The Effective as the Actual and as the Calculable in Jean Cavaillès[1]

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> '[…] [P]ersonally, I am very reluctant to posit anything that would dominate the effective thought of the mathematician. I see an exigency in the problems themselves.'[2]

If Jean Cavaillès' work—incomplete, more a promise than a theory—continues to assert a fascination for philosophers and historians of mathematics it is because he outlined a research programme that would aim to progressively reveal what he designated as the 'objectivity, mathematically founded, of mathematical becoming'.[3] Cavaillès' claim for an identity between the objectivity and the historicity of mathematics is what grounds his reputation as a central figure in the tradition of historical epistemology of mathematics. This same claim harbours an implicit definition of mathematics, that is, of its specificity: mathematics is the sole domain of thought whose rational character lies in its transforming its own objective (symbolic, conceptual, practical) conditions in a regulated manner, whence Cavaillès' characterisation of mathematics as an *experimental* activity. In turn, the singularity of mathematical concepts is to be sought not in their generality, but in the specificities of such transformations, i.e. in the way in which mathematical concepts change in history, in their mode of passage.

The novelty of Cavaillès' project thus consists in his having indicated the practice of a new norm of reading the history of mathematics: to always read the body of already produced mathematical works under the image of *movement*—that is, as indicated above, as a becoming—rather than as a completed foundation. Central to the elaboration of this norm in Cavaillès' truncated *oeuvre* is his enigmatic definition of mathematics as *effective thought* [*la pensée effective*] or *effective work* [*travail effective*]. This designation intermingles philosophical referents—principally via French neo-Kantian reflections on the effective (or the efficacious) as the *actual*—and a precise mathematical or technical history—that is, the development of the concept of *effective calculability,* which runs through the distinct phases of the discourse of the 'French analysts' or 'French empiricists' (principally Borel and Lebesgue), the history of recursive functions, and the formation of the modern notion of computability.

A focus on this concept of the effective highlights how, for all that Cavaillès has a reputation as philosopher of *necessity*, he can equally be read as having advanced a philosophy of *actuality* in light of modern mathematics. In this sense, I agree with Sinaceur that attention to the different modalities of the concept of effectiveness should be a guiding thread for reading Cavaillès. However, I would like to suspend the gesture which grounds Sinaceur's masterful analysis of the role of effectiveness in Cavaillès' 'theory of reason', which is to view Cavaillès' use of the concept as a direct extension of the Hegelian (and post-Hegelian) discourse on *wirklichkeit*.[4] I do this not because I am arguing for any ultimate incompatibility between the Cavaillèsian and Hegelian theories, but because I find it necessary—at least initially—to preserve the autonomy of the referent in Cavaillès' philosophical lexicon if we are to analyse Cavaillès' contribution to 'mathematical philosophy'.[5] With this in mind, three points bear emphasis in what follows:

First, the interest of Cavaillès as a philosopher lies in his rejection of the *a priori,* or, to place the emphasis differently, in his developing a novel concept of the *historical a priori*. This is continuous with Cavaillès' rejection of what he calls 'the Kantian theory of mathematics'. Yet, Cavaillès' relationship to Kantianism in the broad sense, and in particular to the concept of intuition, was not consistent throughout his work. Examining the concept of the effective—and, specifically,

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1 This paper is a development of a presentation given at FPMW 13, hosted at l'Université Côte d'Azur, 7-9 October 2021. I thank the organisers and respondents, whose engagement has improved this work immeasurably.

2 Cavaillès responding to Lautman in 'La pensée mathematique', cited from the *Oeuvres complètes* [Cavaillès 1994] (hereafter *OC*), 628. Translations are my own unless otherwise noted.

3 *OC* 226.

4 C.f. [Sinaceur, 2013, 86]: 'L'effectivité hégélienne […] constitue un fil conducteur essentiel pour lire Cavaillès […].'

5 To resume the polemical term that Cavaillès adopts from Brunschvicg.

the way in which Cavaillès *temporalises* the notion of the effectiveness of mathematics—is essential to periodising Cavaillès' changing views on this matter.

Second, this rejection of the *a priori* is consistent with a central *methodological* tenet of Cavaillès' work and the relation it seeks to exhibit between mathematical work and philosophical practice: the relationship between mathematical and philosophical concepts must not be that the former stand to the latter as *examples*, such that mathematics gives specific content to more general philosophical concepts, i.e. as particulars to universals. Rather, the relation between mathematics and philosophy is conceived according to singular points of contact. The concept of the effective is particularly useful for assessing this aspect of Cavaillès' work in that it is a mathematical referent with which Cavaillès was intimately familiar and which underwent a profound reformalisation in the course of Cavaillès' working life, which forms the background for Cavaillès' extended confrontation with new work by Gödel, Church and Kleene around effectiveness and recursion in *Transfini et continu* (written 1940-1941, hereafter *TC*). The primary aim of this essay is to examine how it is that the encounter with the new formalisation of effective computability informs, first, a shift in Cavaillès' reading of Borel and Lebesgue, and, second, Cavaillès' novel critique of the Kantian theory of mathematics in terms of the latter's conflating of *effective process* [*procès effective*] and *effectuated process* [*procès effectué*].

