Riemann–Weyl in Deleuze's *Bergsonism* and the Constitution of the Contemporary Physico-Mathematical Space

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Abstract

In recent years, the ideas of the mathematician Bernhard Riemann (1826–66) have come to the fore as one of Deleuze's principal sources of inspiration in regard to his engagements with mathematics, and the history of mathematics. Nevertheless, some relevant aspects and implications of Deleuze's philosophical reception and appropriation of Riemann's thought remain unexplored. In the first part of the paper I will begin by reconsidering the first explicit mention of Riemann in Deleuze's work, namely, in the second chapter of *Bergsonism* (1966). In this context, as I intend to show first, Deleuze's synthesis of some key features of the Riemannian theory of multiplicities (manifolds) is entirely dependent, both textually and conceptually, on his reading of another prominent figure in the history of mathematics: Hermann Weyl (1885–1955). This aspect has been largely underestimated, if not entirely neglected. However, as I attempt to bring out in the second part of the paper, reframing the understanding of Deleuze's philosophical engagement with Riemann's mathematics through the Riemann–Weyl conjunction can allow us to disclose some unexplored aspects of Deleuze's further elaboration of his theory of multiplicities (rhizomatic multiplicities, smooth spaces) and profound confrontation with contemporary science (fibre bundle topology and gauge field theory). This finally permits delineation of a correlation between Deleuze's plane of immanence and the contemporary physico-mathematical space of fundamental interactions.

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Bernhard Riemann (1826–66) constitutes one of the main figures of Deleuze's engagement with mathematics. 'The conjunction of Riemann's mathematics and Deleuze's philosophy', Arkady Plotnitsky writes, 'is a remarkable event in the history of twentieth-century philosophy, and it has major implications for our understanding of the relationships between mathematics and Deleuze's thought, and between mathematics and philosophy in general' (Plotnitsky 2009: 190). However, this understanding still remains largely unexplored, beginning with the relationship between Riemann and Deleuze itself.[1]

As is well known, Deleuze's first reference to Riemann is found in the second Chapter of *Bergsonism* (1966). The underlying issue lies in Bergson's distinction between (quantitative and qualitative) multiplicities, and the attempt to overcome the classical dichotomy between the One and the Multiple by affirming a substantive multiplicity. On the background of this philosophical problem, Deleuze synthesises, in a particularly dense passage, one of the key aspects of Riemann's theory of 'multiplicities' (manifolds):

> In fact, this problem dates back to a scholar of genius, Riemann, a physicist and mathematician. Riemann defined as 'multiplicities' those things (*les choses*) that could be determined in terms of their dimensions or their independent variables. He distinguished between *discrete multiplicities* and *continuous multiplicities*. The former contain the principle of their own metric (*portaient le principe de leur métrique*) (the measure of one of their parts being given by the number of elements they contain) (*la mesure d'une de leurs parties étant donnée par le nombre des éléments qu'elles contenaient*). The latter found a metrical principle in something else, even if only in phenomena unfolding in them or in the forces acting in them (*ne fût-ce que dans les phénomènes se déroulant en elles ou dans les forces agissant en elles*). (Deleuze 1988: 39/31–2)[2]

To this passage Deleuze adds a very important footnote. First, he refers to Riemann's famous Habilitation lecture 'On the Hypotheses Which Lie at the Foundation of Geometry', dating to 1854 (but only published posthumously in 1868). As is known, it is in this lecture that Riemann first introduces the concept of manifold or manifoldness (*Mannigfaltigkeit*). Second, he refers to one of the major works of mathematical physics of the early twentieth century: *Space, Time, Matter* (1918), by the German mathematician Hermann Weyl

(1885–1955). Weyl's book constitutes the first detailed exposition of Albert Einstein's general theory of relativity; as well as, starting in subsequent editions – on which the French translation adopted by Deleuze is based – Weyl's original elaboration of the first unified field theory.[3] Finally, Deleuze suggests that even Husserl, although differently than Bergson, was influenced by Riemann's theory.

Now, while the philosophical importance of Riemann for Bergson and Husserl will be frequently repeated in Deleuze's works, except for a passing mention (which we will encounter), Deleuze never referred to Weyl again. Thus, in most research appealing to the significance of Riemann for Deleuze's own theory of multiplicities, the relevance of Weyl has been largely overshadowed (except for Plotnitsky, who indicated this relevance), if not entirely neglected. Nevertheless, as I would like to show, reframing the reconstruction and understanding of the relationship between Deleuze and Riemann on the Riemann–Weyl conjunction allows us not only to reconsider the entire Deleuzian reception of Riemann's figure and work, but also to extrapolate some new aspects and implications of Deleuze's commitment with Riemann in his late works.

