Actions, Enrichments and Universal Enrichments

By

Hyeon Tai Jung

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Hyeon Tai Jung

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Abstract

This thesis studies the relationship between actions of monoidal categories and enrichments in monoidal categories, and uses these ideas to prove the existence of *universal enrichments* of a category.

It first details the correspondence between actions of a monoidal category $\mathcal{V}$ on a category $\mathcal{A}$ which are right-closed, and enrichments of $\mathcal{A}$ in $\mathcal{V}$ which are copowered, and explains how this restricts back to an equivalence between cocomplete enrichments of the cocomplete category $\mathcal{A}$, and right-closed monoidal actions which preserve colimits in the $\mathcal{A}$-variable.

It then considers the locally presentable case, showing that any locally presentable category $\mathcal{A}$ has an enrichment in the locally presentable monoidal category $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ of cocontinuous endofunctors of $\mathcal{A}$. It shows that this is a *universal* cocomplete enrichment of $\mathcal{A}$, in the sense that any other cocomplete enrichment of $\mathcal{A}$ is obtained from the universal one via base-change.

It then considers the case of actions of braided monoidal categories $\mathcal{V}$ on monoidal categories $\mathcal{A}$. It shows that right-closed such actions correspond to enrichments of $\mathcal{A}$ to a *monoidal* $\mathcal{V}$-category. Again, there is a notion of universal enrichment, and the thesis concludes by sketching a proof that a locally presentable monoidal $\mathcal{A}$ has a universal monoidal enrichment in its monoidal centre $\mathcal{Z}(\mathcal{A})$.

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1 Introduction

During the 19th century, many structures in mathematics were abstracted by extracting the fundamental essence of their mathematical properties. One example of this was the idea of symmetry which was formalised as the concept of a group action and generalised as the concept of a monoid action.

Category theory, a highly abstract field of mathematics, was developed in the mid-20th century and using its tools, many parts of mathematics saw further abstraction through a process called *categorification*. One example is the above mentioned concept – *monoid action* – which was generalised to the concept of an *action of a monoidal category $\mathcal{V}$ on a category $\mathcal{A}$*. These can be considered "the same" algebraic structure, just in different settings – one in the symmetric monoidal category **Set** and the other in the symmetric monoidal 2-category **Cat**.

Algebraic structures in different symmetric monoidal categories are formalised by the language of *operads* and *algebras over an operad*. An operad is a gadget parametrising structures involving finitary operations with a notion of associative and unital composition. An algebra over an operad realises these structures where the gadget equips a colored collection of objects with the n-ary operations. Examples include algebras over the little d-disks operad, called *E_d-algebras*. E_1-algebras over **Set** are monoids and E_2-algebras over **Set** are commutative monoids, whereas E_1-algebras over **Cat** are monoidal categories and E_2-algebras over **Cat** are braided monoidal categories.

The operad that governs structures like the action of a monoidal category $\mathcal{V}$ on a category $\mathcal{A}$ is the *Swiss-Cheese operad* SC_d introduced by Voronov in [Vor99]. The algebras over SC_d, also called *Swiss-Cheese algebras*, formalise the concept of an *action of an E_d-algebra on an E_{d-1}-algebra*. In this thesis, we consider two examples over **Cat**:

(a) algebras over SC_1: actions of a monoidal category $\mathcal{V}$ on a category $\mathcal{A}$;

(b) algebras over SC_2: actions of a braided monoidal category $\mathcal{V}$ on a monoidal category $\mathcal{A}$.

Our objective is to find the appropriate conditions on such actions under which they are connected to:

(a) giving the category $\mathcal{A}$ a structure of a $\mathcal{V}$-enriched category; and

(b) giving the monoidal category $\mathcal{A}$ a structure of a monoidal $\mathcal{V}$-category,

respectively, i.e. giving the E_{d-1}-algebra $\mathcal{A}$ in **Cat** a structure of a E_{d-1}-algebra in $\mathcal{V}$-**Cat**.

One example of a connection as in (a) arises in (right) closed monoidal categories $(\mathcal{A}, \otimes, I)$ with internal hom $\underline{\mathcal{A}}(X, Y)$. A (right) closed monoidal category $\mathcal{A}$ has a right-closed monoidal action on itself given by its monoidal product $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. In this situation, $\mathcal{A}$ is enriched over itself and has a structure of an $\mathcal{A}$-category $\underline{\mathcal{A}}$. This is our Example 2.1.1; moreover, it is a copowered enrichment of $\mathcal{A}$ (Example 2.4.2).

A *copower* is one kind of colimit we can have in an enriched category, which generalises copowers in ordinary categories. A *cocomplete $\mathcal{V}$-category* is an enriched category which has all kinds of colimits we can have in an enriched category: not only *copowers* but also *conical colimits*. And thus, we hope to find connections as in (a) which relate to giving $\mathcal{A}$ a structure of a copowered and cocomplete $\mathcal{V}$-category; and similarly connections (b) which relate to giving a copowered and cocomplete monoidal $\mathcal{V}$-category. Thus, we exhibit categorical equivalences that roughly say:

(a′) for a closed monoidal category $\mathcal{V}$ and a cocomplete category $\mathcal{A}$, a monoidal action $\odot\colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ of $\mathcal{V}$ on $\mathcal{A}$ such that each $(-) \odot C\colon \mathcal{V} \to \mathcal{A}$ is equipped with a right adjoint and each $V \odot (-)\colon \mathcal{A} \to \mathcal{A}$ is cocontinuous is equivalent to giving $\mathcal{A}$ a structure of cocomplete $\mathcal{V}$-category.

And when we drop the condition that $V \odot (-)\colon \mathcal{A} \to \mathcal{A}$ be cocontinuous, it is equivalent to giving $\mathcal{A}$ a structure of copowered $\mathcal{V}$-category.

(b′) For a braided and closed monoidal category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ and a cocomplete monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$ whose monoidal product $\boxtimes$ preserves colimits in each variable, an action of $\mathcal{V}$ on $\mathcal{A}$ such that each $(-) \odot C\colon \mathcal{V} \to \mathcal{A}$ is equipped with a right adjoint and each $V \odot (-)\colon \mathcal{A} \to \mathcal{A}$ is cocontinuous is equivalent to giving $\mathcal{A}$ a structure of cocomplete monoidal $\mathcal{V}$-category.

And when we drop the condition that $V \odot (-)\colon \mathcal{A} \to \mathcal{A}$ be cocontinuous, then it is equivalent to giving $\mathcal{A}$ a structure of copowered monoidal $\mathcal{V}$-category.

These results are presented in Chapter 3 and Chapter 6 of the thesis.

In my Master's thesis [Jun18], the Swiss-Cheese algebras (a) and (b) were studied to prove the *Swiss-Cheese Conjecture* proposed by Kontsevich ([Kon99]) in a few specific low-dimensional cases. Readers can check [Jun18] for the details and the raison d'etre for formulating the conjecture, but to describe it briefly, one can fix an E_{d-1}-algebra $\mathcal{A}$ and consider the homotopy category of E_d-algebras $\mathcal{V}$ acting on $\mathcal{A}$, that is, in our case, (a) the category of monoidal categories $\mathcal{V}$ equipped with an action on $\mathcal{A}$ and (b) the category of braided monoidal categories $\mathcal{V}$ acting on $\mathcal{A}$. The Swiss-Cheese conjecture says that these categories have a terminal object $\mathcal{H}(\mathcal{A})$. This means that an algebra over SC_d is uniquely determined by a map of E_d-algebras $\mathcal{V} \to \mathcal{H}(\mathcal{A})$, so for these cases:

(a″) an action of a monoidal category $\mathcal{V}$ on a category $\mathcal{A}$ is equivalent to a monoidal functor $\mathcal{V} \to \mathbf{Cat}(\mathcal{A}, \mathcal{A})$

(b″) an action of a braided monoidal category $\mathcal{V}$ on a monoidal category $\mathcal{A}$ is equivalent to a braided monoidal functor $\mathcal{V} \to \mathcal{Z}(\mathcal{A})$.

This is one way this thesis is related to my Master's thesis; and with this as motivation, we proceed to find the *universal cocomplete enrichment of* $\mathcal{A}$ in the context where $\mathcal{V}$ and $\mathcal{A}$ are locally presentable categories. Our result, roughly, is that

(a‴) any locally presentable category $\mathcal{A}$ has a structure of cocomplete category $\underline{\mathcal{A}}$ enriched over the monoidal category $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ of cocontinuous functors. This is the universal cocomplete enrichment of $\mathcal{A}$ among those over locally presentable closed monoidal categories $\mathcal{V}$.

(b‴) any locally presentable cocomplete monoidal category $\mathcal{A}$ whose monoidal product preserves colimits in each variable has a structure of cocomplete monoidal category $\underline{\mathcal{A}}$ enriched over $\mathcal{Z}(\mathcal{A})$. This is the universal cocomplete monoidal enrichment among those over locally presentable, braided and closed monoidal categories $\mathcal{V}$.

The flow of our argument for (a‴), and similarly for (b‴), is that given a cocomplete enrichment $\underline{\mathcal{A}}_{\mathcal{V}}$ of a locally presentable category $\mathcal{A}$ over a locally presentable closed monoidal category $\mathcal{V}$, that is, via (a′), given by a monoidal action $\odot\colon \mathcal{V}\times\mathcal{A}\to\mathcal{A}$ such that

- each $(-)\odot C\colon \mathcal{V}\to\mathcal{A}$ is equipped with a right adjoint $\underline{\mathcal{A}}_{\mathcal{V}}(C,-)\colon \mathcal{A}\to\mathcal{V}$
- each $V\odot(-)\colon \mathcal{A}\to\mathcal{A}$ is cocontinuous,

we have, by (a″), a cocontinuous monoidal functor $F\colon \mathcal{V}\to\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ where $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ is the category of cocontinuous functors. We take a note of the result proven in [Bir84] that this category $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ is also locally presentable so that by the Adjoint Functor Theorem we have a right adjoint $G\colon \mathrm{Cocts}(\mathcal{A},\mathcal{A})\to\mathcal{V}$. Moreover, by an instance of doctrinal adjunction, this adjunction

$$\mathcal{V} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Cocts}(\mathcal{A},\mathcal{A}). \tag{1.1}$$

is a monoidal adjunction (i.e. an adjunction in the 2-category **MonCat** of monoidal categories, lax monoidal functors and monoidal natural transformations) for which we can then apply the *change of base* along $G\colon \mathrm{Cocts}(\mathcal{A},\mathcal{A})\to\mathcal{V}$ on $\underline{\mathcal{A}}$ from (a‴) to see that $\underline{\mathcal{A}}_{\mathcal{V}}\cong G_*\underline{\mathcal{A}}$ where $G_*\underline{\mathcal{A}}$ is the $\mathcal{V}$-category after the change of base along G. We present these results concretely as a chain of categorical equivalences pseudonatural in $\mathcal{V}$.

As an example, we take the *category of models of a Lawvere theory*. This is a well-known example of a locally presentable category. In particular we look at the category of models of the Lawvere theory generated by an I-indexed set of constants, which is the coslice category $I/\mathbf{Set}$. We choose this example as it is complicated enough to be non-trivial and simple enough to calculate. We calculate the universal cocomplete enrichment $\underline{I/\mathbf{Set}}$ of $I/\mathbf{Set}$ among those over locally presentable right-closed monoidal base categories, that is, the cocomplete enrichment over the monoidal category of endoadjunctions $\mathrm{Cocts}(I/\mathbf{Set},I/\mathbf{Set})\simeq\mathbf{EndAjd}(I/\mathbf{Set})$.

The thesis is constructed as follows. In Chapter 2, we go over some basic aspects of enriched category theory, including the theory of cocomplete enriched categories. In Chapter 3, we present enrichments through monoidal actions. This is where we will develop the connection (a′) between monoidal actions and giving $\mathcal{A}$ a structure of a cocomplete $\mathcal{V}$-category. In Chapter 4, we present the universal cocomplete enrichment for locally presentable categories. This is where we will present the result (a‴). In Chapter 5, we calculate the universal cocomplete enrichment of the locally presentable category $I/\mathbf{Set}$. And in Chapter 6 and Chapter 7, we show the case of higher-tuply monoidal action, as in (b), (b′), (b‴). Finally, we make some further conjectures on possible generalisations of our work.

2 Enriched Category Theory

In this chapter, we summarise the basic elements of *enriched category theory* ([Kel83]). An ordinary locally small category C has hom-sets $C(A, B)$ and composition-functions $C(B, C) \times C(A, B) \to C(A, C)$. In many examples of locally small categories, the hom-sets are also equipped with some additional structure beyond being merely a set.

This leads to the idea of an *enriched category* C, wherein these hom-sets are replaced by hom-*objects* $C(A, B)$ of some monoidal category $(\mathcal{V}, \otimes, I)$ called the **base category**. And likewise, instead of composition-functions, we have composition-*morphisms* $C(B, C) \otimes C(A, B) \to C(A, C)$ of the given base category $\mathcal{V}$. This framework recovers the usual notion of locally small category when the base enriching monoidal category is $(\mathbf{Set}, \times, \{*\})$.

The purpose of this chapter is to introduce the necessary elements of enriched categories by closely following the references [Kel83] and [Rie14]; and except for minor remarks, the material presented in this chapter is purely expository.

Note. Prior to laying out the definitions, we assume that the base category $(\mathcal{V}, \otimes, I)$ is a **locally small, complete and cocomplete monoidal category** unless otherwise stated. We will also use a, r, l for the associators, right unitors and left unitors of the monoidal category $\mathcal{V}$ in contextually obvious cases. Sometimes we add the assumption that $\mathcal{V}$ is symmetric, then write s for the symmetry.

2.1 Enriched Categories

Definition 2.1.1 (Enriched category over $\mathcal{V}$)**.** Given a base category $(\mathcal{V}, \otimes, I)$, an *enriched category C over $\mathcal{V}$*, or simply a *$\mathcal{V}$-category C* consists of

1. a collection $\operatorname{ob} C$ of objects;
2. a hom-object $C(A, B) \in \mathcal{V}$ for each pair $A, B \in \operatorname{ob} C$;
3. a composition-morphism in $\mathcal{V}$ for each triple $A, B, C \in \mathcal{V}$

$$m \colon C(B, C) \otimes C(A, B) \to C(A, C);$$

4. a unit morphism $i_A \colon I \to C(A, A)$ in $\mathcal{V}$ for each object $A \in C$;

such that the following diagrams commute, expressing associativity and unitality.

$$\begin{array}{ccc} C(C,D) \otimes C(B,C) \otimes C(A,B) & \xrightarrow{\mathrm{id} \otimes m} & C(C,D) \otimes C(A,C) \\ \downarrow{\scriptstyle m \otimes \mathrm{id}} & & \downarrow{\scriptstyle m} \\ C(B,D) \otimes C(A,B) & \xrightarrow[m]{} & C(A,D) \end{array}$$

$$\begin{array}{ccccc} C(B,B) \otimes C(A,B) & \xrightarrow{m} & C(A,B) & \xleftarrow{m} & C(A,B) \otimes C(A,A) \\ \uparrow{\scriptstyle i \otimes \mathrm{id}} & \overset{l}{\nearrow}_{\cong} & & \overset{r}{\nwarrow}_{\cong} & \uparrow{\scriptstyle \mathrm{id} \otimes i} \\ I \otimes C(A,B) & & & & C(A,B) \otimes I \end{array}$$

Definition 2.1.2 (Closed monoidal category)**.** A *(right) closed monoidal category* is a monoidal category $(\mathcal{A}, \otimes, I)$ equipped with a right adjoint $[X, -] \colon \mathcal{A} \to \mathcal{A}$ for each functor $(- \otimes X) \colon \mathcal{A} \to \mathcal{A}$

$$\mathcal{A}(A \otimes X, Y) \cong \mathcal{A}(A, [X, Y]). \tag{2.1}$$

The object $[X, Y] \in \mathcal{A}$, which we also write as $\underline{\mathcal{A}}(X, Y) \in \mathcal{A}$, is called the *internal hom* from X to Y and the counits, written as $\mathrm{ev}_{X,Y} \colon [X, Y] \otimes X \to Y$, are called the *evaluation maps*.

A monoidal category $(\mathcal{A}, \otimes, I)$ is *left closed* if each functor $(X \otimes -) \colon \mathcal{A} \to \mathcal{A}$ for each $X \in \mathcal{A}$ admits a right adjoint $[[X, -]] \colon \mathcal{A} \to \mathcal{A}$

$$\mathcal{A}(X \otimes A, Y) \cong \mathcal{A}(A, [[X, Y]]) \tag{2.2}$$

with its counits $\mathrm{ev}'_{X,Y} \colon X \otimes [[X, Y]] \to Y$; and is called *biclosed* if it is left closed and right closed.

In particular, a symmetric monoidal category $\mathcal{A}$ is left closed if and only if it is right closed; and with the isomorphisms $p \colon [X, Y] \xrightarrow{\cong} [[X, Y]]$ obtained from the Yoneda Lemma, it is easy to see that $\mathrm{ev}_{X,B} = \mathrm{ev}'_{X,B} \circ (\mathrm{id} \otimes p) \circ s \colon [X, B] \otimes X \to B$.

Example 2.1.1 (Category enriched over itself)**.** When a monoidal category $(\mathcal{A}, \otimes, I)$, not necessarily symmetric, is monoidally right closed, it is equipped with a canonical structure of an $\mathcal{A}$-enriched category with:

1. objects being the objects of $\mathcal{A}$;
2. the internal hom $\underline{\mathcal{A}}(X, Y) \in \mathcal{A}$ as the hom-object from X to Y;
3. the composition maps $m \colon \underline{\mathcal{A}}(Y, Z) \otimes \underline{\mathcal{A}}(X, Y) \to \underline{\mathcal{A}}(X, Z)$ being the adjunct of the composite

   $$\big(\underline{\mathcal{A}}(Y, Z) \otimes \underline{\mathcal{A}}(X, Y)\big) \otimes X \xrightarrow[\cong]{a} \underline{\mathcal{A}}(Y, Z) \otimes \big(\underline{\mathcal{A}}(X, Y) \otimes X\big) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} \underline{\mathcal{A}}(Y, Z) \otimes Y \xrightarrow{\mathrm{ev}} Z$$

   under the natural bijections of the form $\mathcal{A}(A \otimes B, C) \cong \mathcal{A}(A, \underline{\mathcal{A}}(B, C))$ from the monoidal closedness of $\mathcal{A}$;

4. the units $i_X \colon I \to \underline{\mathcal{A}}(X, X)$ corresponding to the adjunct of the left unitors $l_X \colon I \otimes X \to X$, likewise under the monoidal closedness of $\mathcal{A}$.

Example 2.1.2 (Free $\mathcal{A}$-category)**.** Given any locally small category C and a symmetric closed monoidal category $(\mathcal{A}, \otimes, I)$ with all small coproducts, there is an $\mathcal{A}$-category with:

1. objects being the objects of C;
2. $\coprod_{f \in C(X,Y)} I \in \mathcal{A}$ as the hom-object from X to Y;
3. the composition map $m \colon \left(\coprod_{C(Y,Z)} I\right) \otimes \left(\coprod_{C(X,Y)} I\right) \to \coprod_{C(X,Z)} I$ is easy to see as the monoidal product $\otimes$ preserves coproducts in each variable whenever the symmetric monoidal category $\mathcal{A}$ is monoidally closed and has all small coproducts, i.e.

$$\begin{aligned}\left(\coprod_{C(Y,Z)} I\right) \otimes \left(\coprod_{C(X,Y)} I\right) &\cong \coprod_{C(Y,Z)} \left(I \otimes \left(\coprod_{C(X,Y)} I\right)\right) \\ &\cong \coprod_{C(Y,Z)} \left(\coprod_{C(X,Y)} I \otimes I\right) \\ &\cong \coprod_{C(Y,Z)\times C(X,Y)} I\end{aligned}$$

   and hence the composition map is

$$m \colon \left(\coprod_{C(Y,Z)} I\right) \otimes \left(\coprod_{C(X,Y)} I\right) \cong \coprod_{C(Y,Z)\times C(X,Y)} I \to \coprod_{C(X,Z)} I$$

   whose second part is reindexing along the composition of C.
4. the unit $i_X \colon I \to \coprod_{C(X,X)} I$ is the coprojection $\iota_{\mathrm{id}_X} \colon I \to \coprod_{C(X,X)} I$ indexed at the identity arrow $\mathrm{id}_X \colon X \to X$ of C.

Example 2.1.3 (Opposite $\mathcal{V}$-category)**.** Let $\mathcal{V}$ be a symmetric monoidal category. Given a $\mathcal{V}$-category C, its *opposite $\mathcal{V}$-category* C^{op} has the same objects $\mathrm{ob}\, C^{\mathrm{op}} = \mathrm{ob}\, C$, and hom-objects $C^{\mathrm{op}}(X, Y) = C(Y, X)$ for every pair $X, Y \in C$. The unit map is the same as before and the composition map $m \colon C^{\mathrm{op}}(Y, Z) \otimes C^{\mathrm{op}}(X, Y) \to C^{\mathrm{op}}(X, Z)$ is given by

$$C(Z, Y) \otimes C(Y, X) \xrightarrow[\cong]{s} C(Y, X) \otimes (Z, Y) \xrightarrow{m_C} C(Z, X) = C^{\mathrm{op}}(X, Z).$$

Example 2.1.4 (Tensor product $\mathcal{V}$-category)**.** Let $\mathcal{V}$ be a symmetric monoidal category. Given a pair of $\mathcal{V}$-categories C and $\mathcal{D}$, the *tensor product $\mathcal{V}$-category* $C \otimes \mathcal{D}$ has $\mathrm{ob}(C \otimes \mathcal{D}) = \mathrm{ob}\, C \times \mathrm{ob}\, \mathcal{D}$ and $C \otimes \mathcal{D}\big((C, D), (C', D')\big) = C(C, C') \otimes \mathcal{D}(D, D')$. The composition maps and the unit maps are easy to see from the monoidal symmetry and the unit.

2.2 Enriched Functors and Natural Transformations

The corresponding enrichments of ordinary functors and natural transformations are $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations. Together with $\mathcal{V}$-categories, these form the 2-category $\mathcal{V}$-**Cat**. Here, we introduce these notions.

Definition 2.2.1 ($\mathcal{V}$-functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. A *$\mathcal{V}$-functor* $F\colon \mathcal{C} \to \mathcal{D}$ is given by:

1. a map $\operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$, assigning to every $X \in \operatorname{ob}\mathcal{C}$ some $F(X) \in \operatorname{ob}\mathcal{D}$;
2. a morphism $F_{X,Y}\colon \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ in $\mathcal{V}$ for each pair $X, Y \in \mathcal{C}$;

such that the following diagrams, expressing the preservation of compositions and units, commute:

$$\begin{array}{ccc} \mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y) & \xrightarrow{m} & \mathcal{C}(X,Z) \\ {\scriptstyle F_{Y,Z}\otimes F_{X,Y}}\downarrow & & \downarrow{\scriptstyle F_{X,Z}} \\ \mathcal{D}(FY,FZ)\otimes\mathcal{D}(FX,FY) & \xrightarrow[m]{} & \mathcal{D}(FX,FZ) \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{i} & \mathcal{C}(X,X) \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle F_{X,X}} \\ & & \mathcal{D}(FX,FX) \end{array} \tag{2.3}$$

Remark. Suppose that the base category $\mathcal{V}$ is also monoidally closed so that we can consider it as a $\mathcal{V}$-category enriched over itself (Example 2.1.1). We can express any $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$ differently using the natural bijections of the form $\mathcal{V}(X\otimes Y, Z) \cong \mathcal{V}(X, \underline{\mathcal{V}}(Y,Z))$ from the monoidal closedness of $\mathcal{V}$. Indeed, the family of maps in $\mathcal{V}$

$$F_{A,B}\colon \mathcal{A}(A,B) \to \underline{\mathcal{V}}(F(A),F(B))$$

can be rewritten as a family of maps in $\mathcal{V}$

$$F_{A,B}\colon \mathcal{A}(A,B)\otimes FA \to FB \tag{2.4}$$

and the axioms (2.3) are then expressed by the following commutative diagrams:

$$\begin{array}{ccc} \mathcal{A}(B,C)\otimes\mathcal{A}(A,B)\otimes FA & \xrightarrow{m\otimes\mathrm{id}} & \mathcal{A}(A,C)\otimes FA \\ {\scriptstyle \mathrm{id}\otimes F_{A,B}}\downarrow & & \downarrow{\scriptstyle F_{A,C}} \\ \mathcal{A}(B,C)\otimes FB & \xrightarrow[F_{B,C}]{} & FC \end{array} \tag{2.5}$$

$$\begin{array}{ccc} I\otimes FA & \xrightarrow{i_A\otimes\mathrm{id}} & \mathcal{A}(A,A)\otimes FA \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle F_{A,A}} \\ & & FA. \end{array} \tag{2.6}$$

Example 2.2.1 (Enriched hom-functor)**.** Suppose that the base category $\mathcal{V}$ is symmetric and monoidally closed so that we can again consider it as a $\mathcal{V}$-category enriched over itself. For a $\mathcal{V}$-category $\mathcal{C}$, the *$\mathcal{V}$-enriched hom-functor*

$$\mathrm{hom}\colon \mathcal{C}^{\mathrm{op}}\otimes\mathcal{C} \to \mathcal{V}$$

where the domain is the tensor product $\mathcal{V}$-category (Example 2.1.4), is the $\mathcal{V}$-functor given by

1. the map $(x, y) \mapsto C(x, y)$ on objects;
2. on morphisms, $\hom_{(x,y),(x',y')} : C^{\mathrm{op}}(x, x') \otimes C(y, y') \to \underline{\mathcal{V}}(C(x, y), C(x', y'))$ is given by the adjunct transpose, from the monoidal closedness of $\mathcal{V}$, of the following map

$$\begin{array}{c} C^{\mathrm{op}}(x, x') \otimes C(y, y') \otimes C(x, y) \\ \downarrow{\scriptstyle \mathrm{id} \otimes m} \\ C(x', x) \otimes C(x, y') \xrightarrow[\cong]{s} C(x, y') \otimes C(x', x) \xrightarrow{m} C(x', y'). \end{array}$$

Example 2.2.2 (Representable $\mathcal{V}$-functor)**.** Similarly, suppose that the base category $\mathcal{V}$ is monoidally closed and consider it as a $\mathcal{V}$-category. For a $\mathcal{V}$-category C, the *representable $\mathcal{V}$-functor*

$$C(c, -) : C \to \mathcal{V}$$

is the $\mathcal{V}$-functor defined on the objects by $x \mapsto C(c, x) \in \mathcal{V}$, and on morphisms,

$$C(c, -)_{x,y} : C(x, y) \to \underline{\mathcal{V}}(C(c, x), C(c, y))$$

in $\mathcal{V}$ is given by the adjunct transpose of the composition-morphism $m : C(x, y) \otimes C(c, x) \to C(c, y)$ of the $\mathcal{V}$-category C.

In addition, when $\mathcal{V}$ is also symmetric, the *contravariant representable $\mathcal{V}$-functor* is the $\mathcal{V}$-functor $C(-, c) : C^{\mathrm{op}} \to \mathcal{V}$ defined similarly with

$$C(-, c)_{x,y} : C(x, y) \to \underline{\mathcal{V}}(C(y, c), C(x, c))$$

being the adjunct transpose of the composition map for C^{op} in the Example 2.1.3.

Example 2.2.3 (Partial $\mathcal{V}$-functor)**.** Let the base monoidal category $\mathcal{V}$ be symmetric and consider any two variable $\mathcal{V}$-functor $H : \mathcal{A} \otimes \mathcal{B} \to C$. The *partial $\mathcal{V}$-functor* $H(X, -) : \mathcal{B} \to C$ for H is defined by the following composite of $\mathcal{V}$-functors:

$$\mathcal{B} \xrightarrow{\cong} \mathcal{I} \otimes \mathcal{B} \xrightarrow{X \otimes \mathrm{id}} \mathcal{A} \otimes \mathcal{B} \xrightarrow{H} C$$

where $\mathcal{I}$ is the unit $\mathcal{V}$-category with one object $*$ and where $\mathcal{I}(*, *) = I \in \mathcal{V}$ is the monoidal unit of $\mathcal{V}$, and $X : \mathcal{I} \to \mathcal{A}$ is the obvious $\mathcal{V}$-functor with $* \mapsto X$ and $\mathcal{I}(*, *) = I \xrightarrow{i_X} \mathcal{A}(X, X)$. Similarly for the partial $\mathcal{V}$-functor $H(-, Y) : \mathcal{A} \to C$.

Note that the representable $\mathcal{V}$-functor and the contravariant representable $\mathcal{V}$-functor in Example 2.2.2 are the partial $\mathcal{V}$-functors of the enriched hom-functor in Example 2.2.1.

Definition 2.2.2 ($\mathcal{V}$-natural transformation)**.** Given two $\mathcal{V}$-functors $F, G : C \to \mathcal{D}$, a *$\mathcal{V}$-enriched natural transformation* written $\alpha : F \Rightarrow G : C \to \mathcal{D}$ or

$$C \underset{G}{\overset{F}{\Downarrow \alpha}} \mathcal{D}$$

from F to G is an $(\mathrm{ob}\, C)$-indexed family of morphisms in $\mathcal{V}$

$$\alpha_x \colon I \to \mathcal{D}(Fx, Gx), \tag{2.7}$$

called *components* of α, such that for all objects $x, y \in C$, the following diagram, expressing naturality, commutes:

$$\begin{array}{ccc} C(x,y) & \xrightarrow{F_{x,y}} & \mathcal{D}(Fx,Fy) \\ {\scriptstyle G_{x,y}}\downarrow & & \downarrow{\scriptstyle (\alpha_y)_*} \\ \mathcal{D}(Gx,Gy) & \xrightarrow[(\alpha_x)^*]{} & \mathcal{D}(Fx,Gy). \end{array} \tag{2.8}$$

Here $(\alpha_y)_* = \mathcal{D}(Fx, \alpha_y)$ and $(\alpha_x)^* = \mathcal{D}(\alpha_x, Gy)$ are the following maps called *post- and precomposition* respectively:

$$(\alpha_y)_* \colon \mathcal{D}(Fx,Fy) \cong I \otimes \mathcal{D}(Fx,Fy) \xrightarrow{\alpha_y \otimes \mathrm{id}} \mathcal{D}(Fy,Gy) \otimes \mathcal{D}(Fx,Fy) \xrightarrow{m} \mathcal{D}(Fx,Gy)$$

$$(\alpha_x)^* \colon \mathcal{D}(Gx,Gy) \cong \mathcal{D}(Gx,Gy) \otimes I \xrightarrow{\mathrm{id} \otimes \alpha_x} \mathcal{D}(Gx,Gy) \otimes \mathcal{D}(Fx,Gx) \xrightarrow{m} \mathcal{D}(Fx,Gy).$$

Remark. Given a monoidally closed based category $\mathcal{V}$ and $\mathcal{V}$-valued $\mathcal{V}$-functors $F, G \colon \mathcal{A} \to \mathcal{V}$ given by data (2.4) as in Example 2.1.1, a $\mathcal{V}$-natural transformation $\alpha \colon F \Rightarrow G \colon \mathcal{A} \to \mathcal{V}$ is equally given by a family of maps in $\mathcal{V}$

$$\alpha_A \colon FA \to GA \tag{2.9}$$

via $\mathcal{V}(Fx, Gx) \cong \mathcal{V}(I \otimes Fx, Gx) \cong \mathcal{V}(I, \underline{\mathcal{V}}(Fx, Gx))$, rendering the commutative diagram

$$\begin{array}{ccc} \mathcal{A}(A,B) \otimes FA & \xrightarrow{F_{A,B}} & FB \\ {\scriptstyle \mathrm{id} \otimes \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\ \mathcal{A}(A,B) \otimes GA & \xrightarrow[G_{A,B}]{} & GB \end{array} \tag{2.10}$$

using the naturality of the bijections $\mathcal{V}(A \otimes X, Y) \cong \mathcal{V}(A, \underline{\mathcal{V}}(X,Y))$ in all variables.

Definition 2.2.3 (Enriched adjunction)**.** Let $F \colon C \to \mathcal{D}$ and $G \colon \mathcal{D} \to C$ be a pair of $\mathcal{V}$-functors for a monoidally closed base category $\mathcal{V}$. We say that F is (left) $\mathcal{V}$*-adjoint* to G if we are provided with $\mathcal{V}$-natural isomorphisms $\mathcal{D}(Fx, y) \cong C(x, Gy)$ in $\mathcal{V}$

$$C \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad (\perp)$$

meaning that the following two $\mathcal{V}$-functors are $\mathcal{V}$-naturally isomorphic:

$$C^{\mathrm{op}} \otimes \mathcal{D} \xrightarrow{F^{\mathrm{op}} \otimes \mathrm{id}} \mathcal{D}^{\mathrm{op}} \otimes \mathcal{D} \xrightarrow{\mathrm{hom}} \mathcal{V}$$

$$C^{\mathrm{op}} \otimes \mathcal{D} \xrightarrow{\mathrm{id} \otimes G} C^{\mathrm{op}} \otimes C \xrightarrow{\mathrm{hom}} \mathcal{V}$$

where the first part is the tensor product $\mathcal{V}$-functor and the second is the enriched hom-functor. When $\mathcal{V}$ is also symmetric, this is equivalent to an adjunction in the 2-category $\mathcal{V}$-**Cat**.

$\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations form a 2-category $\mathcal{V}$-**Cat**. In parallel with the tensor product $\mathcal{V}$-categories, given a pair of $\mathcal{V}$-functors $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{D} \to \mathcal{D}'$ for a symmetric monoidal base category $\mathcal{V}$, there is an evident definition of the *tensor product $\mathcal{V}$-functor*

$$\mathcal{F} \otimes \mathcal{G}\colon \mathcal{C} \otimes \mathcal{D} \to \mathcal{C}' \otimes \mathcal{D}'.$$

Likewise for the definition of the *tensor product $\mathcal{V}$-natural transformation*

$$\alpha \otimes \beta\colon F \otimes G \Rightarrow F' \otimes G';$$

giving the 2-functor

$$\otimes\colon \mathcal{V}\text{-}\mathbf{Cat} \times \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$$

when the base monoidal category $\mathcal{V}$ is symmetric.

2.3 Underlying Categories and Change of Base for Enriched Categories

The *set of elements of an object $X \in \mathcal{V}$* is the set $\mathcal{V}(I, X) \in \mathbf{Set}$. The *underlying category* $\mathcal{C}_0$ of an enriched category $\mathcal{C}$ is obtained by taking sets of elements of hom-objects and considering the ordinary category with these hom-sets.

Definition 2.3.1 (Underlying category $\mathcal{C}_0$)**.** For a $\mathcal{V}$-category $\mathcal{C}$, its *underlying category*, which we write as $\mathcal{C}_0$, is an ordinary category with

- the same objects as $\mathcal{C}$;
- for each $X, Y \in \mathcal{C}_0$, the hom-sets $\mathcal{C}_0(X, Y) := \mathcal{V}(I, \mathcal{C}(X, Y)) \in \mathbf{Set}$;
- composition maps $\circ\colon \mathcal{C}_0(Y, Z) \times \mathcal{C}_0(X, Y) \to \mathcal{C}_0(X, Z)$ defined as

$$\mathcal{V}(I, \mathcal{C}(Y, Z)) \times \mathcal{V}(I, \mathcal{C}(X, Y)) \xrightarrow{\otimes} \mathcal{V}(I, \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y)) \xrightarrow{m \circ -} \mathcal{V}(I, \mathcal{C}(X, Z));$$

- for each $X \in \mathcal{C}_0$, the identity morphism $\mathrm{id}_X\colon X \to X$ defined as the unit map $i_X\colon I \to \mathcal{C}(X, X)$ of the $\mathcal{V}$-category $\mathcal{C}$.

Example 2.3.1 (Lemma 3.4.9 [Rie14])**.** For a closed monoidal category $\mathcal{V}$, consider $\underline{\mathcal{V}}$ the $\mathcal{V}$-category enriched over itself as shown in Example 2.1.1. Its underlying category is isomorphic to the original **Set**-category $\mathcal{V}$.

Definition 2.3.2 (Enrichment of a category)**.** We say a $\mathcal{V}$-category $\mathcal{C}$ is an *enrichment* of an ordinary category $\mathcal{A}$ if $\mathcal{C}_0 \cong \mathcal{A}$ via an identity-on-objects functor.

The underlying category is an example of *change of base* along the lax monoidal functor $\mathcal{V}(I, -)\colon \mathcal{V} \to \mathbf{Set}$. Introduced in references such as [EK66] and Chapter 4 of [Cru08], we present the following definitions of base changes along a given lax monoidal functor $F\colon (\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}}) \to (\mathcal{W}, \otimes_{\mathcal{W}}, I_{\mathcal{W}})$.