Third, the concept of the effective is crucial for understanding the object of Cavaillès' final work *Sur la logique et théorie de la science* (written 1942, hereafter LTS). Whereas the doctoral studies are animated by the search for the 'central intuition' that would justify the unity of the theories with which they are concerned, *LTS* presents a thoroughly desubjectivised theory of mathematical history under the name of what Cavaillès will now call *l'enchaînement mathematique* or *l'enchaînement formel.* Cavaillès adopts this notion of formal or mathematical *concatenation* from classical rationalist treatises on deduction, but only defines his own use of the term in one instance in *LTS*, when stating that Carnap 'misrecognises':

> '[W]hat is essential in the formal concatenation [*l'enchaînement formel*], i.e. its progression, which is both necessary and conditioned at every step by the effective [*l'effectif*]. A simultaneous positing of all possibilities […] contradicts the very notion of a formal system: the very meaning of the term implies that it be generated by successive movements of breaking apart and surpassing. The fact that everything cannot be established all at once [*d'un seul coup*] has nothing to do with history, but is the characteristic of the intelligible […].'[6]

On the analysis of Cavaillès' work that I will outline here, the fact that Cavaillès, first, specifies his notion of *concatenation* by stating that its progression is *conditioned by the effective*, and second, states that the result of this conditioning is that 'everything cannot be established all at once' provides the key to understanding both his mature theory and his transformation of the rationalist tradition. Thus, while this essay will be concerned with developments in Cavaillès' work prior to the composition of *LTS*, it should be understood as a contribution to rereading the latter work.

6 *OC* 517-518. Translation from [Cavaillès 2021, 82-83] [translation modified].

I. The French Analysts in Cavaillès' Doctoral Dissertations

> '[T]he analysts, Baire, Borel, and Lebesgue, militant in the concrete theory of sets, which they developed to the point of being new creators, are all empiricists. For them, it is a question of their own work: detached from any speculation, they want to determine which objects and methods they will be allowed to deploy without the risk of running into a contradiction.'[7]

Although Cavaillès' doctoral dissertations—*Remarques sur la formation de la théorie abstraite des ensembles* (1938, hereafter *TAE*) and *Méthode axiomatique et formalisme* (1938, hereafter *MAF*)—are both concerned with more contemporary (and primarily German) developments at what was then the forefront of set theory and the formalist programme respectively, the earlier work of the French analysts played an important role both in framing the epistemological histories that Cavaillès wrote and in delineating the broader conceptual problematics that he extracted from these histories. Two points should be emphasised.

First, the stakes of the 'abstract' set-theory dealt with in *TAE* are to a significant extent defined by opposition to the informal commitment to 'concrete' set theory as espoused by Borel, Lebesgue, et. al. Cavaillès had in mind the following epistemological situation: on one hand, the French analysts made crucial progress in the study of real functions, the theory of measure and the theory of integration by using modern set-theoretical methods, and even extended set theoretical research in important directions; on the other, they expressed explicit methodological scepticism with respect to the autonomy of set theory as a domain of mathematical research, a scepticism which they attempted to ground by reference to a preference for 'concrete' or 'applicable' mathematics.[8] It is this nascent commitment to the ideal of a 'concrete' set theory–initially articulated in terms of an inchoate practical attitude–that Borel and Lebesgue will attempt to specify in the period following the debate of the famous 'Five Letters' (exchanged between Baire, Borel, Hadamard and Lebesgue in 1905 over the reception of Zermelo's Axiom of Choice) by developing (somewhat opposed) quasi-formal notions of the effective.[9] The contested distinction between concrete and abstract set theory was initially understood in the French context in terms of a debate around what constituted an admissible definition in mathematics, and an attempt was made to delimit this in terms of *effective definability*. As we shall see, this notion of the effectively definable was not adequately formalised in the work of Borel and Lebesgue, but they did set a conceptual problematic that is the background for Cavaillès' own notion of effective work, insofar as they attempted to specify one modality of the effective (as 'actual' or practically efficacious mathematics) by reference to another (the effective as the calculable or as what could be concretely 'effectuated' in an unambiguous manner).

Second, Cavaillès' interest in the concept of the effective is connected to the necessity of developing 'a definition of mathematical work in general'.[10] This demand, which is found in both the final sentence of *TAE* and at the end of the opening of *MAF* (following the treatment of Lebesgue), can be read as the conceptual bridge between the two dissertations: the problems posed by the history of abstract set theory cannot be resolved without an interrogation of the concept of mathematical (or effective) work; in turn, it is the history of the debate around formalism and the axiomatic method that will be the essential material for this latter investigation.[11] In order to

7 *OC* 14.

8 For an overview of some of the vagaries of this nascent 'concrete' set theory, c.f. [Murata 1958, 1959a-b, 1960] and [Bertin 2019, 87-91].

9 The 'Five Letters' are initially published as [Baire et. al. 1904] and republished as an appendix to the second and subsequent editions of Borel's *Leçons sur la théorie des fonctions* ([Borel 1914, 150-160]). Translation in [Moore 1982, 311-320].

10 c.f. *OC* 362 and *OC* 29.

