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The class of states of the world as an ∞-groupoid: identifying states of the world and hierarchies of belief

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ABSTRACT

A natural problem in the methodology of Economics is how to model interactions in which the decisions made by agents depend on the beliefs held about both objective aspects of the relevant situation and the beliefs ascribed to other agents. There are two ways of representing such contexts. One is by means of a *hierarchy of beliefs*, the other as a *state of the world*. These two can be seen, from a Systems-theoretical perspective, as representing *streams* generated by a process and its *fixed points*, respectively. These two representations are in fact *equivalent*. The finer question we address here is whether they are *identical*. Resorting to a type-theoretical syntax, we obtain a unified interpretation in which both entities give rise to ∞-groupoids. Voevodsky's univalent axiom then ensures their identity in small types universes.

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1. Introduction

In this paper, we consider the characterization of how agents believe (and therefore end up behaving) in interactive contexts. A *hierarchy of beliefs* over beliefs can be constructed, representing the entire set of relevant information used by the agents to make decisions. Alternatively, we can consider a single global state in which agents believe exactly that this state is the case, called a *state of the world* in the interaction.

The main problem in the characterization of states of the world and belief hierarchies is to ensure that these two are in fact identical, so as to yield the same information held by the agents. This question has been in part answered by various authors over the last four decades, who have provided basically two main types of answer. One is appropriate when beliefs are represented as probability distributions over states of the world. The apparent resulting circularity is resolved by a straightforward application of Kolmogorov's extension theorem (see, for instance, Böge and Eisele 1979; Mertens and Zamir 1984; Brandenburger and Dekel 1993). The other kind of answer is more general and conceives a state of the world as a fixed point of an inductive process of definition of $k+1$-level beliefs in terms of k-level ones (see for instance Vassilakis 1991; Abramsky and Zvesper 2015). Both approaches are based on restricting, rather drastically, the underlying class of descriptions of the objective features of the interactive setting.

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Our take on this issue is to prove the identity of states of the world and hierarchies of belief without any further restriction of the underlying class of *states of nature*. We achieve this by interpreting the hierarchies generated by both forms of inductive process as belonging to an ∞-groupoid, an extension of the mathematical concept of *category* with invertible morphisms.

In this paper, we will show how to make the connection between these apparently disparate notions by means of a specific syntactical "bridge". More precisely, we will first develop a type-theoretical translation of the set-theoretical definition of states of the world. Next, we turn to an alternative semantics of this formalization, in the ∞-groupoid setting. The Univalent foundations of Homotopy Type Theory then yield, almost for free, the desired identity between hierarchies of beliefs and states of the world.

As said, this problem has been addressed in different ways by the literature on the epistemic foundations of Game Theory (Bonanno 1996; Brandenburger and Keisler 2006, among others), but always with a restricted range of validity. The general proof, based on the interpretation of the hierarchies of beliefs and states of the world as elements of an ∞-groupoid, is an original contribution of this work.

From a wider Systems-theoretical perspective, we can see that this problem is just one instance of a more general issue. Hierarchies of belief can be seen as *streams* generated by an underlying *process*, while states of the world are *fixed points* of that process. To determine whether these two characterizations represent identical states is a question that arises in different settings. For instance, it may arise when the process has an impredicative formulation (Louie and Poli 2012) or if it corresponds to an anticipatory system (Neuman 2002; Louie 2011) or if the process generates hyperstructures (Baas 2013, 2016) or when it is defined in reflexive domains (Abramsky and Zvesper 2015; Kauffman 2017). A variant of this problem is even implicit in the characterization of *blockchain* technology (Brooks 2017; Shorish 2018).

General treatments of such *streams* vs. *fixed points* problems are provided by the closely related approaches of *Non-Well-Founded* Set Theory (Barwise and Moss 1996) and the theory of *coalgebras* (Rutten 2000). The former replaces the *Foundation* axiom of Zermelo-Frenkel set theory with an alternative one, called the *Anti-Foundation* axiom, which allows in the theory for the existence of sets with circular definitions or with unending subset nestings. For the latter approach, an object A in a category $\mathcal{C}$ is a coalgebra if, given an endofunctor $F : \mathcal{C} \to \mathcal{C}$ there exists a morphism $\alpha : A \to F(A)$. Such a coalgebra is denoted (A, α). A homomorphism of coalgebras (A, α) and (B, β), for the same endofunctor F is given by a morphism $f : A \to B$ in $\mathcal{C}$ such that α and β commute in the obvious way through $F(f)$. A coalgebra (Z, γ) for F is then said to be *final* if there exists a unique homomorphism of coalgebras such that $! : X \to Z$ for every other coalgebra (X, ϕ) for F. If a process is defined by F, its stream is given by the application of γ while its fixed point is Z (Jacobs 2016).

