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Structures of systems 1. Cohomology of manufacturing and supply network-like systems

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Understanding the relation between large-scale and small-scale functionality is an acute problem in large systems. This paper presents possible structures that capture those relations. Here "structure" means a functor on a category, in this case the category of "combinatorial systems" that model manufacturing systems and supply networks. The details of the dynamics of these systems are frequently the domain of specialists and are ill understood outside such areas. Clustering activities in subsystems introduces covers as scales of analysis. This allows the development of sheaves of schedules and their properties which are analyzed by an adaptation of Čech cohomology. These are applied in supply networks to characterize the fragmentation of these networks and to characterize the conditions for local changes in schedules to extend to the whole system. The intent of this paper is to see sheaves and their cohomology as analytical probes that can enhance our understanding of large-scale systems.

Keywords: combinatorial systems; sheaves of schedules; measures and groupoids; Abelian categories of sheaves; cohomology; manufacturing networks

AMS Subject Classification: 08A99; 18F20; 18G60; 93A30

1. Introduction

Large commercial systems typically associated with a business sector such as manufacturing or supply networks (or "supply chains") have a recognizable structure. The evidence for this is the global trade in software packages for classes of industries such as for manufacturing, student administration, banking, airline booking and so on. For this to be successful some sense of abstract structure must be available to decide which package does or does not fit best. This suggests that there are mathematical categories of types of systems that make the concept of structure precise. The act of choosing a software package from a set of options requires a method of appraisal; this corresponds to a morphism that maps the data-structures and processes of the offered packages to an abstract model of a desired package (the requirements). Package selection can be viewed in terms of category theory.

It is category theory on a heroic scale: the objects are sprawling systems of logic that can required enormous detailed documentation, the details of which is frequently only understood by specialists who have worked on certain areas for years.[1] The mapping multiplies the problems of details: systems can be similar in structure and but quite different in their expression, for example different renditions in computer languages and data-structures. The result is that years can be spent in replacing corporate and governmental software packages with no guarantee of success.[2]

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Package selection is perhaps the costly tip of the iceberg of the problem of complexity inherent in large-scale manufacturing and supply networks. The problem might be called the epistemology of systems; the details are well documented but the overall system has no corresponding expert; the large-scale, managerial or design level understanding does not extent to all parts of the factory floor. One does not need to go to ultra-large-scale systems[3] before process documentation overwhelms. Are there ways of expressing how the large-scale structure, perhaps the management point of view, determines the operational details? This paper takes its inspiration from algebraic geometry and topology and uses sheaf theoretic constructions applied to schedules rather than classes of functions. The schedules can be represented as polynomials for a generic class of structures called combinatorial systems. These systems are in fact quite general and schedules are ubiquitous in business and administration. Understanding the variety of schedules and the way they reflect constraints, both large and small, leads naturally to sheaves and their cohomology. The value of cohomology functors is to give an algebraic measure of the extent to which large-scale structures (in this case schedules) determine the smaller scales. The point of view taken here is that sheaves are analytical probes of large-scale systems and the inputs to cohomology which can build up a different way of understanding these systems.

The paper is divided into two parts. Part 1 defines the relevant categories and develops the sheaves of schedules and their derived properties. It introduces the adaptation of Čech cohomology and gives its properties. Part 2 applies the cohomology of special sheaves to supply networks and to sheaves of adjustments to schedules. Both of these applications are common in industry. The fragmentation of supply networks as the order load overwhelms overall capacity leaving only local capacity intact is given a cohomology treatment. Similarly the question of when changes to schedules in one part of the supply network can be harmonized with the rest of the supply network to provide an overall schedule is re-framed in terms of cohomology of special sheaves of schedules.

2. Part 1. Definitions and concepts

In this paper we introduce the category of combinatorial systems which are modelled on manufacturing processes and supply networks. The objects of this category contain nothing new. All manufacturing processes, either documented for quality assurance certification or represented in information systems or simulations models, list the input to processes and the consequent outputs which can be regarded as work in progress or inventory. Taking assembly as an example, if a number x_i of parts e_i are assembled to make a part e we write this as $\sum_i x_i.e_i \xrightarrow{w} e$. If also $\sum_j y_i.e'_j \xrightarrow{w'} e'$ and $e + e' \xrightarrow{w''} e''$ then we can write $(\sum_i x_i.e_i + \sum_j y_i.e'_j) \xrightarrow{w''\circ(w+w')} e''$. If we define a "type" as an collection of inventory or starting materials that when acted on by a work process, or action, w, the result is a new inventory item or type. For mathematical reasons we expand the set of types to be an integer module generated by inventory items (so allowing inventory to go into "debt") and actions a non-commutative ring that acts on it. In this non-commuting ring $w_1 + w_2$ represents the independent application of actions w_1 and w_2, $w_2 \circ w_1$ is the action of w_2 applied to the outcome of the action w_1. Composition of actions requires that the outcomes or range of w_1 include the inputs or domain of w_2. If this is not the case $w_2 \circ w_1 = w_1 + w_2$. The ring has an additive "do nothing" action 0 and a multiplicative "change nothing" action 1. We shall use the notation W (work, routings) for the ring of actions and Typ for the module of types and so a manufacturing process of many steps is written as an action $\alpha : W \times Typ \to Typ$.

Typically W is not very big; it is easier to manage small collections of work processes in factories and supply networks. The problems of the creation and often frequent change of schedules due to rework, machine breakdown, capacity problems, labor disputes and disrupted supplies

and deliveries gives the reason to separate areas of manufacturing into areas of expertise and management. This leads to considering collections of $W \times Typ \xrightarrow{\alpha} Typ$, each of which is an area of expertise. The full-scale supply network (or set of separately managed factory work areas) is an indexed set of ring actions $\alpha : W \times Typ \to Typ$. Thus we define a *combinatorial system* to be a finite partially ordered set, A, of finitely generated non-commutative rings, $W(a)$ acting on integer modules $Typ(a)$ where $a \in A$:

$$\alpha(a) : W(a) \times Typ(a) \to Typ(a).$$

If a covers b in the order of A then $Typ(a) \cap Typ(b) \neq \emptyset$. This means that $b \prec a$ is a supplier relation. What b supplies to a is in $Typ(a) \cap Typ(b)$ wherein the final types of b, those that are not in the domain of any $w \in W(b)$, are the starting types for a; they are not in the range of any $w' \in W(a)$. Each type $e \in Typ(a) \cap Typ(b)$ is therefore in the range of some $w' \in W(b)$ with $w'\left(\sum_k x_k.e'_k\right) = e$. The set $\left\{x_1.e'_1, x_2.e'_2, \ldots, x_m.e'_m\right\}$ is called the bill of types (or bill of materials) of e in b and denoted $\beta(e, b)$. The bill of types of a in b is the set $\cup\{\beta(e, b) \mid e \in Typ(a) \cap Typ(b)\}$ and is denoted $\beta(a, b)$. This construction also conforms to the idea of systems as communicating objects (as in Goguen 1992).

Finally a combinatorial system has functions that define a sense of progress that reflects a value chain. The first is a *type weighting*: for each $a \in A$ there is a function $\theta(a)$ defined on base types of $Typ(a)$ and extended linearly; $Typ(a)) \xrightarrow{\theta(a)} \mathbb{N}$ (or $\mathbb{Q}, \mathbb{R}$) such that if $w : e_1 \to e_2$ then $\theta(a)(e_2) \geq \theta(a)(e_1)$. Alternatively a combinatorial system has a lead-time function if for all a in A there is a non-zero map

$$L : W(a) \to \mathbb{N}$$

such that $L(w_2 \circ w_2) = L(w_2) + L(w_1)$. For $a \in A$ "lead-time through a" is $max\{L(w) \mid w \in W(a))\}$.

PROPOSITION 2.1 *Every lead-time function is associated with a type weighting and vice versa.*

Proof Given a lead-time function L, if $e = w(e')$ define a type weighting

$$\theta(a)(e) = max\{L(w) \mid w(e') = e\}.$$

Given a type weighting θ define a lead-time function by

$$L(w) = max\left\{\theta(e) - \theta(e') \mid w : e' \to e\right\}.$$

This gives $L(w_2 \circ w_2) = L(w_2) + L(w_1)$ as if e, e' and e'' are chosen to maximize L and $w_1 : e'' \to e'$ and $w_2 : e' \to e$ then $\theta(e) - \theta(e') + (\theta(e') - \theta(e'')) = \theta(e) - \theta(e'')$. □

At this stage we have assumed no limits on capacity. $(W(a), +)$ has arbitrary capacity. Clearly all practical cases will have a limit $N.w$ and one way to enforce this is to make $(W(a), +)$ a $\mathbb{Z}_N$ module. There are many useful cases when $(W(a_i), +)$ is a $\mathbb{Z}_2$ module as will be shown later. In this case the addition $w + w$ as meaning the simultaneous (or overlapping) execution of w cannot happen: thus $w + w = 0$ for all $w \in W(a)$ for all $a \in A$. However $w = w_1 + w_2 + w_3$ is a legitimate non-zero expression. Nevertheless, unless specified, we continue to assume $(W(a), +)$ is a $\mathbb{Z}$ module.

For notational convenience we shall call the information in the functions $W(a) \times Typ(a) \xrightarrow{\alpha(a)} Typ(a)$ and $\theta(a)$ (alternatively $L(a)$) as the "topic a" denoted $Tpc(a)$.

Definition A combinatorial system is a triple

$$\underline{A} = ((A, \prec), \{(\alpha(a) : W(a) \times Typ(a) \to Typ(a), \theta(a)(\text{ or } L(a))) \mid a \in A\}).$$

Let $S(a) = S^1(a) =_{defn} \{b \mid b \prec a\}$. The following notation is used throughout this paper.

$$S^{n+1}(a) = \left\{b \mid \exists b' \in S^n(a) \wedge b \in S(b')\right\}$$

and

$$S^{+(n+1)}(a) = \left\{b \mid \exists b'' \in S^{+n}(a) \wedge b \in S^+(b'')\right\}.$$

Also $S^{\leq n}(a) =_{defn} \underset{0\leq k\leq n}{\cup} S^k(a)$ and $S^{+\leq n}(a) =_{defn} \underset{0\leq k\leq n}{\cup} S^{+k}(a)$ with $S^0(a) = \{a\}$. Whenever A has a maximum or final element it is denoted a^*.

For $U \subseteq A$ an obvious extension of the notation $\alpha(a) : W(a) \times Typ(a) \to Typ(a)$ is

$$\alpha(U) : W(U) \times Typ(U) \to Typ(U).$$

Polynomials of actions defined over sets will be identified with schedules in those sets, and these are the basis of a system's performance. It is this concept of performance which can be good locally and poor overall (globally) that distinguishes systems in any competitive arena.

A combinatorial system in which the objects in $Tpc(a)$ are or have a state in an ordered set and most $w \in W(a)$ decrease the state in their domain while adding to the state in their range is a *supply network* if there is a "replenishment logic" that allows each topic to be used indefinitely.

Examples of combinatorial systems include

(1) "Symbolic" combinatorial system. These are systems in which the domain of map or action, w, is a set of objects or types $\{e_1, e_2, \ldots, e_n\}$ so that we have equations $w(e_1, e_2, \ldots, e_n) = e$. A symbolic combinatorial system replaces w with a "merge" $[e_1, e_2, \ldots, e_n] = e$. All actions become "merges" making it explicit the way objects are "absorbed" in each $a \in Obj(\underline{A})$. Clearly all combinatorial systems have symbolic version. Symbolic combinatorial systems are useful in creating special examples. One can create mathematically legitimate constructions that need not have counterpart in the real world.
(2) Linked Turing machines, or effective definitional schemes (Shepherdson 1985), with each $Tpc(a)$ the definition of a recursive function and each $Tpc(b)$ with $b \in S(a)$ processing data to input to a. Thus b is an oracle (Enderton et al. 1977) for $Tpc(a)$. For each a $Tpc(a)$ can be a different "signature" (Wirsing et al. 1990). Large-scale systems are Turing machines with a very large number of oracles which parametrize responses of the system according to inputs.
(3) Construction projects: the final step is a^*. The $Tpc(a)$ being the sub-projects activities.
(4) Manufacturing and supply networks for consumer items with components drawn from many sources. $Tpc(a)$ corresponds the factories or work centers supplying components.

In all of these cases there is a sense of progressing towards a final result. Something is being defined, be it a product, an artifact or information, by combining other things. Although a combinatorial system can be seen as formal systems or definitional schema the added element of type weightings or lead-times give them concept of progress and dynamics which is the emphasis in this paper. The other aspect that is emphasized here is the combinatorial sub-systems which give a sense of "local" vs. "entire" or "global".

2.1. Morphisms

A morphism of combinatorial systems

$$f : ((A_1, \prec), \{Tpc(a), a \in A_1\}) \to ((A_2, \prec), \{Tpc(b), b \in A_2\})$$

is a family $Tpc(f)(a), a \in A$ such that

(1) $(A_1, \prec) \xrightarrow{f} (A_2 \prec)$ preserves the ordering.
(2) For each $a \in A_1$ there are homomorphisms $W(f)(a) : W(a) \to W(f(a))$ and $Typ(f) : Typ(a) \to Typ(f(a))$ such that the following diagrams commute:

$$\begin{array}{ccc} W(a) \times Typ(a) & \xrightarrow{\alpha(a)} & Typ(a) \\ \downarrow{\scriptstyle W(f)\times Typ(f)} & & \downarrow{\scriptstyle Typ(f)} \\ W(f(a)) \times Typ(f(a)) & \xrightarrow{\alpha(f(a))} & Typ(f(a)). \end{array}$$

LEMMA 2.1 *If f is combinatorial system morphism then*

(1) $a' \in S(a) \Rightarrow f(a') \in S(f(a))$.
(2) $\forall a \in A_1, \theta(a)(e_2) > \theta(a)(e_1) \Rightarrow \theta(Typ(f)(e_1) \geq \theta(Typ(f)(e_2)$.

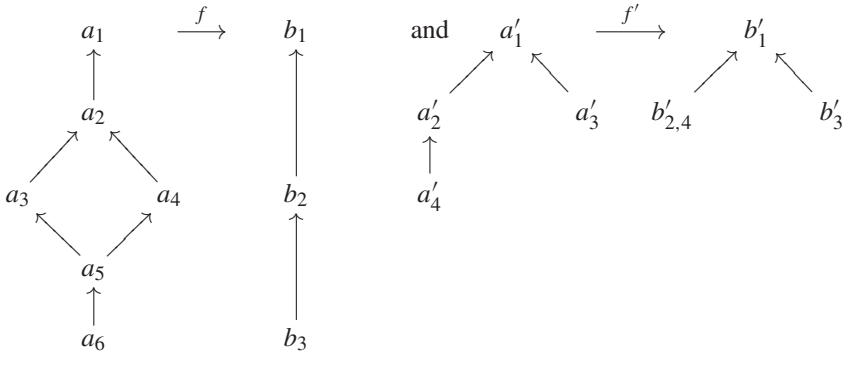
Proof

(1) $a' \in S(a)$ is the same as $a' \prec a$. The condition f preserves $\prec$ means that $f(a') \prec f(a)$ which is equivalent to $f(a') \in S(f(a))$.
(2) If $e_1 \xrightarrow{w} e_2$ in $Tpc(a)$ then

$$\begin{array}{ccc} e_1 & \xrightarrow{w} & e_2 \\ \downarrow{\scriptstyle Typ(f)(a)} & & \downarrow{\scriptstyle Typ(f)(a)} \\ Typ(f)(e_1) & \xrightarrow{W(f)(w)} & Typ(f)(e_2). \end{array}$$

which means that $\theta(a)(e_2) > \theta(a)(e_1)$ implies $\theta(f(a))Typ(f)(e_2) \geq \theta(f(a))$ $Tpc(f)(e_1)$. □

Examples combinatorial system morphisms associated with changes in the partial order of suppliers as follows:



where $f^{-1}(b_2) = \{a_2, a_3, a_4, a_5\}$ and $f'^{-1}(b'_{2,4}) = \{a'_2, a'_4\}$.

Definition A morphism f is a logistic morphism if the following diagram commutes.

$$\begin{array}{ccc} Tpc(f^{-1}(b)) & \xrightarrow{Tpc(f)} & Tpc(b) \\ \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} \\ \mathbb{N} & \xrightarrow{\geq} & \mathbb{N}. \end{array}$$

That is lead-time through b is no greater than lead-time through $f^{-1}(b)$.

Clearly combinatorial systems form a category $Comb$. A simple proof shows that combinatorial systems with logistic morphisms form a subcategory of $Comb$.

A polynomial in $\underline{A}$ is an expression of the form

$$\underline{w} = \sum_j w_{j,m(j)} \circ w_{j,m(j)-1} \circ .. \circ w_{j,2} \circ w_{j,1}$$

An example used later, $\xi(a_{1,3})$ in Section 4.1, is:

$$\begin{aligned}\underline{w} &= w_{1,3} \circ (w_{2,3} \circ ((w_{3,3} + w_{3,4} + w_{3,5}) + w_{3,6})) \\ &= w_{1,3} \circ w_{2,3} \circ w_{3,3} + w_{1,3} \circ w_{2,3} \circ w_{3,4} + w_{1,3} \circ w_{2,3} \circ_{3,5} + w_{1,3} \circ w_{3,6}.\end{aligned}$$

We allow the possibility that there can be j,j',k,k' with $w_{j,k} = w_{j',k'}$ so that the j polynomials expressions can intersect each other. $\underline{w}$ is defined in $W(U)$, $U \subseteq A$ if all domains and ranges of the actions in $\underline{w}$ lie in $Typ(U)$. Call the elements in $w_{j,k}$ the k step actions. A time allocation on $\underline{w}$ is an allocation for actions to be achieved in a series of time intervals or "time-slots". Define a time allocation $s(\underline{w})$ by induction:

(1) The first steps start at time-slot t_1.
(2) $s(\underline{w})(1) = [t_1, t_1 + \left(max\{L(w_{j,1}) \mid w_{j,1} \in \underline{w}\}\right)]$ and put

$$t_2 = t_1 + \left(max\{L(w_{j,1}) \mid w_{j,1} \in \underline{w}\}\right).$$

(3) If $s(\underline{w})(k)$ is defined up to the k step replace $\underline{w}$ with $\underline{w}'$ which is $\underline{w}$ with all k' step actions up to the kth step deleted and then apply the previous induction steps with t_k replaced by t_{k+1}.

The capacity of a is defined as the maximum number of actions that can occur at any one time. Clearly capacity complicates the function s in normal circumstances. It forces choices to be made in the definition of s and so a given polynomial $\underline{w}$ can have more than one time allocation.

Definition A *schedule* in $W(U)$ is a pair $\xi = (\underline{w}, s)$, $\underline{w}$ defined in $W(U)$, and where s maps the actions in $\underline{w}$ to a finite set T of intervals of natural numbers such that if t_1, t_2 are intervals in T then $t_1 < t_2$ if the last number in t_1 precedes the first number in t_2 numbers T. Writing $\mid \xi \mid$ for the set of actions in $\underline{w}$, s is to have the following properties:

(1) If $w \circ w'$ appear in $\underline{w}$ then $s(w') < s(w)$ (w cannot start before w' finishes).
(2) Furthermore $s(w)$ is an interval $[n, n + L(w)]$ in $s(\xi)$.
(3) If $\{a_1, a_2, a_3\} \subseteq U$ and $a_1 \in S(a_2)$ and $a_2 \in S(a_3)$ with $\mid \xi \mid \cap W(a_3) \neq \emptyset$ and $\mid \xi \mid \cap W(a_1) \neq \emptyset$ then $\mid \xi \mid \cap W(a_2) \neq \emptyset$.

The definition of schedule means that we can extend schedules from a to $a \cup S(a)$. If $\underline{w}_1$ is a polynomial in $W(a)$ and $\underline{w}_2$ is a polynomial in $W(b)$ and the last step actions of $\underline{w}_2$ are in the domains of the first step actions of $\underline{w}_2$ then we can create a schedule ξ spanning the polynomial expressions in $\underline{w}_1$ and $\underline{w}_2$. This can be iterated to create schedules in $S^{\infty}(a)$ for any a.

Identifying schedules with their associated polynomials allows us to add them.

LEMMA 2.2 *There is a number $\gamma(a)$ such that a schedule can be written as a set of maps $\xi(a) : T \to W(a)^{\gamma(a)}$.*

Proof Write $T(\xi)$ for $\cup_{w \in \underline{w}} s(w)$. Let $s(\xi) = T(\xi)$ then each w_i in ξ has a value $s(w_i)$ in $T(\xi)$. For a time-slot $t \in T(\xi)$ there is a set $\{w_1, w_2, \ldots, w_{n(t)}\}$ that is $s^{-1}(t)$. Then $s^{-1} : T(\xi) \to W(a_1) \oplus W(a_2) \oplus \ldots \oplus W(a_{n(t)})$ where some of the a_k will be the same. Separating

out the actions for a given a, let $\gamma(a)$ be the maximum of the $t(n)$ for $T(\xi(a))$. We then have $\xi(a) : T(\xi(a) \to W(a)^{\gamma(a)}$. Thus we can write the schedule as a collection $\xi(a) : T \to W(a)^{\gamma(a)}$. □

2.2. Sheaves over combinatorial systems

Schedules are the connecting structures of a combinatorial system. If $\underline{A}$ has tens to hundreds of topics many of which will have scores of suppliers then discerning the possibilities of coordination and fragmentation becomes a problem. The concept of a sheaf and its cohomology has found to be effective in this area. Sheaves have long been used in geometry and topology to express the way local solutions can be patched to provide global solutions (Gunning and Rossi 1965; Spanier 1966; Hartshorne 1977). The development of Elementary Topos theory in the late 1960s with its examples in categories of sheaves provided an abstract stage for talking about "locally" vs. "globally" true (Freyd 1972; Lawvere 1972). The concepts of sheaves influence computer science with papers appearing in the 1990s and continuing ever since (Goguen 1992; Lilius 1993; Ghrist and Hiraoka 2011). Other areas include applications in sensor deployment (de Silva and Ghrist 2007).