Definition 2.3.3 (Change of base – enriched categories)**.** Let there be a lax monoidal functor $F\colon (\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}}) \to (\mathcal{W}, \otimes_{\mathcal{W}}, I_{\mathcal{W}})$. We can induce, from any $\mathcal{V}$-category C, a $\mathcal{W}$-category F_*C as follows:

- objects of F_*C are the same as those of C;
- for each $X, Y \in F_*C$, the hom-object $F_*C(X,Y)$ is $FC(X,Y)$;
- composition maps $m_{\mathcal{W}}\colon F_*C(Y,Z) \otimes_{\mathcal{W}} F_*C(X,Y) \to F_*C(X,Z)$ are defined as

$$FC(Y,Z) \otimes_{\mathcal{W}} FC(X,Y) \xrightarrow{\phi} F\left(C(Y,Z) \otimes_{\mathcal{V}} C(X,Y)\right) \xrightarrow{F(m_{\mathcal{V}})} FC(X,Z)$$

  where ϕ is the lax monoidal constraint of F;
- the unit maps $i_X\colon I_{\mathcal{W}} \to F_*C(X,X)$ are $I_{\mathcal{W}} \xrightarrow{\psi} F(I_{\mathcal{V}}) \xrightarrow{F(i_X)} FC(X,X)$ where ψ is the lax monoidal constraint of F.

Definition 2.3.4 (Change of base – enriched functors)**.** Likewise, from a $\mathcal{V}$-functor $G\colon C \to \mathcal{D}$, the lax monoidal functor F induces a $\mathcal{W}$-functor $F_*G\colon F_*C \to F_*\mathcal{D}$ by the following:

- on objects, it maps the same as G;
- on hom-objects, $F_*G_{X,Y}\colon F_*C(X,Y) \to F_*\mathcal{D}(GX, GY)$ is given by $F(G_{X,Y})$.

Definition 2.3.5 (Change of base – enriched natural transformations)**.** And from a $\mathcal{V}$-natural transformation $\alpha\colon G \Rightarrow H\colon C \to \mathcal{D}$, it induces a $\mathcal{W}$-natural transformation $F_*\alpha\colon F_*G \Rightarrow F_*H\colon F_*C \to F_*\mathcal{D}$ by the components $F_*\alpha_X\colon I_{\mathcal{W}} \to F_*\mathcal{D}(F_*GX, F_*HX)$ defined as:

$$F_*\alpha_X\colon I_{\mathcal{W}} \xrightarrow{\psi} F(I_{\mathcal{V}}) \xrightarrow{F(\alpha_X)} F\mathcal{D}(GX, HX).$$

Example 2.3.2 (Underlying functor G_0 and underlying natural transformation α_0)**.** For a $\mathcal{V}$-functor $G\colon C \to \mathcal{D}$, its *underlying functor*, which we write as $G_0\colon C_0 \to \mathcal{D}_0$, is the ordinary functor through the change of base by $\mathcal{V}(I,-)\colon \mathcal{V} \to \mathbf{Set}$. On objects, G_0 acts the same as G. And G_0 assigns the morphisms by

$$\mathcal{V}(I, C(X,Y)) \xrightarrow{G_{X,Y} \circ -} \mathcal{V}(I, \mathcal{D}(X,Y)).$$

Likewise, the change of base of a $\mathcal{V}$-natural transformation $\alpha\colon G \Rightarrow H\colon C \to \mathcal{D}$ by $\mathcal{V}(I,-)\colon \mathcal{V} \to \mathbf{Set}$ is the *underlying natural transformation* of α, which we write as α_0.

Example 2.3.3. Given a locally small category C and a closed symmetric monoidal category $(\mathcal{A}, \otimes, I)$ with all small coproducts, the free category (Example 2.1.2) is the change of base along the strong monoidal functor $F\colon \mathbf{Set} \to \mathcal{A}$ given by:

- on objects, the following map $X \mapsto \coprod_{x \in X} I$;
- $F_{X,Y} : \mathbf{Set}(X, Y) \to \mathcal{A}(\coprod_X I, \coprod_Y I)$ assigning to a function $f : X \to Y$ the unique map in $\mathcal{A}$
$$F_{X,Y}(f) : \coprod_X I \to \coprod_Y I$$
that reindexes along $f : X \to Y$

where the strong monoidal structure comes from the fact that the monoidal product $\otimes$ of $\mathcal{A}$ naturally commutes with coproducts in each variable.

Theorem 2.3.1 (Theorem 4.2.4 of [Cru08])**.** *Altogether, a lax monoidal functor F induces a* 2*-functor*
$$F_* : \mathcal{V}\text{-}\boldsymbol{Cat} \to \mathcal{W}\text{-}\boldsymbol{Cat}. \tag{2.11}$$

2.4 Cocomplete Enriched Categories: Copowers and Conical Colimits

We now describe what it means for a $\mathcal{V}$-category C to be a *cocomplete $\mathcal{V}$-category* by introducing the kinds of colimits we can have in C: *copowers* and *conical colimits*.

Definition 2.4.1 (Copower)**.** Given a monoidally closed base category $\mathcal{V}$ together with C a $\mathcal{V}$-category, $A \in \mathcal{V}$ and $X \in C$, a *copower* of X by A is an object $A \odot X \in C$ together with isomorphisms $\mathcal{V}$-natural in Y:
$$C(A \odot X, Y) \cong \underline{\mathcal{V}}(A, C(X, Y)). \tag{2.12}$$

Note. By applying the underlying-set functor $\mathcal{V}(I, -) : \mathcal{V} \to \mathbf{Set}$ to (2.12) and using the fact that the underlying category of the $\mathcal{V}$-category $\underline{\mathcal{V}}$ is $\mathcal{V}$ (Example 2.3.1) we get the following natural bijection of sets:
$$C_0(A \odot X, Y) \cong \mathcal{V}(A, C(X, Y)) \tag{2.13}$$

Example 2.4.1. For a **Set**-category C, the copower of $X \in C$ by $A \in \mathbf{Set}$ characterised by
$$C(A \odot X, Y) \cong \mathbf{Set}(A, C(X, Y))$$
is the coproduct $A \odot X = \coprod_{i \in A} X$.

Example 2.4.2. For a closed monoidal category $\mathcal{V}$, consider $\underline{\mathcal{V}}$ the $\mathcal{V}$-category enriched over itself. The monoidal product $\otimes$ of $\mathcal{V}$ is the copower
$$\underline{\mathcal{V}}(A \otimes B, C) \cong \underline{\mathcal{V}}(A, \underline{\mathcal{V}}(B, C)).$$

Remark 2.4.1 ($\mathcal{V}$-functor and $\mathcal{V}$-natural transformation)**.** For a monoidally closed base category $\mathcal{V}$, any $\mathcal{V}$-functor $F : \mathcal{A} \to C$, when C is a $\mathcal{V}$-category with copowers, can be expressed differently with the transposes in the natural bijections (2.13) from the copowers, i.e. with the family of maps
$$\mathcal{A}(a, b) \odot Fa \to Fb \tag{2.14}$$

in C_0.

Similarly, suppose that $\mathcal{D}$ is a $\mathcal{V}$-category with copowers. Then a $\mathcal{V}$-enriched natural transformation $\alpha\colon F \Rightarrow G\colon C \to \mathcal{D}$ can be expressed differently using the natural bijections from the copower (2.13). i.e. (2.7) is equivalently to giving the components in $\mathcal{D}_0$

$$\alpha_x\colon Fx \to Gx \tag{2.15}$$

such that the following adjunct of (2.8)

$$\begin{array}{ccc} C(x,y) \odot Fx & \xrightarrow{F_{x,y}} & Fy \\ {\scriptstyle \mathrm{id}\odot\alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ C(x,y) \odot Gx & \xrightarrow[G_{x,y}]{} & Gy \end{array} \tag{2.16}$$

commutes in $\mathcal{D}_0$.

The expression of a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{V}$ with maps (2.4) and a $\mathcal{V}$-natural transformation $\alpha\colon F \Rightarrow G\colon C \to \mathcal{V}$ with maps (2.9) are examples.

Definition 2.4.2 (Power)**.** The dual notion of the copower is *power* or *cotensor*. Given a monoidally closed base category $\mathcal{V}$ together with C a $\mathcal{V}$-category, $A \in \mathcal{V}$ and $Y \in C$, a *power* of Y by A is an object $A \pitchfork Y \in C$ together with $\mathcal{V}$-natural isomorphisms

$$C(X, A \pitchfork Y) \cong \underline{\mathcal{V}}(A, C(X,Y)). \tag{2.17}$$

Definition 2.4.3 (Conical colimit)**.** Let C be a $\mathcal{V}$-category. Given an ordinary category $\mathcal{A}$ and a functor $D\colon \mathcal{A} \to C_0$, a *conical colimit* for D is an object $\operatorname{colim} D \in C$ together with isomorphisms in $\mathcal{V}$

$$C(\operatorname{colim} D, X) \cong \lim C(D-, X) \tag{2.18}$$

$\mathcal{V}$-natural in X.

Note that since the underlying-set functor $\mathcal{V}(I,-)\colon \mathcal{V} \to \mathbf{Set}$ is a right adjoint functor $\mathcal{V}(\coprod_X I, Y) \cong \mathbf{Set}(X, \mathcal{V}(I,Y))$, it preserves all limits. So by applying it to (2.18), we easily get $C_0(\operatorname{colim} D, X) \cong \lim C_0(D-, X)$. In fact, a conical colimit can also be described as an ordinary colimit for D (i.e. $C_0(\operatorname{colim} D, X) \cong \lim C_0(D-, X)$) which is sent to a limit in $\mathcal{V}$ by each functor $C(-,X)\colon C_0^{\mathrm{op}} \to \mathcal{V}$.

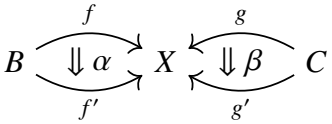
Example 2.4.3. When $\mathcal{V} = \mathbf{Cat}$, C is a 2-category, and $\mathcal{A}$ is a discrete category of 2 objects, a functor $D\colon \mathcal{A} \to C_0$ is just a pair of objects $B, C \in C$. The object $\operatorname{colim} D$ is an object $B + C \in C$ characterised by

$$C(B + C, X) \cong C(B, X) \times C(C, X)$$

in **Cat**. Spelling out: we have $B \xrightarrow{i} B + C \xleftarrow{j} C$ such that

1. for each $B \xrightarrow{f} X \xleftarrow{g} C$, there exists a unique 1-cell $\langle f, g\rangle\colon B + C \to X$ such that $\langle f, g\rangle \circ i = f$ and $\langle f, g\rangle \circ j = g$.

2. for each



there exists a unique 2-cell $\langle \alpha, \beta \rangle \colon \langle f, g \rangle \Rightarrow \langle f', g' \rangle \colon B + C \to X$ such that, by left whiskering with i and j, we have: $\langle \alpha, \beta \rangle \circ i = \alpha$ and $\langle \alpha, \beta \rangle \circ j = \beta$.

Definition 2.4.4 (Cocomplete $\mathcal{V}$-categories)**.** A $\mathcal{V}$-category is *cocomplete* if it has all copowers and small conical colimits.

3

Enrichments through Monoidal Actions

We've seen that a monoidal category $\mathcal{A}$ can be enriched to an $\mathcal{A}$-category $\underline{\mathcal{A}}$ when it is equipped with a closed monoidal structure

$$\mathcal{A}(X \otimes Y, Z) \cong \mathcal{A}(X, \underline{\mathcal{A}}(Y, Z)). \tag{3.1}$$

Moreover, the bijections in (3.1) can be enhanced to $\mathcal{A}$-isomorphisms of the internal-hom $\underline{\mathcal{A}}(-,-)$

$$\underline{\mathcal{A}}(X \otimes Y, Z) \cong \underline{\mathcal{A}}(X, \underline{\mathcal{A}}(Y, Z)). \tag{3.2}$$

In this chapter, we provide a general treatment of this example: *enrichment through monoidal actions*. We explain how a category $\mathcal{A}$ can be enriched (Definition 2.3.2) to a $\mathcal{V}$-category $\underline{\mathcal{A}}$ via an external structure called a *monoidal action of* $\mathcal{V}$ *on* $\mathcal{A}$ and show that categories enriched in this way are, up to isomorphism, the enriched categories with copowers ([JK01]). This connection is a special case of a more general connection, noted in ([GP97]), between closed W-representations and W-categories for a closed bicategory W. And we show that this connection extends to a categorical equivalence between the right-closed monoidal actions of $\mathcal{V}$ on $\mathcal{A}$ and the copowered $\mathcal{V}$-enrichments of $\mathcal{A}$.

We also show a stronger equivalence where another additional condition on the monoidal action gives rise to the enriched categories with copowers and conical colimits, the *cocomplete enriched categories*.

3.1 Monoidal Actions $\mathcal{V} \times \mathcal{A} \to \mathcal{A}$

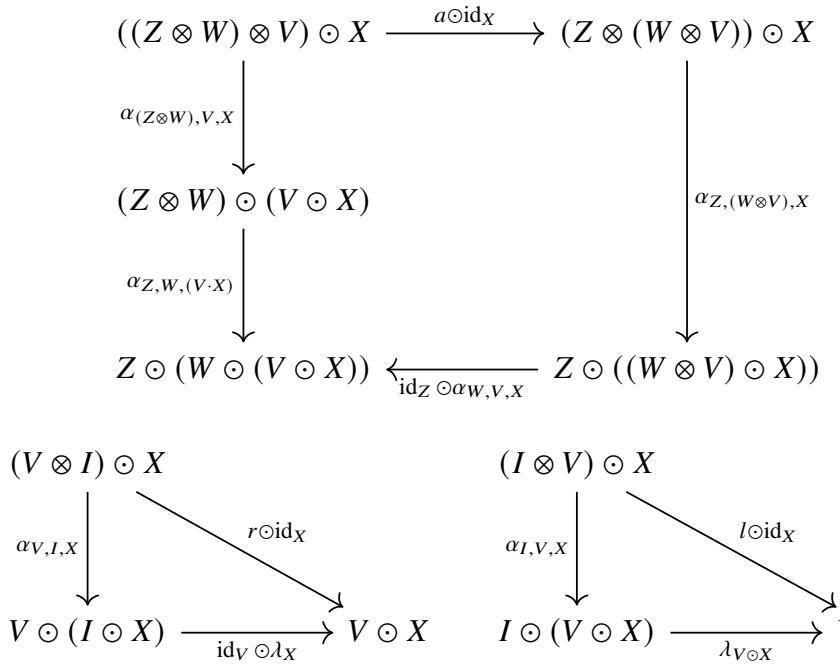
Definition 3.1.1 (Monoidal Action)**.** As introduced in [Bén67] and presented in [JK01], an *action of a monoidal category* $(\mathcal{V}, \otimes, I)$ *on a category* $\mathcal{A}$ is a functor

$$\odot \colon \mathcal{V} \times \mathcal{A} \to \mathcal{A} \tag{3.3}$$

together with two natural isomorphisms in $\mathcal{A}$

$$\begin{aligned} \alpha_{W,V,X} &\colon (W \otimes V) \odot X \cong W \odot (V \odot X) \\ \lambda_X &\colon I \odot X \cong X \end{aligned} \tag{3.4}$$

satisfying the coherence axioms given by commutativity of:



Remark 3.1.1. Under the cartesian closedness $\mathbf{Cat}(\mathcal{V} \times \mathcal{A}, \mathcal{A}) \cong \mathbf{Cat}(\mathcal{V}, \mathbf{Cat}(\mathcal{A}, \mathcal{A}))$ of $\mathbf{Cat}$, the monoidal action $\odot\colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ of $(\mathcal{V}, \otimes, I)$ on $\mathcal{A}$ is equivalent to giving a strong monoidal functor (F, ϕ, φ)

$$F\colon \mathcal{V} \to \mathbf{Cat}(\mathcal{A}, \mathcal{A})$$

where ϕ and φ are the monoidal constraints of F and the endofunctor category $\mathbf{Cat}(\mathcal{A}, \mathcal{A})$ is a strict monoidal category with the composition as its monoidal product.

It is easy to check that giving a natural isomorphism $\alpha_{W,V,X}\colon (W \otimes V) \odot X \cong W \odot (V \odot X)$ natural in $W, V \in \mathcal{V}$ and $X \in \mathcal{A}$ corresponds to giving a natural isomorphism $\phi_{W,V}\colon F_W \circ F_V \cong F_{W \otimes V}$ natural in $W, V \in \mathcal{V}$. And giving a natural isomorphism $\lambda_X\colon I \odot X \cong X$ natural in $X \in \mathcal{A}$ corresponds to giving an isomorphism $\varphi\colon \mathrm{id}_{\mathcal{A}} \cong F_I$ in $\mathbf{Cat}(\mathcal{A}, \mathcal{A})$. The coherence axiom for the monoidal action exactly corresponds to the coherence axiom for the monoidal functor.

Example 3.1.1. The monoidal product of a monoidal category is a monoidal action on itself.

Example 3.1.2. (Strict) monoidal action $\cdot\colon \Delta_+ \times C \to C$ of the augmented simplex category Δ_+ on a category C gives a monad $\langle T, \mu, \eta \rangle$ on C with $T^n X = \underline{n} \cdot x$ and $T^2 X \xrightarrow{\mu_X} TX$ is $(\underline{2} \to \underline{1}) \cdot X$ and $X \xrightarrow{\eta_X} TX$ is $(\underline{0} \to \underline{1}) \cdot X$. The augmented simplex category is the category of all finite ordinals and order-preserving maps; objects are denoted as $\underline{n} = \{1, ..., n\}$ with the empty ordinal $\underline{0} = \emptyset$.

Example 3.1.3. For a symmetric closed monoidal category $\mathcal{V}$, the category $\mathcal{V}\text{-}\mathbf{Cat}(C, \mathcal{D})$ of $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations is equipped with a monoidal action by $\mathcal{V}$

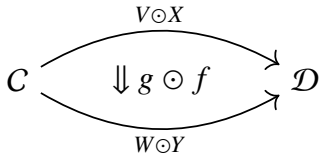
$$\odot\colon \mathcal{V} \times \mathcal{V}\text{-}\mathbf{Cat}(C, \mathcal{D}) \to \mathcal{V}\text{-}\mathbf{Cat}(C, \mathcal{D}) \tag{3.5}$$

whenever $\mathcal{D}$ is a copowered $\mathcal{V}$-category, with chosen copowers $V \odot D \in \mathcal{D}$. Given $\mathcal{V} \in \mathcal{V}$ and $X \in \mathcal{V}\text{-}\mathbf{Cat}(C, \mathcal{D})$, the $\mathcal{V}$-functor $V \odot X \in \mathcal{V}\text{-}\mathbf{Cat}(C, \mathcal{D})$ is defined by

- on objects: $(V \odot X)(A) = V \odot XA$
- on hom-objects (expressed via (2.14) as a map in $\mathcal{D}_0$ and using Lemma 3.3.1 and the monoidal symmetry of $\mathcal{V}$):

$$(V \odot X)_{A,B} \colon \mathcal{C}(A,B) \odot (V \odot XA) \xrightarrow{\cong} V \odot (\mathcal{C}(A,B) \odot XA) \xrightarrow{\mathrm{id} \odot X_{A,B}} V \odot XB$$

and, for all $g \colon V \to W$ in $\mathcal{V}$ and $f \colon X \to Y \in \mathcal{V}\text{-}\mathbf{Cat}(\mathcal{C}, \mathcal{D})$, a $\mathcal{V}$-natural transformation (expressed with (2.15) as a map in $\mathcal{D}_0$)



with components $(g \odot f)_A := g \odot f_A \colon V \odot XA \to W \odot YA$ in $\mathcal{D}_0$.

Definition 3.1.2 (Category of monoidal actions)**.** Given a monoidal category $(\mathcal{V}, \otimes, I)$ and a category $\mathcal{A}$, monoidal actions form a category $\mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})$ with

- objects: the monoidal actions

$$\odot \colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$$

- morphisms: natural transformations $\overline{\sigma} \colon \odot_1 \Rightarrow \odot_2$ such that

$$\begin{array}{ccccc} V \odot_1 (W \odot_1 X) & \xrightarrow{V \odot_1 \overline{\sigma}_{W,X}} & V \odot_1 (W \odot_2 X) & \xrightarrow{\overline{\sigma}_{V, W \odot_2 X}} & V \odot_2 (W \odot_2 X) \\ \cong \downarrow \alpha_1 & & & & \cong \downarrow \alpha_2 \\ (V \otimes W) \odot_1 X & & \xrightarrow[\overline{\sigma}_{V \otimes W, X}]{} & & (V \otimes W) \odot_2 X \end{array} \tag{3.6}$$

$$\begin{array}{ccc} X & & \\ {\scriptstyle \lambda_1} \downarrow \cong & \overset{\lambda_2}{\searrow} & \\ & \cong & \\ I_{\mathcal{V}} \odot_1 X & \xrightarrow[\overline{\sigma}_{I_{\mathcal{V}}, X}]{} & I_{\mathcal{V}} \odot_2 X \end{array} \tag{3.7}$$

3.2 Closed monoidal actions to copowered $\mathcal{V}$-enrichments

Definition 3.2.1 (Category of right-closed monoidal actions)**.** A monoidal action

$$\odot \colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$$

is *(right) closed* if, for each $C \in \mathcal{A}$, the functor $(- \odot C) \colon \mathcal{V} \to \mathcal{A}$ has a right adjoint.
We write $\mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\text{closed}}$ for the category whose objects are the right-closed monoidal actions $\odot \colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ together with a chosen right adjoint $\langle C, - \rangle \colon \mathcal{A} \to \mathcal{V}$ for each $(- \odot C) \colon \mathcal{V} \to \mathcal{A}$ and whose morphisms are as in $\mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})$.

Lemma 3.2.1 (Enrichment through monoidal action)**.** *Consider an object* $\odot\colon \mathcal{V}\times\mathcal{A}\to\mathcal{A}$ *of* $\mathcal{V}$-***Act***$(\mathcal{A})_{\text{closed}}$ *with the chosen right adjoint written* $\langle C,-\rangle\colon \mathcal{A}\to\mathcal{V}$ *for each* $C\in\mathcal{A}$*, yielding isomorphisms*

$$\mathcal{A}(V\odot C, D)\cong\mathcal{V}(V,\langle C,D\rangle) \tag{3.8}$$

for each $V\in\mathcal{V}$ *and* $C,D\in\mathcal{A}$*. Then* $\mathcal{A}$ *can be enriched to a* $\mathcal{V}$*-category* $\underline{\mathcal{A}}$ *(Definition 2.3.2) by taking*

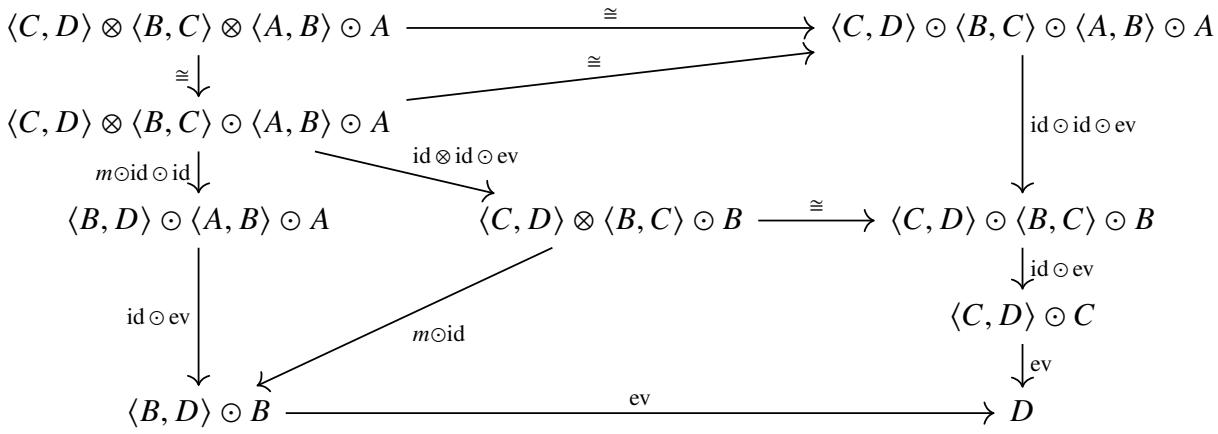
- $\underline{\mathcal{A}}(C,D):=\langle C,D\rangle\in\mathcal{V}$
- *compositions* $m\colon \langle D,E\rangle\otimes\langle C,D\rangle\to\langle C,E\rangle$ *given by the transpose, under (3.8), of*

$$(\langle D,E\rangle\otimes\langle C,D\rangle)\odot C\xrightarrow[\cong]{\alpha}\langle D,E\rangle\odot(\langle C,D\rangle\odot C)\xrightarrow{\mathrm{id}\odot\mathrm{ev}}\langle D,E\rangle\odot D\xrightarrow{\mathrm{ev}}E \tag{3.9}$$

  where $\mathrm{ev}\colon\langle C,D\rangle\odot C\to D$ *is the transpose of* $\mathrm{id}\colon\langle C,D\rangle\to\langle C,D\rangle$*.*

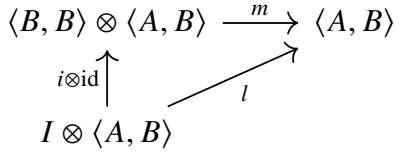
- *And the unit morphisms* $i_C\colon I\to\langle C,C\rangle$ *given by the transpose of* $\lambda_C\colon I\odot C\cong C$*.*

Proof. The associativity diagram of $m\colon\langle D,E\rangle\otimes\langle C,D\rangle\to\langle C,E\rangle$ seen as the transpose under (3.8) is the outer diagram of the below

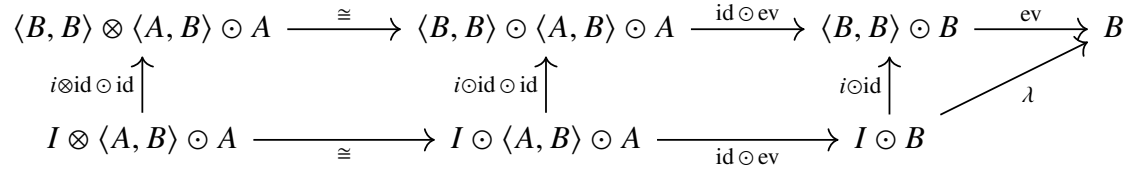


wherein the bottom-right commutes by the definition of $m\colon\langle C,D\rangle\otimes\langle B,C\rangle\to\langle B,D\rangle$ and the rest commutes by the coherence, functoriality and the naturality of the associativity α of the monoidal action $\odot$.

The left unitality condition



transposed under (3.8) is the outer diagram of the below



wherein the right triangle commutes by the definition of the unit-morphism $i\colon I \to \langle B, B\rangle$ and the rest commutes by the naturality of α and the functoriality of the monoidal action $\odot$.

The right unitality condition is seen likewise and the identity-on-object-isomorphism $\underline{\mathcal{A}}_0 \cong \mathcal{A}$ is given by $\mathcal{A}(C,D) \cong \mathcal{A}(I \odot C, D) \cong \mathcal{V}(I, \langle C, D\rangle)$ through (3.8) and the Yoneda lemma with $\lambda\colon I \odot C \cong C$. □

Lemma 3.2.2 (Lemma 2.1 of [JK01])**.** *In the situation of Lemma 3.2.1, suppose the base category $\mathcal{V}$ is also monoidally closed. The natural bijections $\mathcal{A}(V \odot C, D) \cong \mathcal{V}(V, \langle C, D\rangle)$ can be enhanced to $\mathcal{V}$-natural isomorphisms*

$$\underline{\mathcal{A}}(V \odot C, D) \cong \underline{\mathcal{V}}(V, \underline{\mathcal{A}}(C, D)) \tag{3.10}$$

exhibiting $\underline{\mathcal{A}}$ as a $\mathcal{V}$-category with all copowers.

Proof. This is easy to see from

$$\begin{aligned}\mathcal{V}(W, \underline{\mathcal{V}}(V, \langle C, D\rangle)) &\cong \mathcal{V}(W \otimes V, \langle C, D\rangle)\\ &\cong \mathcal{A}((W \otimes V) \odot C, D)\\ &\cong \mathcal{A}(W \odot (V \odot C), D)\\ &\cong \mathcal{V}(W, \langle V \odot C, D\rangle),\end{aligned}$$

giving $\langle V \odot C, D\rangle \cong \underline{\mathcal{V}}(V, \langle C, D\rangle)$ through the Yoneda lemma. We refer to [JK01] for more details. □

Example 3.2.1. The monoidal action in Example 3.1.3 is right-closed as long as C is small and $\mathcal{V}$ is complete, via isomorphisms:

$$\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{C}, \mathcal{D})(V \odot X, Y) \cong \mathcal{V}(V, [\mathcal{C}, \mathcal{D}](X, Y)). \tag{3.11}$$

Here, $[\mathcal{C}, \mathcal{D}](X, Y) \in \mathcal{V}$ is the hom-object of the *functor $\mathcal{V}$-category* $[\mathcal{C}, \mathcal{D}]$, defined in Section 2.2 of [Kel83] as a certain equaliser in $\mathcal{V}$. Indeed, giving a $\mathcal{V}$-natural transformation $f\colon V \odot X \to Y$, i.e. a family of maps $f_A\colon V \odot XA \to YA$ in $\mathcal{D}_0$ for all $A \in \mathcal{A}$ such that

$$\begin{array}{ccccc} \mathcal{C}(A,B) \odot (V \odot XA) & \xrightarrow{\cong} & V \odot (\mathcal{C}(A,B) \odot XA) & \xrightarrow{\mathrm{id} \odot X_{A,B}} & V \odot XB \\ {\scriptstyle \mathrm{id} \odot f_A}\downarrow & & & & \downarrow{\scriptstyle f_B} \\ \mathcal{C}(A,B) \odot YA & & \xrightarrow[Y_{A,B}]{} & & YB \end{array}$$

commutes is equivalently to give, under transposes of the form $\mathcal{D}_0(V \odot A, B) \cong \mathcal{V}(V, \mathcal{D}(A,B))$, a family of maps $\overline{f}_A\colon V \to \mathcal{D}(XA, YA)$ in $\mathcal{V}$ for all $A \in \mathcal{A}$, such that

$$\begin{array}{ccccc} V & \xrightarrow{\overline{f}_B} & \mathcal{D}(XB, YB) & \xrightarrow{\overline{m}} & \underline{\mathcal{V}}(\mathcal{D}(XA, XB), \mathcal{D}(XA, YB)) \\ {\scriptstyle \overline{f}_A}\downarrow & & & & \downarrow{\scriptstyle (X_{A,B})^*} \\ \mathcal{D}(XA, YA) & \xrightarrow[\overline{m_{\mathcal{D}^{\mathrm{op}}}}]{} & \underline{\mathcal{V}}(\mathcal{D}(YA, YB), \mathcal{D}(XA, YB)) & \xrightarrow[(Y_{A,B})^*]{} & \underline{\mathcal{V}}(\mathcal{C}(A,B), \mathcal{D}(XA, YB)) \end{array}$$

commutes, which is again equivalently to give a single map $f\colon V \to \prod_{A \in \mathcal{A}} \mathcal{D}(XA, YA)$ such that

$$V \xrightarrow{f} \prod_{A \in \mathcal{A}} \mathcal{D}(XA, YA) \underset{\varphi}{\overset{\phi}{\rightrightarrows}} \prod_{A,B \in \mathcal{A}} \underline{\mathcal{V}}\,(\mathcal{C}(A,B), \mathcal{D}(XA, YB))$$

commutes where ϕ and φ are induced respectively by

$$\mathcal{D}(XA,YA) \xrightarrow{\overline{m_{\mathcal{D}^{\mathrm{op}}}}} \underline{\mathcal{V}}(\mathcal{D}(YA,YB),\mathcal{D}(XA,YB)) \xrightarrow{(Y_{A,B})^*} \underline{\mathcal{V}}(C(A,B),\mathcal{D}(XA,YB))$$

$$\mathcal{D}(XB,YB) \xrightarrow{\overline{m}} \underline{\mathcal{V}}(\mathcal{D}(XA,XB),\mathcal{D}(XA,YB)) \xrightarrow{(X_{A,B})^*} \underline{\mathcal{V}}(C(A,B),\mathcal{D}(XA,YB)).$$

Hence to give a $\mathcal{V}$-natural transformation $f\colon V \odot X \to Y$ is equivalently to give a map $V \to [C,\mathcal{D}](X,Y)$ in $\mathcal{V}$ where $[C,\mathcal{D}](X,Y) \in \mathcal{V}$ is the equaliser $\int_{A\in\mathcal{A}} \mathcal{D}(XA,YA)$

$$[C,\mathcal{D}](X,Y) \longrightarrow \prod_{A\in\mathcal{A}} \mathcal{D}(XA,YA) \underset{\varphi}{\overset{\phi}{\rightrightarrows}} \prod_{A,B\in\mathcal{A}} \underline{\mathcal{V}}\,(C(A,B),\mathcal{D}(XA,YB))$$

which is exactly the definition of the hom-object $[C,\mathcal{D}](X,Y) \in \mathcal{V}$ given in [Kel83].

While the hom-object $[C,\mathcal{D}](X,Y) \in \mathcal{V}$ can be characterised to be the chosen right adjoint of (3.11), the composition-morphism and the unit-morphism can be given as transposes of the appropriate maps in $\mathcal{V}$-$\mathbf{Cat}(C,\mathcal{D})$, thus characterising the remaining data of the functor-$\mathcal{V}$-category $[C,\mathcal{D}]$ for a copowered $\mathcal{V}$-category $\mathcal{D}$, as the enrichment of $\mathcal{V}$-$\mathbf{Cat}(C,\mathcal{D})$ arising from the monoidal action (3.5).

Moreover, the natural bijection (3.11) can be enhanced to

$$[C,\mathcal{D}](V \odot X, Y) \cong \underline{\mathcal{V}}(V,[C,D](X,Y)) \tag{3.12}$$

exhibiting the functor-$\mathcal{V}$-category $[C,\mathcal{D}]$ as a $\mathcal{V}$-category with all copowers given by the monoidal action (3.5) whenever $\mathcal{D}$ is a copowered $\mathcal{V}$-category, generalising the fact that the category $\mathbf{Cat}(C,\mathcal{D})$ has all small coproducts when $\mathcal{D}$ does.

3.3 Connection to copowered and cocomplete $\mathcal{V}$-enrichments

We've seen in Lemma 3.2.1 and Lemma 3.2.2 that, for a monoidally closed base category $\mathcal{V}$, an object of $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\mathrm{closed}}$ gives a copowered $\mathcal{V}$-enrichment $\underline{\mathcal{A}}$ of $\mathcal{A}$. This has a well-known converse:

Lemma 3.3.1 (Section 6, page 88 of [JK01])**.** *Given a copowered $\mathcal{V}$-category $\mathcal{B}$ with a provided choice of copowers written $V \odot B$, applying the underlying-set functor $\mathcal{V}(I,-)\colon \mathcal{V} \to$ **Set** gives bijections $\mathcal{B}_0(V \odot X, Y) \cong \mathcal{V}(V,\mathcal{B}(X,Y))$ natural in Y, and the unique functor $\odot\colon \mathcal{V}\times\mathcal{B}_0 \to \mathcal{B}_0$ which makes these bijections natural in V,X equips $\mathcal{B}_0$ with a closed monoidal action of $\mathcal{V}$; giving an object of $\odot \in \mathcal{V}$-$\mathbf{Act}(\mathcal{B}_0)_{\mathrm{closed}}$.*

Proof. The hom-object $\mathcal{B}(X,Y) \in \mathcal{V}$ of $\mathcal{B}$ as a result of the unenriched hom functor $\mathcal{B}(-,-)\colon \mathcal{B}_0^{\mathrm{op}} \times \mathcal{B}_0 \to \mathcal{V}$ satisfies

$$\mathcal{B}_0(V \odot X, Y) \cong \mathcal{V}(V,\mathcal{B}(X,Y)) \tag{3.13}$$

through applying the underlying-set functor $\mathcal{V}(I,-)\colon \mathcal{V} \to \mathbf{Set}$ on $\mathcal{B}(V\odot X,Y) \cong \underline{\mathcal{V}}(V,\mathcal{B}(X,Y))$ as mentioned in (2.13). And the unique functor

$$\odot\colon \mathcal{V}\times\mathcal{B}_0 \to \mathcal{B}_0 \tag{3.14}$$

making (3.13) natural in all three variables by the Yoneda lemma is moreover a monoidal action on $\mathcal{B}_0$ through

$$\begin{aligned}\mathcal{B}(V \odot (W \odot X), Y) &\cong \underline{\mathcal{V}}(V, \mathcal{B}(W \odot X, Y)) \cong \underline{\mathcal{V}}(V, \underline{\mathcal{V}}(W, \mathcal{B}(X, Y))) \\ &\cong \underline{\mathcal{V}}(V \otimes W, \mathcal{B}(X, Y)) \cong \mathcal{B}((V \otimes W) \odot X, Y)\end{aligned}$$

and $\mathcal{B}(X, Y) \cong \underline{\mathcal{V}}(I, \mathcal{B}(X, Y)) \cong \mathcal{B}(I \odot X, Y)$ giving natural isomorphisms

$$\alpha \colon (V \otimes W) \odot X \to V \odot (W \odot X) \tag{3.15}$$

$$\lambda \colon I \odot X \to X \tag{3.16}$$

in $\mathcal{B}_0$ by applying $\mathcal{V}(I, -) \colon \mathcal{V} \to \mathbf{Set}$ to those equations and the Yoneda Lemma. The coherence is easy to check. Thus, giving an object of $\mathcal{V}$-$\mathbf{Act}(\mathcal{B}_0)_{\text{closed}}$. □

We now explain the sense in which copowered enrichments and right-closed monoidal actions are equivalent.