The answer to the question of whether streams and fixed points constitute the same set in Non-Well-Founded set theory is trivially positive. But it comes at the price of allowing a definition of "fixed points" that may correspond to non-convergent streams. On the other hand, the coalgebraic approach depends critically on specific features of the category on which the endofunctor is defined, in particular on whether a final coalgebra exists. The approach taken in this paper, instead, combines the generality sought for by Non-Well-Founded set theory with the precision achieved by the theory of coalgebras. The particular

details of the belief hierarchy/states of the world problem in the context of interacting agent economic modeling help in this way to illuminate some of the fine structure of this more general Systems-theoretical issue by providing the occasion to demonstrate a distinctive new type of solution.

The plan of this paper is as follows. Section 2 introduces the notions of *hierarchy of beliefs* and *state of the world* as alternative full descriptions of an interactive agent situation, showing that these two are equivalent. Section 3 then presents the essential ideas required for our proof of the identity of states of the world and hierarchies of beliefs, namely the homotopy interpretation of dependent types and the ensuing structure of identities as ∞-groupoids. In particular, we present the interpretation of a type in this context as a *Kan complex*. Section 4 employs the latter framework to represent in turn the universe in which we prove the main result of this paper. Section 5 draws up our conclusions.

2. States of the world and belief hierarchies

There exist two main alternative ways of defining how players represent a game together with their uncertainties about the rules and their characterizations of the other players. One is in the form of *hierarchies of beliefs* and the other as *states of the world*. For the purposes of our subsequent analysis, we will present both of these approaches in a qualitative setting, that is, instead of representing beliefs as probability distributions (as it is usual in the literature), we will describe them set-theoretically.

For expository reasons we will present first *hierarchies of beliefs* and then *states of the world*. Nevertheless, as basic components of the former we will consider *finite-order* states of the world. The distinction between these latter and full-blown states of the world should be apparent from our characterization.

Consider n agents in an interactive setting (a *game*) such that each agent i has beliefs about such objective aspects as payoffs and game rules as well as about the possible beliefs of the other agents. This distribution of beliefs can be iterated in turn, thus yielding a *hierarchy* of beliefs about the beliefs about the beliefs, etc.[1]

The objective aspects of the game are summarized in a *state of nature*, $s = (s_0, s_1, \ldots, s_n)$, where each s_i, for $i = 1, \ldots, n$ characterizes those features corresponding to player i, while s_0 is some common piece of objective information.

We will, by a slight abuse of language, denote s as $(s'_1, \ldots, s'_n)$, where each $s'_i = \langle s, s_i \rangle$ is a state of agent i, composed of the common objective information, on the one hand, and her individual objective information, on the other.

Example 1: Consider the following interactive situation between two agents 1 and 2. Each has two possible actions of which she has to choose one. In the case of 1 the possible actions are a and b, while 2 has to choose between c and d. They choose independently, and once they have decided, the outcome of their choices generates a pair of *payoffs*, one for each player. The following matrix summarizes the game:

A :

	c	d
a	1, 1	1, 0
b	0, 0	0, 1

Suppose that the choices are, respectively, a and c. Thus, the state of nature will be $s = (s'_1, s'_2)$ where $s'_1 = \langle A, a\rangle$ and $s'_2 = \langle A, c\rangle$ or, for short (since A is redundant) $s = (a, c)$. ◇

The class of states of nature can be identified with a set Ω^0 of *states of the world of order* 0.[2] At s'_i, i can have beliefs about the objective states of the other agents $\mathcal{B}_i(s'_i) \subseteq \Omega^0_{-i}$. Here the *belief function* of agent i is such that $\mathcal{B}_i : \Omega^0_i \to 2^{\Omega^0_{-i}} \setminus \emptyset$, yielding the states of the others that i thinks may be the case. By $\mathcal{B}(s) = (\mathcal{B}_1(s'_1), \ldots, \mathcal{B}_n(s'_n))$ we denote the beliefs held by all the agents giving rise to a state $\omega^1 = \langle s, \mathcal{B}(s)\rangle$, and thus a class of *states of the world of order* 1, denoted Ω^1.

In general, $\mathcal{B}_i : \cup_{k=0}^{\infty}\Omega^k \to \cup_{k=0}^{\infty}\Omega^k$ such that, given a class Ω^k, we can obtain the class $\Omega^{k+1} = \{\omega^{k+1} : \omega^{k+1} = \langle s, \mathcal{B}(\omega^k)\rangle,$ for some $\omega^k \in \Omega^k\}$.

Example 1 (ctd.): Since $s = (a, c)$ we have that Ω^0 will have four elements that can be identified with the entries in matrix A:

$$\Omega^0 = \{(a, c), (a, d), (b, c), (b, d)\}.$$

Suppose that at $s = (a, c)$, $\mathcal{B}_1(a) = \{c, d\}$ while $\mathcal{B}_2(c) = \{a\}$. This can be interpreted as indicating that if 1 chooses action a, she thinks that 2 can choose either c or d, while 2, having chosen c thinks that 1 can only choose a.

