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On varieties and covarieties in a category

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A concept of equation morphism is introduced for every endofuctor F of a cocomplete category C. Equationally defined classes of F-algebras for which free algebras exist are called varieties. Every variety is proved to be monadic over C, and, conversely, every monadic category is equivalent to a variety. The Birkhoff Variety Theorem is also proved for 'Set-like' categories.
By dualising, we arrive at a concept of coequation such that covarieties, that is, coequationally specified classes of coalgebras with cofree objects, correspond precisely to comonadic categories. Natural examples of covarieties are presented.

1. Introduction

This paper, for which an extended abstract appeared in Adámek and Porst (2001), is devoted to a concept of variety of algebras in an arbitrary category C. The basic examples for $\mathsf{C} = \mathsf{Set}$ include

(1) all finitary varieties of classical universal algebra (groups, lattices, and so on),
(2) all infinitary varieties of bounded arities (for example, Boolean σ-algebras, σ-complete lattices), and
(3) those infinitary varieties of unbounded arities that have free algebras, for example, complete semilattices and compact Hausdorff spaces.

It is well known that finitary varieties precisely correspond to finitary monads over Set (Lawvere 1963; Manes 1976) – and it is folklore that this generalises to (some concept of) infinitary varieties and arbitrary monads over Set. The result that we consider surprising is that such a correspondence can be formulated naturally over every cocomplete category C. Recall from Adámek (1974) that for every endofunctor F preserving colimits of ω-chains a free F-algebra $X^\sharp$ on an arbitrary object X can be described as an ω-colimit of the following ω-chain (which mimics the iterative construction of the set of all terms in variables from X):

$$0 \to F0 + X \to F(F0 + X) + X \to \cdots$$

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If F does not preserve ω-colimits, we might iterate further in order to obtain a free algebra $X^\sharp$: for limit steps we form a colimit of the previous chain, and for isolated steps take W_i and answer with $W_{i+1} = FW_i + X$ ($i \in \mathsf{Ord}$). If the connecting morphism $W_i \to W_{i+1}$ is an isomorphism for some i, then $W_i = X^\sharp$ is a free F-algebra. Here, however, we do not restrict ourselves to endofunctors having free algebras: we simply work instead with the whole transfinite construction $(W_i)_{i\in\mathsf{Ord}}$. It has the following property: for every F-algebra A and every morphism $f: X \to A$ (interpretation of variables) we obtain a unique cocone $f_i^\sharp: W_i \to A$ of the above chain such that the second component of $f_{i+1}^\sharp: FW_i + X \to A$ is f (for all ordinals i). We can think of $f_i^\sharp$ as the computation of terms of depth i.

This enables us to define an *equation* as a parallel pair of morphisms whose codomain is one of these W_i's:

$$U \overset{u}{\underset{v}{\rightrightarrows}} W_i \quad (i \in \mathsf{Ord}).$$

An algebra A is said to *satisfy* this equation if for every morphism $f: X \to A$, we have $f_i^\sharp u = f_i^\sharp v$. Moreover, a full subcategory $\mathsf{V} \hookrightarrow \mathsf{Alg}(F)$ of the category of all F-algebras is called a *variety* if

i. V is specified by a class of equations, and
ii. V-free algebras exist, that is, the natural forgetful functor $\mathsf{V} \to \mathsf{C}$ has a left adjoint.

This includes, for example, complete semilattices over Set (since free complete semilattices are the power-sets), but not complete lattices (where a free object on three generators does not exist (Hales 1964)). For any cocomplete category C, varieties are proved to be precisely the monadic categories over C.

The above concept of equation was introduced in Adámek (1983) for $\mathsf{C} = \mathsf{Set}$. A closely related concept, for arbitrary C, was studied by Jan Reiterman in his dissertation, see also Reiterman (1978), where quotient chains of the above chain (W_i) are considered as equations.

Functors F such that every object generates a free F-algebra are called *varietors* in Adámek and Trnková (1990). Although varieties are defined, in general, without this assumption, one can say more about algebras over varietors. For example, one can prove a generalisation of the Birkhoff Variety Theorem for all varietors on Set (Adámek 1983, Theorem 5D5). In the present paper we prove such a theorem for every complete, regularly cowellpowered category with regular factorisations; a similar result can be found in the above mentioned paper by Jan Reiterman. We can also work with coequalisers $c: W_i \to C$ of the above pairs u, v rather than the pairs per se. Then, to satisfy $u = v$ is the same as to be injective with respect to c. A generalisation of varieties using the concept of injectivity was studied in Herrlich and Ringel (1972) and Banaschewski and Herrlich (1976). The generalisation of Birkhoff's Variety Theorem requires a study of limits and colimits, which we also undertake.

All these considerations dualise immediately to *coalgebras* over an endofunctor F of C, since the category $\mathsf{Coalg}(F)$ of coalgebras is dual to $\mathsf{Alg}(F^{op})$. Thus, we obtain a general concept of coequation and coequationally defined classes of coalgebras. In the case of a covarietor F on Set, a coequation is simply an element of a cofree coalgebra, which is the concept studied in Gumm (1999). In the present paper we show that this is just

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a special case of a very general concept: whenever C is a complete category, comonadic categories over C are precisely the covarieties. And for $\mathsf{C} = \mathsf{Set}$ and for covarietors F (which includes, for example, all bounded functors) the Birkhoff Covariety Theorem is proved to hold – thus our concept of covariety is identical to that of Rutten (2000) and to other concepts in the literature, such as the covarieties defined by means of projectivity in Gumm and Schröder (1998), Kurz (2000) and Awoday and Hughes (2000) (that is, a dualisation of the above-mentioned results of Banaschewski and Herrlich). Covarieties satisfying additional assumptions (and so not equivalent to those studied in the present paper) have been studied, for example, in Cîrstea (2000) and Ghani *et al.* (2001).

2. Algebras and coalgebras

2.1. *Standing assumption*

Throughout this paper C denotes a category and F an endofunctor of C. Our main interest will be in $\mathsf{C} = \mathsf{Set}$ or Set^{op}, but in general nothing is assumed about C. The dual of a functor $G\colon \mathsf{C} \to \mathsf{D}$, that is, the functor $\mathsf{C}^{op} \to \mathsf{D}^{op}$ acting on objects and morphisms as G, will be denoted by G^{op}.

2.2. *Category* $\mathsf{Alg}(F)$ *of* F*-algebras*

Objects of $\mathsf{Alg}(F)$, called F-algebras, are pairs (C, α_C) where C is an object of C and $\alpha_C\colon FC \to C$ is a morphism. Morphisms $f\colon (C, \alpha_C) \to (D, \alpha_D)$ of $\mathsf{Alg}(F)$, called *F-algebra homomorphisms*, are morphisms $f\colon C \to D$ of C such that $f \circ \alpha_C = \alpha_D \circ Ff$. Composition and identities in $\mathsf{Alg}(F)$ are those of C. We use

$$U_F\colon \mathsf{Alg}(F) \to \mathsf{C}^{\dagger}$$

to denote the canonical forgetful functor $(C, \alpha_C) \mapsto \mathsf{C}$.

2.3. *Category* $\mathsf{Coalg}(F)$ *of* F*-algebras*

Objects of $\mathsf{Coalg}(F)$, called *F-coalgebras*, are pairs (C, α_C) where C is an object of C and $\alpha_C\colon C \to FC$ is a morphism. Morphisms $f\colon (C, \alpha_C) \to (D, \alpha_D)$ of $\mathsf{Coalg}(F)$, called *F-coalgebra homomorphisms*, are morphisms $f\colon C \to D$ of C such that $Ff \circ \alpha_C = \alpha_D \circ f$. Composition and identities are again those of C. We use

$${}_F U\colon \mathsf{Coalg}(F) \to \mathsf{C}^{\dagger}$$

to denote the canonical forgetful functor $(C, \alpha_C) \mapsto C$.

Lemma 2.1. For any functor $F\colon \mathsf{C} \to \mathsf{C}$ the following hold:

1 $\mathsf{Coalg}(F) = (\mathsf{Alg}(F^{op})^{op})$, that is, F-coalgebras form the dual of the category of F^{op}-algebras;
2 ${}_F U = (U_{F^{op}})^{op}$.

[†] We will omit the subscript F whenever no confusion is likely to arise.

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Examples 2.2.

(1) Algebras on one binary and one nullary operation are F-algebras where F is the endofunctor of Set given by $FX = (X \times X) + 1$.
F-coalgebras, that is, sets C equipped with a function $\alpha_C : C \to C \times C + 1$, can be viewed as dynamic systems with two inputs and deadlocks. That is, C is the set of all states, and for every state $c \in C$ either $\alpha_C(c)$ is the pair of states that are the reactions of c to the two inputs, respectively, or c is a deadlock state. Morphisms are the functions preserving reaction to inputs, and preserving and reflecting deadlock.

(2) Finitary universal algebras of a given (one-sorted) signature $\Omega = (\Omega_n)_{n\in\mathbb{N}}$, where Ω_n is the set of all n-ary operation symbols, can be viewed as F-algebras for the following endofunctor $F = F_\Omega$ of Set: F_Ω assigns to a set X the set $\sum_{n\in\mathbb{N}} \Omega_n \times X^n$. Correspondingly, F_Ω assigns to a map $f : X \to Y$ the map $\sum_{n\in\mathbb{N}} \Omega_n \times f^n$, that is, the map $\sum_{n\in\mathbb{N}} \Omega_n \times X^n \to \sum_{n\in\mathbb{N}} \Omega_n \times Y^n$ mapping a pair $(\omega, (x_1, \ldots, x_n))$ to the pair $(\omega, (fx_1, \ldots, fx_n))$.
F_Ω-coalgebras can be viewed as dynamical systems with several inputs and side effects. This was demonstrated in Rutten (2000).

(3) Many-sorted algebras (of a given S-sorted signature Ω) can, analogously, be described as F_Ω-algebras for the corresponding endofunctors F_Ω of Set^S. For example, the category of multigraphs and homomorphisms is the category of 2-sorted algebras of sorts v (vertices) and e (edges) with two unary operations *target, source*: $e \to v$. This is also a category of coalgebras (for the endofunctor F on $\mathsf{Set} \times \mathsf{Set}$ given by $F(X_v, X_e) = (1, X_v \times X_v)$) – in fact, every category of unary algebras is a category of coalgebras as well.

(4) Using

$$\mathscr{P} : \mathsf{Set} \to \mathsf{Set}$$

to denote the power-set functor, $\mathscr{P}$-algebras are sets A endowed with an operation $\alpha : \mathscr{P}A \to A$. For example, a complete join semilattice is described by $\alpha : M \mapsto \vee M$ (see Example 4.6 below for an equational presentation of complete join semilattices as $\mathscr{P}$-algebras).
A $\mathscr{P}$-coalgebra can be considered as precisely a (notation of a) directed graph (that is, a binary relation) on a set A: given $\alpha : A \to \mathscr{P}A$, then $\alpha(x)$ is the set of all elements related to x. Homomorphisms of $\mathscr{P}$-coalgebras are, however, more specialised than the usual graph homomorphisms: for coalgebras A_i expressed by graphs $R_i \subseteq A_i \times A_i$ (i = 1,2), a homomorphism of $\mathscr{P}$-coalgebras is a function $f : A_1 \to A_2$ such that

(a) f is a graph homomorphism, that is, xR_1y implies $f(x)R_2f(y)$, and

(b) given elements $x \in A_1$ and $y \in A_2$ with $f(x)R_2y$, there exists $x' \in A_1$ with xR_1x' and $f(x') = y$.

(5) Using

$$\mathscr{P}_{fin} : \mathsf{Set} \to \mathsf{Set}$$

to denote the finite-power-set functor, $\mathscr{P}_{fin}$-coalgebras are the directed graphs of finite out-degree, with homomorphisms as defined in (4) above.