In the first part of the paper (sections I–III), I propose a detailed analysis of the passage in *Bergsonism*, firstly to establish its strict dependence on the explanation of Riemannian manifolds and space theory found in Weyl's *Space, Time, Matter*. In the second part (sections IV–VII) is how Deleuze's further theory of multiplicities – within his attempt to overcome Leibniz's closed monadological universe towards an open nomadological 'chaosmos' – implies an intimate confrontation with important twentieth-century evolvements of Riemannian geometry, which can be precisely retraced in Weyl's work and subsequent developments, such as fibre bundle topology and gauge field theory. Finally, these developments allow us to highlight how Deleuze's ultimate philosophical elaboration of Riemann's thought finds a profound correlation between the concept of 'plane of immanence' and the physico-mathematical space of fundamental interactions described by contemporary science.

I. Riemann between Mathematics *and* Physics

The passage in *Bergsonism* opens with Deleuze's presentation of Riemann as a *physicist* and *mathematician*. This characterisation is less obvious than it might seem at first. The importance of Riemann's figure and work has always been primarily recognised for its fundamental

contributions to mathematics (real and complex analysis, number theory, differential geometry, topology) while it is only after its employment in Einstein's theory of general relativity that it has become an integral part of modern mathematical physics. Furthermore, the fact that Riemann was considered primarily as a physicist is far from being evident or even common. Deleuze's statement requires therefore a more attentive consideration.

To begin with, although Deleuze, throughout his work, sometimes refers to Riemann simply as a 'mathematician' (Deleuze and Guattari 1987: 482), it is again to 'physics and mathematics' that he relates what defines the 'Riemann abstract machine' (142). Likewise, although referring to Riemannian spaces or manifolds Deleuze links these to 'mathematics and physics' (Deleuze 1995: 30), elsewhere he reaffirms their relationship to 'the field of physics and mathematics' (Deleuze 2006a: 13). Finally – and even more meaningfully – Deleuze resorts to the compact expression 'the mathematician and physicist Riemann (*le mathématicien–physicien Riemann*)' (Deleuze and Guattari 1987: 32/46), and on the occasion of the translation into English of *Bergsonism*, writes that the term 'multiplicity' is of 'physico-mathematical (Riemann)' origin (Deleuze 2006b: 337). Deleuze's characterisation of Riemann in *Bergsonism* is thus not accidental; on the contrary, it reflects the Deleuzian reception and understanding of Riemann's work as such.

In fact, Deleuze grasps the core of Riemann's 'philosophy of science', that is, the fundamental interrelation between mathematics *and* physics.[4] As it was for Leibniz, Detlef Laugwitz argues, 'Riemann thought that mathematics and physics belong closely together' (Laugwitz 2008: 333). This strict relationship is confirmed by several viewpoints. Riemann devoted during his brief career and life constant and active research to many topics of mathematical physics, such as electrochemistry, electromagnetism, theory of heat, and hydrodynamics. A large part of his writing and teaching activities was focused on the methods of applying differential calculus to physics, in the lineage of Leibniz and Newton and of his great masters Gauss, Dirichlet and Weber, as well as the background of the French tradition of Laplace, Fourier, Poisson and Cauchy. Moreover, several of his writings attest to his attempts to elaborate a unified theory of physical forces, such as gravity, electricity, magnetism and light.

But the more important corroboration of Riemann's physico-mathematical conception resides in the 1854 Habilitation lecture itself. Since it is in this context, as mentioned, that Riemann first

introduces the concept of (*n*-dimensional) manifold, this point is of decisive importance. In fact, if Riemann's lecture establishes the new *mathematical* foundations of modern geometry (as defined by topology or, as it was called at the time, 'analysis situs' and differential geometry), its focal point lies nevertheless in the issue of the *physical* applications of the concept of manifold. In other words, what is at stake in Riemann's lecture is the relationship between *geometry* and *physics*.

Indeed, the profound originality of Riemann's new approach to geometry derives from the fact that, having determined the mathematical condition for the construction of infinitely many spaces, or *n*-dimensional manifolds, he demonstrates the possibility to construct *infinitely many* geometries (see Gray 2010: 201), thus going 'far beyond the intellectual scheme of both Euclidean and non-Euclidean geometry' (Laugwitz 2008: 225). The so-called non-Euclidean geometries are in fact only special cases of Riemann's differential geometry. However, the possibility to construct infinitely many spaces and geometries imposes the most compelling *physical* question of what is the proper geometry of the real space of phenomena. What ultimately interested Riemann, in other words, was the problem of the geometry of *physical space*. As Erhard Scholz stresses, 'The *main goal* of Riemann's inaugural lecture ... was a *reformulation of the conceptual foundations of physical geometry*' (Scholz 1992: 29; emphasis in original). Consequently, the problem that Riemann's theory of manifolds points out is, once again, the relationship between mathematics and physics.