Definition A *cover* $\mathcal{U}$ of $\underline{A}$ is a set of combinatorial subsystems in which every element of A is in at least one element of $\mathcal{U}$ and if U_1 and U_2 are in $\mathcal{U}$ and $U_1 \cap U_2 \neq \emptyset$ then $U_1 \cap U_2 \in \mathcal{U}$.

In this paper a cover defines the primary level of analysis. If we associate $\mathcal{U}$ with its induced set of unions we have a topology: $\mathfrak{T} = \mathcal{U} \cup \{\cup_i U_i \mid U_i \in \mathcal{U}\}$. As $\underline{A}$ is finite such a topology is finite (hence Noetherian) but not necessarily discrete. In fact in this article topology is not a requirement. It is only necessary that if $U_1, U_2, \ldots, U_n$ are in $\mathcal{U}$ and $\cap_{i=1,2,\ldots,n} U_i \neq \emptyset$ then $\cup_{i=1,2,\ldots,n} U_i \in \mathcal{U}$. From here on we assume that our covers have this property. It is sufficient for the definition of sheaves and cohomology. It is always the case that a cover can generate a topology and in the finite case the open (or closed) sets of a topology create a cover.

The data for a presheaf $\mathcal{P}$ defined for, or over, a cover $\mathcal{U}$ is a set of structures (in our case sets or Abelian groups) $\mathcal{P}(U)$ and whenever $V \subset U$ (both in $\mathcal{U}$) there is a *restriction map* (set function or homomorphism) $\mathcal{P}(U) \xrightarrow{\rho(U,V)} \mathcal{P}(V)$. When $W \subset V \subset U$ then $\rho(U, W) = \rho(V, W) \circ \rho(U, V)$. A presheaf is a sheaf if also, for any set U_i (all in $\mathcal{U}$) and for all pairs i, j $\rho(U_i, U_i \cap U_j) = \rho(U_j, U_i \cap U_j)$, there is a map $\rho(\cup_i U_i, U_i)$.

Let $Sch(U)$ be the set of schedules definable on $U \in \mathcal{U}$. If $V \subset U$ there is clearly a restriction of $Sch(U) \to Sch(V)$ so we can define a presheaf Sch of schedules.

PROPOSITION 2.2 *Sch is a sheaf.*

Proof Let ξ_i be defined on U_i such that $\xi_i = \xi_j$ on $U_i \cap U_j$ (the 0-cocycle condition) then as each is well defined on $U_i \setminus (U_i \cap U_j)$ we can define a ξ on $U_i \cup U_j$ that restricts to ξ_i and ξ_j. In this way a schedule ξ can be defined on $U = \cup_i U_i$ such that $\xi \mid U_i = \xi_i$. □

PROPOSITION 2.3 *Sch is a group valued sheaf.*

Proof Given ξ_1 and ξ_2 in $Sch(a)$ then by the lemma we can write $\xi_i(a_{ij}) : T(\xi_i) \to W(a_{ij})^{\gamma(a_{i,j})} \xrightarrow{\Sigma} W(a_{ij})$ where the Σ signifies addition across the separate components of $W(a_{ij})^{\gamma(a_{i,j})}$. If ξ_i is define for $U_i \subset A$ this makes each ξ_i a map $\xi_i : T(\xi_i) \times U_i \to W(U_i)$. Addition $\xi_1 + \xi_2$ is now defined as the map $\xi_1 + \xi_2 : (T(\xi_1) \cup T(\xi_2)) \times (U_1 \cup U_2) \to W(U_1 \cup U_2)$. □

For the subcategory of $Comb$ of logistic morphism we want Sch to be a functor. We need to impose an additional criteria on logistic morphisms.

Definition A morphism $f : \underline{A}_1 \rightarrow \underline{A}_2$ is *uniform* on a cover $\mathcal{U}$ of $\underline{A}$ if for every $U \in \mathcal{U}$ $f^{-1}(f(U)) = U$. This is weaker than "locally bijective" on elements of $\mathcal{U}$ in that the cardinality of $f^{-1}\{b\}$ need not be 1. "Uniform" stops quite distinct U_1 and U_2 with $U_1 \cap U_2 = \emptyset$ mapping to the same cover element in $\underline{A}_2$. Given objects with covers uniform morphisms form a subcategory of $Comb$. The condition $f^{-1}(f(U)) = U$ is not a serious constraint in applications. Most morphism arise by rescaling (lumping sub-systems of topics into a single topic) or finding equivalence classes of types and actions. The importance of uniform morphisms is in the following result.

THEOREM 2.3 *Given $\mathcal{U}$, Sch is a functor from Comb and uniform logistic morphisms to the category of Abelian groups, Ab.*

Proof Writing $Sch(\underline{A})$ for the sheaf of schedules on $\underline{A}$ and $Sch(A)$ for the global sections then we already have defined the functor on objects. Each schedule ξ defined on some $U \subseteq A$ corresponds to a pair $(\underline{w}, s)$. For a logistic morphism $f : \underline{A}_1 \rightarrow \underline{A}_2$ and $a \in U$, $W(f)(a) : W(a) \rightarrow W(f(a))$ is a homomorphism that extends to polynomials and so $W(f)$ extends to polynomials giving $W(f)(a)(\underline{w})$ a polynomial in $W(f(a))$. For each $a' \in S(a)$ the range of the last step actions of $\underline{w}(a')$ must be the domains of first step actions in $W(a)$ so a logistic morphism maintains the connection of a polynomial defined over U. This means we have a polynomial $W(f)(\underline{w})$ defined for $f(U)$. A new $f(s)$ can be recalculated for $W(f)(\underline{w})$: for a k-step action w, $f(s)(W(f)(w))(k) = s(w)$. This defines $Sh(f)(\xi)$.

As the total duration of ξ = range of $s(\xi)$ is the same as $Sch(f)(\xi)$ then the time through U is the same as for $f(U)$. Consequently $Sch(\underline{A}_1) \xrightarrow{Sch(f)} Sch(\underline{A}_2)$ is a well defined homomorphism. □

Definition A combinatorial scheme is a pair $(\underline{A}, \mathcal{P})$ with $\underline{A} \in Comb$ and $\mathcal{P}$ an image of a group-valued sub-sheaf of $Sch(\underline{A})$.

Definition The combinatorial schema of $\underline{A}$ is the finitely complete and cocomplete Abelian category generated by the category of group valued sheaves of schedules. The morphisms of this category are embeddings and epimorphisms constructed from logistic morphisms that induce sheaf homomorphisms over $\underline{A}$. Denote this category as $Comb(\underline{A})$.

Why should we bother about sub-sheaves of $Sch(\underline{A})$ treating them as special? A group valued sub-sheaf of $Sch(\underline{A})$ is usually group valued for some reason. Often because it defines a class of schedules with added properties. These properties are, in turn, properties of the basic structure of $\underline{A}$. Thus an object $\mathcal{P}$ of $Comb(\underline{A})$ can be interpreted as schedules defined in special circumstances or on a *model* of $\underline{A}$. This is a model in terms of model theory and corresponds to an image of a logistic morphism in $Comb$ (Keisler 1977; Lambek and Scott 1986). In terms of applications these become implementations or manifestations of sets of algorithms. Each $\mathcal{P}$ is then a class of dynamics that demonstrate the capability of the system $\underline{A}$ under special constraints or boundary conditions (something made clear in applications discussed below). Thus $\mathcal{P}$ becomes a mode of operation using the conditions on the actions, types and schedules that characterize the sheaf. This finite dynamics is played out in simulations of social and natural systems of many types. In this case each sheaf is a parameterization of the dynamics of $\underline{A}$.

3. Defining classes of descriptions through sheaves

For a given combinatorial system $\underline{A}$ the objects of its combinatorial schema, $Comb(\underline{A})$, will be the main tool to analyze $\underline{A}$. Although $Comb(\underline{A})$ contains all the information carried by sub-sheaves of $Sch(\underline{A})$ we frequently want to consider schedules with properties that are not formulated within the algebra of $\underline{A}$. These arise from the social or business context of $\underline{A}$. This section provides a

general framework that uses the category of set valued sheaves defined on a cover of $\underline{A}$ to define properties and classes of schedules. The advantage of this approach is two-fold:

(1) The definition of properties of the sheaves of $Comb(\underline{A})$ are defined and not just properties of schedules or parts of schedules with the possibility that other parts have contrary properties. If $\mathcal{K}$ is a set valued sheaf defined on a cover of $\underline{A}$ and related to $\mathcal{P}$ via a sheaf $\mathcal{L}$ then for $V \subset U$ properties assigned to $\mathcal{P}(U)$ flow through $\mathcal{K}(U) \to \mathcal{K}(V)$ consistently to $\mathcal{P}(V)$.
(2) Set valued sheaves are a Topos and so have all the power of intuitionist logic (logic without the excluded middle).

In what follows $\xi = (\underline{w}, s)$ will be identified with the polynomial $\underline{w}$ and $\xi_1 < \xi_2$ means $\mid \xi_1 \mid \subset \mid \xi_2 \mid$ and the common actions are active in the same time-slots.

3.1. Defining the sheaf of schedule classes relative to a sheaf $\mathcal{K}$

Let $\mathcal{U}$ be a fixed covering of $\underline{A}$ and let $\mathcal{S}h(\underline{A})$ denote the category of set valued sheaves of $\underline{A}$ with this covering. Let $\mathcal{K} \in \mathcal{S}h(\underline{A})$, a sheaf relation between $\mathcal{P} \in Comb(\underline{A})$ and $\mathcal{K}$ is a sheaf $\mathfrak{L} \in \mathcal{S}h(\underline{A})$ with fixed maps $\mathcal{P} \xleftarrow{proj_1} \mathfrak{L}(\mathcal{P}) \xrightarrow{proj_2} \mathcal{K}$. This is to be a functor

$$[\mathfrak{L}, \mathcal{K}] : Comb(\underline{A}) \to \mathcal{S}h[\mathfrak{L}, \mathcal{K}]$$

The category $\mathcal{S}h[\mathfrak{L}, \mathcal{K}]$ has objects pairs of maps $\mathcal{P} \xleftarrow{proj_1} \mathfrak{L}(\mathcal{P}) \xrightarrow{proj_2} \mathcal{K}$ and maps are commuting diagrams

$$\begin{array}{ccccc} \mathcal{P}_1 & \xleftarrow{proj_{1,1}} & \mathfrak{L}(\mathcal{P}_1) & \xrightarrow{proj_2} & \mathcal{K} \\ \downarrow{\scriptstyle \mathcal{P}(f)} & & \downarrow{\scriptstyle \mathfrak{L}(f)} & & \downarrow{\scriptstyle \mathcal{K}(f)} \\ \mathcal{P}_2 & \xleftarrow{proj_{2,1}} & \mathfrak{L}(\mathcal{P}_2) & \xrightarrow{proj_2} & \mathcal{K} \end{array}$$

Definition In $\mathcal{S}h(\underline{A})$ a $\mathfrak{L}$-compatible map is a either a map $\tilde{x} : \mathcal{K} \to \mathcal{P}$ such that $\tilde{x} \circ proj_2 = proj_1$ or a map $\tilde{y} : \mathcal{P} \to \mathcal{K}$ such that $\tilde{y} \circ proj_1 = proj_2$.

Definition The sheaves $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{K}_{\mathfrak{L}}$, respectively, are defined as the colimits of $\mathfrak{L}$-compatible maps in $\mathcal{S}h(\underline{A})$.

PROPOSITION 3.1 *The sheaf $\mathcal{P}_{\mathfrak{L}}$ is given as follows.*

(1) *For* $\kappa \in \mathcal{K}(U)$, $X(\kappa) =_{defn} \{\xi \in \mathcal{P}(U) \mid (\xi, \kappa) \in \mathfrak{L}(U)\}$.
(2) $\mathcal{P}_{\mathfrak{L}}(U) =_{defn} \{X(\kappa) \mid \exists \xi \in \mathcal{P}(U) \wedge (\xi, \kappa) \in \mathfrak{L}(U)\}$ (*that is* κ *is in the image of* $proj_2$).

Proof The colimits exist in $\mathcal{S}h(\underline{A})$ as this category is an elementary topos and therefore complete and cocomplete. The colimit takes each element of $\mathcal{P}(U)$ to the class defined by all maps with a common origin in $\mathcal{K}$. That is, all the ξ_1 and ξ_2 that are joined by relation of $\mathfrak{L}$ compatible maps $\tilde{x}_1, \tilde{x}_2 : \mathcal{K} \rightrightarrows \mathcal{P}$ wherein for each $U \in \mathcal{U}$, $\tilde{x}_i : \kappa \in \mathcal{K}(U) \mapsto \xi_i$, $i = 1, 2$. This makes sets $\{\xi \in \mathcal{P}(U) \mid \exists \kappa\ \mathfrak{L}(U)(\xi, \kappa) \in \mathfrak{L}(U)\}$ the elements of $\mathcal{P}_{\mathfrak{L}}(U)$. Sheaf maps must preserve the restrictions and patching of sections. Thus for a set of U_i with $U_i \cap U_j \neq \emptyset$ for pairs of i and j and $\mathcal{K}(U_i) \xrightarrow{\tilde{x}_i} \mathcal{P}(U_i)$ with $\tilde{x}_i \mid (U_i \cap U_j) = \tilde{x}_j \mid (U_j \cap U_i)$ the $\tilde{x}_i$ as sheaf maps must preserve the overlaps of sections. Thus for κ_i defined on U_i, equal on the pairs of intersections and therefore having an extension on $U = \cup_i U_i$, the $\tilde{x}_i(\kappa_i) = \tilde{x}_j(\kappa_j)$ are equal on $U_i \cap U_j$ which gives a set $\xi_i = \tilde{x}_i(\kappa_i)$ on the U_i that patch together to make single section $\xi \in \mathcal{P}(U)$. Thus we can define the sheaf $\mathcal{P}_{\mathfrak{L}}$ as stated. □

$X(\kappa)$ can be identified as a property of schedules that can be formulated with reference to the section κ of $\mathcal{K}$. $\mathcal{P}_{\mathfrak{L}}$ is then the set of all $\mathfrak{L}$ compatible classes of the schedules in $\mathcal{P}$ that can be indexed in terms of $\mathcal{K}$.

By the same reasoning we also have

PROPOSITION 3.2 *The sheaf $\mathcal{K}_{\mathfrak{L}}$ is given as follows.*

(1) *For* $\xi \in \mathcal{P}(U)$ $Y(\xi) =_{defn} \{\kappa \in \mathcal{K}(U) \mid (\xi, \kappa) \in \mathfrak{L}\}$.
(2) $\mathcal{K}_{\mathfrak{L}}(U) =_{defn} \{Y(\xi) \mid \exists \kappa \in \mathcal{K}(U) \wedge (\xi, \kappa) \in \mathfrak{L}(U)\}$ (*that is ξ is in the image of* $proj_1$).

COROLLARY 3.1 *The pairing* $[\mathfrak{L}, \mathcal{K}]$ *gives a functor* $()_{\mathfrak{L}} : Comb(\underline{A}) \to \mathcal{S}h(\underline{A})$ *with* $\mathcal{P} \mapsto \mathcal{P}_{\mathfrak{L}}$.

$Y(\xi)$ can be identified as the κ in the sheaf $\mathcal{K}$ that apply to at least one schedule in $\mathcal{P}$ satisfying the relation defined by $\mathfrak{L}$. That is $Y(\xi)$ are all the $\mathcal{K}$ properties that are true of ξ. Each element of $\mathcal{K}_{\mathfrak{L}}$ is then a fiber over ξ of the projection of $\mathfrak{L}$ as is each element of $\mathcal{P}_{\mathfrak{L}}$ a fiber over κ of the projection of $\mathfrak{L}$. If we regard $\mathcal{K}_{\mathfrak{L}}$ as a measure of relevance of $\mathcal{K}$ for the properties of schedules we can ask how well it can discriminate among schedules and, in the case of each $\mathcal{P}$, how well $\mathcal{K}$ discriminates among the schedules of $\mathcal{P}$. This discrimination would fail if for example $\mathcal{P}_{\mathfrak{L}}$ was always the same.

As a test of the discriminating power of the relation $\mathfrak{L}$ we can ask whether there are natural maps $\mathfrak{f} : \mathcal{P}_{\mathfrak{L}} \rightleftharpoons \mathcal{K}_{\mathfrak{L}} : \mathfrak{j}$ such that $\mathfrak{j} \circ \mathfrak{f} = 1_{\mathcal{P}_{\mathfrak{L}}}$ and $\mathfrak{f} \circ \mathfrak{j} = 1_{\mathcal{K}_{\mathfrak{L}}}$.

If $\mathcal{P}_{\mathfrak{L}} \xrightarrow{\mathfrak{f}} \mathcal{K}_{\mathfrak{L}} \xrightarrow{\mathfrak{j}} \mathcal{P}_{\mathfrak{L}} = 1_{\mathcal{P}_{\mathfrak{L}}}$ and $\mathcal{K}_{\mathfrak{L}} \xrightarrow{\mathfrak{j}} \mathcal{P}_{\mathfrak{L}} \xrightarrow{\mathfrak{f}} \mathcal{K}_{\mathfrak{L}} = 1_{\mathcal{K}_{\mathfrak{L}}}$ then $\mathfrak{f}$ and $\mathfrak{j}$ are bijective. This means that if ξ_1 and ξ_2 can be separated by any $\mathfrak{L}$ map $\mathcal{P}' \to \mathcal{P}$ they can be in different classes in $\mathcal{P}_{\mathfrak{L}}$. One of the clearest ways in which schedules can be distinguished is by size $\xi_1 \subset \xi_2$ or (equally) there is a non-zero difference in the associated polynomials. Most maps of sheaves that are induced by a logistic morphisms of $\underline{A}$ will maintain difference in size. Conversely if $\mathfrak{L}$ compatible maps maintain the distinction $\xi_1 \subset \xi_2$ then whatever relation $\mathfrak{L}$ creates with the elements of $\mathcal{K}$ that property is equally distinctive in $\mathcal{K}$. This means the elements in $\mathcal{K}$ and so the classes in $\mathcal{K}_{\mathfrak{L}}$ are as equally divided as they are in $\mathcal{P}_{\mathfrak{L}}$.

Suppose now that a partial ordering can be imposed on sections of $\mathcal{K}$, the indexing of elements in $\mathcal{P}_{\mathfrak{L}}$ define $\mathfrak{L}$ as *order preserving* if:

(1) $(\xi_1, \kappa) \in \mathfrak{L}(U)$ and $\xi_2 > \xi_1$ then $(\xi_2, \kappa) \in \mathfrak{L}(U)$.
(2) $(\xi, \kappa_1) \in \mathfrak{L}(U)$ and $\kappa_1 > \kappa_2$ then $(\xi, \kappa_2) \in \mathfrak{L}(U)$.

LEMMA 3.2 $\mathfrak{L}$ *is order preserving then*

(1) $\kappa_1 < \kappa_2$ *implies* $X(\kappa_2) \subseteq X(\kappa_1)$.
(2) $\xi_1 < \xi_2$ *then* $Y(\xi_1) \subseteq Y(\xi_2)$.

This follows easily from the definitions.

THEOREM 3.3 *For $\mathfrak{L}$ order preserving, if for each $U \in \mathcal{U}$ the partial orders on $\mathcal{P}_{\mathfrak{L}}(U)$ and $\mathcal{K}_{\mathfrak{L}}(U)$ are complete then there exist maps $\mathfrak{j} : \mathcal{P}_{\mathfrak{L}} \to \mathcal{K}_{\mathfrak{L}}$ and $\mathfrak{f} : \mathcal{K}_{\mathfrak{L}} \to \mathcal{P}_{\mathfrak{L}}$ such that $\mathfrak{f} \circ \mathfrak{j} = 1_{\mathcal{K}_{\mathfrak{L}}}$ and $\mathfrak{j} \circ \mathfrak{f} = 1_{\mathcal{P}_{\mathfrak{L}}}$.*

This states there is a Galois Connection (Birkhoff 1967) V.8 or (Davey and Priestley 2002) between the sheaves $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{K}_{\mathfrak{L}}$.

Proof Define $\mathfrak{j} : \mathcal{P}_{\mathfrak{L}} \to \mathcal{K}_{\mathfrak{L}}$ by $\mathfrak{j}(X(\kappa)) = \cap_{\xi \in X(\kappa)} Y(\xi)$. For this to be well defined it should correspond to a unique $Y(\underline{\xi})$. The complete partial order ensures $\underline{\xi}$ exists. For each κ in $Y(\xi)$ $\mathfrak{L}(\xi, \kappa)$ is true and is true for all $\kappa' \leq \kappa$. To pick out a κ that characterizes ξ we want to define $\mathfrak{f} : \mathcal{K}_{\mathfrak{L}} \to \mathcal{P}_{\mathfrak{L}}$ by $\mathfrak{f}(Y(\xi)) = \cap_{\kappa \in Y(\xi)} X(\kappa)$. Again as the order in $\mathcal{P}_{\mathfrak{L}}(U)$ is complete there is a $\bar{\kappa}$

so that $\mathfrak{f}(Y(\xi)) = \cap_{\kappa \in Y(\xi)} X(\kappa) = X(\bar{\kappa})$. $\bar{\kappa}$ is the largest κ for which $\mathfrak{L}(\xi, \kappa)$ is true for the given ξ.