Then Ω^1 will consist of states of the form $\langle(a, c), \mathcal{B}(a, c)\rangle$. Fully spelled out, we see that there are two states in Ω^1:

$$\Omega^1 = \{\omega^1 = \langle(a, c), c, a\rangle, \omega^{1\prime} = \langle(a, c), d, a\rangle\}$$

We can write $\omega^1 = (\omega^1_1, \omega^1_2)$, where $\omega^1_1 = \langle(a, c), c\rangle$ and $\omega^1_2 = \langle(a, c), a\rangle$.

Now suppose that $\mathcal{B}_1(\omega^1_1) = \mathcal{B}_1(\langle(a, c), c\rangle) = \{\langle(a, c), a\rangle\}$ and $\mathcal{B}_2(\omega^1_2) = \mathcal{B}_1(\langle(a, c), a\rangle) = \{\langle(a, c), c\rangle\}$. That is, 1 at state ω^1 thinks that the state of nature is (a, c) and that 2 thinks she has chosen a, while 2 at the same state thinks that the state of nature is (a, c) and that 1 believes she has chosen c.

Then we can define a state

$$\omega^2 = \langle(a, c), \langle(a, c), a\rangle, \langle(a, c), c\rangle\rangle.$$

On the other hand, if $\mathcal{B}_1(\omega^{1\prime}_1) = \mathcal{B}_1(\omega^1_1)$ and since $\omega^1_2 = \omega^{1\prime}_2$, we have that the corresponding state $\omega^{2\prime}$ is equal to ω^2. Then, Ω^2 has only a single state.

Iteratively, we can define Ω^3, etc. ◇

As said, we want to construct hierarchies of beliefs by means of the iterative application of $\mathcal{B}$:

$$\langle s, \omega^1, \omega^2, \ldots, \omega^n, \ldots\rangle = \langle s, \mathcal{B}(s), \mathcal{B}(s, \mathcal{B}(s)), \mathcal{B}(s, \mathcal{B}(s, \mathcal{B}(s))), \ldots, \rangle$$

We impose a pair of conditions on the finite initial segments of a hierarchy $\langle s, \omega^1, \ldots, \omega^n, \ldots\rangle$:

- The *consistency* condition: each order k state ω^k in $\langle s, \omega^1, \ldots, \omega^k, \ldots, \omega^n\rangle$ is the restriction over Ω^k of ω^{n+1}.

- On the other hand, we have a *nesting* condition that

$$\omega^{n+1} = \langle s, \mathcal{B}(\omega^n)\rangle,$$

$$\omega^{n+1} = \langle s, \mathcal{B}(\langle s, \mathcal{B}(\omega^{n-1})\rangle)\rangle,$$

$$\omega^{n+1} = \langle s, \mathcal{B}(\langle s, \mathcal{B}(\langle s, \mathcal{B}(\omega^{n-2})\rangle)\rangle)\rangle,$$

  etc.

These two conditions are captured by a one-to-one function

$$\mathbf{b} : \Omega^{\infty} \to \bigcup_{k=1}^{\infty} \Omega^k$$

where Ω^{∞} is the class of sequences $\langle s, \omega^1, \ldots, \omega^k, \omega^{k+1}, \ldots\rangle$ such that $s \in \Omega^0$ while $\omega^k \in \Omega^k$ for $k \geq 1$. Then, for any $n \geq 0$

$$\mathbf{b}(s, \omega^1, \ldots, \omega^n) = \omega^{n+1}$$

Example 1 (ctd.): Recall that since $|\Omega^2| = 1 = |\Omega^3|$ it follows that $|\Omega^k| = 1$ for every $k > 3$. Then, we have a hierarchy:

$$\langle (a, c), \omega^1, \omega^2, \ldots, \omega^2, \ldots\rangle$$

We can see that this hierarchy satisfies the condition of consistency since both $s = (a, c)$ and $\omega^1 = \langle (a, c), c, a\rangle$ can be naturally seen as restrictions of

$$\omega^2 = \langle (a, c), \langle (a, c), a\rangle, \langle (a, c), c\rangle\rangle$$

on certain of its "coordinates".

On the other hand, we already saw that ω^2 obtains by iterations of $\mathcal{B}$ over (a, c) and then over ω^1. ◇

We can now define the class of *states of the world* $\hat{\Omega}$:

$$\hat{\Omega} = \{\omega : \omega = \langle s, \mathcal{B}(\omega)\rangle\} \tag{1}$$

where $\mathcal{B}$ is, as before, the belief formation operator. That is, any element $\omega \in \hat{\Omega}$ is an ordered pair consisting of an underlying state of nature s and the beliefs about ω, $\mathcal{B}(\omega)$. This non-well-founded definition amounts to consider such ω as a fixed point of $\langle s, \mathcal{B}(\cdot)\rangle : \hat{\Omega} \to \hat{\Omega}$.[3]

$\hat{\Omega}$ comes equipped with the following function:

$$\lambda : \hat{\Omega} \times \mathbb{N} \to \bigcup_{k=0}^{\infty} \Omega^k$$

such that $\lambda(\omega, n) = \omega^n \in \Omega^n$, for any $\omega \in \hat{\Omega}$ and $n \in \mathbb{N}$, satisfying

$$\mathbf{b}(\lambda(\omega, 0), \lambda(\omega, 1), \ldots, \lambda(\omega, n)) = \lambda(\omega, n + 1)$$

The relation between states of the world and hierarchies should be apparent. Expression 1 indicates that in the iterative definition, if $\Omega^{k+1} = \Omega^k$, $\hat{\Omega}$ consists of the k-order states of the world.