This relationship constitutes the fundamental epistemological approach of the entire so-called *Göttingen Tradition* (see Mehra 2001). Within this there is a common orientation of thought leading from Riemann to Felix Klein, David Hilbert, Hermann Minkowski and, not least, precisely to Hermann Weyl. 'No mathematician of the first half of the 20th century', writes Laugwitz, 'cared as much as Weyl for greater depth of thought in analysis, geometry, mathematical physics, and philosophy' (Laugwitz 2008: 274). And in fact, it is Weyl who introduced Riemann's theory into twentieth-century mathematics, both from the purely mathematical (topological) point of view (Riemann surfaces) and for the physico-mathematical implications of Riemannian geometry. It is not surprising then if Weyl, in *Space, Time, Matter*, presents Riemann's ideas as the 'prophetical' (Weyl 1952: 102/88) anticipation of modern mathematical physics, as proven by Einstein's general theory of relativity.

It is in this perspective that Deleuze emphasises the dependence of Einstein's theory on Riemann's theory of manifolds, suggesting

furthermore a hypothesis to found on this basis a possible confrontation between Einstein and Bergson (see Deleuze 1988: 39). If for Deleuze, therefore, Riemann is *inseparably* a physicist *and* a mathematician, as well as his theory of spaces or manifolds is *inseparably* both physical *and* mathematical, it is because of the fact that within the Riemann–Weyl lineage Riemann's radically new conception was acknowledged in its inherent *physico-mathematical* correlation.

II. 'Thing' and Multiplicity: From Riemann to Deleuze

For Riemann, Deleuze states, 'things' are multiplicities. What defines them as such are the parameters (dimensions, independent variables) which they are determined by or depend on. With this general characterisation, Deleuze seems first of all to focalise Riemann's key idea, namely the irreducibility of the manifold concept to any specific domain of application, from which derives its power of determination. In other words, according to Deleuze, any 'object' is, for Riemann, determinable as a multiplicity, regardless of it being a geometric entity, an empirical object, a physical phenomenon or a state of things. The way in which Weyl presents Riemannian manifolds might confirm that Deleuze may have had something similar in mind.

In introducing the principles underpinning Riemann's 'infinitesimal geometry' (differential geometry) Weyl exemplifies the varied use of the concept of manifold from mathematics to the natural sciences (see Weyl 1952: 84/72–3). These examples show that the concept of manifold allows for the definition of geometric entities (e.g. a three-dimensional space), phenomena or states of things (the state space of an ideal gas, for instance), sensible objects (such as pure sounds and colours), the positions of a body, or a physical system (for example, the configuration space of a rigid body or the phase space of a mechanical system). Weyl explains that these mathematical and/or physical 'spaces' as n-dimensional manifolds, are precisely defined by the values of the parameters that determine them, that is, the n coordinates or variables they depend on and which characterise their n dimensions. For example, the state space of an ideal gas is a two-dimensional manifold because it can be determined 'by two independent variables, such as pressure and temperature'; 'colours form a three-dimensional manifold with respect to quality and intensity' and so on.

As I would suggest, some key aspects of Deleuze's further philosophical elaboration of the Idea as pure multiplicity in *Difference and Repetition* are founded on his interpretation of Riemann–Weyl,

notwithstanding the reading of Albert Lautman's works remains, as known, his primary source. In turn, such aspects can clarify Deleuze's claim in the passage of *Bergsonism* that is being analysed. In endorsing the 'Riemannian usage of the word "multiplicity"', in *Difference and Repetition* Deleuze writes: 'Everything (*chaque chose*) is a multiplicity in so far as it incarnates an Idea' (Deleuze 1994: 182/236). This state shows the double meaning of Deleuze's appropriation of Riemann's thought.

First, the concept of multiplicity (as substantive) is raised to the transcendental field of virtual Ideas, gaining its autonomy and irreducibility. However, at the same time it is inseparably related to the empirical determination (or actualisation, as a process of differentiation and genesis) of 'real [spatio-temporal] relations and actual terms' (Deleuze 1994: 183), thus implying its fundamental *immanence*. Weyl's description of Riemann's manifolds exemplifies in the field of mathematics and physics this sense of irreducibility *and* complementarity of the manifold concept. As well as for Deleuze the Idea is a *concrete universal*, a Riemannian manifold is a physico-mathematical Idea, irreducible to, while remaining inseparable from, any empirical effectuation.