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(6) **Labelled transition systems**, which are important both in modal logic and in process algebra, are naturally formalised as coalgebras. For a given set L of labels, a labelled transition system is a set A (of states) together with a binary relation $\xrightarrow{l}$ on A for every label $l \in L$. Put

$$F = \mathscr{P}(- \times L) : \mathsf{Set} \to \mathsf{Set}.$$

Every labelled transition system A defines an F-coalgebra by

$$\alpha_A : A \to \mathscr{P}(A \times L),\ x \longmapsto \{(y,l) \in A \times L;\ x \xrightarrow{l} y\}.$$

This gives rise to an equivalence between $\mathsf{Coalg}(F)$ and the category of labelled transition systems and system morphisms defined analogously to example (4) above. If $L = 1$, we obtain the above case, $F = \mathscr{P}$, and sometimes speak about *non-labelled transition systems.*

(7) Denote by Vec_k the category of vector spaces over a field k and linear functions. Recall that a *k-algebra* is a vector space A together with a bilinear multiplication, that is, a linear function $A \otimes A \to A$ that is associative, and a unit e (which we consider as the corresponding morphism $e : k \to A$) of the multiplication (Sweedler 1969). Thus, k-algebras are F-algebras (satisfying additional axioms) of the functor $F : \mathsf{Vec}_k \to \mathsf{Vec}_k$ given by

$$FA = (A \otimes A) + k.$$

Remark 2.3. The above functors F_Ω are called *polynomial*. They can be introduced, more generally, on every category C with finite products and arbitrary coproducts as the smallest class of endofunctors containing Id_{C} and closed under finite products and all coproducts.

3. Free algebras and cofree coalgebras

Example 3.1. For polynomial endofunctors F_Ω on Set, the concept of free algebra $X^\sharp$ on a set X of generators is well known. We can describe it either recursively as $X^\sharp = \cup_{i<\omega} X^\sharp_i$ where

$$\begin{aligned} X^\sharp_0 &= \Omega_0 + X \\ &= F_\Omega \varnothing + X \quad \text{terms of depths 0 are nullary operations and variables} \\ X^\sharp_{i+1} &= \{(\omega, t_0, \ldots, t_{n-1}) \mid \omega \in \Omega_n, t_0, \ldots, t_{n-1} \in X^\sharp_i\} + X \\ &= F_\Omega X^\sharp_i + X \quad \text{terms of depths i}+1. \end{aligned}$$

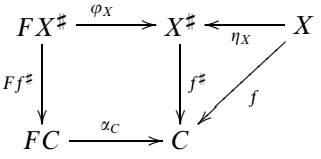
Or, more directly, $X^\sharp$ is the algebra of all finite 'properly' labelled trees. 'Properly' means that a node with $n > 0$ children is labelled by an n-ary operation, and a leaf is labelled by a variable or a nullary operation. We have the universal arrow $\eta_X : X \to X^\sharp$, embedding X into $X^\sharp$.

Remark 3.2. A *free F-algebra* on an object X (of 'variables') of C can be defined as a pair consisting of an F-algebra

$$FX^\sharp \xrightarrow{\varphi_X} X^\sharp \quad \text{and a morphism} \quad \eta_X : X \to X^\sharp$$

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with the universal property that, given an F-algebra (C, α_C) and a morphism $f: X \to C$ of C, there exists a unique F-homomorphism $f^\sharp$ extending f, that is, such that the following diagram



commutes. In other words, η_X is a universal arrow of the forgetful functor $U: \mathsf{Alg}(F) \to \mathsf{C}$.

In particular, if $X = 0$ (an initial object of C), then $0^\sharp$ is an initial F-algebra. Recall that by the famous *Lambek's Lemma* (Lambek 1968), $\varphi_0: F0^\sharp \to 0^\sharp$ is an isomorphism.

Lemma 3.3. Let C have binary coproducts. An initial $F(-) + X$-algebra is equal to a free F-algebra on X.

Corollary 3.4. For a free F-algebra $X^\sharp$ one has $X^\sharp \simeq FX^\sharp + X$

This is Lambek's Lemma applied to $F_X = F(-) + X$.

It is good to have a name for endofunctors F such that every object of C admits a free F-algebra, that is, such that U has a left adjoint.

Definition 3.5 (Adámek and Trnková 1990). An endofunctor $F: \mathsf{C} \to \mathsf{C}$ is called a *varietor* provided that a free F-algebra exists on every C-object.

Examples 3.6.

(1) Polynomial endofunctors on Set are varietors.
(2) The finite-power-set endofunctor of Set is a varietor. This follows from the fact that it is finitary, that is, preserves filtered colimits. Also, the countable-power-set endofunctor of Set is a varietor because it preserves $\aleph_1$-filtered colimits. We will see below that every *λ-accessible* endofunctor of Set, that is, every functor preserving λ-filtered colimits (for some infinite cardinal λ) is a varietor.
(3) The power-set endofunctor $\mathscr{P}$ of Set is not a varietor since an initial $\mathscr{P}$-algebra does not exist.
(4) Generalising (1) above, let C have colimits and finite products such that colimits of ω-chains commute with finite products. Then every polynomial endofunctor F_Ω on C is a varietor. In fact, it is easy to see that, for all Ω, F_Ω preserves colimits of ω-chains. And then free algebras can be obtained by the following

Construction 3.7 (Finitary Free-Algebra Construction – Adámek 1974). This is an application of the famous construction of an *initial F-algebra* (the free F-algebra on 0, an initial object of C) as a colimit of the chain

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \xrightarrow{F^2 0} F^3 0 \cdots$$

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to the functor $F_X = F(-) + X$ (see Lemma 3.3 above). Let C have countable colimits. Given an object X in C, we define an ω-chain $X_i^\sharp (i < \omega)$ as follows:

$$0 \xrightarrow{!} F0 + X \xrightarrow{F!+X} F(F0 + X) + X \xrightarrow{F(F!+X)+X} F(F(F0 + X) + X) + X \cdots$$

That is:

First step: $X_0^\sharp = 0,\ X_1^\sharp = F0 + X$ with $x_{0,1}^\sharp$ the unique morphism $0 \xrightarrow{!} F0 + X$.

Induction step: $X_{i+1}^\sharp = FX_i^\sharp + X$ and $x_{i+1,j+1}^\sharp = Fx_{i,j}^\sharp + X$ for all $i \leqslant j$.

Lemma 3.8. Let C be cocomplete and F preserve colimits of ω-chains. Then for every object X of C,

$$X^\sharp = \operatorname*{colim}_{i<\omega} X_i^\sharp$$

is a free F-algebra on X.

In more detail, let $(X_i^\sharp \xrightarrow{x_i} X^\sharp)$ be a colimit cocone. Since $F(-)+X$ preserves that colimit we have a unique morphism

$$\varphi_X : FX^\sharp + X \to X^\sharp \quad \text{with} \quad \varphi_X \circ (Fx_i + X) = x_{i+1}.$$

The two components $\eta_X : X \to X^\sharp$ and $\alpha_X : FX^\sharp \to X^\sharp$ of φ_X form a free F-algebra on X.

Proof. For every F-algebra (C, α_C) and any morphism ('assignment to variables') $f : X \to C$, define a cocone of the above chain (*computation of terms*) recursively as follows:

$$f_0^\sharp = ! \quad \text{and} \quad f_{i+1}^\sharp = [\alpha_C \circ Ff_i^\sharp, f].$$

Then the (unique) factorisation $X_i^\sharp \xrightarrow{x_i} X^\sharp \xrightarrow{f^\sharp} C = f_i^\sharp$ of that cocone gives the (unique) homomorphism $f^\sharp : (X^\sharp, \alpha_X) \to (C, \alpha_C)$ with $f = f^\sharp \circ \eta_X$. □

Examples 3.9.

(1) For the endofunctor $FY = Y \times Y + 1$ (that is, one constant and one binary operation) on Set, we know that the terms in $X_i^\sharp$ are just the binary trees of depth $\leqslant i$ with leaves labelled in $X + 1$. This corresponds precisely to the construction above.

(2) For the endofunctor $FY = Y^{\mathbb{N}}$ (that is, one ω-ary operation) on Set we again might form the sets $X_i^\sharp$ of terms, but here the colimit after ω steps does not give a free F-algebra, of course: we need ω_1 steps of the following

Construction 3.10 (Free-Algebra Construction). (See either Adámek (1974) or Adámek and Trnková (1990, IV.3.2)) Let C be a cocomplete category. For every endofunctor F on C and every object X ('of variables') in C, define a transfinite chain of objects $X_i^\sharp$ (i any ordinal) and connecting morphisms

$$x_{i,j}^\sharp : X_i^\sharp \to X_j^\sharp \ (i \leqslant j)$$

by the following transfinite induction:

First step: $X_0^\sharp = 0,\ X_1^\sharp = F0 + X$ with $x_{0,1}^\sharp$ the unique morphism $0 \xrightarrow{!} F0 + X$,

Isolated step: $X_{i+1}^\sharp = FX_i^\sharp + X$ for all ordinals i, $x_{i+1,j+1}^\sharp = Fx_{i,j}^\sharp + X$ for all $i \leqslant j$,

Limit step: $X_j^\sharp = \mathsf{colim}_{i<j} X_i^\sharp$ for all limit ordinals j with colimit cocone $x_{i,j}^\sharp$, $i < j$.

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We say that the free algebra construction *stops* after k steps provided that $x^\sharp_{k,k+1}$ is an isomorphism.

Lemma 3.11. Let C be cocomplete and F preserve colimits of λ-chains. Then for every object X of C, $X^\sharp_\lambda$ is a free F-algebra on X.

This is analogous to Lemma 3.7, as we demonstrate next.

Definition 3.12 (Computation of terms). In classical universal algebra, where $X^\sharp_i$ are the terms of depth $< i$ over variables from X, we have the concept of computation of terms in an algebra A: for every interpretation of variables, that is, every function $f: X \to A$, we extend f to a cocone $f^\sharp_i: X^\sharp_i \to A$ of the free-algebra construction by the well-known algebraic recursion. This is a special case of the following:

Let F be an endofunctor of a cocomplete category C. For every F-algebra (A, α_A) and every morphism $f: X \to A$ in C we define a cocone $f^\sharp_i: X^\sharp_i \to A$ of the free-algebra construction as follows:

First step: $f^\sharp_0: 0 \to A$ is the unique morphism.

Isolated step: Given $f^\sharp_i: X^\sharp_i \to A$, let

$$f^\sharp_{i+1}: FX^\sharp_i + X \to A$$

have components $FX^\sharp_i \xrightarrow{Ff^\sharp_i} FA \xrightarrow{\alpha_A} A$ and $X \xrightarrow{f} A$. Observe that the compatibility condition

$$f^\sharp_i = f^\sharp_{i+1} \circ x^\sharp_{i,i+1}$$

is fulfilled (an easy proof by transfinite induction).

Limit step: Given a limit ordinal i and given the cocone $f^\sharp_j: X^\sharp_j \to A$, $j < i$, of the first i steps of the free-algebra construction, there is a unique

$$f^\sharp_i: \operatorname*{colim}_{j<i} X^\sharp_j \to A$$

with the (obligatory) property $f^\sharp_j = f^\sharp_i \circ x^\sharp_{j,i}$ for all $j < i$.

Lemma 3.13. Homomorphisms of F-algebras preserve computation of terms, that is, given a homomorphism $h: (C, \alpha_C) \to (D, \alpha_D)$ and an assignment of variables $f: X \to C$, then, for all ordinals i, $(h \circ f)^\sharp_i = h \circ f^\sharp_i$.

Proof. We proceed by transfinite induction on i. The case $i = 0$ is trivial since the domain of both sides is 0. The limit case is immediate from the universal property of colimits. In the isolated case we assume the above equation for i and derive

$$\begin{aligned}
(h \circ f)^\sharp_{i+1} &= [\alpha_D \circ F(h \circ f)^\sharp_i,\ h \circ f] \\
&= [\alpha_D \circ Fh \circ Ff^\sharp_i,\ h \circ f] \text{ by the induction hypothesis} \\
&= [h \circ \alpha_C \circ Ff^\sharp_i,\ h \circ f] \quad \text{since } h \text{ is a homomorphism} \\
&= h \circ [\alpha_C \circ Ff^\sharp_i,\ f] \\
&= h \circ f^\sharp_{i+1}
\end{aligned}$$

□

Proposition 3.14. If the free-algebra construction stops after k steps, then $X^\sharp_k$ is a free F-algebra on X.