As $\mathfrak{L}$ is order preserving, Lemma 3.2 gives:

(1) $\kappa_1 < \kappa_2$ implies $X(\kappa_2) \subseteq X(\kappa_1)$ and $\mathfrak{j}(X(\kappa_1)) \subseteq \mathfrak{j}(X(\kappa_2))$.
(2) $\xi_1 < \xi_2$ then $Y(\xi_1) \subseteq Y(\xi_2)$ and so $\mathfrak{f}(Y(\xi_2)) \subseteq \mathfrak{f}(Y(\xi_2))$.

Consequently $\mathfrak{j}$ and $\mathfrak{f}$ are order reversing.

Composing these maps, firstly,

$$X(\kappa) \xrightarrow{\mathfrak{j}} Y(\underline{\xi}) \xrightarrow{\mathfrak{f}} \underset{\mathfrak{L}(\underline{\xi}, \kappa')}{\cap} X(\kappa') = X(\bar{\kappa}).$$

If $\xi \in X(\kappa)$ then ξ belongs to all the $Y(\xi)$ determining $\underline{\xi}$ so that ξ is at least as big as $\underline{\xi}$. Hence $\mathfrak{L}(\underline{\xi}, \kappa') \Rightarrow \mathfrak{L}(\xi, \kappa')$ so ξ is in the intersection defining $X(\bar{\kappa})$ so that $X(\kappa) \subseteq X(\bar{\kappa})$. If $\xi' \in X(\bar{\kappa})$ then, as $\mathfrak{L}(\xi', \bar{\kappa})$, $\mathfrak{L}(\xi', \kappa)$ is implied by $\mathfrak{L}(\underline{\xi}, \kappa)$. But $\mathfrak{L}(\underline{\xi}, \kappa)$ so $\mathfrak{L}(\xi', \kappa)$ and ξ' is in $X(\kappa)$. Thus $\mathfrak{f} \circ \mathfrak{j} = 1_{\mathcal{P}_{\mathfrak{L}}}$.

Secondly,

$$Y(\xi) \xrightarrow{\mathfrak{f}} X(\bar{\kappa}) \xrightarrow{\mathfrak{j}} \underset{\mathfrak{L}(\xi', \bar{\kappa})}{\cap} Y(\xi') = Y(\underline{\xi}).$$

Suppose $\kappa' \in Y(\underline{\xi})$. By definition $\mathfrak{L}(\underline{\xi}, \bar{\kappa})$. As $\kappa' < \bar{\kappa}$ so $\mathfrak{L}(\underline{\xi}, \kappa')$. But $\xi > \underline{\xi}$ which implies $\mathfrak{L}(\xi, \kappa')$ and so $\kappa' \in Y(\xi')$ and therefore $Y(\underline{\xi}) \subseteq Y(\xi)$.

If $\kappa'' \in Y(\xi)$ κ'' is in all the $Y(\xi')$ whose intersection gives $X(\bar{\kappa})$. In particular $\kappa'' \leq \bar{\kappa}$ and as $\underline{\xi}$ is the least ξ for which $\mathfrak{L}(\xi', \bar{\kappa})$ and so $\mathfrak{L}(\underline{\xi}, \kappa'')$ and $\kappa'' \in Y(\underline{\xi})$ and so $Y(\xi) \subseteq Y(\underline{\xi})$. This gives $Y(\underline{\xi}) = Y(\xi)$ and so $\mathfrak{j} \circ \mathfrak{f} = 1_{\mathcal{K}_{\mathfrak{L}}}$.

As $\mathfrak{j}$ and $\mathfrak{f}$ are order reversing they give the required duality. □

Treating $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{K}_{\mathfrak{L}}$ as category objects with maps being set inclusions we have:

COROLLARY 3.4 $\mathfrak{f}$ *is the left adjoint of* $\mathfrak{j}$, *that is*

$$\mathcal{P}^{op}_{\mathfrak{L}}(X(\kappa), \mathfrak{f}(Y(\xi))) \cong (\mathcal{K}_{\mathfrak{L}}(\mathfrak{j}(X(\kappa), Y(\xi_1)).$$

This theorem is an example of a class of theorems that establish the limits of discrimination of a language. In this sense it is an epistemological principle: for a given mode of analysis there are limits to discrimination of phenomena. The sheaf relation $\mathfrak{L}$ is an *epistemological operator* and handles the heterogeneous nature of the language of $\underline{A}$. The $[\mathfrak{L}, \mathcal{K}]$ giving a uniform approach to the logical relations arising from the connections of $\underline{A}$.

Not all languages implemented in sheaves $\mathcal{K}$ give rise to ordered classes. We can define a *type protocol* as a property that each $a \in A$ selects a set $E(a) \subset Typ(a)$ such that for $b \in S(a)$ and $E(a) \cap E(b) \subset \beta(a, b)$. We define a set valued sheaf $\mathcal{K}$ defined on the cover $\mathcal{U}$ of $\underline{A}$ such that $\mathcal{K}(U)$ is itself a cover for $E(U) = \cup_{a \in U} E(a)$. For $\kappa \in \mathcal{K}(U)$ define $\mathfrak{L}(\xi, \kappa)(U)$ if and only if for all $w \in range(\xi)$, $dom(w) \subseteq \kappa$. The sheaf nature of $\mathcal{K}$ imposes a connection among the $E(a)$ allowing the definition of protocol observant schedules to be a sub-sheaf of $\mathcal{P}$. Type protocols are found wherever special qualities of materials are required.[4] Similarly we can define *action protocols*.

3.2. *Dynamics: measuring schedules*

Definition A *form* on $Sch(\underline{A})$ is a measure on schedules with values in the real numbers.

Examples of forms are

(1) $\Delta_\theta : Sch(a) \to \mathbb{N}$ gives the increase in type weighting per time-slot so that $\xi \in Sch(\underline{A})$. $\Delta_\theta(\xi)$ can be taken as the per time-slots value added for ξ.
(2) $\mathfrak{a} : Sch(a) \to \mathbb{N}$ the total number of actions of a schedule.
(3) $L : Sch(a) \to \mathbb{N}$ the lead time of the schedule.

We shall assume that each form ϕ comes with an intended measure $m(\phi)$ which is an interval around some desired result. For $U \in \mathcal{U}$, ϕ acting on $\xi \in Sch(U)$, we expect $\phi(\xi) \in m(\phi)(U)$ meaning for $\forall a \in U$, $\phi(\xi(a) \in m(\phi)(a)$. It is a filter for selecting schedules with a desired property.

Let $F(\underline{A}) : \mathcal{U} \to \mathbb{R}$ be a selected ring of real-valued functions defined on each $U \in \mathcal{U}$. This is required to be a ring-valued sheaf.

Let Λ be a finite set of forms

Definition The sheaf $\mathcal{D}[\Lambda]$ of forms of Λ is the module of $F(\underline{A})$ forms generate by the forms $\lambda \in \Lambda$ so that for each $U \in \mathcal{U}$ $\mathcal{D}[\Lambda](U)$ is the set of linear expressions $\phi(U) = \sum_i c_i(U).\lambda_i$ with $\lambda_i \in \Lambda$ and $c_i \in F(U)$ and with the restriction maps $\mathcal{D}[\Lambda](U) \to \mathcal{D}[\Lambda](V)$ given by $\phi(U)(\xi)(U) \mapsto \phi(V)(\xi \mid V) = \sum_i c_i(V).\lambda_i(\xi(V))$ whenever $V \subset U$ in $\mathcal{U}$.

As defined this is a presheaf. Λ is chosen so that that $\mathcal{D}[\Lambda]$ is a sheaf. (We cannot always patch together forms such as the "width", the number of starting types.) The sheaf of forms for a specific combinatorial system, $\underline{A}$, is denoted $\mathcal{D}[\Lambda](\underline{A})$.

From hereon we assume that Λ has been chosen and is fixed. As examples let $c : A \to \mathbb{R}$ (that is $c \in F(\underline{A})$) then two forms have immediate interpretation.

(1) The "advantage" $\vartheta(\mathfrak{a}, c) = \Delta_\theta - c.\mathfrak{a}$, where $c(a)$ is a conversion from the units of the measure $\mathfrak{a}$ in each $a \in A$ to those of the weighting.
(2) If costs are accrued per time $\vartheta(L, c) = \Delta_\theta - c.L$ is a related measure.

If $c(a)$ is cost per time, energy used, power consumption for each $w \in \xi(a)$, $\vartheta(L, c)(a)$ can measure the nett gain in type weighting (as a measure of value) for the time or energy required to produce the value while $\vartheta(\mathfrak{a}, c)(a)$ is the nett gain taking into the cost for each individual action. These forms can be "directed" to measure along a sub-combinatorial system $\underline{A}' \subset \underline{A}$ or modified to count or measure schedules restricted to classes of types.

PROPOSITION 3.3 *$\mathcal{D}$ is a covariant functor on Comb with values in ring-valued sheaves.*

Proof As *Sch* is a covariant functor, $f : \underline{A}_1 \to \underline{A}_2$ in *Comb* such that f maps $\mathcal{U}_1 \to \mathcal{U}_2$ implies $Sch(f) : Sch(\underline{A}_1) \to Sch(\underline{A}_2)$ maps schedules. Given $\phi \in \mathcal{D}(\underline{A})(U)$ define $\mathcal{D}(f)$ by $\mathcal{D}(f)(\phi)(\xi) = \phi(Sch(f)(\xi)(f(U))) \in \mathcal{D}(\underline{A}_2)(f(U)$. □

For a given $\mathcal{P}$ let $\mathfrak{F} \rightarrowtail \mathcal{P} \times \mathcal{D}$ be given by $(\xi, \phi)) \in \mathfrak{F}(U)$ if and only if $\phi(a)(\xi) \in m(\phi)(a)$ is true for all $a \in U$.

LEMMA 3.5 *$\mathfrak{F}$ is a sheaf.*

Proof Let $U_i \in \mathcal{U}$, $U = \cup_i U_i$ and $\xi \mid U_i = \xi_i \in \mathcal{P}(U_i)$, $\xi_i(U_i \cap U_j) = \xi_j(U_i \cap U_j)$ and $\phi_i \in \mathcal{D}(U_i)$. From the fact that the ξ_i are taken from a sheaf $\mathcal{P}$ they will patch together to create a schedule ξ defined in $U = \cup_i U_i$. The ϕ_i likewise patch together and constitute a form on U. (This follow from the product of two sheaves) and consequently we can defined $\phi(U)(\xi(U)) \in \mathfrak{F}$. □

(Alternatively $\mathfrak{F}$ could also be defined as a sub-sheaf. We can define the sub-presheaf as above and then take $\mathfrak{F}$ to be the associated sheaf. This avoids patching the $m(\phi)(U_i)$. The target measures $m(\phi)(a)$ are defined for each a independently but have constraints that are necessary to defined $\mathcal{P}_{\mathfrak{F}}$ below).

3.2.1. *The functors $\mathcal{P}_{\mathfrak{F}}$ and $\mathcal{D}_{\mathfrak{F}}$*

We apply the results of the previous section with the substitutions $\mathfrak{L} \to \mathfrak{F}$, $\mathcal{K} \to \mathcal{D}$. Define the sheaf $\mathcal{P}_{\mathfrak{F}}$ as follows.

(1) For $\phi \in \mathcal{D}(U)$ $F(\phi) =_{defn} \{\xi \in \mathcal{P}(U) \mid \phi(\xi) \in m(\phi)\}$.
(2) $\mathcal{P}_{\mathfrak{F}}(U) =_{defn} \{F(\phi) \mid \phi \in \mathcal{D}(U)\}$.

LEMMA 3.6 *$\mathcal{P}_{\mathfrak{F}}$ can be given a group operation.*

Proof For each $U \in \mathcal{U}$ start with the free group generated by the set of classes $F(\phi) \in \mathcal{P}_{\mathfrak{F}}(U)$ and factor by the sub-group of relations $F(\phi_1) + F(\phi_2) - F(\phi_3)$ for which $\xi \in F(\phi_1) \cap F(\phi_2) \Rightarrow \phi_1(\xi) + \phi_2(\xi) \in m(\phi_3)(U)$. That is as functions $\phi_1 + \phi_2$ defined on $F(\phi_1) \cap F(\phi_2)$ gives the same result on the same domain as ϕ_3 to within the expected results on $\mathcal{P}$. The form $\underline{0} : \xi \mapsto 0$ for all ξ and which has $m(\underline{0}) = \{0\}$ is the zero element. $\forall \xi \in \mathcal{P}(U)$ $\phi_1(\xi) + \phi_2(\xi) \in m(\underline{0}) \Leftrightarrow F(\phi_1) + F(\phi_2) = F(\underline{0})$. $\phi_1(\xi) + \phi_2(\xi) \in m(\underline{0}) = 0$ so $\phi_1 = -\phi_2$ on their target domains. □

The importance will be in equations $\sum_i F(\phi_i)(\xi) = \sum_j F(\phi_j)(\xi)$. If the ϕ_i deal with different aspects of a schedule, and this is where Λ matters, these equations provide a linear equation representation of the relationship among measures. (This is important for the cohomology of $\mathcal{P}_{\mathfrak{F}}$ as will be shown below.)

Definition For $U \in \mathcal{U}$, the sheaf $\mathcal{D}_{\mathfrak{F}}$ is given by

(1) For $\xi \in \mathcal{P}(U)$, $\Delta(\xi) =_{defn} \{\phi \in \mathcal{D}(U) \mid \phi(\xi) \in m(\phi) \text{ in } U\}$.
(2) $\mathcal{D}_{\mathfrak{F}}(U) =_{defn} \{\Delta(\xi)) \mid \xi \in \mathcal{P}(U)\}$.

$\Delta(\xi)$ can be identified as an abstract property associated with the set of measures of ξ with values close to the target value of the measure.

LEMMA 3.7 *$\mathcal{D}_{\mathfrak{F}}$ has a group structure.*

Proof For each $U \in \mathcal{U}$ start with the free group generated by the set of classes $\Delta(\xi) \in \mathcal{D}_{\mathfrak{F}}(U)$ and factor by the sub-group of relations given by $\Delta(\xi_1)(U) + \Delta(\xi_2)(U) - \Delta(\xi_3)(U)$ where for every $\phi_1 \in \Delta(\xi_1)$ and $\phi_2 \in \Delta(\xi_2)$ there is a $\phi' \in \Delta(\xi_3)$ such that $\phi_1(\xi_1) + \phi_2(\xi_2) \in m(\phi')$. If $\bar{0}$ is the "null schedule" for which all ϕ give the value zero, $\Delta(\bar{0})$ is the group identity. Thus if $\Delta(\xi_1) + \Delta(\xi_2) = \Delta(\bar{0})$ then for every pair $(\phi_1, \phi_2) \in \Delta(\xi_1) \times \Delta(\xi_2)$ there is a $\phi' \in \Delta(\bar{0})$ such that $\phi_1(\xi_1) + \phi_2(\xi_2) \in m(\phi')(\bar{0}) = \{0\}$ making $\Delta(\xi_1)$ and $\Delta(\xi_2)$ inverse classes. □

3.3. *Groupoid Fibrations as examples of the $[\mathfrak{L}, \mathcal{K}]$ functors*

A groupoid (Iyanaga and Kawada 1977) can be thought of as a set of reversible transformations some of which compose. When they compose they do so associatively. Within composing subsets the group axioms apply. Groupoids will be important here because changes to schedules form a groupoid.

Let $\mathcal{P} \in Comb(\underline{A})$, a groupoid over $\mathcal{P}$ (as defined in Canonaco 2004) is a sheaf $\mathcal{G}$ in $\mathcal{S}h(\underline{A})$ with maps $s, t : \mathcal{G} \to \mathcal{P}$ (the source and target of the groupoid action on a schedule) with the following properties.

(1) There is an inverse isomorphism map $\iota : \mathcal{G} \to \mathcal{G}$ such that $s \circ \iota = t$ and $t \circ \iota = s$.
(2) g_1, g_2 with $s(g_1) = \xi_1, t(g_1) = \xi_2$ and $s(g_2) = \xi_2, t(g_2) = \xi_3$ then $g_2 \circ g_1$ is defined and $s(g_2 \circ g_1) = s(g_1)$ and $t(g_2 \circ g_1) = s(g_2)$.
(3) s and t are onto and there exists a unit element for which $s = t$.
(4) s and t preserved associativity of $\circ$.

This gives functor $[\mathcal{G}, 1_{Comb(\underline{A})}] : Comb(\underline{A}) \to \mathcal{S}h(\underline{A})[\mathcal{G}, 1_{Comb(\underline{A})}]$ with

$$\mathcal{P} \mapsto \left(\mathcal{P} \xleftarrow{s} \mathcal{G} \xrightarrow{t} \mathcal{P}\right).$$

This can be put in the form $\mathfrak{L}(\mathcal{P}) \rightarrowtail \mathcal{P} \times \mathcal{G}$ with $[\mathfrak{L}, \mathcal{G}]$ given by

$$\mathcal{P} \mapsto \left(\mathcal{P} \xleftarrow{s\circ(proj_2)} \mathfrak{L} \xrightarrow{proj_2} \mathcal{G}\right),$$

so that $proj_1$ is identified with s. Define η as

$$\eta : \left(\mathcal{P} \xleftarrow{s\circ(proj_2)} \mathfrak{L} \xrightarrow{proj_2} \mathcal{G}\right) \mapsto \mathcal{P} \xleftarrow{s} \mathcal{G} \xrightarrow{(s,1)} \mathfrak{L} \xrightarrow{t\circ(proj_2)} \mathcal{P}.$$

This gives a functor η which has an obvious inverse.

$$\begin{array}{ccc} & Comb(\underline{A}) & \\ \swarrow_{[\mathfrak{L},\mathcal{G}]} & & \searrow_{[\mathcal{G},1_{Comb(\underline{A})}]} \\ \mathcal{S}h[\mathfrak{L}, \mathcal{G}] & \xrightarrow{\eta} & \mathcal{S}h[\mathcal{G}, 1_{Comb(\underline{A})}] \end{array}$$

We now have sheaves $\mathcal{P}_{\mathfrak{L}}, \mathcal{G}_{\mathfrak{L}}$ given by:

(1) $X(g) =_{defn} \{\xi \in \mathcal{P}(U) \mid sg = \xi\}$ and so
(2) $\mathcal{P}_{\mathfrak{L}}(U) =_{defn} \{X(g) \mid g \in \mathcal{G}(U) \wedge (sg = \xi) \wedge (g \neq 1)\}$. This is the set of sections of $\mathcal{P}$ that have non-trivial groupoid actions.
(3) $Y(\xi) =_{defn} \{g \in \mathcal{G}(U) \mid (g \neq 1) \wedge (sg = \xi)\}$ and
(4) $\mathcal{G}_{\mathfrak{L}}(U) = \{Y(\xi)(U) \mid \xi \in \mathcal{P}(U)\}$.

$Y(\xi)$ is the set of g that start at ξ and hence can be mapped to the set of ξ' for which $g : \xi \to \xi'$. In Part 2 we shall define this as a section

(1) $G(U)(\xi) = \{\xi' \in \mathcal{P}(U) \mid \exists g \in \mathcal{G}(U)(g \neq 1) \wedge (sg = \xi) \wedge (tg = \xi')\}$ and so
(2) $\mathcal{G}_{\mathfrak{L}}(U) = \{G(U)(\xi) \mid \xi \in \mathcal{P}(U)\}$.

which is equivalent to the definition above. Properties of these functors will be discussed in Part 2.

4. Cohomology

The information in a sheaf $\mathcal{P}$ is in the groups $\mathcal{P}(U), U \in \mathcal{U}$, and in the restriction homomorphisms $\mathcal{P}(U) \xrightarrow{\rho(U,V)} \mathcal{P}(V)$. Cohomology gives information on the relations $\mathcal{P}(U_i) \to \mathcal{P}(U_i \cap U_j)$, $\mathcal{P}(U_i \cap U_j) \to \mathcal{P}(U_i \cap U_j \cap U_k)$ and so on for as many intersections as exist in $\mathcal{U}$. Such relations can only be expressed in the sums and differences amongst the elements of the $\mathcal{P}(U)$ but they contain an algebraic "signature" of the way $\mathcal{P}$ is "assembled". Given a group valued sheaf $\mathcal{P}$ we define the Čech cohomology on $\mathcal{P}$ as a sequence of groups $\check{H}^n(\mathcal{P})$. There are many references to this cohomology with Hartshorne (1977), Spanier (1966), Gunning and Rossi (1965) spanning its use in algebraic geometry, algebraic topology and complex analysis respectively. These techniques are finding their way into other fields (see Gross and Kotiuga 2001; de Silva and Ghrist 2007; Ghrist and Hiraoka 2011).