Second, the Idea, for Deleuze, as a substantive (virtual) multiplicity, and the immanent production of real (actual) objects, involves the complex process of determination of the elements (dx, dy) of the multiplicity (principle of determinability), of their differential relations (dy/dx) or their degrees of variation (principle of reciprocal determination), and of the values of such relations (values of dy/dx) or degrees (principle of complete determination), to which there corresponds a distribution of singular points (see Deleuze 1994: 171, 175). This process of determination, which shows Deleuze's philosophical–mathematical interpretation of the differential calculus in the constitution of the Idea and its internal (immanent) genetic power, is consonant with Weyl's examples. The n-dimensional physico-mathematical 'spaces' (as manifolds) imply the extension of the differential calculus to physics (as well as to geometry), through which actual objects, phenomena, phase transitions (in which singular or critical points play a crucial role) or states of things are determined.

Seen in this Riemann–Weyl perspective, some important aspects of the well-known Deleuzian definition of the Idea in *Difference and Repetition* thus emerge:

> An Idea is an n-dimensional, continuous, defined multiplicity. Colour – or rather, the Idea of colour – is a three-dimensional multiplicity. By dimensions,

> we mean the variables or co-ordinates upon which a phenomenon depends; by continuity, we mean the set of relations between changes in these variables – for example, a quadratic form of the differentials of the co-ordinates; by definition, we mean the elements reciprocally determined by these relations, elements which cannot change unless the multiplicity changes its order and its metric. (Deleuze 1994: 182–3)

The Weylian and Riemannian inspiration for this passage is unequivocal, as attested by the example of colour as a three-dimensional multiplicity[5] or the assertion that phenomena depend on the parameters (variables, coordinates) that determine the dimension of the corresponding multiplicity. As for the example concerning the continuity of a multiplicity, the reference is to the quadratic differential form (that is, the differential expression of the square of the distance *ds* or line element, between infinitesimally neighbouring points) that properly defines a *Riemannian manifold* (or space).[6]

Deleuze's claim in *Bergsonism* means therefore that every 'thing', 'object', 'space' or 'phenomenon' is a multiplicity as it *actualises* its *n*-dimensional continuous Idea. 'Things' are actual multiplicities (a colour, for instance) *dependent* on the virtual multiplicity (the Idea of colour) of differential relations (and singular points) that they embody. In turn, the virtual multiplicity remains *irreducible* to the actual state of things. 'Every «thing»', Deleuze would later affirm, 'is made up so (*Toute «chose» est ainsi faite*)' (Deleuze 2006b: 305/284–5; translation modified).

III. Discrete and Continuous Multiplicities: The *Problem of Space*

As I would like to show, Deleuze derives Riemann's distinction between discrete multiplicities and continuous multiplicities from a close reading of an excerpt from Weyl's *Space, Time, Matter*. Interestingly, such an excerpt consists of a text overlap between Weyl's explanation of the Riemannian distinction and a quotation from Riemann's Habilitation lecture. As we shall see, the meaning of this Riemann–Weyl intersection will be decisive for the Deleuzian elaboration.

Having highlighted that both Euclidean and non-Euclidean geometries are only special cases within Riemann's infinitely many geometries, Weyl stresses the crucial point of Riemann's Habilitation lecture, which lies, he states, in its *final part* (see Weyl 1952: 97/83). According to Weyl, the core issue arising from Riemann's conception – an issue that could only emerge properly and find adequate response with the theory

of general relativity – lies in the connection between the (infinitesimal) geometry of space and the *physical world*. More precisely, once all of the mathematical and geometrical properties of a Riemannian space have been locally and intrinsically determined (that is, in the infinitesimal distance between points and without the assumption of a higher-dimensional embedding space), the question is of its compatibility with the 'real' space. However, as stated previously, as Riemann outlines the conditions for constructing infinitely many spaces and geometries, the issue becomes the relationship between *geometry* and *physics* itself, and hence concerns the very nature of the *physical space*. For Weyl, this is the *problem of space*, which represents Riemann's ultimate question.

It is with the purpose of explaining such a crucial point that Weyl introduces the Riemannian distinction between *discrete manifolds* and *continuous manifolds*:

> I must begin by remarking that Riemann contrasts discrete manifolds, i.e. those composed of single isolated elements, with continuous manifolds. The measure of every part of such a discrete manifold is determined by the number of elements belonging to it (*La mesure d'une partie quelconque d'une multiplicité discrète est donnée par le nombre des éléments qu'elle contient*). Hence, as Riemann expresses it, a discrete manifold has the principle of its metrical relations in itself (*une multiplicité discrète porte le principe de sa métrique*), a priori, as a consequence of the concept of the number. In Riemann's own words: 'The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metrical relations of space. In this question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifold, the principle or character of its metric relations is already given in the notion of the manifold, whereas in a continuous manifold this ground has to be found elsewhere, i.e. has to come from outside (*dans une variété continue, ce principe doit venir d'ailleurs*). Either, therefore, the reality which underlies space must form a discrete manifold, or we must seek the ground of its metric relations (measure-conditions) outside it, in binding forces which act upon it (*dans les forces de liaison qui agissent en lui*). A decisive answer to these questions can be obtained only by starting from the conception of phenomena which has hitherto been justified by experience, to which Newton laid the foundation, and then making in this conception the successive changes required by facts which admit of no explanation on the old theory … This carries us over into the sphere of another science, that of physics.' (Weyl 1952: 97/83–4)

