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Twisted calculus

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ABSTRACT

A twisted ring is a ring endowed with a family of endomorphisms satisfying certain relations. One may then consider the notions of twisted module and twisted differential module. We study them and show that, under some general hypothesis, the categories of twisted modules and integrable twisted differential modules are equivalent. As particular cases, one recovers classical results from the theory of finite difference equations or q-difference equations.

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1. Introduction

Finite difference equations [19] have been used for a long time to approximate solutions of differential equations. There exists also a multiplicative variant called q-difference equations [6]. Both notions, finite difference equations and q-difference equations, can be studied in analogy with the theory of differential equations. It is possible to give a unified treatment to these different theories through noncommutative calculus. We want to mention in particular Yves André's article [1] and the work of Valery Lunts and Alexander Rosenberg [20]. André's theory introduces the notion of noncommutative connections and leads to a beautiful Galois theory. Lunts and Rosenberg are able to define rings of differential operators in noncommutative geometry. These rings appear naturally in representation theory.

Our approach is based on the quantum philosophy: we want to see the theories of finite difference equations and q-difference equations as perturbations of the usual theory of differential equations. In order to do that, we work over a ring A endowed with a family of endomorphisms satisfying some fixed conditions. For example, if A is a ring of functions in one variable x, and we use one endomorphism $\sigma(x) = x + h$ (and no conditions), we recover the theory of finite difference equations; for $\sigma(x) = qx$, we recover the theory of q-difference equations; and, finally, when σ is the identity, we recover the theory of usual differential equations. Even these examples are actually more general than their classical counterpart and encompass the case of positive characteristic as well as the case when q is a root of unity (which is in fact our main concern).

We start with the notion of E-twisted ring (resp. module) which is a ring A (resp. an A-module M) endowed with a family $\underline{\sigma} := \{\sigma_i\}_{i\in E}$ of ring endomorphisms (resp. semi-linear maps) satisfying some conditions such as commutativity, invertibility, existence of roots and so on. One first shows that such a ring (resp. module) may be seen as a G-ring (resp. G-A-module), which is a ring endowed with an action

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of a monoid G by ring endomorphisms (resp. semi-linear maps). In our situation, $G = G(E)$ will be a monoid naturally associated to E and its conditions.

We can do better and introduce the (noncommutative) twisted polynomial ring $A[E]_{\underline{\sigma}}$ associated to A and E (this is the crossed product of A by $G(E)$). Then an E-twisted A-module is nothing but an $A[E]_{\underline{\sigma}}$-module. For example, if $A = \mathbb{C}[x]$ and $\sigma(x) = qx$, giving a σ-twisted module is equivalent to giving a module over the noncommutative polynomial ring $\mathbb{C}[x, y]$ with the commutation rule $yx = qxy$. Our aim with the twisted philosophy is to replace noncommutative objects with twisted objects that seem easier to handle in practice.

Fix now a commutative base ring R. Given a commutative E-twisted R-algebra A, one defines a σ_i-derivation of A (resp. of an A-module M) as an R-linear map D_i satisfying the twisted Leibnitz rule

$$D_i(xs) = D_i(x)s + \sigma_i(x)D_i(s).$$

A twisted derivation of A (resp. of an A-module M) is then a finite sum of such D_i for some of our endomorphisms σ_i.

Twisted calculus concerns the study of A-modules M endowed with an R-linear action of the twisted derivations of A by twisted derivations of M. In order to do that, one can consider the ring of small (or naive) twisted differential operators $\overline{\mathrm{D}}_{\underline{\sigma}}$, which is the smallest ring that contains functions and twisted derivations. Unfortunately, as is already the case in the untwisted situation, the category of $\overline{\mathrm{D}}_{\underline{\sigma}}$-modules will be too small in general (p-curvature phenomenon).

To go further, it is necessary to attach to each endomorphism σ_i a specific σ_i-derivation D_i of A: this is what we call a twisted differential algebra. We may also require some commutation properties that we call the twisted Schwarz conditions. One can then consider A-modules M endowed with an action by σ_i-derivations of those specific σ_i-derivations D_i. We also introduce a notion of twisted Weyl algebra as a filtered analog of the twisted polynomial ring $A[E]_{\underline{\sigma}}$ introduced above, which is a graded ring.

Let us concentrate for a moment (with a harmless but convenient change of notation) on the case we will call n-twisted: $E = \underline{T} := \{T_1, \dots, T_n\}$ endowed with the commutation conditions $T_iT_j = T_jT_i$. Then, one can show that there exists a one-to-one correspondence between twisted differential algebras $(A, \underline{\sigma}, \underline{D})$ that satisfy twisted Schwarz conditions and twisted Weyl algebras $\mathrm{D}_{\underline{\sigma},\underline{D}}$. Moreover, as one of our main results shows, the $\mathrm{D}_{\underline{\sigma},\underline{D}}$-modules correspond exactly to the A-modules with an action of the D_i's by σ_i-derivations that are integrable (another commutation condition).

We want to apply these results when the σ_i-derivations D_i span all twisted derivations of A. This is the case when there exists what we call twisted coordinates $x_1, \dots, x_n$, for example, when A is a twisted localization of the polynomial ring. Then, there exists partial $\underline{\sigma}$-derivations $\partial_1, \dots, \partial_n$ and we may consider the twisted Weyl algebra $\mathrm{D}_{\underline{\sigma},\underline{\partial}}$. Moreover, one can build a canonical map $A[\underline{T}]_{\underline{\sigma}} \to \mathrm{D}_{\underline{\sigma},\underline{\partial}}$ which is an isomorphism when $\underline{\sigma}$ is essentially non trivial. In this case, we obtain an equivalence between A-modules endowed with an R-linear action of twisted derivations that are integrable, on one hand, and twisted A-modules, on the other hand, bringing us back to where we started.

Some of our results may be found, usually in a specific context, elsewhere in the mathematical literature. In fact, sections 1 through 4 are just generalizations of what we have called n-twisted rings and modules, which have been studied since the work of J. Ritt in the 1930's under the names of difference ring and difference modules. The monographs on Difference Algebra by R. M. Cohn [5] and A. B. Levin [18] are excellent references. Sections 5, on twisted Weyl algebra, and 6, where we treat the existence of coordinates that provide a basis of twisted-derivations that behave like partial derivatives, are, we expect, newer. We hope that our abstract and very general description of this twisted calculus will be useful to other mathematicians.

On our side, we plan to apply it to the confluence question: we have shown that, under some general hypothesis, all twisted Weyl algebras are topologically isomorphic and, as a consequence, we proved an equivalence between differential systems and q-difference systems [14.1, 14.2]. We can also recast the quantum Simpson correspondence introduced by the first author and Gros [7] in this new context, by the introduction of twisted divided powers. Ultimately, one also may expect to understand in some cases the classical Simpson's correspondence as a confluence phenomenon from the quantum case. Let us also

mention that some constructions that appear in recent work of Bhatt, Morrow and Scholze on integral p-adic Hodge Theory [3] are examples of the concepts we discuss.

Many thanks to Michel Gros for all our fruitful conversations during the preparation of this article, and also to the referee for the comments and the very useful pointers to the literature.

Throughout the paper, R will denote a commutative ring and our rings will always have an identity. By an *R-algebra*, we mean an R-module A endowed with an R-bilinear multiplication that turns A into a ring. This is equivalent to giving a *central* homomorphism $R \to A$. Note however that homomorphisms of rings that are *not* central also naturally appear in the theory.

2. Twisted modules

In this section, we extend some concepts from the theory of difference rings and modules, as presented for example in [18], to a situation where we do not require the endomorphisms in the so called basic set to commute and to be injective, or this set to be finite. In order to do that, we need first to introduce or recall some notations and vocabulary.

A *word* of length n on a set E, whose elements are usually called *letters*, is a finite sequence $u := i_0 i_1 \cdots i_{n-1}$ of letters of E. More precisely, the set of all words on E is

$$W(E) := \coprod_{n\in\mathbb{N}} \mathrm{Hom}(\{0,\dots,n-1\}, E).$$

By composition, any map $f : E \to E'$ will induce a map $W(f) : W(E) \to W(E')$ and this is clearly functorial. We will denote by 1 the empty word. We will also need the notion of *opposite* to a word $u = i_0 i_1 \cdots i_{n-1}$: this is the word $u^{op} := i_{n-1} i_{n-2} \cdots i_0$ obtained from the symmetry on $W(E)$ induced by all the permutations $k \mapsto n-k-1$ of the sets $\{0,\dots,n-1\}$.

A *(word) condition* on the set E is an ordered pair $(u, v) \in W(E) \times W(E)$ that we will denote by a formal equality "$u = v$" (we will not write the quotes when we believe that there is no ambiguity). A condition of the form "$ij = ji$" is called a *commutation* condition, a condition of the form "$ij = 1$" an *inverting* condition, and a condition of the form "$i^n = j^m$" is called a *root* condition. Finally, the *opposite* to the condition "$u = v$" is the condition "$u^{\mathrm{op}} = v^{\mathrm{op}}$".

By definition, a *set of (word) conditions* on E is a subset $\Gamma \subset W(E) \times W(E)$ (the graph of a relation on $W(E)$). If we are given two sets E and E', both endowed with a set of conditions Γ and Γ' respectively, then an *equivariant map* $f : E \to E'$ is a map that preserves conditions, that is,

$$(W(f) \times W(f))(\Gamma) \subset \Gamma'.$$

Pairs (E, Γ) made of a set E and a set of word conditions on E make a category with equivariant maps as morphisms. When the choice of Γ is clear from the context, we will simply write E and call Γ the *set of E-conditions*. Note that the forgetful functor $(E, \Gamma) \mapsto E$ has an adjoint $E \mapsto E^{\mathrm{triv}} := (E, \emptyset)$ (and also a coadjoint, by the way). In general, when $\Gamma = \emptyset$, we say that E is *free from conditions*.

We will denote by Γ^{op} the set of opposite conditions to those of Γ and $E^{\mathrm{op}} := (E, \Gamma^{\mathrm{op}})$. We will denote by Γ^{ab} the union of Γ and all the commutation conditions and write also $E^{\mathrm{ab}} := (E, \Gamma^{\mathrm{ab}})$. We let $E^{\pm}$ be the disjoint union $E \coprod E^{\mathrm{op}}$ with the conditions of Γ, the conditions of Γ^{op} and all the inverting conditions $ii^{\mathrm{op}} = i^{\mathrm{op}} i = 1$ (where we denote by i^{op} the element i seen as an element of E^{op}). Finally, if S is a set of positive integers, let us write $i^{1/n}$ for the element $(i, n) \in E \times S$. Then we let $E^{1/S} := E \times S$ endowed with the root conditions

$$(i^{1/n})^m = (i^{1/n'})^{m'} \quad \text{whenever} \quad \frac{m}{n} = \frac{m'}{n'},$$

plus all the conditions induced by Γ on each $E^{1/n} \subset E^{1/S}$ (more precisely, we require $i_0^{1/n} \cdots i_{k-1}^{1/n} = j_0^{1/n} \cdots j_{l-1}^{1/n}$ whenever $i_0 \cdots i_{k-1} = j_0 \cdots j_{j-1}$).

When there already exists a multiplication on E, which is usually denoted by using juxtaposition, the word notation might be confusing. This is why we will sometimes apply these considerations to a set of variables $\underline{T} := \{T_i\}_{i\in E}$ rather than to the set E itself. This is just a matter of notations.

Definition 2.1. Let E be a set endowed with a set Γ of (word) conditions. Then, an *E-twisted ring* is a ring A endowed with a family of ring endomorphisms $\underline{\sigma}_A = \{\sigma_{A,i}\}_{i\in E}$ satisfying

$$\sigma_{A,i_0} \circ \cdots \circ \sigma_{A,i_{n-1}} = \sigma_{A,j_0} \circ \cdots \circ \sigma_{A,j_{m-1}} \tag{2.1}$$

for all the conditions $i_0 \cdots i_{n-1} = j_0 \cdots j_{m-1}$ of Γ. It is said to be *inversive* if all $\sigma_{A,i}$'s are bijective. The subring $A^{\underline{\sigma}=\underline{1}} := \{a \in A : \sigma_{A,i}(a) = a \text{ for all } i \in E\}$ is called the ring of *$\underline{\sigma}$-invariants* of A. A *morphism of E-twisted rings* $(A, \underline{\sigma}_A) \to (B, \underline{\sigma}_B)$ is a ring homomorphism $\varphi : A \to B$ such that

$$\forall i \in E, \quad \sigma_{B,i} \circ \varphi = \varphi \circ \sigma_{A,i}.$$

It will be convenient to write $\sigma_{A,u} := \sigma_{A,i_0} \circ \cdots \circ \sigma_{A,i_{n-1}}$ when $u = i_0 \cdots i_{n-1}$ so that the required property becomes $\sigma_{A,u} = \sigma_{A,v}$ for all the conditions $u = v$ of Γ. We also implicitly set $\sigma_{A,1} := \mathrm{Id}_A$.

Of course, E-twisted rings form a category. It is important to emphasize the fact that this definition depends on the set of conditions Γ and we should actually say (E, Γ)-*twisted ring*. On the other hand, when we don't need to specify E, we might simply say *twisted ring*.

Although a twisted ring is a pair $(A, \underline{\sigma}_A)$, we will often only use the first letter A. We will also drop the index A and simply write σ_i for an endomorphism when no confusion can arise.

Observe that, whatever (E, Γ) is, if $\sigma_{A,i} = \mathrm{Id}_A$ for all $i \in E$ we are just looking at usual rings. This allows us to see the category of rings as a full subcategory, that we will call *trivially E-twisted rings*, of the category of E-twisted rings.

Definition 2.2. An *n-twisted ring* is an E^{ab}-twisted ring where $E := \{1, \ldots, n\}$ and $\Gamma = \emptyset$.