Example 1 (ctd.): Consider $\hat{\Omega} = \{\omega^2\}$. We have already shown that $\langle (a, c), \mathcal{B}(\omega^2)\rangle = \omega^2$. $\lambda(\omega^2, \cdot)$ yields the term in the hierarchy of beliefs $\langle (a, c), \omega^1, \omega^2, \omega^2, \ldots, \omega^2, \ldots\rangle$. ◇

As suggested by the example, hierarchies of beliefs and states of the world are indeed closely related:

Proposition 1: *$\hat{\Omega}$ and Ω^∞ are isomorphic.*

***Proof*:** Trivial. Consider $\omega \in \hat{\Omega}$. Given $\lambda(\omega, \cdot) : \mathbb{N} \to \bigcup_{k=0}^{\infty} \Omega^k$ we obtain a class $\{\lambda(\omega, n)\}_{n=0}^{\infty} = \{\omega^n\}_{n=0}^{\infty}$ which satisfies the condition that each $\omega^{k+1} = \mathbf{b}(\omega^0, \ldots, \omega^k)$. This, in turn, gives rise to a sequence $(\omega^0, \omega^1, \ldots, \omega^k, \ldots)$. Since $\mathbf{b}$ is one-to-one, it ensues that $\omega \in \hat{\Omega}$ and $(\omega^0, \omega^1, \ldots, \omega^k, \ldots) \in \Omega^\infty$ are in a one-to-one relation. □

In the example, we saw that a state of the world can be "unfolded" in order to yield the hierarchy of beliefs to which it is isomorphic. Thus, it is natural to conceive these two as in fact representing the *same* object. As previously mentioned, the extant literature presents many proofs of the identity of states of the world and the corresponding hierarchies of beliefs under particular specifications of Ω^0 and $\mathcal{B}$.

At this point in the analysis, however, we are only able to state a significantly weaker claim. The isomorphism in Proposition 1 determines merely an *equivalence* relation between elements in $\hat{\Omega}$ and Ω^∞. Our goal is to strengthen this equivalence to an identity.

3. A brief review of homotopy type theory

In this section, we will review the main mathematical tools we will apply in order to fulfill the task posed at the end of the previous section. Namely, that is, to establish the identity between the class of states of the world $\hat{\Omega}$ and that of hierarchies of beliefs, Ω^∞.

A starting point is to use a type-theoretical language. While there are different varieties, we will focus on *Dependent Type Theory* (Martin-Löf 1975). The basic elements of its syntax are:

- **Types:** $X, Y, \ldots, A \times B, A \to B, \ldots$
- **Terms:** $x : A, b : B, \langle a, b\rangle, \lambda x.b(x), \ldots$
- **Dependent Types:** $x : A \vdash B(x), \sum_{x:A} B(x), \prod_{x:A} B(x)$
- **Equations:** $s = t : A$
- **Inductive Types**, defined by a base case and an inductive step.

Type Theory has traditionally admitted two classical interpretations:

- Types are *sets* and terms are *elements*.
- Types are *propositions* and the terms *proofs*.

where the former is the basis of Bertrand Russell's interpretation while the latter is called the *Curry-Howard correspondence*. But there are indeed other possible interpretations. For instance:

- Types are *problems* and terms *solutions* (due to Kolmogorov).
- Types are seen as *data types* and terms as *instances*.

A new approach, known as *Homotopy Type Theory* (HoTT) is based on the *homotopical* interpretation of types (The Univalent Foundations Program 2013)[4]:

- Types are *spaces* and terms are *points* in them.

This interpretation is particularly useful for the interpretation of dependent types. An expression as $x : A \vdash B(x)$ can be seen as a *fibration* $B \to A$. More precisely, a dependent-type $B(x)$ is the fiber $p : B \to A$ on point $x : A$, i.e. p is a continuous map between B and A (satisfying a certain homotopy lifting property). As a counterpart, we have a *section* $s : A \to B$ such that $p \circ s = 1_A$.

A particularly important dependent type that can be defined is the *identity* on a type A. That is, $x, y : A \vdash \mathrm{Id}_A(x, y)$, which can be interpreted as the *proposition* "x is identical to y". In particular, if two terms of type A are equal ($x = y$) there exists a term $t : \mathrm{Id}_A(x, y)$ (the converse is not in general true for interpretations other than set theory).

The homotopy interpretation of $x, y : A \vdash \mathrm{Id}_A(x, y)$ requires a fibration over $A \times A$. The idea is that a fiber $\mathrm{Id}_A(a, b)$ on $(a, b) \in A \times A$ is the *space of paths* from a to b in A.