Although Weyl points out that quantum mechanics might force us to assume that the answer to the problem of space could be the first hypothesis, i.e. that physical space constitutes a *discrete manifold*, he

regards the second – that physical space forms a *continuous manifold* – as more consistent with Riemann's thought (and with the space–time continuum of General Relativity). He concludes as follows (see also Plotnitsky 2009: 203):

> Riemann rejects the opinion that had prevailed up to his own time, namely, that the metrical structure of space was fixed and inherently independent of the physical phenomena for which it serves as a background (*indépendante des phénomènes physiques qui se déroulent dans son sein*), and that the real content takes possession of it as with residential flats. He asserts, on the contrary, that space in itself is nothing more than a three-dimensional manifold devoid of all form (*une multiplicité tridimensionnelle amorphe*); and that it acquires a definite form only through the advent of the material content filling it and determining its metric relations. (Weyl 1952: 98/84)

Now, compared with the passage in *Bergsonism*, it is evident that Deleuze summarises the Riemannian distinction by extrapolating nearly literally from this lengthy excerpt of Weyl, in which, it is noteworthy, is inserted a quotation of Riemann. This is one of Riemann's most celebrated quotations, taken from the *concluding part* of his Habilitation lecture (see Riemann 2007: 661). The synthesis devised by Deleuze, derives therefore from this intersection between Weyl and Riemann, from which results – as proof of the arguments outlined above – the very Deleuzian understanding of Riemann's thought.

Indeed, while devoting the first and second part of his lecture to the definition of the mathematical concept of *n*-dimensional manifold and to the local and intrinsic differential geometry which it is possible to construct upon it, Riemann dedicates the third and last part – from which Weyl quotes – to addressing the issue of its *applications to space* (see Riemann 2007: 659): which geometry of space is consistent with the empirical world? It is in relation to this question that Riemann's distinction between discrete and continuous manifolds – corresponding to the distinction between numerical or quantitative magnitudes and spatial or topological magnitudes –[7] gains its decisive relevance. Contrary to the Newtonian absolute space (and time) and the Euclidean geometry that founds it, Riemann's reply is that the geometry of physical space can not be determined a priori. The 'real' space is just a special case of manifold (three-dimensional only by 'hypothesis'), which is purely informal (topological) in itself ('devoid of all form', as Weyl says),[8] before any metric determination (that is, before the introduction of the ds^2 form). Therefore, in the passage quoted by Weyl, Riemann implies that, as far as the metric relations of space with respect to the

validity of differential geometry are concerned, the problem remains within *geometry* and is hence un-decidable. From the metric properties determined on a manifold, it is indeed impossible to derive a priori *which* manifold corresponds to the physical space. As a consequence, the geometry of space can only be determined in relation to *physical forces*. The question about the discrete or continuous nature of the physical space must hence necessarily be addressed to another science, namely, to *physics*. As a result, space (and time) ceases to be axiomatically presumed as the homogeneous (Euclidean), static (invariable) and matter-indifferent (absolute) arena in which material bodies would be placed; on the contrary, it becomes the dynamic (variable) expression of the fundamental *physical interactions* themselves.

Through his Riemann–Weyl synthesis in *Bergsonism*, Deleuze thus grasps the double meaning of Riemann's physico-mathematical theory: the pure informal (topological) nature and autonomy of the Idea of multiplicity, and its essential correlation with the geometry of space (time) and physical forces. Developing his rhizomatic multiplicities and smooth spaces in *A Thousand Plateaus*, Deleuze reaffirms this twofold character, precisely resuming Riemann and Weyl. Like a Riemannian space (or rather, its underlying topological space), a smooth space is 'an *amorphous*, nonformal (*informel*) space' (Deleuze and Guattari 1987: 477/595). At the same time, 'The metrical principle' of rhizomatic multiplicities 'is not to be found in a homogeneous milieu but resides elsewhere, in forces at work within them, in the physical phenomena inhabiting them (*dans les forces qui agissent en elles, dans les phénomènes physiques qui les occupent*)' (31/44).