In the following we use the convention that if $U_{i_j} \in \mathcal{U}$ and $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_n}$ is non-empty then $U[i_k] = \cap_{j \neq k} U_{i_j}$. As usual denote the cochains by

$$C^n = C^n(\mathcal{P}, \mathcal{U}) = \prod_{i_0, i_1, \ldots, i_n} \mathcal{P}(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_n}) \tag{1}$$

where the product is over $n+1$ tuples $i_0 < i_1 < \ldots < i_n$ giving n intersections. Define the *coboundary* map $\partial^n : C^n \longrightarrow C^{n+1}$ by setting

$$\partial^n(\underline{\xi})(U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_n} \cap U_{i_{n+1}}) = \sum_{k=0}^{n+1} (-1)^k . \xi_{i_k} \mid (U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_n} \cap U_{i_{n+1}}) \tag{2}$$

where $\underline{\xi} =< \xi_{i_k} > \in \Pi_k \mathcal{P}(U[i_k])$, $k = 0, 1, \ldots, n+1$.

Notation:

(1) The subgroup of n-cocycles in C^n, $Z^n(\mathcal{P})$, is the group $ker(\partial^n)$. We write Z^n when $\mathcal{P}$ is implied.
(2) The subgroup of n-co-boundaries in C^n, $B^n(\mathcal{P})$, is the group $im(\partial^{n-1})$. We write B^n when $\mathcal{P}$ is implied.

Definition The group of n cohomology classes, or the nth cohomology group is $\check{H}^n(\mathcal{P}) =_{defn} ker(\partial^n)/im(\partial^{n-1}) = Z^n/B^n$.

By convention B^0 is set to $\{0\}$. Sometimes an *augmentation* of the cochain is defined by an augmentation map:

$$\varepsilon : \mathcal{P}'(A) \to C^0(\mathcal{V}, \mathcal{P}) = \prod_i \mathcal{P}(P_i).$$

where $\mathcal{P}' \subset \mathcal{P}$ and $\varepsilon(\xi)$ is $\xi \mid U$ in each $\mathcal{P}(U)$. B^0 is then taken to be defined as the image of ε. $\check{H}^0(\mathcal{P}) = Z^0/B^0$ is then seen to be the classes of global sections relative to those of $\mathcal{P}'$.

We shall presently show that $\check{H}$, despite its name, is in this context a covariant functor. But first we shall get a taste of cohomology.

4.1. An example

The choice of the cover $\mathcal{U}$ for a combinatorial system $\underline{A}$ has a significant effect on the cohomology as the following result shows.

THEOREM 4.1 *Let $\mathcal{U} = \{U_1, U_2, \ldots, U_N\}$ be a cover for $\underline{A}$ such that there is number $n_0 < N$ such that any set of $n > n_0$ intersections $U_{i_0} \cap \ldots \cap U_{i_n}$ is equal to the intersections of a subset of those sets with no more than n_0 members. If $\mathcal{P}$ is defined on $\mathcal{U}$ then $H^{n_0+k}(\mathcal{P}) = 0$ for $k \geq 1$.*

Proof This follows as any cocycle on more than $n > n_0 + 1$ intersections is an equation on $n-1$ intersections and hence can be represented as coboundaries. Specifically, let $< \xi_j >$ form a n-cocycle. ξ_j is defined on $U_{i_0} \cap \ldots \hat{U}_{i_j} \cap \ldots \cap U_{i_n}$. Hence $\Sigma(-1)^j \xi_j = 0$ on $U_{i_0} \cap \ldots \cap U_{i_n}$. For $n > n_0 + 1$ we need to show this is a coboundary. If $n > n_0 + 1$ then for each j there is an l such that $U_{i_0} \cap \ldots \cap \hat{U}_{i_j} \cap \ldots \cap U_{i_n} = U_{i_0} \cap \ldots \cap \hat{U}_{i_j} \cap \ldots \cap \hat{U}_{i_l} \cap .. \cap U_{i_n}$ so $\mathcal{P}(U_{i_0} \cap \ldots \cap \hat{U}_{i_j} \cap \ldots \cap U_{i_n}) = \mathcal{P}(U_{i_0} \cap \ldots \cap \hat{U}_{i_j} \cap \ldots \cap \hat{U}_{i_l} \cap \ldots \cap U_{i_n})$. This means each ξ_j can be written as an alternating sum of schedules defined on fewer intersections of the cover and hence can be expressed as a coboundary. Thus for $n > n_0$ $\check{H}^n(\mathcal{P}) = 0$. □

For this reason we use a "standard" cover: $\mathcal{U}$ is the set of sets $S^k(a)$, $k > 1$, for all $a \in A$ together with their intersections. This is a very redundant cover and many useful covers will have fewer sets.

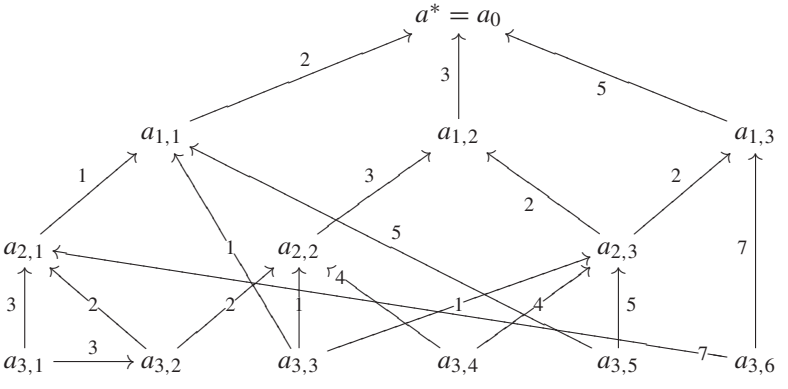
The following example was constructed to

(1) be large enough that a covering, $\mathcal{U}$ could be defined in terms of sets $S^2(a)$. This means that A has to have a minimum of three levels;
(2) not to have a morphism to a simpler system; and
(3) the sheaf $\mathcal{P}$ is constructed to have a non-trivial cohomology to $H^2(\mathcal{P}, \mathcal{U})$.

This last condition means that $\mathcal{P}$ is not a soft (equiv. "flasque") sheaf (Gunning and Rossi 1965, Chapter 6; Hartshorne 1977, Chapter 3) in which everything is determined by the global sections.

We use the terms supplier and customers when topics have relation $a' \in S(a)$.

$(A, \prec)$ is given in the following diagram:



An arrow from $a_{k,l}$ to $a_{i,j}$ signifies $a_{k,l} \in S(a_{i,j})$. Numbers on the arrows are the lead-times.

Arrows can be considered as the flow of goods. No topic $a_{i,j}$ has only one supplier unless the supplier provides goods to other customers. If this were so the system $\underline{A}$ could be mapped to $\underline{A}'$ absorbing the supplier in its customer without changing the cohomology. Similarly no supplier has only one customer for the same reasons.

Taking the simplest non-trivial case we assume that $W(a_{ij})$ is generated by the one action: $\{e_1, e_2, \ldots, e_{n(i,j)}\} \xrightarrow{w_{i,j}} e_{i,j}$ where the e_i come from the types of the set of suppliers $S(a)$ and $n(i, j)$ is the number of such suppliers. The system $\underline{A}$ is a model of a piecework or craft-work supply network in which only one item of a particular type can be produced per action. (This contrasts with batch processes which one action can produce many items.). This means that for $w_{i,j} \in W(a_{ij})$, $w_{i,j} + w_{i,j}$, being the simultaneous or overlapping execution of two instances of $w_{i,j}$, is not possible. Thus we assert $w_{i,j} + w_{i,j} = 0$ and set $W(a_{ij}) = \mathbb{Z}_2$. On the other hand a schedule ξ, which we shall represent as a sum of terms $(w_{i,j}, s(i, j))$, $s(i, j)$ the starting time-slot for the particular instance of $w_{i,j}$, which might be one of many but not at the same time. The addition of schedules means that $(w, s) \pm (w, s) = 0$.

Let $\mathcal{U}$ be all the sets $S^2(a_{ij})$ together their intersections and A so

(1) $S^2(a_0) = \{a_0, a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}, a_{2,3}\}$.
(2) $S^2(a_{1,1}) = \{a_{1,1}, a_{2,1}, a_{3,1}, a_{3,2}, a_{3,3}, a_{3,5}, a_{3,6}\}$.
(3) $S^2(a_{1,2}) = \{a_{1,2}, a_{2,2}, a_{2,3}, a_{3,2}, a_{3,3}, a_{3,4}, a_{3,5}\}$.
(4) $S^2(a_{1,3}) = \{a_{1,3}, a_{2,3}, a_{3,3}, a_{3,4}, a_{3,5}, a_{3,6}\}$.

Table 1. Basic schedules, no shared resources.

Time-slot	$\xi(a_{11})$, $\mathcal{P}(S^2(a_{1,1}))$	$\xi(a_{12})$, $\mathcal{P}(S^2(a_{1,2}))$	$\xi(a_{13})$, $\mathcal{P}(S^2(a_{1,3}))$
1	w_{36}	w_{31},w_{34},w_{35}	w_{35},w_{36}
2	w_{31},w_{36}	w_{31},w_{34},w_{35}	w_{34},w_{35},w_{36}
3	w_{31},w_{36}	w_{31},w_{34},w_{35}	w_{34},w_{35},w_{36}
4	w_{31},w_{35},w_{36}	w_{32},w_{33},w_{34},w_{35}	w_{34},w_{35},w_{36}
5	w_{31},w_{36},w_{35}	w_{32},w_{34},w_{35}	w_{33},w_{34},w_{35},w_{36}
6	w_{31},w_{32},w_{36},w_{35}	w_{22},w_{34}	w_{23},w_{36}
7	w_{31},w_{32},w_{36},w_{35}	w_{22},w_{34}	w_{23},w_{36}
8	w_{21},w_{33},w_{35}	w_{22},w_{34},w_{33}	w_{13}
9	w_{11}	w_{23}	w_{13}
10	w_{11}	w_{23}	w_{13}
11	a_0	w_{12}	w_{13}
12	–	w_{12}	w_{13}
13	–	w_{12}	a_0
14	–	a_0	–

(5) $S^2(a_0) \cap S^2(a_{1,1}) = \{a_{2,1}, a_{3,5}\}$.
(6) $S^2(a_0) \cap S^2(a_{1,2}) = \{a_{2,2}, a_{2,3}\}$.
(7) $S^2(a_0) \cap S^2(a_{1,3}) = \{a_{2,3}, a_{3,6}\}$.
(8) $S^2(a_{1,1}) \cap S^2(a_{1,2}) = \{a_{3,2}, a_{3,3}, a_{3,5}\}$.
(9) $S^2(a_{1,2}) \cap S^2(a_{1,3}) = \{a_{2,3}, a_{3,3}, a_{3,4}, a_{3,5}\}$.
(10) $S^2(a_{1,1}) \cap S^2(a_{1,3}) = \{a_{3,3}, a_{3,5}, a_{3,6}\}$.
(11) $S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3}) = \{a_{3,3}, a_{3,5}\}$.
(12) $S^2(a_0) \cap S^2(a_{1,1}) \cap S^2(a_{1,2}) = \emptyset$.
(13) $S^2(a_0) \cap S^2(a_{1,1}) \cap S^2(a_{1,3}) = \emptyset$.
(14) $S^2(a_0) \cap S^2(a_{1,2}) \cap S^2(a_{1,3}) = \{a_{2,3}\}$.

For a sheaf $\mathcal{P}$ this gives the cochains:

(1) $C^0(\mathcal{P}, \mathcal{U}) = \mathcal{P}(S^2(a_{1,1})) \oplus \mathcal{P}(S^2(a_{1,2})) \oplus \mathcal{P}(S^2(a_{1,3})) \oplus \mathcal{P}(S^2(a_0))$.
(2) $C^1(\mathcal{P}, \mathcal{U}) = \mathcal{P}(S^2(a_{1,1}) \cap S^2(a_{1,2})) \oplus \mathcal{P}(S^2(a_{1,2}) \cap S^2(a_{1,3}))$
$\oplus \mathcal{P}(S^2(a_{1,1}) \cap S^2(a_{1,3})) \oplus \mathcal{P}(S^2(a_0) \cap S^2(a_{1,1}))$
$\oplus \mathcal{P}(S^2(a_0) \cap S^2(a_{1,2})) \oplus \mathcal{P}(S^2(a_0) \cap S^2(a_{1,3}))$.
(3) $C^2(\mathcal{P}, \mathcal{U}) = \mathcal{P}(S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3}) \oplus \mathcal{P}(S^2(a_0) \cap S^2(a_{1,2}) \cap S^2(a_{1,3}))$.
(4) $C^p(\mathcal{P}, \mathcal{U}) = 0$ for $p > 2$.

$Sch(\underline{A})$ contains all possible schedules including those which have a very leisurely time-table. Instead the example will use a sheaf $\mathcal{P}$ that is constructed as follows. All schedules are defined in terms of the same time-slots that are sufficient for the longest non-repeating global section defined below. This avoids schedules being different by virtue of different starting times. $\mathcal{P}(A)$ is to be generated by a small set of global schedules. The restrictions of these global schedules populate the $\mathcal{P}(U)$ for each $U \in \mathcal{U}$. Local options or variations are introduced and will be described below. The definition of $\mathcal{P}$ is similar to giving operational guidelines and as long as it is consistent with lead-times and the algebra of the system $\underline{A}$ there is freedom to choose. The tables below define $\mathcal{P}$ for the sets $S^2(a_{i,j})$.[5]

Table 1 gives schedules for each $S^2(a_{i,j})$ that minimize time but do not share resources.

Note that each time an action w_{ij} is run it uses its domain types so for example, as $w_{2,3}$ is actioned twice, it has to be supplied twice by $w_{3,4}$. A comparisons of the three "generic" $S^2(a_{ij})$

schedules, $\xi(a_{i,j})$ shows that there is considerable overlap in the requirements for certain long lead-time actions such as $w_{3,4}$ and $w_{3,6}$. This makes the schedules linearly independent but are not restrictions of a global schedule. For these reason we list three global schedules, each of which gives priority to one of the $S^2(a_{i,j})$

$\tilde{\xi}_1$ which gives priority to $S^2(a_{11})$ and then $S^2(a_{12})$ actions.

Time-slot	$S^2(a_{11})$ action	$S^2(a_{12})$ action	$S^2(a_{13})$ action
1	w_{35},w_{36}	w_{31},w_{34}	–
2	w_{35},w_{36}	w_{31},w_{34}	–
3	w_{35},w_{36}	w_{31},w_{34}	–
4	w_{31},w_{35},w_{36}	w_{34},w_{32},w_{33}	–
5	w_{31},w_{35},w_{36}	w_{32}	–
6	w_{31},w_{36}	w_{22},w_{35}	–
7	w_{31},w_{36},w_{32}	w_{22},w_{35}	–
8	w_{31},w_{32}	w_{22},w_{35}	w_{36}
9	w_{31}	w_{35},w_{33}	w_{36},w_{34}
10	w_{21},w_{33}	w_{35}	w_{36},w_{34}
11	$w_{11}-$	w_{23}	w_{36},w_{35},w_{34}
12	w_{11}	w_{23}	w_{36},w_{35},w_{34}
13	–	w_{12}	w_{36},w_{35}
14	–	w_{12}	w_{36},w_{35}
15	–	w_{12}	w_{35},w_{33}
16	–	–	w_{23}
17	–	–	w_{23}
18	–	–	w_{13}
19	–	–	w_{13}
20	–	–	w_{13}
21	–	–	w_{13}
22	–	–	w_{13}
23	a_0	a_0	a_0

$\tilde{\xi}_2$ gives priority to $S^2(a_{12})$ and $S^2(a_{13})$ actions.

Time-slot	$S^2(a_{11})$ action	$S^2(a_{12})$ action	$S^2(a_{13})$ action
1	–	w_{34},w_{31},w_{35}	w_{36}
2	–	w_{34},w_{31},w_{35}	w_{36}
3	–	w_{34},w_{31},w_{35}	w_{36}
4	–	$w_{34},w_{32},w_{35},w_{33}$	w_{36}
5	–	w_{32},w_{34},w_{35}	w_{36}
6	–	w_{22},w_{34}	w_{36},w_{35}
7	–	w_{22},w_{34}	w_{36},w_{35}
8	w_{31},w_{36}	w_{22},w_{33},w_{34}	w_{35}
9	w_{31},w_{36}	w_{23}	w_{35},w_{34}
10	w_{31},w_{36}	w_{23}	w_{35},w_{34}
11	$w_{31},w_{32},w_{35},w_{36}$	w_{12}	w_{34}

Time-slot	$S^2(a_{11})$ action	$S^2(a_{12})$ action	$S^2(a_{13})$ action
12	w_{31},w_{32},w_{35},w_{36}	w_{12}	w_{34},w_{33}
13	w_{31},w_{35},w_{36}	w_{12}	w_{23}
14	w_{21},w_{35},w_{36}	–	w_{23}
15	w_{35},w_{33}	–	w_{13}
16	w_{11}	–	w_{13}
17	w_{11}	–	w_{13}
18	–	–	w_{13}
19	–	–	w_{13}
20	a_0	a_0	a_0

$\tilde{\xi}_3$ gives priority to $S^(a_{13})$

Time-slot	$S^2(a_{11})$ action	$S^2(a_{12})$ action	$S^2(a_{13})$ action
1	–	–	w_{36},w_{35}
2	–	–	w_{36},w_{34},w_{35}
3	–	–	w_{36},w_{34},w_{35}
4	–	–	w_{36},w_{34},w_{35}
5	–	–	w_{36},w_{34},w_{35},w_{33}
6	–	w_{31},w_{34}	w_{36},w_{23}
7	–	w_{31},w_{34},w_{35}	w_{36},w_{23}
8	w_{36}	w_{31},w_{34},w_{35}	w_{13}
9	w_{31},w_{36}	w_{32},w_{34},w_{35},w_{33}	w_{13}
10	w_{31},w_{36}	w_{32},w_{34},w_{35}	w_{13}
11	w_{31},w_{36}	w_{22},w_{34},w_{35}	w_{13}
12	w_{32},w_{31},w_{35},w_{36}	w_{22},w_{34},	w_{13}
13	w_{32},w_{31},w_{35},w_{36}	w_{22},w_{34},w_{33}	–
14	w_{31},w_{35},w_{36}	w_{23}	–
15	w_{21},w_{33},w_{35}	w_{23}	–
16	w_{35}	w_{12}	–
17	w_{11}	w_{12}	–
18	w_{11}	w_{12}	–
19	a_0	a_0	a_0

These are our three global sections. Along with the restrictions of the $\xi(a_{i,j})$ their restrictions populate all the $\mathcal{P}(U)$ groups with an initial set of schedules. These will be the only set of schedules in $\mathcal{P}(S^2(a_0))$. We are still free to define the schedules in $\mathcal{P}(U)$ in the way that represents the operational options of $\underline{A}$.

There are a number of degrees of freedom (options on start time in order for all the processed types to reach a_0), particularly the start times of $w_{3,3}$, $w_{3,4}$ in $S^2(a_{1,3})$ and the start times of $S^2(a_{1,1})$, $S^2(a_{1,2})$, $S^2(a_{1,3})$ relative to each other. These generate the degrees of freedom in the $\mathcal{P}(S^2(a_{i,j}))$ in the $C^0(\mathcal{P},\mathcal{U})$.

We now define our sheaf on intersections. $\mathcal{P}(S^2(a_{i,j})\cap S^2(a_{k,l}))$. In all cases these "inherit" the restrictions of the global schedules as well as those from $\mathcal{P}(S^2(a_{i,j}))$ and $\mathcal{P}(S^2(a_{k,l}))$. For each of these $\mathcal{P}(S^2(a_{i,j})\cap S^2(a_{k,l}))$ there are three restrictions from $\mathcal{P}(A)$ and one each from $\mathcal{P}(S^2(a_{i,j}))$ and $\mathcal{P}(S^2(a_{k,l}))$ giving five schedules each providing a $\mathbb{Z}_2$ dimension to start with. But there is no reason that these are the only schedules in $\mathcal{P}(S^2(a_{i,j})\cap S^2(a_{k,l}))$. After all $S^2(a_{i,j})\cap S^2(a_{k,l})$

simply produces what is required of it as it can. Thus on top of the restrictions of schedules imposed on $\mathcal{P}(S^2(a_{i,j}) \cap S^2(a_{k,l}))$ we allow some variations.

(1) $S^2(a_{1,1}) \cap S^2(a_{1,3}) = \{a_{3,3}, a_{3,5}, a_{3,6}\}$. We allow the variations:

   (a) First $w_{3,5}$ (block) starts at slot 1 then the second $w_{3,5}$ (block) has 5 starting options.
   (b) $w_{3,5}$ starts at slot 2 then the second $w_{3,5}$ has 4 options.
   (c) $w_{3,5}$ starts at slot 3 then the second $w_{3,5}$ has 3 options.
   (d) $w_{3,5}$ starts at slot 4 then the second $w_{3,5}$ has 2 options.
   (e) and $w_{3,5}$ can start at slot 5. Giving 15 options in all.
   (f) $w_{3,3}$ is given no additional options beyond those that come from restrictions of the global schedules and $S^2(a_{1,1})$ and $S^2(a_{1,3})$. Similarly there are no options for when to run $w_{3,6}$. We have schedules $\eta_1, \eta_2, \ldots \eta_{15}$ the sum of any pair of which eliminates all the $w_{3,6}$ and some of the $w_{3,5}$ and is not another schedule η_i. Thus the $\eta_i, i = 1, 2, \ldots 15$ is the basis of a 15 dimensional vector space over the field $\mathbb{Z}_2$ and so $\mathcal{P}(S^2(a_{1,1}) \cap S^2(a_{1,3})) \cong (\mathbb{Z}_2)^5 \oplus (\mathbb{Z}_2)^{15}$.