If we have two terms $f, g : \mathrm{Id}_A(a, b)$, f is identical to g iff there exists a term t of type $\mathrm{Id}_{\mathrm{Id}_A(a,b)}(f, g)$. The interpretation of this somewhat recursive notation is that f and g are paths from a to b in space A, while t is a *homotopy* between f and g.[5]

We can thus generate a hierarchy of identity types:

$$a, b : A$$

$$f, g : \mathrm{Id}_A(a, b)$$
$$s, t : \mathrm{Id}_{\mathrm{Id}_A(a,b)}(f, g)$$
$$\alpha, \beta : \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A(a,b)}(f,g)}(s, t)$$
$$\dots$$

The first two levels of the hierarchy of identity types can be modeled as a *groupoid*, i.e. a category in which all the morphisms are invertible (Hofmann and Streicher 1998). This allows for the interpretation of all terms of order 0 and 1 (*points* and *paths*):

- A term $f : \mathrm{Id}_A(a, a)$ yields the *reflexivity* of $\to$ between objects in the groupoid: $a \to a$.
- A term g of type $\mathrm{Id}_A(a, b) \to \mathrm{Id}_A(b, a)$ corresponds to the *symmetry* $a \rightleftarrows b$.
- A term h of type $\mathrm{Id}_A(a, b) \times \mathrm{Id}_A(b, c) \to \mathrm{Id}_A(a, c)$ is associated to the *transitivity* $a \to b \to c$.

But this doesn't cover all the possibilities. In fact, what is needed is an ∞-*groupoid*: in addition to objects and invertible morphisms, we need invertible morphisms between morphisms, invertible morphisms between morphisms between morphisms, etc.

In an ∞-groupoid we have representations for terms of *all* orders (*points*, *paths*, *homotopies of paths*, etc.). So, in general we can have that a path f from a to b in space A composed with its inverse f^{-1} yields a closed path I_a, not directly but homotopically. That is, $t : \mathrm{Id}_{\mathrm{Id}_A(a,a)}(f^{-1} \circ f, I_a)$.

The system of identities of all orders

$$A \leftleftarrows \mathrm{Id}_A \leftleftarrows \mathrm{Id}_{\mathrm{Id}_A} \leftleftarrows \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A}} \leftleftarrows \dots$$

where $\leftleftarrows$ indicates that each leg is a fiber over the cartesian product of the type on the left side, constitutes the ∞-**fundamental groupoid** of type A.

Then we have

Theorem 1 (Lumsdaine 2010): *The system of identities of all orders on a given type is an ∞-groupoid.*

An alternative characterization shows that an ∞-groupoid G is an $(\infty, 0)$-category. Besides objects, G has morphisms of all orders $k \geq 1$ (a morphism of order $k+1$ is a morphism between morphisms of order k) and morphisms of order k are *iso* for $k > 0$.[6]

Another way of conceiving an ∞-groupoid is as a *Kan complex*.[7] To elaborate on this concept, consider the category Δ of *simplices* in which (Friedman 2012),

- objects are non-empty totally ordered sets, $[0] = \{0\}$, $[1] = \{0, 1\}$, $[2] = \{0, 1, 2\}$, etc.
- morphisms are order-preserving functions.

A *simplicial set* is a *presheaf* from Δ to the category of small sets , i.e. a functor $S : \Delta^{\mathrm{op}} \rightarrow \mathbf{Set}$. Suppose given X , a space of interest. A simplicial set can be obtained assigning a simplex to a family of subsets of X: to $[k]$ of Δ it assigns a family $S[k] \subseteq 2^X$, such that each of its elements are of cardinality $k+1$.[8] The contravariant definition of a simplicial set means that given two simplices $[k-1]$ and $[k]$ in Δ, their assigned sets are such that all the faces of the $S[k]$s (the restrictions of the functor on subsets of cardinality $k-1$) are in $S[k-1]$.

In more intuitive terms, a simplicial set yields, for every k, a covering of X by simplices of order k (i.e. elements of $S[k]$) with the property that the higher the k, the coarser the covering of X.

By a slight abuse of language, we call Δ^k a generic simplex in $S[k]$ while Λ^k_j is, for $0 \leq j \leq k$, constituted by Δ^{k-1}s containing the jth vertex of Δ^k. Λ^k_j is called a *horn*.

Consider now a simplicial set $\mathfrak{X}$ on X. $\mathfrak{X}$ is then a *Kan complex* (Lurie 2008) if for every k and every j with $0 \leq j \leq k$, Λ^k_j can be extended to a Δ^k. That is, each simplex in the covering "fills in" the skeleton made by its horns. Formally, it means that being given $s : \Lambda^k_j \rightarrow \mathfrak{X}$ yields the $k+1$ k-simplexes in $\mathfrak{X}$ that can be seen as the faces of Δ^k that belong to the horn. Furthermore, given a family of *degeneracy maps* $d_j : \Delta^k \rightarrow S[k-1]$, $j = 1, \ldots, k+1$, that yield the jth face of Δ^k, we have that $d_i s_j = d_{j-1} s_i$ for $i < j$ and $i, j \neq k$.