IV. Riemann's Revolution in the Lineage of Leibniz

Through the idea of *n*-dimensional (continuous) manifold, then, Riemann found both the *mathematical space* and the *physical space*, stating also their necessary interrelation. This was the starting point of the process of *geometrisation of physics* that will be fulfilled by Einstein's theory, and that is at the core of contemporary mathematical physics, especially within what has been called 'the gauge field programme' for fundamental interactions (see Cao 1997: ch. 9–11). The Riemann–Weyl lineage allows us to bring out some important aspects of such developments and show Deleuze's philosophical elaboration of them.

For Riemann, writes Laugwitz, 'the true laws of nature are inherent in the infinitely small' (Laugwitz 2008: 261). This claim, for our purposes,

has two interrelated physico-mathematical meanings, which reveal Riemann's deep relationship to Leibniz.[9] First, it means that physical laws are fundamentally expressible through differential equations, that is, through the differential calculus, of which Leibniz (along with Newton) is, as is known, the founder. For Riemann, in other words, the true mathematics for physics is indisputably the differential calculus. Analysis (complex and real) has for him the pre-eminence in all mathematics, and even his concept of manifold, hence his 'analysis situs' and differential geometry, emerge as an extension of the calculus. Thus, according to Riemann, natural laws must be searched in a strict interrelation between analysis, geometry and physics, and therefore inherently described by means of differential equations.

Second, the claim above implies the idea, already underlined by Leibniz, that natural forces can only act by continuous action; that is, by local (infinitesimal) contact, and by their progressive transmission through the surrounding space. For this reason, as Riemann specifies, natural laws are 'in general partial differential equations' (cited in Laugwitz 2008: 260), since they involve a change in physical quantities dependent on several variables, and not merely one, as with ordinary differential equations of classical mechanics (of which Newton's laws of motion are the model). This viewpoint proves that, in contrast to the idea assumed in Newtonian mechanics of action at a distance between bodies (particles or material points), namely, the idea of forces acting instantaneously regardless of the distance, Riemann conceived of mathematical physics as a *field physics* (see Laugwitz 2008: 257), although he did not arrive at the concept of the field itself, which was introduced later by Faraday and Maxwell.

Now, both aspects of Riemann's view of natural laws in the lineage of Leibniz are crucial to Deleuze's philosophical elaboration. With regard to the first, it is at this point that we encounter Deleuze's second (and otherwise unique) explicit mention of Weyl in his later works. In *The Fold. Leibniz and the Baroque* Deleuze ascribes to Weyl the statement for which 'a law of Nature is necessarily a differential equation' (Deleuze 1993: 47). However, as has just been seen, such a formulation can be traced back precisely to Riemann, revealing thus its physico-mathematical sense. Indeed, in one of his seminars on Leibniz, Deleuze stated that 'one cannot understand anything about infinitesimal analysis if one does not see that all physical equations are by nature differential equations' (Deleuze 1980b). Therefore, as I would suggest, it is again within the Riemann–Weyl lineage that Deleuze maintains that the differential calculus (through the differential relation, dy/dx) is

what 'made possible this kind of co-penetration of physical reality and mathematical calculus' (Deleuze 1980a). Consequently, what Deleuze seems to rigorously grasp from Riemann and Weyl is that natural laws are 'necessarily' conceived as differential equations, and thus as inherently infinitesimal (differential), because in the lineage of Leibniz modern analysis, geometry and physics have become the differential 'sufficient reason' of mathematical physics, that is to say, of the physico-mathematical explanation of Nature.

This leads us to the second aspect of Riemann's conception of the laws of nature. As remarked by Weyl himself (see Weyl 1952: 66), Riemann's affinity with Leibniz's continuous and infinitesimal vision implies a clear opposition to the Newtonian mechanical model of physical interactions and, conversely, a profound proximity to the *field theories* of Faraday and Maxwell: 'The principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts is the mainspring of the theory of knowledge in infinitesimal physics (*la physique des actions de contact*) as in Riemann's geometry' (Weyl 1952: 92/79). In this sense, continues Weyl, 'The transition from Euclidean geometry to that of Riemann is founded in principle on the same idea as that which led from physics based on action at a distance to physics based on infinitely near action' (91). In other words, whereas Euclidean geometry and Newtonian mechanics are based on global axioms and instantaneous interaction at a distance (as in Newton's law of universal gravitation), *Riemannian geometry* and the *field physics* of Faraday and Maxwell are based on the local, intrinsic construction of space and on progressive contact interactions, propagating through space (as in Faraday's 'lines of force', or rather, field lines, and in Maxwell's equations for electromagnetism).[10]

Riemann's rejection of action at a distance in favour of a field–theoretical approach in the lineage of Leibniz is thus evidence that the problem of interactions (forces) is shifted from bodies, and their bi-univocal relation (see Deleuze and Guattari 1987: 370), to the continuous space (field) *between* bodies. This fundamental displacement discloses the 'continuous–discontinuous' dichotomy, dating back in its basic outline to Leibniz and Newton, which characterised nineteenth-century science, as well as its reductionist tendencies (see Deleuze and Guattari 1987: 370), becoming finally, as is known, one of the main problems of twentieth-century physics after Einstein's quantum revolution: namely, the wave–particle duality. Ultimately, with his theory of continuous manifolds, Riemann was looking to overcome the Euclidean–Newtonian framework of space, time and matter, and to

provide an entirely new description of the physical space of fundamental interactions.