Examples

1. A 1-twisted ring is a pair (A, σ_A) where A is a ring and σ_A is a ring endomorphism of A: one can use the one element set $\{T\}$ with no condition at all. Notice that this is nothing but a difference ring in the sense of [5]. Adding the condition $T^2 = 1$ (resp. $T^2 = T$) for example corresponds to the requirement that σ_A is a symmetry (resp. a projector).
2. An n-twisted ring is a ring A endowed with n *commuting* ring endomorphisms $\sigma_{A,1}, \ldots, \sigma_{A,n}$: we may use the family $\{T_1, \ldots, T_n\}$ with all the commutation conditions $T_iT_j = T_jT_i$. More generally, when E is free from conditions, then an E^{ab}-ring is a ring endowed with a family $\{\sigma_{A,i}\}_{i\in E}$ of commuting endomorphisms. In other words, a difference ring in the sense of [18], but we prefer to use a name more compatible with the general situation.
3. An inversive 1-twisted ring is a pair (A, σ_A) where A is a ring and σ_A is a ring *automorphism* of A, and the full subcategory of inversive 1-twisted rings is equivalent to the category of $T^{\pm}$-twisted rings, where $T^{\pm} := \{T, T^{-1}\}$ with the relations $TT^{-1} = T^{-1}T = 1$. Thus, as in [5], where Cohn first introduced the name inversive, we actually find triples $(A, \sigma_A, \sigma_A^{-1})$. More generally, the category of $E^{\pm}$-twisted rings is equivalent to the category of inversive E-twisted rings for any (E, Γ).
4. If S is a set of positive integers, then $T^{1/S} := \{T^{\frac{1}{n}}\}_{n\in S}$ is endowed with the root conditions

$$(T^{\frac{1}{n}})^m = (T^{\frac{1}{n'}})^{m'} \quad \text{whenever} \quad \frac{m}{n} = \frac{m'}{n'}.$$

If $1 \in S$, then a $T^{1/S}$-twisted ring is essentially a 1-twisted ring endowed with a system of n-th roots $\sigma_{A,n}$ of σ_A for $n \in S$ (in the sense that $\sigma_{A,n}^m$ only depends on $r := \frac{m}{n} \in \mathbb{Q}$ and $\sigma_{A,n}^n = \sigma_A$ [17]). Again, one can extend this construction to families.

Recall that we always denote by R a fixed commutative ring. Then, we can give some more explicit examples:

Examples

1. If $q_i, h_i \in R$ for $i = 1, \dots, n$, then the endomorphisms of the R-algebra $R[x_1, \dots x_n]$ given by
$$\sigma_i(x_j) = \begin{cases} q_i x_i + h_i \text{ if } j = i \\ x_j \text{ otherwise} \end{cases}$$
turns $R[x_1, \dots x_n]$ into an n-twisted ring.
2. Let S be a nonempty set of positive integers and $\{q_n\}_{n\in S}$ a *system of roots* in R (see [17] again). We let σ_n denote the unique endomorphism of the R-algebra $R[x]$ such that $\sigma_n(x) = q_n x$. By construction, $R[x]$ endowed with the σ_n's, is a $T^{1/S}$-twisted ring. Of course, we can mix this construction with the previous example.
3. An n-twisted structure on a smooth R-algebra A is called a *q-deformation* (*of the trivial n-structure*) if there exists étale coordinates $x_1, \dots, x_n$ on A such that
$$\sigma_i(x_j) = x_j \quad \text{whenever} \quad i \neq j, \quad \sigma_i(x_i) = q x_i \quad \text{and} \quad \sigma_i \equiv \mathrm{Id}_A \mod 1 - q.$$
Actually, there always exists a unique q-deformation as long as A is complete for the $(q-1)$-adic topology. This is the situation considered in [3] (see also [23] or [21]).

When σ is an endomorphism of a ring A, we will consider the *pull back functor*
$$M \mapsto \sigma^* M := A \otimes_{A,\sigma} M$$
along σ on left A-modules. Unless otherwise specified, we only consider *left* A-modules.

Definition 2.3. Let A be an E-twisted ring. Then, an *E-twisted A-module* is an A-module M endowed with a family of A-linear maps $\{\sigma_{M,i} : \sigma_{A,i}^* M \to M\}_{i\in E}$ satisfying
$$\sigma_{M,i_0} \circ \sigma_{A,i_0}^*(\sigma_{M,i_1}) \circ \cdots \circ \sigma_{A,i_0\cdots i_{n-2}}^*(\sigma_{M,i_{n-1}}) \circ \sigma_{A,i_0\cdots i_{n-1}}^*(\sigma_{M,i_n}) =$$
$$\sigma_{M,j_0} \circ \sigma_{A,j_0}^*(\sigma_{M,j_1}) \circ \cdots \circ \sigma_{A,j_0\cdots j_{m-2}}^*(\sigma_{M,j_{m-1}}) \circ \sigma_{A,j_0\cdots j_{m-1}}^*(\sigma_{M,j_m})$$
for all the E-conditions $i_0 \cdots i_n = j_0 \cdots j_m$. An E-twisted A-module is said to be *inversive* if all $\sigma_{M,i}$'s are bijective. A *morphism of E-twisted A-modules* $(M, \underline{\sigma}_M) \to (N, \underline{\sigma}_N)$ is an A-linear map $u : M \to N$ such that
$$\forall i \in E, \quad \sigma_{N,i} \circ \sigma_{A,i}^*(u) = u \circ \sigma_{M,i}.$$

Of course, E-twisted A-modules form a category that will be shown later (Corollary 3.11) to be abelian with sufficiently many injective and projective objects.

Examples

1. If A is an E-twisted ring, then A itself, together with the identity maps (and *not* with the σ_i's), is an E-twisted A-module. This is even an inversive E-twisted A-module.
2. When E is finite and we consider only the commutation conditions, we are of course looking at the difference modules studied in [18].

Recall that if σ is an endomorphism of a ring A, then the above extension functor $M \mapsto \sigma^* M := A \otimes_{A,\sigma} M$ has a right restriction adjoint $M \mapsto \sigma_* M := {}^{\sigma}M$. As an abelian group, ${}^{\sigma}M$ is identical to M but the action of A is obtained by composition with σ. An A-linear map $M \to {}^{\sigma}M$ is a *σ-linear map*: an additive map $\sigma_M : M \to M$ such that
$$\forall x \in A, \forall s \in M, \quad \sigma_M(xs) = \sigma(x)\sigma_M(s).$$

Definition 2.4. If A is an E-twisted ring, then a $\underline{\sigma}_A$*-module* is a pair $(M, \underline{\sigma}_M)$ where M is an A-module and $\underline{\sigma}_M = \{\sigma_{M,i}\}_{i\in E}$ is a family of $\sigma_{A,i}$-linear endomorphisms of M satisfying

$$\sigma_{M,i_0} \circ \cdots \circ \sigma_{M,i_{n-1}} = \sigma_{M,j_0} \circ \cdots \circ \sigma_{M,j_{m-1}}$$

for all E-conditions. A *morphism of* $\underline{\sigma}_A$*-modules* $(M, \underline{\sigma}_M) \to (N, \underline{\sigma}_N)$ is an A-linear map $u : M \to N$ such that

$$\forall i \in E, \quad \sigma_{N,i} \circ u = u \circ \sigma_{M,i}.$$

Again, $\underline{\sigma}_A$-modules form a category that we will denote by $\underline{\sigma}_A-$Mod.
The following result follows inmediatly from adjointness.

Proposition 2.5. *If A is an E-twisted ring, then the category of E-twisted A-modules is equivalent (and actually isomorphic) to the category of $\underline{\sigma}_A$-modules.* □

Thus, we might as well call a $\underline{\sigma}_A$-module M an E-twisted A-module and conversely. In particular, we will call a $\underline{\sigma}_A$-module *inversive* when the corresponding E-twisted A-module is. Be careful that, when M is inversive, the semi-linear maps $\sigma_{M,i} : M \to M$ need not be bijective unless A itself is inversive.

If $A \to B$ is a morphism of E-twisted rings, then, as in the n-twisted case, both extension and restriction of scalars preserve E-twisted modules. Also, if $f : E \to E'$ is an equivariant map, then there exists an obvious pull-back functor from E'-twisted rings (resp. modules) to E-twisted rings (resp. modules).

As we already mentioned (and will prove below), if A is an E-twisted ring, then the category of $\underline{\sigma}_A$-modules is abelian with sufficiently many projective and injective objects and we may therefore define for a $\underline{\sigma}_A$-module M,

$$\mathrm{R}\Gamma_{\underline{\sigma}}(M) := \mathrm{RHom}_{\underline{\sigma}_A-\mathrm{Mod}}(A, M) \quad \text{and} \quad \mathrm{H}^i_{\underline{\sigma}}(M) := \mathrm{Ext}^i_{\underline{\sigma}_A-\mathrm{Mod}}(A, M).$$

Examples

1. We have

$$\mathrm{H}^0_{\underline{\sigma}}(M) \simeq \cap_{i\in E}\mathrm{H}^0_{\sigma_i}(M) \simeq \{s \in M, \forall i \in E, \sigma_i(s) = s\}.$$

2. If A is a 1-twisted ring and M is a 1-twisted A-module, then one can show that

$$\mathrm{R}\Gamma_\sigma(M) \simeq \left[M \xrightarrow{1-\sigma} M \right].$$

This is a complex concentrated in degree 0 and 1 whose cohomology is given by

$$\mathrm{H}^0_\sigma(M) \simeq \{s \in M, \sigma(s) = s\} \quad \text{and} \quad \mathrm{H}^1_\sigma(M) \simeq M/\{s - \sigma(s), s \in M\}.$$

3. More generally, if A is an n-twisted ring, then $\mathrm{R}\Gamma_{\underline{\sigma}}(M)$ is the Koszul complex on $(1-\sigma_1, \ldots, 1-\sigma_n)$.

Until the end of this section, we will assume (for simplicity) that all the rings are *commutative*.

If A is an E-twisted commutative ring, then there always exists an internal tensor product in $\underline{\sigma}_A-$Mod: if M, N are two $\underline{\sigma}_A$-modules, one can endow $M \otimes_A N$ with the σ_i-linear endomorphisms defined by

$$\sigma_i(s \otimes t) = \sigma_i(s) \otimes \sigma_i(t).$$

Alternatively, one can use the canonical isomorphism

$$A\ {}_{\sigma_i}\otimes_A (M \otimes_A N) \simeq (A\ {}_{\sigma_i}\otimes_A M) \otimes_A (A\ {}_{\sigma_i}\otimes_A N). \tag{2.2}$$

However, in order to turn $\mathrm{Hom}_A(M,N)$ into a $\underline{\sigma}_A$-module in general, we need M to be *inversive* and we may then send u to the composite

$$\sigma_i(u) : M \xleftarrow{\simeq} A \otimes_{A,\sigma_i} M \xrightarrow{\mathrm{Id}\otimes u} A \otimes_{A,\sigma_i} N \longrightarrow N.$$

In other words, $\sigma_i(u)$ will be characterized by the identity $\sigma_i(u) \circ \sigma_{M,i} = \sigma_{N,i} \circ u$.

Definition 2.6. Let A be an E-twisted commutative ring. Then an E-twisted A-algebra (resp. a $\underline{\sigma}_A$-*algebra*) is an A-algebra B endowed with a family of A-algebra homomorphism $\sigma_{B,i} : A \otimes_{A,\sigma_i} B \to B$ (resp. $\sigma_{A,i}$-linear ring endomorphisms $\sigma_{B,i}$) satisfying the E-conditions.

Again, we obtain two *equivalent* categories (with A-linear ring endomorphisms that are compatible with the data). Actually, giving such a structure is also equivalent to giving a central morphism of E-twisted rings $A \to B$.

As a particular case, we will consider the category of E-twisted R-algebras where R is our trivially twisted commutative base ring. In other words, an *E-twisted R-algebra* is an R-algebra A endowed with a family of R-linear ring endomorphisms $\sigma_{A,i}$ satisfying the E-conditions.

It should also be noticed that if $A \to A'$ is any morphism of E-twisted commutative rings, then any E-twisted A'-algebra has a natural structure of E-twisted A-algebra. Conversely, if B is an E-twisted A-algebra, then $\underline{\sigma}_{A'} \otimes \underline{\sigma}_B$ turns $A' \otimes_A B$ into an E-twisted A'-algebra.

Once more, if B and C are two E-twisted A-algebras, then $B \otimes_A C$ is naturally an E-twisted A-algebra.

Definition 2.7. Let A be an E-twisted commutative ring.

1. An A-algebra B is an *E-twisted quotient* of A if there exists an isomorphism of A-algebras $B \simeq A/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal in A, such that for all $i \in E$, $\sigma_i(\mathfrak{a}) \subset \mathfrak{a}$.
2. An A-algebra B is an *E-twisted localization* of A if there exists an isomorphism of A-algebras $B \simeq S^{-1}A$ where S is a multiplicative submonoid of A satisfying for all $i \in E$, $\sigma_i(S) \subset S$.

It is clear that if $A \to B$ is a morphism of E-twisted commutative rings which is also a quotient (resp. localization) morphism, then B is a E-twisted quotient (resp. localization) of A. Conversely, the following proposition follows immediately from the universal property of quotients (resp. localization).

Proposition 2.8. *Let A be an E-twisted commutative ring and B an E-twisted quotient (resp. localization) of A. Then there exists a unique structure of (commutative) E-twisted A-algebra on B.* □

Examples

1. If R is an integral domain with fraction field K, then any E-twisted structure on an R-algebra A extends uniquely to $K \otimes_R A$.
2. If R is an integral domain with fraction field K, and σ is a non-constant R-algebra endomorphism of $R[x]$ (which means that $\sigma(x) \notin R$), then it extends uniquely to $K(x)$.
3. The n-twisted structure given by $\sigma_i(x) = q_i x$ for some $q_i \in R^\times$ on $R[x_1, \ldots, x_n]$, extends uniquely to $R[x_1^{\pm}, \ldots, x_n^{\pm}]$.
4. The endomorphism given by $\sigma(x) = x + 1$ of $R[x]$ extends uniquely to $R[x, \frac{1}{x}, \frac{1}{x+1}, \ldots, \frac{1}{x+p-1}]$ when $\mathrm{Char}(R) = p > 0$.

3. Twisted polynomial rings

We will show in this section that twisted modules may always be seen as usual modules over a suitable ring. Since most of the results below are well known when G is a group (see for example Chapter VIII of [14]), we will only give a few proofs.

Recall that a monoid G is a set endowed with an associative multiplication and a two-sided unit and that morphisms of monoids preserve all finite products (in particular the unit, which is the empty product).

When G is a monoid, we may endow the set $E_G := \{T_g\}_{g\in G}$ with the set Γ_{tot} of all conditions $T_{g_0}\cdots T_{g_{n-1}} = T_{h_0}\cdots T_{h_{m-1}}$ whenever we have an equality $g_0\cdots g_{n-1} = h_0\cdots h_{m-1}$ in G. Clearly the map $G \mapsto (E_G, \Gamma_{\mathrm{tot}})$ is functorial in G.

Recall that if E is any set, then $W(E)$ is a monoid for *concatenation* and the functor $E \mapsto W(E)$ is adjoint to the forgetful functor from monoids to sets.

Now, assume that E is endowed with a set Γ of (word) conditions. This defines a relation $\mathcal{R}$ on E and we may consider the monoidal equivalence relation $\bar{\mathcal{R}}$ generated by $\mathcal{R}$. We set

$$G(E,\Gamma) := W(E)/(\Gamma) := W(E)/\bar{\mathcal{R}},$$

which is a monoid for the quotient structure, and we obtain a functor $E \mapsto G(E,\Gamma)$ which is adjoint to the above functor $G \mapsto (E_G, \Gamma_{\mathrm{tot}})$.