(2) $S^2(a_{1,2}) \cap S^2(a_{1,3}) = \{a_{2,3}, a_{3,3}, a_{3,4}, a_{3,5}\}$. We allow

   (a) From $\tilde{\xi}_1$ $w_{3,4}$ can start two slots later giving two new options.
   (b) From $\tilde{\xi}_2$ $w_{3,5}$ can start two slots later giving two new options.
   (c) From $\tilde{\xi}_3$ $w_{3,4}$ can start a slot earlier $S^2(a_{1,3})$ which can bring forward the dependent starts of $w_{2,3}$ in $S^2(a_{1,2})$ or they can stay put. Gives one new option on $w_{2,3}$.

   Thus we set $\mathcal{P}(S^2(a_{1,2}) \cap S^2(a_{1,3})) \cong \mathbb{Z}_2^5 \oplus \mathbb{Z}_2^5$.

(3) $S^2(a_{1,1}) \cap S^2(a_{1,2}) = \{a_{3,2}, a_{3,3}, a_{3,5}\}$. All time-slots have $w_{3,5}$ and no $w_{3,2}$ can occur earlier than slot 4. $\mathcal{P}(S^2(a_{1,1}) \cap S^2(a_{1,2})$ will have the following options not shared with the other $\mathcal{P}(S^2(a_{i,j}) \cap S^2(a_{k,l})$.

   (a) From $\tilde{\xi}_1$ $w_{2,2}$ can move up to 2 slots later as can $w_{3,2}$ which can also stay where it is. Each time $w_{3,2}$ changes $w_{3,3}$ can be in either of the $w_{3,2}$ slots. This gives 5 new variations one of which comes from changing $w_{3,3}$ when $w_{3,2}$ does not change.

   Thus we set $\mathcal{P}(S^2(a_{1,2}) \cap S^2(a_{1,3})) \cong \mathbb{Z}_2^5 \oplus \mathbb{Z}_2^5$.

(4) $\mathcal{P}(S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3})) = \{w_{3,3}, w_{3,5}\}$. The schedules inherited from the supersets of $S^2(a_{1,2}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3})$ put $w_{3,5}$ in all the time-slots from 1 to 16 and $w_{3,3}$ in time-slots of $\{4, 5, 6, 7, 8, 9, 10, 12, 13, 15\}$. This leaves a number of possible variations. Leaving the slots 4 to 8 unchanged allowing $w_{3,3}$ in slots 9, 10, 12, 13, 15 to move a slot forward or stay put introduces 32 variations. Consequently $\mathcal{P}(S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3})) \cong \mathbb{Z}_2^{32}$.

This "specification" of $\mathcal{P}$ justifies the term "combinatorial schema". $\underline{A}$ gives the system but $\mathcal{P}$ and, by implication, $\mathcal{U}$ give the mode of operational or the class of dynamics of $\underline{A}$.

Calculating $\check{H}^0(\mathcal{P}, \mathcal{U})$):

$C^0 = C^0(\mathcal{P}, \mathcal{U}) = \underset{U \in \mathcal{U}}{\oplus} \mathcal{P}(U) = \mathcal{P}(S^2(a_0)) \oplus (\underset{a_{i,j}}{\oplus} \mathcal{P}(S^2(a_{i,j}))$. Let $\zeta(i, j) \in \mathcal{P}(S^2(a_{i,j}))$. In particular ζ_0 will be the restriction of one of the three global sections on $S^2(a_0)$ and $\mathcal{P}(S^2(a_{i,j}))$ will similarly contain the restrictions of the three global sections.

$$\partial^0(\zeta(0), \zeta(1,1), \zeta(1,2), \zeta(1,3)) = (\zeta(0) - \zeta(1,1)) \mid (S^2(a_0) \cap S^2(a_{1,1})) + (\zeta(0) - \zeta(1,2)) \mid (S^2(a_0) \cap S^2(a_{1,1}))$$

$$
\begin{aligned}
&+ (\zeta(0) - \zeta(1,3)) \mid (S^2(a_0) \cap S^2(a_{1,3}))\\
&+ (\zeta(1,1) - \zeta(1,2)) \mid (S^2(a_{1,1}) \cap S^2(a_{1,2}))\\
&+ (\zeta(1,1) - \zeta(1,3)) \mid (S^2(a_{1,1}) \cap S^2(a_{1,3}))\\
&+ (\zeta(1,2) - \zeta(1,3)) \mid (S^2(a_{1,2}) \cap S^2(a_{1,3})).
\end{aligned}
$$

$\xi(i,j) - \xi(k,l)$ are not equal on $S^2(a_{i,j}) \cap S^2(a_{j,k})$ and contribute no cocycles. The cocycles will occur when $(\zeta(i,j) \mid (S^2(a_{i,j}))) \cap S^2(a_{k,l}) = (\zeta(k,l) \mid (S^2(a_{i,j}) \cap S^2(a_{k,l})))$ which will only occur if $\zeta(i,j) = \tilde{\xi}_n \mid S^2(a_{i,j})$ and $\zeta(k,l) = \tilde{\xi}_n \mid S^2(a_{k,l})$ for one of the global sections $\tilde{\xi}_n$. Thus $Z^0(\mathcal{P},\mathcal{U}) = \mathcal{P}(A)$ and is generated by the three global sections $\tilde{\xi}_n$, $n = 1, 2, 3$. $B^0(\mathcal{P},\mathcal{U})$ is $\{0\}$ by convention[6] and $\check{H}^0(\mathcal{P},\mathcal{U}) = \mathcal{P}(A)$ which we can take as generated by $\tilde{\xi}[n]$, $n = 1, 2, 3$;

$$\check{H}^0(\mathcal{P},\mathcal{U})) \cong \mathbb{Z}_2^3.$$

$\check{H}^1(\mathcal{P},\mathcal{U})$.

Evaluating $\partial^1 : C^1(\mathcal{P},\mathcal{U}) \to C^2(\mathcal{P},\mathcal{U})$ we write $\xi(i,j;k,l)$ for ξ defined on component $\mathcal{P}(S^2(a_{i,j}) \cap S^2(a_{k,l}))$ of $C^1(\mathcal{P},\mathcal{U})$. Then

$$
\begin{aligned}
&\partial^1(\xi(1,1,1,2), \xi(1,1,1,3), \xi(1,2,1,3), \xi(0,1,1), \xi(0,1,2), \xi(0,1,3))\\
&\quad = \xi(1,1,1,2) - \xi(1,1,1,3) + \xi(1,2,1,3) \mid S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3})\\
&\qquad + \xi(0,1,3) - \xi(0) + \xi(0,1,2) \mid S^2(a_0) \cap S^2(a_{1,2}) \cap S^2(a_{1,3})\\
&\quad = \xi(1,1,1,2) - \xi(1,1,1,3) + \xi(1,2,1,3) \mid \{a_{3,3}, a_{3,5}\}\\
&\qquad + \xi(0,1,3) - \xi(0) + \xi(0,1,2) \mid \{a_{2,3}\}
\end{aligned}
$$

This is a cocycle if each component is zero. That is:

(1) $\xi(1,1,1,3)(\{a_{3,3}, a_{3,5}\}) = \xi(1,1,1,2)(\{a_{3,3}, a_{3,5}\}) + \xi(1,2,1,3)(\{a_{3,3}, a_{3,5}\})$ and
(2)
$$\xi(0)(\{a_{2,3}\}) = \xi(0,1,3)(\{a_{2,3}\}) + \xi(0,1,2)(\{a_{2,3}\})$$

Solutions for the first cocycle:

(1) $\xi(1,1,1,2)$ requires $w_{3,5}$ and $w_{3,3}$ to produce the types for $w_{2,3}$ to produce for $a_{1,2}$ and similarly for $a_{1,1}$.
(2) $\xi(1,2,1,3)$ requires $w_{3,5}$ and $w_{3,3}$ for both $w_{2,3}$ going to $a_{1,2}$ and $a_{1,3}$.
(3) $\xi(1,1,1,3)$ requires $w_{3,5}$ and $w_{3,3}$ to produce the types for $a_{1,1}$ and $w_{2,3}$ for $a_{1,3}$.

Putting $\xi(1,1,1,2) = (w_{3,5},1) + (w_{3,3},4)$ directed to $a_{1,1}$ and after that, $\xi(1,2,1,3) = (w_{3,5},6) + (w_{3,3},8)$ giving $\xi(1,1,1,3) = (w_{3,5},1) + (w_{3,3},4) + (w_{3,5},6) + (w_{3,3},8)$. $\xi(i,j,k,l)$ can be any of variations allowed in their specifications given above. These give a set of possibilities $\xi(1,1,1,3) = (w_{3,5},s) + (w_{3,3},t) + (w_{3,5},s') + (w_{3,3},t')$ with $t < s' \le s + 5 + m < t'$ and $1 \le m$ with $s + 5 + m \le 15$ which is the last run of $w_{3,5}$ in any of the global sections. $(w_{3,3},t)$ can have $4 \le t \le 7$ while $(w_{3,3},t')$ with $t' = 15$ comes from global schedules $\tilde{\xi}_{[3]}$ and $\tilde{\xi}_{[1]}$. This gives 4 possible solutions. The schedules of $\mathcal{P}(S^2(a_{1,1})) \cap (S^2(a_{1,3})))$ require two blocks of five continuous slots for $w_{3,5}$ in at most 15 slots. In fact we consider only starting times $\{1, 6, 11\}$. This contributes $\mathbb{Z}_2^8$ to $Z^1(\mathcal{P},\mathcal{U})$.

Solutions for the second cocycle:

(1) $\xi(0)(\{a_{2,3}\}) = \xi(0,1,3)(\{a_{2,3}\}) + \xi(0,1,2)(\{a_{2,3}\})$
$\xi(0)(\{a_{2,3}\})$ has the form $(w_{2,3},s) + (w_{2,3},s')$, $|s' - s| > 2$ to produce supplies for $a_{1,2}$ and $a_{1,3}$.
(2) $\xi(0,1,3)(\{a_{2,3}\})$ has the form $(w_{2,3},s)$ to produce supplies for $a_{1,3}$.

(3) $\xi(0,1,2)(\{a_{2,3}\})$ has the form $(w_{2,3}, s')$ to produce supplies for $a_{1,2}$.

The global schedules provide $\{(11,14),(9,13),(14,6)\}$ as solutions for (s,s') giving three cocycle solutions.

These equations characterize elements in $C^1(\mathcal{P},\mathcal{U})$ that give $Z^1(\mathcal{P},\mathcal{U}) \cong \mathbb{Z}_2^8 \oplus \mathbb{Z}_3^2$.

Which of these cocycles is a coboundary? A 1-coboundary has the form $\xi(i,j) - \xi(k,l)$ on $S^2(a_{i,j}) \cap S^2(a_{k,l})$. This means that as far as the relevant actions go a coboundary ς on $S^2(a_{1,1}) \cap S^2(a_{1,3})$ has the form

$$(w_{3,5}(1,1), s) - (w_{3,5}(1,3), s') + (w_{3,3}(1,1), t) \\ - (w_{3,3}(1,3), t') + (w_{3,6}(1,1), u) - (w_{3,3}(1,3), u').$$

where the (i,j) after the action indicates the intended topic. Now if $\xi(1,1,1,3) = \varsigma$ then $\xi(1,1,1,3)+(w_{3,5}(1,3), s')+(w_{3,3}(1,3), t')+(w_{3,6}(1,3), u') = (w_{3,5}(1,1), s)+(w_{3,3}(1,1), t) + (w_{3,6}(1,1), u)$. Putting

$$\xi(1,1,1,3) = (w_{3,5}, n_1) + (w_{3,3}, n_2) + (w_{3,6}, n_3) + (w_{3,5}, n_1') + (w_{3,3}, n_2') + (w_{3,6}, n_3')$$

where $|n_1' - n_1| \geq 6$ and $|n_3' - n_1| > 7$ and $|n_2' - n_2| \geq 1$ we get

$$(w_{3,5}, n_1) + (w_{3,5}, n_1') + (w_{3,3}, n_2) + (w_{3,3}, n_2') + (w_{3,6}, n_3)(w_{3,6}, n_3') \\ + [(w_{3,5}(1,3), s') + (w_{3,3}(1,3), t') + (w_{3,6}(1,3), u')] \\ = (w_{3,5}(1,1), s) + (w_{3,3}(1,1), t) + (w_{3,6}(1,1), u).$$

Suppose the n_i are the times for the $S^2(a_{i,3})$ actions in $\xi(1,1,1,3)$; putting $s' = n_1$, $t = n_2$, $u = n_3$ and $n_1' = s$, $n_2' = t$, $n_3' = u$ and using the $\mathbb{Z}_2$ addition, $\xi(1,1,1,3)$ can be represented as difference of schedules.

Similarly suppose $\xi(1,1,1,2) = (w_{3,5}, n_1) + (w_{3,3}, n_2) + (w_{3,2}, n_3) = \varsigma$. Retracing the argument:

$$(w_{3,5}, n_1) + (w_{3,3}, n_2) + (w_{3,2}, n_3) + [(w_{3,5}(1,2), s') + (w_{3,3}(1,2), t') + (w_{3,2}(1,2), u')] \\ = (w_{3,5}(1,1), t) + (w_{3,3}(1,1), t) + (w_{3,2}(1,1), u)$$

This time we have an imbalance of $w_{3,5}$, $w_{3,3}$ and $w_{3,2}$ that cannot use the 2-adic cancellation. So $\xi(1,1,1,2)$ and, by the same reasoning, $\xi(1,2,1,3)$ cannot be represented by a coboundary. This means $(\xi(1,1,1,3)\xi(1,1,1,2), \xi(1,2,1,3)) \in C^1$ is not in the image of $\partial^0 : C^0 \to C^1$.

$\xi(0)(\{a_{2,3}\}) = \xi(0,1,3)(\{a_{2,3}\}) + \xi(0,1,2)(\{a_{2,3}\})$. If $\xi(0) = (w_{2,3}, s) + (w_{2,3}, s') = \varsigma' = (w_{2,3}, t) - (w_{2,3}, t')$ then we get as before

$$(w_{2,3}, s) + (w_{2,3}, s') + (w_{2,3}, t') = (w_{2,3}, t)$$

which is true if $s = t'$ and $s' = t$. Again, if $\xi(0,1,3) = (w_{2,3}, s) = (w_{2,3}, t) - (w_{2,3}, t')$ then

$$(w_{2,3}, s) + (w_{2,3}, t') = (w_{2,3}, t)$$

which cannot be true. The same argument holds for $\xi(0,1,2)$. Hence $(\xi(0), \xi(0,1,2), \xi(0,1,3)) \in C^1$ is not a coboundary.

We conclude that all the cocycles contribute to $\check{H}^1(\mathcal{P},\mathcal{U})$ hence we have

$$\check{H}^1(\mathcal{P},\mathcal{U}) \cong \mathbb{Z}_2^8 \oplus \mathbb{Z}_3^2 = \mathbb{Z}_2^{10}.$$

Calculating $\check{H}^2(\mathcal{P},\mathcal{U})$):

As $C^3(\mathcal{P},\mathcal{U}) = \{0\}, ker(\partial^2) = Z^2(\mathcal{P},\mathcal{U}) = C^2(\mathcal{P},\mathcal{U}) = \mathcal{P}(S^2(a_{1,1})\cap S^2(a_{1,2})\cap S^2(a_{1,3}))\oplus \mathcal{P}(S^2(a_0)\cap S^2(a_{1,2})\cap S^2(a_{1,3}))$. $B^2(\mathcal{P},\mathcal{U}) = im(C^1 \xrightarrow{\partial^1} C^2)$ only affects the schedules as defined in $\{a_{23}\}\cup\{a_{3,3}, a_{3,5}\}$). $B^2(\mathcal{P},\mathcal{U})$ is the expressions of the form:

(1) $\xi(1,1,1,2) - \xi(1,1,1,3) + \xi(1,2,1,3) \mid \{a_{3,3}, a_{3,5}\}$ and $\xi(0,1,1) - \xi(0,1,3) + \xi(0,1,2) \mid \{a_{2,3}\}$.
(2) The first group can be characterized as schedules $((w_{3,5}, s(1112) - (w_{3,5}, s(1113)) + (w_{3,5}, s(1213)) + (w_{3,3}, s'(1112)) - (w_{3,3}, s'(1113)) + ((w_{3,3}, s'(1213)))$ with the starting times inherited from schedules defined over $S^2(a_{1,1})\cap S^2(a_{1,2})$, $S^2(a_{1,2})\cap S^2(a_{1,3})$ and $S^2(a_{1,1})\cap S^2(a_{1,3})$. These schedules cover all the time-slots for $w_{3,5}$ but $\mathcal{P}((S^2(a_0)\cap S^2(a_{1,2})\cap S^2(a_{1,3})$ was given additional options for the placement of the $w_{3,3}$. The addition and subtraction of schedules does not change the time-slots of these "free" $w_{3,3}$ actions so they not included in $B^2(\mathcal{P},\mathcal{U})$. Thus the free component of $\mathcal{P}((S^2(a_{11})\cap S^2(a_{1,2})\cap S^2(a_{1,3})$ contributes a component of $\mathbb{Z}_2^{32}$ to the $\check{H}^2(\mathcal{P},\mathcal{U})$.
(3) An arbitrary schedule of $\zeta \in \mathcal{P}(S^2(a_0)\cap S^2(a_{1,2})\cap S^2(a_{1,3})) = \mathcal{P}(\{a_{2,3}\})$ has $w_{2,3}$ starting in places 11 and 16, 9 and 13, 6 and 14 inherited from $\mathcal{P}(S^2(a_0))$ (and hence the global schedules). The second component in $B^2(\mathcal{P},\mathcal{U})$ group can be characterized as schedules $\xi_1 - \xi_2 + \xi_3$ with the starting times inherited from the relevant $a_{2,3}$ schedules.

   (a) $\zeta \in \mathcal{P}(S^2(a_0)\cap S^2(a_{1,2})\cap S^2(a_{1,3}) = \mathcal{P}(\{a_{2,3}\})$ has two instances of $w_{2,3}$. Put the $\xi_1 = (w_{2,3}, s) = \xi_2$ and $\xi_3 = \zeta$.
   (b) ζ has only one instance of $w_{2,3}$. As above.

   All the schedules can be represented in $B^2(\mathcal{P},\mathcal{U})$ and so $\mathcal{P}(S^2(a_0)\cap S^2(a_{1,2})\cap S^2(a_{1,3})$ contributes no homology classes to $\check{H}^2(\mathcal{P},\mathcal{U})$.

We conclude that $\check{H}^2(\mathcal{P},\mathcal{U}) = \mathbb{Z}_2^{32}$.

Thus we have

$$\check{H}^0(\mathcal{P},\mathcal{U}) = \mathbb{Z}_2^3,\ \check{H}^1(\mathcal{P},\mathcal{U}) = \mathbb{Z}_2^{10},\ \check{H}^2(\mathcal{P},\mathcal{U}) = \mathbb{Z}_2^{32} \text{ and } \check{H}^p(\mathcal{P},\mathcal{U}) = 0 \text{for } p > 2.$$

What does this tell us? There are three independent classes of global schedules as indicated by $\check{H}^0(\mathcal{P},\mathcal{U})$ but knowing what they are does not tell everything there is to be known about $\mathcal{P}$. Each of the $\check{H}^1(\mathcal{P},\mathcal{U})$ and $\check{H}^2(\mathcal{P},\mathcal{U})$ tells us that independent local schedules exist and in what number. This has been built into the example for the purposes of illustration. Although the example is typically of piece work, or one at a time batch processes or even stages of mastery it illustrates how $\mathcal{P}$ defines the operating mode. This is a tiny network in comparison to "industrial strength" networks mentioned below in the introduction to Part 2. It is obvious the further developments will be to create programs which will aid in the calculation of cohomology as is already the case for cohomology of finite groups (Carlson 2005).

4.2. Further examples

In the rest of the paper we adopt the conventions:

(1) In an expression such as $U_{i_0}\cap U_{i_0}\cap\ldots\cap\widehat{U_{i_k}}\cap\ldots U_{i_n}$, $\widehat{\ }$ denotes omission of the set, in this case U_{i_k}.
(2) A cocycle equation $\xi_0 - \xi_1 + \ldots + (-1)^n\xi_n) \mid U = 0$ will usual be written as $(\xi_1 + \xi_3 + \ldots + \xi_{2k-1}) \mid U = \xi_0 + \xi_2 + \ldots + \xi_{2k} \mid U$ or $\Sigma_{k\ odd}\xi_k = \Sigma_{k\ even}\xi_k$ or a similar variation.

The example just given gives but a hint of the way cohomology uses the algebra of actions. The following illustrates how equations among types and actions affect cohomology.