Example 2: Consider $\mathfrak{X}$, defined over a class of elements $\{x, y, z\}$ and, given $[0]$, $[1]$ and $[2]$ a presheaf S such that $S[0] = \{\{x\}, \{y\}, \{z\}\}$, $S[1] = \{\{x, y\}, \{x, z\}, \{y, z\}\}$ and $S[2] = \{\{x, y, z\}\}$. It can be easily checked that $\mathfrak{X}$ is a simplicial set.

Let us denote with Δ^2 the only element in $S[2]$. And the degeneracy maps defined on Δ^2 as d_1, d_2 and d_3 can be, by a slight abuse of language, identified with the faces $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$, respectively. Then consider the horn $\Lambda^2_3 = \{\{x, y\}, \{x, z\}\}$.

Now consider $s : \Lambda^2_3 \rightarrow \mathfrak{X}$ yielding s_1 and s_2 that can be associated to $\{x, y\}$ and $\{x, z\}$, respectively. Then, $d_1 s_2 = d_1 s_1 = \{x\}$. Similarly, this holds for all other elements in $\mathfrak{X}$, and we can thus see that it is a Kan complex.$\diamond$

The Kan complex $\mathfrak{X}$ is a model of the *fundamental ∞-groupoid* of the space X (Joyal 2005). This has as a particular consequence that the coverings are mutually compatible, due to the isomorphisms at all levels in an ∞-groupoid. That is, the coverings by k and $k-1$

simplices (faces, which belong to the horns, are simplices of order $k-1$) are compatible and, by the defining property of a Kan complex, they are also compatible with the covering of level $k-2$ (the vertices of the covering by k-simplices).

These compatibility properties matter for the definition of the *identity* of types in the *universe of small types*[9] $\mathcal{U}$ (Pelayo and Warren 2014):

- A type $A:\mathcal{U}$ is of level $h=0$ if it is contractible (i.e. can be seen as homotopically equivalent to a point).
- A type $A:\mathcal{U}$ is of level $h=k+1$ if the type Id_A is of level k.
- A type $A:\mathcal{U}$ is of level $h=k$ if it has the homotopy type of types at level $h=k-2$.
- For any k, if $A:\mathcal{U}$ or $B:\mathcal{U}$ are of level $h=k+1$, $\mathrm{Id}(A,B)$ is a type of level $h=k$.

Here h defines a *filtration* at all levels in $\mathcal{U}$. A condition that ensures the existence of such filtration is Pelayo and Warren (2014)[10]:

- **Univalence axiom**: *the function* $\iota : Id_{\mathcal{U}}(A,B) \to Eq_{\mathcal{U}}(A,B)$ *is an equivalence, for any pair of types* A, B *in* $\mathcal{U}$.

This axiom can be understood as saying that *equality expands to cover equivalences*. This means that if we consider a partition given by equivalences, we treat all the elements of any class of the partition as being equal.

The question of whether there exists a universe $\mathcal{U}$ in which the Univalence axiom is satisfied was answered by Vladimir Voevodsky:

Theorem 2 (Kapulkin, Lumsdaine, and Voevodsky 2014): *If there exist at least two inaccessible cardinals* $\phi < \psi$, *there exists a* $\mathcal{U}_\phi$ *which is a* universe *in which the univalence axiom is satisfied.*

4. Main results

We are now in condition to restate the main statements of Section 2 in the language of type theory, assuming the homotopical interpretation of types:

(1) $\langle s,\ldots,\omega^k,\ldots\rangle : \Omega^\infty$, i.e. a hierarchy of beliefs $\langle s,\ldots,\omega^k,\ldots\rangle$ is a term of type Ω^∞.
(2) $\omega : \hat{\Omega}$, i.e. a state of the world ω is of type $\hat{\Omega}$.
(3) $\omega : \hat{\Omega} \vdash \langle s,\ldots,\omega^k,\ldots\rangle : \Omega^\infty$. This is the gist of Proposition 1.

Recall that Condition 3 can be interpreted as the existence of a *fibration* $\Omega^\infty \to \hat{\Omega}$ that defines $\langle s,\ldots,\omega^k,\ldots\rangle$ on a point ω in $\hat{\Omega}$. Intuitively, the corresponding section unfolds ω in the full sequence $\langle s,\ldots,\omega^k,\ldots\rangle$.

Furthermore, again according Proposition 1, we have that at any small types universe $\mathcal{U}$ we have

$$\Omega^\infty, \hat{\Omega} : \mathcal{U} \vdash \mathrm{Eq}_{\mathcal{U}}(\Omega^\infty, \hat{\Omega})$$

and

$$\alpha : \mathrm{Eq}_{\mathcal{U}}(\Omega^\infty, \hat{\Omega}) \tag{2}$$

i.e. the dependent type of equivalences between Ω^∞ and $\hat{\Omega}$ is populated by a term, which we denote α.