When Γ is understood from the context, we will simply write $G(E)$ and denote by g_i the class of $i \in E$ in $G(E)$. However, when we work with a family $\underline{T} := \{T_i\}_{i\in E}$ of variables, we'd rather denote the class of T_i by i. The notation we use should in each case be clear from the context.

Definition 3.1. Let G be a monoid, then a *G-ring* is a ring A endowed with an action of G by ring endomorphisms. A *morphism of G-rings* is a ring homomorphism $\varphi : A \to B$ which is compatible with the actions.

As an immediate consequence of the universal property of $G(E)$, we get:

Proposition 3.2. *If E is a set with conditions and $G := G(E)$, then the category of E-twisted rings is equivalent (and even isomorphic) to the category of G-rings.* □

It follows that the category of E-twisted rings only depends on $G(E)$.

Examples

1. When E is free from conditions, we have $G(E) = W(E)$. Giving a $W(E)$-ring is equivalent to giving a ring A together with a family $\{\sigma_{A,i}\}_{i\in E}$ of endomorphisms of A.
2. As a particular case, we find $G(T) = W(T) \simeq \mathbb{N}$. With the condition $T^2 = 1$, we get $\mathbb{Z}/2$, and with the condition $T^2 = T$, we find $(\{0,1\},\times)$. Giving an $\mathbb{N}$-ring (resp. a $\mathbb{Z}/2$-ring, a $\{0,1\}$-ring) is equivalent to giving a ring A together with a ring endomorphism (resp. a ring symmetry, a ring projector) σ_A.
3. If $T^{\pm}$ denotes the set $\{T, T^{-1}\}$ with the inverting conditions as usual, we see that $G(T^{\pm}) \simeq \mathbb{Z}$. More generally, $G(E^{\pm})$ is always the fraction group of the monoid $G(E)$. Giving a $\mathbb{Z}$-ring is equivalent to giving a ring A together with a ring automorphism.
4. If we endow $\{T_1,\dots,T_n\}$ with the commutation conditions, we find $G(T_1,\dots,T_n) \simeq \mathbb{N}^n$. More generally, $G(E^{\mathrm{ab}})$ is always the abelian quotient of $G(E)$. Giving an $\mathbb{N}^n$-ring (resp. a $\mathbb{Z}^n$-ring) is equivalent to giving a ring A together with n commuting endomorphisms (resp. automorphisms) $\sigma_{A,i}$.
5. Let S be a set of positive integers, then $G(T^{1/S})$ is isomorphic to the submonoid $N := \mathbb{N}\frac{1}{S}$ of $\mathbb{Q}_{\geq 0}$ generated by the set $\{\frac{1}{n}\}_{n\in S}$. If N is a submonoid of $\mathbb{Q}_{>0}$ containing 1, giving an N-ring is equivalent to giving a ring A together with a ring endomorphism σ_A, and a compatible family of n-roots $\sigma_{A,n}$ of σ_A whenever $\frac{1}{n} \in N$.

If A is ring and E is any set, we denote by $AE := \oplus_{i\in E} Ai$ the free A-module on the basis E.

Assume now that A is a G-ring for some monoid G. Then, in particular, we can see A as a G-monoid (for multiplication) and consider the semi-direct product $H := A \rtimes_\sigma G$ which is a monoid with $A \times G$ as

underlying set. The canonical map $H \to AG$ extends uniquely to a surjective additive homomorphism $\pi : \mathbb{Z}H \to AG$, whose kernel is easily checked to be a two-sided ideal.

Proposition 3.3. *If A is a G-ring, then there exists a unique bilinear multiplication on the underlying additive abelian group of AG that extends the multiplication of A and G and such that*

$$\forall a \in A, g \in G, \quad ga = \sigma_g(a)g.$$

This multiplication turns AG into a ring.

Proof. It is enough to observe that this multiplication is induced by the quotient map $\pi : \mathbb{Z}H \to AG$. □

Definition 3.4.
1. If A is a G-ring, then the *crossed product* of A by G is the free A-module AG endowed with the ring structure of the proposition.
2. If A is an E-twisted ring, then the *E-twisted polynomial ring* $A[E]_{\underline{\sigma}}$ is the crossed product of A by $G(E)$.

Actually, we should (and might) write $A[E, \Gamma]_{\underline{\sigma}}$ and not merely $A[E]_{\underline{\sigma}}$ because this ring strongly depends on the word conditions. When A is trivially twisted, we will simply write $A[E]$ or $A[E, \Gamma]$.

Remarks
1. When $\Gamma = \emptyset$, our conventions say that $A[E]$ denotes the ring of *noncommutative* polynomials on E. On the other hand, the usual polynomial ring, which is generally denoted by $A[E]$, is actually for us $A[E^{\mathrm{ab}}]$. We might nevertheless use the same notation for both notions when we believe that there is no risk of confusion.
2. In [22], Claude Sabbah uses some kind of twisted polynomial rings to in order to study q-difference equations. More precisely, he uses the ring $A := K[x_1, \dots, x_n]$ with $\sigma_i(x_j) = q^{\delta_{i,j}} x_j$, where K is a field of characteristic zero, q a non-zero element of K, and $\delta_{i,j}$ is Kronecker's delta. Note that he also considers the case $A = K[x_1^{\pm}, \dots, x_n^{\pm}]$ and $\sigma_i^{\pm}(x_j) = q^{\pm\delta_{i,j}} x_j$.

Examples
1. If A is a 1-twisted ring, then $A[T]_\sigma$ is the noncommutative polynomial ring in one variable over A with the commutation rule $Tx = \sigma(x)T$ for $x \in A$ (the Ore extension of A by σ and 0).
2. As a particular case, if $A = R[S]$ is endowed with $\sigma(S) = qS$, then $A[T]_\sigma$ is the standard *noncommutative polynomial ring* $R[T, S]_q$ over R with the commutation rule $TS = qST$ which is used to define the quantum plane.
3. If A is an n-twisted ring, then $A[T_1, \cdots, T_n]_{\underline{\sigma}}$ is the free A module with basis all $T_1^{k_1} \cdots T_n^{k_n}$ for $k_1, \dots, k_n \in \mathbb{N}$ endowed with the usual multiplication rules for the variables and $T_i x = \sigma_i(x) T_i$ for $x \in A$ and $i = 1, \dots, n$. We call it the *twisted n-polynomial ring* on A (note that the commutation of variables is built-in).
4. If A is an inversive 1-twisted ring, then $A[T^{\pm}]_\sigma$ is the noncommutative ring of Laurent polynomials with the commutation rules $Tx = \sigma(x)T$ and $T^{-1}x = \sigma^{-1}(x)T^{-1}$ for $x \in A$. This extends to several variables and gives rise to *twisted Laurent polynomials*.
5. Let S be a set of positive integers. Then $A[T^{1/S}]_\sigma$ is the noncommutative ring of Puiseux polynomials with powers in $\mathbb{N}\frac{1}{S}$, with the commutation rules $T^r x = \sigma_n^m(x) T^r$ for $x \in A$ and $r = \frac{m}{n}$. Note that there exists an isomorphism

$$A[T^{1/S}]_\sigma \simeq \varinjlim_{\{T \to T^n\}_{n \in S}} A[T]_{\sigma_n}$$

This description extends also to several variables and gives rise to *twisted Puiseux polynomials*.

From the various universal properties that occur in the construction of $A[E]_{\underline{\sigma}}$, we get the following universal property of twisted polynomial rings:

Proposition 3.5. *Let A be an E-twisted ring. Let $\varphi : A \to B$ be a ring homomorphism and $\{y_i\}_{i\in E}$ a family of elements of B such that $y_{i_0} \dots y_{i_{n-1}} = y_{j_0} \dots y_{j_{m-1}}$ for any E-condition $i_0 \cdots i_n = j_0 \cdots j_n$. If we have*

$$\forall i \in E, \forall x \in A, \quad y_i \varphi(x) = \varphi(\sigma_i(x)) y_i,$$

then there exists a unique ring homomorphism $\Phi : A[E]_{\underline{\sigma}} \to B$ that extends φ and such that for all $i \in E$, we have $\Phi(g_i) = y_i$. □

Proposition 3.6. *If $A \to B$ is a morphism of E-twisted rings, then there exists a canonical isomorphism*

$$B \otimes_A A[E]_{\underline{\sigma}} \simeq B[E]_{\underline{\sigma}}.$$

Proof. Both rings share the same universal property. □

As an illustration, we can compute the center of $A[T]_\sigma$ when A is an integral domain:

Proposition 3.7. *Let A be a 1-twisted integral domain. Then,*
1. *If there exists $p > 0$ such that $\sigma^p = \mathrm{Id}_A$, and p is the smallest such integer, then the centralizer of A in $A[T]_\sigma$ is $A[T^p]$ and the center of $A[T]_\sigma$ is equal to $A^{\sigma=1}[T^p]$.*
2. *Otherwise, the centralizer of A in $A[T]_\sigma$ is equal to A and the center of $A[T]_\sigma$ is equal to $A^{\sigma=1}$.*

Of course, with some extra conventions, we can see the second statement as a particular case of the first one (case $p = 0$).

Proof. If $\sum x_k T^k \in A[T]_\sigma$ commutes with all $x \in A$, we must have for all $k \in \mathbb{N}$, $x_k \sigma^k(x) = x_k x$. Since we assume that A is an integral domain, this can happen only if $x_k = 0$ or $\sigma^k = \mathrm{Id}$. Thus, we see that the centralizer of A in $A[T]_\sigma$ is reduced to A unless there exists $k > 0$ such that $\sigma^k = \mathrm{Id}$. If p is the smallest such integer, then necessarily, $k \equiv 0 \mod p$ and the assertions on the centralizer of A are proved. Now, an element $\sum x_k T^k \in A[T]_\sigma$ is in the center of the ring if and only if it is in the centralizer of A and commutes with T. This last condition means that for all $k \in \mathbb{N}$, x_k commutes with T and this is equivalent to $\sigma(x_k) = x_k$. In other words, we get that $x_k \in A^{\sigma=1}$. □

Definition 3.8. Let A be a G-ring. Then, a *G-A-module* is an A-module M endowed with an action of G by semi-linear maps. A *morphism of G-A-modules* is a homomorphism $u : M \to N$ which is compatible with the actions of elements of G.

Semi-linearity means that, beside the standard properties

$$\forall g, h \in G, s \in M, \quad (gh).s = g.(h.s) \quad \text{and} \quad \forall s \in M, \quad 1.s = s,$$

we must also have

$$\forall g \in G, s, t \in M, \quad g.(s + t) = g.s + g.t \quad \text{and} \quad \forall g \in G, x \in A, s \in M, g.(xs) = (g.x)(g.s).$$

The compatibility condition means that

$$\forall g \in G, x \in A, \quad u(g.x) = g.u(x).$$

From the universal property of $G(E)$, we get:

Proposition 3.9. *If E is a set with conditions and $G := G(E)$, then the category of E-twisted A-modules is equivalent (and even isomorphic) to the category of G-A-modules.* □

In particular, we see that the category of E-twisted A-modules only depends on $G(E)$, and a formal proof gives:

Proposition 3.10. *Let A be an E-twisted ring. Then the category of E-twisted A-modules is equivalent (and even isomorphic) to the category of $A[E]_{\underline{\sigma}}$-modules.* □

Corollary 3.11. *If A is an E-twisted ring, then the category of E-twisted A-modules is an abelian category with sufficiently many projective and injective objects.* □

It follows from Proposition 3.10 that if M is an E-twisted A-module, then

$$\mathrm{R}\Gamma_{\underline{\sigma}}(M) = \mathrm{RHom}_{A[E]_{\underline{\sigma}}}(A, M).$$

Examples

1. One may use the complex
$$\left[A[T]_\sigma \xrightarrow{1-T} A[T]_\sigma\right],$$
which is a free left resolution of A, to describe $H^i_\sigma(M)$ when M is a 1-twisted A-module.
2. More generally, for an n-twisted A-module, one can use the complex
$$\bigotimes_{i=1}^{n}\left[A[T_i]_{\sigma_i} \xrightarrow{1-T_i} A[T_i]_{\sigma_i}\right].$$
Notice (Lemma 4.4 below) that $1-\sigma_i$ is a derivation on A, and a straightforward calculation shows that $1-T_i$ is a $(1-\sigma_i)$-derivation on $A[T_i]_{\sigma_i}$. Thus, we are dealing with twisted differential modules (Definition 5.5) and this is a Spencer complex (see [11] for an elegant homological interpretation of Spencer's theory of linear differential operators [24]).

Remark

From section 6.3 of Peter Hendriks' thesis [9], one obtains a beautiful classification of 1-twisted modules on $\mathbb{C}(z)$ when $\sigma(z) = qz$ with q a root of unity.

4. Twisted derivations

In this section, we generalize to the case of twisted algebras the notions of derivation and small (or naive) differential operator. From now on, we stick to the *commutative* case.

Let A be a commutative R-algebra. If M and N are two A-modules, we will denote by $\mathrm{Hom}_R(M,N)$ the R-module of all R-linear maps from M to N. Recall that $\mathrm{Hom}_R(M,N)$ has two A-module structures coming from the A-modules structures of M and N respectively. In the particular case $M = N$, we will write $\mathrm{End}_R(M)$ which is actually an R-algebra for composition. We will implicitly consider multiplication by $x \in A$ as an endomorphism of M (or N). Then, the A-modules structures on $\mathrm{Hom}_R(M,N)$ are given by composition on the left or composition on the right with these morphisms. We will also implicitly consider an element $s \in M$ as an A-linear map $A \to M$ (the unique A-linear map sending 1 to s).

All this applies in particular to the case $M = N = A$ so that $\mathrm{End}_R(A)$ will denote the R-algebra of all R-*linear* endomorphisms of A (and *not* only R-algebra endomorphisms). We will identify A with an R-subalgebra of $\mathrm{End}_R(A)$ (sending $x \in A$ to multiplication by x inside A).

Definition 4.1. Let A be a commutative R-algebra, σ a ring endomorphism of A and M, N two A-modules. If $\varphi \in \mathrm{Hom}_R(M,N)$ and $x \in A$, then the σ*-twisted bracket* of φ and x is

$$[\varphi, x]_\sigma = \varphi \circ x - \sigma(x) \circ \varphi \in \mathrm{Hom}_R(M,N).$$

A straightforward proof gives:

Lemma 4.2. *Let A be a commutative R-algebra, σ a ring endomorphism of A and M, N two A-modules. Then, for fixed $x \in A$ the map $\varphi \mapsto [\varphi, x]_\sigma$, from $\mathrm{Hom}_R(M,N)$ to itself, is A-linear both on the left and on the right.* □

We will now extend the classical notion of σ-derivation [18] to an E-twisted situation. Notice that the definition is not much more general than the one we could give in the n-twisted case: the notion of $\underline{\sigma}$-derivation strongly depends on E and not only on $G(E)$ since, in fact, the word conditions do not play any role.