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In more detail, denoting the components of $x_{k,k+1}^{-1}: FX_k^\sharp + X \to X_k^\sharp$ by

$$\alpha_X : FX_k^\sharp \to X_k^\sharp \quad \text{and} \quad \eta_X : X \to X_k^\sharp$$

respectively, these form a free F-algebra on X.

Proof. Let (C, α_C) be an F-algebra $f: X \to C$ a morphism. From the equation

$$f_k^\sharp = f_{k+1}^\sharp \circ x_{k,k+1} = [\alpha_C \circ Ff_k^\sharp, f] \circ [\alpha_X, \eta_X]^{-1},$$

it follows that $f_k^\sharp$ is a homomorphism extending f. The uniqueness is easy to verify. □

Definition 3.15 (Adámek and Trnková 1990). A functor $F: \mathsf{C} \to \mathsf{C}$ with C cocomplete is called a *constructive varietor* provided that its free-algebra construction stops for each object X in C.

Remark 3.16. A functor F is said to be *accessible* if it preserves, for some infinite cardinal λ, all λ-filtered colimits (we then speak more precisely of λ-accessiblity). Accessible Set-functors are precisely the bounded functors of Kawahara and Mori (2000), as proved in Adámek and Porst (2001). Since colimits of λ-chains are λ-filtered, we get the following corollary.

Corollary 3.17. Every accessible endofunctor of a cocomplete category is a constructive varietor.

A full characterisation of the endofunctors of Set that are varietors was presented in Adámek and Trnková (1990). We call an endofunctor *trivial* if its restriction to all non-empty sets and functions is constant. (For example, the functor $\varnothing \mapsto \varnothing$ and $X \mapsto 1$ for all $X \neq \varnothing$ is trivial.) An object X such that FX is isomorphic to X is called a *fixed point* of F.

Theorem 3.18. The following conditions on an endofunctor F of Set are equivalent:

(i) F is a varietor.
(ii) F is a constructive varietor.
(iii) F has arbitrarily large fixed points, or F is trivial.

For each such functor F the category of F-algebras is cocomplete.

Proof. See Adámek and Trnková (1990, IV.43 and IV.8.2). □

Remark 3.19. An analogous result holds for Vec_k: (constructive) varietors are precisely the endofunctors that have arbitrarily large fixed points, or are constant (Adámek and Trnková 1990, IV.5.5).

Example 3.20. We give here an example of a varietor on Set that is not accessible. Assume the generalised continuum hypothesis. For a class M of cardinals with $0, 1 \in M$, we use

$$\mathscr{P}_M : \mathsf{Set} \to \mathsf{Set}$$

to denote the functor assigning to every set A the set of all subsets $B \subseteq A$ with $\mathsf{card} B \in M$. Then $\mathscr{P}_M$ assigns to a morphism $f: A \to A'$ the function defined as follows:

$$B \mapsto f[B] \text{ if } f \text{ restricted to } B \text{ is one-to-one, otherwise } B \mapsto \varnothing.$$

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It is easy to see that $\mathscr{P}_M$ is accessible iff M is a set. And an infinite cardinal α is a fixed point of $\mathscr{P}_M$ iff $\alpha \notin M$. (The proof is easy using the fact that $\alpha^\beta = \alpha$ for all infinite cardinals $\beta < \alpha$ (Aczel *et al.* 2002).)

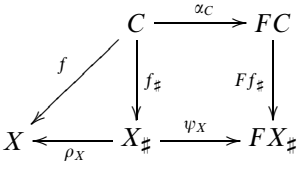
Now choose a class M of cardinals such that neither M nor its complement $\mathsf{card} \setminus M$ is a set. Then $\mathscr{P}_M$ is a varietor because it has arbitrarily large fixed points (namely, all those whose cardinality does not lie in M), but it is not accessible.

Example 3.21. We next give an example of a (trivial) non-constructive varietor. Let C be the usual ordered class of all ordinals extended by a largest element, $\top$, considered as a category (with $\vee$ as binary coproduct). The functor $F : \mathsf{C} \longrightarrow \mathsf{C}$ with

$$F(i) = i + 1 \text{ for all ordinals } i \text{ and } F(\top) = \top$$

is a varietor: $\mathsf{Alg}(F)$ consists of a single algebra $(\top, \alpha_\top)$. However, for $X = 0$, we get $X_i^\sharp = i$ for every ordinal i, thus, the free-algebra construction never stops.

Remark 3.22. *Cofree coalgebras* are the corresponding dualisation of free algebras. A cofree F-coalgebra on an object X ('of colours') of C is a coalgebra $\psi_X : X_\sharp \to FX_\sharp$ together with a ('colouring') morphism $\rho_X : X_\sharp \to X$ having the universal property that given an F-coalgebra (C, α_C) and a morphism $f : C \to X$ of C, there exists a unique F-coalgebra homomorphism $f_\sharp$, such that the following diagram commutes:



In other words, ρ_X is a couniversal arrow of the forgetful functor $U : \mathsf{Coalg}(F) \to \mathsf{C}$.

Definition 3.23. An endofunctor $F : \mathsf{C} \to \mathsf{C}$ is called a *covarietor* if there exists a cofree F-coalgebra on every C-object.

This terminology is justified by the following remark based on Lemmas 2.1 and 3.3.

Remark 3.24. The following are equivalent for any $F : \mathsf{C} \to \mathsf{C}$:

(i) F is a covarietor.
(ii) F^{op} is a varietor.

When C has finite products, another equivalent condition is:

(iii) For every object X in C, the functor $F^X = F(-) \times X$ has a final (= terminal) coalgebra.

Example 3.25. Every accessible endofunctor of Set (in fact, of any locally presentable category) is a covarietor, as shown in Barr (1993).

A dualisation of the free-algebra construction above gives the following construction.

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Construction 3.26 (Cofree Coalgebra Construction). Let C be a complete category. For every endofunctor F on C and every object X ('of colours') in C, define a transfinite cochain of objects $X^i_\sharp$ (i any ordinal) and connecting morphisms $x^{i,j}_\sharp : X^i_\sharp \to X^j_\sharp$ ($i \geqslant j$) as follows (where 1 denotes a terminal object of C):

First step: $X^0_\sharp = 1$, $X^1_\sharp = F1 \times X$ with $x^{1,0}_\sharp : F1 \times X \xrightarrow{!} 1$ the unique morphism,
Isolated step: $X^{i+1}_\sharp = FX^i_\sharp \times X$ for all ordinals i, $x^{i+1,j+1}_\sharp = Fx^{i,j}_\sharp \times X$ for all $i \geqslant j$,
Limit step: $X^j_\sharp = \lim_{i<j} X^i_\sharp$ for all limit ordinals j with limit cone $x^{j,i}_\sharp$, $i < j$.

If this cochain construction *stops after k steps*, that is, $x^{k+1,k}_\sharp : FX^k_\sharp \times X \to X^k_\sharp$ is an isomorphism, for some ordinal k, then $X^k_\sharp$ is a cofree F-coalgebra on X. In more detail, denoting the components of the inverse of $x^{k+1,k}_\sharp$ by

$$\alpha_X : X^k_\sharp \to FX^k_\sharp \quad \text{and} \quad \rho_X : X^k_\sharp \to X,$$

these form a cofree F-coalgebra on X. For an F-coalgebra (C, α_C) and a morphism $f : C \to X$, the extension $f_\sharp$ of f is the k-th member of the cocone $f^i_\sharp : C \to X^i_\sharp$, which is defined dually to Definition 3.12.

Definition 3.27. A functor $F : \mathsf{C} \to \mathsf{C}$ is called *constructive covarietor* provided that the above Construction 3.26 stops for each C-object X.

Examples 3.28.

(1) Let $F = \mathrm{Id}_{\mathsf{Set}}$. For every set X (of colours) we form the first steps of the cofree-coalgebra construction:

$$1 \leftarrow 1 \times X \leftarrow 1 \times X \times X \leftarrow 1 \times X \times X \times X \leftarrow \dots$$

where each of the connecting maps is the projection (forgetting the last component). A limit of this ω^{op}-sequence is $X_\sharp = X^\omega$, the set of all infinite words over X (with projections as limit maps).
For a given unary (co-)algebra (A, α_A) and a given colouring $f : A \to X$, the unique homomorphism $f_\sharp : A \to X_\sharp$ assigns to every element $a \in A$ the word of colours of the elements $a, \alpha_A a, \alpha_A^2 a, \dots$

(2) For polynomial endofunctors F_Ω of Set, a description of a final coalgebra $1_\sharp$ (which can also be derived from the cofree coalgebra construction) is as follows: the elements of the coalgebra $1_\sharp$ are finite and infinite Ω-labelled trees (where Ω-labelled means that a node with k children is labelled by an element of Ω_k for all $k = 0, 1, 2 \dots$). The structure map

$$\alpha_{1_\sharp} : 1_\sharp \to \sum_{k \in \mathbb{N}} \Omega_k \times (1_\sharp)^k$$

assigns to a tree t whose root is labelled by $\omega \in \Omega_k$ the k-tuple of the subtrees given by the sons of the root (that is, the corresponding element of $(1_\sharp)^k$ in the summand indexed by ω); that is, the coalgebra structure is the inverse of tree tupling.

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From Corollary 3.4, we obtain immediately a description of $X_\sharp$ for any set X (of colours). Since $F_\Omega(-) \times X \simeq F_{\Omega^X}(-)$ for the signature $\Omega^X = (\Omega_n \times X)_{n\in\mathbb{N}}$, we conclude the following:

$X_\sharp$ is the coalgebra of finite and infinite trees with a node having k children labelled in $\Omega_k \times X$ for all $k = 0, 1, 2, \ldots$.

(3) For $\mathscr{P}_{fin}$, the final-coalgebra construction stops after $\omega + \omega$ steps (Worell 2000).

Theorem 3.29. Every covariator in Set is constructive.

In fact, to prove this, it is, due to Remark 3.24, sufficient to prove that whenever F has a final coalgebra, the cofree-coalgebra construction for 1 stops; for this see Adámek and Koubek (1995).

A full characterisation of covarietors F in Set is not known. Let us call a cardinal number α an *exponential fixed point* of F provided that every set X whose cardinality lies between α and 2^α is a fixed point of F.

Theorem 3.30 (Adámek and Koubek 1995). Every endofunctor of Set with arbitrarily large exponential fixed points is a covarietor.

Theorem 3.31 (Worell 2000). Every λ-accessible endofunctor of Set is a covarietor, and the cofree-coalgebra construction stops in $\lambda + \lambda$ steps.

Example 3.32. of a covarietor on Set that is not accessible. Assuming the Generalised Continuum Hypothesis, let M be a class of cardinals containing $0, 1$ such that M is not a set, but there exist arbitrarily large cardinals α with $M \cap [\alpha, 2^\alpha] = \varnothing$. Then $\mathscr{P}_M$ (see Example 3.20) is a covarietor because it has arbitrarily large exponential fixed points. However, $\mathscr{P}_M$ is not accessible.

4. Varieties and covarieties

In classical finitary algebra, an *equation* in variables from a set X is a pair of terms, in other words, elements of a free algebra $X^\sharp$ on X. That is, a parallel pair

$$u, v : 1 \to X^\sharp$$

of morphisms in Set. An algebra A *satisfies* that equation provided that for every interpretation of variables in A, that is, every function $f : X \to A$, the unique homomorphism $f^\sharp : X^\sharp \to A$ extending f (that is, the function computing terms in A) satisfies

$$f^\sharp u = f^\sharp v.$$

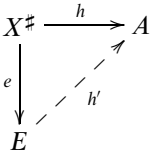
This is equivalent to saying that $f^\sharp$ factorises through a coequaliser

$$e : X^\sharp \to E$$

of u and v.