V. Beyond Leibniz: Riemann's Demand of Opening

Despite the profound affinity between Leibniz and Riemann, there is nevertheless a crucial point of divergence. In *A Thousand Plateaus* and *The Fold*, Deleuze argues that Riemann's conception, as drawn and developed in contemporary mathematics and physics, implies a fundamental change of Leibniz's thought, which is consonant with the demand of contemporary philosophy (by Whitehead, for instance, but even more so in Deleuze's own philosophy) to overcome Leibniz's 'monadological' metaphysics. In order to underline Riemann's contribution to these *neo*-Leibnizian tendencies, and thus to show Riemann's detachment from Leibniz, Deleuze draws heavily on the mathematician and philosopher Gilles Châtelet (1944–99). To comprehend Deleuze's ultimate philosophical elaboration of Riemann's ideas, it is thus necessary to consider some aspects of Châtelet's writings.

In his paper entitled *Sur une petite phrase de Riemann*,[11] Châtelet focalises the critical point posed by Riemann in the conclusion of his Habilitation lecture, of which we have already seen the relevance for Weyl and Deleuze.[12] Interweaving the 'continuous–discontinuous' dichotomy implied in Riemann's text with Leibniz's concept of monad, Châtelet observes:

> How to found the metric of physical space? If this space were discrete, a natural class of metrics would be imposed immediately because the discrete tolerate only a limited proximity for each of the elements. But space is given as a *continuous* manifold in mathematical physics, *if one admits the possibility of communication between monads*, communication provided by the transmission of signals that follow the path of greatest proximity. (Châtelet 2010: 91; emphasis in original on the word 'continuous')

With this argument, Châtelet underlines a decisive problem. Once accepted the Riemannian hypothesis for which physical space is a continuous manifold indiscernible from physical phenomena, it becomes necessary to admit local *interactions* (or 'communication', as Châtelet terms it), which can be transmitted through space. Châtelet has here clearly in mind Maxwell's and Einstein's theories. 'General relativity', he writes, indeed provides 'an answer to Riemann's question' (Châtelet 2010: 92).

But the problem lies elsewhere. As we have seen, the admission of local interactions corresponds to Riemann's field–theoretical approach to mathematical physics. This entails the idea of action by contact, contrary to the idea of action at a distance assumed in Newtonian mechanics. Nevertheless, and here is the point, to admit local interactions involves at the same time a criticism addressed to one of the fundamental conditions of Leibniz's monadology: the condition of absolute *closure* which defines the monads (see Deleuze 1993: 22–26, 81–2). Contrary to Newton (or more prudently, to the Newtonians), Leibniz advocated the idea of action by contact, as something expressing the physical mechanism of *extrinsic* movement. However, since each monad, in its closure, integrally includes the totality of the world, all action must necessarily be *internal* to the monad. Consequently, in Leibniz's metaphysics there is by no means, in the narrow sense, a proper idea of *interaction* between monads.

It follows that Riemann's hypothesis, regained and proven in modern physics, compels the rejection of the condition of *closure* which defines Leibnizian monads, demanding on the contrary a condition of *opening*. This 'openness' implies, according to Châtelet, 'communication' and 'coexistence', founded in the infinitesimal (differential) proximity of interactions. In other words, and in terms later resumed by Deleuze (see Deleuze and Guattari 1987: 485, 493, 574), 'differentiation', Châtelet writes, 'propagates step-by-step (*de proche en proche*) in this community of monads which involve *tactile relations* described by a kind of tension which might be called *connection*' (Châtelet 2010: 90; emphases in original). Now, without entering into technical details, it is nevertheless important to understand the origin and meaning of the term 'connection' as used here by Châtelet for describing interactions.

VI. The Genesis of Fibre Bundle Topology and Gauge Field Theory

In his paper on Riemann, Châtelet employs the modern mathematical and physical notion of 'connection' in its current understanding – as is meant in the mathematical *theory of fibre bundles* (bundle connection) and in physical *gauge field theories* (gauge connection).[13] Despite the independent development of these theories (at least to a certain extent, as we will see), I wish to first of all emphasise that their historical genesis can be traced back precisely to the works of Weyl, and specifically, to his extensions of Riemannian geometry and Einstein's general theory of relativity.