Definition 3.3.1. Given a category $\mathcal{A}$, let $\mathcal{V}$-$\mathbf{Cat}(\mathcal{A})$ be the category of its $\mathcal{V}$-enrichments with

- objects: $\mathcal{V}$-categories $\underline{\mathcal{A}}$ equipped with an identity-on-objects isomorphism of categories $\theta \colon \underline{\mathcal{A}}_0 \xrightarrow{\cong} \mathcal{A}$.
- arrows $(\underline{\mathcal{A}}^1, \theta_1) \to (\underline{\mathcal{A}}^2, \theta_2)$ are: $\mathcal{V}$-functors $\sigma \colon \underline{\mathcal{A}}^1 \to \underline{\mathcal{A}}^2$ such that

$$\begin{array}{ccc} \underline{\mathcal{A}}^1_0 & \xrightarrow{\sigma_0} & \underline{\mathcal{A}}^2_0 \\ & {\scriptstyle\theta_1}\searrow^{\cong} \quad \swarrow^{\cong}{\scriptstyle\theta_2} & \\ & \mathcal{A} & \end{array} \tag{3.17}$$

  commutes.

  Here $\sigma_0 \colon \underline{\mathcal{A}}^1_0 \to \underline{\mathcal{A}}^2_0$ is of course the underlying functor of σ as in Example 2.3.2 above.

Definition 3.3.2. For a closed monoidal base category $\mathcal{V}$, denote **copower**-$\mathcal{V}$-$\mathbf{Cat}(\mathcal{A})$ to be the category whose objects $(\underline{\mathcal{A}}, \odot, \theta)$ are the copowered $\mathcal{V}$-categories $\underline{\mathcal{A}}$, with a provided choice of copowers $V \odot X$, that are the $\mathcal{V}$-enrichments θ of $\mathcal{A}$; and whose morphisms are as in $\mathcal{V}$-$\mathbf{Cat}(\mathcal{A})$.

Definition 3.3.3. For a closed monoidal base category $\mathcal{V}$ and a cocomplete category $\mathcal{A}$, denote **cocomp**-$\mathcal{V}$-$\mathbf{Cat}(\mathcal{A})$ to be the category whose objects $(\underline{\mathcal{A}}, \odot, \theta)$ are the cocomplete $\mathcal{V}$-categories $\underline{\mathcal{A}}$, with a provided choice of copowers $V \odot X$, that are the $\mathcal{V}$-enrichments θ of $\mathcal{A}$; and whose morphisms are as in $\mathcal{V}$-$\mathbf{Cat}(\mathcal{A})$.

For a closed monoidal base category $\mathcal{V}$, we have shown in Lemma 3.2.1 and Lemma 3.2.2 that an object $\odot \in \mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\text{closed}}$ gives a copowered enrichment $(\underline{\mathcal{A}}, \odot, \eta)$ of $\mathcal{A}$ with the isomorphism $\eta \colon \underline{\mathcal{A}}_0 \cong \mathcal{A}$ that is identity on objects and is

$$\eta_{X,Y} \colon \mathcal{V}(I, \underline{\mathcal{A}}(X, Y)) \cong \mathcal{A}(I \odot X, Y) \cong \mathcal{A}(X, Y) \tag{3.18}$$

on morphisms. We now show that this assignment on objects gives a categorical equivalence:

$$\mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\text{closed}} \to \mathbf{copower}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})^{\text{op}}. \tag{3.19}$$

We make use of the *Weak $\mathcal{V}$-Yoneda lemma*.

Lemma 3.3.2 (Weak $\mathcal{V}$-Yoneda Lemma, 1.9 [Kel83])**.** *For a closed monoidal base category $\mathcal{V}$, let C be a $\mathcal{V}$-category. For any $\mathcal{V}$-functor $X \in \mathcal{V}\text{-}\boldsymbol{Cat}(C, \mathcal{V})$ and any object $A \in C$, there is a bijection*

$$[C, \mathcal{V}]_0 \left(C(A, -), X\right) \cong \mathcal{V}(I, XA).$$

Proof. A $\mathcal{V}$-natural transformation $\alpha \colon C(A, -) \Rightarrow X$ gives

$$\eta \colon I \xrightarrow{i} C(A, A) \xrightarrow{\alpha_A} XA. \tag{3.20}$$

Inversely, a map $\eta \colon I \to XA$ gives a $\mathcal{V}$-natural transformation $\alpha \colon C(A, -) \Rightarrow X$ with components

$$\alpha_B \colon C(A, B) \cong C(A, B) \otimes I \xrightarrow{\mathrm{id} \otimes \eta} C(A, B) \otimes XA \xrightarrow{X_{A,B}} XB. \tag{3.21}$$

□

Proposition 3.3.3. *For a closed monoidal base category $\mathcal{V}$, the connections (Lemma 3.2.1, Lemma 3.2.2, Lemma 3.3.1) between the closed monoidal actions by $\mathcal{V}$ and the copowered $\mathcal{V}$-enrichments yields an equivalence of categories*

$$\mathcal{V}\text{-}\boldsymbol{Act}(\mathcal{A})_{\text{closed}} \simeq \boldsymbol{copower}\text{-}\mathcal{V}\text{-}\boldsymbol{Cat}(\mathcal{A})^{\text{op}}. \tag{3.22}$$

Proof. Given an object $(\underline{\mathcal{A}}, \odot, \theta) \in \textbf{copower-}\mathcal{V}\textbf{-Cat}(\mathcal{A})$, its underlying category $\underline{\mathcal{A}}_0$ with its copower operation $\odot$ gives an object of $\mathcal{V}\textbf{-Act}(\underline{\mathcal{A}}_0)_{\text{closed}}$ (Lemma 3.3.1) and we get the below functor

$$\begin{array}{ccc} \mathcal{V} \times \underline{\mathcal{A}}_0 & \xrightarrow{\quad\odot\quad} & \underline{\mathcal{A}}_0 \\ {\scriptstyle \mathcal{V}\times\theta^{-1}}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \theta} \\ \mathcal{V} \times \mathcal{A} & & \mathcal{A}. \end{array} \tag{3.23}$$

This functor $\theta \circ \odot \circ (\mathrm{id}_{\mathcal{V}} \times \theta^{-1}) \colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ is a monoidal action where the unitality and associativity are seen from the below isomorphisms

$$\mathcal{A}(X, Y) \overset{\theta^{-1}}{\cong} \underline{\mathcal{A}}_0(X, Y) := \mathcal{V}(I, \underline{\mathcal{A}}(X, Y)) \cong \underline{\mathcal{A}}_0(I \odot X, Y) \overset{\theta}{\cong} \mathcal{A}(I \odot X, Y) \tag{3.24}$$

and

$$\begin{aligned} \mathcal{A}(V \odot (W \odot X), Y) &\overset{\theta^{-1}}{\cong} \underline{\mathcal{A}}_0(V \odot (W \odot X), Y) \cong \mathcal{V}(V, \underline{A}(W \odot X, Y)) \\ &\cong \mathcal{V}(V, \underline{\mathcal{V}}(W, \underline{\mathcal{A}}(X, Y))) \cong \mathcal{V}(V \otimes W, \underline{\mathcal{A}}(X, Y)) \\ &\cong \underline{\mathcal{A}}_0((V \otimes W) \odot X, Y) \overset{\theta}{\cong} \mathcal{A}((V \otimes W) \odot X, Y) \end{aligned} \tag{3.25}$$

giving the natural isomorphisms $I \odot X \cong X$ and $(V \otimes W) \odot X \cong V \odot (W \odot X)$ in $\mathcal{A}$.

And it is a right-closed monoidal action through

$$\mathcal{A}(V \odot A, B) \overset{\theta^{-1}}{\cong} \underline{\mathcal{A}}_0(V \odot A, B) \cong \mathcal{V}(V, \underline{\mathcal{A}}(A, B)) \tag{3.26}$$

giving us an object of $\mathcal{V}\textbf{-Act}(\mathcal{A})_{\text{closed}}$ which is sent by the assignment $\mathcal{V}\textbf{-Act}(\mathcal{A})_{\text{closed}} \to \textbf{copower-}\mathcal{V}\textbf{-Cat}(\mathcal{A})^{\text{op}}$ to the copowered enrichment $(\underline{\mathcal{A}}, \odot, \eta)$ back to the given $\underline{\mathcal{A}}$ and $\odot$ by applying what's been shown in the last paragraph of Section 6 of [JK01]. It remains to show

that η from (3.18) agrees with θ. The bijections $\eta_{X,Y}\colon \underline{\mathcal{A}_0}(X,Y) \to \mathcal{A}(X,Y)$ are obtained through composing the map $\mathcal{V}(I, \underline{\mathcal{A}}(X,Y)) \to \mathcal{A}(I \odot X, Y)$ from $(3.26)^{-1}$ with the map $\mathcal{A}(I \odot X, Y) \to \mathcal{A}(X,Y)$ from $(3.24)^{-1}$ which is easily seen to be the given $\theta_{X,Y}$; thus $\eta = \theta$, showing the surjectivity of (3.19).

Now, given a morphism $\overline{\sigma}\colon \odot_2 \Rightarrow \odot_1$ in $\mathcal{V}$-**Act**$(\mathcal{A})_{\text{closed}}$, which is a natural transformation such that the diagrams

$$\begin{array}{ccccc} V \odot_2 (W \odot_2 X) & \xrightarrow{V \odot_2 \overline{\sigma}_{W,X}} & V \odot_2 (W \odot_1 X) & \xrightarrow{\overline{\sigma}_{V, W \odot_1 X}} & V \odot_1 (W \odot_1 X) \\ \cong \downarrow a_2^{-1} & & & & a_1^{-1} \downarrow \cong \\ (V \otimes W) \odot_2 X & & \xrightarrow[\overline{\sigma}_{V \otimes W, X}]{} & & (V \otimes W) \odot_1 X \end{array} \tag{3.27}$$

$$\begin{array}{ccc} X & & \\ \cong \downarrow & \searrow^{\cong} & \\ I_{\mathcal{V}} \odot_2 X & \xrightarrow[\overline{\sigma}_{I_{\mathcal{V}}, X}]{} & I_{\mathcal{V}} \odot_1 X \end{array} \tag{3.28}$$

commute, there is a unique collection (Yoneda lemma)

$$\sigma_{A,B}\colon \underline{\mathcal{A}^1}(A,B) \to \underline{\mathcal{A}^2}(A,B) \tag{3.29}$$

for all $A, B \in \underline{\mathcal{A}}$ such that

$$\begin{array}{ccc} \mathcal{A}(V \odot_1 A, B) & \xrightarrow{\cong} & \mathcal{V}(V, \underline{\mathcal{A}^1}(A,B)) \\ {\scriptstyle -\circ \overline{\sigma}_{V,A}} \downarrow & & \downarrow {\scriptstyle \sigma_{A,B} \circ -} \\ \mathcal{A}(V \odot_2 A, B) & \xrightarrow[\cong]{} & \mathcal{V}(V, \underline{\mathcal{A}^2}(A,B)) \end{array} \tag{3.30}$$

commutes. The map $\sigma_{A,B}$ is of course the transpose of $\underline{\mathcal{A}^1}(A,B) \odot_2 A \xrightarrow{\overline{\sigma}} \underline{\mathcal{A}^1}(A,B) \odot_1 A \xrightarrow{\epsilon_B} B$

$$\begin{array}{ccccc} \underline{\mathcal{A}^1}(A,B) \odot_1 A \xrightarrow{\epsilon_B} B & \longleftarrow\!\shortmid & & \shortmid\!\longrightarrow & \underline{\mathcal{A}^1}(A,B) \xrightarrow{\mathrm{id}} \underline{\mathcal{A}^1}(A,B) \\ \downarrow & & & & \downarrow \\ \underline{\mathcal{A}^1}(A,B) \odot_2 A \xrightarrow{\overline{\sigma}} \underline{\mathcal{A}^1}(A,B) \odot_1 A \xrightarrow{\epsilon_B} B & & \longmapsto & & \underline{\mathcal{A}^1}(A,B) \xrightarrow{\sigma_{A,B}} \underline{\mathcal{A}^2}(A,B) \end{array} \tag{3.31}$$

seen from this chase in the above diagram (3.30). And these $\sigma_{A,B}$'s, together with identity on objects, will give a map $\sigma\colon (\underline{\mathcal{A}^1}, \eta_1) \to (\underline{\mathcal{A}^2}, \eta_2)$ in **copower**-$\mathcal{V}$-**Cat**$(\mathcal{A})$. The $\mathcal{V}$-functoriality

$$\begin{array}{ccc} \underline{\mathcal{A}^1}(B,C) \otimes \underline{\mathcal{A}^1}(A,B) & \xrightarrow{m} & \underline{\mathcal{A}^1}(A,C) \\ {\scriptstyle \sigma \otimes \sigma} \downarrow & & \downarrow {\scriptstyle \sigma} \\ \underline{\mathcal{A}^2}(B,C) \otimes \underline{\mathcal{A}^2}(A,B) & \xrightarrow[m]{} & \underline{\mathcal{A}^2}(A,C) \end{array} \tag{3.32}$$

transposed under the form $\mathcal{A}(V \odot_2 A, B) \cong \mathcal{V}(V, \underline{\mathcal{A}}^2(A,B))$ is

$$\begin{array}{ccccc}
(\underline{\mathcal{A}}^1(B,C) \otimes \underline{\mathcal{A}}^1(A,B)) \odot_2 A & \xrightarrow{m\odot_2 A} & \underline{\mathcal{A}}^1(A,C) \odot_2 A & \xrightarrow{\overline{\sigma}} & \underline{\mathcal{A}}^1(A,C) \odot_1 A \\
\downarrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \epsilon_C} \\
\underline{\mathcal{A}}^1(B,C) \odot_2 (\underline{\mathcal{A}}^1(A,B) \odot_2 A) & & & & C \\
\downarrow{\scriptstyle \mathrm{id}\odot_2 \overline{\sigma}} & & & \nearrow{\scriptstyle \epsilon_C} & \\
\underline{\mathcal{A}}^1(B,C) \odot_2 (\underline{\mathcal{A}}^1(A,B) \odot_1 A) & & & & \\
\downarrow{\scriptstyle \mathrm{id}\odot_2 \epsilon_B} & & & & \\
\underline{\mathcal{A}}^1(B,C) \odot_2 B & \xrightarrow[\overline{\sigma}]{} & \underline{\mathcal{A}}^1(B,C) \odot_1 B & &
\end{array} \tag{3.33}$$

which commutes since, in the below diagram,

$$\begin{array}{ccccc}
(\underline{\mathcal{A}}^1(B,C) \otimes \underline{\mathcal{A}}^1(A,B)) \odot_2 A & \xrightarrow{m\odot_2 A} & \underline{\mathcal{A}}^1(A,C) \odot_2 A & \xrightarrow{\overline{\sigma}} & \underline{\mathcal{A}}^1(A,C) \odot_1 A \\
\downarrow{\scriptstyle \cong} \quad \searrow{\scriptstyle \overline{\sigma}} & & & \nearrow{\scriptstyle m\odot_1 A} & \downarrow{\scriptstyle \epsilon_C} \\
\underline{\mathcal{A}}^1(B,C) \odot_2 (\underline{\mathcal{A}}^1(A,B) \odot_2 A) & & (\underline{\mathcal{A}}^1(B,C) \otimes \underline{\mathcal{A}}^1(A,B)) \odot_1 A & & C \\
\downarrow{\scriptstyle \mathrm{id}\odot_2 \overline{\sigma}} & & \downarrow{\scriptstyle \cong} & & \\
\underline{\mathcal{A}}^1(B,C) \odot_2 (\underline{\mathcal{A}}^1(A,B) \odot_1 A) & \xrightarrow{\overline{\sigma}} & \underline{\mathcal{A}}^1(B,C) \odot_1 (\underline{\mathcal{A}}^1(A,B) \odot_1 A) & & \\
\downarrow{\scriptstyle \mathrm{id}\odot_2 \epsilon_B} & & \downarrow{\scriptstyle \mathrm{id}\odot_1 \epsilon_B} & \nearrow{\scriptstyle \epsilon_C} & \\
\underline{\mathcal{A}}^1(B,C) \odot_2 B & \xrightarrow[\overline{\sigma}]{} & \underline{\mathcal{A}}^1(B,C) \odot_1 B & &
\end{array} \tag{3.34}$$

the left inner pentagon commutes by (3.27), the right inner pentagon commutes by the definition of $m : \underline{\mathcal{A}}^1(B,C) \otimes \underline{\mathcal{A}}^1(A,B) \to \underline{\mathcal{A}}^1(A,C)$ and the two remaining inner diagrams commute by the naturality of $\overline{\sigma}$.

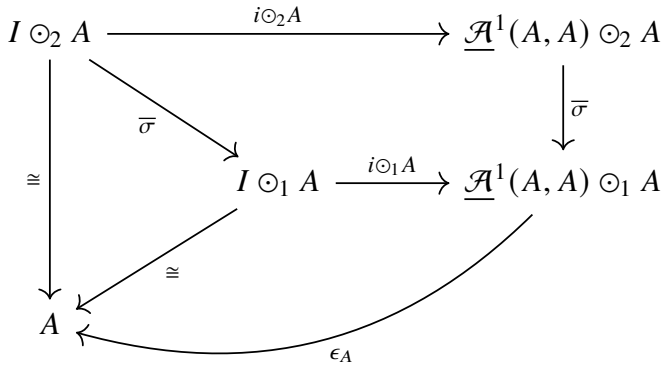
The unit axiom

$$\begin{array}{ccc}
I & \xrightarrow{i} & \underline{\mathcal{A}}^1(A,A) \\
& \searrow{\scriptstyle i} & \downarrow{\scriptstyle \sigma} \\
& & \underline{\mathcal{A}}^2(A,A)
\end{array} \tag{3.35}$$

transposed under the form $\mathcal{A}(V \odot_2 A, B) \cong \mathcal{V}(V, \underline{\mathcal{A}}^2(A,B))$ is

$$\begin{array}{ccc}
I \odot_2 A & \xrightarrow{i\odot_2 A} & \underline{\mathcal{A}}^1(A,A) \odot_2 A \\
& & \downarrow{\scriptstyle \overline{\sigma}} \\
& & \underline{\mathcal{A}}^1(A,A) \odot_1 A \\
& \overset{\cong}{\searrow} & \downarrow{\scriptstyle \epsilon_A} \\
& & A
\end{array} \tag{3.36}$$

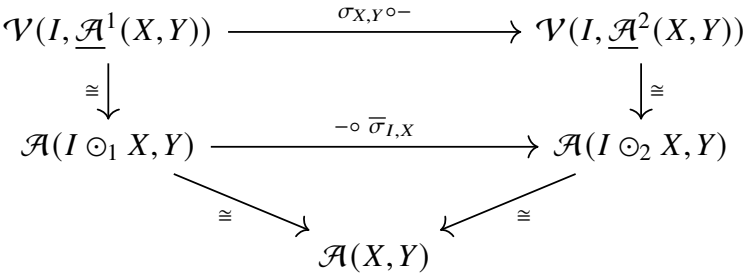
which commutes since, in the below diagram,



(3.37)

the left inner triangle commutes by (3.28) and the other two inner diagrams commute by the naturality of $\overline{\sigma}$ and by the definition of $i\colon I \to \underline{\mathcal{A}}^1(A,A)$. Hence we indeed have a $\mathcal{V}$-functor $\sigma\colon \underline{\mathcal{A}}^1 \to \underline{\mathcal{A}}^2$, that is identity on objects.

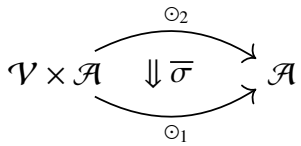
The condition (3.17) requires σ to be identity on objects (which we already have) and the outer diagram below



(3.38)

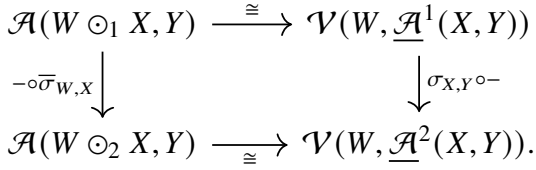
to commute. It commutes if and only if the lower inner triangle commutes which is precisely the given condition (3.28). Hence we have, from a morphism $\overline{\sigma}\colon \odot_2 \Rightarrow \odot_1$ in $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\text{closed}}$, a morphism $\sigma\colon (\underline{\mathcal{A}}^1, \eta_1) \to (\underline{\mathcal{A}}^2, \eta_2)$ in **copower**-$\mathcal{V}$-$\mathbf{Cat}(\mathcal{A})$.

This assignment has a bijective inverse, that is, given a map $\sigma\colon (\underline{\mathcal{A}}^1, \eta_1) \to (\underline{\mathcal{A}}^2, \eta_2)$ in **copower**-$\mathcal{V}$-$\mathbf{Cat}(\mathcal{A})$, we have a unique natural transformation



(3.39)

such that for all $W \in \mathcal{V}$ and $X \in \mathcal{A}$, the diagram (3.30)



commutes through (3.26). The map $\overline{\sigma}_{W,X}\colon W \odot_2 X \to W \odot_1 X$ is of course the transpose of

$$W \xrightarrow{\eta^1_W} \underline{\mathcal{A}}^1(X, W \odot_1 X) \xrightarrow{\sigma_{X, W \odot_1 X}} \underline{\mathcal{A}}^2(X, W \odot_1 X) \tag{3.40}$$

as seen from the below chase

$$\begin{array}{ccccc} W \odot_1 X & \xrightarrow{\mathrm{id}} & W \odot_1 X & \longmapsto & W \xrightarrow{\eta^1_W} \underline{\mathcal{A}}^1(X, W \odot_1 X) \\ \downarrow & & & & \downarrow \\ W \odot_2 X & \xrightarrow{\overline{\sigma}_{W,X}} & W \odot_1 X & \longmapsto & W \xrightarrow{\eta^1_W} \underline{\mathcal{A}}^1(X, W \odot_1 X) \xrightarrow{\sigma} \underline{\mathcal{A}}^2(X, W \odot_1 X) \end{array} \tag{3.41}$$

in the diagram above. And $\overline{\sigma}$ will satisfy (3.27) and (3.28) and hence be a map in $\mathcal{V}$-**Act**$(\mathcal{A})_{\text{closed}}$.

Since $\sigma\colon (\underline{\mathcal{A}}^1, \eta_1) \to (\underline{\mathcal{A}}^2, \eta_2)$ is a map in **copower**-$\mathcal{V}$-**Cat**$(\mathcal{A})$, the outer diagram (3.38) commutes and we've already mentioned that this means the condition (3.28) holds.

The condition (3.27) holds if and only if

$$\begin{array}{ccc} \mathcal{A}((V \otimes W) \odot_1 X, Y) & \xrightarrow{\mathcal{A}(\overline{\sigma}, Y)} & \mathcal{A}((V \otimes W) \odot_2 X, Y) \\ {\scriptstyle \mathcal{A}(a_1^{-1}, Y)}\downarrow & & \downarrow{\scriptstyle \mathcal{A}(a_2^{-1}, Y)} \\ \mathcal{A}(V \odot_1 (W \odot_1 X), Y) & & \mathcal{A}(V \odot_2 (W \odot_2 X), Y) \\ & {\scriptstyle \mathcal{A}(\overline{\sigma}, Y)}\searrow \quad \nearrow{\scriptstyle \mathcal{A}(V \odot_2 \overline{\sigma}, Y)} & \\ & \mathcal{A}(V \odot_2 (W \odot_1 X), Y) & \end{array}$$

commutes. And this commutes if and only if the outer diagram shown below

$$\begin{array}{ccc} \mathcal{V}(V \otimes W, \underline{A}^1(X,Y)) & \xrightarrow{\mathcal{V}(V \otimes W, \sigma)} & \mathcal{V}(V \otimes W, \underline{\mathcal{A}}^2(X,Y)) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathcal{V}(V, \underline{\mathcal{V}}(W, \underline{\mathcal{A}}^1(X,Y))) & \xrightarrow{\mathcal{V}(V, \underline{\mathcal{V}}(W, \sigma))} & \mathcal{V}(V, \underline{\mathcal{V}}(W, \underline{\mathcal{A}}^2(X,Y))) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathcal{V}(V, \underline{\mathcal{A}}^1(W \odot_1 X, Y)) & & \mathcal{V}(V, \underline{\mathcal{A}}^2(W \odot_2 X, Y)) \\ & {\scriptstyle \mathcal{V}(V, \sigma)}\searrow \quad \nearrow{\scriptstyle \mathcal{V}(V, \underline{\mathcal{A}}^2(\overline{\sigma}, Y))} & \\ & \mathcal{V}(V, \underline{\mathcal{A}}^2(W \odot_1 X, Y)) & \end{array} \tag{3.42}$$

commutes. And since its upper inner rectangle commutes, the outer diagram commutes if and only if the lower inner diagram commutes which is again precisely saying that

$$\begin{array}{ccc} \underline{\mathcal{V}}(W, \underline{\mathcal{A}}^1(X,Y)) & \xrightarrow{\underline{\mathcal{V}}(W, \sigma)} & \underline{\mathcal{V}}(W, \underline{\mathcal{A}}^2(X,Y)) \\ {\scriptstyle \alpha^1_Y}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle \alpha^2_Y} \\ \underline{\mathcal{A}}^1(W \odot_1 X, Y) & & \underline{\mathcal{A}}^2(W \odot_2 X, Y) \\ & {\scriptstyle \sigma}\searrow \quad \nearrow{\scriptstyle \underline{\mathcal{A}}^2(\overline{\sigma}, Y)} & \\ & \underline{\mathcal{A}}^2(W \odot_1 X, Y) & \end{array} \tag{3.43}$$

commutes in $\mathcal{V}$.

Note (from the Weak $\mathcal{V}$-Yoneda, Lemma 3.3.2) that the $\mathcal{V}$-natural isomorphism $\alpha^1 \colon \underline{\mathcal{A}}^1(W \odot_1 X, -) \xRightarrow{\cong} \underline{\mathcal{V}}(W, \underline{\mathcal{A}}^1(X, -))$ is determined by

$$\overline{\eta}^1 \colon I \xrightarrow{i} \underline{\mathcal{A}}^1(W \odot_1 X, W \odot_1 X) \xrightarrow{\alpha^1_{W \odot_1 X}} \underline{\mathcal{V}}(W, \underline{\mathcal{A}}^1(X, W \odot_1 X)) \tag{3.44}$$

with components

$$\alpha^1_Y \colon \underline{\mathcal{A}}^1(W \odot_1 X, Y) \xrightarrow{\mathrm{id} \otimes \overline{\eta}^1} \underline{\mathcal{A}}^1(W \odot_1 X, Y) \otimes \underline{\mathcal{V}}(W, \underline{\mathcal{A}}^1(X, W \odot_1 X)) \xrightarrow{\mathrm{ev}} \underline{\mathcal{V}}(W, \underline{\mathcal{A}}^1(X, Y)). \tag{3.45}$$

Here $\overline{\eta}^1$ (3.44) transposed under

$$\mathcal{V}(W, \underline{\mathcal{A}}^1(X, W \odot_1 X)) \cong \mathcal{V}(I, \underline{\mathcal{V}}(W, \underline{\mathcal{A}}^1(X, W \odot_1 X)))$$

is the $\eta^1_W \colon W \to \underline{\mathcal{A}}^1(X, W \odot_1 X)$ from (3.41). And note that α^1_Y (3.45) transposed under the form $\mathcal{V}(A \otimes B, C) \cong \mathcal{V}(A, \underline{\mathcal{V}}(B, C))$ is

$$\underline{\mathcal{A}}^1(W \odot_1 X, Y) \otimes W \xrightarrow{\mathrm{id} \otimes \eta^1_W} \underline{\mathcal{A}}^1(W \odot_1 X, Y) \otimes \underline{\mathcal{A}}^1(X, W \odot_1 X) \xrightarrow{\circ} \underline{\mathcal{A}}^1(X, Y).$$

Using this, the diagram (3.43) transposed under the form $\mathcal{V}(A \otimes B, C) \cong \mathcal{V}(A, \underline{\mathcal{V}}(B, C))$ gives

$$\begin{array}{ccccc}
\underline{\mathcal{A}}^1(W \odot_1 X, Y) \otimes W & \xrightarrow{\mathrm{id} \otimes \eta^1_W} & \underline{\mathcal{A}}^1(W \odot_1 X, Y) \otimes \underline{\mathcal{A}}^1(X, W \odot_1 X) & \xrightarrow{\circ} & \underline{\mathcal{A}}^1(X, Y) \\
\downarrow{\scriptstyle \sigma \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \sigma_{X,Y}} \\
\underline{\mathcal{A}}^2(W \odot_1 X, Y) \otimes W & & & & \underline{\mathcal{A}}^2(X, Y) \\
\downarrow{\scriptstyle \underline{\mathcal{A}}^2(\overline{\sigma}, Y) \otimes \mathrm{id}} & & & & \uparrow{\scriptstyle \circ} \\
\underline{\mathcal{A}}^2(W \odot_2 X, Y) \otimes W & \xrightarrow[\mathrm{id} \otimes \eta^2_W]{} & \underline{\mathcal{A}}^2(W \odot_2 X, Y) \otimes \underline{\mathcal{A}}^2(X, W \odot_2 X) & &
\end{array}$$

which commutes since, in the below diagram,

$$\begin{array}{ccccc}
\underline{\mathcal{A}}^1(W \odot_1 X, Y) \otimes W & \xrightarrow{\mathrm{id} \otimes \eta^1_W} & \underline{\mathcal{A}}^1(W \odot_1 X, Y) \otimes \underline{\mathcal{A}}^1(X, W \odot_1 X) & \xrightarrow{\circ} & \underline{\mathcal{A}}^1(X, Y) \\
\downarrow{\scriptstyle \sigma \otimes \mathrm{id}} & & \downarrow{\scriptstyle \sigma \otimes \sigma} & & \downarrow{\scriptstyle \sigma_{X,Y}} \\
\underline{\mathcal{A}}^2(W \odot_1 X, Y) \otimes W & & \underline{\mathcal{A}}^2(W \odot_1 X, Y) \otimes \underline{\mathcal{A}}^2(X, W \odot_1 X) & \xrightarrow{\circ} & \underline{\mathcal{A}}^2(X, Y) \\
 & & \uparrow{\scriptstyle \mathrm{id} \otimes \underline{\mathcal{A}}^2(X, \overline{\sigma})} & & \uparrow{\scriptstyle \circ} \\
\underline{\mathcal{A}}^2(W \odot_1 X, Y) \otimes W & \xrightarrow{\mathrm{id} \otimes \eta^2_W} & \underline{\mathcal{A}}^2(W \odot_1 X, Y) \otimes \underline{\mathcal{A}}^2(X, W \odot_2 X) & & \\
\downarrow{\scriptstyle \underline{\mathcal{A}}^2(\overline{\sigma}, Y) \otimes \mathrm{id}} & & \downarrow{\scriptstyle \underline{\mathcal{A}}^2(\overline{\sigma}, Y) \otimes \mathrm{id}} & & \\
\underline{\mathcal{A}}^2(W \odot_2 X, Y) \otimes W & \xrightarrow[\mathrm{id} \otimes \eta^2_W]{} & \underline{\mathcal{A}}^2(W \odot_2 X, Y) \otimes \underline{\mathcal{A}}^2(X, W \odot_2 X) & &
\end{array}$$

the lower left inner rectangle commutes from the functoriality of the monoidal product $\otimes$ and the two inner diagrams on the right commute from the $\mathcal{V}$-functoriality of σ and the associativity of composition in the $\mathcal{V}$-category $\underline{\mathcal{A}}^2$. And lastly, the upper left inner diagram commutes from the definition of $\overline{\sigma}_{W,X}$ being the transpose of (3.40)

$$W \xrightarrow{\eta^1_W} \underline{\mathcal{A}}^1(X, W \odot_1 X) \xrightarrow{\sigma_{X,W\odot_1 X}} \underline{\mathcal{A}}^2(X, W \odot_1 X)$$

where $\eta^2_W \colon W \to \underline{\mathcal{A}}^2(X, W\odot_2 X)$ exhibits $W\odot_2 X$ as free on W with respect to $\underline{\mathcal{A}}^2(X,-)\colon \mathcal{A} \to \mathcal{V}$

$$\begin{array}{ccc} W & \xrightarrow{\eta^1_W} & \underline{\mathcal{A}}^1(X, W \odot_1 X) \\ {\scriptstyle \eta^2_W}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \underline{\mathcal{A}}^2(X, W \odot_2 X) & \xrightarrow[\underline{\mathcal{A}}^2(X,\overline{\sigma}_{W,X})]{} & \underline{\mathcal{A}}^2(X, W \odot_1 X). \end{array}$$

Hence the condition (3.27) holds and $\overline{\sigma}$ (3.39) is a map in $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\text{closed}}$. □

In addition, we show the stronger case that, for a cocomplete category $\mathcal{A}$, to give an enrichment of $\mathcal{A}$ to a cocomplete $\mathcal{V}$-category $\underline{\mathcal{A}}$ is equivalent to giving a right-closed monoidal action $\mathcal{V} \times \mathcal{A} \to \mathcal{A}$ cocontinuous in the $\mathcal{A}$-variable.

Lemma 3.3.4. *For a closed monoidal base category $\mathcal{V}$ and a cocomplete category $\mathcal{A}$, consider an object $\odot \in \mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\text{closed}}$ and let $\underline{\mathcal{A}}$ be the corresponding copowered $\mathcal{V}$-enrichment of $\mathcal{A}$ under (3.22). The $\mathcal{V}$-category $\underline{\mathcal{A}}$ has all small conical colimits if and only if the functors $V \odot -\colon \mathcal{A} \to \mathcal{A}$ for all $V \in \mathcal{V}$ are cocontinuous.*

Proof. First, we show the 'if' part. Given a functor $D\colon \mathcal{I} \to \underline{\mathcal{A}}_0$ from a small category $\mathcal{I}$, since $\underline{\mathcal{A}}_0 \cong \mathcal{A}$, the category $\underline{\mathcal{A}}_0$ is cocomplete and so we can take the ordinary colimit $\operatorname{colim} D \in \underline{\mathcal{A}}_0$. We see from

$$\begin{aligned} \mathcal{V}(V, \underline{\mathcal{A}}(\operatorname{colim} D, X)) &\cong \mathcal{A}(V \odot \operatorname{colim} D, X) && \text{(closedness of monoidal action)} \\ &\cong \mathcal{A}(\operatorname{colim}(V \odot (D-)), X) && (V \odot - \text{ is cocontinuous}) \\ &\cong \lim \mathcal{A}(V \odot (D-), X) \\ &\cong \lim \mathcal{V}(V, \underline{\mathcal{A}}(D-, X)) && \text{(closedness of monoidal action)} \\ &\cong \mathcal{V}(V, \lim \underline{\mathcal{A}}(D-, X)). && (\mathcal{V} \text{ assumed to be complete}) \end{aligned}$$

and the Yoneda lemma, that

$$\underline{\mathcal{A}}(\operatorname{colim} D, X) \cong \lim \underline{\mathcal{A}}(D-, X)$$

and the $\mathcal{V}$-naturality in X expressed with (2.10) is the below diagram in $\mathcal{V}$

$$\begin{array}{ccc} \underline{\mathcal{A}}(X,Y) \otimes \underline{\mathcal{A}}(\operatorname{colim} D, X) & \xrightarrow{m} & \underline{\mathcal{A}}(\operatorname{colim} D, Y) \\ {\scriptstyle \mathrm{id}\otimes\cong}\downarrow & & \downarrow{\scriptstyle \cong} \\ \underline{\mathcal{A}}(X,Y) \otimes \lim \underline{\mathcal{A}}(D-, X) & \xrightarrow[\lim \underline{\mathcal{A}}(D-,-)_{X,Y}]{} & \lim \underline{\mathcal{A}}(D-, Y) \end{array}$$

which commutes since the below diagram commutes

$$\begin{array}{ccc} \underline{\mathcal{A}}(X,Y) \otimes \underline{\mathcal{A}}(\operatorname{colim} D, X) & \xrightarrow{m} & \underline{\mathcal{A}}(\operatorname{colim} D, Y) \\ \downarrow{\scriptstyle \mathrm{id} \otimes \underline{\mathcal{A}}(\iota_i, X)} & & \downarrow{\scriptstyle \underline{\mathcal{A}}(\iota_i, Y)} \\ \underline{\mathcal{A}}(X,Y) \otimes \underline{\mathcal{A}}(Di, X) & \xrightarrow[m]{} & \underline{\mathcal{A}}(Di, Y) \end{array}$$

where $\iota_i \colon Di \to \operatorname{colim} D$ is the coprojection map. And thus, $\underline{\mathcal{A}}$ has all small conical colimits.