11 I take this as a reason why Cavaillès published *MAF* as the sequel to *TAE*. While *MAF* is Cavaillès' *thèse principale* and *TAE* is the *thèse complémentaire*, *TAE* appeared as volumes 606 and 607 of the collection *Actualités scientifiques et techniques* published by Hermann, and *MAF* as volumes 608-610. Cavaillès states that *MAF* is the 'continuation' of *TAE* (c.f. *OC* 15).

understand why these problematics are connected, it should be emphasised that the attempt to delimit 'concrete' mathematics in terms of an informal notion of effective definability was unsuccessful on the terms in which the French analysts posed it, insofar as, despite the expressed scruples of Borel, Lebesgue et. al., the theories that they had developed in order to advance mathematical analysis were shown to depend on the very aspects of set theory which they contested.[12] In the introduction to *TAE*, Cavaillès frames the 'paradox' posed by set theory in terms of 'the ever more widespread recourse to a contested theory within incontestable mathematics', and expands on this by stating that '[t]he theoretical crisis which commenced in 1905 has only embarrassed restrictions and scruples concerning set theoretical modes of reasoning without arresting their internal power of expansion.'[13] We can understand this 'paradox' to be that the practical efficacy of set theory had proved it to be an essential matter for mathematical research even in domains and amongst mathematicians who had wished to delimit its content in purely instrumental terms. This is a paradigmatic example for what Cavaillès will call in the introduction to *TAE* the 'problem of explaining the unequal fecundity of methods.'[14] On Cavaillès' reading, the French analysts had posed this problem for themselves without having the adequate formal means to resolve it. He saw Hilbertian proof theory, and in particular the broad history of the expanded notion of recursive or effective definitions as it had been developed following Dedekind's work, as being essentially continuous with this unresolved problematic in Borel and Lebesgue. As such, interrogation of these two concepts of *the effective* and of *mathematical work* should be seen to inform each other more or less *mutatis mutandis* throughout Cavaillès' oeuvre.

Two points should be emphasised in Cavaillès' reading of this epistemological conjuncture. In the first place, Cavaillès progressively imbues the concept of the effective with a *temporal* connotation, viz. as a problematic of the relation between *already completed* (or effectuated) mathematics and future mathematical production. This means that Cavaillès' wish to do justice to the open-ended *effectiveness* of mathematical work is inseparable from his ultimate rejection of any *a priori* theory of mathematics, and his partial sympathy with an 'empiricist' project (in the sense indexed to the French analysts).[15] Second, the dialectic of effectiveness and effectuation at play here must be indexed to Cavaillès' changing views on the idea of an intuitive basis for mathematical construction and hence to a shifting relation both to French neo-Kantianism and to the Kantian underpinnings of finitist theories in mathematics.

Turning to the 'Five Letters', the debate there, which sets the terms for the notions of the effective subsequently developed by Borel and Lebesgue, concerned what constitutes an admissible definition in mathematics. In this respect, a shared reference for the French analysts is a prior distinction drawn by Tannery in his 1897 review of Couturat's 1896 *De l'infini mathématique*, in which Tannery affirmed the validity of mathematical correspondences that could be *determined* but which could not be *described*.[16] Tannery's position on such correspondences is framed as an explicitly Cartesian commitment to the 'idea of determination', i.e. that we can intuit mathematical determinations (or laws determining correspondences) independently of the possible realisation (or description) of such correspondences:

> '[I]t is important to observe that the idea of determination is independent of the possibility of formulating what this determination consists of. It is not necessary, for a set to be determined, that one can actually identify with respect to [*effectivement reconnaître sur*] a given number whether it [actually] belongs or does not belong to that set […].'[17]

12 For an analysis of the history of the relevant results here, c.f. [Moore 1983].
13 *OC* 224.
14 *OC* 227.
15 The relevant distinction between 'empiricism' and 'idealism' at stake here has its roots in du Bois-Reymond. See the commentary in [Brunschvicg 1912, 527-533].
16 [Tannery 1897, 132], reviewing [Couturat 1896].
17 [Tannery 1897, 134].

We can thus gloss this distinction as that between *abstract*—or *postulated*—*correspondences* (which can only be 'determined' but not 'described'), and *effective correspondences* (correspondences that can be concretely 'described'). In the 'Five Letters', Hadamard, Borel, Baire and Lebesgue each invoke this distinction, but in opposed terms. Indeed, the shifting lexical terrain around this referent in the discussion can be read as the sign of a contested concept.[18] For Hadamard, abstract or postulated *determination* is sufficient for mathematical *definition*; for Borel and Lebesgue *definition* or *determination* must equally entail (*effective*) *description*. In this sense, the later moves by Borel and Lebesgue to transform the 'effective' into a quasi-formal notion can be understood as differing attempts to give a *descriptive definition of definition* in mathematics (it is in this sense that we are discussing the genesis of descriptive set theory). The different proposals made by Borel and Lebesgue respectively push in *constructive* and purely *descriptive* directions. The two definitions they advance are as follows:

> Borel's definition of the *effectivement énumérable* (1908): A set is '*effectively enumerable*', if it is possible to 'state, by means of a finite number of words, a definite procedure for unambiguously attributing a determined rank to each of its elements.'[19]

> Lebesgue's definition of the *nommable* or *l'effectif* (1905): 'An object is defined or given when a finite number of words applying to this object and to this object alone have been uttered; that is, when a characteristic property of the object has been *named*.'[20]

In Borel's case, it is clear that what he has in mind is the individuation of mathematical objects. He will often speak of certain individual 'incommensurable numbers' such as π for which we can nevertheless determine the decimal representation up to any desired degree of approximation (these will be 'effectively enumerable' in a way that, e.g., 'the set of all the incommensurable numbers' is not). Borel intends this stricture to formally characterise the epistemological notion of a mathematical object being agreed upon in an unambiguous manner by working mathematicians. He thus wishes to characterise mathematical objectivity by reference to intersubjectivity–i.e. that any given mathematician will be able to determine the 'same' object to the same degree–and in turn to specify this notion of the intersubjective availability of the object by reference to calculability. This is clear when Borel later develops an even more restrictive notion of 'calculable numbers' or 'calculable functions': 'We state that number α is calculable when, given an arbitrary whole number n, we can obtain a rational number which differs from α by less that $1/n$.'[21] Again, the privileged referent for Borel in specifying his notion of effectiveness or calculability is the possibility of decimal expansion. In a footnote to this definition of calculable numbers, Borel then develops what I will call Borel's Thesis, i.e. that effectively calculable mathematics coincides with (or exhausts) applicable mathematics:

> Borel's Thesis: 'I am intentionally leaving aside questions concerning the greater or lesser practical length of the operations; what is essential is that each of these operations is executable in a finite time, by a certain and unambiguous method. […] From the practical point of view, it may be said that the numbers effectively needed [*effectivement besoin*] can, in general, be effectively calculated [*effectivement*

18 See [Desanti 1965] for an analysis of the same debate as an exemplary case for what Desanti calls an 'epistemological problem' in mathematics.

19 [Borel 1914, 165]. Reprint of [Borel 1908], wherein the constraint is initially articulated negatively in response to Richard's paradox. Cited at *OC* 17.

20 [Lebesgue 1905, 205].

21 Appendix to [Borel 1914, 219], reworking [Borel, 1912]. The definition of 'calculable function' given immediately afterwards follows the same schema.

calculés] to the desired degree of approximation […].'[22]

I call this Borel's Thesis because there is an identification at stake, one which crosses our two modalities of the effective, as the actual and as the calculable. The thesis is that all of the mathematical objects that practising mathematicians are going to need for what Borel will think of as *actual* mathematical work (i.e. which are of relevance to 'concrete' or 'applicable' mathematics) are those which can be *effectively calculated* (i.e. given in a computational manner). The identification at stake is thus between two different senses of the same term, as neatly condensed in the above citation: Borel is attempting to give a univocal interpretation to the word *effectivement* by using a formal notion (*effectivement calculés*) to specify an informal or conceptual one (those mathematics which are *effectivement besoin* 'from a practical point of view').

It is this identification that Cavaillès will oppose by giving a temporal reading of the restrictions at stake: for Cavaillès, Borel's definitions are constructive—and the Kroneckerian legacy is clear here, in that we are discussing *constraints* placed on mathematical reasoning—insofar as they are *prescriptive* with respect to *future* mathematical work. By contrast, Cavaillès takes Lebesgue's definition of effective mathematics (above) to be more *open* (or less restrictive) in that definitions are to be introduced according to the immanent demands of specific theories, rather than as a propodeautic to their development.[23] As Cavaillès summarises Lebesgue's position, 'it is the notion of definition that has changed here: far from demanding [qua Borel] a possible construction solely based on a type of objects posed once and for all, the effective [*l'effectif*, i.e. Lebesgue's notion] on the contrary demands that only that which is important for the present reasoning [*le raissonnement présent*] appears in the characterisation of the object.'[24] Thus for Lebesgue, all we need to do to define an object that we wish to describe is to *have named* its characteristic properties (and the tense is crucial here), but there is no *particular type* of (practically effectuable) object which is being prescribed (nor proscribed) by this definition of the effective, and thus there is no prescription with respect to future mathematical work. The characteristic example here is Lebesgue's definition of measure, viz. that to every set of points E we can associate a non-negative number $m(E)$ that satisfies the following three conditions:

> '(I) Any two equal sets have the same measure;
> (II) A set that is the sum of a finite or of a denumerable infinity of sets, without any points in common between them, has as its measure the sum of the measures of these sets;
> (III) The measure of the set of all the points (0,1) is 1.'[25]

This definition simply defines the properties that we wish the concept of measure to have, but does not prescribe anything in particular about 'calculation': the stated stricture tells us exactly the conditions that concept of measure is to fulfil, and *nothing more*, i.e. it does not extend to delimiting the modes of effectuation involved. Cavaillès thus expresses his sympathy for Lebesgue's project (at least at this point) in terms of the attempt to honour the *immanence* of modes of definition in mathematics to what Cavaillès will call *effective work*. Having cited the above definition of measure and then referenced the general advances made as a result in the theory of

22 [Borel 1914, 219].

