answer to the following:

Open problem: Is every variety equivalent to the dual of a covariety?

Another open problem: can the characterization of functor H such that **Coalg** H is a presheaf category (Theorem 3.9) be generalized to many-sorted sets?

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