The 'only if' part is seen simply by rearrangement of the equation

$$\begin{aligned} \mathcal{A}(V \odot \operatorname{colim} D, X) &\cong \mathcal{V}(V, \underline{\mathcal{A}}(\operatorname{colim} D, X)) && \text{(closedness of monoidal action)} \\ &\cong \mathcal{V}(V, \lim \underline{\mathcal{A}}(D-, X)) && (\underline{\mathcal{A}} \text{ has all small conical colimits}) \\ &\cong \lim \mathcal{V}(V, \underline{\mathcal{A}}(D-, X)) && \text{(right adjoint preserves limits)} \\ &\cong \lim \mathcal{A}(V \odot (D-), X) && \text{(closedness of monoidal action)} \\ &\cong \mathcal{A}(\operatorname{colim}(V \odot (D-)), X). && (\mathcal{A} \text{ is cocomplete}) \end{aligned}$$

showing the cocontinuity of the functors $V \odot - \colon \mathcal{A} \to \mathcal{A}$ by the Yoneda lemma. □

Definition 3.3.4. We write $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}}$ for the full subcategory of $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\text{closed}}$ with those that are also cocontinuous in $\mathcal{A}$-variable. i.e. monoidal actions $\odot \colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ such that

- each $(-) \odot C \colon \mathcal{V} \to \mathcal{A}$ is equipped with a right adjoint $\langle C, - \rangle \colon \mathcal{A} \to \mathcal{V}$
- each $V \odot (-) \colon \mathcal{A} \to \mathcal{A}$ is cocontinuous.

Proposition 3.3.5. *For a closed monoidal base category $\mathcal{V}$ and a cocomplete category $\mathcal{A}$, the assignment from $\mathcal{V}$-**Act**$(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}}$ to **cocomp**-$\mathcal{V}$-**Cat**$(\mathcal{A})^{\text{op}}$ explained by Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.3.4 yields an equivalence of categories*

$$\mathcal{V}\text{-}\boldsymbol{Act}(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}} \simeq \boldsymbol{cocomp}\text{-}\mathcal{V}\text{-}\boldsymbol{Cat}(\mathcal{A})^{\text{op}} \tag{3.46}$$

Proof. The surjectivity is trivial from Proposition 3.3.3 and Lemma 3.3.4. Fully-faithfulness remains exactly the same as shown in Proposition 3.3.3. □

4

Universal Cocomplete Enrichment for Locally Presentable Categories

In this chapter, we work with locally presentable categories and cocontinuous functors. A locally presentable category C is one which is "algebraic" enough to ensure that various categorical constructions in the category C work as smoothly as they do in the category of sets. In particular, any functor defined on C which you think should have an adjoint does in fact have an adjoint:

Theorem 4.0.1 (Adjoint Functor Theorem for locally presentable categories)**.** *A functor between locally presentable categories has a right adjoint if and only if it is cocontinuous.*

Among the locally presentable categories are the algebraic varieties, i.e., the category of models of a Lawvere theory as discussed in Chapter 5 below. This means that, for example, each of the following categories is locally presentable: groups, rings, vector spaces over a field, Boolean algebras, and so on. However, locally presentable categories are more general than this in two ways. On the one hand, we can look at possibly infinitary algebraic theories, such as the theory of ω-complete partial orders, i.e., partially ordered sets with joins of all ω-chains. On the other hand, we can look not just at algebraic theories (Lawvere theories) but also at "essentially-algebraic theories". For example, there is an essentially algebraic theory whose models are small categories. Other examples include sets equipped with a binary relation; sets equipped with a partial binary operation; presheaves on any small category; sheaves on a topological space; and so on.

In general, we have:

Definition 4.0.1. Let κ be an infinite regular cardinal. A category C is said to be *locally κ-presentable* if there exists a small category $\mathcal{T}$ with κ-small limits such that $C \simeq \kappa\text{-Lim}(\mathcal{T}, \mathbf{Set})$, the category of κ-small-limit preserving functors $\mathcal{T} \to \mathbf{Set}$. A category C is *locally presentable* if it is locally κ-presentable for some regular cardinal κ.

In this chapter, we specialise our previous results to the locally presentable case. For a closed monoidal base category $\mathcal{V}$ and a cocomplete category $\mathcal{A}$, we have shown the connection between objects of $\mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}}$, i.e. monoidal actions $\odot\colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ such that

(i) each $(-)\odot C\colon \mathcal{V}\to\mathcal{A}$ is equipped with a right adjoint $\langle C,-\rangle\colon \mathcal{A}\to\mathcal{V}$

(ii) each $V\odot(-)\colon \mathcal{A}\to\mathcal{A}$ is cocontinuous

and cocomplete $\mathcal{V}$-enrichments of $\mathcal{A}$. In the context of locally presentable categories $\mathcal{V}$ and $\mathcal{A}$, it suffices by Adjoint Functor Theorem for $\odot$ to be cocontinuous in each variable, so we will simply write $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\text{cocts}}$ to mean $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}}$.

An important result for us is that the category of cocontinuous functors between given locally presentable categories is also locally presentable. This appears in [Bir84]:

Theorem 4.0.2 (Chapter 3, Corollary 3.18 [Bir84])**.** *If $\mathcal{E}$ and $\mathcal{F}$ are locally presentable, then* $\mathrm{Cocts}(\mathcal{E},\mathcal{F})$ *is also locally presentable.*

We show that a locally presentable category $\mathcal{A}$ has a cocomplete enrichment over $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ and that this enrichment is universal among cocomplete enrichments $\underline{\mathcal{A}}_{\mathcal{V}}$ of $\mathcal{A}$ in locally presentable right-closed monoidal base categories $\mathcal{V}$'s.

4.1 Enrichment of locally presentable category $\mathcal{A}$ over $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$

We start by proving that any locally presentable category $\mathcal{A}$ has a cocomplete enrichment over the locally presentable closed monoidal category $\mathcal{V}=\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ of cocontinuous functors $\mathcal{A}\to\mathcal{A}$ and natural transformations, where the monoidal structure of $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ is given by the composition of cocontinuous functors.

Proposition 4.1.1. *Any locally presentable category $\mathcal{A}$ has an enrichment, which we write as $\underline{\mathcal{A}}$, over the locally presentable monoidal category* $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ *of cocontinuous functors $\mathcal{A}\to\mathcal{A}$, by Lemma 3.2.1, through the monoidal action*

$$*\colon\ \mathrm{Cocts}(\mathcal{A},\mathcal{A})\times\mathcal{A}\to\mathcal{A} \tag{4.1}$$

given by the evaluation $(F,X)\mapsto F(X)$.

Proof. By Lemma 3.2.1, we want to show that, for each fixed $X\in\mathcal{A}$, the functor

$$\begin{aligned}(-)*X=\mathrm{ev}_X\colon\ \mathrm{Cocts}(\mathcal{A},\mathcal{A})&\to\mathcal{A}\\ F&\mapsto F(X)\end{aligned} \tag{4.2}$$

has a right adjoint $\underline{\mathcal{A}}(X,-)\colon\mathcal{A}\to\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ providing a natural isomorphism

$$\mathcal{A}(F(X),Y)\cong\mathrm{Cocts}(\mathcal{A},\mathcal{A})(F,\underline{\mathcal{A}}(X,Y)). \tag{4.3}$$

By the *Adjoint Functor Theorem for locally presentable categories* (Theorem 4.0.1), a functor between locally presentable categories has a right adjoint if and only if it is cocontinuous. So it suffices to prove that the functor (4.2) is cocontinuous. This is true because $\mathcal{A}$, a locally presentable category, is cocomplete and colimits in $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ are computed pointwise.

Indeed, given a functor $D\colon\mathcal{I}\to\mathrm{Cocts}(\mathcal{C},\mathcal{D})$, the pointwise colimit $(\mathrm{colim}\,D)\colon\mathcal{C}\to\mathcal{D}$ is a cocontinuous functor as given a diagram $F\colon\mathcal{J}\to\mathcal{C}$ for small $\mathcal{J}$, we have:

$$\begin{aligned}(\mathrm{colim}\,D)(\mathrm{colim}_j(F_j)) &:= \mathrm{colim}_i\left(D_i\left(\mathrm{colim}_j(F_j)\right)\right)\\ &\cong \mathrm{colim}_i\left(\mathrm{colim}_j\left(D_i(F_j)\right)\right) && (D_i \text{ is cocontinuous})\\ &\cong \mathrm{colim}_j\left(\mathrm{colim}_i\left(D_i(F_j)\right)\right) && (\text{colimits commute with colimits})\\ &= \mathrm{colim}_j\left((\mathrm{colim}\,D)(F_j)\right).\end{aligned} \tag{4.4}$$

Consequently, by the definition of colim D, the evaluation functors $\mathrm{ev}_X\colon \mathrm{Cocts}(C,\mathcal{D}) \to \mathcal{D}$ for all $X \in C$ are cocontinuous and this is in particular true for (4.2). And the identity-on-objects isomorphism of categories $\theta\colon \underline{\mathcal{A}}_0 \cong \mathcal{A}$ comes from

$$\begin{aligned}\theta_{X,Y}\colon \underline{\mathcal{A}}_0(X,Y) &:= \mathrm{Cocts}(\mathcal{A},\mathcal{A})(\mathrm{id}_{\mathcal{A}}, \underline{\mathcal{A}}(X,Y)) \\ &\cong \mathcal{A}(\mathrm{id}_{\mathcal{A}}(X), Y) \\ &= \mathcal{A}(X,Y).\end{aligned}$$

□

Proposition 4.1.2. *The enrichment $\underline{\mathcal{A}}$ in Proposition 4.1.1 is a cocomplete enrichment of $\mathcal{A}$ over the locally presentable closed monoidal category* $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$.

Proof. It is easy to see that $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ is a closed monoidal category; the precomposition functor $(-\circ G)\colon \mathrm{Cocts}(\mathcal{A},\mathcal{A}) \to \mathrm{Cocts}(\mathcal{A},\mathcal{A})$ is cocontinuous since colimits in $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$ are computed pointwise and thus has a right adjoint by the Adjoint Functor Theorem for locally presentable categories (Theorem 4.0.1). Note that it is also biclosed: $(G\circ -)\colon \mathrm{Cocts}(\mathcal{A},\mathcal{A}) \to \mathrm{Cocts}(\mathcal{A},\mathcal{A})$ is cocontinuous likewise, since G is so.

Finally, the above enrichment of $\mathcal{A}$ via (4.1)

$$*\colon \mathrm{Cocts}(\mathcal{A},\mathcal{A}) \times \mathcal{A} \to \mathcal{A}$$

is cocontinuous in the $\mathcal{A}$-variable, giving $\mathcal{A}$ a cocomplete $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$-enrichment $\underline{\mathcal{A}}$ by Proposition 3.3.5. □

4.2 $\underline{\mathcal{A}}$ as the universal cocomplete enrichment of $\mathcal{A}$

Given a locally presentable category $\mathcal{A}$ and a locally presentable closed monoidal category $\mathcal{V}$, suppose that $\mathcal{A}$ has a cocomplete $\mathcal{V}$-enrichment $\underline{\mathcal{A}}_{\mathcal{V}}$ via a monoidal action

$$\odot\colon \mathcal{V} \times \mathcal{A} \to \mathcal{A} \tag{4.5}$$

in $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$, i.e. such that

(i) each $(-)\odot C\colon \mathcal{V} \to \mathcal{A}$ is equipped with a right adjoint $\underline{\mathcal{A}}_{\mathcal{V}}(C,-)\colon \mathcal{A} \to \mathcal{V}$;

(ii) each $V \odot (-)\colon \mathcal{A} \to \mathcal{A}$ is cocontinuous.

Under $\mathbf{Cat}(\mathcal{V}\times\mathcal{A},\mathcal{A}) \cong \mathbf{Cat}(\mathcal{V},\mathbf{Cat}(\mathcal{A},\mathcal{A}))$, the monoidal action (4.5) satisfying condition (ii) corresponds precisely to a strong monoidal functor (Remark 3.1.1)

$$F\colon \mathcal{V} \to \mathrm{Cocts}(\mathcal{A},\mathcal{A}). \tag{4.6}$$

This is cocontinuous by condition (i) and therefore has a right adjoint G

$$G\colon \mathrm{Cocts}(\mathcal{A},\mathcal{A}) \to \mathcal{V} \tag{4.7}$$

by the Adjoint Functor Theorem for locally presentable categories. The right adjoint G is lax monoidal which corresponds to F, the strong monoidal functor, seen as an oplax functor

[1]. Moreover, by an instance of doctrinal adjunction ([Kel74]), a left adjoint functor which is strong monoidal, like F, precisely means that it is a monoidal adjunction, i.e. an adjunction in the 2-category **MonCat** of monoidal categories, lax monoidal functors and monoidal natural transformations:

$$\mathcal{V} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Cocts}(\mathcal{A},\mathcal{A}). \tag{4.8}$$

Considering the lax monoidal functor $G\colon \mathrm{Cocts}(\mathcal{A},\mathcal{A}) \to \mathcal{V}$, we can apply change of base (Definition 2.3.3) to see the following result:

Proposition 4.2.1. *Let $G_*\underline{\mathcal{A}}$ be the $\mathcal{V}$-category acquired from the* $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$*-enriched category $\underline{\mathcal{A}}$ of Proposition 4.1.1 through change of base along the lax monoidal functor $G\colon \mathrm{Cocts}(\mathcal{A},\mathcal{A}) \to \mathcal{V}$ of (4.8). This is $\mathcal{V}$-isomorphic to the $\mathcal{V}$-category $\underline{\mathcal{A}}_{\mathcal{V}}$ of (4.5):*

$$\underline{\mathcal{A}}_{\mathcal{V}} \cong G_*\underline{\mathcal{A}}.$$

Proof. We have a chain of natural isomorphisms:

$$\begin{aligned}
\mathcal{V}(V, \underline{\mathcal{A}}_{\mathcal{V}}(A,B)) &\cong \mathcal{A}(V \odot A, B) && \text{(from (4.5) condition (i))}\\
&= \mathcal{A}\big(F(V)(A),\ B\big) && \text{(from (4.6))}\\
&\cong \mathrm{Cocts}(\mathcal{A},\mathcal{A})\ \big(F(V),\ \underline{\mathcal{A}}(A,B)\big) && \text{(from (4.3))}\\
&\cong \mathcal{V}\ \big(V,\ G\underline{\mathcal{A}}(A,B)\big) && \text{(from (4.8))}
\end{aligned} \tag{4.9}$$

and so, by the Yoneda lemma, a family of isomorphisms $H_{A,B}\colon \underline{\mathcal{A}}_{\mathcal{V}}(A,B) \to G\underline{\mathcal{A}}(A,B)$ in $\mathcal{V}$, where:

$$\begin{array}{ccccccc}
\mathcal{V}(\underline{\mathcal{A}}_{\mathcal{V}}(A,B),\ \underline{\mathcal{A}}_{\mathcal{V}}(A,B)) & \cong & \mathcal{A}(\underline{\mathcal{A}}_{\mathcal{V}}(A,B) \odot A,\ B) & \cong & \mathcal{V}(\underline{\mathcal{A}}_{\mathcal{V}}(A,B),\ G\underline{\mathcal{A}}(A,B))\\
\mathrm{id}_{\underline{\mathcal{A}}_{\mathcal{V}}(A,B)} & \mapsto & \mathrm{ev}_{A,B} & \mapsto & H_{A,B}.
\end{array} \tag{4.10}$$

So we can define the data of a $\mathcal{V}$-functor $H\colon \underline{\mathcal{A}}_{\mathcal{V}} \to G_*\underline{\mathcal{A}}$ which is identity on objects, and given by $H'_{A,B}s$ on hom-objects. It remains to check $\mathcal{V}$-functoriality of H.

Note that the lax monoidal unit constraint $I_{\mathcal{V}} \to G(\mathrm{id}_{\mathcal{A}})$ of G is given by the transpose of the oplax monoidal constraint $\lambda\colon F(I_{\mathcal{V}}) \xrightarrow{\cong} \mathrm{id}_{\mathcal{A}}$ of F provided by the monoidal action (4.5), that is:

$$I_{\mathcal{V}} \xrightarrow{\eta} G(F(I_{\mathcal{V}})) \xrightarrow{G(\lambda)} G(\mathrm{id}_{\mathcal{A}}).$$

And hence the unit morphism $i\colon I_{\mathcal{V}} \to G\underline{\mathcal{A}}(A,A)$ for $G_*\underline{\mathcal{A}}$ is given by

$$I_{\mathcal{V}} \xrightarrow{\eta} G(F(I_{\mathcal{V}})) \xrightarrow{G(\lambda)} G(\mathrm{id}_{\mathcal{A}}) \xrightarrow{G(i_{\underline{\mathcal{A}}})} G\underline{\mathcal{A}}(A,A)$$

and transposing this map through the isomorphisms (4.9) gives $\lambda_A\colon I_{\mathcal{V}} \odot A \xrightarrow{\cong} A$ in $\mathcal{A}$ as:

$$\begin{array}{ccccccc}
\mathcal{V}(I_{\mathcal{V}},\ G\underline{\mathcal{A}}(A,A)) & \xrightarrow{\cong} & \mathrm{Cocts}(\mathcal{A},\mathcal{A})(F(I_{\mathcal{V}}),\ \underline{\mathcal{A}}(A,A)) & \xrightarrow{\cong} & \mathcal{A}(I_{\mathcal{V}} \odot A, A).\\
G(i_{\underline{\mathcal{A}}} \circ \lambda) \circ \eta & \mapsto & i_{\underline{\mathcal{A}}} \circ \lambda & \mapsto & \lambda_A
\end{array}$$

[1]Given an adjunction $\langle F \dashv G, \eta, \epsilon\rangle\colon \mathcal{M} \to \mathcal{N}$ between monoidal categories $\mathcal{M}$ and $\mathcal{N}$, the left adjoint functor $F\colon \mathcal{M} \to \mathcal{N}$ is an oplax monoidal functor if and only if $G\colon \mathcal{N} \to \mathcal{M}$ is lax monoidal. If F is oplax monoidal by $F(A \otimes B) \xrightarrow{\phi_{A,B}} F(A) \otimes F(B)$ and $F(I) \xrightarrow{\psi} I$, then G is lax monoidal by the adjunct transpose of $F(GA \otimes GB) \xrightarrow{\phi} FG(A) \otimes FG(B) \xrightarrow{\epsilon_A \otimes \epsilon_B} A \otimes B$ and $F(I) \xrightarrow{\psi} I$.

That is, by $G(A) \otimes G(B) \xrightarrow{\eta} GF(GA \otimes GB) \xrightarrow{G(\phi)} G(FG(A) \otimes FG(B)) \xrightarrow{G(\epsilon_A \otimes \epsilon_B)} G(A \otimes B)$ and $I \xrightarrow{\eta} G(FI) \xrightarrow{G(\psi)} G(I)$.

Hence, expressing in the form (2.14) in $\mathcal{A}$ with

$$\mathrm{ev}_{A,B} \colon \underline{\mathcal{A}}_{\mathcal{V}}(A,B) \odot A \to B$$

in $\mathcal{A}$, the $\mathcal{V}$-functoriality of H is expressed by the below diagrams

$$\begin{array}{ccc}
\Big(\underline{\mathcal{A}}_{\mathcal{V}}(B,C) \otimes \underline{\mathcal{A}}_{\mathcal{V}}(A,B)\Big) \odot A & \xrightarrow{m \odot \mathrm{id}_A} & \underline{\mathcal{A}}_{\mathcal{V}}(A,C) \odot A \\
\downarrow \cong & & \\
\underline{\mathcal{A}}_{\mathcal{V}}(B,C) \odot \Big(\underline{\mathcal{A}}_{\mathcal{V}}(A,B) \odot A\Big) & & \Big\downarrow \mathrm{ev}_{A,C} \\
\Big\downarrow \mathrm{id} \odot \mathrm{ev}_{A,B} & & \\
\underline{\mathcal{A}}_{\mathcal{V}}(B,C) \odot B & \xrightarrow[\mathrm{ev}_{B,C}]{} & C
\end{array}$$

$$\begin{array}{ccc}
I \odot A & \xrightarrow{i \odot \mathrm{id}_A} & \underline{\mathcal{A}}_{\mathcal{V}}(A,A) \odot A \\
& \overset{\lambda_A}{\underset{\cong}{\searrow}} & \Big\downarrow \mathrm{ev}_{A,A} \\
& & A.
\end{array}$$

By transposing the above diagrams in the appropriate bijections of the form $\mathcal{A}(V \odot A,\ B) \xrightarrow{\cong} \mathcal{V}(V,\ \underline{\mathcal{A}}_{\mathcal{V}}(A,B))$, it is easy to see that they commute by the definition of the composition morphisms m (3.9) and the unit morphisms i for $\underline{\mathcal{A}}_{\mathcal{V}}$. Hence H is a $\mathcal{V}$-functor.

This gives the required $\mathcal{V}$-isomorphism $H \colon \underline{\mathcal{A}}_{\mathcal{V}} \cong G_* \underline{\mathcal{A}}$. □

We now show that this $\mathcal{V}$-isomorphism $H \colon \underline{\mathcal{A}}_{\mathcal{V}} \xrightarrow{\cong} G_* \underline{\mathcal{A}}$ is an isomorphism in the category **cocomp**-$\mathcal{V}$-**Cat**$(\mathcal{A})$. We first define *change of base for cocomplete enrichments of* $\mathcal{A}$ along a monoidal adjunction between locally presentable closed monoidal categories.

Definition 4.2.1 (Change of base for cocomplete enrichments in the context of locally presentable categories)**.** Given a locally presentable category $\mathcal{A}$ and a monoidal adjunction $\langle F \dashv G, \eta, \epsilon \rangle \colon \mathcal{V} \to \mathcal{W}$ between locally presentable closed monoidal categories $\mathcal{W}, \mathcal{V}$, the *change of base functor for cocomplete enrichments of* $\mathcal{A}$

$$G_* \colon \textbf{cocomp}\text{-}\mathcal{W}\text{-}\textbf{Cat}(\mathcal{A}) \to \textbf{cocomp}\text{-}\mathcal{V}\text{-}\textbf{Cat}(\mathcal{A}) \tag{4.11}$$

rendered by the functor in Theorem 2.3.1 is defined by

- assigning on objects $(\mathcal{D},\ \theta \colon \mathcal{D}_0 \xrightarrow{\cong} \mathcal{A}) \in \textbf{cocomp}\text{-}\mathcal{W}\text{-}\textbf{Cat}(\mathcal{A})$ the object $(G_*\mathcal{D},\ G_*\theta \colon (G_*\mathcal{D})_0 \xrightarrow{\cong} \mathcal{A}) \in \textbf{cocomp}\text{-}\mathcal{V}\text{-}\textbf{Cat}(\mathcal{A})$, where the identity-on-objects isomorphism $G_*\theta$ is given on maps by

$$\begin{aligned}
(G_*\theta)_{X,Y} \colon (G_*\mathcal{D})_0(X,Y) &:= \mathcal{V}(I, G\mathcal{D}(X,Y)) \\
&\cong \mathcal{W}(FI, \mathcal{D}(X,Y)) \\
&\cong \mathcal{W}(I, \mathcal{D}(X,Y)) \\
&:= \mathcal{D}_0(X,Y) \\
&\overset{\theta_{X,Y}}{\cong} \mathcal{A}(X,Y).
\end{aligned} \tag{4.12}$$

  Note that the copower of $X \in G_*\mathcal{D}$ by $V \in \mathcal{V}$ is given by $F(V) \circledast X \in G_*\mathcal{D}$ where $\circledast$ is the copower of the $\mathcal{W}$-category $\mathcal{D}$.

- And on morphisms $\mu\colon (\mathcal{D}_1, \theta_1) \to (\mathcal{D}_2, \theta_2)$ in **cocomp**-$\mathcal{W}$-**Cat**$(\mathcal{A})$, the $\mathcal{V}$-functor $G_*\mu\colon G_*\mathcal{D}_1 \to G_*\mathcal{D}_2$ is a map in **cocomp**-$\mathcal{V}$-**Cat**$(\mathcal{A})$ since

$$\begin{array}{ccc}
\mathcal{V}(I, G\mathcal{D}_1(X,Y)) & \xrightarrow{G\mu_{X,Y}\circ -} & \mathcal{V})(I, G\mathcal{D}_2(X,Y)) \\
\cong\downarrow & & \downarrow\cong \\
\mathcal{W}(FI, \mathcal{D}_1(X,Y)) & \xrightarrow{\mu_{X,Y}\circ -} & \mathcal{W}(FI, \mathcal{D}_2(X,Y)) \\
\cong\downarrow & & \downarrow\cong \\
\mathcal{W}(I, \mathcal{D}_1(X,Y)) & \xrightarrow{\mu_{X,Y}\circ -} & \mathcal{W}(I, \mathcal{D}_2(X,Y)) \\
& {\scriptstyle\theta_{1X,Y}}\searrow^{\cong} \quad \swarrow^{\cong}{\scriptstyle\theta_{2X,Y}} & \\
& \mathcal{A}(X,Y) &
\end{array}$$

  commutes by the naturality of the adjunction $F \dashv G$ and the fact that $\mu\colon (\mathcal{D}_1, \theta_1) \to (\mathcal{D}_2, \theta_2)$ is a map in **cocomp**-$\mathcal{W}$-**Cat**$(\mathcal{A})$.

Proposition 4.2.2 (($\underline{\mathcal{A}}, \theta$) as universal cocomplete enrichment)**.** *Consider the cocomplete* $\mathrm{Cocts}(\mathcal{A},\mathcal{A})$*-enrichment* $(\underline{\mathcal{A}}, \theta) \in$ ***cocomp***-$\mathrm{Cocts}(\mathcal{A},\mathcal{A})$-***Cat***$(\mathcal{A})$ *from Proposition 4.1.1 and let* $(G_*\underline{\mathcal{A}}, G_*\theta)$ *be the* $\mathcal{V}$*-enrichment acquired from* $(\underline{\mathcal{A}}, \theta)$ *through the change of base (Definition 4.2.1) along the monoidal adjunction* $\langle F \dashv G\rangle\colon \mathcal{V} \to \mathrm{Cocts}(\mathcal{A},\mathcal{A})$ *from (4.8). This is isomorphic in* ***cocomp***-$\mathcal{V}$-***Cat***$(\mathcal{A})$ *to the given* $\mathcal{V}$*-enrichment* $(\underline{\mathcal{A}}_{\mathcal{V}}, \eta)$ *from (4.5)*

$$(\underline{\mathcal{A}}_{\mathcal{V}}, \eta) \cong (G_*\underline{\mathcal{A}}, G_*\theta) \tag{4.13}$$

by the $\mathcal{V}$*-isomorphism from Proposition 4.2.1.*

Proof. The isomorphism in **cocomp**-$\mathcal{V}$-**Cat**$(\mathcal{A})$ is given by the $\mathcal{V}$-isomorphism $H\colon \underline{\mathcal{A}}_{\mathcal{V}} \xrightarrow{\cong} G_*\underline{\mathcal{A}}$ in Proposition 4.2.1.

Note that the identity-on-objects isomorphism of categories $\theta\colon \underline{\mathcal{A}}_0 \xrightarrow{\cong} \mathcal{A}$ comes from

$$\begin{aligned}
\theta_{X,Y}\colon \underline{\mathcal{A}}_0(X,Y) &:= \mathrm{Cocts}(\mathcal{A},\mathcal{A})(\mathrm{id}_{\mathcal{A}}, \underline{\mathcal{A}}(X,Y)) \\
&\cong \mathcal{A}(\mathrm{id}_{\mathcal{A}}(X), Y) \\
&= \mathcal{A}(X,Y).
\end{aligned} \tag{4.14}$$

To show we have a map $H\colon (\underline{\mathcal{A}}_{\mathcal{V}}, \eta) \xrightarrow{\cong} (G_*\underline{\mathcal{A}},\ G_*\theta)$ in **cocomp**-$\mathcal{V}$-**Cat**$(\mathcal{A})$, we must show the diagram

$$\begin{array}{ccc}
\left(\underline{\mathcal{A}}_{\mathcal{V}}\right)_0 & \xrightarrow{H_0} & \left(G_*\underline{\mathcal{A}}\right)_0 \\
& {\scriptstyle\eta}\searrow \quad \swarrow{\scriptstyle G_*\theta} & \\
& \mathcal{A} &
\end{array} \tag{4.15}$$

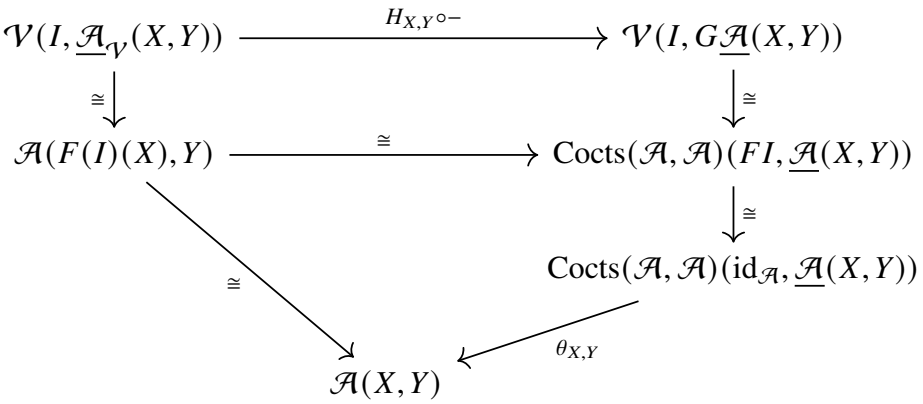
commutes. This is clear on objects since all maps are identities. And on hom-sets, we must check commutativity in:

$$\begin{array}{ccc}
\mathcal{V}(I, \underline{\mathcal{A}}_{\mathcal{V}}(X,Y)) & \xrightarrow{H_{X,Y}\circ -} & \mathcal{V}(I, G\underline{\mathcal{A}}(X,Y)) \\
& {\scriptstyle\eta_{X,Y}}\searrow \quad \swarrow{\scriptstyle (G_*\theta)_{X,Y}} & \\
& \mathcal{A}(X,Y) &
\end{array} \tag{4.16}$$

where $(G_*\theta)_{X,Y}$ is the map from (4.12) and $\eta_{X,Y}$ is the map

$$\eta_{X,Y} \colon \mathcal{V}(I, \underline{\mathcal{A}}_{\mathcal{V}}(X,Y)) \cong \mathcal{A}(F(I)(X), Y) \cong \mathcal{A}(X,Y). \tag{4.17}$$

Now (4.16) commutes since in the diagram:



the bottom-rectangle commutes by the naturality of (4.3) and the upper-rectangle commutes by the definition of $H_{X,Y}$. □

4.3 MonAdj$(\mathcal{V}, \mathrm{Cocts}(\mathcal{A},\mathcal{A})) \simeq$ cocomp-$\mathcal{V}$-Cat$(\mathcal{A})^{\mathrm{op}}$

Given a locally presentable category $\mathcal{A}$ and a locally presentable closed monoidal category $\mathcal{V}$, consider the category which we denote as $\mathrm{Cocts}(\mathcal{A},\mathcal{A})^{(\mathrm{MonAdj})\mathcal{V}}$ or **MonAdj**$(\mathcal{V}, \mathrm{Cocts}(\mathcal{A},\mathcal{A}))$, with as objects monoidal adjunctions

$$\langle F, G, \eta, \epsilon \rangle \colon \mathcal{V} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Cocts}(\mathcal{A},\mathcal{A}), \tag{4.18}$$

and as arrows the *conjugate pairs* of natural transformations $\sigma \colon F \Rightarrow F'$, $\tau \colon G' \Rightarrow G$

$$\langle F, G, \eta, \epsilon \rangle \xrightarrow{(\sigma,\tau)} \langle F', G', \eta', \epsilon' \rangle \tag{4.19}$$

such that $\sigma \colon F \Rightarrow F'$ is a monoidal natural transformation.

Remark 4.3.1 (Conjugate)**.** The ordinary adjunctions between two ordinary categories, not necessarily monoidal, form a category **Adj**$(C, \mathcal{D})$. A morphism between two adjunctions $\langle F, G, \eta, \epsilon \rangle \colon C \to \mathcal{D}$ and $\langle F', G', \eta', \epsilon' \rangle \colon C \to \mathcal{D}$, is a pair of natural transformations $\sigma \colon F \Rightarrow F'$ and $\tau \colon G' \Rightarrow G$ satisfying the following condition

$$\begin{array}{ccc} \mathcal{D}(F'A, B) & \xrightarrow{\cong} & C(A, G'B) \\ {\scriptstyle -\circ\sigma_A = \mathcal{D}(\sigma_A, B)}\downarrow & & \downarrow{\scriptstyle C(A,\tau_B) = \tau_B \circ -} \\ \mathcal{D}(FA, B) & \xrightarrow{\cong} & C(A, GB). \end{array} \tag{4.20}$$

Such pair (σ, τ) are said to be *conjugate* ([Mac13] IV.7).

Lemma 4.3.1 ([Mac13] Theorem 2, IV.7)**.** *The following conditions are equivalent to (4.20):*

(a) $G' \overset{\tau}{\Rightarrow} G \quad = \quad G' \overset{\eta G}{\Longrightarrow} GFG' \overset{G\sigma G'}{\Longrightarrow} GF'G' \overset{G\epsilon'}{\Longrightarrow} G$

(b) $F \overset{\sigma}{\Rightarrow} F' \quad = \quad F \overset{F\eta'}{\Longrightarrow} FG'F' \overset{F\tau F'}{\Longrightarrow} FGF' \overset{\epsilon F'}{\Longrightarrow} F'$

(c) $FG' \overset{F\tau}{\Longrightarrow} FG \overset{\epsilon}{\Rightarrow} \mathrm{id}_{\mathcal{D}} \quad = \quad FG' \overset{\sigma G'}{\Longrightarrow} F'G' \overset{\epsilon'}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$

(d) $\mathrm{id}_C \overset{\eta}{\Rightarrow} GF \overset{G\sigma}{\Longrightarrow} GF' \quad = \quad \mathrm{id}_C \overset{\eta'}{\Rightarrow} G'F' \overset{\tau F'}{\Longrightarrow} GF'$

Also, given a conjugate pair (σ, τ), it is easy to see that σ is a natural isomorphism if and only if τ is. And given two adjunctions $\langle F, G, \eta, \epsilon\rangle$ and $\langle F', G', \eta', \epsilon'\rangle \in \boldsymbol{Adj}(\mathcal{C}, \mathcal{D})$, to give a morphism (σ, τ) in $\boldsymbol{Adj}(\mathcal{C}, \mathcal{D})$ is equivalently to give just a single conjugate component $\sigma\colon F \Rightarrow F'$ where $\tau\colon G' \Rightarrow G$ defined as (a) is the unique one that makes (σ, τ) a conjugate pair. Likewise, given a single natural transformation $\tau\colon G' \Rightarrow G$, there is a unique natural transformation $\sigma\colon F \Rightarrow F'$ such that (σ, τ) is a conjugate pair, which is the natural transformation σ defined as (b).

We now show that $\mathbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A}))$ and $\mathbf{cocomp}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}$ are categorically equivalent by a chain of equivalences through $\mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$. Moreover, we show that this is a chain of pseudonatural equivalences.