Definition 4.3. Let A be a commutative R-algebra and M an A-module. If σ is an endomorphism of A, then a *σ-derivation of A into M* is an element $D \in \mathrm{Hom}_R(A,M)$ such that

$$\forall x \in A, \quad [D,x]_\sigma = D(x).$$

More generally, if A is an E-twisted commutative R-algebra, then a *$\underline{\sigma}$-derivation of A into M* is a finite sum $D := \sum_{i\in E} D_i$ where D_i is a σ_i-derivation of M. In the case $M = A$, we call D a *$\underline{\sigma}$-derivation of A over R*.

For later use, note that an R-linear map D is a σ-derivation if and only if it satisfies the *σ-Leibnitz rule*

$$\forall x, y \in A, \quad D(xy) = yD(x) + \sigma(x)D(y). \tag{4.1}$$

Note also that, as for $\sigma = \mathrm{Id}_A$, if D is a $\underline{\sigma}$-derivation, then we have $D(a) = 0$ for all $a \in R$.

Examples

1. If σ is any endomorphism of the polynomial R-algebra $A := R[x]$, then there exists a unique σ-derivation ∂_σ of A/R such that $\partial_\sigma(x) = 1$. It is given by
$$\partial_\sigma(x^n) = \sum_{i+j+1=n} x^i \sigma(x)^j. \tag{4.2}$$
2. Assume that $A := R[x_1, \dots, x_n]$ is endowed with n endomorphisms σ_i such that $\sigma_i(x_j) = x_j$ whenever $i \neq j$. Then, for each $i = 1, \dots, n$, there exists a unique σ_i-derivation ∂_i of A/R such that
$$\forall j = 1, \dots, n, \quad \partial_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$
Actually, if we let $R_i := R[x_1, \dots, \widehat{x_i}, \dots, x_n]$, this is the unique σ_i-derivation of A/R_i satisfying $\partial_i(x_i) = 1$.
3. If K is a field and σ is field endomorphism of $K(x)/K$, then there exists a unique σ-derivation on $K(x)/K$ with $\partial_\sigma(x) = 1$. It is given by
$$\partial_\sigma\left(\frac{y}{z}\right) = \frac{\partial_\sigma(y)z - y\partial_\sigma(z)}{z\sigma(z)} \tag{4.4}$$
for $y, z \in K[x]$, K-linearity and formula (4.2) for positive powers of x. This case extends to several variables as well.
4. For this example, we recall from [17], when $q \in R$ and $m \in \mathbb{N}$, the *quantum integer*
$$(m)_q := 1 + q + \cdots q^{m-1}.$$
Now, assume that $A := R[x]$ and let $\sigma(x) = qx$, then the unique σ-derivation on A with $\partial_\sigma(x) = 1$ is given by
$$\partial_\sigma(x^n) = (n)_q x^{n-1} \quad \text{for } n \in \mathbb{N}. \tag{4.5}$$
Again, this extends to more variables.

5. When $q \in R^\times$ and $m \in \mathbb{N}$, we may also consider the *quantum integer*
$$(-m)_q := -\frac{1}{q} - \frac{1}{q^2} - \cdots - \frac{1}{q^m}.$$
If A $:= R[x, x^{-1}]$ and we let $\sigma(x) = qx$, then there exists again a unique σ-derivation on A with $\partial_\sigma(x) = 1$. It is given by formula (4.5) with $n \in \mathbb{Z}$, and this works as well with more variables.
6. If $A := R[x]$ and we let $\sigma(x) = x + h$ with $h \in R$, then the unique σ-derivation with $\partial_\sigma(x) = 1$ is given by
$$\partial_\sigma(x^n) = \sum_{i+j+1=n} \binom{n}{i} h^j x^i.$$

The next lemma, whose proof is straightforward, gives a generic example:

Lemma 4.4. *If σ is an endomorphism of a commutative R-algebra A, then the operator $1 - \sigma$ is a σ-derivation of A/R.* □

The asymmetry in the σ-derivation formula has strong consequences:

Lemma 4.5. *Let σ be an endomorphism of a commutative R-algebra A and $D : A \to M$ a σ-derivation.*
1. *Assume that there exists $x \in A$ such that $D(x)$ is regular in M (not a torsion element), then*
$$\forall z \in A, \quad D(z) = 0 \Rightarrow \sigma(z) = z.$$
2. *Assume that there exists $x \in A$ such that $y := x - \sigma(x)$ is regular on M (not a zero divisor and not zero either), then*
$$\forall z \in A, \quad \sigma(z) = z \Rightarrow D(z) = 0.$$

Proof. We have for all $x, z \in A$,
$$xD(z) + \sigma(z)D(x) = D(xz) = D(zx) = zD(x) + \sigma(x)D(z).$$
Thus, if we set $y := x - \sigma(x)$, we see that $yD(z) = (z - \sigma(z))D(x)$ and both assertions follow. □

If $R \to R'$ is any homomorphism, then a σ-derivation $A \to M$ over R extends by linearity to a $1 \otimes \sigma$-derivation $R' \otimes_R A \to R' \otimes_R M$ over R'. Conversely if we are given some homomorphism $R_0 \to R$, then any σ-derivation of A over R may be seen as a σ-derivation over R_0. Also, if $E' \to E$ is any injective map, then any $\underline{\sigma}_{|E'}$-derivation gives rise to a $\underline{\sigma}$-derivation.

Proposition 4.6. *If A is an E-twisted R-algebra and B is an E-twisted localization of A, then any σ-derivation $D : A \to M$ will extend uniquely to a σ-derivation (still denoted) $D : B \to B \otimes_A M$.*

Proof. We can write $B = S^{-1}A$ were S is a submonoid of $A \setminus \{0\}$ such that $\sigma(S) \subset S$. For $x \in S$, we must have
$$0 = D(1) = D\left(x\frac{1}{x}\right) = xD\left(\frac{1}{x}\right) + \frac{1}{\sigma(x)}D(x) \tag{4.6}$$
and consequently $D\left(\frac{1}{x}\right) = -\frac{D(x)}{x\sigma(x)}$. It follows that if $y \in A$, we will have
$$D\left(\frac{y}{x}\right) = yD\left(\frac{1}{x}\right) + \frac{1}{\sigma(x)}D(y) = -y\frac{D(x)}{x\sigma(x)} + \frac{1}{\sigma(x)}D(y) = \frac{xD(y) - yD(x)}{x\sigma(x)}. \tag{4.7}$$
This proves uniqueness, and existence can then be checked directly. □

If A is an E-twisted commutative R-algebra, we will denote by $\mathrm{Der}_{R,\underline{\sigma}}(A,M)$ the set of all $\underline{\sigma}$-derivations of A into M and let

$$\mathrm{T}_{A/R,\underline{\sigma}} := \mathrm{Der}_{R,\underline{\sigma}}(A,A).$$

Note that if $R \to R'$ is any map and $A' := R' \otimes_R A$, there exists a canonical map

$$\mathrm{Der}_{R,\underline{\sigma}}(A,M) \to \mathrm{Der}_{R',\underline{\sigma}}(A', A' \otimes_A M).$$

Also, as usual, when A is trivially twisted, we will drop the prefix $\underline{\sigma}$ everywhere.

Proposition 4.7. *If A is an E-twisted R-algebra and M is an A-module, then $\mathrm{Der}_{R,\underline{\sigma}}(A,M)$ is an A-submodule of $\mathrm{Hom}_R(A,M)$ for the action on the left.*

Proof. We have by definition

$$\mathrm{Der}_{R,\underline{\sigma}}(A,M) = \sum_{i\in E} \mathrm{Der}_{R,\sigma_i}(A,M),$$

and we may therefore assume that we are in the 1-twisted case. Then our assertion follows immediately from the left A-linearity of the maps $D \mapsto [D,x]_\sigma$. □

Remarks

1. In general, $\mathrm{Der}_{R,\underline{\sigma}}(A,M)$ is *not* an A-submodule for the action on the right.
2. The construction of $\mathrm{Der}_{R,\underline{\sigma}}(A,M)$ is not functorial in A and does not even commute with extensions of R in general.

The following result follows from Lemma 4.5:

Proposition 4.8. *Let σ be an endomorphism of an R-algebra A and $R' := A^{\sigma=1}$. Let M be an A-module. Assume that there exists $x \in A$ such that $y := x - \sigma(x)$ is regular on M. Then, we have*

$$\mathrm{Der}_{R,\sigma}(A,M) = \mathrm{Der}_{R',\sigma}(A,M). \quad \square$$

Using this, we can give a concrete example:

Example Assume that R is an integral domain and that $A := R[x_1,\dots,x_n]$ is endowed with n endomorphisms σ_i such that $\sigma_i(x_j) = x_j$ whenever $i \neq j$ $\sigma_i(x_i) \neq x_i$ for all $i = 1\dots,n$. Then, $\mathrm{T}_{A/R,\underline{\sigma}}$ is a free module of rank n. More precisely, a basis is given by by the ∂_i's of (4.3).

Although the module of $\underline{\sigma}$-derivations does not have a very nice functorial behavior, we have the following:

Proposition 4.9. *If A is an E-twisted R-algebra of finite type, B is an E-twisted localization of A and M is an A-module, then there exists an isomorphism*

$$B \otimes_A \mathrm{Der}_{R,\underline{\sigma}}(A,M) \simeq \mathrm{Der}_{R,\underline{\sigma}}(B, B \otimes_A M).$$

Proof. Writing again $B = S^{-1}A$, it follows from Proposition 4.6 that there exists a map

$$B \otimes_A \mathrm{Der}_{R,\underline{\sigma}}(A,M) \to \mathrm{Der}_{R,\underline{\sigma}}(B, B \otimes_A M).$$

which is easily seen to be A-linear and injective (since B is flat over A). Surjectivity will follow if we show that any $\underline{\sigma}$-derivation D' into $B \otimes_A M$ has the form $\frac{1}{y}D$ where D is a $\underline{\sigma}$-derivation into M and $y \in S$. But if we choose a set of generators $\{x_1,\dots,x_n\}$ of A, we will have $D'(x_i) \in \frac{1}{y_i}M$ for some $y_i \in S$ and we can take $y := \prod_{i=1}^n y_i$. □

Recall that, if A is a commutative R-algebra, then an *R-linear action* of an A-module T on an A-module M is a biadditive map

$$T \times M \to M, \quad (D, s) \mapsto Ds.$$

such that

$$\forall D \in T, \forall s \in M, \quad \forall x \in A, \quad (xD)s = x(Ds) \quad \text{and} \quad \forall a \in R, \quad D(as) = a(Ds).$$

We will usually denote by $D_M : M \to M$, the map defined by $D_M(s) = Ds$.

If both M and N are endowed with an R-linear action of T, then a *T-horizontal map* is an A-linear map $u : M \to N$ such that

$$\forall D \in T, \forall m \in M, \quad Du(m) = u(Dm).$$

The category obtained this way is equivalent to the category of (left) U-modules where U denotes the tensor algebra of T over A. In particular, this is an abelian category with sufficiently many injective and projective objects.

For example, an E-twisted commutative R-algebra A is naturally endowed with an action of $\mathrm{T}_{A/R,\underline{\sigma}}$.

Definition 4.10. Let A be an E-twisted commutative R-algebra and M an A-module endowed with an *R-linear action* of $\mathrm{T}_{A/R,\underline{\sigma}}$. Then, a *horizontal section* of M is an element s such that

$$\forall D \in \mathrm{T}_{A/R,\underline{\sigma}}, \quad Ds = 0.$$

Clearly, the set

$$M' = \bigcap_{D \in \mathrm{T}_{A/R,\underline{\sigma}}} \ker D_M.$$

of all horizontal sections of M is an R-submodule that may be identified with the set of all horizontal maps $A \to M$.

Proposition 4.11. *If A is an E-twisted commutative R-algebra, then the horizontal sections of A form an R-subalgebra A' of A which is contained in the ring $A^{\underline{\sigma}=\underline{1}}$ of invariants of A.*

Proof. The first assertion follows from the σ-Leibnitz rule (4.1) and the second one from Lemma 4.4 which tell us that for all $i \in E$, the map $1 - \sigma_i$ is a $\underline{\sigma}$-derivation of A. □

Examples

1. Recall from [17] that if $q \in R$, then the *q-characteristic* of R is the smallest positive integer $p =: q\text{−char}(R)$ such that $1 + q + \cdots q^{p-1} = 0$ if it exists, and zero otherwise. Also, R is said to be *q-flat* (resp. *q-divisible*) if $1 + q + \cdots + q^{m-1}$ is regular (resp. invertible) in R or equal to 0.

   Now, we let σ be the endomorphism of $R[x]$ defined by $\sigma(x) = qx$ and we assume that R is q-flat. Then,

   (a) If $q\text{−char}(R) = 0$, then the set A' of horizontal sections of A is exactly R.

   (b) If $q\text{−char}(R) = p > 0$, we have $A' = R[x^p]$.

   Note that, in both cases, we will actually have $A' = A^{\sigma=1}$ if we also assume that $1 - q$ is not a zero-divisor.
2. We let σ be the endomorphism of $R[x]$ defined by $\sigma(x) = x + 1$.

   (a) If $\mathrm{Char}(R) = 0$, then the set A' of horizontal sections of A is exactly R.

   (b) If $\mathrm{Char}(R) = p > 0$, we have $A' = R[x^p - x]$.

   Again, in both cases, we have $A' = A^{\sigma=1}$.

Definition 4.12. Let A be a commutative R-algebra, σ a ring endomorphism of A and M an A-module. An R-linear endomorphism D_M of M is called a *σ-derivation of M* if there exists $D_A \in \mathrm{T}_{A/R,\sigma}$ such that

$$\forall x \in A, \quad [D_M, x]_\sigma = D_A(x).$$

More generally, if A is an E-twisted commutative R-algebra, then a *$\underline{\sigma}$-derivation of an A-module M* is a finite sum $D = \sum_{i\in E} D_i$ of σ_i-derivations of M. Finally, a *twisted differential A-module* is an an A-module M endowed with an *R-linear action* of $\mathrm{T}_{A/R,\underline{\sigma}}$ by $\underline{\sigma}$-derivations of M.