Thus, every equation gives us a regular quotient $e : X^\sharp \to E$ of the free algebra $X^\sharp$ defining satisfaction in A by requesting that every homomorphism $h : X^\sharp \to A$ factors through e:

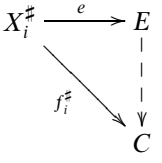
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Conversely, given a regular quotient $e: X^\sharp \to E$, satisfaction of e in the above sense is equivalent to the satisfaction of the 'classical' equations $u = v$ for all pairs, $u, v: 1 \to X^\sharp$ with $eu = ev$. This leads us to the following concepts, which are closely related to those introduced in Adámek (1983) and Banaschewski and Herrlich (1976).

Definition 4.1. Let F be an endofunctor of a cocomplete category C. Using the notation $X_i^\sharp$ and $f_i^\sharp$ as in Construction 3.10 and Definition 3.12, we define:

1. An *equation arrow over* X is a regular epimorphism $e: X_i^\sharp \to E$ for some ordinal i. An F-algebra (C, α_C) is said to *satisfy* e provided that for every morphism $f: X \to C$ the morphism $f_i^\sharp$ factors through e:



2. For any class $\mathscr{E}$ of equation arrows, $\mathsf{Alg}(F, \mathscr{E})$ denotes the full subcategory of $\mathsf{Alg}(F)$ formed by all F-algebras satisfying every $e \in \mathscr{E}$. Such categories are called *equational categories* (*of* F*-algebras*) over C.

Examples 4.2.

(1) Varieties of finitary algebras are precisely the equational categories

$$\mathsf{Alg}(\mathsf{F}_\Omega, \mathscr{E}).$$

Here, equations $e: X_i^\sharp \to E$ can always be substituted by equations

$$e: X^\sharp (= X_\omega^\sharp) \to E,$$

as we see below. Moreover, for every variety we only need a single equation-morphism $e: X^\sharp \to E$ with X a countable set: just consider the smallest equivalence $\sim$ on $X^\sharp$ with $u \sim v$ for every equation $u = v$ in the presentation of the variety. This is generalised in Corollary 4.10 below.

(2) $\mathsf{Alg}(F)$ is, of course, an equational class for any functor F: put $\mathscr{E} = \varnothing$.

(3) If C has coequalisers, then for every parallel pair

$$u, v: C \to X_i^\sharp \quad (i \in \mathsf{Ord})$$

of morphisms a coequaliser of u and v is an equational arrow. An F-algebra A satisfies this arrow iff

$$f_i^\sharp u = f_i^\sharp v \quad \text{for all} \quad f: C \to A \qquad (\star)$$

Conversely, if C has kernel pairs, every equational arrow $e: X_i^\sharp \to E$ is a coequaliser of its kernel pair $u, v: C \to X_i^\sharp$, so satisfaction can be defined *via* $(\star)$.

We use the usual notation $u = v$ for the equation morphism that is the coequaliser of u and v.

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Remark 4.3. We want to distinguish between equational classes of algebras and varieties. The latter are required to have the (important) property that free algebras exist on every object C of variables. This is not automatic, for example, $\mathsf{Alg}(\mathscr{P})$, for the power-set functor $\mathscr{P}$, is an equational class but not a variety.

Definition 4.4. A *variety* of F-algebras is an equational class $\mathsf{Alg}(F, \mathscr{E})$ such that the forgetful functor into C (a restriction of U_F) has a left adjoint.

Example 4.5. 'Classical' finitary varieties are precisely varieties of F-algebras for polynomial endofunctors F_Ω of Set.

Example 4.6 (Complete semilattices). The category of complete join semilattices has as objects all posets with joins and as morphisms all functions preserving joins. The objects can be described as precisely those $\mathscr{P}$-algebras $\alpha_A : \mathscr{P}A \to A$ for which the (join-) operation α_A satisfies:

(i) $\alpha_A\{a\} = a$ for every $a \in A$, and
(ii) $\alpha_A\{\alpha_A(A_i) \mid i \in I\} = \alpha_A \bigcup_{i\in I} A_i$ for all collections $A_i \subseteq A,\ i \in I$.

We can express (i) and (ii) by equation-arrows as follows.

ad i. Put $X = \{x\}$ and observe that we have

$$\begin{aligned} X_0^\sharp &= \varnothing \\ X_1^\sharp &= \mathscr{P}\varnothing + X = \{\varnothing, x\} \\ X_2^\sharp &= \mathscr{P}\{\varnothing, x\} + X = \{\varnothing, x, \{x\}, \{\varnothing\}, \{\varnothing, x\}\}. \end{aligned}$$

Condition (i) is expressed by the equation

$$\{x\} = x$$

or, more precisely, by the equation-arrow $e : X_2^\sharp \to X_2^\sharp/\!\sim$, which is the canonical map of the equivalence $\sim$ whose only non-singleton equivalence class is $\{x, \{x\}\}$. In fact, a $\mathscr{P}$-algebra (A, α_A) satisfies $x = \{x\}$ iff for every morphism $f : X \to A$ (that is, every element $a = f(x)$ of A) the extension $f_2^\sharp : X_2^\sharp \to A$ satisfies $f_2^\sharp(\{x\}) = f_2^\sharp(x)$. Equivalently: $\alpha_A\{a\} = a$.

ad ii. Put $X = A$ and observe that $\bigcup_{i\in I} A_i$ is an element of $X_2^\sharp$ and $\{A_i \mid i \in I\}$ an element of $X_3^\sharp$. Consider the equation

$$\{A_i \mid i \in I\} = \bigcup_{i\in I} A_i,$$

or, more precisely, the equation-arrow $\bar{e} : X_3^\sharp \to X_3^\sharp/\!\sim$ where $\sim$ is the equivalence whose only non-singleton equivalence class is $\{\{A_i \mid_{i\in I}\}, \bigcup_{i\in I} A_i\}$. The satisfaction of the equation arrow $\bar{e}$ implies (ii) trivially: just use $f = \mathrm{id} : X \to A$. Conversely, every $\mathscr{P}$-algebra satisfying (i) and (ii) above satisfies the equation arrow $\bar{e}$.

Remark 4.7. Free complete semilattices exist: they are the powerset-semilattices. Thus complete semilattices form a variety of $\mathscr{P}$-algebras.

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Lemma 4.8. For arbitrary ordinals $i \leqslant j$, any equation

$$u = v \quad \text{with} \quad u, v : C \to X_i^\sharp$$

is equivalent to the equation

$$x_{ij} u = x_{ij} v.$$

That is, an F-algebra satisfies the first one iff it satisfies the latter.

Proof. This follows from $f_j^\sharp x_{ij} = f_i^\sharp$, see Lemma 3.13. □

Corollary 4.9. If F is a constructive varietor, we only need to consider equations $u = v$ for morphisms $u, v : C \to X^\sharp$.

Corollary 4.10. For every accessible endofunctor F of Set and any equational class V of F-algebras, there is a single equation arrow e presenting V, that is, such that $\mathsf{V} = \mathsf{Alg}(F, \{e\})$.

In fact, if F is λ-accessible, it is easy to see that every equation $u = v$ for elements $u, v \in X^\sharp$ (see Corollary 4.9 and Remark 3.16) is equivalent to an equation $u' = v'$ for suitable elements $u', v' \in M^\sharp$, where M is an (arbitrary) set of cardinality λ. Thus, every equational class is presented by a set pairs of elements of $M^\sharp$. For each of these pairs, form a coequaliser and obtain a set of quotients of $M^\sharp$ in Set. Forming a pushout, e, of that set of (regular) epimorphisms, we obtain $\mathsf{V} = \mathsf{Alg}(F, \{e\})$.

By formally dualising Definition 4.1, see Lemma 2.1, we obtain the following definition.

Definition 4.11. Let F be an endofunctor of a complete category C.

1. A *coequation arrow over* X is a regular monomorphism $m : M \to X_\sharp^i$ for some ordinal i. An F-coalgebra (C, α_C) is said to *satisfy* m provided that for every morphism $f : C \to X$ the morphism $f_\sharp^i$ factors through m.
2. For any class $\mathscr{M}$ of coequation arrows, we use $\mathsf{Coalg}(F, \mathscr{M})$ to denote the full subcategory of $\mathsf{Coalg}(F)$ formed by all F-coalgebras satisfying every $m \in \mathscr{M}$. Such categories are called *coequational categories (of F-coalgebras)* over C.

Notation 4.12. For $\mathsf{C} = \mathsf{Set}$, given an element $d \in X_\sharp^i$, we use $\neg d$ to denote the coequation arrow

$$X_\sharp^i \setminus \{d\} \hookrightarrow X_\sharp^i.$$

Thus, a coalgebra A satisfies $\neg d$ iff for every colouring $f : A \to X$, we have

$$f_\sharp^i(a) \neq d \text{ for all } a \in A.$$

Observe that we do not need any more general coequations: instead of a coequation arrow $m : M \to X_\sharp^i$, we can always consider the coequations $\neg d$ for all $d \in X_i^\sharp \setminus m[M]$.

Example 4.13 (Non-labelled transition systems (= graphs)). Here $F = \mathscr{P}$, the power-set functor. We give some examples of covarieties.

(a) **No terminals.** A state s of a transition system A is *terminal* if $s \to t$ holds for no state t. The non-existence of terminal states is expressed by the coequation $\neg\varnothing$ for

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$\varnothing \in \mathscr{P}1 = 1^1_\sharp$. In fact, we have $f^0_\sharp : A \to 1$, the constant function, and $f^1_\sharp : A \to \mathscr{P}1 = \{\varnothing, 1\}$ taking value $\varnothing$ for terminal states and 1 for non-terminal states.

(b) **One-step reachability of terminals.** Consider

$$1^2_\sharp = \mathscr{P}\mathscr{P}1 = \{\varnothing, \{\varnothing\}, \{1\}, \{\varnothing, 1\}\}.$$

Given a state s then $f^2_\sharp(s) = \varnothing$ iff s is terminal, $f^2_\sharp(s) = \{\varnothing\}$ iff s is non-terminal but only terminal states can be reached from s directly, and $f^2_\sharp(s) = \{1\}$ iff s is non-terminal and only non-terminal states can be reached from s directly. Thus, the coequation $\neg\{1\}$ means: from every state a terminal state is reachable in at most one step.

(c) **Reachability of terminals.** In the cofree-coalgebra sequence

$$1 \leftarrow \mathscr{P}1 \leftarrow \mathscr{P}\mathscr{P}1 \leftarrow \dots$$

consider the elements $d_n \in \mathscr{P}^n 1$ where d_0 is the unique element of 1, and

$$d_{n+1} = \{d_n\}.$$

The sequence $d = (d_n) = (d_0, 1, \{1\}, \{\{1\}\} \dots)$ is compatible, that is, is an element of the limit

$$1^\omega_\sharp = \lim_{n<\omega} 1^n_\sharp.$$

For a state s we have $f^\omega_\sharp(s) = d$ iff $f^1_\sharp(s) = \{1\}$ (that is, only non-terminals are reached in one step) and $f^2_\sharp(s) = \{1\}$ (that is, only non-terminals are reached in two steps), and so on. Thus, $\neg d$ expresses the fact that from each state a terminal state is reachable.

(d) **Reflexivity.** That is, the fact that $s \to s$, for each state s, can be easily expressed using two colours. Put $2 = \{0, 1\}$ and consider the cofree-coalgebra sequence

$$1 \leftarrow \mathscr{P}(1) \times 2 \leftarrow \mathscr{P}(\mathscr{P}(1) \times 2) \times 2 \leftarrow \dots$$

Given a colouring $f : A \to \{0, 1\}$, we have $f^2_\sharp(s) = (M, f(s))$, where $M \subseteq \mathscr{P}(1) \times 2$ is the set of all pairs (B, i) where, for some state s' reachable from s, we have $f(s') = i$ and, either $B = \varnothing$ if s' is a terminal state, or $B = 1$, otherwise.
Suppose that $f^2_\sharp(s) = (\{1, 0\}, 1)$. Then s has colour 1 but every reachable state has colour 0. Thus, s is not reflective. It is easy to see that the coequation $\neg(\{1, 0\}, 1)$ means: for every state s either $s \to s$, or s is terminal. Thus, reflexivity is axiomatised by the two coequations $\neg\varnothing$ and $\neg(\{1, 0\}, 1)$.