The main question we intend to answer is whether

$$\Omega^{\infty}, \hat{\Omega} : \mathcal{U} \vdash \mathrm{Id}_{\mathcal{U}}(\Omega^{\infty}, \hat{\Omega})$$

and

$$\beta : \mathrm{Id}_{\mathcal{U}}(\Omega^{\infty}, \hat{\Omega})$$

i.e. whether there exists an identity β between Ω^{∞} and $\hat{\Omega}$. In particular, we want to see if $\langle s, \ldots, \omega^k, \ldots \rangle$ is *identical* to ω.

The question of the equality of a state of the world ω and its unfolding $\langle s, \ldots, \omega^k, \ldots \rangle$ can be indeed be posed and answered in the aforementioned terms noticing that $\hat{\Omega}$ and Ω^{∞} can be seen, rather trivially, as ∞-groupoids. To show this, we will appeal to the concept of *Kan complexes*.

Let us start by defining two simplicial sets $\mathfrak{O}$ and $\mathfrak{O}^{\infty}$ corresponding to $\hat{\Omega}$ and Ω^{∞}, respectively.

- $\mathfrak{O}$
  We start with $\hat{\Omega}$, defining a partial ordering over states of the world: given $\omega, \omega' \in \hat{\Omega}$, we say that $\omega \leq \omega'$ iff
  - $\lambda(\omega, 0)$ is a *state of nature s*.
  - there exists $K \geq 0$ such that $\{\lambda(\omega, j)\}_{j=0}^{K} \subseteq \{\lambda(\omega', j)\}_{j=0}^{\infty}$

  Then we can define $S : \Delta^{\mathrm{op}} \to \hat{\Omega}$ as follows. Given any $[k]$ in Δ, $k \geq 0$, each $\Delta^k \in S[k]$ is a sequence $(\hat{\omega}_0, \hat{\omega}_1, \ldots, \hat{\omega}_k)$, where $\hat{\omega}_i \in \hat{\Omega}$ for $i = 1, \ldots, k$ and $\hat{\omega}_i \leq \hat{\omega}_j$ for $0 \leq i < j \leq k$. That is, Δ^k is a chain of states of the world, ordered by $\leq$, of length $k + 1$.
  The faces of Δ^k are sequences $(\bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_{k-1})$, where each $\bar{\omega}_i \in \hat{\Omega}$ and $\bar{\omega}_i \leq \bar{\omega}_j$ for $0 \leq i < j \leq (k-1)$ is such that for each $i = 0, \ldots, (k-1)$, there exists $j \in \{0, \ldots, k\}$ satisfying that $\bar{\omega}_i = \hat{\omega}_j$. That is, each face of Δ^k is a subchain of length $k - 1$ of the chain corresponding to Δ^k.
- $\mathfrak{O}^{\infty}$
  Analogously to the definition of $\mathfrak{O}$, we consider an ordering $\leq^{\infty}$ over Ω^{∞}. That is, given two elements $\langle \omega^0, \ldots, \omega^k, \ldots \rangle$ and $\langle \omega'^0, \ldots, \omega'^k, \ldots \rangle$ in Ω^{∞}, we say that $\langle \omega^0, \ldots, \omega^k, \ldots \rangle \leq^{\infty} \langle \omega'^0, \ldots, \omega'^k, \ldots \rangle$, iff
  - ω^0 is a *state of nature s*.
  - there exists $K \geq 0$ such that $\{\omega^j\}_{j=0}^{K} \subseteq \{\lambda(\omega^j)\}_{j=0}^{\infty}$

  Each Δ^k can be seen here as a chain of $k + 1$ hierarchies of beliefs, ordered by $\leq^{\infty}$, while its faces are its $k + 1$ subchains of length k.

These two structures satisfy the following:

Proposition 2: *$\mathfrak{O}$ and $\mathfrak{O}^{\infty}$ are Kan complexes.*

***Proof*:** We can check this for $\mathfrak{O}$ since the argument for $\mathfrak{O}^{\infty}$ is strictly analogous. By our definition

- $\langle \hat{\Omega}, \leq \rangle$ is a partially ordered set. In fact, it is easy to check that $\leq$ is a reflexive (when K is infinite) and transitive relation.

- The chains of $k+1$ elements ordered by inclusion are understood as simplexes corresponding to $[k]$ and each subchain with one element eliminated (a chain of length k) is a face.

This construction is the *nerve* of $\langle \hat{\Omega}, \leq \rangle$. The nerve of any poset (and actually of any small category) is a simplicial set (see Section 1.4 in Joyal and Tierney (2008)).