Note that if D_M is a *σ-derivation of M*, then D_A is uniquely determined by D_M. When we want to emphasize the role of D_A, we will call D_M a *D_A-derivation*. Note that, by definition, D_M is a D_A-derivation if and only if it satisfies the *twisted Leibnitz rule*:

$$\forall x \in A, \forall s \in M, \quad D_M(xs) = D_A(x)s + \sigma(x)D_M(s). \tag{4.8}$$

It should also be remarked that in the case $M = A$, this is compatible with Definition 4.3.

It sounds natural to define a twisted differential operator as an operator that can be built by standard operations from functions and twisted derivations. However, as it is already the case in the untwisted case, we obtain a ring which is smaller than expected in general.

Definition 4.13. Let A be an E-twisted commutative R-algebra. Then, the ring of *small twisted differential operators* $\overline{\mathrm{D}}_{A/R,\underline{\sigma}}$ is the subring of $\mathrm{End}_R(A)$ generated by A and $\mathrm{T}_{A/R,\underline{\sigma}}$.

Examples

1. Assume that $R = \mathbb{C}$ and $A = \mathbb{C}[x]$.
   (a) If $\sigma = \mathrm{Id}_A$, then $\overline{\mathrm{D}}_{A/R,\sigma}$ is nothing but the usual ring of differential operators on the affine line (the Weyl algebra in one variable).
   (b) If $\sigma(x) = x + h$ where $0 \neq h \in \mathbb{C}$, then $\overline{\mathrm{D}}_{A/R,\sigma}$ is the usual ring of finite difference operators on $\mathbb{C}[x]$.
   (c) If $\sigma(x) = qx$ where $q \in \mathbb{C}$ satisfies $|q| \neq 0, 1$, then $\overline{\mathrm{D}}_{A/R,\sigma}$ is the usual ring of q-difference operators on $\mathbb{C}[x]$ (the quantum Weyl algebra).
2. There are however some situations where the ring of *small* twisted differential operators deserves its name.
   (a) As it is well known, if $R = \mathbb{F}_2$, $A = \mathbb{F}_2[x]$ and $\sigma = \mathrm{Id}_A$, then $\overline{\mathrm{D}}_{A/R,\sigma} = A \oplus \mathrm{T}_{A/R,\sigma}$. In particular, this is *not* the Weyl algebra.
   (b) In the same situation $R = \mathbb{F}_2$, $A = \mathbb{F}_2[x]$ but $\sigma(x) = x + 1$, we also get $\overline{\mathrm{D}}_{A/R,\sigma} = A \oplus \mathrm{T}_{A/R,\sigma}$.
   (c) We want to emphasize the fact that this phenomenon also occurs in characteristic zero: if $R = \mathbb{C}$, $A = \mathbb{C}[x]$ and $\sigma(x) = -x$, then we have $\overline{\mathrm{D}}_{A/R,\sigma} = A \oplus \mathrm{T}_{A/R,\sigma}$ again. We understand that this is a consequence of the q-characteristic being positive.
3. Assume that $R = \mathbb{C}[t, t^{-1}]$ and $A = R[x_1, \cdots, x_n]$ is endowed with the endomorphisms given by $\sigma_i(x_j) = \delta_{ij} t x_j$ where δ denotes the Kronecker symbol. Then $\overline{\mathrm{D}}_{A/R,\underline{\sigma}}$ is identical to the n-th quantized Weyl algebra denoted by $A_n^{(t)}$ defined by Backelin on page 319 of [2].
4. Assume that $R = K$ is a field containing $\mathbb{Q}(q)$ with $q \notin \mathbb{Q}^{\mathrm{alg}}$ and let $A = K[x]$. We endow the set $E = \{0, 1, -1\}$ with the conditions $T_0 = 1$ and $T_1 T_{-1} = T_{-1} T_1 = 1$, and the ring A with the endomorphisms given by $\sigma_0(x) = x$, $\sigma_1(x) = qx$ and $\sigma_{-1}(x) = \frac{1}{q}x$. Then $\overline{\mathrm{D}}_{A/R,\underline{\sigma}}$ is exactly the ring D_q described in the article [10] (they use the letter R for what we call A).
5. Let σ be an endomorphism of an R-algebra A and $\{\sigma_n\}_{n\in S}$ be a (compatible) system of roots of σ. Then, we do *not* have in general

$$\overline{\mathrm{D}}_{A/R,\underline{\sigma}} = \cup_{n\in S} \overline{\mathrm{D}}_{A/R,\sigma_n}.$$

It works fine in the case $R = \mathbb{Q}(t)$, $A = R[x]$, $S = \{1, 2\}$, $\sigma(x) = t^2 x$ and $\sigma_2(x) = tx$ because we will have

$$\partial_\sigma = \partial_{\sigma_2} + \frac{t(t-1)}{t+1} x \partial_{\sigma_2}^2$$

as one can easily check. But the same formula shows that the result will not hold, for example, if we restrict to $R = \mathbb{Q}[t]$, because of the existence of a denominator.

Remark Notice that, If A is an E-twisted commutative R-algebra and M is a $\overline{\mathrm{D}}_{A/R,\underline{\sigma}}$-module, then the induced action of $\mathrm{T}_{A/R,\underline{\sigma}}$ on M is an R-linear action by $\underline{\sigma}$-derivations.

The converse is false in general, since there already exists a problem in the trivially twisted situation in positive characteristic (p-curvature phenomenon). It is also important to notice that $\overline{\mathrm{D}}_{A/R,\sigma}$ does not commute with extensions of R in general, as it is already the case in the trivially twisted situation also.

5. Twisted differential algebras

We introduce in this section the notion of twisted differential algebra. This is a twisted algebra enhanced with a family of twisted derivations. The n-twisted case is treated in [12]. Although our results are more general, we can unfortunately not avoid requiring $E = E^{\mathrm{ab}}$ in Proposition 5.2, or E finite in Proposition 5.4.

Definition 5.1. An *E-twisted differential R-algebra* is an E-twisted commutative R-algebra $(A, \underline{\sigma}_A)$ endowed with a family of $\sigma_{A,i}$-derivations $D_{A,i}$ for all $i \in E$. It is said of *Schwarz type* if

$$\forall i, j \in E, \quad D_{A,i} \circ D_{A,j} = D_{A,j} \circ D_{A,i} \quad \text{and} \quad i \neq j \Rightarrow \sigma_{A,i} \circ D_{A,j} = D_{A,j} \circ \sigma_{A,i}. \tag{5.1}$$

A *morphism* between two E-twisted differential R-algebras A and B is a morphism $\varphi : A \to B$ of E-twisted R-algebras such that

$$\forall i \in E, \forall x \in A, \quad \varphi(D_{A,i}(x)) = D_{B,i}(\varphi(x)).$$

Clearly, we obtain a category. Although a twisted differential R-algebra is a triple $(A, \underline{\sigma}_A, \underline{D}_A)$, we will normally use only the first letter A to denote it. Also we will drop the index A when there is no risk of confusion and write D_i instead of $D_{A,i}$. Note, and this is important, that this notion strongly depends on the choice of E and *not* only on $G(E)$.

Examples

1. If the R-algebra $A := R[x_1, \ldots, x_n]$ is endowed with n endomorphisms σ_i satisfying $\sigma_i(x_j) = x_j$ for $j \neq i$, we may use the σ_i-derivations ∂_i of (4.3). This defines an n-twisted differential algebra of Schwarz type.
2. If $R = K$ is a field and $A := K(x)$ is endowed with an endomorphism σ, then we may use the unique σ-derivation ∂_σ such that $\partial_\sigma(x) = 1$ in order to turn A into a 1-twisted differential algebra. This may be extended to several variables.
3. If $A := R[x, x^{-1}]$ and $\sigma(x) = qx$ where $q \in R^\times$ then we may use the σ-derivation ∂_σ given by $\partial_\sigma(x^n) = (n)_q x^{n-1}$. Again, it works with several variables as well.
4. Let S be a set of positive integers (with $1 \in S$, as usual), $A = R[x]$, $\{\sigma_n\}_{n \in S}$ a compatible system of roots of some R-endomorphism σ of an R-algebra A and for each $n \in S$, ∂_n the unique σ_n-derivation of A such that $\partial_n(x) = 1$. Then, the S-twisted R-algebra $(A, \underline{\sigma}, \underline{D})$ is not of Schwarz type (at all) in general.

There exists a base change mechanism for twisted differential algebras: if A is an E-twisted differential R-algebra (of Schwarz type) and $R \to R'$ is any homomorphism, then $A' := R' \otimes_R A$ has a natural structure of E-twisted differential R'-algebra (of Schwarz type). And conversely, there exists an obvious restriction functor. Also, if $E' \to E$ is any equivariant map, then there exists an obvious restriction functor from E-twisted differential R-algebras (of Schwarz type) to E'-twisted differential algebras (of Schwarz type). Finally, any E-twisted localization of an E-twisted differential algebra is naturally an E-twisted differential algebra.

Proposition 5.2. *Assume $E = E^{\mathrm{ab}}$. If A is an E-twisted differential algebra of Schwarz type, then any E-twisted localization B of A is also of Schwarz type.*

Proof. In order to lighten the notations, we do not mention the indexes A and B in the σ_i's. We write $B = S^{-1}A$ where S is a submonoid of $A \setminus \{0\}$ and we let $x \in S$ and $y \in A$. Then, for $i \neq j$, we have, thanks to formula (4.7):

$$\sigma_i\left(D_j\left(\frac{y}{x}\right)\right) = \sigma_i\left(\frac{xD_j(y) - yD_j(x)}{x\sigma_j(x)}\right) = \frac{\sigma_i(x)\sigma_i(D_j(y)) - \sigma_i(y)\sigma_i(D_j(x))}{\sigma_i(x)\sigma_i(\sigma_j(x))}$$

and

$$D_j\left(\sigma_i(\frac{y}{x})\right) = D_j\left(\frac{\sigma_i(y)}{\sigma_i(x)}\right) = \frac{\sigma_i(x)D_j(\sigma_i(y)) - \sigma_i(y)D_j(\sigma_i(x))}{\sigma_i(x)\sigma_j(\sigma_i(x))}.$$

Thus, we see that the second condition of (5.1) is satisfied in B. The first condition is shown to hold exactly in the same way. □

Definition 5.3. The module of *derivations* of an E-twisted differential R-algebra $(A, \underline{\sigma}, \underline{D})$ is the A-submodule $\mathrm{T}_{A/R,\underline{\sigma},\underline{D}}$ of $\mathrm{End}_R(A)$ generated by the family $\underline{D}$. Given such an $(A, \underline{\sigma}, \underline{D})$, we will denote by $\overline{\mathrm{D}}_{A/R,\underline{\sigma},\underline{D}}$ the subring of $\mathrm{End}_R(A)$ generated by A and $\mathrm{T}_{A/R,\underline{\sigma},\underline{D}}$. Finally, the *module of differential forms* of $(A, \underline{\sigma}, \underline{D})$ is the A-submodule $\Omega^1_{A/R,\underline{\sigma},\underline{D}}$ of A^E generated by the image of $\underline{D} : A \to A^E$.

Proposition 5.4. *Let A be an E-twisted differential R-algebra. If E is finite and $\Omega^1_{A/R,\underline{\sigma},\underline{D}}$ is projective, then there exists an isomorphism*

$$\mathrm{Hom}_A(\Omega^1_{A/R,\underline{\sigma},\underline{D}}, A) \simeq \mathrm{T}_{A/R,\underline{\sigma},\underline{D}}.$$

Proof. Since we assume that E is finite, the image of the natural map

$$\mathrm{Hom}_A(A^E, A) \to \mathrm{End}_R(A), u \mapsto u \circ \underline{D},$$

is exactly $\mathrm{T}_{A/R,\underline{\sigma},\underline{D}}$. Now, by definition, u is in the kernel of this map if and only if its restriction to $\Omega^1_{A/R,\underline{\sigma},\underline{D}}$ is zero. Finally, since we assume that $\Omega^1_{A/R,\underline{\sigma},\underline{D}}$ is projective, the restriction map

$$\mathrm{Hom}_A(A^E, A) \to \mathrm{Hom}_A(\Omega^1_{A/R,\underline{\sigma},\underline{D}}, A)$$

is surjective and we obtain the expected identification. □

Definition 5.5. Let A be an E-twisted differential R-algebra. Then, a *$\underline{D}_A$-module* is an A-module M endowed with a family of $D_{A,i}$-derivations $D_{M,i}$. It is called *integrable* if the $D_{M,i}$ commute with each other. A *morphism of $\underline{D}_A$-modules* is an A-linear map $u : M \to N$ such that

$$\forall i \in E, \forall s \in M, \quad u(D_{M,i}(s)) = D_{N,i}(u(s)). \tag{5.2}$$

Clearly, with this definition, $\underline{D}_A$-modules form a category $\underline{D}_A-\mathrm{Mod}$. We will denote the subcategory of integrable modules as $\underline{D}_A-\mathrm{Mod}^{\mathrm{int}}$. Note that A itself is an E-twisted differential A-module and that it is integrable when A is of Schwarz type.

Remark If A is an E-twisted differential R-algebra and M and N are two E-twisted differential A-modules, then there are natural structures of E-twisted differential A-module on $M \otimes_A N$ and $\mathrm{Hom}_A(M, N)$ given by the same construction as in the n-twisted case [12]: for all $i \in E$,

$$D_{M\otimes N,i}(m \otimes n) = D_{M,i}(m) \otimes n + m \otimes D_{N,i}(n),$$

$$D_{\mathrm{Hom}_A(M,N),i}(f)(m) = D_{N,i}(f(m)) - f(D_{M,i}(m)). \tag{5.3}$$

As we will see below, if A is an n-twisted differential R-algebra of Schwarz type, then $\underline{D}_A-\mathrm{Mod}$ is an abelian category with enough injectives. If M is a $\underline{D}_A$-module, one sets

$$\mathrm{R}\Gamma_{\underline{D}}(M) := \mathrm{RHom}_{\underline{D}_A-\mathrm{Mod}}(A, M) \quad \text{and} \quad \mathrm{H}^i_{\underline{D}}(M) := \mathrm{Ext}^i_{\underline{D}_A-\mathrm{Mod}}(A, M).$$

Examples

1. We have
$$\mathrm{H}^0_{\underline{D}}(M) \simeq \cap_{i\in E} \ker D_{M,i}.$$
2. If (A, σ, D) is a 1-twisted differential R-algebra and (M, D) is a 1-twisted differential A-module, then one can show that
$$\mathrm{R}\Gamma_D(M) \simeq \left[M \xrightarrow{D} M \right].$$
This is a complex concentrated in degree 0 and 1 whose cohomology is given by
$$\mathrm{H}^0_D(M) \simeq \ker D \quad \text{and} \quad \mathrm{H}^1_D(M) \simeq M/\mathrm{Im}\, D.$$
3. More generally, in the n-twisted situation, when M is integrable, then $\mathrm{R}\Gamma_{\underline{D}}(M)$ is the twisted de Rham complex of M.