Proposition 4.3.2. *Given a locally presentable closed monoidal category $\mathcal{V}$ and a locally presentable category $\mathcal{A}$, there is a chain of equivalences of categories between $\boldsymbol{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A}))$ and $\boldsymbol{cocomp}\text{-}\mathcal{V}\text{-}\boldsymbol{Cat}(\mathcal{A})^{\mathrm{op}}$:*

$$\boldsymbol{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \xrightarrow{\simeq} \mathcal{V}\text{-}\boldsymbol{Act}(\mathcal{A})_{\mathrm{cocts}} \xrightarrow{\simeq} \boldsymbol{cocomp}\text{-}\mathcal{V}\text{-}\boldsymbol{Cat}(\mathcal{A})^{\mathrm{op}}. \tag{4.21}$$

Proof. The second equivalence comes from Proposition 3.3.5. The first equivalence

$$\mathbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \xrightarrow{\simeq} \mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}} \tag{4.22}$$

is easy to see. On objects, a monoidal adjunction $\langle F, G, \eta, \epsilon\rangle$ gives a strong monoidal cocontinuous functor $F\colon \mathcal{V} \to \mathrm{Cocts}(\mathcal{A}, \mathcal{A})$ which is same as a monoidal action $\odot\colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ cocontinuous in each variable, giving an object of $\mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$. This assignment

$$\langle F, G, \eta, \epsilon\rangle \mapsto \odot \tag{4.23}$$

is surjective (as we've seen in (4.5) and (4.8)). Moreover, a map

$$(\sigma, \tau)\colon \langle F_1, G_1, \eta_1, \epsilon_1\rangle \to \langle F_2, G_2, \eta_2, \epsilon_2\rangle \tag{4.24}$$

in $\mathbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A}))$, which is a conjugate pair of natural transformation $\sigma\colon F_1 \Rightarrow F_2$ and $\tau\colon G_2 \Rightarrow G_1$ such that σ is a monoidal natural transformation, is precisely equivalent to a single monoidal natural transformation $\sigma\colon F_1 \Rightarrow F_2$ (Lemma 4.3.1). This is again precisely a natural transformation

$$\overline{\sigma}_{V,A}\colon V \odot_1 A \cong V \odot_2 A \tag{4.25}$$

satisfying (3.6) and (3.7) as the monoidal axioms of the monoidal natural transformation $\sigma\colon F_1 \Rightarrow F_2$

$$\begin{array}{ccc} F_1(V) \circ F_1(W) & \xrightarrow{\sigma_V \circ \sigma_W} & F_2(V) \circ F_2(W) \\ \cong\downarrow & & \downarrow\cong \\ F_1(V \otimes W) & \xrightarrow[\sigma_{V\otimes W}]{} & F_2(V \otimes W) \end{array} \tag{4.26}$$

$$\begin{array}{ccc} \mathrm{id}_{\mathcal{A}} & & \\ \cong\downarrow & \searrow^{\cong} & \\ F_1(I_{\mathcal{V}}) & \xrightarrow[\sigma_{I_{\mathcal{V}}}]{} & F_2(I_{\mathcal{V}}) \end{array} \tag{4.27}$$

exactly correspond to (3.6) and (3.7).

□

We now wish to show that the chain of categorical equivalences in Proposition 4.3.2 are pseudonatural in $\mathcal{V}$. To this end, consider the 2-category **MonAdj** of locally presentable closed monoidal categories, monoidal adjunctions $\langle F \dashv G, \eta, \epsilon\rangle\colon \mathcal{V} \to \mathcal{W}$ as 1-cells and conjugate pairs of natural transformations $(\sigma, \tau)\colon \langle F \dashv G, \eta, \epsilon\rangle \to \langle F' \dashv G', \eta', \epsilon'\rangle$ such that $\sigma\colon F \Rightarrow F'$ is a monoidal natural transformation as 2-cells. For a fixed locally presentable category $\mathcal{A}$, the categories **cocomp-**$\mathcal{V}$**-Cat**$(\mathcal{A})^{\mathrm{op}}$ are the values of a 2-functor from the opposite 2-category **MonAdj**$^{\mathrm{op}}$, here in which only the 1-cells are reversed.

Definition 4.3.1. We have the 2-functor

$$\textbf{cocomp-}(-)\textbf{-Cat}(\mathcal{A})^{\mathrm{op}}\colon \textbf{MonAdj}^{\mathrm{op}} \to \textbf{Cat} \tag{4.28}$$

given by

- on objects: $\mathcal{V} \mapsto$ **cocomp-**$\mathcal{V}$**-Cat**$(\mathcal{A})^{\mathrm{op}}$
- on 1-cells: $\langle F \dashv G, \eta, \epsilon\rangle\colon \mathcal{V} \to \mathcal{W}$ gets mapped to the change of base functor defined in Definition 4.2.1

$$G_*\colon \textbf{cocomp-}\mathcal{W}\textbf{-Cat}(\mathcal{A})^{\mathrm{op}} \to \textbf{cocomp-}\mathcal{V}\textbf{-Cat}(\mathcal{A})^{\mathrm{op}} \tag{4.29}$$

- on 2-cells: the pair $(\sigma, \tau)\colon \langle F_1, G_1\rangle \to \langle F_2, G_2\rangle$

$$\mathcal{V} \;\underset{\langle F_2, G_2\rangle}{\overset{\langle F_1, G_1\rangle}{\Downarrow (\sigma,\tau)}}\; \mathcal{W}$$

is sent to the natural transformation

$$\textbf{cocomp-}\mathcal{W}\textbf{-Cat}(\mathcal{A})^{\mathrm{op}} \;\underset{G_{2*}}{\overset{G_{1*}}{\Downarrow \tau_*}}\; \textbf{cocomp-}\mathcal{V}\textbf{-Cat}(\mathcal{A})^{\mathrm{op}} \tag{4.30}$$

with component at $(\mathcal{D},\theta) \in \mathbf{cocomp}\text{-}\mathcal{W}\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}$ given by the map $\tau_{*(\mathcal{D},\theta)} : (G_{2*}\mathcal{D}, G_{2*}\theta) \to (G_{1*}\mathcal{D}, G_{1*}\theta)$ in $\mathbf{cocomp}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})$ which is the identity on objects and $G_2\mathcal{D}(X,Y) \xrightarrow{\tau_{\mathcal{D}(X,Y)}} G_1\mathcal{D}(X,Y)$ on hom-$\mathcal{V}$-objects. This $\mathcal{V}$-functor is indeed a map in $\mathbf{cocomp}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})$, i.e.

$$\begin{array}{ccc} (G_{2*}\mathcal{D})_0 & \xrightarrow{(\tau_{*(\mathcal{D},\theta)})_0} & (G_{1*}\mathcal{D})_0 \\ & {\scriptstyle G_{2*}\theta}\searrow \quad \swarrow{\scriptstyle G_{1*}\theta} & \\ & \mathcal{A} & \end{array}$$

commutes since on objects, they are all identities, and on morphisms the diagram:

$$\begin{array}{ccc} \mathcal{V}(I, G_2\mathcal{D}(X,Y)) & \xrightarrow{\tau_{\mathcal{D}(X,Y)}\circ -} & \mathcal{V}(I, G_1\mathcal{D}(X,Y)) \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{W}(F_2 I, \mathcal{D}(X,Y)) & \xrightarrow{-\circ\sigma_I} & \mathcal{W}(F_1 I, \mathcal{D}(X,Y)) \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{W}(I, \mathcal{D}(X,Y)) & = & \mathcal{W}(I, \mathcal{D}(X,Y)) \\ & {\scriptstyle\cong}\ {\scriptstyle\theta_{X,Y}}\searrow \quad \swarrow{\scriptstyle\theta_{X,Y}}\ {\scriptstyle\cong} & \\ & \mathcal{A}(X,Y) & \end{array}$$

commutes from the fact that (σ,τ) is a conjugate pair of natural transformations and σ is a monoidal natural transformation.

To prove the pseudonaturality of (4.21), we make use of the *Bicategorical Yoneda Lemma*. This is the generalisation of the Yoneda Lemma to the setting of bicategories, pseudofunctors, pseudonatural transformations and modifications. For a bicategory C, consider the 2-category $\mathbf{Bicat}(C^{\mathrm{op}}, \mathbf{Cat})$ with pseudofunctors $C^{\mathrm{op}} \to \mathbf{Cat}$ as objects, pseudonatural transformations as 1-cells and modifications as 2-cells. Note that for bicategories C and $\mathcal{D}$, this $\mathbf{Bicat}(C,\mathcal{D})$ is a bicategory but is a 2-category when $\mathcal{D}$ is, like the 2-category $\mathbf{Bicat}(C^{\mathrm{op}}, \mathbf{Cat})$. The details of the Bicategorical Yoneda Lemma are originally introduced in [Str80] and are also well presented in many accounts such as Section 8.3 of [JY20] and [Bak] stating:

Lemma 4.3.3 (Bicategorical Yoneda Lemma)**.** *For each $A \in C$ and each pseudofunctor $F : C^{\mathrm{op}} \to \boldsymbol{Cat}$, we have an equivalence of categories*

$$\boldsymbol{Bicat}(C^{\mathrm{op}}, \boldsymbol{Cat})(C(-,A), F) \simeq FA \tag{4.31}$$

pseudonatural in $A \in C$ and $F \in \boldsymbol{Bicat}(C^{\mathrm{op}}, \boldsymbol{Cat})$ where $C(-,A) : C^{\mathrm{op}} \to \boldsymbol{Cat}$ is the representable pseudofunctor.

The evaluation functor $e : \mathbf{Bicat}(C^{\mathrm{op}}, \mathbf{Cat})(C(-,A), F) \to FA$ is presented in Definition 8.3.2 of [JY20] and its essential surjectivity and the proof is shown in Definition 8.3.9, Lemma 8.3.11 and Lemma 8.3.16 of [JY20].

Proposition 4.3.4. *The equivalence of categories*

$$\boldsymbol{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A},\mathcal{A})) \simeq \boldsymbol{cocomp}\text{-}\mathcal{V}\text{-}\boldsymbol{Cat}(\mathcal{A})^{\mathrm{op}}$$

in Proposition 4.3.2 is pseudonatural in $\mathcal{V} \in \boldsymbol{MonAdj}$, assembling to form a pseudonatural equivalence.

Proof. Using the Bicategorical Yoneda lemma, we have

$$\begin{aligned}&\mathbf{Bicat}(\mathbf{MonAdj}^{\mathrm{op}}, \mathbf{Cat})\Big(\mathbf{MonAdj}(-, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})),\ \mathbf{cocomp}\text{-}(-)\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}\Big)\\ &\simeq \mathbf{cocomp}\text{-}\mathrm{Cocts}(\mathcal{A}, \mathcal{A})\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}.\end{aligned}$$

Thus the element $(\underline{\mathcal{A}}, \theta) \in \mathbf{cocomp}\text{-}\mathrm{Cocts}(\mathcal{A}, \mathcal{A})\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}$ from Proposition 4.1.1 corresponds to a pseudonatural transformation

$$\alpha \colon \mathbf{MonAdj}(-, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \Longrightarrow \mathbf{cocomp}\text{-}(-)\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}} \tag{4.32}$$

on the left-side of the Bicategorical Yoneda lemma, with components:

$$\alpha_{\mathcal{V}} \colon \mathbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \longrightarrow \mathbf{cocomp}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}} \tag{4.33}$$

for each $\mathcal{V} \in \mathbf{MonAdj}$, defined ([JY20] Definition 8.3.9) as follows:

- on objects: for each monoidal adjunction

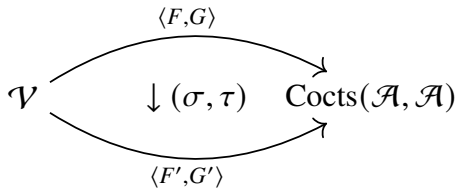
$$\langle F, G\rangle \colon \mathcal{V} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Cocts}(\mathcal{A}, \mathcal{A}) \tag{4.34}$$

  we take $\alpha_{\mathcal{V}}(\langle F, G\rangle) = G_*(\underline{\mathcal{A}}, \theta) = (G_*\underline{\mathcal{A}},\ G_*\theta)$ where G_* is the functor

$$G_* \colon \mathbf{cocomp}\text{-}\mathrm{Cocts}(\mathcal{A}, \mathcal{A})\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}} \to \mathbf{cocomp}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}$$

  we've seen in (4.29).

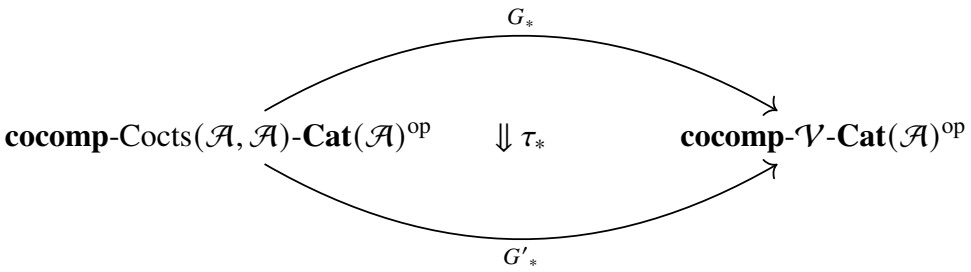
- on morphisms: given a pair $(\sigma, \tau) \colon \langle F, G\rangle \to \langle F', G'\rangle$



  we take the map

$$\tau_* \colon (G'_*\underline{\mathcal{A}},\ G'_*\theta) \to (G_*\underline{\mathcal{A}},\ G_*\theta) \tag{4.35}$$

  in **cocomp**-$\mathcal{V}$-**Cat**$(\mathcal{A})$ which is the component of the natural transformation (4.30)



  at $(\underline{\mathcal{A}}, \theta)$.

To show the equivalence of Proposition 4.3.2 is pseudonatural in $\mathcal{V}$, it thus suffices to show that each functor

$$\alpha_{\mathcal{V}} \colon \mathbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \to \mathbf{cocomp}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}$$

is naturally isomorphic to the categorical equivalence in our Proposition 4.3.2. For each monoidal adjunction (4.34), we have the isomorphism

$$H \colon (\underline{\mathcal{A}}_{\mathcal{V}}, \eta) \xrightarrow{\cong} (G_* \underline{\mathcal{A}}, G_* \theta) \tag{4.36}$$

from Proposition 4.2.2.

To show that this is a natural isomorphism, the diagram below in $\mathbf{cocomp}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}$ must commute

$$\begin{array}{ccc} (\underline{\mathcal{A}}_{\mathcal{V}}, \eta) & \xleftarrow{H^{\mathrm{op}}} & (G_* \underline{\mathcal{A}},\ G_* \theta) \\ {\scriptstyle \underline{\sigma}^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle \tau_*^{\mathrm{op}}} \\ (\underline{\mathcal{A}'}_{\mathcal{V}}, \eta') & \xleftarrow[H'^{\mathrm{op}}]{} & (G'_* \underline{\mathcal{A}},\ G'_* \theta). \end{array}$$

On objects, they are all the identity and on hom-objects, it is the below diagram in $\mathcal{V}$

$$\begin{array}{ccc} \underline{\mathcal{A}}_{\mathcal{V}}(A,B) & \xrightarrow{H_{A,B}} & G\underline{\mathcal{A}}(A,B) \\ {\scriptstyle \underline{\sigma}_{A,B}}\uparrow & & \uparrow{\scriptstyle \tau_{\langle A,B\rangle}} \\ \underline{\mathcal{A}'}_{\mathcal{V}}(A,B) & \xrightarrow[H'_{A,B}]{} & G'\underline{\mathcal{A}}(A,B) \end{array}$$

where the $\underline{\sigma}_{A,B}$'s are the maps (3.29). It commutes by chasing elements in the commutative diagram below, consisting of the equation (4.9)

$$\begin{array}{ccccc} \mathcal{V}(\underline{\mathcal{A}'}_{\mathcal{V}}(A,B), \underline{\mathcal{A}'}_{\mathcal{V}}(A,B)) & \xrightarrow{\cong} & \mathrm{Cocts}(\mathcal{A},\mathcal{A})(F'(\underline{\mathcal{A}'}_{\mathcal{V}}(A,B)), \underline{\mathcal{A}}(A,B)) & \xrightarrow{\cong} & \mathcal{V}(\underline{\mathcal{A}'}_{\mathcal{V}}(A,B), G'\underline{\mathcal{A}}(A,B)) \\ {\scriptstyle \underline{\sigma}_{A,B}\circ -}\downarrow & & \downarrow{\scriptstyle -\circ\sigma_{\underline{\mathcal{A}'}_{\mathcal{V}}(A,B)}} & & \downarrow{\scriptstyle \tau_{\underline{\mathcal{A}}(A,B)}\circ -} \\ \mathcal{V}(\underline{\mathcal{A}'}_{\mathcal{V}}(A,B), \underline{\mathcal{A}}_{\mathcal{V}}(A,B)) & \xrightarrow[\cong]{} & \mathrm{Cocts}(\mathcal{A},\mathcal{A})(F(\underline{\mathcal{A}'}_{\mathcal{V}}(A,B)), \underline{\mathcal{A}}(A,B)) & \xrightarrow[\cong]{} & \mathcal{V}(\underline{\mathcal{A}'}_{\mathcal{V}}(A,B), G\underline{\mathcal{A}}(A,B)) \end{array}$$

by starting from the top-right with the element $H'_{A,B}$:

$$\begin{array}{ccc} \mathrm{id}_{\underline{\mathcal{A}'}_{\mathcal{V}}(A,B)} & \longmapsfrom & H'_{A,B} \\ \downarrow & & \downarrow \\ \underline{\sigma}_{A,B} & \longmapsto & \tau_{\langle A,B\rangle} \circ H'_{A,B} = H_{A,B} \circ \underline{\sigma}_{A,B}. \end{array}$$

□

We now show that the individual categorical equivalences in Proposition 4.3.2

$$\mathbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A},\mathcal{A})) \xrightarrow{\simeq} \mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}} \xrightarrow{\simeq} \mathbf{cocomp}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}$$

are pseudonatural equivalences in $\mathcal{V}$. Having established that their composite

$$\mathbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A},\mathcal{A})) \simeq \mathbf{cocomp}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})^{\mathrm{op}}$$

is a pseudonatural equivalence, it suffices to show that one of them is a pseudonatural equivalence.

Definition 4.3.2. We have the 2-functor

$$(-)\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}} \colon \mathbf{MonAdj}^{\mathrm{op}} \to \mathbf{Cat} \tag{4.37}$$

given by

- on objects: $\mathcal{V} \mapsto \mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$
- on 1-cells: $\langle F \dashv G, \eta, \epsilon \rangle \colon \mathcal{V} \to \mathcal{W}$ gets mapped to the functor

$$F^* \colon \mathcal{W}\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}} \longrightarrow \mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}} \tag{4.38}$$

which is pre-composition by $(F \times \mathrm{id}_{\mathcal{A}})$; that is, on objects, F^* maps a $\mathcal{W}$-action $\odot \colon \mathcal{W} \times \mathcal{A} \to \mathcal{A}$ to a $\mathcal{V}$-action given by

$$\mathcal{V} \times \mathcal{A} \xrightarrow{F \times \mathrm{id}_{\mathcal{A}}} \mathcal{W} \times \mathcal{A} \xrightarrow{\odot} \mathcal{A}$$

where the associativity and unitality of the monoidal action is

$$\begin{aligned} &F(V \otimes W) \odot X \cong (FV \otimes FW) \odot X \cong FV \odot (FW \odot X) \\ &\text{and} \quad FI \odot X \cong I \odot X \cong X, \end{aligned} \tag{4.39}$$

and on morphisms of $\mathcal{W}$-$\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$, maps the natural transformation

$$\mathcal{W} \times \mathcal{A} \;\; \overset{\odot_1}{\underset{\odot_2}{\Downarrow \beta}} \;\; \mathcal{A}$$

to the natural transformations $FV \odot_1 X \xrightarrow{\beta} FV \odot_2 X$ obtained by left-whiskering

$$\mathcal{V} \times \mathcal{A} \xrightarrow{F \times \mathrm{id}_{\mathcal{A}}} \mathcal{W} \times \mathcal{A} \;\; \overset{\odot_1}{\underset{\odot_2}{\Downarrow \beta}} \;\; \mathcal{A}.$$

This is a map in $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$ since the outside of the two diagrams below commute – showing that (3.6) and (3.7) hold. This is from the fact that the upper-inner diagrams commute since $\beta \colon \odot_1 \Rightarrow \odot_2$, as a map in $\mathcal{W}$-$\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$, satisfies (3.6) and (3.7) and from the fact that the bottom-inner diagrams commute by naturality.

$$\begin{array}{ccccc}
FV \odot_1 (FW \odot_1 X) & \xrightarrow{\mathrm{id} \odot_1 \beta} & FV \odot_1 (FW \odot_2 X) & \xrightarrow{\beta} & FV \odot_2 (FW \odot_2 X) \\
\cong \downarrow & & & & \downarrow \cong \\
(FV \otimes FW) \odot_1 X & & \xrightarrow{\qquad\qquad \beta \qquad\qquad} & & (FV \otimes FW) \odot_2 X \\
\cong \downarrow & & & & \downarrow \cong \\
F(V \otimes W) \odot_1 X & & \xrightarrow[\qquad\qquad \beta \qquad\qquad]{} & & F(V \otimes W) \odot_2 X
\end{array}$$

$$\begin{array}{ccc} X & & \\ {\scriptstyle\cong}\downarrow & \searrow^{\cong} & \\ I \odot_1 X & \xrightarrow[\beta]{} & I \odot_2 X \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ FI \odot_1 X & \xrightarrow[\beta]{} & FI \odot_2 X \end{array}$$

- on 2-cells: assigning to the pair $(\sigma, \tau) \colon \langle F_1, G_1 \rangle \to \langle F_2, G_2 \rangle$

$$\mathcal{V} \;\overset{\langle F_1, G_1 \rangle}{\underset{\langle F_2, G_2 \rangle}{\Downarrow (\sigma, \tau)}}\; \mathcal{W}$$

a natural transformation

$$\mathcal{W}\text{-}\mathbf{Act}(\mathcal{A})_{\text{cocts}} \;\overset{F_1{}^*}{\underset{F_2{}^*}{\Downarrow \sigma^*}}\; \mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\text{cocts}} \tag{4.40}$$

with component at $(\mathcal{W} \times \mathcal{A} \xrightarrow{\odot} \mathcal{A}) \in \mathcal{W}\text{-}\mathbf{Act}(\mathcal{A})_{\text{cocts}}$ given by the natural transformation $F_1 V \odot X \xrightarrow{\sigma_V \odot \mathrm{id}} F_2 V \odot X$ obtained by right-whiskering

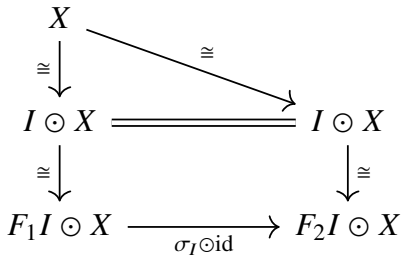
$$\mathcal{V} \times \mathcal{A} \;\overset{F_1 \times \mathrm{id}_{\mathcal{A}}}{\underset{F_2 \times \mathrm{id}_{\mathcal{A}}}{\Downarrow \sigma \times \mathrm{id}}}\; \mathcal{W} \times \mathcal{A} \xrightarrow{\odot} \mathcal{A}.$$

This is a map in $\mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\text{cocts}}$ since the outer diagram of the below, showing the condition (3.6), commutes

$$\begin{array}{ccccc} F_1 V \odot (F_1 W \odot X) & \xrightarrow{\mathrm{id} \odot (\sigma_W \odot \mathrm{id})} & F_1 V \odot (F_2 W \odot X) & \xrightarrow{\sigma_V \odot (\mathrm{id} \odot \mathrm{id})} & F_2 V \odot (F_2 W \odot X) \\ {\scriptstyle\cong}\downarrow & & & & \downarrow{\scriptstyle\cong} \\ (F_1 V \otimes F_1 W) \odot X & & \xrightarrow{(\sigma_V \otimes \sigma_W) \odot \mathrm{id}} & & (F_2 V \otimes F_2 W) \odot X \\ {\scriptstyle\cong}\downarrow & & & & \downarrow{\scriptstyle\cong} \\ F_1(V \otimes W) \odot X & & \xrightarrow[\sigma_{V \otimes W} \odot \mathrm{id}]{} & & F_2(V \otimes W) \odot X \end{array}$$

from the fact that the upper-inner rectangle commutes by the naturality of the associativity of the monoidal action $\odot \colon \mathcal{W} \times \mathcal{A} \to \mathcal{A}$ and the bottom-inner rectangle commutes by the natural transformation $\sigma \colon F_1 \Rightarrow F_2$ being monoidal.

Likewise, the outer diagram of the below, showing the condition (3.7), commutes



also from the fact that the natural transformation $\sigma\colon F_1 \Rightarrow F_2$ is monoidal.

Proposition 4.3.5. *The chain of categorical equivalences in Proposition 4.3.2*

$$\textbf{\textit{MonAdj}}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \xrightarrow{\simeq} \mathcal{V}\textbf{-\textit{Act}}(\mathcal{A})_{\mathrm{cocts}} \xrightarrow{\simeq} \textbf{\textit{cocomp}}\text{-}\mathcal{V}\textbf{-\textit{Cat}}(\mathcal{A})^{\mathrm{op}}.$$

is a chain of equivalences pseudonatural in $\mathcal{V} \in$ ***MonAdj***.

Proof. It suffices to show that the equivalence of categories between

$$\textbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \simeq \mathcal{V}\textbf{-Act}(\mathcal{A})_{\mathrm{cocts}}$$

is pseudonatural in $\mathcal{V} \in$ **MonAdj**. Then, the pseudonaturality of

$$\mathcal{V}\textbf{-Act}(\mathcal{A})_{\mathrm{cocts}} \simeq \textbf{cocomp}\text{-}\mathcal{V}\textbf{-Cat}(\mathcal{A})^{\mathrm{op}}$$

also automatically holds.

Again, using the Bicategorical Yoneda lemma, we have

$$\begin{aligned}&\textbf{Bicat}(\textbf{MonAdj}^{\mathrm{op}}, \textbf{Cat})\Big(\textbf{MonAdj}(-, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})),\ (-)\textbf{-Act}(\mathcal{A})_{\mathrm{cocts}}\Big)\\ &\simeq \mathrm{Cocts}(\mathcal{A}, \mathcal{A})\textbf{-Act}(\mathcal{A})_{\mathrm{cocts}}.\end{aligned}$$

So taking the element $(\mathrm{Cocts}(\mathcal{A}, \mathcal{A}) \times \mathcal{A} \xrightarrow{\mathrm{ev}} \mathcal{A}) \in \mathrm{Cocts}(\mathcal{A}, \mathcal{A})\textbf{-Act}(\mathcal{A})_{\mathrm{cocts}}$ from (4.1) of Proposition 4.1.1, this gives the pseudonatural transformation

$$\alpha\colon \textbf{MonAdj}(-, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \Longrightarrow (-)\textbf{-Act}(\mathcal{A})_{\mathrm{cocts}} \tag{4.41}$$

on the left-side of the Bicategorical Yoneda lemma, with components:

$$\alpha_{\mathcal{V}}\colon \textbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \longrightarrow \mathcal{V}\textbf{-Act}(\mathcal{A})_{\mathrm{cocts}} \tag{4.42}$$

that is, a functor $\alpha_{\mathcal{V}}$, for each $\mathcal{V} \in$ **MonAdj**, defined as follows:

- on objects: for each monoidal adjunction

$$\langle F, G\rangle\colon \mathcal{V} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Cocts}(\mathcal{A}, \mathcal{A}).$$

  $\alpha_{\mathcal{V}}(\langle F, G\rangle) = F^*(\mathrm{ev})$ where $F^*\colon \mathrm{Cocts}(\mathcal{A}, \mathcal{A})\textbf{-Act}(\mathcal{A})_{\mathrm{cocts}} \longrightarrow \mathcal{V}\textbf{-Act}(\mathcal{A})_{\mathrm{cocts}}$ is the functor (4.38). That is,

$$\mathcal{V} \times \mathcal{A} \xrightarrow{F\times \mathrm{id}} \mathrm{Cocts}(\mathcal{A}, \mathcal{A}) \times \mathcal{A} \xrightarrow{\mathrm{ev}} \mathcal{A}. \tag{4.43}$$

- on morphisms: assigning to a pair $(\sigma, \tau) \colon \langle F_1, G_1 \rangle \to \langle F_2, G_2 \rangle$

$$\mathcal{V} \;\underset{\langle F_2, G_2 \rangle}{\overset{\langle F_1, G_1 \rangle}{\Downarrow (\sigma,\tau)}}\; \mathrm{Cocts}(\mathcal{A}, \mathcal{A})$$

a map in $\mathcal{V}$-$\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$ given by the natural transformation obtained by right-whiskering

$$\mathcal{V} \times \mathcal{A} \;\underset{F_2 \times \mathrm{id}_{\mathcal{A}}}{\overset{F_1 \times \mathrm{id}_{\mathcal{A}}}{\Downarrow \sigma \times \mathrm{id}}}\; \mathrm{Cocts}(\mathcal{A}, \mathcal{A}) \times \mathcal{A} \xrightarrow{\;\mathrm{ev}\;} \mathcal{A}.$$

which is the component of the natural transformation (4.40)

$$\mathrm{Cocts}(\mathcal{A}, \mathcal{A})\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}} \;\underset{F_2^*}{\overset{F_1^*}{\Downarrow \sigma^*}}\; \mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$$

at $(\mathrm{Cocts}(\mathcal{A}, \mathcal{A}) \times \mathcal{A} \xrightarrow{\mathrm{ev}} \mathcal{A}) \in \mathrm{Cocts}(\mathcal{A}, \mathcal{A})\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$.

This functor $\alpha_{\mathcal{V}} \colon \mathbf{MonAdj}(\mathcal{V}, \mathrm{Cocts}(\mathcal{A}, \mathcal{A})) \to \mathcal{V}\text{-}\mathbf{Act}(\mathcal{A})_{\mathrm{cocts}}$ is exactly the same as the first equivalence (4.22), showing the pseudonaturality of the first categorical equivalence; and hence the pseudonaturality of both. □

5

Example: the coslice category $I/\mathbf{Set}$

In this Chapter, we consider an example of a locally presentable category: the category of models of a Lawvere theory. In particular, we look at the coslice category $I/\mathbf{Set}$ and calculate the universal cocomplete enrichment of $I/\mathbf{Set}$ over $\mathrm{Cocts}(I/\mathbf{Set}, I/\mathbf{Set})$. By the adjoint functor theorem, this is equally a cocomplete enrichment over the monoidal category of endoadjunctions $\mathbf{EndAdj}(I/\mathbf{Set})$.

Our results extend the paper [Jib95] which sketches a calculation of the monoidal category $\mathbf{EndAdj}(I/\mathbf{Set})$ but says nothing about enrichment.

5.1 Review of Lawvere theories

We begin by recalling *Lawvere theories*, a categorical treatment of equational algebraic systems.

It is classical to describe a variety of algebraic objects, such as groups, rings, R-modules and so on, as a set X together with a collection Q of operators satisfying a list E of equations between derived operations. And, for given Q and E, the collection of structure-preserving maps between such (Q, E)-algebras X and Y (i.e. functions $g\colon X \to Y$ that preserve all the operations from Q) gives a category **(Q,E)-Alg**; for example $\mathbf{Grp}, \mathbf{Rng}$, $\mathbf{Mod}_R$ and many more arise in this way. It is also well known that, for a given algebraic system (Q, E), there is the free-forgetful adjunction $U\colon$ **(Q,E)-Alg** $\to$ **Set** where U takes the underlying set and the left adjoint $F\colon$ **Set** $\to$ **(Q,E)-Alg** forms free (Q, E)-algebras.

The idea of *Lawvere theories* is to give these concepts a categorical treatment allowing for abstraction into more general contexts than **Set**. The idea is to encode algebraic structure, like (Q, E), in a category $\mathbb{T}$ called a *Lawvere theory* whose *algebras* in a category $\mathcal{X}$ will be certain functors $\mathbb{T} \to \mathcal{X}$.

The notion of Lawvere theory was introduced in [Law63] and has been discussed in various other references such as [ARV10] and [Buc08]. For a quick and concise reference covering the essentials we will need, we direct the reader to the article [Buc08] from which we adopt our notations in the below definition:

Definition 5.1.1. A category $\mathbb{T}$ with finite products is a *Lawvere theory* when its objects are

precisely the finite products $\prod_{i=0}^{n} c$ of a given generating object $c \in \mathbb{T}$. For a category $\mathcal{X}$ with finite products, we call a finite product preserving functor $F\colon \mathbb{T} \to \mathcal{X}$ a *model of* $\mathbb{T}$ *in* $\mathcal{X}$ and we write $\mathrm{Prod}(\mathbb{T}, \mathcal{X})$ for the category of such functors and natural transformations.

Let $c \in \mathbb{T}$ be the generating object of a Lawvere theory $\mathbb{T}$. The finite product preservation part of a model $F\colon \mathbb{T} \to \mathcal{X}$ encodes the existing commutative diagrams in $\mathbb{T}$ into the context of $\mathcal{X}$, giving $Fc \in \mathcal{X}$ the structure of the algebraic system of $\mathbb{T}$ in $\mathcal{X}$. And the natural transformation $\alpha\colon F \Rightarrow G\colon \mathbb{T} \to \mathcal{X}$ specifies the morphisms of algebras F and G. By considering the product projection maps of $\mathbb{T}$, it is easy to see that the components $\alpha_n\colon (Fc)^n \to (Gc)^n$ in $\mathcal{X}$ are uniquely determined by a map $\alpha_1\colon Fc \to Gc$ which preserves the operators.

We also recall some well-known results. Firstly, the category $\mathrm{Prod}(\mathbb{T}, \mathbf{Set})$ of $\mathbb{T}$-algebras is complete and cocomplete (section 5.2 [Buc08]). Secondly, there is the free-forgetful adjunction (section 5.3 [Buc08]). The forgetful functor $U\colon \mathrm{Prod}(\mathbb{T}, \mathbf{Set}) \to \mathbf{Set}$ sends a $\mathbb{T}$-algebra G to its underlying set $G(c) \in \mathbf{Set}$ where $c \in \mathbb{T}$ is the generating object of the Lawvere theory $\mathbb{T}$ and sends a $\mathbb{T}$-algebra morphism $\alpha\colon G \Rightarrow G'$ to its determining morphism $\alpha_1\colon Gc \to G'c$ in $\mathbf{Set}$. Its left adjoint $F\colon \mathbf{Set} \to \mathrm{Prod}(\mathbb{T}, \mathbf{Set})$ is called the free $\mathbb{T}$-algebra functor, sending a set to the free $\mathbb{T}$-algebra.

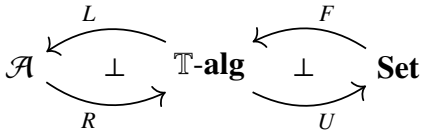
5.2 Category of endoadjunctions of coslice category $I/\mathbf{Set}$

Note that the forgetful functor $U\colon \mathrm{Prod}(\mathbb{T}, \mathbf{Set}) \to \mathbf{Set}$ is represented by the free $\mathbb{T}$-algebra $F(1)$ as

$$\mathrm{Prod}(\mathbb{T}, \mathbf{Set})(F(1), C) \cong \mathbf{Set}(1, UC) \cong UC \tag{5.1}$$

where $1 = \{*\} \in \mathbf{Set}$ is the one-element-set.

Consider a pair of adjunctions $L \dashv R$



where $\mathbb{T}$-**alg** = $\mathrm{Prod}(\mathbb{T}, \mathbf{Set})$ and $\mathcal{A}$ is a cocomplete category. It is easy to see from (5.1) that for all $X \in \mathcal{A}$

$$UR(X) \cong \mathbb{T}\text{-}\mathbf{alg}(F(1), R(X)) \cong \mathcal{A}(LF(1), X). \tag{5.2}$$

This means that the model $R(X)\colon \mathbb{T} \to \mathbf{Set}$ gives $R(X)(c) = \mathcal{A}(LF(1), X)$ a structure of a $\mathbb{T}$-algebra in $\mathbf{Set}$ where $c \in \mathbb{T}$ is the generator for the Lawvere theory $\mathbb{T}$. Now, since

$$\begin{aligned} \mathbf{Cat}(\mathcal{A}, \mathbb{T}\text{-}\mathbf{alg}) &= \mathbf{Cat}(\mathcal{A}, \mathrm{Prod}(\mathbb{T}, \mathbf{Set})) \\ &\cong \mathrm{Prod}(\mathbb{T},\ \mathbf{Cat}(\mathcal{A}, \mathbf{Set})), \end{aligned} \tag{5.3}$$

given a right adjoint $R \in \mathbf{Cat}(\mathcal{A}, \mathbb{T}\text{-}\mathbf{alg})$, and considering the equivalent $\overline{R} \in \mathrm{Prod}(\mathbb{T},\ \mathbf{Cat}(\mathcal{A}, \mathbf{Set}))$, we get $LF(1) \in \mathcal{A}$ with $\mathbb{T}$-algebra structure $\overline{R}$ on $\overline{R}(a) = \mathcal{A}(LF(1), -)\colon \mathcal{A} \to \mathbf{Set}$ in $\mathbf{Cat}(\mathcal{A}, \mathbf{Set})$. From the Yoneda Lemma and cocompleteness of $\mathcal{A}$, this is same as $\mathbb{T}$-algebra structure on $LF(1) \in \mathcal{A}^{\mathrm{op}}$; i.e. $\mathbb{T} \to \mathcal{A}^{\mathrm{op}}$ with $a \mapsto LF(1)$.