To see that $\mathfrak{O}$ is also a Kan complex, consider a horn Λ^k_j of a Δ^k in $\mathfrak{O}$. It is constituted by simplices in $S[k-1]$ (call them $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{k+1}$). By definition, each $s_i \in \Lambda^k_j$ is a subchain of length k of the chain Δ^k of length $k+1$. Then degeneracy maps d_i and d_{j-1} (which extract the ith and the $(j-1)$th elements in an ordered chain, respectively) and considering s_j and s_i that add an element at the jth and ith positions of a chain of length $k+1$ we can see that $d_i s_j = d_{j-1} s_i$, yielding the same chain, for $i < j$, with $i, j \neq k$. That is, $\mathfrak{O}$ satisfies the Kan condition. □

We are now finally in the position to prove the identity of $\hat{\Omega}$ and Ω^∞:

Theorem 3: *If there exist at least two inaccessible cardinals $\phi < \psi$, each state of the world ω is equal to its unfolding $\langle s, \ldots, \omega^k, \ldots \rangle$.*

***Proof*:** We have that $\hat{\Omega}$ and Ω^∞, represented by $\mathfrak{O}$ and $\mathfrak{O}^\infty$, respectively, can be seen as types in a small types universe $\mathcal{U}_\phi$. Furthermore, since they are equivalent (according to Proposition 1), by Theorem 2 we can define a filtration h in which this is equivalent to $\mathrm{Id}_{\mathcal{U}_\phi}(\Omega^\infty, \Omega)$. In particular, we see that given ω in $\hat{\Omega}$ and its unfolding $\langle s, \ldots, \omega^k, \ldots \rangle$ in Ω^∞, these terms are identical in $\mathcal{U}_\phi$. □

5. Conclusion

We have shown that one instance of the more general Systems-theoretical problem of streams vs. fixed points that resides in the epistemic foundations of Economics, namely the equality of states of the world in interactions and their unfoldings as hierarchies of belief can be analyzed fruitfully in the context of Homotopy Type Theory. More precisely, we first presented the classes of states of the world and of hierarchies of belief, $\hat{\Omega}$ and Ω^∞, respectively, and proved that they are isomorphic. Then, interpreting these as types and resorting to their interpretation as spaces, we showed that they can be construed as Kan complexes, and thus as ∞-groupoids. Finally, the Univalent axiom then allowed us to show that their equivalence is itself equivalent to the epistemic identity of $\hat{\Omega}$ and Ω^∞.

Notes

1. This approach to the treatment of uncertainty in games was introduced in the literature by Harsanyi (1967) and has been extended to treatments of uncertainty about the actions that will be chosen by the players (see for instance Brandenburger and Keisler 2006; Brandenburger, Friedenberg, and Keisler 2008). The logic of this construction and how it leads to some kind of "fixed point" is discussed in Bonanno (1996) and Abramsky and Zvesper (2015). Our presentation is related to that in Tohmé (2005) and in Fagin et al. (1999).
2. $\Omega^0 = \prod_{i=1}^n \Omega^0_i$ is, from the standpoint of agent i, denoted $\Omega^0 = \Omega^0_i \times \Omega^0_{-i}$.
3. The first proofs of existence of this subclass of states of the world were restricted to a probabilistic framework (see, for instance, Böge and Eisele 1979; Mertens and Zamir 1984; Brandenburger and Dekel 1993). More qualitative presentations, in modal frameworks generalizing the probabilistic formalism, were developed later (Bonanno 1996; Fagin et al.

1999; Brandenburger and Keisler 2006, among others). Finally, in recent years these two approaches were unified in a *coalgebraic* language (Moss and Viglizzo 2006; Abramsky and Zvesper 2015). The view of states of the world as non-well-founded sets was developed in Heifetz (1996) and Tohmé (2005).

4. A closely related alternative to this approach is based on considering *cubical sets* (Bezem, Coquand, and Huber 2014).
5. As shown in Ladyman and Presnell (2015) this amounts to the *substitution salva veritate* principle. That is, *a* an *b* are identical if one can be substituted for the other in any statement without changing the truth of that statement. This is exactly the epistemic requirement for the identity of states of the world and hierarchies of beliefs.
6. The expression ∞-*category* is reserved for $(\infty, 1)$-categories. That is, for all those in which morphisms of order $k \geq 2$ are invertible.
7. See Section 1.2.5 in Lurie (2009).
8. This is easy to depict for sets in Euclidean spaces. For instance $S[2]$ may include the segment joining $(1, 0)$ and $(0, 1)$, its vertices, in $\mathbb{R}^2$. On the other hand $S[3]$ can include the triangle on vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in $\mathbb{R}^3$, where the segments that join the vertices are the *faces* of the simplex.
9. That is, $\mathcal{U}$ itself is a type and its objects are types. $\mathcal{U}$ is such that it cannot be subject to a type-theoretic version of Russell's paradox.
10. A (weak) equivalence between two types $C : \mathcal{U}$ and $D : \mathcal{U}$ is a *homotopy equivalence* between C and D. That is, there exist continuous functions $f : C \to D$ and $g : D \to C$ such that the composition $f \circ g$ is homotopic to the identity of D and $g \circ f$ is homotopic to the identity of C. The type of the equivalences between C and D is denoted $\mathrm{Eq}_{\mathcal{U}}(C, D)$.

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