Let ξ_k be defined on $U[k] = \cap_{i \neq k} U_i$ and let the total time-slots available in $U = \cap_i U_i$ be the set $T(U) = \{t_1, t_2, \ldots, t_n\}$. On U the equation

$$\sum_{k\ even} \xi_k(t_i) = \sum_{k\ odd} \xi_k(t_i);\ t_i \in T(U) \tag{3}$$

holds. Let $\xi_k(t_i) = \sum_j w_{k,j}(t_i)$ in the above equation giving

$$\sum_{k\ even} \sum_j w_{k,j}(t_i) = \sum_{k\ odd} \sum_j w_{k,j}(t_i). \tag{4}$$

Even in the simplest case of Z^0 with $\xi_0 = \xi_1$ on $U = U_0 \cap U_1$ these equations imply $\Sigma w_{0,j}(t_i) = \Sigma w_{1,j}(t_i)$. The equations in actions translate to equations in types. Suppose $w_0(t)(e_1, e_2) = e$ and $w_0 = w_{1,1} + w_{1,2}$ on U so that $(w_{1,1} + w_{1,2})(e_1, e_2) = e$. This forces a decomposition of $e = e' + e''$ and corresponding breakup of (e_1, e_2) to $e_i = \sum a_{i,l} e_{i,l}$. These equations can change from time-slot to time-slot. For a cocycle not to be a coboundary then at least one of these equations has to be specific to U and not in all of the U_k.

Illustrations.

(1) 1 cocycle. $Typ(U) = \{e_0, e_2, e_3\}$ and ξ_0 in U is $w_0(e_1, e_2) = e'_0$, ξ_2 in U is $w_2(e_0, e_1) = e'_2$, ξ_1 in U is $w_1(e_0, 2.e_1, e_2) = e'_1$, then $e_1 = e_0 + e_2$ means $\xi_1 = \xi_0 + \xi_2$ in U. This becomes a matrix equation in the ring $GL(\mathbb{Z}, 3)$ giving the possibility with links to other parts of algebra.

(2) 2 cocycle: $\xi_0 + \xi_2 = \xi_1 + \xi_3$ in U. $Typ(U) = \{e_0, e_2, e_3, e_4\}$.

   (a) ξ_0 in U is $w_0(e_1, e_3) = e'_0$, ξ_2 in U is $w_2(e_1, e_3) = e'_2$, ξ_1 in U is $w_1(e_0, e_2) = e'_1$, ξ_3 in U is $w_1(e_0, e_2) = e'_3$, then $e_1 + e_3 = e_0 + e_2$ and $e'_1 + e'_3 = e'_0 + e'_2$ means $\xi_1 + \xi_3 = \xi_0 + \xi_2$ in U.

   (b) ξ_0 in U is $w_0(e_1, e_2, e_3) = e'_0$, ξ_2 in U is $w_2(e_1, e_3) = e'_2$, ξ_1 in U is $w_1(e_0, e_1, e_2) = e'_1$, ξ_3 in U is $w_1(e_0, e_2) = e'_3$, then $2e_1 + e_2 + 2.e_3 = 2.e_0 + e_1 + 2.e_2$ and $e'_1 + e'_3 = e'_0 + e'_2$ means $\xi_1 + \xi_3 = \xi_0 + \xi_2$ in U. A solution is $2.e_0 = e_1$ and $e_2 = e_3$.

The schedules in the coboundaries groups $B^k(\mathcal{P}, \mathcal{U})$ contain only the information available from restriction of schedules defined on larger domains. If the restriction map $\mathcal{P}(U_1) \to \mathcal{P}(U_2)$ is an epimorphism then given $\xi \in \mathcal{P}(U_2)$ there are always η_1, η_2 such that $\xi(U_2) = \eta_1 \mid U_2 - \eta_2 \mid U_2$ where the η_i are defined on U_1.[7] If $U = \cap_i U_i = \cap_k U[k]$, $i, k = 1 \ldots n$ and the image of the restriction maps

$$\oplus_k \mathcal{P}(U[k]) \xrightarrow{\oplus \rho(U[k], U)} \mathcal{P}(U)$$

is not onto then the image of ∂, so B^n, is also not all of C^n. This is a necessary condition for the existence of cohomology groups. This can arise when some of the $\xi \in \mathcal{P}(U)$ can create larger quantities of a type than can be absorbed or used in elements of $a \in U[k] \backslash U$ for which $U \cap S^\infty(a)$ is not empty. In such cases $\xi \in \mathcal{P}(U)$ can produce beyond what a can absorb then ξ cannot be the difference of two schedules defined on $U[k]$ and consequently $\mathcal{P}(U[k])$ cannot generate $\mathcal{P}(U)$. This is true whether the limits on a are in terms of quantities or rates of processing. This also occurs when $\mathcal{P}(U)$ is generated by a single schedule ζ which cannot be decomposed. Such schedules exist for monopolies, single source of expertise or key ideas. In such cases $(W(a), +)$ splits into the direct sum of an integer module and a $\mathbb{Z}_2$ module.

4.3. The Cohomology of $\mathcal{P}_{\mathfrak{F}}$

In this section we investigate $\check{H}^n(\mathcal{P}_{\mathfrak{F}})$. Although the groups $\check{H}^n$ have been defined on group valued sub-sheaves of Sch they can be defined on any group valued pre-sheaf defined for some appropriate cover of $\underline{A}$. Depending on $\mathcal{D}[\Lambda]$ we shall show that the cohomology gives significant information as to the relations among the properties defined by $F(\phi)$. In particular this cohomology can be seen as the cohomology of measurements on $\mathcal{P}$. To illustrate this we start with cocyles. As usual $U[k] = U_{i_0} \cap U_{i_1} \cap \ldots \widehat{U_{i_k}} \ldots \cap U_{i_n}$ and ϕ_k is defined on $U[k]$. A cocycle becomes an equation of the type $\Sigma_{k\ even} F(\phi_k)(\xi) = \Sigma_{k\ odd} F(\phi_k)(\xi)$ both sides restricted to $U = \cap_k U[k]$.

The simplest non trivial case being: $F(\phi_0) + F(\phi_2) = F(\phi_1)$. This indicates that $[\Lambda]$ has sufficient measures so the equation holds and that for any test schedule ξ defined on U the measures $\phi_0(\xi) + \phi_2(\xi)$ are "close to" (within $m(\phi_1)$) of $\phi_1(\xi)$. Thus the cohomology of $\mathcal{P}_{\mathfrak{F}}$ gives equations among the forms that apply to the schedules of $\mathcal{P}$. This then characterizes the objects of $Comb(\underline{A})$ in a functorial way.

This is just the start of a series of questions:

(1) Can classes of sheaves $\Gamma(\Lambda)$ of $Comb(\underline{A})$ be classified by sufficient large sets of measures Λ such that within $\Gamma(\Lambda)$ the $\mathcal{P}$ are defined by cohomology of their corresponding $\mathcal{P}_{\mathfrak{F}}$? As the cohomology groups $\check{H}^n(\mathcal{P}_{\mathfrak{F}})$ are the classes of cocycles $\Sigma_{k\ even} F(\phi_k)(\xi) = \Sigma_{k\ odd} F(\phi_k)(\xi)$ that hold locally the equations become equations among the forms of Λ that apply to $\mathcal{P}_{\mathfrak{F}}$ so charactering the sheaves $\Gamma(\Lambda)$ in terms of "hyperplanes" on the multidimensional lattice $\mathbb{Z}^{\Lambda}$.

(2) Let $\Lambda_0 \subset \Lambda_0 \subset \ldots. \subset \Lambda_n \subset \ldots$ be a sequence of forms such that $d_p^{\pm} : \mathcal{D}[\Lambda_p] \to \mathcal{D}[\Lambda_{p\pm1}]$ form exact sequences sheaves: d^- chains, d^+ cochains. Can such structures be the basis of "resolutions" of a given $\mathcal{P}$ and so that the usual algebraic or algebraic geometry constructions as given Eilenberg and Cartan (1956) or Hartshorne (1977) Chapter 3, be applied to relational derived series $\mathcal{P}_{\mathfrak{F}(p)}$ where $\mathcal{P} \leftarrow \mathfrak{F}(p)^{\pm} \to \mathcal{D}[\Lambda_p]$?

4.4. Functorial properties

Given a logistic morphism $f : \underline{A}_1 \to \underline{A}_2$ and $\mathcal{P}_2$ a sheaf on $\underline{A}_2$ defined for covering $\mathcal{U}_2$, then the pullback of $\mathcal{P}_2$, $f^{-1}(\mathcal{P}_2)$ defined on $f^{-1}(\mathcal{U})$ is a sheaf on $\underline{A}_1$. But it need not be in $Comb(\underline{A})$. For example, if $U \subset A_1$ is mapped by f to a single element $b \in A_2$. If $\{b\}$ is in $\mathcal{U}$ it is not clear how to interpret a schedule in $\mathcal{P}_2(b)$ to a schedule in $\mathcal{P}_1(U)$. For this reason this paper concentrates on cohomology as a *covariant* functor. In this case $f : \underline{A}_1 \to \underline{A}_2$ must preserve intersections: for $U_1, U_2, \ldots, U_n$ all in $\mathcal{U}$, $f(U_1 \cap U_2 \cap \ldots \cap U_n) = f(U_1) \cap f(U_2) \cap \ldots \cap f(U_n)$. This ensures that the elements of cochains $C^n(\mathcal{P})$ map appropriately and $Sch(f)$ maps schedules to schedules giving the covariant functor $\check{H}(f) : \check{H}^n(\mathcal{P}_1) \to \check{H}^n(\mathcal{P}_2)$. Preserving intersections is the same constraints on morphisms as being uniform. (Proof: Assume f is not uniform and pick $U \in \mathcal{U}$ such that $f^{-1}(f(U)) \neq U$. Let $V = f^{-1}(f(U)) \setminus U \neq \emptyset$ and $f(V) \subseteq f(U)$. But $U \cap V = \emptyset$ hence $f(V \cap U) = \emptyset$ but $f(U) \cap f(V) = f(V)$ a non-empty subset of $f(U)$. So f does not preserve intersections. But $(\neg(f \text{ is uniform }) \Rightarrow \neg(f \text{ preserve intersections})$. By modus tollens this is (f preserves intersections $\Rightarrow$ f is uniform). Now f is uniform implies preserves intersections: it is always the case that $f(U_1 \cap U_2) \subseteq f(U_1) \cap f(U_2)$. Assume f is uniform. Let $x \in f(U_1) \cap f(U_2)$. Suppose $f(y) = x$ then $y \in f^{-1}(f(U_1) \cap f(U_2)) = f^{-1}(f(U_1)) \cap f^{-1}(f(U_2)) = U_1 \cap U_2$ as f^{-1} preserves $\cap$. Therefore $f(y) = x \in f(U_1 \cap U_2)$ and so $f(U_1) \cap f(U_2) \subseteq f(U_1 \cap U_2)$. Thus uniform implies preserves intersections and the two conditions are equivalent. This can be proved in any topos.)

Under the constraint that f is uniform (or preserves intersections) the usual cohomology properties apply to the Čech cohomology (see Spanier 1966). In particular, for an exact sequence

of sheaves the "long exact sequence" holds:

$$\ldots \to \check{H}^n(\mathcal{P}') \to \check{H}^n(\mathcal{P}) \to \check{H}^n(\mathcal{P}'') \to \check{H}^{n+1}(\mathcal{P}') \to \ldots \tag{5}$$

In the following a combinatorial scheme $(\underline{A}, \mathcal{P})$ is given with a covering $\mathcal{U}$, $U \in \mathcal{U}$ and $\underline{B}$ is a subobject of $\underline{A}$. Define presheaves $\mathcal{P}^B$ and $\mathcal{P}_B$ by

$$\mathcal{P}^B(U) = \mathcal{P}(U) \qquad U \cap B = \emptyset \tag{6}$$

$$\mathcal{P}^B(U) = 0 \qquad U \cap B \neq \emptyset \tag{7}$$

and

$$\mathcal{P}_B(U) = \mathcal{P}(U) \qquad U \cap B \neq \emptyset \tag{8}$$

$$\mathcal{P}_B(U) = 0 \qquad U \cap B = \emptyset \tag{9}$$

If we call the closure of B the set $\cup_{b \in B}(S(b) \cup \{b\} \cup S^+(b))$ then $\mathcal{P}^B$ zeroes everything inside this closure while $\mathcal{P}^B$ zeroes everything outside the closure.

These definitions allow the definition of a short exact sequence of presheaves defined on $\underline{A}$:

$$0 \to \mathcal{P}^B \to \mathcal{P} \to \mathcal{P}_B \to 0.$$

Making the definitions

$$\check{H}^n(\underline{B}, \mathcal{P}) = \check{H}^n(\underline{A}, \mathcal{P}_B) \tag{10}$$

$$\check{H}^n(\underline{A}, \underline{B}, \mathcal{P}) = \check{H}^n(\underline{A}, \mathcal{P}^B). \tag{11}$$

allows us to apply the long exact sequence to get relative cohomology:

$$\ldots \to \check{H}^{n-1}(\underline{B}) \to \check{H}^n(\underline{A}, \underline{B}) \to \check{H}^n(\underline{A}) \to \check{H}^{n+1}(\underline{A}, \underline{B}) \to \ldots$$

(Here the sheaves are implicit). These results follow from standard (co)homological algebra and will not be proved here.

The Excision theorem also has a version here. In topology, given $B_1 \subset B \subset A$ the Excision theorem requires that the closure of B_1 is in the interior of B. For combinatorial system with $B_1 \subset B \subset A$ we require that all $b \in B_1$ $S(b) \cup S^+(b) \subseteq B$ then

$$\check{H}^n(\underline{A} \setminus \underline{B}_1, \underline{B} \setminus \underline{B}_1) = \check{H}^n(\underline{A}, \underline{B}). \tag{12}$$

These results show that the cohomology developed here justifies it claim as a such a functor even though it is covariant on its domain category.

5. Part 2. Applications: manufacturing and supply networks

This part of the paper addresses applications in more detail. Because of length of the paper only one class of applications is selected to illustrate the following points.

(1) The application of combinatorial systems that uses the full set of definitions including replenishment.
(2) The application of the $[\mathfrak{L}, \mathcal{K}]$ in real circumstances.
(3) The economic importance of the concepts.

For these reasons the section is on manufacturing networks. Advanced technology manufacturing is characterized by the scale of detailed documentation,[8] its highly systematized processes

(algorithms) to ensure specifications are met and elaborate connections of engineering, physical, chemical and marketing reasons for doing things.

6. Manufacturing and supply networks

Manufacturing networks, also called supply networks and (most commonly and misleadingly called) supply chains were the first motivating examples for combinatorial systems.[9] These networks marshal the material (types) and expertise (actions) required to make something. They are the international face of the integration of expertise across the world; they can be defined very widely indeed and are of paramount importance to modern economies. The scale of manufacturing networks is considerable: according to Ford CEO Alan Mulally (quoted in "Time" 6 September 2010, 28) a car has 10,000 parts and an airliner 4 million.

Manufacturing networks have an extensive history of mathematical modeling. There are many themes or perspectives. The majority of this research has been in optimization and simulation and many recent surveys, (Vidal and Goetschalckx 1997; Beamon 1998; Sarmiento and Nagi 1999; Ganeshan et al. 2000; Smit 2001; Min and Zhou 2002; Kleijnen 2003; Shen 2007), give a guide to that literature. A related perspective considers the design of supply chains as a social structure of obligations and is motivated by Game Theory and procurements strategies (Cachon and Netessine 2004). Optimal solutions, simulation and contractual games will give structures to aim for. Attaining and keeping the optimal conditions can be extremely difficult especially when managing very large numbers of variables such as supplier prices and raw material transport costs. The difficulty of controlling variables and the growth of inter-company IT networks has resulted in studies of the effects information sharing and show that information sharing reduces the fluctuations that arise in response to changing demand (Joshi 2000; Cannellaa and Cianciminoa 2010). This has prompted the approach characterized as the collaborative planning, forecasting and replenishment or "CPFR" (Knolmayer, Mertens, and Zeier 2002) which emphasizes coordinated replenishment.

From now on our combinatorial system $\underline{A}$ will be a supply network. $Typ(a)$ will be a $\mathbb{Z}$ module as usual but the expression $x_1.e_1 + x_2.e_2 + \ldots x_n.e_n$ can be interpreted as inventory: a quantity x_i of e_i. Furthermore $\underline{A}$ operates a replenishment logic. This means that the lowest topics of the network are the source of stock for inventory but lead times continue to operate. The replenishment logic of a supplier a vary from the simple: if e goes below an amount $q_{min}(e)$ order enough to get back to $q_{max}(e)$, to elaborate and strategically important algorithms for forecasting, automated purchasing and sharing information with relevant suppliers in the network. The significance and variety of replenishment methods cannot be underestimated. It cannot be given due treatment here.

The objective of this part of the paper is to give two areas of applications of epistemological operators to supply networks. The first application is to questions of demand in which sheaves of classes of schedules that can satisfy requirements act as "boundary conditions." The second case is to sheaves of classes of schedules that can changed or "adjusted".

7. Scenarios, capability and fragmentation

Definition A *scenario* is a schedule that fulfills the sequence of requests or orders to a manufacturing network. It starts with

(1) a sequence of orders or "requests" $r_1, r_2, \ldots, r_n$ issued at time $t_1, t_2, \ldots, t_n$,
(2) a starting state $q(t_0) \in Typ(A)$ with $t_0 \leq t_1$, and
(3) the requests are to be fulfilled at $t'_1, t'_2, \ldots, t'_n$ respectively. Alternatively this can be stated that there is a requirement $\sigma_r : T \to Typ(a^*)$ where T is time.

Let $scn(\underline{A})$ denote the set of scenario requests that can be formulated in $\underline{A}$. A solution of the scenario is any schedule that can be constructed within the logic of the manufacturing network to deliver r_i on or before the t_i'. What is the class of sheaves that can fulfill a request?

Definition The sheaf of scenarios, denoted $Scn(\underline{A})$ is defined in a number of steps as follows.

(1) Define the "inventory sheaf" Inv on $\underline{A}$ as $Inv(a) = Typ(a) \times \mathbb{Z}$ which is a sheaf in $\mathbb{Z}$ modules. Here $\mathbb{Z}$ is the constant sheaf $\mathbb{Z}(U) = \mathbb{Z}$. Because of the independence of states among the suppliers Inv is trivially a sheaf.
(2) Give a scenario σ define $(\sigma(a), t)$ to be the types required in $Typ(a)$ at time t. The condition on this is that the types $\sigma(a^*)$ must be in $Typ(a^*)$ at times t_i as in the request σ_r of the scenario and if $(\sigma(a), t)$, $b \in S(a)$ and $(\sigma(b), t')$ then $t' < t - 1$. This gives a map:

$$Scn(\underline{A}) \xrightarrow{\mathfrak{s}} Inv;$$
$$\mathfrak{s}(\sigma)(U) =_{defn} \{(\sigma(a), t) \in Inv(a) \mid a \in U, t \text{ constrained as above}\}.$$

(3) $Scn(\underline{A})(U)$ is the image of $\mathfrak{s}(U)$ and is clearly a presheaf.

$Scn(\underline{A})$ is in fact a sheaf because of weak constraints on sections.

Definition The addition of scenarios. For any σ_1 and σ_2 in $scn(\underline{A})$ the requested goods and inventory can easily be added and union of the times provides the delivery constraints. This gives $\sigma_1 + \sigma_2$ in $scn(\underline{A})$. Using the $\mathfrak{s}(\sigma_1 + \sigma_2)$ we define the sum of $(\sigma_1 + \sigma_2)$ in $Scn(U)$.

PROPOSITION 7.1 *The addition of scenarios makes Scn a group valued sheaf.*

The proof is left to the reader. $Scn(\underline{A})$ has few constraints. We want to link it with $Comb(\underline{A})$. The following concepts can all be put in terms of sheaf operations:

(1) Define $inv : \mathcal{P} \to Inv$ by $inv(\xi)(a)$ is the set $(q(a), t)$ if ξ leaves $Typ(a)$ in the state $q(a)$ at time-slot t.
(2) Inv has a sheaf ordering given by $(q, t) < (q', t') \Leftrightarrow (q \leq q') \wedge (t = t')$.
(3) Define a sheaf relation $\Rightarrow$ by

$$\Rightarrow (U) =_{defn} \{(\xi, \sigma) \mid (\xi \in \mathcal{P}(U)) \wedge (\mathfrak{s}(U)(\sigma) \leq inv(\xi))\}.$$

This means that if $a^* \in U$, $inv(\xi)$ must equal or exceed the quantities in the requests of σ at the time that they are required and the times in $\mathfrak{s}(U)(\sigma)$ match those in $inv(\xi)$ and so correspond to a real schedule in U.

As $[\Rightarrow, Scn]$ is in the form of $[\mathfrak{L}, \mathcal{K}]$; writing $\Rightarrow (\xi, \sigma)$ as $\xi \Rightarrow \sigma$ we can define:

(1) The set-valued sheaf $Scn_{\Rightarrow}$ given by

   (a) For $\xi \in \mathcal{P}(U)$, $S(\xi) =_{defn} \{\sigma \in Scn(U) \mid \xi \Rightarrow \sigma\}$.
   (b) $Scn_{\Rightarrow}(U) =_{defn} \{S(\xi) \mid \xi \in \mathcal{P}(U)\}$.

(2) The set-valued sheaf $\mathcal{P}_{\Rightarrow}$ given by

   (a) For a scenario σ and $U \subseteq A$, $Z(\sigma) =_{defn} \{\xi \in \mathcal{P}(U) \mid \xi \Rightarrow \sigma(U)\}$.
   (b) For $U \in \mathcal{U}$, $\mathcal{P}_{\Rightarrow}(U) =_{defn} \{Z(\sigma)(U) \mid \sigma \text{ is a scenario}\}$.