(e) **Symmetry.** That is, the fact that

$$s \to t \text{ implies } t \to s \quad \text{(for all states } s, t\text{)}.$$

Here we use

$$f^3_\sharp : A \to \mathscr{P}(\mathscr{P}(\mathscr{P}(1) \times 2) \times 2) \times 2$$

given by $f^3_\sharp(s) = (R, f(s))$ where $R \subseteq \mathscr{P}(\mathscr{P}(\mathscr{P}(1) \times 2) \times 2) \times 2$ is the set of all $f^2_\sharp(t)$ for edges $s \to t$. Suppose that for a colouring $f : A \to \{0, 1\}$ and a state s we have $f^2_\sharp(s) = (R, 0)$ where $R \subseteq \mathscr{P}(\mathscr{P}(1) \times 2) \times 2$ contains an element of the form $(-, 1)$. Every element $r \in R$ (a pair consisting of a subset of $\mathscr{P}(1) \times 2$ and a colour) corresponds to a state t with $s \to t$ in the sense that $f^1_\sharp(t) = r$. Thus, if some element $r \in R$ has the form

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$r = (Q, 1)$, we have a state t directly reachable from s of colour 1 such that $f^2_\sharp(t) = Q$. Symmetry then guarantees that Q contains $(1, 0)$: in fact, we have the edge $t \to s$ where $f^1_\sharp(s) = (1, 0)$. Thus, in a symmetric transition system $f^3_\sharp(s)$ never has the form $(R, 0)$ where some element of R is $(Q, 1)$ with $(1, 0) \notin Q$. Symmetry is axiomatised by the following coequations:

$$\neg(R, 0) \quad \text{for all } R \subseteq \mathscr{P}(\mathscr{P}(1) \times 2) \times 2 \text{ containing some } (Q, 1) \text{ with } (1, 0) \notin Q.$$

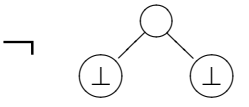
Example 4.14 (Dynamical systems on two inputs). Here we consider the functor $FX = X \times X + 1$ of Example 2.2(1), thus, coalgebras are dynamical systems on two inputs with deadlock states. Recall that the final coalgebra consists of binary trees with leaves labelled by $\bot$.

(a) The coequation



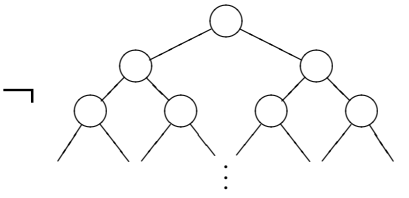
means: no deadlock states.

(b) The coequation



means: every non-deadlock state can continue indefinitely (without encountering a deadlock state).

(c) The coequation

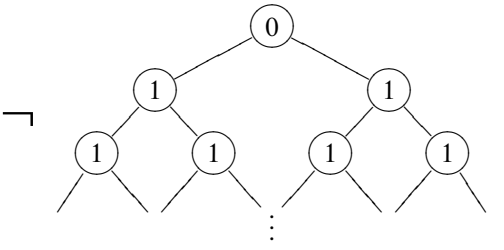


means: from every state a deadlock state is reachable.

(d) For $2 = \{0, 1\}$ we have

$2^{\sharp} =$ coalgebra of binary trees with leaves labelled by $(\bot, 0)$ or $(\bot, 1)$ and internal nodes labelled by 0 or 1.

The coequation



means: from every non-periodic state a deadlock state is reachable. (A state is periodic if the system can return to that state after $n > 0$ steps.)

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Example 4.15 (Dynamical system on one input (without deadlock)). Here $F = \mathrm{Id}_{\mathsf{Set}}$, and coalgebras are just unary algebras. We have $X_\sharp = X^\omega$, the set of all infinite words on X.

Given an element a of a (co)algebra A, we say that *a lies over a cycle* of length n provided that n is the smallest number with $\alpha_A^k(a) = \alpha_A^{k+n}(a)$ for some $k \in \mathbb{N}$. The smallest such k is called the *distance* of a from the cycle.

(a) For $2 = \{0,1\}$ consider

$$d = 010101\ldots = (01)^\omega.$$

An algebra A satisfies the coequation $\neg d$ iff every element lies over an odd cycle.
In fact, A satisfies $\neg d$ iff for every colouring $f : A \to \{0,1\}$, given an element $a \in A$, the word

$$(f(a), f(\alpha_A(a)),\ f(\alpha_A^2(a))\ldots)$$

is distinct from d. This is certainly true if a lies over an odd cycle.
Conversely, if a lies over an even cycle, or does not lie over any cycle (that is, the elements $\alpha_A^k(a)$ are pairwise distinct), we can certainly choose a colouring $f : A \to \{0,1\}$ with

$$f(\alpha_A^k(a)) = \begin{cases} 0 & k \text{ even} \\ 1 & k \text{ odd.} \end{cases}$$

Then $f_\sharp(a) = 010101\ldots$.

(b) For

$$d = \underbrace{00\ldots0}_{s}\ 010101\ldots = 0^s(01)^\omega,$$

an algebra A satisfies $\neg d$ iff every element either lies over an odd cycle, or it lies over an even one and has distance less than s from it. Or, more briefly, no element has distance s from an even cycle, and every element lies over a cycle.

(c) Analogously, for

$$d = \underbrace{00\ldots0}_{s}\ \underbrace{00\ldots0}_{p}\ 1\ \underbrace{00\ldots0}_{p}\ 1\ \ldots = 0^s(0^p1)^\omega,$$

an algebra satisfies $\neg d$ iff no element has distance s from a cycle whose length is a multiple of p, and every element lies over a cycle.

Remark 4.16. The above example shows that for every set $S \subseteq \mathbb{N} \times \mathbb{N}$ we have the following covariety C_S of unary algebras:

$$A \in \mathsf{C}_S, \text{iff given } (s,p) \in S:$$

— no element of A has distance s from a cycle of length in divisible by p and
— every element lies over a cycle.

One can show that these are precisely all the covarieties in $\mathsf{Coalg}(Id)$.

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5. Limits and colimits

In this section we summarise results on limits and colimits of (co)algebras needed for the proof of Birkhoff's Variety Theorem in the next section. Most of these results can be found in the existing literature, although, for example, the Colimit Theorem below seems to be new (and easy).

Construction 5.1 (Products of F-algebras). These are easy to construct: suppose C has products and let (A_i, α_i) be a collection of F-algebras ($i \in I$). Form a product $A = \prod_{i \in I} A_i$ in C with projections $\pi_i : A \longrightarrow A_i$. Then there exists a unique $\alpha : FA \to A$ turning each π_i into a homomorphism, *viz.*, $\alpha = \langle \alpha_i \circ F\pi_i \rangle : FA \to \prod A_i$. And it is easy to see that (A, α) is a product of the given F-algebras in $\mathsf{Alg}(F)$.

Fact 5.2 (Limits of F-algebras). The above construction is a special case of the following fact: the forgetful functor

$$U : \mathsf{Alg}(F) \to \mathsf{C}$$

creates limits. This means that, given a diagram D in $\mathsf{Alg}(F)$ and forming a limit cone $(A \xrightarrow{\pi_i} UD_i)$ of UD in C, there exists a unique $\alpha : FA \to A$ turning each π_i into a homomorphism. And (A, α) is, then, a limit of D. The proof is easy, see, for example, Adámek and Trnková (1990).

Corollary 5.3. If C is complete, categories of F-algebras are complete.

Corollary 5.4. A homomorphism in $\mathsf{Alg}(F)$ is an isomorphism (or a monomorphism) iff its underlying morphism is an isomorphism (or a monomorphism, respectively) in C.

If C has kernel pairs and F preserves coequalisers of kernel pairs, a homomorphism in $\mathsf{Alg}(F)$ is a regular epimorphism iff its underlying morphism is a regular epimorphism in C.

In fact, a functor creating pullbacks preserves and reflects monomorphisms, isomorphisms and kernel pairs. And regular epimorphisms are precisely the coequalisers of their kernel pairs.

Since the underlying functor of $\mathsf{Alg}(F)$ allows for the unique transport of structures and every C-object carries only a set of F-algebras, we get the following corollary.

Corollary 5.5. If C is wellpowered (that is, every object has only a set of subobjects), then so are the categories of F-algebras.

5.1. *Colimits of F-algebras*

This is a more difficult topic. First, colimits need not exist, in general. For example, the power-set functor $\mathscr{P} : \mathsf{Set} \to \mathsf{Set}$ does not have any fixed point. Thus, by Lambek's Lemma, an initial $\mathscr{P}$-algebra (that is, the empty colimit in $\mathsf{Alg}\ (\mathscr{P})$) does not exist.

One easy case is that of colimit types preserved by F.

Proposition 5.6. For every type of colimit in C that F preserves, the category of F-algebras has colimits of that type, which are created by the forgetful functor.

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Proof. Given a diagram $D: \mathsf{D} \to \mathsf{Alg}(F)$ such that C has and F preserves colimits over D, form a colimit $(A_d \xrightarrow{c_d} A)_{d\in\mathsf{D}}$ of UD in C, where $UDd = A_d$, and observe that, since $(FA_d \xrightarrow{Fc_d} FA)_{d\in\mathsf{D}}$ is a colimit of FUD, there exists a unique $\alpha: FA \to A$ turning each c_d into a homomorphism. It is easy to see that (A, α) together with c_d $(d \in \mathsf{D})$ is a colimit of D in $\mathsf{Alg}(F)$. □

Example 5.7. The polynomial functors $F_\Omega: \mathsf{Set} \to \mathsf{Set}$ preserve filtered colimits. This is true because:

(i) each functor $X \mapsto X^n$ (n a natural number) preserves filtered colimits, since finite products and filtered colimits commute in Set and

(ii) F_Ω is a coproduct of functors (i), and coproducts commute with (filtered) colimits.

Therefore, filtered colimits of 'classical' algebras are created by the forgetful functor.

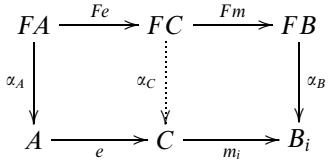
Remark 5.8. Recall that a category C is said to have *regular factorisations* if every morphism of C factorises as a regular epimorphism followed by a monomorphism. A functor $U: \mathsf{A} \to \mathsf{C}$ is said to *create* regular factorisations if for every morphism $f: A \to B$ in A with a regular factorisation $Uf = m \circ e$ in C, there exist unique morphisms m_0 and e_0 in A with $m = Um_0$ and $e = Ue_0$, and, moreover, m_0 is a monomorphism and e_0 is a regular epimorphism with $f = m_0 \circ e_0$.

Proposition 5.9. Let C have pullbacks and regular factorisations. If F preserves regular epimorphisms, then $\mathsf{Alg}(F)$ has regular factorisations created by the forgetful functor.

Remark 5.10. We will prove a more general statement, which concerns *regular cone-factorisations*. Consider a cone, that is, a (possibly large or empty) collection of morphisms $f_i: A \to B_i$ $(i \in I)$. A regular cone-factorisation is a regular epimorphism $e: A \to A'$ and a collectively monomorphic cone $m_i: A' \to B_i$ $(i \in I)$ with $f_i = m_i e$ for all $i \in I$. If a cowellpowered category C has regular factorisations and products, then it has regular cone-factorisations. (In fact, for small cones, $f_i: A \to B_i$ just form a regular factorisation of $\langle f_i \rangle_{i\in I}: A \to \prod_{i\in I} B_i$. For large ones, use cowellpoweredness on the coimages of $f_i, i \in I$, to reduce to a small cone.) Note that this condition in the case of an empty cone simply means that there is a regular epimorphism $A \xrightarrow{e} B$ such that any two maps $r, s: B' \to B$ coincide. In Set, a regular factorisation of a cone $f_i: A \to B_i$ can be obtained by factorising each f_i over A/ρ where ρ is the intersection of the kernels of all f_i.

We prove that whenever C has pullbacks and regular cone-factorisations, $\mathsf{Alg}(F)$ has regular cone-factorisations created by the forgetful functor.