6. Twisted Weyl algebras

In this section, we introduce the notion of twisted Weyl algebra which is attached to a twisted differential algebra in the same way as a a twisted polynomial ring is attached to a twisted ring.

We first recall some basic facts on filtrations and graduations. We call *graded R-algebra* (resp. *filtered R-algebra*) an R-algebra B endowed with a graduation (resp. an increasing filtration) by R-submodules which is indexed by a monoid G endowed with a preorder compatible with multiplication on both sides (for example, the natural preorder defined by the conditions $g \leq gh$ and $h \leq gh$ for all $g, h \in G$).

More precisely, a structure of graded R-algebra is given by a family of R-submodules Gr^g such that

$$B = \oplus_{g\in G} \mathrm{Gr}^g,$$

with the extra property that

$$\forall \varphi \in \mathrm{Gr}^g, \psi \in \mathrm{Gr}^h, \quad \varphi\psi \in \mathrm{Gr}^{gh} \quad \text{and} \quad 1 \in \mathrm{Gr}^1.$$

A structure of filtered R-algebra is given by a family of R-submodules Fil^g such that

$$\forall g \leq h, \quad \mathrm{Fil}^g \subset \mathrm{Fil}^h \quad \text{and} \quad B = \cup_{g\in G} \mathrm{Fil}^g,$$

with the extra properties that

$$\forall \varphi \in \mathrm{Fil}^g, \psi \in \mathrm{Fil}^h, \quad \varphi\psi \in \mathrm{Fil}^{gh} \quad \text{and} \quad 1 \in \mathrm{Fil}^1.$$

Any graded R-algebra may be seen as a filtered R-algebra by setting $\mathrm{Fil}^g := \oplus_{h\leq g} \mathrm{Gr}^h$.

Proposition 6.1. *Let E be a set with conditions and $G := G(E)$.*

1. *If A is an E-twisted R-algebra, then $A[E]_{\underline{\sigma}}$ is a G-graded R-algebra for the grading $A[E]_{\underline{\sigma}} = \oplus_{g\in G} Ag$ and G is a submonoid of (the multiplicative monoid of) $A[E]_{\underline{\sigma}}$.*
2. *Conversely, let B be a G-graded R-algebra with grading $B = \oplus_{g\in G} Ag$ where G is a submonoid of (the multiplicative monoid of) B. Then, there exists a unique structure of E-twisted R-algebra on A such that $B \simeq A[E]_{\underline{\sigma}}$ (as graded R-algebras).*

Proof. By definition, if A is an E-twisted R-algebra, then

$$A[E]_{\underline{\sigma}} = \oplus_{g\in G} Ag$$

as an A-module, G is a submonoid for multiplication, and if we are given $g, h \in G$ and $x, y \in A$, we have $(xg)(yh) = x\sigma_g(y)gh$ which shows that multiplication is compatible with the grading.

Conversely, assume $B = \oplus_{g\in G} Ag$ is a graded R-algebra where G is a submonoid of B. If $x \in \mathrm{Gr}^1 = A$ and $g \in G$, we must have $gx \in \mathrm{Gr}^g$ which means that $gx = \sigma_g(x)g$ for some $\sigma_g(x) \in A$. Distributivity implies that σ_g is an additive map, associativity that it is multiplicative, and the unit property shows that σ_g is actually a ring homomorphism. Associativity again shows that this defines an action of G on A (or, equivalently, an E-twisted structure). The universal property of $A[E]_{\underline{\sigma}}$ provides a ring homomorphism $A[E]_{\underline{\sigma}} \to B$ which is clearly bijective. □

Now, we want to relax the grading condition and replace it by a filtration condition:

Definition 6.2. Let E be a set with conditions and $G := G(E)$. An *E-twisted Weyl algebra* is a filtered R-algebra B such that there exists a commutative R-algebra A and a morphism of monoids $G \to B, g \mapsto \partial_g$ inducing an isomorphism $\oplus_{g\in G} Ag \simeq B$ of filtered A-modules. We call $\{\partial_i\}_{i\in E}$ a *set of generators* of B over A and $\{\partial_g\}_{g\in G}$ a *Weyl basis* of B over A.

This definition only depends on G and not on E. Note however that giving the map $G \to B$ is equivalent to giving a set of generators $\{\partial_i\}_{i\in E}$ satisfying the E-conditions in B.

The next result is in some sense a generalization of Proposition 1.4 of [4] (see also Theorem 1.7.1 of [13]) and may possibly be deduced from it but we'd rather give a direct proof.

Theorem 6.3. *Let B be an n-twisted Weyl algebra over R with generators $\partial_1, \dots, \partial_n$. Then, there exists a unique n-twisted differential R-algebra $(A, \underline{\sigma}, \underline{D})$ of Schwarz type such that for all $i = 1, \dots, n$, we have*

$$\forall x \in A, \quad \partial_i x = D_i(x) + \sigma_i(x)\partial_i. \tag{6.1}$$

Conversely, given an n-twisted differential R-algebra of Schwarz type $(A, \underline{\sigma}, \underline{D})$, there exists, up to isomorphism, a unique n-twisted Weyl algebra B over R with generator $\partial_1, \dots, \partial_n$ over A satisfying (6.1)

Proof. Let us begin with the first assertion. Since multiplication respects the filtration, we can write for each $i = 1, \dots, n$, a relation (6.1) for some maps $\sigma_i, D_i : A \to A$. Moreover, distributivity in B implies that both maps are additive. Now, we have on one hand for $x, y \in A$,

$$(\partial_i x)y = (D_i(x) + \sigma_i(x)\partial_i)y = D_i(x)y + \sigma_i(x)\partial_i y$$

$$= D_i(x)y + \sigma_i(x)(\sigma_i(y)\partial_i + D_i(y)) = D_i(x)y + \sigma_i(x)D_i(y) + \sigma_i(x)\sigma_i(y)\partial_i$$

and on the other hand

$$\partial_i(xy) = D_i(xy) + \sigma_i(xy)\partial_i.$$

The multiplication law in B being associative, we must have

$$D_i(xy) = D_i(x)y + \sigma_i(x)D_i(y) \quad \text{and} \quad \sigma_i(xy) = \sigma_i(x)\sigma_i(y).$$

Thus we see that σ_i is multiplicative and D_i satisfies the σ_i-Leibnitz rule. Also, since B is an R-algebra, we must have

$$\forall a \in R, \quad a\partial_i = \partial_i a = D_i(a) + \sigma_i(a)\partial_i$$

It follows that $\sigma_i(a) = a$ and $D_i(a) = 0$ whenever $a \in R$. Then R-linearity follows for both maps. Now, for any $i, j \in \{1, \dots, n\}$ and $x \in A$, we have

$$\partial_i \partial_j x = \partial_i D_j(x) + \partial_i \sigma_j(x) \partial_j = D_i(D_j(x)) + \sigma_i(D_j(x))\partial_i + D_i(\sigma_j(x))\partial_j + \sigma_i(\sigma_j(x))\partial_i \partial_j$$

Since ∂_i and ∂_j commute, we must have

$$D_i \circ D_j = D_j \circ D_i \quad \text{and} \quad \sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$$

but also when $i \neq j$,

$$\sigma_i \circ D_j = D_j \circ \sigma_i.$$

Thus we see that all the expected commutation rules are satisfied.

The converse requires more work even if the uniqueness of B is clearly automatic. In order to show the existence of B, it is actually sufficient to build a *big* R-algebra C where all the expected relations hold and define B as the subalgebra generated by A and $\partial_1, \dots, \partial_n$. It will then automatically follow that any element of B can be written as a finite sum $\sum x_{\underline{k}} \partial^{\underline{k}}$ with $x_{\underline{k}} \in A$ (we use the standard multiindex notation), and it will only remain to check that this expression is unique. In other words, we will have to verify that $\sum x_{\underline{k}} \partial^{\underline{k}} \neq 0$ unless all $x_{\underline{k}} = 0$ (this is what *big* means here).

Let M be the free A-module on $\{e_{\underline{k}}\}_{\underline{k} \in \mathbb{N}^n}$ and $C = \mathrm{End}_R(M)$. We may clearly consider A as a subring of C and define ∂_i as the unique R-linear endomorphism of M such that

$$\forall x \in A, \forall \underline{k} \in \mathbb{N}^n, \quad \partial_i(x e_{\underline{k}}) = D_i(x) e_{\underline{k}} + \sigma_i(x) e_{\underline{k}+\underline{1}_i},$$

where we denote by $\underline{1}_i$ the element of $\mathbb{N}^n$ that has 1 in position i and 0 elsewhere. Let us check that formula (6.1) holds in C. If $x, y \in A$, we have

$$(\partial_i \circ x)(y e_{\underline{k}}) = \partial_i(xy e_{\underline{k}}) = D_i(xy) e_{\underline{k}} + \sigma_i(xy) e_{\underline{k}+\underline{1}_i} = (D_i(x)y + \sigma_i(x) D_i(y)) e_{\underline{k}} + \sigma_i(x)\sigma_i(y) e_{\underline{k}+\underline{1}_i}$$

$$= D_i(x) y e_{\underline{k}} + \sigma_i(x)(D_i(y) e_{\underline{k}} + \sigma_i(y) e_{\underline{k}+\underline{1}_i}) = D_i(x) y e_{\underline{k}} + \sigma_i(x) \partial_i(y e_{\underline{k}})$$

and it follows that

$$\partial_i \circ x = D_i(x) + \sigma_i(x) \circ \partial_i.$$

Also, for all $i, j \in \{1, \dots, n\}$ and $x \in A$, we have

$$\partial_i(\partial_j(x e_{\underline{k}})) = \partial_i(D_j(x) e_{\underline{k}} + \sigma_j(x) e_{\underline{k}+\underline{1}_j})$$

$$= D_i(D_j(x)) e_{\underline{k}} + \sigma_i(D_j(x)) e_{\underline{k}+\underline{1}_i} + D_i(\sigma_j(x)) e_{\underline{k}+\underline{1}_j} + \sigma_i(\sigma_j(x)) e_{\underline{k}+\underline{1}_i+\underline{1}_j}.$$

With our commutation assumptions, this is symmetric in i and j and it follows that ∂_i and ∂_j commute with each other. Finally, we have for all $i = 1, \dots, n$,

$$\partial_i(e_{\underline{k}}) = D_i(1) e_{\underline{k}} + \sigma_i(1) e_{\underline{k}+\underline{1}_i} = e_{\underline{k}+\underline{1}_i}$$

and it follows that for all for all $\underline{l} \in \mathbb{N}^n$, we have

$$\left(\sum x_{\underline{k}} \partial^{\underline{k}}\right)(e_{\underline{l}}) = \sum x_{\underline{k}} e_{\underline{k}+\underline{l}}.$$

Thus, we see that the condition $\sum x_{\underline{k}} \partial^{\underline{k}} = 0$ implies that $x_{\underline{k}} = 0$ for all $\underline{k} \in \mathbb{N}^n$. □

Definition 6.4. If $(A, \underline{\sigma}, \underline{D})$ is an n-twisted differential R-algebra of Schwarz type, the unique n-twisted Weyl algebra $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}$ over R with generator $\partial_1, \dots, \partial_n$ over A such that formulas (6.1) hold, is called the *Ore extension* of A by $\underline{\sigma}$ and $\underline{D}$.

The following examples show that this notion unifies several constructions from the literature.

Examples

1. When $A = R[x_1, \dots, x_n]$, all $\sigma_i = \mathrm{Id}_A$ and $D_i = \partial/\partial x_i$, then $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}$ is the usual Weyl algebra $A_n(R)$ over R.
2. More generally, when A is a smooth R-algebra with étale coordinates $x_1, \dots, x_n$, all $\sigma_i = \mathrm{Id}_A$ and $D_i = \partial/\partial x_i$, then $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}$ is the ring of differential operators *of level* 0 of A/R (not Grothendieck's ring of differential operators) that appears in crystalline cohomology. Actually, when R is a $\mathbb{Q}$-algebra, it is identical to the usual ring of differential operators.
3. Assume that $A = R[x]$ is endowed with an R-endomorphism σ and that $D = \partial_\sigma$ is given by formula (4.2) above. Then, $\mathrm{D}_{A/R,\sigma,D}$ is the noncommutative polynomial ring in two variables $R[x, \partial]$ with the commutation condition
$$\partial x = \sigma(x)\partial + 1.$$
In the particular case $R = \mathbb{Z}[q]$ and $\sigma(x) = qx$, this is the quantum Weyl algebra used by Gros and the first author in [7], definition 2.
4. Assume that $R = \mathbb{C}$ and $A = \mathbb{C}[x_1, \cdots, x_n]$ is endowed with
$$\sigma_i(x_j) = \delta_{ij} q x_j \quad \text{and} \quad D_i(x^{\underline{k}}) = (k_i)_q x^{\underline{k}-\underline{1}_i}$$
for some $q \neq 0$. Then $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}$ is identical to the n-th quantized Weyl algebra denoted by $A_n^{(q)}$ on page 319 in [2].

Remark Assume $R = K$ is a field containing $\mathbb{Q}(t)$, $A = K[x]$ and $\sigma(x) = tx$. Then the ring of quantum differential operators over $K[x]$ introduced by Lunts and Rosenberg in [20] and studied by Iyer and McCune in [10] is *not* the Weyl algebra in our sense. It is bigger in general since it always contains usual differential operators. Our ring is actually the $\sharp$ version introduced in section 1.2 of [20].

The n-twisted Weyl algebra satisfies the following universal property:

Proposition 6.5. *Let A be an n-twisted differential R-algebra of Schwarz type. Given a morphism of R-algebras $\varphi : A \to B$, and n commuting elements $y_1, \dots, y_n \in B$ such that*
$$\forall i = 1, \dots, n, \forall x \in A, \quad y_i \varphi(x) = \varphi(D_i(x)) + \varphi(\sigma_i(x)) y_i, \tag{6.2}$$
there exists a unique morphism of R-algebras $\Phi : \mathrm{D}_{A/R,\underline{\sigma},\underline{D}} \to B$ that extends φ and sends ∂_i to y_i for all $i = 1, \dots, n$.