Note that if ξ and ξ' imply the same set of local scenarios then $S(\xi) = S(\xi')$.

Let $\sigma_1 < \sigma_2$ if the requests of σ_1 are a subset of the requests of σ_2 but the times of delivery are the same. This means that if ξ_2 can satisfy σ_2 there is a ξ_1 that satisfies σ_1 and $\xi_1 \subseteq \xi_2$. With

scenarios and schedules finite partially ordered sets they are order complete and we can apply Theorem 3.3.

COROLLARY 7.1 *If the partial orders given by inclusion on the sets $Z(\sigma)$ and $S(\xi)$ are complete then there are sheaf maps $\mathfrak{j} : \mathcal{P}_{\Rightarrow} \to Scn_{\Rightarrow}$ and $\mathfrak{f} : Scn_{\Rightarrow} \to \mathcal{P}_{\Rightarrow}$ that are inverses.*

Group addition for the sheaves $\mathcal{P}_{\Rightarrow}$ and $Scn_{\Rightarrow}$ is defined as follows

(1) $\mathcal{P}_{\Rightarrow}$. Define an addition, denoted by $\tilde{+}$, at each U by the local relations $Z(\sigma_1)\tilde{+}Z(\sigma_2) =_{defn} Z(\sigma_1 + \sigma_2)$.
(2) $Scn_{\Rightarrow}$. The addition denoted by $\breve{+}$ is defined at each U by the local relations $S(\xi_1) \breve{+} S(\xi_2) =_{defn} S(\xi_1 + \xi_2)$.

As defined these are sheaves as they are defined in terms of schedules and scenarios which are already sheaf sections in existing sheaves. As $S(\xi_1 + \xi_2)$ contains both $S(\xi_1)$ and $S(\xi_2)$ while $Z(\sigma_1)$ and $Z(\sigma_2)$ both contain $Z(\sigma_1 + \sigma_2)$ the respective definitions of addition do not indicate set identification. This is added structure.

Definition A manufacturing supply network $\underline{A}$ operates on *make to schedule quantity* if orders are accepted only in quantities that fit a range of schedules. This includes "assemble to order". This means that whenever a set of orders (scenarios) σ_i with $\sigma_i(a) = \Sigma n_{ij}.e_{ij}$ have make to order schedules $\underline{\xi}_i$ then there are no common components or processes that can be batched or combined. In this case $\Sigma\sigma_i(a)$ has a least schedule $\underline{\xi} = \Sigma\underline{\xi}_i$ in $\Sigma\mathfrak{j}(Z(\sigma_i))$ giving $\Sigma\mathfrak{j}(Z(\sigma_i)) = \mathfrak{j}(\Sigma(Z(\sigma_i)))$.[10] The effect of making to schedule quantity with schedules having minimum orders is illustrated by the following example. Suppose all schedules ξ produce a certain item e in multiples of 5. σ_1 wants 6 of e and σ_2 wants 11. $\underline{\xi}_1 + \underline{\xi}_2$ produce 25 between them but $\underline{\xi}$ need only produce 11. If $\underline{A}$ operates on minimum order sizes then $\underline{\xi}_1 + \underline{\xi}_2$ produces the same amount as $\sigma_1 + \sigma_2$. This is equivalent to $\underline{\xi}_i$ producing exactly what σ wants and the same for $\underline{\xi}$.

PROPOSITION 7.2 *If $\underline{A}$ operates on make to schedule quantity then $\mathfrak{j}$ and $\mathfrak{f}$ are homomorphisms*

Proof Write $\mathfrak{j}(Z(\sigma))$ as $S(\underline{\xi})$ where $\underline{\xi}$ is the least ξ with $\xi \Rightarrow \sigma$. Then $\mathfrak{j}(Z(\sigma_1)\tilde{+}Z(\sigma_2)) = \mathfrak{j}(Z(\sigma_1 + \sigma_2))$ which is to equal $\mathfrak{j}(Z(\sigma_1))\breve{+}\mathfrak{j}(Z(\sigma_2))$.

$$\mathfrak{j}(Z(\sigma_1)) \breve{+} \mathfrak{j}(Z(\sigma_2)) = S(\underline{\xi}_1) \breve{+} S(\underline{\xi}_2) = S(\underline{\xi}_1 + \underline{\xi}_2) \text{ (by definition).}$$

$\mathfrak{j}(Z(\sigma_1 + \sigma_2)) = S(\underline{\xi})$ where $\underline{\xi}$ is the greatest lower bound with $\underline{\xi} \Rightarrow \sigma_1 + \sigma_2$. As $\underline{\xi}_i \Rightarrow \sigma_i$ it is natural to identify $\underline{\xi}_1 + \underline{\xi}_2$ with $\underline{\xi}$ in which case $\mathfrak{j}$ would conserve addition. $\underline{A}$ makes to schedule quantities implies item for item $\underline{\xi}$ matches $\underline{\xi}_1 + \underline{\xi}_2$ and so $S(\underline{\xi}) = S(\underline{\xi}_1 + \underline{\xi}_2)$ and $\mathfrak{j}$ preserves the addition. That $\mathfrak{j}$ takes 0 to 0 and inverses to inverses is obvious so $\mathfrak{j}$ is a homomorphism.

Again write $\mathfrak{f}(S(\xi)) = Z(\hat{\sigma})$ where $\hat{\sigma}$ is the least upper bound of σ with $\xi \Rightarrow \sigma$. We want to show

$$\mathfrak{f}(S(\xi_1) \breve{+} S(\xi_2)) = \mathfrak{f}(S(\xi_1 + \xi_2)) \text{ is equal to } \mathfrak{f}(S(\xi_1)) \tilde{+} \mathfrak{f}(S(\xi_2)).$$

But

$$\mathfrak{f}(S(\xi_1)) \tilde{+} \mathfrak{f}(S(\xi_2)) = Z(\hat{\sigma}_1) \tilde{+} Z(\hat{\sigma}_1).$$

$\hat{\sigma}_i$ is the least upper bound for ξ_i. Hence $\xi_1 + \xi_2 \Rightarrow \hat{\sigma}_1 + \hat{\sigma}_2$. Using the make to schedule quantities requested in scenarios and schedules the least upper bound in both cases is the same and so $\mathfrak{f}(S(\xi_1)\breve{+}S(\xi_2)) = Z(\hat{\sigma}_1)\tilde{+}Z(\hat{\sigma}_1)$. The remaining details to show $\mathfrak{f}$ is a homomorphism are clear. □

The condition "make to schedule quantities" is crucial here as the proof indicates. Its effect is to match schedules for all scenarios that can be accomplished.

PROPOSITION 7.3 *If $\underline{A}$ operates on make to schedule quantities the cohomology of the sheaves $\mathcal{P}$, $\mathcal{P}_{\Rightarrow}$ and $Scn_{\Rightarrow}$ are identical.*

Proof In $C^n(\mathcal{P}_{\Rightarrow})$ an n-cocycle requires

$$Z(\sigma_0)\tilde{+}Z(\sigma_2)\tilde{+}\ldots\tilde{+}Z(\sigma_m) = Z(\sigma_1)\tilde{+}Z(\sigma_3)\tilde{+}\ldots\tilde{+}Z(\sigma_{m'})$$

where m, m' are n and $n-1$ depending on whether n is even or odd. By definition this is $Z(\sigma_0+\sigma_2+\ldots+\sigma_m) = Z(\sigma_1+\sigma_3+\ldots+\sigma_{m'})$ and corresponds to $\sigma_0+\sigma_2+\ldots+\sigma_m$ in $C^n(\tilde{S})$. It also corresponds to

$$S(\underline{\xi}_0)\check{+}S(\underline{\xi}_2)\check{+}\ldots\check{+}S(\underline{\xi}_m) = S(\underline{\xi}_1)\check{+}S(\underline{\xi}_3)\check{+}\ldots\check{+}S(\underline{\xi}_{m'})$$

$$\text{that is } S(\underline{\xi}_0+\underline{\xi}_2+\ldots+\underline{\xi}_m) = S(\underline{\xi}_1+\underline{\xi}_3+\ldots+\underline{\xi}_{m'}).$$

As the schedules effectively match the scenarios we have

$$\underline{\xi}_0+\underline{\xi}_2+\ldots+\underline{\xi}_m = \underline{\xi}_1+\underline{\xi}_3+\cdots+\underline{\xi}_{m'}.$$

which gives a cocycle in $C^n(\mathcal{P})$. □

A similar argument applies to coboundaries. The "make to schedule quantities" condition is really a statement asserting a close match between scenarios and schedules; it means that any schedule can appear as a $S(\xi)$ as we simply adjust scenarios to require it. This way the three cohomologies are the same.

7.1. Fragmentation

Let $\sigma(\underline{r},\underline{q})$ be an order to deliver to a^* a set $\underline{r}$ of r_i goods no later than t_i[11] and let $\underline{q} \in Inv(A)$ be the starting inventory given by $\underline{q}(a)$ for each $a \in \underline{A}$.

Definition Define $Sch(\underline{r},\underline{q})$ to be the group-valued sheaf generated by schedules that satisfy $\sigma(\underline{r},\underline{q})$.

As a sheaf $Sch(\underline{r},\underline{q})(U)$ is the set of schedules generated by $Z(\sigma(\underline{r},\underline{q})) \in Sch_{\Rightarrow}(U)$ and so we can write $Sch(\sigma(\underline{r},\underline{q})) \Rightarrow \sigma(\underline{r},\underline{q})$. $Z(\sigma(\underline{r},\underline{q}))$ is a semi-group; the sum of any two schedules that satisfy $\underline{r},\underline{q}$ again satisfy the scenario. $Sch(\underline{r},\underline{q})(U)$ is the smallest group valued sheaf containing $Z(\sigma(\underline{r},\underline{q}))$. As every schedule in $Sch(\underline{r},\underline{q})$ fulfills $\sigma(\underline{r},\underline{q})$ then every restriction to $\xi \in Sch(\underline{r},\underline{q})(U)$, $U \in \mathcal{U}$ extends to a full schedule that fulfills $\sigma(\underline{r},\underline{q})$. This means $Sch(\underline{r},\underline{q})$ is a soft sheaf: $Sch(\underline{r},\underline{q})(A) \twoheadrightarrow Sch(\underline{r},\underline{q})(U)$ and consequently $\check{H}^n(Sch(\underline{r},\underline{q})) = 0$ for $n > 0$. We can use cohomology to mark the difference between scenarios that can be satisfied by $Sch(\underline{r},\underline{q}))$ and those that cannot.

Definition The *frontier* of $\underline{A}$ is the set $(\underline{r},\underline{q})$ such that for any pair $(\delta r, \delta q) \in Typ(a^*) \times Typ(A)$, δr and δq positive, $\check{H}^0(Sch(\underline{r},\underline{q}))$ is non trivial and $\check{H}^0(Sch(\underline{r}+\delta r, \underline{q}-\delta q) = 0$.

The frontier divides the scenario space of elements $(\underline{r},\underline{q})$ into demand that can be accomplished and demand for which there is no global schedule. If $\check{H}^0(Sch(\underline{r},\underline{q}))$ is non trivial there is a global section and so at least one schedule that fulfills $\sigma(\underline{r},\underline{q})$. $\check{H}^0(Sch(\underline{r}+\delta r, \underline{q}-\delta q) = 0$ means no such schedule exists. Instead $\check{H}^0(Sch(\underline{r}+\delta r, \underline{q}-\delta q) = 0$ for small δr or δq and some of the groups $\check{H}^n(Sch(\underline{r}+\delta r, \underline{q}-\delta q) \neq 0$ as only local areas have capacity to fulfill more than $\sigma(\underline{r},\underline{q})$.

(1) If $\mathcal{P}$ is soft then $\check{H}^n(\mathcal{P}) = 0$ which means the $Z^n(\mathcal{P}) = B^n(\mathcal{P})$ and no new phenomena happens among the n intersections.

(2) Given $j : Sch(\underline{r}_1, \underline{q}) \hookrightarrow Sch(\underline{r}_2, \underline{q})$ as in $r_1 > r_2$ then

$$\check{H}^k(Sch(\underline{r}_1, \underline{q})) \xrightarrow{\check{H}^k(j)} \check{H}^k(Sch(\underline{r}_2, \underline{q})).$$

(3) The equations among the cocycles hold for both sheaves. Let z be cocycle in $Z^k(Sch(\underline{r}_1, \underline{q}))$. If $\check{H}^k(j)(z) = 0$ then $C^k(j)(z) \in B^k(Sch(\underline{r}_2, \underline{q}))$.

We have:

$$\check{H}^k(j)(z) = 0 \Leftrightarrow z \in B^k(Sch(\underline{r}_2, \underline{q}))$$

so that

$$ker(\check{H}^k(j)) = \check{H}(j)^{-1}(B^k(Sch(\underline{r}_2, \underline{q})))/B^k(Sch(\underline{r}_1, \underline{q}))$$

which gives a cohomology dynamics of the way sheaves of schedules *fragment* as order load exceeds capacity.

8. The coordination of change

In this section the sheaf, $\mathcal{G}_{\mathfrak{L}}$ (introduced in Section 3) is applied to manufacturing networks. This object is a preoccupation of managers of jobbing shops across the world. Not, of course, in this form. The concepts are commonplace. We abstract the possible changes, "adjustments", to a class of schedules. Adjustments to schedules can be composed and in this context reversed. Adjustments to one schedule need not apply to another. This makes adjustments a groupoid in the sense Brandt (see Iyanaga and Kawada 1977 or Ivan 2002). The machinery of groupoid fibrations as in Section 3.3. provides a general passage to sheaf theoretic concepts and opens up the possible applications of homotopy concepts introduced in Gabriel and Zisman (1967).

8.1. Groupoid classes and the $J(\mathcal{G}, \mathcal{P})$ functor

Section 3.3 presented a groupoid action on a sheaf of schedules giving sheaves $\mathcal{P}_{\mathfrak{L}}$ and $\mathcal{G}_{\mathfrak{L}}$. In this section $\mathcal{G}_{\mathfrak{L}}$ will be our main interest. To emphasis this as a functor from groupoid fibrations over sheaves in $Comb(\underline{A})$ we shall write $\mathcal{G}_{\mathfrak{L}}$ as $J(\mathcal{G}, \mathcal{P})$ and $J(\mathcal{G})$ when the $\mathcal{P}$ is understood.

From Section 3.3 $J(\mathcal{G}, \mathcal{P})$ is given by

(1) $G(U)(\xi) = \{\xi' \in \mathcal{P}(U) \mid \exists g \in \mathcal{G}(U)(g \neq 1) \wedge (sg = \xi) \wedge (tg = \xi')\}$ and so
(2) $J(\mathcal{G}, \mathcal{P})(U) \equiv \mathcal{G}_{\mathfrak{L}}(U) = \{G(U)(\xi) \mid \xi \in \mathcal{P}(U)\}$.

An arbitrary section of $J(\mathcal{G}, \mathcal{P})$ will be written $G(\xi)$ and can be thought of as all the possible adjustments to ξ.

There are many classes of groupoids that are relevant to changing schedules in an industrial setting.

(1) Adjustments for schedules that do not change elements $w \in W(a)$ that use large amounts of an expensive and not always available resource.
(2) Adjustments associated with a time cutoff so that all the adjustments take place after the cutoff. This leaves the early part of schedules unchanged.
(3) Schedules that depend on a set of variables or parameters that can be changed are a source of groupoids and their associated equivalences classes. The groupoid acts on schedules through the schedule's dependence on parameters that can be changed. If $\xi = \xi(v_1, v_2, \ldots, v_n)$ where $v_i \in V_i$ such that if g acts on each V_i then g changes ξ by the action $g\xi(\underline{v}) = \xi(g(\underline{v}))$ for some $\underline{v} \in \oplus_i V_i$. For this to be well defined there must be a $\xi' = \xi(g(\underline{v}))$.

Definition of the group valued sheaf of adjustment classes

The structure of a given $G(\xi)$ can be analyzed in terms of its subgroupoids. If $g : \xi_1 \to \xi_2$ there is an associated set function $\tilde{g} : G(\xi_2) \to G(\xi_1)$ given by $\tilde{g}(h) = h \circ g$. It is not a groupoid homomorphism as in Ivan (2002) but allows the decomposition of groupoids in a way that reflects the hierarchy of equivalence relations among the schedules connected by adjustments. For example given

$$\xi_1 \xleftarrow{g_1} \xi_0 \xrightarrow{g_2} \xi_2$$

where g_1 and g_2 are the only adjustments from ξ_0 to ξ_1 and ξ_2 then $G(\xi_0)$ decomposes as $G(\xi_1) \sqcup G(\xi_2)$. Furthermore any equivalence relation between a ξ' connected to ξ_1 and ξ'' connected to ξ_2 must be related through ξ_0.

Definition The relationship of $G(\xi_0)$ and $G(\xi_1) \sqcup G(\xi_2)$ is denoted "$\approx$" so $G(\xi_0) \approx G(\xi_1) \sqcup G(\xi_2)$ means any relation that ξ_0 holds with any other schedule arises through adjustments through one of ξ_1 or ξ_2.

Examples

(1) If ξ_0 is the disjoint union of ξ_1 and ξ_2 as sets of actions and $\mathcal{G}$ is defined in a way that there are adjustments that apply to ξ_1 and ξ_2 but not necessarily ξ_0 then $G(\xi_0) \approx G(\xi_1) \sqcup G(\xi_2)$.
(2) If $g_1 : \xi_1 \to \xi_2$, $g_2 : \xi_2 \to \xi_3$, $g_3 = g_2 \circ g_1$ and g_2 cannot apply to ξ_1 and g_1 cannot apply to ξ_2 then $G(\xi_1) \approx G(\xi_2) \sqcup G(\xi_3)$.

Definition Define addition on $J(\mathcal{G}, \mathcal{P})$ as the free group generated by the elements of the set $\{G(\xi) \mid \xi \in \mathcal{P}(U)\}$ factored by the subgroup generated by the set of relations

$$\{G(\xi_1) + G(\xi_2) - G(\xi_3) \mid G(\xi_3) \approx G(\xi_1) \sqcup G(\xi_2)\}.$$

Given $G(\xi)$ denote its class in $J(\mathcal{G}, \mathcal{P})$ as $[G(\xi)]$.

This is a standard Grothendieck Group construction (Lang 1969). The action of the adjustments is fully consistent with sheaf structure in the sheaf of sets. This additional structure is defined in terms of sections and so it gives a group structure on the sheaf $J(\mathcal{G}, \mathcal{P})$.

For the rest of this section assume that among the possible groupoids of adjustments, $\mathcal{G}(\xi)$ is the full set of adjustments that are relevant to the ξ. The phrase "relevant to" is to imply that a category of groupoids has been chosen to reflect the type of adjustments that are to be studied. This means that $\mathcal{G}$ contains all the decompositions of groupoids. If $\mathcal{G}_1$ and $\mathcal{G}_2$ act on ξ with $g_1 \in \mathcal{G}_1$, $g_2 \in \mathcal{G}_2$, then $g_2 \circ g_1^{-1} \in \mathcal{G}_2$, $g_1 \circ g_2^{-1} \in \mathcal{G}_1$ so that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a subgroupoid of the set of all adjustments on ξ.

8.2. Adjustments and forms; optimization

Let Λ be a form or forms that are desired measures of schedules. Then, as both $\mathcal{D}[\Lambda]$ and $\mathcal{G}$ are sheaves defined on $\underline{A}$, so therefore is their product $\mathcal{K} = (\mathcal{D}[\Lambda] \times \mathcal{G})$. Assume that for any ϕ in $\mathcal{D}[\Lambda]$ with $\phi(\xi)$ positive (ϕ has a positive norm or is in the positive cone of $\mathcal{D}[\Lambda]$) then it is desirable that $\phi(\xi)$ is as small as possible. Define a relationship sheaf $\mathfrak{G} \rightarrowtail \mathcal{P} \times (\mathcal{D} \times \mathcal{G})$ by $\mathfrak{G}(U)(\xi, <\phi, g>) \Leftrightarrow ((\phi(U)(g(\xi)(U))) \leq (\phi(U)(\xi(U))))$.[12] $\mathcal{P}_{\mathfrak{G}}$ is then given by

$$\mathcal{P}_{\mathfrak{G}}(U) = \{X(\phi, g) \mid \exists \xi \in \mathcal{P}(U) \text{ and } \mathfrak{G}(\xi, (\phi, g))\}$$

where $X(\phi, g) = \{\xi \in \mathcal{P}(U) \mid \mathfrak{G}(\xi, (\phi, g))\}$

$$\mathcal{K}_{\mathfrak{G}}(U) = (\mathcal{D}[\Lambda] \times \mathcal{G})_{\mathfrak{G}}(U) = \{Y(\xi) \mid \exists \xi \in \mathcal{P}(U)\}$$

and $Y(\xi) = \{(\phi, g) \in (\mathcal{D}[\Lambda] \times \mathcal{G})(U) \mid \phi(g.\xi) \leq \phi(\xi)\}$. This gives the existence of improved outcomes resulting from the adjustments in $\mathcal{G}$ defined for $\mathcal{P}$.