Proof. Given a cone $f_i: (A, \alpha_A) \to (B_i, \alpha_{B_i})$ in $\mathsf{Alg}(F)$ and given a regular cone factorisation $f_i = m_i e$ in C with $e: A \to C$, there exists



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a unique $\alpha_C : FC \to C$ turning e and each m_i into homomorphisms. This follows from the diagonal fill-in property of the regular epimorphism Fe and the (possibly empty) monomorphic cone m_i. Since m_i is monomorphic in C, it follows easily that $m_i : (C, \alpha_C) \to (B, \alpha_B)$ is monomorphic in $\mathsf{Alg}(F)$. Let us form a kernel pair $k_1, k_2 : K \to A$ of e (that is, a pullback of e and e). Then since e is a regular epimorphism, it is a coequaliser of k_1 and k_2 in C. Now, U creates pullbacks, thus, we have homomorphisms $k_1, k_2 : (K, \alpha_K) \to (A, \alpha_A)$, and it is easy to see that e is a coequaliser of k_1 and k_2 in $\mathsf{Alg}(F)$. Thus, e is a regular epimorphism in $\mathsf{Alg}(F)$. □

Corollary 5.11. For every endofunctor F of Set, the category of F-algebras has regular factorisations created by the forgetful functor.

In fact, regular epimorphisms in Set are just the split epimorphisms that all functors preserve.

Theorem 5.12 (Colimit Theorem). Let C have pullbacks and regular cone-factorisations. For every varietor F preserving regular epimorphisms, $\mathsf{Alg}(F)$ has colimits of all types existing in C.

Proof. Let $D : \mathsf{D} \to \mathsf{Alg}(F)$ be a diagram with $UDd = A_d$ $(d \in \mathsf{D})$ such that UD has a colimit

$$(A_d \xrightarrow{c_d} C)_{d \in \mathsf{D}}$$

in C. Let $\eta_C : C \to UC^\sharp$ be a free F-algebra on C. Form the cone of all homomorphisms $r : C^\sharp \to R$ in $\mathsf{Alg}(F)$ with the property that each $Ur \cdot \eta_C \cdot c_d$ is a homomorphism from Dd to R $(d \in \mathsf{D})$. Let

$$C^\sharp \xrightarrow{e} E \xrightarrow{r'} R = C^\sharp \xrightarrow{r} R$$

be a regular factorisation of that cone in $\mathsf{Alg}(F)$ (see Remark 5.10). Then each

$$\bar{c}_d = Ue \circ \eta_C \circ c_d : Dd \to E \qquad (d \in \mathsf{D})$$

is a homomorphism. In fact, for every r the following square

$$\begin{array}{ccccccccc}
FA_d & \xrightarrow{Fc_d} & FC & \xrightarrow{F\eta_C} & FUC^\sharp & \xrightarrow{FUe} & FUE & \xrightarrow{FUr'} & FUR \\
{\scriptstyle \alpha_{A_d}}\downarrow & & & & & & \vdots\,{\scriptstyle \alpha_E} & & \downarrow{\scriptstyle \alpha_R} \\
A_d & \xrightarrow[c_d]{} & FC & \xrightarrow[\eta_C]{} & UC^\sharp & \xrightarrow[Ue]{} & UE & \xrightarrow[Ur']{} & UR
\end{array}$$

commutes because $Ur \cdot \eta_C \cdot c_d$ is a homomorphism. Since the cone of all Ur' is collectively monomorphic, it follows that the left-hand subsquare commutes.

We claim that $(Dd \xrightarrow{\bar{c}_d} (E, \alpha_E))_{d \in \mathsf{D}}$ is a colimit of D in $\mathsf{Alg}(F)$. It is obviously a cocone of D, because (c_d) is a cocone of UD. Let $f_d : Dd \to B$ be another cocone of D. Then there exists a unique $f : C \to UB$ in C with $Uf_d = f \circ c_d$ $(d \in D)$. The unique homomorphism $r : C^\sharp \to B$ with $f = Ur \circ \eta_C$ is a member of the above cone, so we have a factorisation $r = r' \circ m$. The homomorphism $r' : E \to B$ fulfills $f_d = r' \circ \bar{c}_d$ for all d because

$$Uf_d = f \circ c_d = Ur \circ \eta_C \circ c_d = Ur' \circ U\bar{c}_d.$$

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And r' is unique: suppose that a homomorphism $h: E \to B$ also fulfills $f_d = h \circ \bar{c}_d$ for all d, then we have

$$Uh \circ Ue \circ \eta_c \circ c_d = Ur' \circ Ue \circ \eta_C \circ c_d \quad (d \in D)$$

which, due to the universal property of colimits, implies

$$Uh \circ Ue \circ \eta_C \;=\; Ur' \circ Ue \circ \eta_C$$

and this implies $h \circ e = r' \circ e$. Since e is an epimorphism, this proves $h = r$. □

Corollary 5.13. Let C be a complete, cocomplete, regularly cowellpowered category with regular factorisations. Then for every varietor F preserving regular epimorphisms, the category of F-algebras is cocomplete.

Theorem 5.14. For every functor $F: \mathsf{Set} \to \mathsf{Set}$, we have $\mathsf{Alg}(F)$ is cocomplete, provided F is a varietor.
The reverse implication holds under the generalised Continuum Hypothesis.

Proof. The implication above is a consequence of Corollary 5.13: Set satisfies all the hypotheses, and F preserves regular (= split) epimorphisms automatically.

The reverse implication follows from Adámek *et al.* (1972, Theorem III.4), which states a necessary and sufficient condition of cocompleteness of categories of bialgebras, $A(F, G)$. The choice of $G = Id_{\mathsf{Set}}$ yields

$$\mathsf{Alg}(F) = A(F, \mathrm{Id}_{\mathsf{Set}})$$

and the theorem states that $\mathsf{Alg}(F)$ has colimits iff F preserves unions or is non-excessive, which means that for arbitrarily large sets X we have $\mathsf{card}FX \leqslant \mathsf{card}X$. Now every set functor preserving unions is easily seen to be non-excessive. And non-excessive functors are varietors (Adámek and Trnková 1990, 4.3). □

Remark 5.15. Another important criterion for cocompleteness of categories of algebras follows from the theory of locally presentable categories. Recall that a category C is *locally λ-presentable* provided that it is cocomplete and has a set A of λ-presentable objects (that is, objects A such that the hom-functor of A preserves λ-filtered colimits) such that all objects of C are λ-filtered colimits of objects of A. For example, $\mathsf{Set}, \mathsf{Vec}_k, \mathsf{Pos}$ are locally $\aleph_0$-presentable, the category of $\mathsf{CPO}'\mathsf{s}$ is locally $\aleph_1$-presentable, but Set^{op} is not locally presentable.

Theorem 5.16. For every accessible endofunctor F of a locally presentable category C the category of F-algebras has colimits. In fact, $\mathsf{Alg}(F)$ is locally presentable.

Proof. See Adámek and Rosický (1994, 2.75). □

Using Lemma 2.1, we obtain the following dualisations of the above results.

Theorem 5.17. The forgetful functor of the category of F-coalgebras creates colimits and all limits preserved by F.

Corollary 5.18. If C is cocomplete, all categories of coalgebras are cocomplete.

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Corollary 5.19. A homomorphism in $\mathsf{Coalg}(F)$ is an isomorphism (or an epimorphism) iff its underlying morphism is an isomorphism (an epimorphism, respectively) in C. If C has cokernel pairs and F perserves equalisers of cokernel pairs, then a homomorphism in $\mathsf{Coalg}(F)$ is a regular monomorphisms iff its underlying morphism is a regular monomorphism.

Corollary 5.20. If C is cowellpowered, so are all categories of coalgebras.

Proposition 5.21. (See also Gumm and Schröder (2001)) For every covarietor F in Set the category of F-coalgebras is complete.

Proof. If F preserves regular monomorphisms, this follows from the dual of Theorem 5.12.

Now recall that regular monomorphisms in Set are just one-to-one functions, and that all except those with empty domains are split (and thus preserved by any functor)! To complete the proof it remains to apply the following result due to V. Trnková (Adámek and Trnková 1990) – see Remark 5.23. □

Lemma 5.22. For every functor $F:\mathsf{Set}\to\mathsf{Set}$ that does not preserve monomorphisms there exists a functor $F':\mathsf{Set}\to\mathsf{Set}$ preserving monomorphisms, and such that

$$\begin{aligned} FX &= F'X \quad \text{for all sets } X \neq \emptyset \\ Ff &= F'f \quad \text{for all functions } f:X\to Y \text{ with } X\neq\emptyset \end{aligned}$$

and $$F\emptyset \neq \emptyset \neq F'\emptyset.$$

Proof. See Adámek and Trnková (1990, III.4.5–6). □

Remark 5.23. Now, given F and F' as in the above lemma, we have

$$\mathsf{Alg}(F) = \mathsf{Alg}(F')$$

because no F-algebra (and no F'-algebra) has underlying set $\emptyset$. And

$$\mathsf{Coalg}(F) \cong \mathsf{Coalg}(F')$$

because the two categories of coalgebras only differ in the name of the codomain of the unique coalgebra $(\emptyset, \alpha_\emptyset)$.

In particular, if $\mathsf{Coalg}(F')$ is complete, so is $\mathsf{Coalg}(F)$.

Remark 5.24. It is not known whether completeness of $\mathsf{Coalg}(F)$, where F is a Set-functor, implies that F is a covarietor.

Remark 5.25. Dualising Remark 5.8, we see that for C with pushouts and *coregular factorisations*, that is, factorisations as an epimorphism followed by a regular monomorphism, the following holds: $\mathsf{Coalg}(F)$ has coregular factorisations created by U whenever F preserves regular monomorphisms. This also generalises to factorisations of cocones. In Set a coregular factorisation of a cocone $f_i:B_i\to A$ can be obtained by factorising each f_i over the subset $X\subseteq A$ that is the union of the images of all f_i. Combining this with Lemma 5.22, we obtain the following corollary.

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Corollary 5.26. If $\mathsf{C} = \mathsf{Set}$, all categories of coalgebras have coregular factorisations created by the forgetful functor.

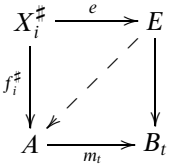
6. Birkhoff Variety (and Covariety) Theorem

Observation 6.1. Suppose that F is an endofunctor of a complete category; then the following hold for every equational class V of F-algebras.

(1) V is closed under products, that is, if $A_t \in \mathsf{V}$ for $t \in T$, then $\prod A_t \in \mathsf{V}$.

(2) V is closed under subalgebras, that is, given a monomorphism $m: A \to B$ in $\mathsf{Alg}(F)$ with $B \in \mathsf{V}$, we have $A \in \mathsf{V}$.

(3) Generalising (1) and (2), V is closed under *monomorphic cones* in $\mathsf{Alg}(F)$. That is, if $m_t: A \to B_t$ is a monomorphic family of homomorphisms in $\mathsf{Alg}(F)$, and if $B_t \in \mathsf{V}$ for all t, then $A \in \mathsf{V}$.
In fact, if $e: X_i^\sharp \to E$ is an equation arrow that each B_t satisfies, then A satisfies e, too: for every $f: X \to A$ factor $(m_t f)_i^\sharp = m_t f_i^\sharp$ (see Lemma 3.13) through e and use the diagonal fill-in



Note that here the indices might even range over a class T.

(4) V is closed under *retract-carried quotient algebras* in $\mathsf{Alg}(F)$, that is, given a homomorphism $q: A \to B$ in $\mathsf{Alg}(F)$ and a morphism $m: UB \to UA$ in C with $Uq \circ m = id$, we have if $A \in V$, then $B \in V$. In fact, if A satisfies $e: X_i^\sharp \to E$, then B also satisfies it: for every $f: X \to B$ we have a factorisation of $(mf)_i^\sharp: X_i^\sharp \to A$ through e, say, $(mf)_i^\sharp = ge$. From Lemma 3.13 we get $(qg)e = q(mf)_i^\sharp = (qmf)_i^\sharp = f_i^\sharp$, showing that $f_i^\sharp$ factors through e.