23 Lebesgue makes this clear when, in the 2nd edition of the [Lebesgue 1904], published in 1928, he adds a footnote to the effect that Hilbert's approach in the *Grundlagen der Geometrie* exemplifies the axiomatic method insofar as it presents an axiomatisation as a 'completed whole' prior to the elaboration of the theory, whereas 'descriptive definitions laid down in the course of the development of a theory, such as the [i.e. Lebegue's] definition of the integral, do not claim to enumerate all the axioms on which they are based; they do not form a complete whole and cannot be isolated from the rest of the theory.' [Lebesgue 1928, 207].

24 *OC* 23-24. Cavaillès is selective in his treatment of Lebesgue here. Lebesgue uses both descriptive and constructive definitions, and sometimes positions his work on measure as orientated towards producing constructive definitions of the central concepts of the theory of integration that would be *equivalent* to previously given descriptive ones. C.f. [Lebesgue 1904, 100], and the commentary in [Michel 1992, 85].

25 [Lebesgue 1904, 103].

integration, Cavaillès writes:

> 'They [these advances] stem from the simplifications achieved thanks to the descriptive character of the definition: all that matters is what is named there, whereas a construction that has nothing to do with the problem posed [again, i.e. Borel's strictures] could only be a burden or an arbitrary restriction. In a way, what is affirmed here is the homogeneity of the materials of an enterprise, the simultaneity of mathematics with the work before it: this remains empiricism, since it is only a question of describing effective work [*le travail effectif*], but it is an empiricism of actual thought [*la pensée en acte*], with no other reference than to the unpredictable becoming of mathematics.'[26]

Once again, in praising this more capacious definition of effective work (or actual thought), Cavaillès has progressively imbued the notion of effectiveness with a temporal connotation. This informs Cavaillès' qualified aligning of himself with an 'empiricist' project for the description of 'mathematical experience', and against the possibility of an *a priori* theory of science.[27] Indeed, this conviction constitutes the core of the Cavaillèsian programme: the effectiveness of mathematics is defined in terms of the *unpredictability* of its development; the rejection of the *a priori* is a rejection of *prediction*.

26 *OC* 25-26.

27 In a letter to Paul Labérenne (written end of 1938), Cavaillès claims explicitly that it is the 'same empiricism' of the French analysts which, in manner more radical than he had expressed in *MAF*, allows Cavaillès to see that 'all recourse to any *a priori* whatsoever is dishonest: thus my complete break with idealism, even of the Brunschvicgian variety', given that 'there is no kind of way to pre-emptively dominate or decide the effective work of the mind—that is, of human technique—in experience.' Cited from [Mougin 1945, 79].

II. Effective Process and Effectuated Process in *Transfini et Continu*

Written in 1940-1941 and published posthumously in 1947, *Transfini et continu* marks a new moment, both in Cavaillès' thinking on the concept of the effective and in the broader mathematical history of the concept. The essay is predominantly concerned with Cavaillès' technical exposition of two new mathematical developments: first, the formalisation of Church-Kleene notion of effective calculability (or computability) in the papers they authored from 1935 to 1938, i.e. the formation of the so-called Church-Turing thesis; second, Gödel's 1938/1940 results on the consistency of the continuum hypothesis.[28] Cavaillès then concludes with a dense few pages in which he develops a new critique of Kant. The point I wish to focus on is in what way Cavaillès' use of the concept of the effective with respect to the earlier debate between the French analysts is transformed by Cavaillès' engagement with the Herbrand-Gödel notion of recursive function and the Church-Kleene formalisation of effective calculability in this new moment. For our purposes here, it is significant that Cavaillès directly relates Borel and Lebesgue's notions of the effective to the Church's recasting of the concept of 'effective procedure (or process)' by introducing a split between what Cavaillès will call 'effective calculation' and 'effectuated calculation', which then informs his critique of Kant using the notions of 'effective process' [*procès effectif*] and 'effectuated process' [*procès effectué*].

The fact that Cavaillès understands Church and Kleene to be formalising the same as Borel and Lebesgue merits comment, given that neither Church's own papers nor prominent commentaries on the Church-Turing hypothesis tend to mention this connection, and instead solely index Church, Kleene and Turing's formalisations to the general pre-history of the notion of recursive definition and the search for a resolution to the 'decision problem', whose major points include the introduction of the chain rule in [Dedekind 1888], Hilbert's use of transfinite recursion and the epsilon calculus in the attempted proof of the continuum hypothesis in [Hilbert 1925], work by Hilbert, Ackermann and Bernays on the *Entscheidungsproblem*, and the Herbrand-Gödel notion of quasi-recursive function.[29] These are doubtlessly the major references explicitly made by Church. It is important to note, however, that Church repeatedly refers to his and Turing's definitions as providing a formalisation of an already widespread 'ordinary (not explicitly defined)' or 'vague' notion of effective calculability (indeed, the status of the Church-Turing thesis as a 'thesis' is that it suggests such an identification between a formal and an informal notion, and thus is not 'provable' in formal terms).[30] I take Cavaillès to have seen this informal notion to be precisely what was at stake in the debates over the notion of effective definability among the French analysts. Indeed, in having seen this connection, Cavaillès in *TC* is to an extent anticipating the history of effective descriptive set theory.[31]