Inversely, let $A \in \mathcal{A}^{\mathrm{op}}$ and a $\mathbb{T}$-algebra structure $F\colon \mathbb{T} \to \mathcal{A}^{\mathrm{op}}$ on $A \in \mathcal{A}^{\mathrm{op}}$, we retrieve the right adjoint $R\colon \mathcal{A} \to \mathbb{T}\text{-}\mathbf{alg}$ by

$$Y \longmapsto \left(\mathbb{T} \xrightarrow{F} \mathcal{A}^{\mathrm{op}} \xrightarrow{\mathcal{A}(-,Y)} \mathbf{Set}\right). \tag{5.4}$$

Thus, we have:

Lemma 5.2.1. *There is a correspondence, up to isomorphism, between models of* $\mathbb{T}$ *in* $\mathcal{A}^{\mathrm{op}}$ *and right adjoint functors* $\mathcal{A} \to \mathbb{T}$-**alg**.

The specific case of this observation for $\mathcal{A} = \mathbb{T}$-**alg** can be found in [Jib95] as:

Proposition 5.2.2 (Proposition 0 of [Jib95])**.** $\mathbf{EndAdj}(\mathbb{T}\text{-}\mathbf{alg}) \simeq \mathrm{Prod}(\mathbb{T}, \mathbb{T}\text{-}\mathbf{alg}^{\mathrm{op}})^{\mathrm{op}}$ *where* $\mathbf{EndAdj}(C)$ *is the category of endoadjunctions of a category* C.

We now turn our attention to a particular example of the category of endoadjunctions $\mathbf{EndAdj}(\mathbb{T}\text{-}\mathbf{alg})$ specified in Proposition 1 of [Jib95]. The Lawvere theory $\mathbb{T}$ used in the mentioned proposition is generated by a set

$$\{f_i \colon t \to a \mid i \in I\}$$

of constants, for some fixed set I where $a \in \mathbb{T}$ is the generating object and $t \in \mathbb{T}$ is the terminal object. It is easy to see that a $\mathbb{T}$-algebra $F \colon \mathbb{T} \to \mathbf{Set}$ is equivalent to giving a set $X \in \mathbf{Set}$ and map $f \colon I \to X$ in $\mathbf{Set}$ (i.e. I-indexed family of maps $\{F f_i \colon \{*\} \to X\}$ in $\mathbf{Set}$) and that the category $\mathbb{T}$-**alg** of $\mathbb{T}$-algebras is the coslice category $I/\mathbf{Set}$. We will write the objects as $X_f \in I/\mathbf{Set}$ and morphisms as $\alpha \colon X_f \to Y_g$, meaning a function $\alpha \colon X \to Y$ such that $\alpha \circ f = g$.

Proposition 1 of [Jib95] states that, as a consequence of Proposition 5.2.2, there is an equivalence of categories between $\mathbf{EndAdj}(I/\mathbf{Set})$ and **ConstFact** where **ConstFact** is the category with

- objects: being the pairs of maps $I \xrightarrow{f} X \xrightarrow{e} \mathbf{Set}(I, I)$ such that they factorise the constant map $\mathrm{const} \colon I \to \mathbf{Set}(I, I)$ defined by $\mathrm{const}(i)(j) = i$:

$$e \circ f = \mathrm{const}$$

- morphisms $(f, e) \to (f', e')$: are maps $\alpha \colon X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccccc} I & \xrightarrow{f} & X & \xrightarrow{e} & I^I \\ & {\scriptstyle f'} \searrow & \downarrow{\scriptstyle \alpha} & \nearrow {\scriptstyle e'} & \\ & & Y & & \end{array}$$

It further describes the monoidal structure of **ConstFact** corresponding to the monoidal structure of $\mathbf{EndAdj}(I/\mathbf{Set})$ given by the composition of endoadjunctions.

We elaborate, in two parts, the proof of Proposition 1 of [Jib95], which was only given as a sketch in the paper.

Proposition 5.2.3 (Proposition 1 of [Jib95])**.** $\mathbf{EndAdj}(I/\mathbf{Set}) \simeq \mathbf{ConstFact}$

Proof. Proposition 5.2.2 with our Lawvere theory $\mathbb{T}$ generated by an I-indexed set of constants gives:

$$\mathbf{EndAdj}(I/\mathbf{Set}) \simeq \mathrm{Prod}(\mathbb{T},\ (I/\mathbf{Set})^{\mathrm{op}})^{\mathrm{op}}. \tag{5.5}$$

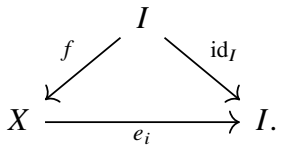
On the right, a $\mathbb{T}$-algebra structure $F\colon \mathbb{T} \to (I/\mathbf{Set})^{\mathrm{op}}$ on $X_f \in I/\mathbf{Set}$ essentially gives a set of maps

$$\{e_i^{\mathrm{op}}\colon I_{\mathrm{id}_I} \to X_f \mid i \in I\}$$

in $(I/\mathbf{Set})^{\mathrm{op}}$ where I_{id_I} represents the initial object $\mathrm{id}\colon I \to I$ in $I/\mathbf{Set}$. This is a collection of functions

$$\{e_i\colon X \to I \mid i \in I\} \tag{5.6}$$

in $\mathbf{Set}$ such that

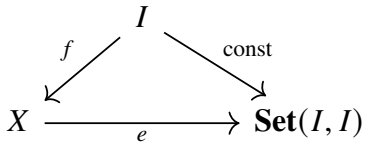


(5.7)

Expressing (5.6) as a single map $e\colon X \times I \to I$ by $(x, i) \mapsto e_i(x)$ or equivalently as a map

$$e\colon X \to \mathbf{Set}(I, I) \tag{5.8}$$

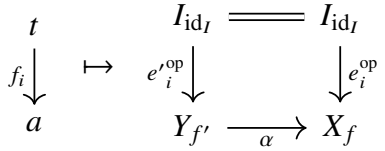
a $\mathbb{T}$-algebra $F\colon \mathbb{T} \to (I/\mathbf{Set})^{\mathrm{op}}$ corresponds to giving a pair of maps $f\colon I \to X$ and $e\colon X \to \mathbf{Set}(I, I)$ such that the following diagram commutes



(5.9)

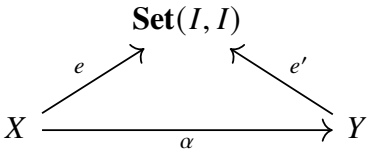
factorising the map $\mathrm{const}\colon I \to \mathbf{Set}(I, I)$ where const maps an element $j \in I$ to the constant map $\mathrm{const}(j)\colon I \to I$.

Consider another $\mathbb{T}$-algebra $G\colon \mathbb{T} \to (I/\mathbf{Set})^{\mathrm{op}}$ corresponding, by the above, to a pair $f'\colon I \to Y$ and $e'\colon Y \to \mathbf{Set}(I, I)$ and consider a $\mathbb{T}$-algebra morphism $\alpha^{\mathrm{op}}\colon F \Rightarrow G$ in $\mathrm{Prod}(\mathbb{T},\ (I/\mathbf{Set})^{\mathrm{op}})^{\mathrm{op}}$. Then by considering the naturality squares of $\alpha\colon G \Rightarrow F\colon \mathbb{T} \to (I/\mathbf{Set})^{\mathrm{op}}$, i.e. the diagrams in $(I/\mathbf{Set})^{\mathrm{op}}$ for all $i \in I$:



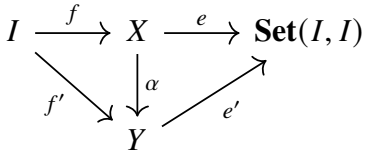
(5.10)

and expressing these in terms of the single maps e and e', we get a map $\alpha\colon X_f \to Y_{f'}$ in $I/\mathbf{Set}$ such that the below diagram in $\mathbf{Set}$



(5.11)

commutes. We thus see that a $\mathbb{T}$-algebra morphism corresponds to giving a function $\alpha\colon X \to Y$ in $\mathbf{Set}$ such that the diagram



(5.12)

commutes where the left triangle simply expresses that the map $\alpha\colon X_f \to Y_{f'}$ is a map in $(I/\mathbf{Set})$. □

Definition 5.2.1 ($R_{(e,f)}$ and $L_{(e,f)}$)**.** Consider again the specific case of (5.4) for our Lawvere theory $\mathbb{T}$ and $\mathcal{A} = \mathrm{Prod}(\mathbb{T},\ \mathbf{Set}) = I/\mathbf{Set}$. Given a $\mathbb{T}$-algebra $F\colon \mathbb{T} \to (I/\mathbf{Set})^{\mathrm{op}}$, i.e. an object $(e\colon X \to I^I,\ f\colon I \to X) \in \mathbf{ConstFact}$, we recover $R_{(e,f)}\colon I/\mathbf{Set} \to I/\mathbf{Set}$ by

$$S_s \xmapsto{R_{(e,f)}} \Big(\mathbb{T} \xrightarrow{F} (I/\mathbf{Set})^{\mathrm{op}} \xrightarrow{I/\mathbf{Set}(-,\ S_s)} \mathbf{Set}\Big). \tag{5.13}$$

That is, $R_{(e,f)}(S_s) = I/\mathbf{Set}(-,\ S_s) \circ F :\ \mathbb{T} \longrightarrow \mathbf{Set}$ maps

$$\begin{array}{ccc} t & & I_{\mathrm{id}} \\ \downarrow{\scriptstyle f_i} & \overset{F}{\mapsto} & \downarrow{\scriptstyle e_i{}^{\mathrm{op}}} \\ a & & X_f \end{array} \qquad \xmapsto{I/\mathbf{Set}(-,\ S_s)} \qquad \begin{array}{c} I/\mathbf{Set}(I_{\mathrm{id}},\ S_s) = \{s\colon I \to S\} \\ \downarrow{\scriptstyle -\circ e_i} \\ I/\mathbf{Set}(X_f,\ S_s) \end{array} \tag{5.14}$$

and it is easy to see that $R_{(e,f)}(S_s) \in I/\mathbf{Set}$ corresponds to the set $I/\mathbf{Set}(X_f,\ S_s)$ together with the map $t\colon I \to I/\mathbf{Set}(X_f,\ S_s)$ sending $i \mapsto (s \circ e_i\colon X \to S)$.

Its left adjoint $L_{(e,f)}\colon I/\mathbf{Set} \to I/\mathbf{Set}$ is defined on objects Y_{f_Y} by

$$\begin{array}{ccc} I & & I \\ \downarrow{\scriptstyle f_Y} & \overset{L_{(e,f)}}{\longmapsto} & \downarrow{\scriptstyle f_\wedge} \\ Y & & X \wedge_I Y \end{array} \tag{5.15}$$

where $X \wedge_I Y = X \times Y/(x, f_Y(i)) \sim (f_X(e(x)(i)), y)$ and $f_\wedge$ maps $i \mapsto f_X(i) \wedge f_Y(i)$, the equivalence class of $(f_X(i), f_Y(i))$.

We elaborate the sketch given in [Jib95] of the proof of the following result, describing the monoidal structure of **ConstFact**.

Proposition 5.2.4 (Proposition 1 of [Jib95])**.** *For 2 objects $(I \xrightarrow{f_X} X \xrightarrow{e} I^I)$ and $(I \xrightarrow{f_Y} Y \xrightarrow{d} I^I)$ of **ConstFact**, their induced monoidal product is given by:*

$$(I \xrightarrow{f_\wedge} X \wedge_I Y \xrightarrow{e_\wedge} I^I)$$

where $e_\wedge\colon X \wedge_I Y \to I^I$ maps $(x \wedge y) \mapsto e_X(x) \circ e_Y(y)$ and the monoidal unit is $(I \xrightarrow{\mathrm{const}} \mathrm{const}(I) \cup \{\mathrm{id}_I\} \xrightarrow{\subset} I^I)$.

Proof. Consider a composite of adjunctions

$$I/\mathbf{Set} \underset{R_{(e,f_X)}}{\overset{L_{(e,f_X)}}{\rightleftarrows}} I/\mathbf{Set} \underset{R_{(d,f_Y)}}{\overset{L_{(d,f_Y)}}{\rightleftarrows}} I/\mathbf{Set}$$

Let $R_{(d,f_Y)}$ be the right adjoint corresponding to endowing $Y_{f_Y} \in I/\mathbf{Set}$ with $\mathbb{T}$-algebra structure in $(I/\mathbf{Set})^{\mathrm{op}}$ by $F\colon \mathbb{T} \to (I/\mathbf{Set})^{\mathrm{op}}$, i.e. $(I \xrightarrow{f_Y} Y \xrightarrow{d} I^I) \in \mathbf{ConstFact}$ from which we recover $R_{(d,f_Y)}\colon I/\mathbf{Set} \to I/\mathbf{Set}$ by

$$A \xmapsto{R} \Big(\mathbb{T} \xrightarrow{F} (I/\mathbf{Set})^{\mathrm{op}} \xrightarrow{I/\mathbf{Set}(-,A)} \mathbf{Set}\Big).$$

Then since

$$\begin{aligned} UR_{(d,f_Y)}R_{(e,f_X)}(X) &= I/\mathbf{Set}(Y, R_{(e,f_X)}X) \\ &= I/\mathbf{Set}(L_{(e,f_X)}Y, X) \end{aligned}$$

the composite right adjoint $R_{(d,f_Y)} \circ R_{(e,f_X)}$ corresponds to endowing $L_{(e,f_X)}(Y_{f_Y})$ with $\mathbb{T}$-model structure $G\colon \mathbb{T} \to (I/\mathbf{Set})^{\mathrm{op}}$ in $(I/\mathbf{Set})^{\mathrm{op}}$.

Note that the composite

$$I/\mathbf{Set} \xrightarrow{R_{(e,f_X)}} I/\mathbf{Set} \xrightarrow{R_{(d,f_Y)}} I/\mathbf{Set}$$

is given by:

$$\begin{aligned} A \xmapsto{R_{(e,f_X)}} R_{(e,f_X)}(A) \xmapsto{R_{(d,f_Y)}} & \left(\mathbb{T} \xrightarrow{F} (I/\mathbf{Set})^{\mathrm{op}} \xrightarrow{I/\mathbf{Set}(-,R_{(e,f_X)}A)} \mathbf{Set}\right) \\ &\cong \left(\mathbb{T} \xrightarrow{F} (I/\mathbf{Set})^{\mathrm{op}} \xrightarrow{I/\mathbf{Set}(L_{(e,f_X)}-,A)} \mathbf{Set}\right) \\ &\cong \left(\mathbb{T} \xrightarrow{F} (I/\mathbf{Set})^{\mathrm{op}} \xrightarrow{L^{\mathrm{op}}_{(e,f_X)}} (I/\mathbf{Set})^{\mathrm{op}} \xrightarrow{I/\mathbf{Set}(-,A)} \mathbf{Set}\right). \end{aligned}$$

And the $\mathbb{T}$-algebra $G\colon \mathbb{T} \to (I/\mathbf{Set})^{\mathrm{op}}$ corresponding to the composite right adjoint $R_{(d,f_Y)} \circ R_{(e,f_X)}$ is $G = L^{\mathrm{op}}_{(e,f_X)} \circ F$.

Now, consider the left adjoint $L_{(e,f_X)}\colon I/\mathbf{Set} \to I/\mathbf{Set}$. On the initial object $(I \xrightarrow{\mathrm{id}_I} I)$ of $I/\mathbf{Set}$, the left adjoint maps to $L_{(e,f_X)}(I \xrightarrow{\mathrm{id}_I} I) = (I \xrightarrow{\mathrm{id}_\wedge} X \wedge I)$ where $\mathrm{id}_\wedge$ maps $i \mapsto f_X(i) \wedge i$. Since left adjoint preserves colimits, $(I \xrightarrow{\mathrm{id}_\wedge} X \wedge I)$ is also the initial object in $I/\mathbf{Set}$: for any object $(I \xrightarrow{f_Y} Y) \in I/\mathbf{Set}$, the function

$$\begin{aligned} \alpha\colon\ & X \wedge I \longrightarrow Y \\ & (x \wedge i) \longmapsto f_Y(e(x)(i)) \end{aligned} \tag{5.16}$$

given by $\alpha(x \wedge i) = f_Y(e(x)(i))$ makes $\alpha \circ \mathrm{id}_\wedge = f_Y$ as $\alpha(f_X(i) \wedge i) = f_Y(e(f_X(i))(i)) = f_Y(\mathrm{const}(i)(i)) = f_Y(i)$. And its uniqueness is easy to see as for any $\beta\colon X \wedge I \to Y$ such that $\beta \circ \mathrm{id}_\wedge = f_Y$, using the equivalence relation, we have $\beta(x \wedge i) = \beta(x \wedge \mathrm{id}(i)) = \beta(f_X(e(x)(i)) \wedge j) = f_Y(e(x)(i)) = \alpha(x \wedge i)$.

The $\mathbb{T}$-algebra $G = L^{\mathrm{op}}_{(e,f_X)} \circ F$ corresponding to $R_{(d,f_Y)} \circ R_{(e,f_X)}$ maps to

$$\begin{array}{ccc} t & & I_{\mathrm{id}} \\ \downarrow{\scriptstyle f_i} & \xmapsto{F} & \downarrow{\scriptstyle d_i{}^{\mathrm{op}}} \\ a & & Y_{f_Y} \end{array} \qquad \xmapsto{L^{\mathrm{op}}_{(e,f_X)}} \qquad \begin{array}{c} (X \wedge I)_{\mathrm{id}_\wedge} \\ \downarrow{\scriptstyle (\mathrm{id}_X \wedge d_i)^{\mathrm{op}}} \\ (X \wedge Y)_{f_\wedge} \end{array} \tag{5.17}$$

giving the set of maps

$$\{\mathrm{id}_X \wedge d_i\colon (X \wedge Y)_{f_\wedge} \to (X \wedge I)_{\mathrm{id}_\wedge} \mid i \in I\}$$

in $(I/\mathbf{Set})$. And by the unique map from (5.16) which exhibits, in the right triangle of the below diagram, as the unique isomorphism between two initial objects

$$\begin{array}{ccccc} & & I & & \\ & {\scriptstyle f_\wedge}\swarrow & \downarrow{\scriptstyle \mathrm{id}_\wedge} & \searrow{\scriptstyle !} & \\ X \wedge Y & \xrightarrow[\mathrm{id}_X \wedge d_i]{} & X \wedge I & \xrightarrow[\alpha = \mathrm{id}(e(-)(-))]{} & I \end{array}$$

it is again equivalent to giving the set of maps

$$\{X \wedge Y \xrightarrow{e(-)(d_i(-))} I \mid i \in I\} \tag{5.18}$$

in **Set** such that

$$\begin{array}{ccc} & I & \\ {\scriptstyle f_\wedge}\swarrow & & \| \\ X \wedge Y & \xrightarrow[e(-)(d_i(-))]{} & I. \end{array}$$

Expressing this set of maps (5.18) by the single map $e_\wedge \colon X \wedge Y \to \mathbf{Set}(I, I)$

$$x \wedge y \longmapsto \left(I \xrightarrow{d(y)} I \xrightarrow{e(x)} I \right)$$

we can see that the induced monoidal product $(I \xrightarrow{f_\wedge} X \wedge_I Y \xrightarrow{e_\wedge} I^I)$ is shown as required. □

5.3 Universal cocomplete enrichment $\underline{I/\mathbf{Set}}$ of $I/\mathbf{Set}$

Using the above, we now calculate the universal cocomplete enrichment of $I/\mathbf{Set}$, which is, the enrichment over $\mathbf{EndAjd}(I/\mathbf{Set}) \simeq \mathbf{ConstFact}$ by Proposition 4.1.1. With $L_{(e,f)}$ defined in (5.15), we have the monoidal action of **ConstFact** on $I/\mathbf{Set}$ defined as:

$$\begin{aligned} *\colon \mathbf{ConstFact} \times I/\mathbf{Set} &\longrightarrow I/\mathbf{Set} \\ (I \xrightarrow{f} X \xrightarrow{e} I^I),\ Y_s &\longmapsto L_{(e,f)}(Y_s) \end{aligned} \tag{5.19}$$

and we want for all $Y_{f_Y}, Z_{f_Z} \in I/\mathbf{Set}$, an object $\langle Y_{f_Y},\ Z_{f_Z} \rangle \in \mathbf{ConstFact}$ such that

$$I/\mathbf{Set}(L_{(e,f)}(Y_{f_Y}),\ Z_{f_Z}) \cong \mathbf{ConstFact}\left((I \xrightarrow{f} X \xrightarrow{e} I^I),\ \langle Y_{f_Y},\ Z_{f_Z} \rangle \right); \tag{5.20}$$

this will provide the homs of the **ConstFact**-enriched category $\underline{I/\mathbf{Set}}$.

So consider the left hand side of (5.20) and its adjunct transpose $L_{(e,f)} \dashv R_{(e,f)}$

$$I/\mathbf{Set}(L_{(e,f)}(Y_{f_Y}),\ Z_{f_Z}) \cong I/\mathbf{Set}(Y_{f_Y},\ R_{(e,f)}(Z_{f_Z})). \tag{5.21}$$

A map $\theta \in I/\mathbf{Set}(Y_{f_Y},\ R_{(e,f)}(Z_{f_Z}))$ is a map $\theta \colon Y \to I/\mathbf{Set}(X_{f_X}, Z_{f_Z})$ such that

$$\begin{array}{ccc} & I & \\ {\scriptstyle f_Y}\swarrow & & \searrow{\scriptstyle t} \\ Y & \xrightarrow[\theta]{} & I/\mathbf{Set}(X_{f_X}, Z_{f_Z}) \end{array} \tag{5.22}$$

commutes. This reduces to giving a map

$$Y \xrightarrow{\theta} \mathbf{Set}(X, Z) \tag{5.23}$$

satisfying the conditions $\theta(y)(f_X(i)) = f_Z(i)$ which asserts that $\theta(y)\colon X \to Z$ is a map in $I/\mathbf{Set}(X_{f_X}, Z_{f_Z})$, and $\theta(f_Y(i))(x) = f_Z\,(e(x)(i))$ which asserts commutativity of the diagram (5.22).

Writing this in different form, this reduces to giving a map

$$X \xrightarrow{\theta} \mathbf{Set}(Y, Z) \tag{5.24}$$

such that

$$\begin{aligned} &1.\ \ \theta(f_X(i))(y) = f_Z(i) \\ &2.\ \ \theta(x)(f_Y(i)) = f_Z(e(x)(i)). \end{aligned} \tag{5.25}$$

Definition 5.3.1 (Internal hom $\langle Y_{f_Y},\ Z_{f_Z}\rangle \in$ **ConstFact**)**.** Define for any $Y_{f_Y}, Z_{f_Z} \in I/\mathbf{Set}$, an object $\langle Y_{f_Y},\ Z_{f_Z}\rangle \in$ **ConstFact** to be

$$I \xrightarrow{f'} \mathbf{Set^2}(I \xrightarrow{f_Y} Y, I \xrightarrow{f_Z} Z) \xrightarrow{e'} I^I \tag{5.26}$$

where the category $\mathbf{Set^2}$ of functors $\mathbf{2} \to \mathbf{Set}$ is the arrow category whose objects are arrows in **Set** and whose morphisms are pairs of maps giving commuting squares in **Set**. The function $f'\colon I \to \mathbf{Set^2}(I \xrightarrow{f_Y} Y, I \xrightarrow{f_Z} Z)$ is defined as $i \longmapsto (I \xrightarrow{\text{const}(i)} I,\ Y \xrightarrow{\text{const}(f_Z(i))} Z)$ where $\text{const}(f_Z(i))\colon Y \to Z$ maps any $y \mapsto f_Z(i)$ giving the commuting square

$$\begin{array}{ccc} I & \xrightarrow{f_Y} & Y \\ {\scriptstyle \text{const}(i)}\downarrow & & \downarrow{\scriptstyle \text{const}(f_Z(i))} \\ I & \xrightarrow[f_Z]{} & Z. \end{array}$$

The function $e'\colon \mathbf{Set^2}(I \xrightarrow{f_Y} Y, I \xrightarrow{f_Z} Z) \to I^I$ is defined as $(I \xrightarrow{k} I,\ Y \xrightarrow{\alpha} Z) \longmapsto k$.

Consider the right hand side

$$\mathbf{ConstFact}\left((I \xrightarrow{f} X \xrightarrow{e} I^I),\ \langle Y_{f_Y},\ Z_{f_Z}\rangle\right) \tag{5.27}$$

of (5.20). A morphism $h\colon (I \xrightarrow{f} X \xrightarrow{e} I^I) \to \langle Y_{f_Y},\ Z_{f_Z}\rangle$ in **ConstFact** is a function $h\colon X \to \mathbf{Set^2}(I \xrightarrow{f_Y} Y, I \xrightarrow{f_Z} Z)$ such that

$$\begin{array}{ccccc} I & & \xrightarrow{f} & X & \xrightarrow{e} & I^I \\ & {\scriptstyle f'}\searrow & & \downarrow{\scriptstyle h} & \nearrow{\scriptstyle e'} & \\ & & \mathbf{Set^2}(I \xrightarrow{f_Y} Y, I \xrightarrow{f_Z} Z). & & & \end{array} \tag{5.28}$$

Now, writing $h(x) = (I \xrightarrow{h_1(x)} I,\ Y \xrightarrow{h_2(x)} Z)$, the data from the map h to make the right triangle commute necessarily requires $h_1(x) = e(x)\colon I \to I$ and thus reduces to giving a map $h_2\colon X \to \mathbf{Set}(Y, Z)$ such that

$$\begin{array}{ccc} I & \xrightarrow{f_Y} & Y \\ {\scriptstyle e(x)}\downarrow & & \downarrow{\scriptstyle h_2(x)} \\ I & \xrightarrow[f_Z]{} & Z \end{array} \tag{5.29}$$

i.e. $h_2(x)(f_Y(i)) = f_Z(e(x)(i))$; and then the remaining data from the left triangle requires $h_2(f(i)) = \mathrm{const}(f_Z(i))$, i.e. $h_2(f(i))(y) = f_Z(i)$.

Thus, giving a morphism $h\colon (I \xrightarrow{f} X \xrightarrow{e} I^I) \to \langle Y_{f_Y},\ Z_{f_Z}\rangle$ in **ConstFact** amounts to the data (5.24) and (5.25), hence giving

$$I/\mathbf{Set}(Y_{f_Y},\ R_{(e,f)}(Z_{f_Z})) \cong \mathbf{ConstFact}\left((I \xrightarrow{f} X \xrightarrow{e} I^I),\ \langle Y_{f_Y},\ Z_{f_Z}\rangle\right)$$

and so, the desired bijection (5.20)

$$I/\mathbf{Set}(L_{(e,f)}(Y_{f_Y}),\ Z_{f_Z}) \cong \mathbf{ConstFact}\left((I \xrightarrow{f} X \xrightarrow{e} I^I),\ \langle Y_{f_Y},\ Z_{f_Z}\rangle\right).$$

Composition morphism of $\underline{I/\mathbf{Set}}$

From this structure, we obtain a cocomplete **ConstFact**-enrichment $\underline{I/\mathbf{Set}}$ with the shown hom-object. The composition morphism

$$m\colon \langle Y_{f_Y},\ Z_{f_Z}\rangle \otimes \langle X_{f_X},\ Y_{f_Y}\rangle \to \langle X_{f_X},\ Z_{f_Z}\rangle$$

is the transpose of

$$\begin{array}{l}\big(\langle Y_{f_Y},\ Z_{f_Z}\rangle \otimes \langle X_{f_X},\ Y_{f_Y}\rangle\big) * X_{f_X}\\ \qquad\qquad \cong \big\downarrow a\\ \langle Y_{f_Y},\ Z_{f_Z}\rangle * \big(\langle X_{f_X},\ Y_{f_Y}\rangle * X_{f_X}\big) \xrightarrow{\mathrm{id}*\mathrm{ev}} \langle Y_{f_Y},\ Z_{f_Z}\rangle * Y_{f_Y} \xrightarrow{\mathrm{ev}} Z_{f_Z}\end{array}$$

under (5.20) where $\mathrm{ev}\colon \langle X_{f_X},\ Y_{f_Y}\rangle * X_{f_X} \to Y_{f_Y}$ is given by

$$\begin{array}{c} \overset{f_\wedge}{\swarrow} I \overset{f_Y}{\searrow}\\ \mathbf{Set}^{\mathbf{2}}(I \xrightarrow{f_X} X, I \xrightarrow{f_Y} Y) \wedge X \xrightarrow[\mathrm{ev}]{} Y\\ (k,\alpha) \wedge x \xmapsto{\mathrm{ev}} \alpha(x)\end{array}$$

as the transpose of $\mathrm{id}\colon \langle X_{f_X},\ Y_{f_Y}\rangle \to \langle X_{f_X},\ Y_{f_Y}\rangle$. Thus,

$$m\colon \langle Y_{f_Y},\ Z_{f_Z}\rangle \otimes \langle X_{f_X},\ Y_{f_Y}\rangle \to \langle X_{f_X},\ Z_{f_Z}\rangle \tag{5.30}$$

in **ConstFact** is the function

$$\mathbf{Set}^{\mathbf{2}}(I \xrightarrow{f_Y} Y, I \xrightarrow{f_Z} Z) \wedge \mathbf{Set}^{\mathbf{2}}(I \xrightarrow{f_X} X, I \xrightarrow{f_Y} Y) \xrightarrow{m} \mathbf{Set}^{\mathbf{2}}(I \xrightarrow{f_X} X, I \xrightarrow{f_Z} Z)$$

given by

$$(I \xrightarrow{l} I, Y \xrightarrow{\beta} Z) \wedge (I \xrightarrow{k} I, X \xrightarrow{\alpha} Y) \xmapsto{m} (I \xrightarrow{l\circ k} I, X \xrightarrow{\beta\circ\alpha} Z) \tag{5.31}$$

which makes the below diagram commute

$$\begin{array}{ccccc} I & \xrightarrow{f_\wedge} & \mathbf{Set}^{\mathbf{2}}(I \xrightarrow{f_Y} Y, I \xrightarrow{f_Z} Z) \wedge \mathbf{Set}^{\mathbf{2}}(I \xrightarrow{f_X} X, I \xrightarrow{f_Y} Y) & \xrightarrow{e_\wedge} & I^I\\ & \searrow^{f'} & \big\downarrow m & \nearrow_{e'} & \\ & & \mathbf{Set}^{\mathbf{2}}(I \xrightarrow{f_Y} Y, I \xrightarrow{f_Z} Z). & & \end{array} \tag{5.32}$$

The unit morphism is also easy to see.

Example 5.3.1. When $I = \{*\}$,

$$I/\mathbf{Set} = \mathbf{Set}_* = \mathbf{ConstFact}$$

and $\mathbf{ConstFact} = \mathbf{Set}_*$ is the monoidal category with monoidal structure given by the smash product. This case gives the familiar result that is already well known which is that the category $\mathbf{Set}_*$ is enriched over itself with the hom-objects given by

$$\Big(\mathbf{Set}_*\big((X,x),(Y,y)\big),\ \ X \xrightarrow{\text{const}(y)} Y\Big) \in \mathbf{Set}_*.$$

General ConstFact-enriched category C

A general **ConstFact**-enriched category C consists of

- a set of objects $\mathrm{ob}\, C$
- for all $X, Y \in \mathrm{ob}\, C$, the hom-object

$$(I \xrightarrow{f^{XY}} C(X,Y) \xrightarrow{e^{XY}} I^I) \in \mathbf{ConstFact}$$

- for all $X, Y, Z \in \mathrm{ob}\, C$, the composition map in **ConstFact**, which is a function

$$C(Y,Z) \wedge_I C(X,Y) \xrightarrow{m} C(X,Z) \tag{5.33}$$

such that

$$\begin{array}{ccccc} & & I & & \\ & {\scriptstyle f_\wedge}\swarrow & & \searrow{\scriptstyle f^{XZ}} & \\ C(Y,Z) \wedge_I C(X,Y) & & \xrightarrow{\quad m \quad} & & C(X,Z) \\ & {\scriptstyle e_\wedge}\searrow & & \swarrow{\scriptstyle e^{XZ}} & \\ & & I^I & & \end{array} \tag{5.34}$$

- for all $X \in \mathrm{ob}\, C$, the unit map, which is a function

$$I \coprod \{*\} \xrightarrow{i} C(X,X) \tag{5.35}$$

such that

$$\begin{array}{ccccc} & & I & & \\ & \text{inclusion}\swarrow & & \searrow{\scriptstyle f^{XX}} & \\ I \coprod \{*\} & & \xrightarrow{\quad i \quad} & & C(X,X) \\ & \text{(const,id)}\searrow & & \swarrow{\scriptstyle e^{XX}} & \\ & & I^I & & \end{array} \tag{5.36}$$

- satisfying the associativity and unitality axioms.

Remark. Giving the composition map is to give a function

$$C(Y,Z) \times C(X,Y) \xrightarrow{m} C(X,Z)$$

such that

1. $m(f^{YZ}(i), h) = f^{XZ}(i)$
2. $m(k, f^{XY}(i)) = f^{XZ}(e^{YZ}(k)(i))$
3. $e^{XZ}(m(k,h)) = e^{YZ}(k) \circ e^{XY}(h)$.

3. comes from the bottom triangle of the equation (5.34) and 1. and 2. come from the upper triangle of the equation (5.34) which is a map

$$L_{(e^{YZ}, f^{YZ})}(C(X,Y)_{f^{XY}}) \to C(X,Z)_{f^{XZ}}$$

in $I/\mathbf{Set}$. This is equivalent by the adjunction to a map

$$C(X,Y)_{f^{XY}} \to R_{(e^{YZ}, f^{YZ})}(C(X,Z)_{f^{XZ}})$$

which reduces to the data of the form (5.24) and (5.25) that gives 1. and 2.. It is easy to see that to give the unit map is to give an element

$$i(*) = \mathrm{id}_X \in C(X,X) \tag{5.37}$$

such that

4. $e^{XX}(\mathrm{id}_X) = \mathrm{id}_I$.

All in all, a **ConstFact**-enriched category C is the same as:

(i) an ordinary category C (coming from (5.33) and (5.37) alone satisfying the associativity and unitality axioms)

(ii) a functor $e\colon C \to \sum \mathbf{Set}(I,I)$ (coming from 3. and 4.) where $\sum \mathbf{Set}(I,I)$ is the full subcategory of **Set** containing only one object $I \in \mathbf{Set}$

(iii) maps $f^{XY}(i)\colon X \to Y$ in C for all $X, Y \in C$ and $i \in I$ such that

$$e^{XY}(f^{XY}(i)) = \mathrm{const}(i)$$

(iv) such that $f^{YZ}(i) \circ h = f^{XZ}(i)$ and $k \circ f^{XY}(i) = f^{YZ}(e^{YZ}(k)(i))$ (which is the data 1. and 2.)

Intuitively, a **ConstFact**-enriched category C can be thought of as a category where objects, although they aren't necessarily sets, can be considered like sets which have an I-indexed family of "basepoints" where for a map $h\colon X \to Y$ in C, the map $e^{XY}(h)\colon I \to I$ shows how h "acts on the basepoints". With this idea, $f^{XY}(i)\colon X \to Y$ is a map "sending every point of X to the i-th basepoint of Y". This intuition is apparent in our **ConstFact**-enriched category $\underline{I/\mathbf{Set}}$:

Example 5.3.2. The **ConstFact**-enriched category $\underline{I/\mathbf{Set}}$ in these terms is

(i) the full subcategory ${}^{I}\mathbf{Set}^{\mathbf{2}}$ of $\mathbf{Set}^{\mathbf{2}}$ whose objects are of the form $I \xrightarrow{f_X} X$ in **Set**. Arrows are in the form of pairs $a = (I \xrightarrow{a_1} I, X \xrightarrow{a_2} Y)$ making the diagram commute

$$\begin{array}{ccc} I & \xrightarrow{f_X} & X \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ I & \xrightarrow[f_Y]{} & Y \end{array}$$

(ii) The functor e maps

$$\begin{array}{ccc} I & \xrightarrow{f_X} & X \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle a_2} \\ I & \xrightarrow[f_Y]{} & Y \end{array} \quad \overset{e}{\mapsto} \quad \begin{array}{c} I \\ {\scriptstyle a_1}\downarrow \\ I \end{array}$$

revealing how $a_2 \colon X \to Y$ acts on basepoints.

(iii) for all objects $(I \xrightarrow{f_X} X)$ and $(I \xrightarrow{f_Y} Y)$ in ${}^I\mathbf{Set}^{\mathbf{2}}$, the map $f^{XY}(i) = (I \xrightarrow{\text{const}(i)} I,\ X \xrightarrow{\text{const}(f_Y(i))} Y)$ in ${}^I\mathbf{Set}^{\mathbf{2}}$ giving

$$\begin{array}{ccc} I & \xrightarrow{f_X} & X \\ {\scriptstyle \text{const}(i)}\downarrow & & \downarrow{\scriptstyle \text{const}(f_Y(i))} \\ I & \xrightarrow[f_Y]{} & Y \end{array}$$

(iv) which satisfies the required equations $f^{YZ}(i) \circ h = f^{XZ}(i)$ and $k \circ f^{XY}(i) = f^{YZ}(e^{YZ}(k)(i))$.