Theorem 6.2 (The Birkhoff Variety Theorem). Let C be a complete, regularly cowellpowered category with regular factorisations. For every constructive varietor F preserving regular epimorphisms, varieties of F-algebras are precisely the full subcategories of $\mathsf{Alg}(F)$ closed under:

(i) products,
(ii) subalgebras, and
(iii) retract-carried quotient algebras.

Remark 6.3. We will also prove that every variety is reflective in $\mathsf{Alg}(F)$ with reflection arrows being regular epimorphisms. Thus, varieties are complete, and when C is cocomplete, they are also cocomplete (see Theorem 5.12)

Proof. If V is a variety, then it is closed under (i)–(iii) by Observation 6.1. Conversely, let V be closed under (i)–(iii).

(a) Observe that $\mathsf{Alg}(F)$ has regular cone-factorisations created by the forgetful functor – see Remark 5.10. Since $\mathsf{Alg}(F)$ is regularly cowellpowered (which is obvious from

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Proposition 5.9), any full subcategory V of $\mathsf{Alg}(F)$ closed under products and subalgebras is closed under monomorphic cones – see Adámek *et al.* (1990, 16.8). It follows that any such V is reflective in $\mathsf{Alg}(F)$. A reflection of an F-algebra A is obtained from the cone of all morphisms with domain A and codomain in V, $f_t: A \to B_t$ $(t \in T)$: let $e: A \to A'$ be a regular epimorphism forming a regular factorisation of that cone, that is, we have a monomorphic cone $m_t: A' \to B_t$ with $f_t = m_t \circ e$. Since $B_t \in \mathsf{V}$ for each t, we conclude that $A' \in \mathsf{V}$. It follows that $e: A \to A'$ is a reflection of A.

(b) Let $\mathscr{E}$ be the class of all equation arrows $e: X^\sharp \to A'$ where X is an object of C, $A = X^\sharp$ $(= X_i^\sharp$ for some $i)$ is a free F-algebra on X, and $e: A \to A'$ is a reflection of A in V. Then we prove

$$\mathsf{V} = \mathsf{Alg}(F, \mathscr{E}).$$

(α) Every algebra $B \in \mathsf{V}$ satisfies each of the equation arrows $e: X^\sharp \to A'$ above. In fact, for every morphism $f: X \to A$, the unique homomorphism $f^\sharp: X^\sharp \to B$ with $f = f^\sharp \circ \eta_X$ factors through e (since $f^\sharp = f_t$ for some t, and $f^\sharp = m_t \circ e$).

(β) Every algebra C satisfying each $e \in \mathscr{E}$ lies in V. Put $X = UC$ and use $h: X^\sharp \to C$ to denote the unique homomorphism with $h \circ \eta_X = id_X$. For the reflection $e: X^\sharp \to A'$ we know that, since C satisfies e, the homomorphism h factors through e, say, $h = ke$. Then $k: A' \to C$ is a regular epimorphism carried by a split epimorphism in C (since h is), and thus from $A' \in \mathsf{V}$ we conclude that $C \in \mathsf{V}$. □

Remark 6.4. *Regular quotients* of A in $\mathsf{Alg}(F)$ are those B in $\mathsf{Alg}(F)$ that admit a regular epimorphism $e: A \to B$. For $\mathsf{C} = \mathsf{Set}$ the regular quotients in $\mathsf{Alg}(F)$ are precisely those carried by (split) epimorphisms in Set (that is, the 'quotients' or 'homomorphic images' in the usual sense), see Corollary 5.4, since coequalisers of kernel pairs in Set are absolute, that is, preserved by every functor. Thus, the following is a corollary to Theorem 6.2, which, for polynomial functors F, specialises to the classical Birkhoff Variety Theorem.

Corollary 6.5. For every varietor F over Set, varieties of F-algebras are precisely the full subcategories closed in $\mathsf{Alg}(F)$ under products, subalgebras and regular quotients.

Notation 6.6. For a class $\mathscr{E}$ of morphisms in a category C, we use $\mathsf{Inj}\mathscr{E}$ to denote the full subcategory of all objects C injective with respect to any member $e: A \to A'$ of $\mathscr{E}$ (that is, for every morphism $f: A \to C$ there exists $f': A' \to C$ with $f = f'e$). We call $\mathsf{Inj}\mathscr{E}$ the *injectivity class* of $\mathscr{E}$.

Corollary 6.7. Under the assumptions of Theorem 6.2, varieties of F-algebras are precisely the injectivity classes $\mathsf{Inj}\mathscr{E}$ in $\mathsf{Alg}(F)$, where $\mathscr{E}$ is a collection of regular quotients of free F-algebras.

In fact, in the proof of Theorem 6.2 we have found, for every variety V, the corresponding collection $\mathscr{E}$, *viz.*, all reflections of free F-algebras in V. Conversely, it is sufficient to verify that $\mathsf{Inj}\mathscr{E}$, for a class $\mathscr{E}$ as above, is closed under

(i) products – this is trivial,

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(ii) subalgebras – this follows from the diagonal fill-in between monomorphisms and regular epimorphisms, and
(iii) retract-carried quotients.

To show (iii), let $q: R \to Q$ be a homomorphism with $Uq \circ m = id$ for some $m: UQ \to UR$. If R is $\mathscr{E}$-injective, then so is Q: given $e: X^\sharp \to E$ in $\mathscr{E}$ and $h: X^\sharp \to Q$, we form for $g = U(mh)\eta_X: X \to UR$ the unique homomorphism $g^\sharp: X^\sharp \to R$ with $g = Ug^\sharp \cdot \eta_X$. There exists a homomorphism $k: E \to R$ with $g^\sharp = ke$. We claim that $qk: E \to Q$ is the desired factorisation of h through e, that is, $qke = h: X^\sharp \to Q$. Since both sides are homomorphisms (recall $ke = g^\sharp$), it is sufficient to observe that

$$U(qke) \circ \eta_X = Uq \circ g = Uq \circ Um \circ Uh \circ \eta_X = Uh \circ \eta_X.$$

Using Corollary 4.10, we also obtain the following corollary.

Corollary 6.8. Let F be an accessible endofunctor of Set and V be an equational class of F-algebras. Then there is a single regular quotient e of a free F-algebra $X^\sharp$ such that $\mathsf{V} = \mathsf{Inj}\{e\}$.

Remark 6.9. Recall that an object A is called a *regular projective* provided it is projective (the dual to the above injective) with respect to to all regular epimorphisms.

For example, every free F-algebra $X^\sharp$ for an endofunctor F of Set is a regular projective. In fact, given a regular epimorphism $e: A \to B$ in $\mathsf{Alg}(F)$, we have that e is a (split) epimorphism in Set, and choose $m: UB \to UA$ with $Ue \circ m = id$. For every homomorphism $h: X^\sharp \to B$ form $k = (m \circ Uh \circ \eta_X)^\sharp: X^\sharp \to A$, then by Lemma 3.13, we have

$$e \circ k = (Ue \circ m \circ Uh \circ \eta_X)^\sharp = (Uh \circ \eta_X)^\sharp = h.$$

Corollary 6.10. For a varietor F of Set, varieties of F-algebras are precisely the injectivity classes of regular quotients of regular projectives in $\mathsf{Alg}(F)$.

In fact, every variety is such an injectivity class by Corollary 6.7 and Remark 6.9. Conversely, let $\mathscr{E}$ be a collection of regular quotients of regular projectives. Then $\mathsf{Inj}\mathscr{E}$ is closed under products and subalgebras (the argument is precisely as in Corollary 6.7) and it is closed under regular quotients. The argument for the latter is also analogous to that in Corollary 6.7: let $q: R \to Q$ with $Uq \circ m = id$ and $e: E \to E'$ in $\mathscr{E}$ be given, with E a regular projective. For $h: E \to Q$, we have, since q is a regular epimorphisms (Corollary 5.11), $k: E \to R$ with $h = qk$. From the fact that R is e-injective, it now easily follows that so is Q.

Remark 6.11. It might be instructive to show how to obtain, in $\mathsf{Alg}(F)$, from a regular quotient of a regular projective, e, a regular quotient of a free algebra, $\bar{e}$, such that injectivity with respect to e and $\bar{e}$ is equivalent. In fact, consider such an epimorphism, $e: (D, \alpha_D) \to (E, \alpha_E)$. Since the homomorphism $id^\sharp: (D^\sharp, \varphi_D) \to (D, \alpha_D)$ is a regular epimorphism in $\mathsf{Alg}(F)$, and (D, α_D) is regularly projective, we have a homomorphism

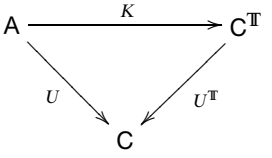
$$m: (D, \alpha_D) \to (D^\sharp, \varphi_D) \quad \text{with} \quad id^\sharp \circ m = id.$$

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Choose a pair of homomorphisms u, v with coequaliser e in $\mathsf{Alg}(F)$. Then an algebra (C, α_C) is orthogonal to e iff for every homomorphism $h:(D, \alpha_D) \to (C, \alpha_C)$ we have $h \circ u = h \circ v$. This is equivalent to stating that for every homomorphism $k:(D^\sharp, \varphi_D) \to (C, \alpha_C)$ we have $k \circ (m \circ u) = k \circ (m \circ v)$: given k, put $h = k \circ m$, and given h, put $k = h \circ id^\sharp$. Thus, if $\bar{e}:(D^\sharp, \varphi_D) \to (\bar{E}, \alpha_{\bar{E}})$ denotes a coequaliser of $m \circ u$ and $m \circ v$ in $\mathsf{Alg}(F)$, then injectivity to $\bar{e}$ and e, respectively, is equivalent. (And the former can be substituted by the equations $u_0 = v_0$ obtained by the kernel pair of $\bar{e}$.)

Remark 6.12. The above Corollary is much in the spirit of Herrlich and Ringel (1972) and Banaschewski and Herrlich (1976).

In the following result we fully characterise varieties as *monadic categories* over C, that is, such concrete categories $\mathsf{A} \xrightarrow{U} \mathsf{C}$ for which U is a right adjoint and the comparison functor K



of the corresponding monad $\mathbb{T}$ over C is an equivalence functor.

In general, we call concrete categories (A, U) and (A', U') *concretely equivalent* provided that there exists an equivalence functor $K:\mathsf{A} \to \mathsf{A}'$ with $U = U'K$.

Theorem 6.13 (Monadicity Theorem). Let C be a cocomplete category. The monadic categories over C are, up to concrete equivalence, precisely the varieties.

Proof.

I. Let $\mathsf{V} = \mathsf{Alg}(F, \mathscr{E})$ be a variety of F-algebras. By definition, the forgetful functor $U:\mathsf{V} \to \mathsf{C}$ is a right adjoint. By Proposition 5.17, U creates all colimits that F preserves, in particular, U creates absolute coequalisers. Consequently, (V, U) is monadic by Beck's Theorem (Mac Lane 1971).