When Cavaillès introduces the notion of effective calculation in *TC*—and the immediate reference here is to Gödel, but the middle section of *TC* will be concerned with the Church-Kleene formalisation in a way that shows that Cavaillès is connecting the two developments—he does so by immediate reference to the same point which framed the introduction to *TAE*, i.e. the discussion between Borel and Lebesgue around 1905 concerning 'the notions of the nameable and of the effective.'[32] He then cites Lebesgue's definition of this notion (see above), before remarking that:

'The vagueness of this requirement—which was intended to limit the domain of

28 The essay was intended for *Revue philosophique*, but the journal was abolished by the Nazi occupying forces before it could be published. The relevant papers cited by Cavaillès in *TC* are [Kleene 1935a, 1935b, 1938], [Church 1936, 1938] and [Church and Kleene 1937]. Cavaillès does not seem to have read Turing.

29 Exceptions include [Avigad & Brattka 2014] and [Dean & Walsh 2017], as well as some Russian histories, probably owing to the influence of Lusin. C.f. [Uspensky & Semenov 1981]. The major points of this 'pre-history' are all surveyed by Cavaillès at an earlier moment in chapter III of *MAF*. Whilst Church doesn't directly index the notion of effective procedure to Borel et. al., he does cite Borel's notion of the effectively enumerable in [Church 1936b,].

30 C.f. [Church 1937, 43] and [Church 1938, 226-227].

31 I thank Brice Halimi and Walter Dean for clarifying conversations on this point.

32 *OC* 462.

> authentic mathematics—is due to the indeterminacy of [the notion of] *words*. If these include “all objects [such] that... there is an object that...”, where is the finitude? The development of formalisms, in particular thanks to Gödel's methods, makes it possible to avoid this pitfall. By obliging a calculation to be effective [*être effectif*], Lebesgue was led to transform it into an effectuated calculation [*calcul effectué*], [at least] if he wanted to chase away the indeterminacy of “a calculation”. But this would have made all progress impossible. The correlative notions of the restricted [functional] calculus [*calcul régulier*] and of quasi-recursive functions introduced thanks to Gödel's work escape this dilemma.'[33]

What is going on here? The point concerning the 'indeterminacy of words' is clear: if we say that all we need to do in order to have defined a mathematical object is to name a characteristic property as the definiens of the definiendum, without having placed any constraints on the acceptable modes of definition involved, then nothing prevents us from naming precisely those 'paradoxical' properties that the notion of effectiveness was intended to prohibit, i.e. the problems of impredicativity return in full force. The only practical means advanced for avoiding this problem at Lebesgue's time was to (implicitly or explicitly) refer back to the kind of constraints on calculability that characterised Borel's notion of the effectively enumerable. This explains the ambiguity of Lebesgue's position, wherein he both seems to have a wider notion of the effective than Borel and states that he is attempting to express the same constraint.[34] From Cavaillès' perspective this means that *in practice* Lebesgue's openness to the demands of effective work collapses into constraining mathematics solely to the domain of *effectuated* calculations, i.e. that Lebesgue's quasi-formal notion of effectiveness is only clarified by reference to a particular class of (already) effectuated objects. Lebesgue's position thus collapses back into Borel's in the course of its elaboration. Cavaillès' claim is that first Gödel and then Church-Kleene, in formalising the modern notion of effective computability, allow us to avoid this pitfall:

> 'By means of this numbering and these recursive relations, there is a permanent control of the effectiveness of the calculation [*l'effectivité du calcul*]: each of these steps falls under our grasp since we are sure to reach it by means of a canonical sequence of steps. The determination lies in the process of recursion. The truth is that this is not fixed once and for all, which is precisely what prevents [future] mathematics from being limited to [already completed] contemporary mathematics [*A la vérité celui-ci n'est pas fixé une fois pour toutes, ce qui évite justement de borner les mathématiques aux mathématiques actuelles*].'[35]

The development here is dialectically subtle. As Cavaillès notes, it is Gödel's work that inspired Kleene and Church to develop what Church calls an *absolute* definition of constructibility or effectiveness, where 'absolute' is understood to signify that the notion is 'independent of reference to a given formal system'.[36] This absoluteness or independence of the new formal definition of effectiveness is what makes it adequate as a proposed formalisation of the informal notion. Yet the inverse of this movement of absolutisation is that the notion of an effectuable object is now purely *relative* to what Cavaillès will call a 'given conceptual system' (i.e. to a given formalism).[37] In terms

33 *OC* 462-463.

34 E.g. Lebesgue frames his commitment to descriptive definitions as being distinct from Borel's constructive definitions; on the other hand, immediately preceding his definition of the effective Lebesgue writes: 'I shall try never to speak of a function without effectively defining it; I shall thus adopt a point of view very similar to that taken by M. Borel in his *Leçons sur la théorie des fonctions.*' [Lebesgue 1905, 205]

35 *OC* 462.