6

Monoidal Enrichments through $\mathcal{V}$-actions on $\mathcal{A}$

We investigate our findings with one step higher-tuply monoidal categories where $\mathcal{V}$ is a braided monoidal category, with braidings written with b, and $\mathcal{A}$ is a monoidal category.

In parallel with Chapter 3, we consider the action of a braided monoidal category $\mathcal{V}$ on a monoidal category $\mathcal{A}$ and show its connections to giving $\mathcal{A}$ a structure of a copowered and cocomplete monoidal $\mathcal{V}$-category. We find the appropriate conditions for such connections and present the properties of monoidal enrichments that arise in this way through action of $\mathcal{V}$ on $\mathcal{A}$.

6.1 Action of a braided monoidal category $\mathcal{V}$ on a monoidal category $\mathcal{A}$

Definition 6.1.1 ($\mathcal{V}$-action on $\mathcal{A}$)**.** An *action of a braided monoidal category* $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ *on a monoidal category* $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$, or simply a *$\mathcal{V}$-action on $\mathcal{A}$*, is a monoidal action (Definition 3.1.1) of $\mathcal{V}$ on $\mathcal{A}$ such that

- $\odot\colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ is a strong opmonoidal functor, i.e. comes equipped with

  (i) $(V \otimes W) \odot (A \boxtimes B) \xrightarrow[\cong]{\phi} (V \odot A) \boxtimes (W \odot B)$ natural in $V, W \in \mathcal{V}$ and $A, B \in \mathcal{A}$; and

  (ii) $I_{\mathcal{V}} \odot I_{\mathcal{A}} \xrightarrow[\cong]{\psi} I_{\mathcal{A}}$

  satisfying the coherence axioms for an opmonoidal functor;

- $\alpha_{W,V,X}\colon (W \otimes V) \odot X \cong W \odot (V \odot X)$ is a monoidal natural transformation, i.e. the

following diagrams commute:

$$\begin{array}{ccc}
((W \otimes V) \odot A) \boxtimes ((W' \otimes V') \odot A') & \xrightarrow{\alpha \boxtimes \alpha} & (W \odot (V \odot A)) \boxtimes (W' \odot (V' \odot A')) \\
\uparrow{\scriptstyle \phi} & & \uparrow{\scriptstyle \phi} \\
((W \otimes V) \otimes (W' \otimes V')) \odot (A \boxtimes A') & & (W \otimes W') \odot ((V \odot A) \boxtimes (V' \odot A')) \\
\uparrow{\scriptstyle (W \otimes b \otimes V') \odot (A \boxtimes A')} & & \uparrow{\scriptstyle (W \otimes W') \odot \phi} \\
((W \otimes W') \otimes (V \otimes V')) \odot (A \boxtimes A') & \xrightarrow[\alpha]{} & (W \otimes W') \odot ((V \otimes V') \odot (A \boxtimes A'))
\end{array} \tag{6.1}$$

$$\begin{array}{ccc}
I_{\mathcal{A}} & \xleftarrow{\psi} & \\
\uparrow{\scriptstyle \psi} & & \\
I_{\mathcal{V}} \odot I_{\mathcal{A}} & & I_{\mathcal{V}} \odot I_{\mathcal{A}} \\
\uparrow{\scriptstyle \cong \odot I_{\mathcal{A}}} & & \uparrow{\scriptstyle I_{\mathcal{V}} \odot \psi} \\
(I_{\mathcal{V}} \otimes I_{\mathcal{V}}) \odot I_{\mathcal{A}} & \xrightarrow[\alpha]{} & I_{\mathcal{V}} \odot (I_{\mathcal{V}} \odot I_{\mathcal{A}})
\end{array} \tag{6.2}$$

- $\lambda_X \colon I \odot X \cong X$ is a monoidal natural transformation, i.e. the following diagrams commute:

$$\begin{array}{ccc}
(I_{\mathcal{V}} \odot A) \boxtimes (I_{\mathcal{V}} \odot A') & \xrightarrow{\lambda \boxtimes \lambda} & A \boxtimes A' \\
\uparrow{\scriptstyle \phi} & & \| \\
(I_{\mathcal{V}} \otimes I_{\mathcal{V}}) \odot (A \boxtimes A') & & \| \\
\uparrow{\scriptstyle \cong \odot (A \boxtimes A')} & & \| \\
I_{\mathcal{V}} \odot (A \boxtimes A') & \xrightarrow[\lambda]{} & A \boxtimes A'
\end{array} \tag{6.3}$$

$$\begin{array}{ccc}
I_{\mathcal{A}} & & \\
\uparrow{\scriptstyle \psi} & \searrow\!\!\searrow & \\
I_{\mathcal{V}} \odot I_{\mathcal{A}} & \xrightarrow[\lambda]{} & I_{\mathcal{A}}.
\end{array} \tag{6.4}$$

Definition 6.1.2 (Category $\mathcal{V}$-$\mathbf{Act}_2(\mathcal{A})$ of $\mathcal{V}$-actions on $\mathcal{A}$)**.** Given a braided monoidal category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ and a monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$, $\mathcal{V}$-actions on $\mathcal{A}$ form a category $\mathcal{V}$-$\mathbf{Act}_2(\mathcal{A})$ with

- objects: *the $\mathcal{V}$-actions on $\mathcal{A}$*
$$\odot \colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$$
- morphisms: *monoidal* natural transformations $\overline{\sigma} \colon \odot_1 \Rightarrow \odot_2$ such that (3.6) and (3.7) hold, i.e.

$$\begin{array}{ccccc}
V \odot_1 (W \odot_1 X) & \xrightarrow{V \odot_1 \overline{\sigma}_{W,X}} & V \odot_1 (W \odot_2 X) & \xrightarrow{\overline{\sigma}_{V, W \odot_2 X}} & V \odot_2 (W \odot_2 X) \\
\cong \downarrow{\scriptstyle \alpha_1} & & & & \cong \downarrow{\scriptstyle \alpha_2} \\
(V \otimes W) \odot_1 X & & \xrightarrow[\overline{\sigma}_{V \otimes W, X}]{} & & (V \otimes W) \odot_2 X
\end{array} \tag{6.5}$$

$$\begin{array}{ccc} X & & \\ {\scriptstyle\lambda_1}\downarrow{\scriptstyle\cong} & \overset{\lambda_2}{\searrow}\ \cong & \\ I_{\mathcal{V}} \odot_1 X & \xrightarrow[\overline{\sigma}_{I_{\mathcal{V}},X}]{} & I_{\mathcal{V}} \odot_2 X \end{array} \tag{6.6}$$

commute.

Monoidality says that the below 2 diagrams commute also:

$$\begin{array}{ccc} (V \odot_1 A) \boxtimes (W \odot_1 B) & \xrightarrow{\overline{\sigma}\boxtimes\overline{\sigma}} & (V \odot_2 A) \boxtimes (W \odot_2 B) \\ {\scriptstyle\phi_1}\uparrow & & \uparrow{\scriptstyle\phi_2} \\ (V \otimes W) \odot_1 (A \boxtimes B) & \xrightarrow[\overline{\sigma}]{} & (V \otimes W) \odot_2 (A \boxtimes B) \end{array} \tag{6.7}$$

$$\begin{array}{ccc} I_{\mathcal{A}} & \overset{\psi_2}{\nwarrow} & \\ {\scriptstyle\psi_1}\uparrow & & \\ I_{\mathcal{V}} \odot_1 I_{\mathcal{A}} & \xrightarrow[\overline{\sigma}]{} & I_{\mathcal{V}} \odot_2 I_{\mathcal{A}} \end{array} \tag{6.8}$$

6.2 Closed $\mathcal{V}$-actions on $\mathcal{A}$ to copowered monoidal $\mathcal{V}$-enrichments preserving copowers

Definition 6.2.1 (Category of right-closed $\mathcal{V}$-actions on $\mathcal{A}$)**.** We write $\mathcal{V}\text{-}\mathbf{Act}_2(\mathcal{A})_{\text{closed}}$ for the full subcategory of $\mathcal{V}\text{-}\mathbf{Act}_2(\mathcal{A})$ consisting of those right-closed $\mathcal{V}$-actions on $\mathcal{A}$ $\odot\colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ with a chosen right adjoint $\langle C, -\rangle\colon \mathcal{A} \to \mathcal{V}$ for each $(- \odot C)\colon \mathcal{V} \to \mathcal{A}$.

Proposition 6.2.1 (Enrichment through $\mathcal{V}$-actions on $\mathcal{A}$)**.** *Consider a braided monoidal category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ and a monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$. Suppose the braided monoidal category $\mathcal{V}$ is also monoidally closed and consider an object $\odot\colon \mathcal{V}\times\mathcal{A} \to \mathcal{A}$ of $\mathcal{V}\text{-}\mathbf{Act}_2(\mathcal{A})_{\text{closed}}$ with the chosen right adjoint written $\underline{\mathcal{A}}(C, -)\colon \mathcal{A} \to \mathcal{V}$ for each $C \in \mathcal{A}$ yielding isomorphisms*

$$\mathcal{A}(V \odot A, B) \cong \mathcal{V}(V, \underline{\mathcal{A}}(A, B)) \tag{6.9}$$

for each $\mathcal{V} \in \mathcal{V}$ and $A, B \in \mathcal{A}$. Then the copowered $\mathcal{V}$-category $\underline{\mathcal{A}}$, that is, the enrichment of $\mathcal{A}$ via Proposition 3.3.3, is a monoidal $\mathcal{V}$-category by the $\mathcal{V}$-functor,

$$\boxtimes\colon \underline{\mathcal{A}} \otimes \underline{\mathcal{A}} \to \underline{\mathcal{A}} \tag{6.10}$$

(where $\underline{\mathcal{A}} \otimes \underline{\mathcal{A}}$ is the tensor product $\mathcal{V}$-category) defined as:

- *on objects:* $(A, B) \mapsto A \boxtimes B$
- *on hom-objects:*

$$\boxtimes_{(A,B),(A',B')}\colon \underline{\mathcal{A}}(A, A') \otimes \underline{\mathcal{A}}(B, B') \longrightarrow \underline{\mathcal{A}}(A \boxtimes B, A' \boxtimes B') \tag{6.11}$$

is defined as the transpose of

$$\begin{array}{ccc} (\underline{\mathcal{A}}(A, A') \otimes \underline{\mathcal{A}}(B, B')) \odot (A \boxtimes B) & & (A' \boxtimes B') \\ {\scriptstyle\phi}\downarrow & \overset{\text{ev}\boxtimes\text{ev}}{\nearrow} & \\ (\underline{\mathcal{A}}(A, A') \odot A) \boxtimes (\underline{\mathcal{A}}(B, B') \odot B) & & \end{array} \tag{6.12}$$

under (6.9).

The coherence $\mathcal{V}$-natural isomorphisms of this monoidal structure have components

- $a\colon I_{\mathcal{V}} \to \underline{\mathcal{A}}((A \boxtimes B) \boxtimes C, A \boxtimes (B \boxtimes C))$
- $r\colon I_{\mathcal{V}} \to \underline{\mathcal{A}}(A \boxtimes I_{\mathcal{A}}, A)$
- $l\colon I_{\mathcal{V}} \to \underline{\mathcal{A}}(I_{\mathcal{A}} \boxtimes A, A)$

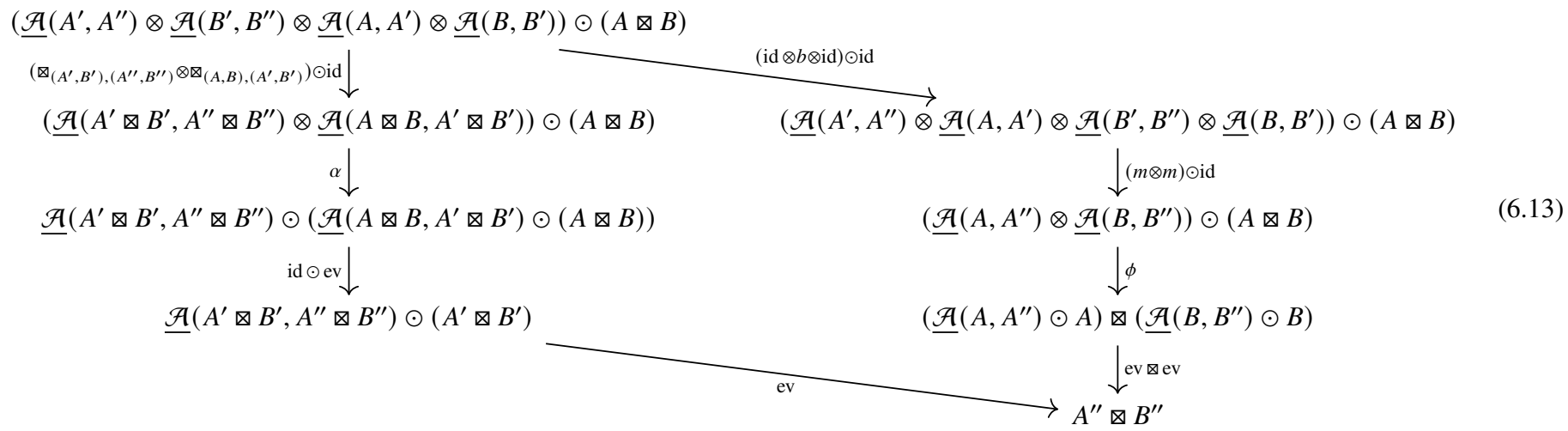
given by transposing the monoidal associativity and unitality of $\mathcal{A}$ under the bijections

$$\mathcal{A}(X,Y) \cong \mathcal{A}(I_{\mathcal{V}} \odot X, Y) \cong \mathcal{V}(I_{\mathcal{V}}, \underline{\mathcal{A}}(X,Y))$$

(i.e. (3.18) from $\underline{\mathcal{A}}_0 \cong \mathcal{A}$).

(The proof starts on the next page due to landscape format.)

Proof. The preservation of compositions for (6.10), expressed through tranposes under (6.9), is the below diagram



(6.13)

which commutes since, starting from the right-hand side, we have that

$$
\begin{aligned}
(\mathrm{ev}\boxtimes\mathrm{ev})\circ\phi\circ((m\otimes m)\odot\mathrm{id})\circ((\mathrm{id}\otimes b\otimes\mathrm{id})\odot\mathrm{id}) &= (\mathrm{ev}\boxtimes\mathrm{ev})\circ((m\odot\mathrm{id})\boxtimes(m\odot\mathrm{id}))\circ\phi\circ((\mathrm{id}\otimes b\otimes\mathrm{id})\odot\mathrm{id}) && \text{(from naturality of } \phi)\\
&= (\mathrm{ev}\boxtimes\mathrm{ev})\circ((\mathrm{id}\odot\mathrm{ev})\boxtimes(\mathrm{id}\odot\mathrm{ev}))\circ(\alpha\boxtimes\alpha)\circ\phi\circ((\mathrm{id}\otimes b\otimes\mathrm{id})\odot\mathrm{id}) && \text{(from definition of } m)\\
&= (\mathrm{ev}\boxtimes\mathrm{ev})\circ((\mathrm{id}\odot\mathrm{ev})\boxtimes(\mathrm{id}\odot\mathrm{ev}))\circ\phi\circ(\mathrm{id}\odot\phi)\circ\alpha && \text{(from 6.1)}\\
&= (\mathrm{ev}\boxtimes\mathrm{ev})\circ\phi\circ((\mathrm{id}\otimes\mathrm{id})\odot(\mathrm{ev}\boxtimes\mathrm{ev}))\circ(\mathrm{id}\odot\phi)\circ\alpha && \text{(from naturality of } \phi)
\end{aligned}
$$

which is the composite

$$
\begin{array}{ccccc}
(\underline{\mathcal{A}}(A',A'')\otimes\underline{\mathcal{A}}(B',B'')\otimes\underline{\mathcal{A}}(A,A')\otimes\underline{\mathcal{A}}(B,B'))\odot(A\boxtimes B) & \xrightarrow{\alpha} & (\underline{\mathcal{A}}(A',A'')\otimes\underline{\mathcal{A}}(B',B''))\odot\Big((\underline{\mathcal{A}}(A,A')\otimes\underline{\mathcal{A}}(B,B'))\odot(A\boxtimes B)\Big) & & \\
 & & \xleftarrow{\mathrm{id}\odot\phi} & & \\
(\underline{\mathcal{A}}(A',A'')\otimes\underline{\mathcal{A}}(B',B''))\odot\Big((\underline{\mathcal{A}}(A,A')\odot A)\boxtimes(\underline{\mathcal{A}}(B,B')\odot B)\Big) & & \xrightarrow{(\mathrm{id}\otimes\mathrm{id})\odot(\mathrm{ev}\boxtimes\mathrm{ev})} & & (\underline{\mathcal{A}}(A',A'')\otimes\underline{\mathcal{A}}(B',B''))\odot(A'\boxtimes B') \\
 & & & \xleftarrow{\phi} & \\
(\underline{\mathcal{A}}(A',A'')\odot A')\boxtimes(\underline{\mathcal{A}}(B',B'')\odot B') & & \xrightarrow{\qquad\qquad\mathrm{ev}\boxtimes\mathrm{ev}\qquad\qquad} & & A''\boxtimes B''
\end{array}
$$

which is by the definition of (6.11) equal to

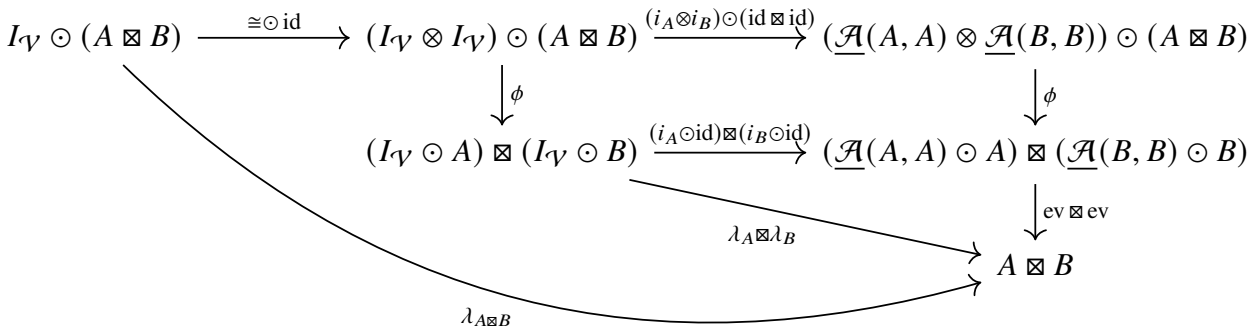
$$
\begin{array}{ccccc}
(\underline{\mathcal{A}}(A',A'')\otimes\underline{\mathcal{A}}(B',B'')\otimes\underline{\mathcal{A}}(A,A')\otimes\underline{\mathcal{A}}(B,B'))\odot(A\boxtimes B) & \xrightarrow{\alpha} & (\underline{\mathcal{A}}(A',A'')\otimes\underline{\mathcal{A}}(B',B''))\odot\Big((\underline{\mathcal{A}}(A,A')\otimes\underline{\mathcal{A}}(B,B'))\odot(A\boxtimes B)\Big) & & \\
 & & \xleftarrow{\boxtimes_{(A',B'),(A'',B'')}\odot(\boxtimes_{(A,B),(A',B')}\odot\mathrm{id})} & & \\
\underline{\mathcal{A}}(A'\boxtimes B',A''\boxtimes B'')\odot\Big(\underline{\mathcal{A}}(A\boxtimes B,A'\boxtimes B')\odot(A\boxtimes B)\Big) & & \xrightarrow{\qquad\mathrm{id}\odot\mathrm{ev}\qquad} & & \underline{\mathcal{A}}(A'\boxtimes B',A''\boxtimes B'')\odot(A'\boxtimes B') \\
 & & & \xleftarrow{\mathrm{ev}} & \\
A''\boxtimes B''. & & & &
\end{array}
$$

And this is, by the naturality of α, equal to the left-hand side of (6.13).

The preservation of units is the below diagram

$$
\begin{array}{ccccc}
I_{\mathcal{V}} & \xrightarrow{\cong} & I_{\mathcal{V}}\otimes I_{\mathcal{V}} & \xrightarrow{i_A\otimes i_B} & \underline{\mathcal{A}}(A,A)\otimes\underline{\mathcal{A}}(B,B) \\
 & \searrow^{i_{A\boxtimes B}} & & & \downarrow{\scriptstyle\boxtimes_{(A,B),(A,B)}} \\
 & & & & \underline{\mathcal{A}}(A\boxtimes B,A\boxtimes B)
\end{array}
\tag{6.14}
$$

which transposes to the outside of the following diagram under (6.9):



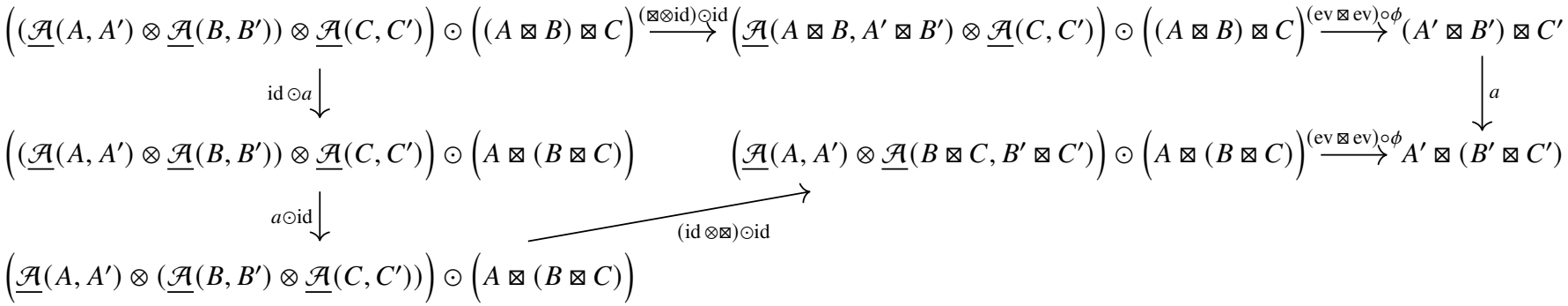
(6.15)

which commutes by the naturality of ϕ, the definition of i_A and i_B and by λ being a monoidal natural transformation (6.3).

We now show that the components $a\colon I_{\mathcal{V}} \to \underline{\mathcal{A}}((A \boxtimes B) \boxtimes C, A \boxtimes (B \boxtimes C))$ found as the transposes, under (6.9), of the monoidal associativity constraints of $\mathcal{A}$

$$a\colon (A \boxtimes B) \boxtimes C \xrightarrow{\cong} A \boxtimes (B \boxtimes C)$$

are $\mathcal{V}$-natural. The $\mathcal{V}$-naturality transposed to $\mathcal{A}$ is given by



(6.16)

which commutes, since, starting from the right-hand side, we have

$$\begin{aligned} a \circ (\mathrm{ev} \boxtimes \mathrm{ev}) \circ \phi \circ ((\boxtimes \otimes \mathrm{id}) \odot (\mathrm{id} \boxtimes \mathrm{id})) &= a \circ (\mathrm{ev} \boxtimes \mathrm{ev}) \circ ((\boxtimes \odot \mathrm{id}) \boxtimes (\mathrm{id} \odot \mathrm{id})) \circ \phi && \text{(from naturality of } \phi) \\ &= a \circ ((\mathrm{ev} \boxtimes \mathrm{ev}) \boxtimes \mathrm{ev}) \circ (\phi \boxtimes \mathrm{id}) \circ \phi && \text{(by definition of ev and } \boxtimes_{(A,B),(A',B')} \text{ from (6.11))} \\ &= (\mathrm{ev} \boxtimes (\mathrm{ev} \boxtimes \mathrm{ev})) \circ a \circ (\phi \boxtimes \mathrm{id}) \circ \phi && \text{(from naturality of } a) \end{aligned}$$

which is the composite:

$$\begin{array}{c} \Big((\underline{\mathcal{A}}(A,A') \otimes \underline{\mathcal{A}}(B,B')) \otimes \underline{\mathcal{A}}(C,C')\Big) \odot \Big((A \boxtimes B) \boxtimes C\Big) \xrightarrow{\ \phi\ } \Big((\underline{\mathcal{A}}(A,A') \otimes \underline{\mathcal{A}}(B,B')) \odot (A \boxtimes B)\Big) \boxtimes \Big(\underline{\mathcal{A}}(C,C') \odot C\Big) \\ \xleftarrow{\ \phi \boxtimes \mathrm{id}\ } \\ \Big((\underline{\mathcal{A}}(A,A') \odot A) \boxtimes (\underline{\mathcal{A}}(B,B') \odot B)\Big) \boxtimes \Big(\underline{\mathcal{A}}(C,C') \odot C\Big) \xrightarrow{\ a\ } (\underline{\mathcal{A}}(A,A') \odot A) \boxtimes \Big((\underline{\mathcal{A}}(B,B') \odot B) \boxtimes (\underline{\mathcal{A}}(C,C') \odot C)\Big) \\ \xleftarrow{\ \mathrm{ev} \boxtimes (\mathrm{ev} \boxtimes \mathrm{ev})\ } \\ A' \boxtimes (B' \boxtimes C'). \end{array} \tag{6.17}$$

Now the left-hand side of (6.16) becomes

$$\begin{aligned} (\mathrm{ev} \boxtimes \mathrm{ev}) \circ \phi \circ ((\mathrm{id} \otimes \boxtimes) \odot (\mathrm{id} \boxtimes \mathrm{id})) \circ (a \odot a) &= (\mathrm{ev} \boxtimes \mathrm{ev}) \circ ((\mathrm{id} \odot \mathrm{id}) \boxtimes (\boxtimes \odot \mathrm{id})) \circ \phi \circ (a \odot a) && \text{(from naturality of } \phi) \\ &= (\mathrm{ev} \boxtimes (\mathrm{ev} \boxtimes \mathrm{ev})) \circ (\mathrm{id} \boxtimes \phi) \circ \phi \circ (a \odot a) && \text{(by definition of } \boxtimes_{(B,C),(B',C')} \text{ from (6.11) and ev)} \end{aligned}$$

which is:

$$\begin{array}{c} \Big((\underline{\mathcal{A}}(A,A') \otimes \underline{\mathcal{A}}(B,B')) \otimes \underline{\mathcal{A}}(C,C')\Big) \odot \Big((A \boxtimes B) \boxtimes C\Big) \xrightarrow{\ a \odot a\ } \Big(\underline{\mathcal{A}}(A,A') \otimes (\underline{\mathcal{A}}(B,B') \otimes \underline{\mathcal{A}}(C,C'))\Big) \odot \Big(A \boxtimes (B \boxtimes C)\Big) \\ \xleftarrow{\ \phi\ } \\ \Big(\underline{\mathcal{A}}(A,A') \odot A\Big) \boxtimes \Big((\underline{\mathcal{A}}(B,B') \otimes \underline{\mathcal{A}}(C,C')) \odot (B \boxtimes C)\Big) \xrightarrow{\ \mathrm{id} \boxtimes \phi\ } \Big(\underline{\mathcal{A}}(A,A') \odot A\Big) \boxtimes \Big((\underline{\mathcal{A}}(B,B') \odot B) \boxtimes (\underline{\mathcal{A}}(C,C') \odot C)\Big) \\ \xleftarrow{\ \mathrm{ev} \boxtimes (\mathrm{ev} \boxtimes \mathrm{ev})\ } \\ A' \boxtimes (B' \boxtimes C'). \end{array} \tag{6.18}$$

And (6.17) and (6.18) are equal since

$$\begin{array}{ccc}
\Big((\underline{\mathcal{A}}(A,A')\otimes\underline{\mathcal{A}}(B,B'))\otimes\underline{\mathcal{A}}(C,C')\Big)\odot\Big((A\boxtimes B)\boxtimes C\Big) & \xrightarrow{a\odot a} & \Big(\underline{\mathcal{A}}(A,A')\otimes(\underline{\mathcal{A}}(B,B')\otimes\underline{\mathcal{A}}(C,C'))\Big)\odot\Big(A\boxtimes(B\boxtimes C)\Big) \\
\Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\phi} \\
\Big((\underline{\mathcal{A}}(A,A')\otimes\underline{\mathcal{A}}(B,B'))\odot(A\boxtimes B)\Big)\boxtimes\Big(\underline{\mathcal{A}}(C,C')\odot C\Big) & & \Big(\underline{\mathcal{A}}(A,A')\odot A\Big)\boxtimes\Big((\underline{\mathcal{A}}(B,B')\otimes\underline{\mathcal{A}}(C,C'))\odot(B\boxtimes C)\Big) \\
\Big\downarrow{\scriptstyle\phi\boxtimes\mathrm{id}} & & \Big\downarrow{\scriptstyle\mathrm{id}\boxtimes\phi} \\
\Big((\underline{\mathcal{A}}(A,A')\odot A)\boxtimes(\underline{\mathcal{A}}(B,B')\odot B)\Big)\boxtimes\Big(\underline{\mathcal{A}}(C,C')\odot C\Big) & \xrightarrow[a]{} & \Big(\underline{\mathcal{A}}(A,A')\odot A\Big)\boxtimes\Big((\underline{\mathcal{A}}(B,B')\odot B)\boxtimes(\underline{\mathcal{A}}(C,C')\odot C)\Big)
\end{array}$$

commutes from the coherence axiom for $\odot\colon \mathcal{V}\times\mathcal{A}\to\mathcal{A}$ to be a strong opmonoidal functor.

We also have a $\mathcal{V}$-natural isomorphism $r\colon (-\boxtimes I_{\mathcal{A}})\Rightarrow \mathrm{id}\colon \underline{\mathcal{A}}\to\underline{\mathcal{A}}$

$$\begin{array}{ccccc}
 & & (-\boxtimes I_{\mathcal{A}}) & & \\
 & \nearrow & r\Downarrow & \searrow & \\
\underline{\mathcal{A}} & & & & \underline{\mathcal{A}} \\
 & \searrow & & \nearrow & \\
 & & \mathrm{id} & &
\end{array} \tag{6.19}$$

where $(-\boxtimes I_{\mathcal{A}})\colon \underline{\mathcal{A}}\to\underline{\mathcal{A}}$ is the partial $\mathcal{V}$-functor (Example 2.2.3) of $\boxtimes\colon \underline{\mathcal{A}}\otimes\underline{\mathcal{A}}\to\underline{\mathcal{A}}$ (6.10), that is, on objects it maps $A\mapsto A\boxtimes I_{\mathcal{A}}$ and on hom-objects it is

$$\underline{\mathcal{A}}(A,B)\xrightarrow[\cong]{r^{-1}}\underline{\mathcal{A}}(A,B)\otimes I_{\mathcal{V}}\xrightarrow{\mathrm{id}\otimes i_I}\underline{\mathcal{A}}(A,B)\otimes\underline{\mathcal{A}}(I_{\mathcal{A}},I_{\mathcal{A}})\xrightarrow{\boxtimes_{(A,I),(B,I)}}\underline{\mathcal{A}}(A\boxtimes I_{\mathcal{A}},B\boxtimes I_{\mathcal{A}}).$$

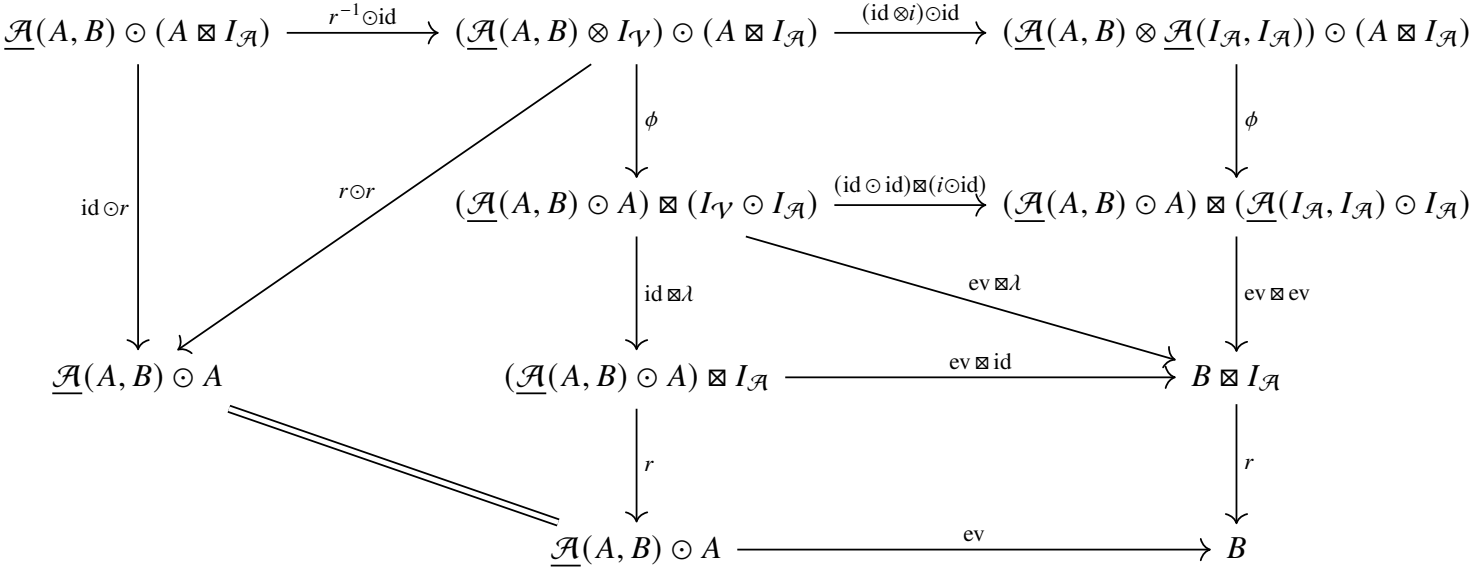
Its components $r\colon I_{\mathcal{V}}\to\underline{\mathcal{A}}(A\boxtimes I_{\mathcal{A}},A)$ in $\mathcal{V}$ are given by the transposes under (6.9)

$$\mathcal{A}(A\boxtimes I_{\mathcal{A}},A)\cong\mathcal{A}(I_{\mathcal{V}}\odot(A\boxtimes I_{\mathcal{A}}),A)\cong\mathcal{V}(I_{\mathcal{V}},\underline{\mathcal{A}}(A\boxtimes I_{\mathcal{A}},A))$$

of the monoidal right unitors of $\mathcal{A}$,

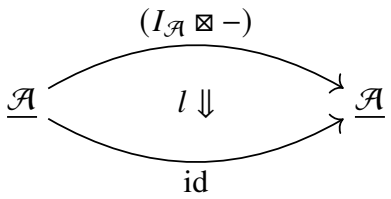
$$r\colon A\boxtimes I_{\mathcal{A}}\to\mathcal{A}.$$

The $\mathcal{V}$-naturality transposed to $\mathcal{A}$ is the outer diagram below



which commutes since the right inner diagrams commute by the naturality of ϕ, the definition of $i\colon I_{\mathcal{V}} \to \underline{\mathcal{A}}(I_{\mathcal{A}}, I_{\mathcal{A}})$ and $\mathrm{ev}\colon \underline{\mathcal{A}}(A,B) \odot A \to B$ and by the naturality of r. And the left inner diagram commutes by $\lambda = \psi$ from (6.4) and by the coherence axiom for $\odot\colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ to be an opmonoidal functor.

Likewise, we have the $\mathcal{V}$-natural isomorphism $l\colon (I_{\mathcal{A}} \boxtimes -) \Rightarrow \mathrm{id}\colon \underline{\mathcal{A}} \to \underline{\mathcal{A}}$



(6.20)

given by the components $l\colon I_{\mathcal{V}} \to \underline{\mathcal{A}}(I_{\mathcal{A}} \boxtimes A, A)$ as the transposes of the monoidal left unitor

$$l\colon I_{\mathcal{A}} \boxtimes A \to A$$

in $\mathcal{A}$. Its $\mathcal{V}$-naturality is obtained in the similar manner.

So the copowered $\mathcal{V}$-category $\underline{\mathcal{A}}$ is a monoidal $\mathcal{V}$-category. □

This enrichment $\underline{\mathcal{A}}$ of a given monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$ via a $\mathcal{V}$-action on $\mathcal{A}$ satisfies several properties. First, it is a *monoidal enrichment of* $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$. Second, the monoidal $\mathcal{V}$-functor $\boxtimes\colon \underline{\mathcal{A}} \otimes \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ preserves copowers in each variable. We outline precisely what this means.