We consider two ways to give $\mathcal{P}_{\mathfrak{G}}$ a group structure. $\mathcal{P}_{\mathfrak{G}}$ is given the following group operation. For every U $\mathcal{P}_{\mathfrak{G}}(U)$ is, as usual, the free group of the elements $X(\phi, g))$ but factored as follows:

(1) Factored by the relation $X(\phi_1, g_1) \dot{+} X(\phi_2, g_2) = X(\phi_3, g_3)$ if and only if $\xi_3 \in X(\phi_3, g_3)$ which is acted on by $g_3 \in \mathcal{G}(\xi_3) \approx \mathcal{G}(\xi_1) \sqcup \mathcal{G}(\xi_2)$. This means ξ_3 is a mixture of ξ_1 and ξ_2 and these components can be acted on by $\mathcal{G}$ separately. g_3 can be separated into g_1 and g_2 so the equation will only hold if $\phi_i(\xi_i)$ is appropriated for the segments of the schedule ξ_3 and $\phi_1(g_1.\xi_1) + \phi_1(g_1.\xi_1) \in m(\phi_3)$.
Denote this group structure as $(\mathcal{P}_{\mathfrak{G}}, \dot{+})$.

(2) $X(\phi_1, g_1) \boxplus X(\phi_2, g_2) = X(\phi_3, g_3)$ if and only if $g_3 = g_1 \circ g_2$ or $g_3 = g_2 \circ g_1$ and $\| \phi_3 \| \leq \| (\phi_1 \wedge \phi_2) \|$, where $\wedge$ is the lattice operation of greatest lower bound.
Denote this group structure as $(\mathcal{P}_{\mathfrak{G}}, \boxplus)$.

The second relation means that $\xi_i \in X(\phi_i, g_i)$, $i = 1, 2$ are in $X(\phi_3, g_3)$ as a ξ_3 for which $\phi_3(g_i \circ g_j)(\xi_3) \leq \phi_3(\xi_3) \leq \phi_i(\xi_i)$. That is both ξ_1 and ξ_2 satisfy $\phi_3(g_i \circ g_j)(\xi_i) \leq \phi_3(\xi_i) \leq \phi_i(\xi_i)$. That is $g_1 : \xi_1 \to \xi_1'$, $g_2 : \xi_2 \to \xi_2'$ and $g_2(\xi_1') = \xi_3$ and $g_1(\xi_2') = \xi_3' \in G(\xi_3)$. The case above using $\mathcal{G}(\xi_1) \sqcup \mathcal{G}(\xi_2)$ could be a subclass of this in which case this second method is a quotient of that group.

LEMMA 8.1 *If ξ_3 is a mixture of ξ_1 and ξ_2 and these components can be acted on by g_1 and g_2 in $\mathcal{G}(U)$ separately then it is also true that $g_1 : \xi_1 \to \xi_1'$, $g_2 : \xi_1 \to \xi_2'$ implies $g_2(\xi_1') \in G(\xi_3)$ and $g_1(\xi_2') \in G(\xi_3)$.*

Proof ξ_3 can be considered $\xi_1 \sqcup \xi_2$ wherein each segment has their respective g_i acting on them to improve their measure by their respective forms ϕ_i. ϕ_3 measures the improvement of g_3 on ξ_3 which is both of ξ_1 and ξ_2. g_3 acts on the domains of g_1 and g_2 so g_3 as $g_1 \sqcup g_2$ can be represented as $g_2 \circ g_1$ or $g_1 \circ g_2$ each acting on separate domains. □

COROLLARY 8.2 $\dot{+} \Rightarrow \boxplus$ *hence* $(\mathcal{P}_{\mathfrak{G}}, \boxplus)$ *is a quotient of* $(\mathcal{P}_{\mathfrak{G}}, \dot{+})$.

Proof The relations defining $\boxplus$ and $\dot{+}$ from the initial free group reduce the number of dimensions of the group. As $\dot{+} \Rightarrow \boxplus$, $\dot{+}$ reduces the free group less than the relation $\boxplus$ and this gives the result. □

The cohomology of $(\mathcal{P}_{\mathfrak{G}}, \boxplus)$ (or $(\mathcal{P}_{\mathfrak{G}}, \dot{+})$) gives the local improvements in the measure of Λ. A cocycle equation $\Sigma_{k\ even} X(\phi_k, g_k) = \Sigma_{k\ odd} X(\phi_k, g_k)$ with $k = 0, 1, \cdots m$ that is a class in $\check{H}^m(\mathcal{P}_{\mathfrak{G}}, \boxplus)$ means that there two ways in the intersection $U = \cap_{k=0,1,\ldots m} U[k]$ for adjusting the schedules and locally improving their measures simultaneously. This is most clear when $\Lambda = \{\phi\}$ has only one measure. In this case the schedules

$$\xi \in X(\phi, g) = \sum_{k\ even} X(\phi, g_k) = \sum_{k\ odd} X(\phi, g_k)$$

have been subject to the adjustments of the $g_k \in \mathcal{G}(U)$ with each g_k reducing the measure ϕ. The existence of $\check{H}^p(\mathcal{P}_{\mathfrak{G}}, \boxplus)$ (or $\check{H}^p(\mathcal{P}_{\mathfrak{G}}, \dot{+})$) with a high p means there are schedules that can be improved over $p/2$ iterations which might correspond to distinct dimensions of adjustments. Conversely, $\check{H}^p(\mathcal{P}_{\mathfrak{G}}, \boxplus)$ ($\check{H}^p(\mathcal{P}_{\mathfrak{G}}, \dot{+})$) is zero for $p > 2$ or 3 implies little ability to improve the measures of schedules.

8.3. Application to the worked example

To apply $\mathcal{P}_{\mathfrak{G}}$ to worked example of Section 4.1 we define

(1) $\Lambda = \{\phi\}$. Define $\phi(\xi)$ to be the (inventory) holding time of the schedule ξ. This is the total number of time-slots between the end of any action w and the start of the action that uses its output (w's "target"). In cases where $V \subset U$ and ϕ is defined on schedules in $\mathcal{P}(U)$ but $\mathcal{P}(V)$ has no expressions of the form $w_2 \circ w_1$ (as will be the case for $S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3})$) then ϕ must be defined on $\mathcal{P}(V)$ by using $\mathcal{P}(U) \xrightarrow{\rho(U,V)} \mathcal{P}(V)$.
(2) Define the groupoid $\mathcal{G}$ of adjustments that move individual actions to an earlier or later time-slot. In particular $\mathcal{G}$ contains only adjustments that move actions to their best position rather than move them through all possible intermediate steps.
(3) Classes $X(\phi, g) \in \mathcal{P}_{\mathfrak{G}}$ are the elements of $\mathcal{G}$ that do not increase holding time of at least one schedule: $\exists \xi \phi(U)(g(\xi)(U))) \leq (\phi(U)(\xi(U)))$.
(4) Using the definitions in Sections 8.2 and 8.1, $X(\phi, g_1) \dot{+} X(\phi, g_2) = X(\phi, g_3)$ implies that $g_3 = g_2 \circ g_1$, but g_2 cannot apply to the domain of g_1 and vice-versa. If g_1 and g_2 satisfy the ϕ criterion then $X(\phi, g_2 \circ g_1) = X(\phi, g_1) \dot{+} X(\phi, g_2)$. If g_1 and g_2 just move (say) $w_{3,3}$ one time-slot later when the holding time can be improved by two time-slots then addition corresponds to composition. If $g_2 \circ g_1$ overshoots then the target has to wait and the holding time is increased. This means $X(\phi, g_2 \circ g_1)$ does not exist and $X(\phi, g_1) \dot{+} X(\phi, g_2) = 0$.

What are the possible classes in $\mathcal{P}_{\mathfrak{G}}(U)$?

(1) $U = S^2(a_{1,1})$. Holding time for $\xi(a_{1,1})$ is already minimal so $\mathcal{P}_{\mathfrak{G}}(S^2(a_{1,1})$ has only one class $X(\phi, 1)$.
(2) $U = S^2(a_{1,2})$. In $\xi(a_{1,2})$, the first $w_{3,3}$ is not in an optimal slot. If $g(12)$ moves it two slots later then $X(\phi, g(12))$ is a class in $\mathcal{P}_{\mathfrak{G}}(S^2(a_{1,2})$.
(3) $U = S^2(a_{1,3})$. Holding time for $\xi(a_{1,3})$ is already minimal so $\mathcal{P}_{\mathfrak{G}}(S^2(a_{1,3})$ has only one class $X(\phi, 1)$.
(4) $\mathcal{P}(S^2(a_{1,2}) \cap S^2(a_{1,3}))$ has schedules to $a_{2,3}$ requiring $\{w_{3,3}, w_{3,4}, w_{3,5}\}$ each of which has to occur twice giving a minimum of 10 time-slots. $w_{3,4}$ has two options for each run with the same holding time of one (because of the 5 slot lead time of $w_{3,2} \circ w_{3,1}$ or $w_{3,5}$) and $w_{3,3}$ gives no increase in holding time except if $w_{3,3}$ is run at the last time-slot for $w_{3,5}$. If the movements of $w_{3,4}$ correspond to h_1 and h_2 then $\mathcal{G}((S^2(a_{1,2}) \cap S^2(a_{1,3}))$ is $\langle h_1, h_2 \rangle$ and we have two classes $X(\phi, h_1)$ and $X(\phi, h_2)$ in $\mathcal{P}_{\mathfrak{G}}((S^2(a_{1,2}) \cap S^2(a_{1,3}))$.
(5) The restrictions of $\tilde{\xi}_1$, $\tilde{\xi}_2$ and $\tilde{\xi}_3$ to $S^2(a_{1,2})$ give rise to three possible adjustments of $w_{3,3}$ in $S^2(a_{1,2})$ segment that improve holding time. The two from $\tilde{\xi}_1$, $\tilde{\xi}_2$ to the first $w_{3,3}$ are the same, one from $\tilde{\xi}_3$ and one from $\tilde{\xi}_1$ for the last $w_{3,3}$ giving three adjustments denoted $\tilde{g}_1$, $\tilde{g}_2$ and $\tilde{g}_3$.
(6) $S^2(a_0)$ does not include the actions $w_{3,k}$ actions.

Cohomology calculations.
$\check{H}^0(\mathcal{P}_{\mathfrak{G}})$.
$C^0(\mathcal{P}_{\mathfrak{G}}) = \oplus_{ij} \mathcal{P}_{\mathfrak{G}}(S^2(a_{i,j}))$. Cocycles require $X(\phi, g(ij)) \in \mathcal{P}_{\mathfrak{G}}(S^2(a_{i,j}))$, $X(\phi, g(kl)) \in \mathcal{P}_{\mathfrak{G}}(S^2(a_{k,l}))$ such that $X(\phi, g(ij)) \mid S^2(a_{i,j}) \cap S^2(a_{k,l}) = X(\phi, g(kl)) \mid S^2(a_{i,j}) \cap S^2(a_{k,l})$. The only possible cases are those from the global schedules $\tilde{\xi}_i$ restricted to $S^2(a_{1,2}) \cap S^2(a_{1,3})$ and one from $\xi(a_{1,2})$ that is the same as the first from $X(\phi, \tilde{g}_1)$. These all give independent restrictions and so $\check{H}^0(\mathcal{P}_{\mathfrak{G}})$ is generated by the classes of $X(\phi, \tilde{g}_1)$, $X(\phi, \tilde{g}_2)$ and $X(\phi, \tilde{g}_3)$.

$\check{H}^1(\mathcal{P}_{\mathfrak{G}})$.

$$C^1(\mathcal{P}_{\mathfrak{G}}) = \mathcal{P}_{\mathfrak{G}}(S^2(a_{1,1}) \cap S^2(a_{1,2})) \oplus \mathcal{P}_{\mathfrak{G}}(S^2(a_{1,1}) \cap S^2(a_{1,3})) \oplus \mathcal{P}_{\mathfrak{G}}(S^2(a_{1,2}) \cap S^2(a_{1,3}))$$
$$\cong \mathcal{P}_{\mathfrak{G}}(S^2(a_{1,2}) \cap S^2(a_{1,3}))$$

To look for cocycles we need $X(\phi, g(ijkl)) \in \mathcal{P}_{\mathfrak{G}}(S^2(a_{i,j}) \cap S^2(a_{k,l}))$ such that there is at least one $\xi(ijkl)$ for which $g(ijkl)(\xi(ijkl))$ evaluated on $S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3})$ does not increase the holding time of $\xi(ijkl)$. As listed above we have two classes $X(\phi, h_1)$ and $X(\phi, h_2)$ in $\mathcal{P}_{\mathfrak{G}}(S^2(a_{1,2}) \cap S^2(a_{1,3}))$. We need:

$$X(\phi, g(1113)) \mid (S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3}))$$
$$= (X(\phi, g(1112)) \dot{+} X(\phi, g(1213))) \mid (S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3})).$$

The only possible $g(1113) \in \mathcal{P}_{\mathfrak{G}}(S^2(a_{1,1}) \cap S^2(a_{1,3}))$ and $g(1112) \in \mathcal{P}_{\mathfrak{G}}(S^2(a_{1,1}) \cap S^2(a_{1,3}))$ are identities. No cocycle exists and so $\check{H}^1(\mathcal{P}_{\mathfrak{G}}) = 0$.

$\check{H}^2(\mathcal{P}_{\mathfrak{G}})$.

$C^2(\mathcal{P}_{\mathfrak{G}}) = \mathcal{P}_{\mathfrak{G}}(S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3}))$ and ∂^2 is zero so all elements in $\mathcal{P}_{\mathfrak{G}}(S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3}))$ are cocycles. The only possibility for cocycles from $\mathcal{P}(S^2(a_{1,1}) \cap S^2(a_{1,2}) \cap S^2(a_{1,3})) = \mathcal{P}(\{a_{3,3}, a_{3,6}\})$ are adjustments to $w_{3,3}$ or $w_{3,6}$. $w_{3,6}$ cannot be adjusted so any g is applied to $w_{3,3}$. In $\{a_{3,3}, a_{3,6}\}$, ϕ must be defined in terms on what time-slot the "target" was. This comes from the image in $C^1(\mathcal{P}_{\mathfrak{G}})$. This is $im(\partial^1(C^1(\mathcal{P}_{\mathfrak{G}}))$.

Proof Any equation $X(\phi, g) = X(\phi, g(1112)) - X(\phi, g(1113)) \dot{+} X(\phi, g(1213))$ has $g(1112) = g(1113) = 1$ (i.e no change), $X(\phi, g) = X(\phi, g(1213))$ which is necessarily the case as this is the only non-trivial class that can come from $C^1(\mathcal{P}_{\mathfrak{G}})$. Consequently all cocycles are coboundaries and $\check{H}^2(\mathcal{P}_{\mathfrak{G}}) = 0$. □

$\mathcal{G}$ was chosen to focus on those changes that are registered by ϕ. The cohomology of $\mathcal{P}_{\mathfrak{G}}$ gives no sense of scale interactions. This is because there is no coordination among adjustments and they derive from adjustments that affect the global schedules. Adjustments that are changes that move a number of actions in a coordinated manner will have a much richer cohomology.

9. Conclusion

The mastery of complexity required for the successful creation and management of commercial and infrastructure systems is likely to require news ways of understanding the interaction of all scales of the system. Cohomology functors of sheaves are likely to play a significant part in these efforts. This is a broad area of research combining what might be called dynamical sheaves such as schedules with sheaves of desired qualities (commercial, legal or social) to understand the scale (local and global) features of large systems. Here the dynamics is inherent in the concept of combinatorial systems the schedules of which arise from natural questions about polynomials. The desired qualities arise from the reasons for the systems; the way they are to be implemented or manifested.

Exploiting the abstraction of combinatorial systems raises many questions about the nature and scope of structures we seek in them.

(1) The cohomologies of $\mathcal{P}$ and $\mathcal{P}_{\mathfrak{L}}$ are tied to the choice of cover $\mathcal{U}$ which defines a "scale of analysis." As $\mathcal{U}$ grows so does the effort to specify both $\mathcal{P}$ and $\mathcal{P}_{\mathfrak{L}}$ hence $\mathcal{L}$ and $\mathcal{K}$. The test of how much information can be extracted from cohomology will rely on efficient ways of computing the cohomology. This is likely to be through a combination of further mathematics and computer programs.

(2) The totality of constructions $\mathcal{P}_{\mathfrak{L}}$ added to $Comb(\underline{A})$ builds a "conceptual context" of the system $\underline{A}$, effectively linking it to its intended applications or manifestations in society. If the $\mathfrak{L}$ and its associate $\mathcal{K}$ determines a class of sheaves Γ the conceptual context is then a new category defined by the Γ classes. It becomes an *epistemological structure* of a class of systems related to $\underline{A}$. What information does this contain about the way we think about these activities?

These are fertile areas of research for systems. The concepts discussed in this paper are highly abstract but are in fact part of the daily work of business systems analysts in innumerable businesses.

10. Table of symbols

Symbols	Meaning	Sections
A	The index set of the topics in a combinatorial system	2
$a, b, a_1, a_2, \ldots$	Elements of A	2
$Tpc(a), Tc(b), Tpc(a_1),'\ldots$	Topics (categories) of elements of A	2
$e, e', e_1, e_2, \ldots$	Objects of $Tpc(a), a \in A$	2
$w, w', w_1, w_2, \ldots$	Maps of $Tpc(a), a \in A$	2
$\underline{A}, \underline{A}_1, \ldots$	Combinatorial systems	2
Cat	A category of small categories	2
$Typ(a)$	Types of a combinatorial system	2
$act(a), W(a)$	Actions and ring of actions of a combinatorial system	2
$f : \underline{A}_1 \to \underline{A}_2, Tpc(f)$	Combinatorial system morphism	2.1
$Comb$	The category of combinatorial systems	2.1
$CombMod$	The category of combinatorial systems	2.1
$\xi, \xi', \xi_1, \ldots\eta, \zeta$	Schedules	2.1
$U, U_i\ \mathcal{U}$	Subsets of A, covering of A	2.1
Sch	The sheaf of schedules	2.1
$[\mathfrak{L}, \mathcal{K}]$	The relative sheaf construction	3.1
$\mathcal{P}_{\mathcal{L}}, \mathcal{K}_{\mathcal{L}}$	Class sheaves of $\mathfrak{L}$	3.1
$\Lambda, \mathcal{D}[\Lambda]$	Sets of forms, the sheaf of Λ forms	3.1
$\mathfrak{F}, \mathcal{P}_{\mathfrak{F}}, \mathcal{D}[\Lambda]_{\mathfrak{F}}$	Class sheaves for forms	3.2
$\mathcal{G}, s, t : \mathcal{G} \rightrightarrows \mathcal{P}$	Groupoid fibration over $\mathcal{P}$	3.3
$C^n(\mathcal{P}, \mathcal{U}), C^n$	Cochains for the sheaf $\mathcal{P}$ and cover $\mathcal{U}$	4
$\partial^n : C^n \to C^{n+1}$	The cochain homomorphism	4
$Z^n, Z^n(\mathcal{P}, \mathcal{U}), B^n, B^n(\mathcal{P}, \mathcal{U})$	The nth cocycle and coboundary groups	4
$\check{H}^n, \check{H}^n(\mathcal{P}, \mathcal{U})$	The nth cohomology group	4
$Inv, Scn(\underline{A})$	The "inventory" sheaf and sheaf of scenarios of $\underline{A}$	7
$\mathcal{P}_{\Rightarrow}, Scn_{\Rightarrow}$	Class sheaves of schedules satisfying scenarios	7.1
$Sch(\sigma(\underline{r}, \underline{q}))(= Sch(\underline{r}, \underline{q}))$	Sheaf of schedules fulfilling $\sigma(\underline{r}, \underline{q})$	7.1
$J(\mathcal{G}, \mathcal{P})$	The class sheaf functor of the groupoid $\mathcal{G} \rightrightarrows \mathcal{P}$	8.1
$G(\xi_3) \approx G(\xi_1) \sqcup G(\xi_2)$	Groupoid actions on ξ_3 decompose into actions on ξ_1, ξ_2	8.1
$\mathfrak{G} \rightarrowtail \mathcal{P} \times (\mathcal{D} \times \mathcal{G})$	"Optimizing" relationship sheaf	8.2

Notes

1. This complexity recalls the story of blind men describing an elephant after feeling parts of it.
2. A recent example is the failure of the USDoD attempt to replace its inventory systems. See http://www.eaconsult.com/2012/12/18/when-cots-costs-too-much-oracle-the-us-air-force-and-a-1-billion-project-failure/
3. See the Carnegie Mellon Software Engineering Institute website http://www.sei.cmu.edu/uls/?location=secondary-nav&source=5754
4. Such as straight "A" students, high-purity waters in medical compounds destined for immune compromised patients.
5. In the tables we drop the "," in the subscripts of the actions.
6. We are not introducing an augmentation in this example.
7. This is the reason that soft sheaves have trivial cohomology for $n > 0$.
8. "When the documentation is as heavy as the aircraft you know you are getting it right" Quoted from TV series "How to make a" episode of "How to make the wing of a super-jumbo (A380)" SBS (Australia) 18 August 2012 7:30 pm.
9. Manufacturing networks are most relevant here because of the emphasis on algorithms. Supply networks typically emphasis inventory and logistics.
10. "Lean manufacturing" has this linearity as a goal. It is the same as "scaling without drama". A similar concept is minimum order quantities to fit jumps in scheduling.
11. interpreted as $Inv(a^*)$ will have r_i goods available at t_i.
12. We skip the details of defining $\mathfrak{G}$ in terms of category of sheaves. This category being a topos allows the definition of order relations and groupoid actions and a precise definition so can be easily accomplished.

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