36 *OC* 463. Cavaillès is here paraphrasing [Church 1938, 224]: 'Much of the interest of the proposed definition lies […] in its absoluteness, and would be lost if it could be shown that it was in any essential sense relative to a particular scheme of notation or a particular formal system of logic.'

37 *OC* 472.

of the temporalised reading that we have been discussing, Cavaillès' is asserting that it is precisely *because* the notion of effectiveness in the Church setting is solely a notion of *procedure* or *process* that it doesn't attempt to limit mathematics to any particular class of effectively or intuitively graspable objects, and thus is more supple with respect to the demands of mathematical theories that have *not yet* been developed.

Thus the novel point here—all of which comes to depend 'on the definition of effective process [*processus effectif*]'[38]—is that the same movement in the history of the concept of the effective can at once be read as an *absolutisation* of the notion of constructibility and a *relativisation* of the notion of intuition. The notion of intuition thus no longer refers to any given class of 'evident' objects upon which construction could be founded, but rather, in Cavaillès' terms, *intuition is internal to concatenation*. It is this inversion which grounds Cavaillès' new critique of Kant, or, more precisely, of the Kantian underpinnings of competing projects of mathematical finitism:

> 'To posit an irreducible intuition is only to come to a thoughtless halt. The Kantian theory of mathematics […] conflates the dialectical moment of the positing of the concept and the transcendental moment of its schematisation. […] Moreover, the misunderstanding appears which takes the construction itself to be representable; this is the confusion already noted between effective process and effectuated process [*procés effectif et procés effectué*]. […] The fundamental ambiguity of transcendental philosophy is that it poses the conditions of knowledge as relative to a synthetic activity which is defined and directly apprehended in a knowledge that itself escapes from critique: the condition of representation is conflated with the representations of this condition.'[39]

To summarise, the problem with Borel's position was that his proto-constructivist notion of the effectively enumerable sought to condition mathematics by something extrinsic to mathematics, i.e. the notion of a finite number of words or characters. This is connected to the notion of an intuitive basis on which (future) mathematical work could be grounded. In turn, Lebesgue's more liberal 'descriptive' notion of the effective was tied to the same strictures. We can say that Borel and Lebesgue share a certain 'Kantianism' that we also find in Kronecker, Hilbert, Brouwer and other finitist projects (albeit in very different ways) insofar as they make a double attempt, at once descriptive and constrictive: the first is to describe the constructive (or intuitive) activity of mathematical understanding as it actually (or effectively) proceeds; the second is that they aim to ground this constructive activity in certain well-defined or intuitively graspable class of effectuated objects (or intuitions), that is to say well-defined (or individuated) with respect to *our consciousness* and its limitations.

Hilbert's methodological comments in the 1927 Hamburg address on 'The Foundations of Mathematics' are exemplary in this regard. On the one hand, he positions the project of proof theory in descriptive terms: 'The fundamental idea of my proof theory is none other than to describe the activity [*Tätigkeit]* of our understanding, to make a protocol of the rules according to which our thinking actually [*tatsätlich*] proceeds.'[40] On the other, this ambition to describe construction is prefaced by the assertion that 'as a condition for the use of logical inferences and the performance of logical operations, something must already be given to us in our faculty of representation [*in der Vorstellung*], *certain extralogical concrete objects that are intuitively* [*anschauchlich*] *present as immediate experience prior to all thought*.'[41] What Cavaillès argues, to the contrary, is that by delinking effectiveness and effectuation, Gödel and Church allow mathematical philosophy to maintain its descriptive ambition without its constrictive supplement. This is founded in the

38 *OC* 464.
39 *OC* 469-470.
40 [Hilbert 1928]. Translation from [van Heijenoort 1967, 475], emphasis mine.
41 Ibid., 464.

recognition that the meanings given to any type of object that we might wish to serve as a foundation or as an intuitive basis for mathematical reasoning (such as points, marks, whole numbers, the continuum) are continually altered and rewritten by the formalisms (or concatenations) that specify them. There is nothing prior to thought: this is the consequence of Cavaillès' characteristic conjunction 'mathematical experience'.

Thus, what is at stake in the critique of the Kantian theory of mathematics in the conclusion to *TC* is Cavaillès' confrontation with various projects of finitism and constructivism spanning the preceding 50 years, one that is remarkable because it displaces certain technical problematics from the history of mathematical finitism onto Cavaillès' own philosophy of mathematical history. The new theory of intuition that is outlined here is thus Cavaillès' contribution to a relativisation or historicisation of the transcendental:

> 'What marks history is the submission of the transcendental to its stages […]. The necessity appears after the fact [*aprés coup*]. No analysis of the consciousness of acts permits prediction, any more than it can procure an arbitrary permanence. Intuitive is a synonym for effective (or effectuating) consciousness, transcendental for constitutive (or constituting) consciousness, relative to given conceptual system. No more than there is a closed system is there an absolute transcendental: formal unity is the continuity of a concatenation of acts, the concrete is the novelty of unifying a multiplicity that is still conceptually recognised as such.'[42]