Definition 6.2.2 (Monoidal enrichment of a monoidal category)**.** For a braided monoidal category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$, we say a monoidal $\mathcal{V}$-category $(C, \boxtimes, I)$ is a *monoidal enrichment* of an ordinary monoidal category $(\mathcal{A}, \bullet, I_{\mathcal{A}})$ if $\theta\colon C_0 \cong \mathcal{A}$ via an identity-on-objects strict monoidal functor θ. Strict monoidality means that:

$$\begin{array}{ccccc} C_0 \times C_0 & \xrightarrow{\chi} & (C \otimes C)_0 & \xrightarrow{\boxtimes_0} & C_0 \\ {\scriptstyle \theta\times\theta}\downarrow & & & & \downarrow{\scriptstyle \theta} \\ \mathcal{A} \times \mathcal{A} & & \xrightarrow[\bullet]{} & & \mathcal{A} \end{array} \tag{6.21}$$

commutes and

$$\theta(I) = I_{\mathcal{A}}. \tag{6.22}$$

Remark 6.2.1 (Theorem 5.7.1 of [Cru08])**.** For a braided monoidal base category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ and a monoidal $\mathcal{V}$-category $(C, \boxtimes, I)$, the underlying category C_0 is a monoidal category by the functor

$$C_0 \times C_0 \xrightarrow{\chi} (C \otimes C)_0 \xrightarrow{\boxtimes_0} C_0, \tag{6.23}$$

where $\chi\colon C_0 \times C_0 \to (C \otimes C)_0$ is identity on objects and on hom-sets, maps the pair

$$(I_{\mathcal{V}} \xrightarrow{f} C(X, X'),\ \ I_{\mathcal{V}} \xrightarrow{g} C(Y, Y'))$$

to

$$I_{\mathcal{V}} \cong I_{\mathcal{V}} \otimes I_{\mathcal{V}} \xrightarrow{f\otimes g} C(X, X') \otimes C(Y, Y').$$

This is an application of the result (Theorem 5.7.1) in Chapter 5 of [Cru08]: if $F\colon \mathcal{V} \to \mathcal{W}$ is a braided lax monoidal functor between braided monoidal base categories, then the change of base 2-functor from Theorem 2.3.1

$$F_*\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{W}\text{-}\mathbf{Cat}$$

is a (lax) monoidal 2-functor, and hence takes monoidal $\mathcal{V}$-categories to monoidal $\mathcal{W}$-categories.

In our case, for any braided monoidal category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$, the lax monoidal functor

$$\mathcal{V}(I_{\mathcal{V}}, -)\colon \mathcal{V} \to \mathbf{Set}$$

is braided and hence the change of base

$$(\ )_0\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathbf{Cat}$$

is a monoidal 2-functor, thus sending monoidal $\mathcal{V}$-categories to monoidal categories.

Proposition 6.2.2. *The monoidal $\mathcal{V}$-category $(\underline{\mathcal{A}}, \boxtimes, I_{\mathcal{A}})$ from Proposition 6.2.1 is a monoidal enrichment of the given monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$.*

Proof. We already have the identity-on-objects isomorphism of categories $\theta\colon \underline{\mathcal{A}}_0 \cong \mathcal{A}$ that is on hom-sets given by

$$\theta_{X,Y}\colon \mathcal{V}(I, \underline{\mathcal{A}}(X,Y)) \cong \mathcal{A}(I \odot X, Y) \cong \mathcal{A}(X,Y).$$

We now show (6.21) commutes, i.e. that

$$\begin{array}{ccccc} \underline{\mathcal{A}}_0 \times \underline{\mathcal{A}}_0 & \xrightarrow{\chi} & (\underline{\mathcal{A}} \otimes \underline{\mathcal{A}})_0 & \xrightarrow{\boxtimes_0} & \underline{\mathcal{A}}_0 \\ \downarrow{\scriptstyle \theta\times\theta} & & & & \downarrow{\scriptstyle \theta} \\ \mathcal{A} \times \mathcal{A} & & \xrightarrow[\boxtimes]{} & & \mathcal{A} \end{array}$$

commutes. This is obvious on objects $(X,Y) \mapsto X \boxtimes Y$. And on morphisms, we must show commutativity in:

$$\begin{array}{ccc} \mathcal{V}(I, \underline{\mathcal{A}}(X,X')) \times \mathcal{V}(I, \underline{\mathcal{A}}(Y,Y')) & \xrightarrow{\chi} & \mathcal{V}(I, \underline{\mathcal{A}}(X,X') \otimes \underline{\mathcal{A}}(Y,Y')) \\ & & \downarrow{\scriptstyle \boxtimes_{(X,Y),(X',Y')} \circ -} \\ \downarrow{\scriptstyle \theta\times\theta} & & \mathcal{V}(I, \underline{\mathcal{A}}(X \boxtimes Y, X' \boxtimes Y')) \\ & & \downarrow{\scriptstyle \theta} \\ \mathcal{A}(X,X') \times \mathcal{A}(Y,Y') & \xrightarrow[\boxtimes]{} & \mathcal{A}(X \boxtimes Y, X' \boxtimes Y'). \end{array}$$

The top-right composites maps the pair $(I \xrightarrow{f} \underline{\mathcal{A}}(X,X'),\ I \xrightarrow{g} \underline{\mathcal{A}}(Y,Y'))$ to

$$\begin{array}{l} X \boxtimes Y \xrightarrow{\lambda^{-1}} I \odot (X \boxtimes Y) \xrightarrow{\cong \odot \mathrm{id}} (I \otimes I) \odot (X \boxtimes Y) \xrightarrow{(f\otimes g)\odot \mathrm{id}} (\underline{\mathcal{A}}(X,X') \otimes \underline{\mathcal{A}}(Y,Y')) \odot (X \boxtimes Y) \\ \qquad\qquad \downarrow{\scriptstyle \phi} \\ \qquad\qquad (\underline{\mathcal{A}}(X,X') \odot X) \boxtimes (\underline{\mathcal{A}}(Y,Y') \odot Y) \\ \qquad\qquad \downarrow{\scriptstyle \mathrm{ev} \boxtimes \mathrm{ev}} \\ \qquad\qquad X' \boxtimes Y' \end{array}$$

and the bottom-left composites maps (f,g) to

$$X \boxtimes Y \xrightarrow{\lambda^{-1}\boxtimes\lambda^{-1}} (I \odot X) \boxtimes (I \odot Y) \xrightarrow{\overline{f}\boxtimes\overline{g}} X' \boxtimes Y'.$$

These are equal by commutativity of the diagram:

$$\begin{array}{l} X \boxtimes Y \xrightarrow{\lambda^{-1}} I \odot (X \boxtimes Y) \xrightarrow{\cong \odot \mathrm{id}} (I \otimes I) \odot (X \boxtimes Y) \xrightarrow{(f\otimes g)\odot \mathrm{id}} (\underline{\mathcal{A}}(X,X') \otimes \underline{\mathcal{A}}(Y,Y')) \odot (X \boxtimes Y) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \phi} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \phi} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \mathrm{ev} \boxtimes \mathrm{ev}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad X' \boxtimes Y' \end{array}$$

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \phi} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \mathrm{ev} \boxtimes \mathrm{ev}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad X' \boxtimes Y' \end{array}$$

wherein the inner rectangle commutes by naturality of ϕ and the left-upper triangle commutes by (6.3).

Now (6.22) is clear, since both monoidal $\mathcal{V}$-categories use the same unit $I_{\mathcal{A}}$. □

Definition 6.2.3 (Preserves copowers)**.** Let the base category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ be a braided and closed monoidal category and let $(C, \boxtimes, I)$ be a copowered monoidal $\mathcal{V}$-category with a provided choice of copowers written $V \odot X$. The tensor $\mathcal{V}$-functor

$$\boxtimes \colon C \otimes C \to C$$

preserves copowers in each variable if

- $$\mu \colon V \odot (A \boxtimes B) \to (V \odot A) \boxtimes B \tag{6.24}$$
  given by the transpose of
  $$V \xrightarrow{\eta_{V,A}} C(A, V \odot A) \xrightarrow{(-\boxtimes B)_{A, V\odot A}} C(A \boxtimes B, (V \odot A) \boxtimes B) \tag{6.25}$$
  under the bijection $C_0(V \odot X, Y) \cong \mathcal{V}(V, C(X, Y))$ is an isomorphism in C_0
- and
  $$\pi \colon V \odot (A \boxtimes B) \to A \boxtimes (V \odot B) \tag{6.26}$$
  given by the transpose of
  $$V \xrightarrow{\eta_{V,B}} C(B, V \odot B) \xrightarrow{(A\boxtimes -)_{B, V\odot B}} C(A \boxtimes B, A \boxtimes (V \odot B)) \tag{6.27}$$
  under the bijection $C_0(V \odot X, Y) \cong \mathcal{V}(V, C(X, Y))$ is an isomorphism in C_0

where $V \xrightarrow{\eta_{V,A}} C(A, V \odot A)$ here is the transpose of $\mathrm{id} \colon V \odot A \to V \odot A$ in C_0.

Proposition 6.2.3. *In Definition 6.2.3, μ and π defined by (6.24) and (6.26) are compatible with $\lambda \colon I_{\mathcal{V}} \odot X \cong X$ in C_0 of the monoidal action $\odot \in \mathcal{V}$-**Act**$(C_0)_{\text{closed}}$ shown in Lemma 3.3.1, i.e.*

$$\begin{array}{ccc} I_{\mathcal{V}} \odot (A \boxtimes B) & \xrightarrow{\mu} & (I_{\mathcal{V}} \odot A) \boxtimes B \\ & \searrow^{\lambda} & \downarrow{\scriptstyle \lambda\boxtimes \mathrm{id}} \\ & & A \boxtimes B \end{array} \tag{6.28}$$

and

$$\begin{array}{ccc} I_{\mathcal{V}} \odot (A \boxtimes B) & \xrightarrow{\pi} & A \boxtimes (I_{\mathcal{V}} \odot B) \\ & \searrow^{\lambda} & \downarrow{\scriptstyle \mathrm{id}\boxtimes\lambda} \\ & & A \boxtimes B \end{array} \tag{6.29}$$

commute.

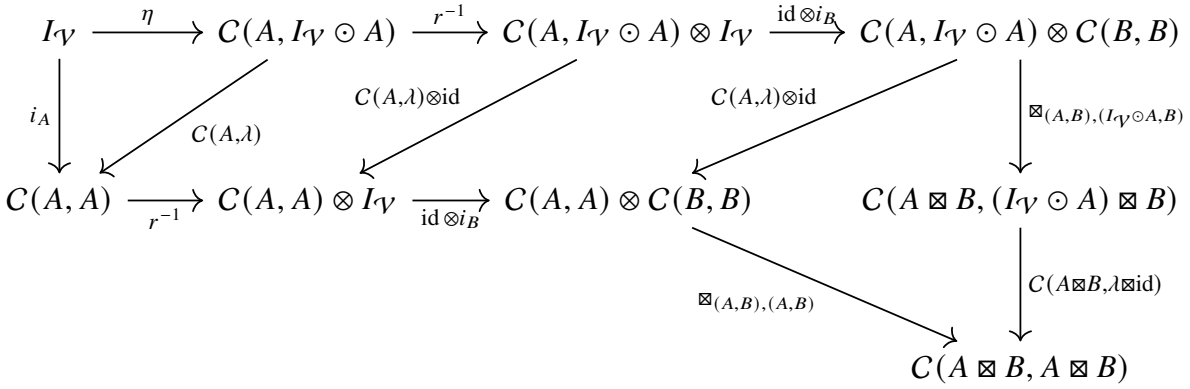
Proof. (6.28) transposed under the bijection of the form $C_0(V \odot X, Y) \cong \mathcal{V}(V, C(X, Y))$ gives

$$\begin{array}{ccccccc} I_{\mathcal{V}} & \xrightarrow{\eta} & C(A, I_{\mathcal{V}} \odot A) & \xrightarrow{r^{-1}} & C(A, I_{\mathcal{V}} \odot A) \otimes I_{\mathcal{V}} & \xrightarrow{\mathrm{id}\otimes i_B} & C(A, I_{\mathcal{V}} \odot A) \otimes C(B, B) \\ & \searrow^{i} & & & & & \downarrow{\scriptstyle \boxtimes_{(A,B),(I_{\mathcal{V}}\odot A, B)}} \\ & & & & & & C(A \boxtimes B, (I_{\mathcal{V}} \odot A) \boxtimes B) \\ & & & & & & \downarrow{\scriptstyle C(A\boxtimes B, \lambda\boxtimes \mathrm{id})} \\ & & & & & & C(A \boxtimes B, A \boxtimes B) \end{array} \tag{6.30}$$

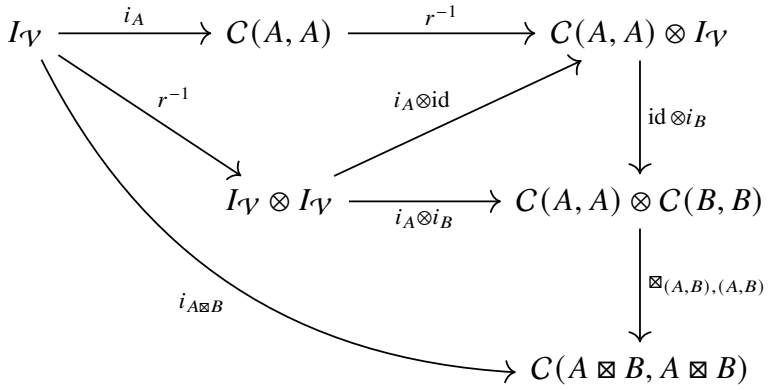
and the composite on the right-side becomes

$$\boxtimes_{(A,B),(A,B)} \circ (\mathrm{id} \otimes i_B) \circ r^{-1} \circ i_A \colon I_{\mathcal{V}} \to C(A \boxtimes B, A \boxtimes B)$$

via the commutative diagram



using the $\mathcal{V}$-functoriality of $\boxtimes \colon C \otimes C \to C$, functoriality of $\otimes$, naturality of r and the definition of $\lambda \colon I_{\mathcal{V}} \odot A \cong A$. This again becomes $i_{A\boxtimes B}$ via the commutative diagram:



by the naturality of r and the preservation of units of the $\mathcal{V}$-functor $\boxtimes \colon C \otimes C \to C$; thus showing that (6.30) commutes and hence (6.28) commutes.

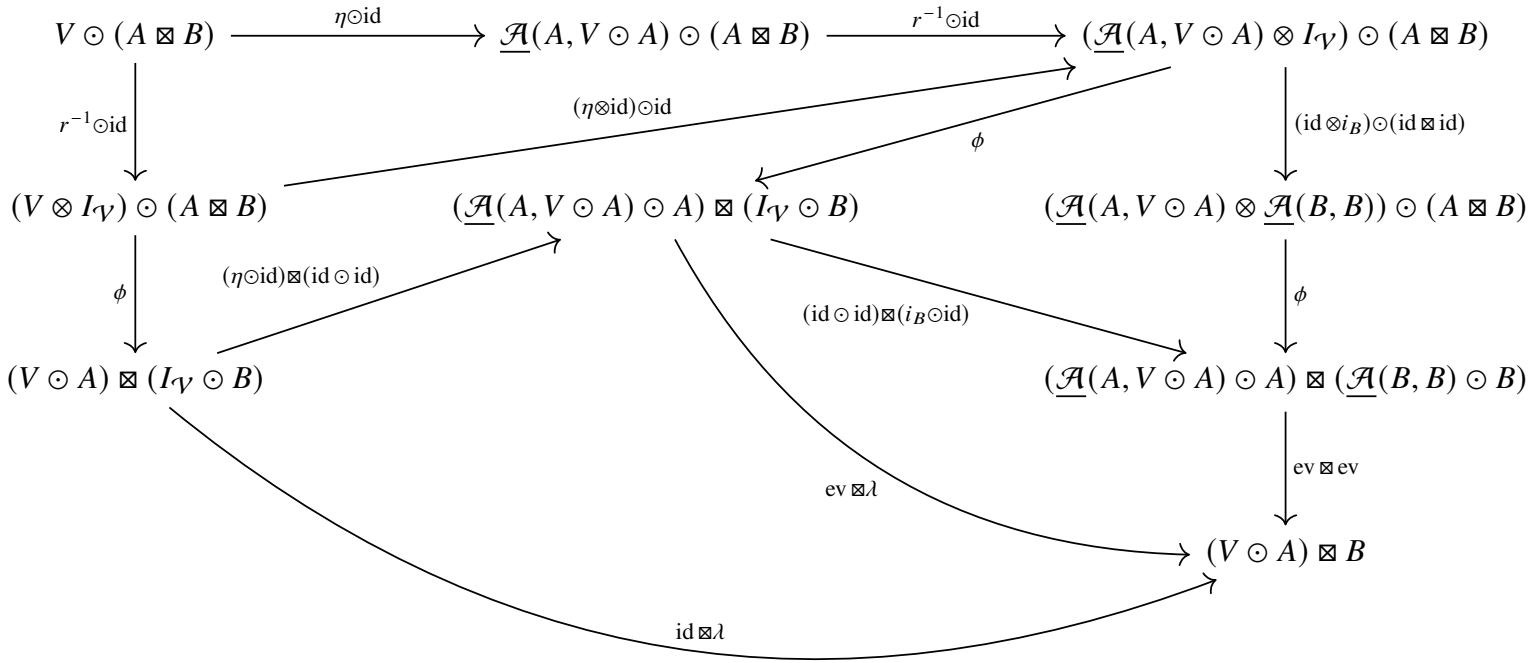
The condition (6.29) can be shown in a similar way. □

Proposition 6.2.4. *The tensor $\mathcal{V}$-functor of the monoidal $\mathcal{V}$-category $(\underline{\mathcal{A}}, \boxtimes, I_{\mathcal{A}})$ from Proposition 6.2.1 preserves copowers in each variable.*

Proof. (6.25) is

$$\begin{array}{ccccc} V & \xrightarrow{\eta} & \underline{\mathcal{A}}(A, V \odot A) & \xrightarrow{r^{-1}} & \underline{\mathcal{A}}(A, V \odot A) \otimes I_{\mathcal{V}} \\ & & & & \downarrow{\scriptstyle \mathrm{id} \otimes i_B} \\ \underline{\mathcal{A}}(A \boxtimes B, (V \odot A) \boxtimes B) & \xleftarrow[\boxtimes_{(A,B),(V\odot A,B)}]{} & & & \underline{\mathcal{A}}(A, V \odot A) \otimes \underline{\mathcal{A}}(B, B) \end{array}$$

and μ is the transpose of (6.25) which is the composite of the right-side of the below diagram:



which is equal to the composite on the left-most side of the diagram

$$\mu \colon V \odot (A \boxtimes B) \xrightarrow[\cong]{r^{-1} \odot \mathrm{id}} (V \otimes I_{\mathcal{V}}) \odot (A \boxtimes B) \xrightarrow[\cong]{\phi} (V \odot A) \boxtimes (I_{\mathcal{V}} \odot B) \xrightarrow[\cong]{\mathrm{id} \boxtimes \lambda} (V \odot A) \boxtimes B \tag{6.31}$$

after applying the naturality of ϕ and r, the definition of $i_B \colon I_{\mathcal{V}} \to \underline{\mathcal{A}}(B,B)$ and the triangle identity

$$\begin{array}{ccc} V \odot A & \xrightarrow{\eta \odot \mathrm{id}} & \underline{\mathcal{A}}(A, V \odot A) \odot A \\ & \searrow & \downarrow \mathrm{ev} \\ & & V \odot A. \end{array}$$

Likewise, $\pi \colon V \odot (A \boxtimes B) \xrightarrow{\cong} A \boxtimes (V \odot B)$ given by the transpose of (6.27) is equal to

$$\pi \colon V \odot (A \boxtimes B) \xrightarrow[\cong]{l^{-1} \odot id} (I_{\mathcal{V}} \otimes V) \odot (A \boxtimes B) \xrightarrow[\cong]{\phi} (I_{\mathcal{V}} \odot A) \boxtimes (V \odot B) \xrightarrow[\cong]{\lambda \boxtimes \mathrm{id}} A \boxtimes (V \odot B). \tag{6.32}$$

□

6.3 Connection to copowered and cocomplete monoidal $\mathcal{V}$-enrichments

We saw in the previous section that, for a braided and closed monoidal base category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ and a monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$, an object of $\mathcal{V}$-$\mathbf{Act}_2(\mathcal{A})_{\text{closed}}$ gives a copowered *monoidal enrichment* $(\underline{\mathcal{A}}, \boxtimes, I_{\mathcal{A}})$ of $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$ that *preserves copowers in each variable*. In this section, we show that this has a converse.

Proposition 6.3.1. *Let there be a given braided and closed monoidal base category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ and a copowered monoidal $\mathcal{V}$-category $(\mathcal{B}, \boxtimes, I)$ with a provided choice of copowers, written $V \odot B$, such that the tensor product of $\mathcal{B}$ preserves copowers in each variable (Definition 6.2.3) with notations:*

$$\mu \colon (V \odot A) \boxtimes B \xrightarrow{\cong} V \odot (A \boxtimes B) \tag{6.33}$$

$$\pi \colon A \boxtimes (V \odot B) \xrightarrow{\cong} V \odot (A \boxtimes B). \tag{6.34}$$

Then the closed monoidal action $\odot \colon \mathcal{V} \times \mathcal{B}_0 \to \mathcal{B}_0$ of $\mathcal{V}$-$\boldsymbol{Act}(\mathcal{B}_0)_{\text{closed}}$ from Lemma 3.3.1 is also a $\mathcal{V}$-action on $\mathcal{A}$, i.e. an object of $\mathcal{V}$-$\boldsymbol{Act}_2(\mathcal{B}_0)_{\text{closed}}$.

Proof. The monoidal action $\odot \colon \mathcal{V} \times \mathcal{B}_0 \to \mathcal{B}_0$ is a strong opmonoidal functor by

$$\phi \colon (V \otimes W) \odot (A \boxtimes B) \xrightarrow{\alpha} V \odot (W \odot (A \boxtimes B)) \xrightarrow{V \odot \pi^{-1}} V \odot (A \boxtimes (W \odot B)) \xrightarrow{\mu^{-1}} (V \odot A) \boxtimes (W \odot B) \tag{6.35}$$

and

$$\psi \colon I_{\mathcal{V}} \odot I_{\mathcal{A}} \xrightarrow{\lambda} I_{\mathcal{A}}. \tag{6.36}$$

With these definitions of ϕ and ψ, it is easy to check (6.1), (6.2), (6.3) and (6.4) to see that $\alpha_{W,V,X} \colon (W \otimes V) \odot X \cong W \odot (V \odot X)$ and $\lambda_X \colon I \odot X \cong X$ are monoidal natural transformations. □

Definition 6.3.1. Given a monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$ and a braided monoidal base category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$, let **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})$ be the category of the monoidal enrichments of $\mathcal{A}$ over $\mathcal{V}$, that is:

- objects: monoidal $\mathcal{V}$-categories $(\underline{\mathcal{A}}, \bullet, I_{\mathcal{A}})$ equipped with an identity-on-objects strict monoidal isomorphism $\theta \colon \underline{\mathcal{A}}_0 \cong \mathcal{A}$.
- arrows: $(\underline{\mathcal{A}}^1, \bullet_1, I_{\mathcal{A}}, \theta_1) \xrightarrow{\sigma} (\underline{\mathcal{A}}^2, \bullet_2, I_{\mathcal{A}}, \theta_2)$ are monoidal $\mathcal{V}$-functors $\sigma \colon (\underline{\mathcal{A}}^1, \bullet_1, I_{\mathcal{A}}) \to (\underline{\mathcal{A}}^2, \bullet_2, I_{\mathcal{A}})$ such that

$$\begin{array}{ccc} \underline{\mathcal{A}}^1_0 & \xrightarrow{\quad\sigma_0\quad} & \underline{\mathcal{A}}^2_0 \\ & {\scriptstyle\theta_1}\searrow \quad \swarrow{\scriptstyle\theta_2} & \\ & \mathcal{A} & \end{array} \tag{6.37}$$

  commutes.

Remark 6.3.1. Note that on objects, strict monoidality of θ implies that $X \bullet Y = X \boxtimes Y$ and (6.37) implies that σ is identity on objects and that $\sigma \colon \underline{\mathcal{A}}^1 \to \underline{\mathcal{A}}^2$ is in fact strict monoidal.

Definition 6.3.2. For a braided and closed monoidal base category $\mathcal{V}$, let **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})_{\text{copower}}$ be the category with as objects the copowered monoidal $\mathcal{V}$-enrichments $(\underline{\mathcal{A}}, \bullet, I_{\mathcal{A}}, \theta)$ with a provided choice of copowers $V \odot X$, such that the tensor $\mathcal{V}$-functor $\bullet\colon \underline{\mathcal{A}} \otimes \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ preserves copowers in each variable (Definition 6.2.3). The morphisms of **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})_{\text{copower}}$ are as in **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})$.

Definition 6.3.3. For a braided and closed monoidal base category $\mathcal{V}$ and a cocomplete monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$ whose monoidal product $\boxtimes$ preserves colimits in each variable, let **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})_{\text{cocomp}}$ be the full subcategory of **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})_{\text{copower}}$ with objects $(\underline{\mathcal{A}}, \bullet, I_{\mathcal{A}}, \theta, \odot)$ that are the cocomplete monoidal $\mathcal{V}$-enrichments $(\underline{\mathcal{A}}, \bullet, I_{\mathcal{A}}, \theta)$ such that the tensor $\mathcal{V}$-functor $\bullet\colon \underline{\mathcal{A}} \otimes \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ preserves all $\mathcal{V}$-colimits (copowers and conical colimits) in each variable.

We now have generalisations of Proposition 3.3.3 and Proposition 3.3.5 to the higher-tuply monoidal setting.

Proposition 6.3.2. *For a braided and closed monoidal base category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ and a monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$, there is an equivalence of categories*

$$\mathcal{V}\text{-}\boldsymbol{Act}_2(\mathcal{A})_{\text{closed}} \simeq \boldsymbol{Mon}\text{-}\mathcal{V}\text{-}\boldsymbol{Cat}(\mathcal{A})^{\text{op}}_{\text{copower}}. \tag{6.38}$$

Proof. This is a straightforward modification of Proposition 3.3.3 using the observations in this Chapter. Proposition 6.2.1 assigns an object of $\mathcal{V}$-**Act**$_2(\mathcal{A})_{\text{closed}}$ to a copowered monoidal $\mathcal{V}$-enrichment preserving copowers in each variable (Proposition 6.2.2 and Proposition 6.2.4).

Essential surjectivity is shown by Proposition 6.3.1. That is, given an object $(\underline{\mathcal{A}}, \bullet, I_{\mathcal{A}}, \theta, \odot) \in$ **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})_{\text{copower}}$, its underlying monoidal category $\underline{\mathcal{A}}_0$ with its copower operation $\odot$ gives an object of $\mathcal{V}$-**Act**$_2(\underline{\mathcal{A}}_0)_{\text{closed}}$. We then get an object of $\mathcal{V}$-**Act**$_2(\mathcal{A})_{\text{closed}}$ by transporting across the isomorphism $\theta\colon \mathcal{A} \cong \underline{\mathcal{A}}_0$, and the image of this object under $\mathcal{V}$-**Act**$_2(\mathcal{A})_{\text{closed}} \to$ **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})^{\text{op}}_{\text{copower}}$ is isomorphic to the given $(\underline{\mathcal{A}}, \bullet, I_{\mathcal{A}}, \theta, \odot)$. Given a morphism $\overline{\sigma}\colon \odot_2 \Rightarrow \odot_1$ in $\mathcal{V}$-**Act**$_2(\mathcal{A})_{\text{closed}}$ the $\sigma'_{A,B}s$ from (3.29) together with identity on objects will give a map $\sigma\colon (\underline{\mathcal{A}}^1, \bullet_1, I_{\mathcal{A}}, \theta_1) \to (\underline{\mathcal{A}}^2, \bullet_2, I_{\mathcal{A}}, \theta_2)$ in **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})_{\text{copower}}$. And given a map $\sigma\colon (\underline{\mathcal{A}}^1, \bullet_1, I_{\mathcal{A}}, \theta_1) \to (\underline{\mathcal{A}}^2, \bullet_2, I_{\mathcal{A}}, \theta_2)$ in **Mon**-$\mathcal{V}$-**Cat**$(\mathcal{A})_{\text{copower}}$, its inverse $\overline{\sigma}\colon \odot_2 \Rightarrow \odot_1$ in $\mathcal{V}$-**Act**$_2(\mathcal{A})_{\text{closed}}$ is obtained by (3.39). □

Definition 6.3.4. We write $\mathcal{V}$-**Act**$_2(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}}$ for the full subcategory of $\mathcal{V}$-**Act**$_2(\mathcal{A})_{\text{closed}}$ on those objects whose actions are also cocontinuous in the $\mathcal{A}$-variable. i.e. $\mathcal{V}$-actions $\odot\colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ on $\mathcal{A}$ such that

- each $(-) \odot C\colon \mathcal{V} \to \mathcal{A}$ is equipped with a right adjoint $\langle C, -\rangle\colon \mathcal{A} \to \mathcal{V}$;
- each $V \odot (-)\colon \mathcal{A} \to \mathcal{A}$ is cocontinuous.

Proposition 6.3.3. *For a braided and closed monoidal base category $(\mathcal{V}, \otimes, I_{\mathcal{V}}, b)$ and a cocomplete monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$ whose monoidal product $\boxtimes$ preserves colimits in each variable, there is an equivalence of categories*

$$\mathcal{V}\text{-}\boldsymbol{Act}_2(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}} \simeq \boldsymbol{Mon}\text{-}\mathcal{V}\text{-}\boldsymbol{Cat}(\mathcal{A})^{\text{op}}_{\text{cocomp}} \tag{6.39}$$

Proof. This is a modification of Proposition 3.3.5. We leave the proof to the reader. □

7 Future Directions

We now work in the context where $\mathcal{V}$ and $\mathcal{A}$ in Proposition 6.3.3 are both locally presentable categories. Our objective is to describe the universal cocomplete monoidal enrichment $\underline{\mathcal{A}}$ of $\mathcal{A}$ among those over locally presentable braided and right-closed monoidal base categories $\mathcal{V}$'s. We present these results without proof; although we have checked most of the details, we leave these proofs for future work. We also state a number of conjectures describing proposed future generalisations of our work.

For a locally presentable monoidal category $(\mathcal{A}, \boxtimes, I_{\mathcal{A}})$ whose monoidal product $\boxtimes$ preserves colimits in each variable, the centre of $\mathcal{A}$ is the category $\mathcal{Z}(\mathcal{A})$ which has

- objects: objects $Z \in \mathcal{A}$ equipped with a natural isomorphism

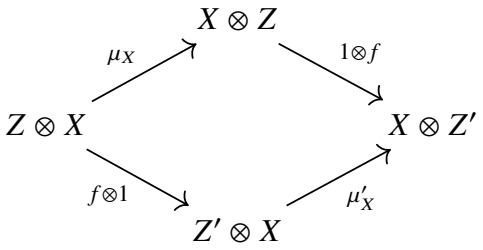
$$\mu \colon (Z \otimes -) \Rightarrow (- \otimes Z) \colon \mathcal{A} \to \mathcal{A} \tag{7.1}$$

such that the following diagrams commute:

$$\begin{array}{ccc} Z \otimes I & \xrightarrow{\mu_I} & I \otimes Z \\ & {\scriptstyle r_Z} \searrow \quad \swarrow {\scriptstyle l_Z} & \\ & Z & \end{array}$$

$$\begin{array}{ccccc} & & Z \otimes (X \otimes Y) & \xrightarrow{\mu_{X \otimes Y}} & (X \otimes Y) \otimes Z \\ & {\scriptstyle a_{Z,X,Y}} \nearrow & & & \searrow {\scriptstyle a_{X,Y,Z}} \\ (Z \otimes X) \otimes Y & & & & X \otimes (Y \otimes Z). \\ & {\scriptstyle \mu_X \otimes 1_Y} \searrow & & & \nearrow {\scriptstyle 1_X \otimes \mu_Y} \\ & & (X \otimes Z) \otimes Y & \xrightarrow[a_{X,Z,Y}]{} & X \otimes (Z \otimes Y) \end{array} \tag{7.2}$$

- morphisms: $f\colon (Z,\mu) \to (Z',\mu')$ are arrows $f\colon Z \to Z'$ in $\mathcal{A}$ such that



(7.3)

  commutes for all $X \in \mathcal{A}$.

With some work, one may show that $\mathcal{Z}(\mathcal{A})$ is a locally presentable, braided and closed monoidal category. Moreover, there is a $\mathcal{Z}(\mathcal{A})$-action on $\mathcal{A}$ given by the monoidal product of $\mathcal{A}$:

$$\begin{aligned} \boxtimes\colon \mathcal{Z}(\mathcal{A}) \times \mathcal{A} &\to \mathcal{A} \\ ((Z,\mu), X) &\mapsto Z \boxtimes X. \end{aligned} \tag{7.4}$$

This is cocontinuous in each variable from the fact that the monoidal product $\boxtimes$ of $\mathcal{A}$ is cocontinuous in each variable, so giving an object of $\mathcal{Z}(\mathcal{A})$-$\mathbf{Act}_2(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}}$ with a choice of a right adjoint $\underline{\mathcal{A}}(C,-)\colon \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ for each $(-)\boxtimes C\colon \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$. This gives a cocomplete monoidal enrichment $\underline{\mathcal{A}}$ of $\mathcal{A}$ by Proposition 6.3.3 which we claim is the universal cocomplete monoidal enrichment of $\mathcal{A}$ among those over locally presentable braided and right-closed monoidal categories $\mathcal{V}$'s. We now sketch an explanation why this should be true.

We point out that a general $\mathcal{V}$-action on $\mathcal{A}$, with $\mathcal{V}$ and $\mathcal{A}$ not necessarily locally presentable, is a *Swiss-Cheese algebra* of dimension $d = 2$ in **Cat**. We direct the reader to Chapter 6 of my Master's thesis [Jun18] for a concise introduction to *Swiss-Cheese operads*, *Swiss-Cheese algebras* and the *Swiss-Cheese Conjecture* by Kontsevich. *A Swiss-Cheese algebra of dimension $d = 2$ in **Cat** is equivalent to a braided monoidal functor $\mathcal{V} \to \mathcal{Z}(\mathcal{A})$*; this statement is one special case of the Swiss-Cheese Conjecture which has been shown in Chapter 5 of [Jun18] and also in [Idr17], where (7.4) is the terminal object in the category of Swiss-Cheese algebras on $\mathcal{A}$.

From this, an object of $\mathcal{V}$-$\mathbf{Act}_2(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}}$ is equivalent to a braided monoidal and cocontinuous functor $\mathcal{V} \to \mathcal{Z}(\mathcal{A})$. And in the case of locally presentable $\mathcal{V}$ and $\mathcal{A}$, we now have a chain of categorical equivalences:

$$\mathbf{BrMonAdj}(\mathcal{V}, \mathcal{Z}(\mathcal{A})) \xrightarrow{\simeq} \mathcal{V}\text{-}\mathbf{Act}_2(\mathcal{A})_{\text{closed},\mathcal{A}\text{-coct}} \xrightarrow{\simeq} \mathbf{Mon}\text{-}\mathcal{V}\text{-}\mathbf{Cat}(\mathcal{A})^{\text{op}}_{\text{cocomp}} \tag{7.5}$$

where the equivalence on the right comes from the previous chapter.

Note also that a general monoidal action of $\mathcal{V}$ on $\mathcal{A}$ is a Swiss-Cheese algebra of dimension $d = 1$ in **Cat** which is same as a monoidal functor $\mathcal{V} \to \mathbf{Cat}(\mathcal{A},\mathcal{A})$. This statement is also a case of the Swiss-Cheese Conjecture. We thus make the following conjectures:

Conjecture 1. *To give a Swiss-Cheese action of $\mathcal{V}$ on $\mathcal{A}$ of dimension d in **Cat**, where $\mathcal{V}$ is a closed E_d-algebra and $\mathcal{A}$ is a cocomplete E_{d-1}-algebra preserving colimits in each variable, such that the action is right-closed and cocontinuous in $\mathcal{A}$-variable is, up to isomorphism, equivalent to giving $\mathcal{A}$ a structure of a cocomplete E_{d-1}-algebra in $\mathcal{V}$-**Cat**.*

Example 7.0.1. For $d = 1$ this is Proposition 3.3.5 and for $d = 2$ this is Proposition 6.3.3.

Conjecture 2. *For a locally presentable E_{d-1}-algebra $\mathcal{A}$ in **Cat** preserving colimits in each variable, the terminal object $\mathcal{H}(\mathcal{A})$ in the category of Swiss-Cheese algebras of locally presentable and closed E_d-algebra $\mathcal{V}$ on $\mathcal{A}$ gives $\mathcal{A}$ a universal structure of a cocomplete E_{d-1}-algebra in $\mathcal{V}$-**Cat**.*

Example 7.0.2. For $d = 1$, this is given by $\mathcal{H}(\mathcal{A}) = \mathrm{Cocts}(\mathcal{A}, \mathcal{A})$ and for $d = 2$, this is given by $\mathcal{H}(\mathcal{A}) = \mathcal{Z}(\mathcal{A})$.

The theory of bicategories enriched in a monoidal bicategory was developed in [GS16]. We also conjecture that there are generalisations to these statements in higher categorical setting in arbitrary dimension.

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