Proof. By definition, we must have $\Phi(\sum x_{\underline{k}} \partial^{\underline{k}}) := \sum \varphi(x_{\underline{k}}) y^{\underline{k}}$ and uniqueness follows. It only remains to show that the map is multiplicative. This follows easily by induction from condition (6.2) since the y_i's commute with each other. □

Proposition 6.6. *Let A be an n-twisted differential R-algebra of Schwarz type. If M is a $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}$-module, then the map*
$$\begin{array}{ccc} M & \xrightarrow{D_{M,i}} & M \\ s & \longmapsto & \partial_i s \end{array}$$
turns M into an integrable n-twisted differential A-module and we obtain an equivalence (an isomorphism) of categories
$$\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}\text{–Mod} \simeq \underline{D}_A\text{–Mod}^{\mathrm{int}}.$$

Proof. This result follows immediately from the universal property of $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}$ applied to the canonical morphism $A \to B := \mathrm{End}_R(M)$ when M is an A-module. More precisely, a structure of $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}$-module

on M is given by n commuting R-endomorphism $D_{M,i}$ of M that satisfy

$$\forall i = 1, \dots, n, \forall x \in A, \forall s \in M, \quad D_{M,i}(xs) = D_i(x)s + \sigma_i(x)D_{M,i}(s).$$

This is exactly what we want. □

It follows from the proposition that if M is an integrable $\underline{D}_A$-module, then

$$\mathrm{R}\Gamma_{\underline{D}}(M) = \mathrm{RHom}_{\mathrm{D}_{A/R,\sigma,\underline{D}}}(A, M).$$

Examples

1. If (A, σ, D) is a 1-twisted differential R-algebra and (M, D) is a 1-twisted differential A-module, one may use the complex

$$\left[\mathrm{D}_{A/R,\sigma,D} \xrightarrow{\partial} \mathrm{D}_{A/R,\sigma,D}\right],$$

which is a free left resolution of A, to describe $H^i_\sigma(M)$ when M is a 1-twisted A-module.

1. More generally, in the n-twisted situation, one can use the Spencer complex of $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}$ with respect to $(\partial_1, \dots, \partial_n)$.

As a consequence of the proposition, we also see that, if M is any n-twisted integrable differential A-module, then the canonical morphism of R-algebras $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}} \to \mathrm{End}_R(M)$ takes values into the R-subalgebra generated by A and the $\underline{D}$-derivations of M. If we apply these considerations to the case $M = A$, we obtain:

Corollary 6.7. *There exists a morphism of R-algebras*

$$\mathrm{D}_{A/R,\underline{\sigma},\underline{D}} \to \overline{\mathrm{D}}_{A/R,\underline{\sigma}}$$

from the Weyl algebra to the ring of small twisted differential operators whose image is exactly the subring $\overline{\mathrm{D}}_{A/R,\underline{\sigma},\underline{D}}$ generated by A and the derivations (Definition 5.3). □

Corollary 6.8. *The category of integrable n-twisted differential A-modules is an abelian category with sufficiently many projective and injective objects.* □

Unlike the ring of small twisted differential operators, which does not even commute with extensions of R, the twisted Weyl algebra commutes with both extensions of R and A:

Proposition 6.9. 1. *Let A be an n-twisted differential R-algebra of Schwarz type, $R \to R'$ be a ring homomorphism and $A' := R' \otimes_R A$. Then there exists a canonical isomorphism*

$$R' \otimes_R \mathrm{D}_{A/R,\underline{\sigma},\underline{D}} \simeq \mathrm{D}_{A'/R',\underline{\sigma},\underline{D}}.$$

2. *If $A \to B$ is a morphism of n-twisted differential R-algebras of Schwarz type, then there exists a canonical isomorphism*

$$B \otimes_A \mathrm{D}_{A/R,\underline{\sigma},\underline{D}} \simeq \mathrm{D}_{B/R,\underline{\sigma},\underline{D}}.$$

Proof. In both cases, we can use the universal property of $\mathrm{D}_{A/R,\underline{\sigma},\underline{D}}$ in order to build the map and then compare basis on both sides. □

7. Twisted coordinates

In this last section, we are interested in twisted differential algebras, where the twisted derivations may be seen as twisted partial derivatives.

We denote by $(A, \underline{\sigma})$ an n-twisted commutative R-algebra: it means that we are given n commuting endomorphisms $\sigma_1, \dots, \sigma_n$ of a commutative R-algebra.

Definition 7.1. We say that $x_1, \dots, x_n \in A$ are *$\underline{\sigma}$-coordinates* for A/R if
1. there exists for each $i = 1, \dots, n$, a unique σ_i-derivation ∂_i of A/R such that

$$\forall j = 1, \dots, n, \quad \partial_i(x_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases} \tag{7.1}$$

2. for any $\underline{\sigma}$-derivation D of A/R, we have

$$D = \sum_{i=1}^{n} D(x_i)\partial_i. \tag{7.2}$$

Then, $(A, \underline{\sigma}, \underline{\partial})$ is called a *standard n-twisted differential R-algebra*. The $\underline{\sigma}$-coordinates are said to be *of Schwarz type* if the twisted differential algebra $(A, \underline{\sigma}, \underline{\partial})$ is of Schwarz type.

Alternatively, in the definition of $\underline{\sigma}$-coordinates, one may require that the evaluation map

$$\begin{aligned} T_{A/R,\underline{\sigma}} &\longrightarrow A^n \\ D &\longmapsto (D(x_1), \dots, D(x_n)) \end{aligned} \tag{7.3}$$

is bijective and that, for each $i = 1, \dots, n$, the unique $\underline{\sigma}$-derivation satisfying (7.1) is actually a σ_i-derivation.

Examples
1. When $n = 1$, an $x \in A$ is a σ-coordinate for A if and only if the evaluation map $T_{A/R,\sigma} \to A$ is bijective, and ∂ will then denote the unique σ-derivation such that $\partial(x) = 1$.
2. Assume that R is an integral domain and that $A := R[x_1, \dots, x_n]$ is endowed with n endomorphisms σ_i such that $\sigma_i(x_j) = x_j$ whenever $i \neq j$ and $\sigma_i(x_i) \neq x_i$ for all $i = 1 \dots, n$. Then one can check that $x_1, \dots, x_n$ are $\underline{\sigma}$-coordinates of Schwarz type for A.
3. If A is a smooth R-algebra and $\sigma_i = \mathrm{Id}_A$ for $i = 1, \dots, n$, then $x_1, \dots, x_n$ are $\underline{\sigma}$-coordinates for the n-twisted R-algebra A if and only they are étale coordinates for A over R (they define an étale map between the polynomial algebra and A). Étale coordinates are automatically of Schwarz type thanks to the usual Schwarz integrability conditions.
4. More generally, any q-deformation of the trivial n-structure of A turns A into a standard n-twisted differential algebra of Schwarz type. In particular, the methods developed here apply to the situation considered in [3].

By specializing the endomorphisms, one may recover the classical finite difference algebras and q-difference algebras, as well as usual differential algebras.

Definition 7.2. Let $x_1, \dots, x_n$ be $\underline{\sigma}$-coordinates for A.
1. If for all $i = 1, \dots, n$, we have $\sigma_i(x_i) = x_i + h_i$ with $h_i \in R \setminus \{0\}$ and $\sigma_i(x_j) = x_j$ for $i \neq j$, we call A a *finite difference algebra* and the corresponding modules *finite difference modules*.
2. If for all $i = 1, \dots, n$, we have $\sigma_i(x_i) = q_i x_i$ with $q_i \in R \setminus \{1\}$ and $\sigma_i(x_j) = x_j$ for $i \neq j$, we call A a *q-difference algebra* and the corresponding modules *q-difference modules*.
3. If for all $i, j \in \{1, \dots, n\}$, we have $\sigma_i(x_j) = x_j$, we call A a *(usual) differential algebra* and talk about *(usual) differential module*.

The existence of the twisted partial derivative has some important consequences as we will see shortly.

Lemma 7.3. *Let $x_1, \dots, x_n$ be $\underline{\sigma}$-coordinates for A. Then, if we set $y_i := x_i - \sigma_i(x_i)$ for all $i = 1, \dots, n$, we have*
1. $\forall j = 1, \dots, n, \quad i \neq j \Rightarrow \big(\sigma_i(x_j) = x_j \text{ and } \sigma_i(y_j) = y_j\big)$.
2. *if $\underline{x}$ is of Schwarz type, then* $\forall j = 1, \dots, n, \quad i \neq j \Rightarrow \partial_i(y_j) = 0$.

Proof. It follows from Lemma 4.5 that $\sigma_i(x_j) = x_j$ for $j \neq i$. Moreover, since σ_i and σ_j commute with each other, we also have

$$\sigma_i(y_j) = \sigma_i(x_j - \sigma_j(x_j)) = \sigma_i(x_j) - \sigma_i\sigma_j(x_j) = x_j - \sigma_j\sigma_i(x_j) = x_j - \sigma_j(x_j) = y_j.$$

Finally, if $\underline{x}$ is of Schwarz type, then we have $\partial_i\sigma_j = \sigma_j\partial_i$ for $j \neq i$, and it follows that

$$\partial_i(y_j) = \partial_i(x_j - \sigma_j(x_j)) = -\partial_i\sigma_j(x_j) = -\sigma_j\partial_i(x_j) = 0. \qquad \square$$

In a standard n-twisted differential algebra, we can recover the family of endomorphisms $\underline{\sigma}$ from the family of twisted derivations $\underline{\partial}$. More precisely, we have:

Proposition 7.4. *If $x_1, \dots, x_n$ are $\underline{\sigma}$-coordinates for A and we set $y_i := x_i - \sigma_i(x_i)$ for all $i = 1, \dots, n$, then we have*

$$\sigma_i = 1 - y_i\partial_i. \tag{7.4}$$

Proof. Since $1 - \sigma_i$ is a $\underline{\sigma}$-derivation thanks to Lemma 4.4, we have

$$1 - \sigma_i = \sum_{j=1}^{n} (x_j - \sigma_i(x_j))\partial_j = y_i\partial_i \tag{7.5}$$

because $\sigma_i(x_j) = x_j$ for $j \neq i$. $\square$

Definition 7.5. The $\underline{\sigma}$-coordinates $x_1, \dots, x_n$ are said to be *strong* if for all $i = 1, \dots, n$, we have $y_i := x_i - \sigma_i(x_i) \in A^\times$. We will also say that the n-twisted R-algebra is *strong*.

Examples
1. Assume that $n = 1$ and $A = R[x]$. Then x is a strong σ-coordinate for A if and only if $\sigma(x) = x + h$ with $h \in R^\times$.
2. Assume that $n = 1$ and $A = K(x)$, where $R = K$ is a field. Then x is a strong σ-coordinate for A if and only if $\sigma \neq \mathrm{Id}_A$.
3. Assume that $n = 1$, $A = R[x, x^{-1}]$ and $\sigma(x) = qx$ with $q \in R^\times$. Then x is a strong σ-coordinate for A if and only if $1 - q \in R^\times$. This is the case for example if $R = K$ is a field and $q \neq 0, 1$ or more generally, if R contains a field K with $q \in K \setminus \{0, 1\}$.

When A is strong, we can recover the twisted partial derivative ∂_i from the endomorphism σ_i by the formula

$$\partial_i = \frac{1}{y_i}(1 - \sigma_i). \tag{7.6}$$

The classical operators provide examples (we write them only for $n = 1$ to simplify notations).

Examples
1. In the finite difference situation, when $h \in R^\times$, the coordinate is strong. Writing then Δ_h instead of ∂, formula (7.6) gives back the classical formula for the finite difference operator:

$$\Delta_h(f)(x) = \frac{f(x + h) - f(x)}{h}.$$

2. In the q-difference situation, assuming $1-q \in R^\times$ and $x \in A^\times$, the coordinate is strong. Writing δ_q instead of ∂, formula (7.6) gives back the classical formula for the q-difference operator:
$$\delta_q(f)(x) = \frac{f(qx)-f(x)}{qx-x}.$$
3. When A is a (usual) differential algebra, we do write ∂, and we have the classical formulas (whenever they mean something):
$$\partial(f)(x) = \lim_{h\to 0} \Delta_h(f)(x) = \lim_{q\to 1} \delta_q(f)(x).$$

Many more examples come from the following result:

Proposition 7.6. *If B is an n-twisted localization of A and $x_1,\dots,x_n$ are $\underline{\sigma}$-coordinates for A/R, then their images in B are also $\underline{\sigma}$-coordinates for B/R.*

Proof. It follows from Proposition 4.9 that $T_{B/R,\underline{\sigma}} \simeq B\otimes_A T_{A/R,\underline{\sigma}}$ and the evaluation map (7.3) therefore stays an isomorphism after extension to B (and the ∂_i's are σ_i-derivations of B/R). □

Recall from Section 4 that we can associate to the n-twisted algebra $(A,\underline{\sigma})$ its module of $\underline{\sigma}$-derivations $\mathrm{T}_{A/R,\underline{\sigma}}$ and its ring of small twisted differential operators $\overline{\mathrm{D}}_{A/R,\underline{\sigma}}$. On the other hand, if we have $\underline{\sigma}$-coordinates, we may also consider the module of derivations $\mathrm{T}_{A/R,\underline{\sigma},\underline{\partial}}$ and the ring of small differential operators $\overline{\mathrm{D}}_{A/R,\underline{\sigma},\underline{\partial}}$ associated to the n-twisted differential algebra $(A,\underline{\sigma},\underline{\partial})$ (Definition 5.3). It should be clear that, in both cases, the latter, which is always contained in the former, is actually equal to it. Note also that we have here $\Omega^1_{A,\underline{\sigma},\underline{\partial}} = A^n$. More important for us, we have the following:

Proposition 7.7. *If A is a standard n-twisted differential R-algebra, then the category of A-modules endowed with an R-linear action by derivation of $\mathrm{T}_{A/R,\underline{\sigma}}$ is equivalent (and even isomorphic) to the category $\underline{\partial}_A-$Mod.*

Proof. This follows from the fact that $\{\partial_1,\dots,\partial_n\}$ is a basis of $\mathrm{T}_{A/R,\underline{\sigma}}$. □

Definition 7.8. If A is a standard n-twisted differential algebra, we call an A-module with a linear action of $\mathrm{T}_{A/R,\underline{\sigma}}$ by derivation *integrable* if the corresponding $\underline{\partial}_A$-module is integrable.

They form a category that we will denote by $\mathrm{MIC}(A,\underline{\sigma})$. Note that this is not an intrinsic definition since it seems to rely on the choice of the $\underline{\sigma}$-coordinates.

Definition 7.9. If $x_1,\dots,x_n$ are $\underline{\sigma}$-coordinates of Schwarz type for A/R, then the *standard twisted Weyl algebra* associated to $\underline{x}$ is the Ore extension $\mathrm{D}_{A/R,\underline{\sigma},\underline{\partial}}$ of A by $\underline{\sigma}$ and $\underline{\partial}$.