Mapping this passage onto the one cited just above, the effective or effectuating pole is placed on the side of 'effective process', while the transcendental or constitutive pole is placed on the side of 'effectuated process'. For Cavaillès, the (temporal) disjunction here between the effective (or effectuating) and the effectuated poles is not something that is contingent or *extrinsic* to the mathematical concept, but is something that is *constitutive* of the concept itself. The new concept of concept presented here is continuous with Cavaillès' redefinition of the notion of the effective, which now names the whole array of symbolic and conceptual material available to a working mathematician at a given moment. In other words, the effective becomes the site of what Petitot has called Cavaillès' theory of the '*continuous actualisation* [actualisation continue] of intuition', an actualisation which is rather a perpetual reactualisation that is each time mediated by the specificities of mathematical work, and not through any reference to the putatively *a priori* structure of consciousness.[43] The conclusion *TC* should then be read as the transition to the object of Cavaillès' mature work in *LTS*, namely that of providing a theory of the logic of the passages between the positing of objects and their transformations in new formal settings, which is to say a theory of *logical time* that is no longer to be indexed to the time of consciousness, as with the Kantian and Husserlian theories.

42 *OC* 471-472.

43 C.f. [Petitot 2014, 2]. On Petitot's reading, Cavaillès' gesture in *TC* is in fact to simultaneously historicise the two Kantian regimes of intuition and the concept: On the one hand, 'intuition (i.e. the transcendental aesthetic) is historicised through reactualised intuitive superposition, [that is] intuition is connected to effective practices [*pratiques effectives*] and no longer to an original donation as with Husserl'; on the other 'categoricity (i.e. the transcendental analytic) is historicised through the dialectic of the concept.' (ibid.) In other word, what interests Petitot in *TC* is that in the same movement by which Cavaillès abolishes the idea of an external donation which is to act as the basis for mathematical synthesis, he reworks the doctrine of the schematism as *internal* to mathematical work.

Conclusion

In designating *LTS* as a theory of logical time, and positioning *TC* as the transitional text for the development of that theory, I am importing a concept from outside of Cavaillès' own discourse.[44] This framing is intended to capture what Cavaillès himself states in more aporetic terms in the course of his analysis of Bolzano: '[…] science moves outside of time—if by time we mean reference to the lived experience [*vécu*] of a consciousness.'[45] This formula is paradoxical: if science moves outside of time, what is the medium of this movement? I suggest that *LTS* is to be understood as providing a theory of this time of construction, understood not as the time *internal* to the construction of a mathematical theory, but rather as the time of the passage *between* incommensurable mathematical theories, or, as Cavaillès will sometimes say (whence his confrontation with Husserl), between *consciousnesses*. As I have indicated, Cavaillès' transition to consideration of this logical time passes in an essential manner through reflection on the mathematical history of the effective or (it comes to the same thing) the problem of recursive definitions. In this respect, it is important to note that the kernel of this new theoretical horizon was already present in *MAF* when Cavaillès treats Herbrand's definition of intuitionistic or finitistic reasoning in 'On the consistency of arithmetic' (1931), i.e. the exact definition that spurred Gödel to develop his concept of general recursive functions.[46] With respect to this definition, Cavaillès states:

> 'Indeterminacy now only remains with respect to the construction: it can be understood as a progression of stages, each of which is finite in the sense mentioned above. There is no longer a way to encompass all of these steps in a single glance **[***Il n'y a plus moyen d'embrasser du regard toutes les démarches***]**—if consciousness ceases to be coextensive with their succession, it is at least present in each of them.'[47]

This remarkable formulation anticipates the definition of the formal concatenation with which I began: the characteristic of the intelligible is that not everything can be established at once, i.e. that no single consciousness (for which we can read: no single formalism, no single theory) can simultaneously survey the elements of mathematical work. This insight that modern mathematics dissolves the ambition to enumerate a unitary structure of consciousness then comes to ground Cavaillès' final break with Husserl in the conclusion to *LTS*, on the grounds that in the passage between an old and a new mathematical theory '[t]here is more consciousness in it—*and not the same consciousness as before*. The term ‘consciousness’ does not entail univocity of application any more than ‘thing’ does an isolable unity.'[48] Yet, between the composition of these two texts in 1938 and 1942, there has been a crucial development: we have moved from the assertion that consciousness is *present at each step* to the assertion that *at each step it cannot be the same consciousness that is present.* My claim is that this modification in Cavaillès' view coincides with his treatment of the mathematical developments around the concept of the effective that we have traced, and is continuous with his profound reorientation with respect to the concept of intuition: if the new formalisation of effective procedure destroys the idea that mathematics is to be grounded in any *particular* intuitive basis, it equally shows that there can only be a multiplicity of consciousnesses (or transcendentals), which are each immanent to the progress of mathematical work. This discovery, which implies that there can only ever be a progression of incommensurable formalisations, is understood by Cavaillès not as a consequence of the limitations of our faculties, but rather as a characterisation of reason itself.

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44 The conjunction 'temps logique' is found in both [Lacan 1945] and [Bachelard 1949, 60, 96].
45 *OC* 504, Translation from [Cavaillès 2021, 65].
46 Footnote 5 in [Herbrand 1971, 288-289].
47 *OC* 161.
48 *OC* 560, translation from [Cavaillès 2021, 136].

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