II. For the converse, it suffices to show that, for any monad $\mathbb{T} = (T, \eta, \mu)$ on a cocomplete category C, the Eilenberg–Moore category $\mathsf{C}^{\mathbb{T}}$ of $\mathbb{T}$-algebras coincides with the subcategory $\mathsf{Alg}(T, \mathscr{E})$ of $\mathsf{Alg}(T)$ for a suitable class $\mathscr{E}$ of equation arrows. To do this, consider, for every C-object X, the coproduct

$$X \xrightarrow{m_{i+1}} TX_i^\sharp + X = X_{i+1}^\sharp \xleftarrow{n_{i+1}} TX_i^\sharp.$$

A class $\mathscr{E}_1$ of equation arrows is now defined as follows: for every C-object X let $e_X:X_2^\sharp \to E_X$ be a coequaliser of the pair $m_2, n_2 \circ \eta_{X_1^\sharp} \circ m_1$. A T-algebra (C, α_C) satisfies e_X iff, for every morphism $f:X \to C$, the morphism

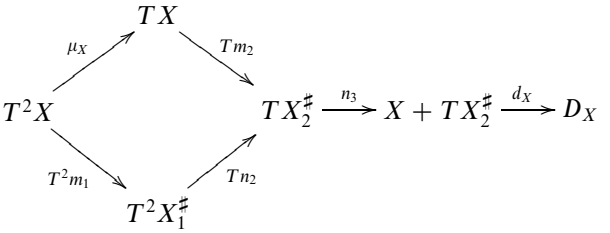
$$f_2^\sharp = [\alpha_C \circ Tf_1^\sharp, f]:TX_1^\sharp + X \to C$$

satisfies $f_2^\sharp \circ m_2 = f_2^\sharp \circ n_2 \circ \eta_{X_1^\sharp} \circ m_1$ or, equivalently, $f = \alpha_C \circ Tf_1^\sharp \circ \eta_{X_1^\sharp} \circ m_1$. Since η is natural, this is equivalent to $f = \alpha_C \circ \eta_C \circ f$, which, for $X = C$ and $f = 1_C$ yields

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satisfaction of the $\mathbb{T}$-algebra axiom $\alpha_C \circ \eta_C = 1_C$. Conversely, $\alpha_C \circ \eta_C = 1_C$ yields $f = \alpha_C \circ \eta_C \circ f$ by composition with f. Thus, the satisfaction of $\mathscr{E}_1$ is equivalent to $\alpha_C \circ \eta_C = 1_C$.

Next we define a class $\mathscr{E}_2$ of equation arrows as follows: for every C-object X, let $d_X : X + TX_2^\sharp = X_3^\sharp \to D_X$ be a coequaliser of the pair $n_3 \circ Tm_2 \circ \mu_X, n_3 \circ Tn_2 \circ T^2 m_1$.



A T-algebra (C, α_C) satisfies d_X iff, for every morphism $f : X \to C$, the morphism

$$f_3^\sharp = [\alpha_C \circ T[\alpha_C \circ Tf_1^\sharp, f], f] : TX_2^\sharp + X \to C$$

satisfies $f_3^\sharp \circ n_3 \circ Tn_2 \circ T^2 m_1 = f_3^\sharp \circ n_3 \circ Tm_2 \circ \mu_X$. This is equivalent to $\alpha_C \circ T\alpha_C \circ T^2 f = \alpha_C \circ Tf \circ \mu_X$, or, since μ is natural, to $\alpha_C \circ T\alpha_C \circ T^2 f = \alpha_C \circ \mu_C \circ T^2 f$. Choosing $f = 1_C$, this yields satisfaction of the $\mathbb{T}$-algebra axiom $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$. Conversely, $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$ yields $\alpha_C \circ T\alpha_C \circ T^2 f = \alpha_C \circ \mu_C \circ T^2 f$ by composition with $T^2 f$. Thus, the satisfaction of $\mathscr{E}_2$ is equivalent to $\alpha_C \circ T\alpha_C = \alpha_C \circ \mu_C$.

Choosing $\mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2$, one thus gets $\mathsf{C}^{\mathbb{T}} = \mathsf{Alg}(T, \mathscr{E})$. □

Example 6.14 (Infinitary universal algebra). Generalising the finitary Ω-algebras of Example 2.2(2), we introduce here a *large signature* as a collection $\Omega = (\Omega_n)_{n \in \mathsf{card}}$ of sets Ω_n (of n-ary operation symbols) indexed by cardinal numbers n. An Ω-*algebra* is a set A together with an n-ary operation $\omega_A : A^n \to A$ for all $n \in \mathsf{card}$ and $\omega \in \Omega_n$. We use Ω-Alg to denote the category of all Ω-algebras and homomorphisms. Well, strictly speaking, this is not a category (having 'too many' objects) but a quasicategory: if $\Omega_n \neq \varnothing$ for arbitrarily large cardinal numbers n, then the collection of all Ω-algebras on the set $\{0, 1\}$ is bigger than a proper class. However, we are actually interested in varieties of Ω-algebras, and those are legitimate categories, as we show below.

The concept of Ω-terms over a set X of variables (disjoint from any Ω_n) is introduced as usual: the class $T_\Omega X$ of all terms is the smallest class containing X and containing $\omega(t_i)_{i<n}$ for any $n \in \mathsf{card}, \omega \in \Omega_n$, and $t_i \in T_\Omega X$ for $i < n$. Computation of terms in an Ω-algebra A is also standard: given $f : X \to A$, extend it to $f^\sharp : T_\Omega X \to A$ by $f^\sharp(\omega(t_i)_{i<n}) = \omega_A(f^\sharp(t_i))_{i<n}$. If Ω is small, then $T_\Omega X$ is a free Ω-algebra on X, if Ω is large, no such free algebra over X exists (even for $X = 1$). By a *variety of* Ω-*algebras* we mean an equationally defined full subcategory V of Ω-Alg having free algebras, that is, such that the obvious forgetful functor $U : \mathsf{V} \to \mathsf{Set}$ has a left adjoint.

Although Ω-Alg is, for Ω large, not expressible by any functor (because $\mathsf{Alg}(F)$ is a category for any F, and Ω-Alg is not), every variety V of Ω-algebras is a monadic category over Set. This follows from Beck's theorem (see the argument in Mac Lane (1971, VI.8)).

By dualising the above results, using Lemma 2.1, we obtain the following result, where *quotients* of a coalgebra A are represented by epimorphisms $e : A \to B$ in $\mathsf{Coalg}(F)$, and

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coretract-carried subcoalgebras of A are carried by monomorphisms in $\mathsf{Coalg}(F)$ that split in C. *Regular subcoalgebras* are subcoalgebras represented by regular monomorphisms.

Theorem 6.15 (Birkhoff Covariety Theorem). Let C be a cocomplete, regularly wellpowered category with coregular factorisations. For every constructive covarietor F preserving regular monomorphisms, covarieties of F-coalgebras are precisely the full subcategories of $\mathsf{Coalg}(F)$ closed under

i. coproducts,
ii. quotient coalgebras, and
iii. coretract-carried subcoalgebras.

Remark 6.16. *Regular subcoalgebras* of A in $\mathsf{Coalg}(F)$ are those B in $\mathsf{Coalg}(F)$ that admit a regular monomorphism $m: B \to A$. For $\mathsf{C} = \mathsf{Set}$, the regular subcoalgebras in $\mathsf{Coalg}(F)$ are precisely those carried by monomorphisms in Set (that is, the 'subcoalgebras' in the usual sense; monomorphisms are more general in Coalg(F)!); this follows from Corollary 5.19 if F preserves monomorphisms, since equalisers of cokernel pairs of functions $f: B \to A$ with $B \neq \emptyset$ in Set are absolute, that is, preserved by every functor (and every empty homomorphism $(\emptyset, \alpha_\emptyset) \to (A, \alpha_A)$ is an equaliser of the two coproduct injections of $(A, \alpha_A) + (A, \alpha_A)$). If F does not preserve monomorphisms, apply Lemma 5.22.

Thus, since every covariety contains the empty coalgebra, the following corollary is an immediate consequence of Theorem 6.15 for functors F that preserve monomorphisms; if F does not preserve monomorphisms, apply Lemma 5.22 again.

Corollary 6.17. For every covarietor F over Set, covarieties of F-coalgebras are precisely the full subcategories closed in $\mathsf{Coalg}(F)$ under coproducts, regular subcoalgebras and quotient coalgebras.

Corollary 6.18. Under the assumptions of the Birkhoff Covariety Theorem, covarieties of F-coalgebras are precisely the projectivity classes of regular subcoalgebras of cofree F-coalgebras.

Corollary 6.19. For a covarietor F over Set, covarieties of F-coalgebras are precisely the projectivity classes of regular subobjects of regular injectives in $\mathsf{Coalg}(F)$.

For the sake of completeness we recall the following result from Gumm (1999); note that this is *not* the dual of Corollary 6.8 (though embeddings of subcoalgebras are precisely the regular monomorphisms)!

Let C be a covariety of F-coalgebras for a bounded Set-functor. Then C is the projectivity class of a single embedding of a subcoalgebra into a cofree F-coalgebra.

Example 6.20. Given a category Vec_k of vector spaces over a field k we use F to denote the endofunctor of Vec_k sending a space V to $k \times (\mathsf{V} \otimes \mathsf{V})$. F is a constructive covarietor (Porst 2001). k-coalgebras in the traditional sense (see, for example, Sweedler (1969)) obviously form a full subcategory of $\mathsf{Coalg}(F)$. By straightforward diagram chasing, one can prove that this subcategory satisfies the condition of the Birkhoff Covariety Theorem

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(Theorem 6.15) (Porst 2001). Thus *the category of k-coalgebras is a covariety of F-coalgebras.*

Theorem 6.21 (Comonadicity Theorem). Let C be a complete category. Then comonadic categories over C are, up to concrete equivalence, precisely the covarieties.

While the last theorem shows that the dual of a covariety over Set is a variety over Set^{op} it also implies the following additional dualisation principle.

Proposition 6.22. The dual of a covariety over Set is equivalent to a variety over Set.

Proof. Let C be a covariety over Set with underlying functor U. By means of the contravariant power-set functor $\mathscr{P}'$, the category Set^{op} is monadic over Set. Let $V:\mathsf{C}\to\mathsf{Set}$ be the composite of U^{op} and $\mathscr{P}'$. We need to show that V is monadic. Since V has a left adjoint and creates limits, it suffices to prove that V creates coequalisers of congruence relations (= kernel pairs). Hence, let $r,s:(C,\alpha_C)\to(D,\alpha_D)$ be a pair of C-morphisms such that Vr, Vs is a congruence relation, and let $q:\mathscr{P}'(D)\to X$ be its coequaliser. Since $\mathscr{P}'$ reflects congruence relations and creates their coequalisers, there is a unique Set^{op}-morphism $q':D\to X'$ with $\mathscr{P}'(q')=q$, and this is a coequaliser of the congruence relation Ur, Us. If $X'\neq\varnothing$, this will even be a split coequaliser such that U creates from it a coequaliser of r,s. The remaining case $X'=\varnothing$ is trivial: the unique F-coalgebra structure on $\varnothing$ obviously does the job. □

7. Conclusions and future research

As mentioned in the introduction, our paper has been inspired by a number of interesting articles devoted to covarieties of coalgebras in the category Set. We somehow felt that we were missing the 'beginning of the story': whereas the concept of a variety has been inspired by examples such as groups and lattices, and developed further through Birkhoff's classical variety theorem to the more general categorical concepts, in the realm of coalgebras all this has been missing. In the process of asking ourselves what the analogue of equations of universal algebra for coalgebras is, we have, to our surprise, observed that the situation is not clear enough for varieties either: the equivalence between (infinitary) varieties and monadicity over Set did not seem to have an adequate formalisation over other categories. This led us to the concept of an equation based on the free-algebra construction for a functor F that, in fact, does not have free algebras. The fact then that the slogan

$$\text{varietal} = \text{monadic}$$

holds for all cocomplete categories seems to support our approach.

To dualise the story, and to get the concepts of coequation and covariety so that the slogan

$$\text{covarietal} = \text{comonadic}$$

holds in complete categories was an easy way to the original goal of our paper. But then we were surprised of how scarce the concrete examples of covarieties were in the literature. In the present paper we have shown several natural examples of covarieties. Some of them, for example, reflexivity and symmetry for transition systems, are somehow reminiscent of modal logic, but we have not pursued this in any way.

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In order to prove an analogue of Birkhoff's (Co) Variety Theorem, much more than just limits and colimits needs to be assumed. But we hope that the side conditions presented in our paper are close to the minimum of what one should ask for. It would be worthwhile to supplement the above results with counter-examples demonstrating that this or that side condition is indispensible.

Our paper is an extension and refinement of the extended abstract Adámek and Porst (2001), except that we have omitted the last part of that abstract, in which

$$\text{accessible} = \text{bounded}$$

is proved for all set functions. The reason is that we are preparing a paper (Adámek and Porst (2002)) devoted to the properties of the categories of coalgebras on accessible functors, where this result fits much better. (The main property is that if F is λ-accessible, then F-algebras form a λ-accessible category; this holds 'almost' but not quite.)

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