We will also denote by $\partial_1,\dots,\partial_n$ the corresponding generators of $\mathrm{D}_{A/R,\underline{\sigma},\underline{\partial}}$. They should not be confused with the $\underline{\sigma}$-derivations that we also call $\partial_1,\dots,\partial_n$. If necessary, we will write $\partial_{A,i}$ when we need to emphasize that we are considering the σ_i-derivation of A (and not one of the generators of the n-twisted Weyl algebra). Also, if we write as usual $y_i := x_i - \sigma_i(x_i)$, we will set below $\sigma_i := 1 - y_i\partial_i \in \mathrm{D}_{A/R,\underline{\sigma},\underline{\partial}}$. Then, we will insist on the notation $\sigma_{A,i}$ when we want to emphasize the fact that we are considering the endomorphism of A (and not the element of the twisted Weyl algebra). This being said, we see that, by definition, an element of the standard n-twisted Weyl algebra $\mathrm{D}_{A/R,\underline{\sigma},\underline{\partial}}$ may be uniquely written as a finite sum $\sum z_{\underline{k}}\underline{\partial}^{\underline{k}}$, with $z_{\underline{k}} \in A$, and multiplication is characterized by the properties
$$\partial_i z = \partial_{A,i}(z) + \sigma_{A,i}(z)\partial_i \quad \text{for } z\in A, \text{ and} \quad \partial_i\partial_j = \partial_j\partial_i.$$

Proposition 7.10. *If A is a standard n-twisted differential R-algebra of Schwarz type, then there exists a canonical surjective map*

$$\begin{aligned} D_{A/R,\underline{\sigma},\underline{\partial}} &\twoheadrightarrow \overline{D}_{A/R,\underline{\sigma}} \\ \partial_i &\longmapsto \partial_{A,i}. \end{aligned} \tag{7.7}$$

Moreover, if we set $y_i = x_i - \sigma_i(x_i)$ and $\sigma_i := 1 - y_i\partial_i \in \mathrm{D}_{A/R,\underline{\sigma}}$, then the image of σ_i is $\sigma_{A,i}$.

Proof. Follows from Corollary 6.7 and Proposition 7.4 where the equality is meant to hold in $\mathrm{T}_{A/R,\underline{\sigma}} \subset \overline{\mathrm{D}}_{A/R,\underline{\sigma}}$. □

Lemma 7.11. *Let $x_1, \dots, x_n$ be $\underline{\sigma}$-coordinates of Schwarz type for A. If for all $i = 1, \dots, n$, we let $y_i = x_i - \sigma_i(x_i) \in A$ and $\sigma_i := 1 - y_i\partial_i \in \mathrm{D}_{A/R,\underline{\sigma},\underline{\partial}}$, then we have*

1. $\forall z \in A, \quad \sigma_i z = \sigma_i(z)\sigma_i$
2. $\forall j = 1, \dots, n, \quad i \neq j \Rightarrow \sigma_i\partial_j = \partial_j\sigma_i$
3. $\forall j = 1, \dots, n, \quad \sigma_i\sigma_j = \sigma_j\sigma_i$

Proof. The first equality is easily checked. In order to prove the second identity, we compute

$$\sigma_i\partial_j = (1 - y_i\partial_i)\partial_j = \partial_j - y_i\partial_i\partial_j,$$

and

$$\partial_j\sigma_i = \partial_j(1 - y_i\partial_i) = \partial_j - \partial_j y_i\partial_i = \partial_j - \partial_j(y_i)\partial_i - y_i\partial_j\partial_i = \partial_j - y_i\partial_j\partial_i.$$

The last identity then easily follows. □

Theorem 7.12. *Let $(A, \underline{\sigma})$ be an n-twisted commutative R-algebra. If $x_1, \dots, x_n$ are $\underline{\sigma}$-coordinates of Schwarz type for A, then there exists a unique A-linear homomorphism of filtered R-algebras*

$$\begin{aligned} A[\underline{T}]_{\underline{\sigma}} &\longrightarrow \mathrm{D}_{A/R,\underline{\sigma},\underline{\partial}} \\ T_i &\longmapsto \sigma_i := 1 - (x_i - \sigma_i(x_i))\partial_i \end{aligned} \tag{7.8}$$

If the $\underline{\sigma}$-coordinates are strong, this map is an isomorphism.

Proof. Both uniqueness and existence of the homomorphism will follow from the universal property of $A[\underline{T}]_{\underline{\sigma}}$ once we have checked that

$$\forall i, j \in \{1, \dots, n\}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i$$

and

$$\forall i = 1, \dots, n, \forall z \in A, \quad \sigma_i z = \sigma_i(z)\sigma_i.$$

This was shown in Lemma 7.11. When $(A, \underline{\sigma})$ is strong, one can show in the same way that there exists a unique A-linear homomorphism of R-algebras

$$\begin{aligned} \mathrm{D}_{A/R,\underline{\sigma},\underline{\partial}} &\longrightarrow A[\underline{T}]_{\underline{\sigma}} \\ \partial_i &\longmapsto \tfrac{1}{y_i}(1 - T_i) \end{aligned} \tag{7.9}$$

with $y_i := x_i - \sigma_i(x_i)$. This time, we use the universal property of $\mathrm{D}_{A/R,\underline{\sigma},\underline{\partial}}$. We have to check that

$$\forall i, j \in \{1, \dots, n\}, \quad \frac{1}{y_i}(1 - T_i)\frac{1}{y_j}(1 - T_j) = \frac{1}{y_j}(1 - T_j)\frac{1}{y_i}(1 - T_i) \tag{7.10}$$

and

$$\forall i = 1, \dots, n, \quad \forall z \in A, \quad \frac{1}{y_i}(1 - T_i)z = \partial_{A,i}(z) + \sigma_{A,i}(z)\frac{1}{y_i}(1 - T_i). \tag{7.11}$$

The identity (7.10) is trivial when $j = i$. When $j \neq i$, it follows from Lemma 7.3 that $\sigma_i(\frac{1}{y_j}) = \frac{1}{y_j}$. Therefore, $\frac{1}{y_j}$ commutes with T_i and the left-hand side is symmetric on i and j. In order to show the identity (7.11), we first use formula (7.6) which tells us that $\partial_{A,i}(z) = \frac{1}{y_i}(z - \sigma_{A,i}(z))$, and the condition can be rewritten as

$$(1 - T_i)z = z - \sigma_{A,i}(z) + \sigma_{A,i}(z)(1 - T_i),$$

which in turn reduces to $T_i z = \sigma_i(z)T_i$. Finally, using the universal properties of both R-algebras, it is clear that the morphisms of (7.8) and (7.9) are inverse to each other. □

Corollary 7.13. *Let $(A, \underline{\sigma})$ be an n-twisted commutative R-algebra and $x_1, \dots, x_n$ $\underline{\sigma}$-coordinates of Schwarz type for A. If an A-module M is endowed with an integrable action of $\mathrm{T}_{A,\underline{\sigma}}$, then for all $i = 1, \dots, n$, the endomorphism defined by*

$$\forall s \in M, \quad \sigma_{M,i}(s) = s - (x_i - \sigma_{A,i}(x_i))\partial_{M,i}(s),$$

is $\sigma_{A,i}$-linear and we obtain a functor

$$\mathrm{MIC}(A, \underline{\sigma}) \to \underline{\sigma}_A\text{−Mod}.$$

This is an equivalence of categories when A is strong.

Of course, when A is strong, if we are given a $\underline{\sigma}_A$-module M, then the corresponding action will be given by

$$\forall s \in M, \quad \partial_{M,i}(s) = \frac{s - \sigma_{M,i}(s)}{x - \sigma_{A,i}(x)}.$$

Proof. As we saw in Proposition 3.10, there exists an equivalence of categories between $A[\underline{T}]_{\underline{\sigma}}$-modules and n-twisted A-modules, which in turn is equivalent to the category of $\underline{\sigma}_A$−Mod. We also showed in Proposition 6.6 that there exists an equivalence of categories between $\mathrm{D}_{A/R,\underline{\sigma},\underline{\partial}}$-modules and $\underline{\partial}_A$−Mod$^{\mathrm{int}}$, which by definition is equivalent to $\mathrm{MIC}(A, \underline{\sigma})$. The assertion therefore follows from Theorem 7.12. □

Remark Using this dictionary, we see that the modules considered by Claude Sabbah in [22] correspond to integrable twisted differential modules on $A := K[x_1, \dots, x_n]$ with respect to $\partial_{A,i}(\underline{x}^{\underline{k}}) = q^{k_i}\underline{x}^{\underline{k}-\underline{1}_i}$.

Example We can illustrate this equivalence of categories with the following classical case:

$$A = R[x, x^{-1}] \quad \text{and} \quad \sigma(x) = qx \text{ with } q, 1 - q \in R^\times.$$

Thus, we have

$$\forall n \in \mathbb{Z}, \quad \sigma(x^n) = q^n x^n \quad \text{and} \quad \partial(x^n) = (n)_q x^{n-1}.$$

Then, there exists an isomorphism

$$\begin{array}{ccc} R[x, x^{-1}][T]_\sigma & \xrightarrow{\simeq} & \mathrm{D}_{R[x,x^{-1}]/R,\sigma,\partial} \\ T & \longmapsto & 1 - (1 - q)x\partial \\ \frac{1-T}{(1-q)x} & \longleftarrow & \partial \end{array}$$

inducing an equivalence of categories

$$\left\{1-\text{Twisted differential modules on } R[x,x^{-1}]\right\} \simeq \left\{1-\text{Twisted modules on } R[x,x^{-1}]\right\}.$$

It is given by

$$\sigma_M(s) = s - (1-q)x\partial_M(s) \quad \text{and} \quad \partial_{M,\sigma}(s) = \frac{s-\sigma_M(s)}{(1-q)x}.$$

We want to stress out the fact that we do not make any assumption on R which may as well be a ring of characteristic $p > 0$ nor on q which may also be a p-th root of unity. And of course, this example extends to higher dimension.

In the situation of the corollary, we will always have a canonical map $\mathrm{R}\Gamma_{\underline{\partial}}M \to \mathrm{R}\Gamma_{\underline{\sigma}}M$ and this is an isomorphism when M is strong.

Examples

1. In the one dimensional case, one may use the notion of *derived division* and show that, when M has no y-torsion, then there exists an isomorphism

$$\mathrm{R}\Gamma_{\partial}M \simeq \mathrm{L}\eta_y \mathrm{R}\Gamma_{\sigma}M.$$

   Actually, the left-hand side is nothing but $M \xrightarrow{\partial} M$ and the right-hand side is $M \xrightarrow{(1-\sigma)/y} M$.
2. In general, if we assume that for each $i = 1,\dots,n$, $\sigma_i(x_i) = qx_i$ with x_i invertible and there is no $(1-q)$-torsion, then we obtain exactly as in [3], an isomorphism

$$\mathrm{R}\Gamma_{\underline{\partial}}M \simeq \mathrm{L}\eta_{1-q}\mathrm{R}\Gamma_{\underline{\sigma}}M.$$

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References

[1] André, Y. (2001). Différentielles noncommutatives et théorie de Galois différentielle ou aux différences. *Ann. Sci. École Norm. Sup. (4)* 34(5):685–739.
[2] Backelin, E. (2011). Endomorphisms of quantized Weyl algebras. *Lett. Math. Phys.* 97(3):317–338.
[3] Bhatt, B., Morrow, M., Scholze, P. (2016). Integral p-adic Hodge theory. arXiv:1602.03148v1 [math.AG].
[4] Bourbaki, N. (2012). *Éléments de Mathématique. Algèbre. Chapitre 8. Modules et Anneaux Semi-simples*. Berlin: Springer. Second revised edition of the 1958 edition.
[5] Cohn. (1965). *Difference Algebra*. New York: Interscience.
[6] Di Vizio, L., Ramis, J.-P., Sauloy, J., Zhang, C. (2003). Équations aux q-différences. *Gaz. Math.* 96:20–49.
[7] Gros, M., Le Stum, B. (2014). Une neutralisation explicite de l'algèbre de Weyl quantique complétée. *Commun. Algebra* 42(5):2163–2170.
[8] Gros, M., Le Stum, B., Quirós, A. (2017). Twisted divided powers and applications. arXiv:1711.01907v1 [math.AG]
[9] Hendriks, P. (1996). *Algebraic aspects of linear differential and difference equations*. PhD thesis, Rijksuniversiteit Groningen.
[10] Iyer, U. N., McCune, T. C. (2002). Quantum differential operators on $k[x]$. *Int. J. Math.* 13(4):395–413.
[11] Johnson, J. (1971). On Spencer's cohomology theory for linear partial differential operators. *Trans. Am. Math. Soc.* 154:137–149.
[12] Johnson, J. (1971). Extensions of differential modules over formal power series rings. *Am. J. Math.* 93(3):731–741.
[13] Kassel, C. (1995). Quantum groups. Volume 155 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.

[14] Kondrateva, M. V., Levin, A. B., Mikhalev, A. V., Pankratev, E. V. (1998). *Differential and Difference Dimension Polynomials*. Dordrecht: Kluwer Academic Publishers.
[15] Le Stum, B., Quirós, A. (2017). Twisted calculus on affinoid algebras. arXiv: 1712.03745v1 [math.AG]
[16] Le Stum, B., Quirós, A. (2018). Formal confluence of quantum differential operators. *Pacific J. Math.* 292(2):427–478.
[17] Le Stum, B., Quirós, A. (2015). On quantum integers and rationals. In: Fernando, C., Jordi, G., Antonio, R., JoséMaría, T., Ed., *Trends in Number Theory—Sevilla 2013*. Contemporary Math., Providence, RI: American Mathematical Society p. 107–130.
[18] Levin, A. B. (2008). *Difference Algebra*. New York: Springer.
[19] Levy, H., Lessman, F. (1992). *Finite Difference Equations*. New York: Dover Publications Inc.. Reprint of the 1961 edition.
[20] Lunts, V. A., Rosenberg, A. L. (1997). Differential operators on noncommutative rings. *Selecta Math. (N.S.)* 3(3):335–359.
[21] Pridham, J. (2016). On q-de Rham cohomology via Λ-rings. arXiv:1608.07142v2 [math.AG].
[22] Sabbah, C. (1993). Systèmes holonomes d'équations aux q-différences. In: Masaki, K., Teresa, M.F., Pierre, S., Ed. *D-Modules and Microlocal Geometry (Lisbon, 1990)*. Berlin: de Gruyter, pp. 125–147.
[23] Scholze, P. (2017). Canonical q-deformations in arithmetic geometry. *Ann. Fac. Sci. Toulouse Math*. 26(5):1163–1192.
[24] Spencer, D. C. (1969). Overdetermined systems of linear partial differential equations. *Bull. Am. Math. Soc.* 75:179–239.