The original notion of 'connection', first introduced in 1917 by Levi-Civita in his work on the parallel transport of vectors in Riemannian geometry, was further developed by Weyl – and also in the fundamental works of the French mathematician Élie Cartan (1869–1951) – in his 'purely infinitesimal geometry' (see Scholz 2001). Weyl started by liberating Levi-Civita's notion of parallel transport from its dependence on a metric, and the embedding of the Riemannian manifold on which parallelism is defined into an external Euclidean tangent space. Levi-Civita's parallelism was, in fact, an *extrinsic* notion, in conflict with Riemann's conception of geometry. Indeed, as underlined by Lautman, one of the decisive aspects of Riemannian geometry is that it is an *intrinsic* differential geometry. The space (as manifold) and its properties (curvature, for instance) are defined without any reference to an external ambient space, a 'universal container' (Lautman 2011: 112), as Newton's absolute space. Following Cartan, Lautman shows how Levi-Civita's notion of parallelism (which Cartan called 'Euclidean connection') on Riemannian manifolds entailed instead the introduction of the extrinsic and external viewpoint considering the manifold 'as embedded in a Euclidean space to a sufficient number of dimensions' (113). Indeed, Lautman points out, 'the exterior tangent planes, the projections and the rotations implied by the parallelism of Levi-Civita only make sense with respect to the space in which the manifold is embedded' (115).

Weyl is credited with having achieved the first intrinsic formulation of parallelism by elaborating the concept of 'affine connection' (now called 'linear connection') on (differentiable) manifolds, subsequently extended to the 'projective' and 'conformal' cases. The concept of 'affine connection' was conceived for the mathematical (geometrical) description of (continuous) infinitesimal displacements of vectors in tangent (affine) spaces over manifolds. However, in Weyl's main intentions, such a concept had first of all a physical meaning within the framework of his attempt to formulate a *unified field theory* of the fundamental interactions known at the time (gravity and electromagnetism), thus also incorporating Einstein's theory of general relativity.[14]

In this context, according to Weyl, for reaching a 'purely' local and intrinsic infinitesimal geometry it was first necessary to remove from Riemannian geometry the presence of what he understood as a last vestige of the Euclidean 'geometry at a distance' (Weyl 1952: 102), something also present in Einstein's theory, namely, the possibility of a direct comparison of lengths (specifically, the length of vectors) located

at distant points. The most important consequence of Weyl's approach was that it involved a path-dependence of spatial and temporal lengths; that is, a local change in the length scale from point to point of space–time. This locality entailed a *relativity* of the standard of length (or unit of scale), and therefore the existence of variable scale factors. This is what Weyl expressed by introducing the idea of a *gauge system* (see Cao 1997: 105–6, 271–73), for which he required a necessary *gauge invariance*, that is, an independence with respect to any particular choice of unit length (or gauge). Thereby, Weyl's unified field theory represents the first formulation of a *local gauge theory*.

Nevertheless, although Weyl's theory was applauded at first, above all by Einstein himself, the latter addressed soon thereafter some cogent criticisms, showing that this was in conflict with physical observations, hence revealing its failure. Towards the end of the 1920s, Weyl's further research, partly solicited by the discovery of two new fundamental interactions (the weak and the strong), convinced him to utilise the gauge idea in the context of quantum mechanics, although in a suitably modified manner (precisely, not as a scale factor but as a phase factor of the wave function of quantum particles). This reformulation is the foundation of the modern meaning of the *gauge principle* (see Cao 1997: 271–5, 332–8), which stimulated new programmes of unification for the fundamental interactions within the framework of *quantum field theory*. Starting from these developments, a decisive step was achieved in the 1950s by the elaboration of modern *gauge field theories* (the so-called Yang–Mills' theories) that constitute today the core of the Standard Model of elementary particle physics.

From the mathematical point of view, starting from the 1930s, Weyl's (and Cartan's) general concept of 'connection' on manifolds was extended and reworked in the theory of *topological fibred spaces*, thanks to the major contributions to algebraic and differential topology provided by mathematicians like Hopf, Seifert, Whitney, Serre and Ehresmann.[15] The last had by the mid-1940s introduced the concept of 'infinitesimal connection' on fibred spaces, leading to the first 'global theory of connections'. Finally, Norman Steenrod published in 1951 the first monograph on the topology of fibre bundles (see Steenrod 1951), providing what has since become the modern meaning of this field of mathematics.[16]

Intuitively, a *fibre bundle* can be thought as a kind of 'enlarged' topological space composed by a *base space B* (an *n*-dimensional manifold) and a *total space E* (or bundle space) 'above' the base space (generally, the total space is endowed with a group structure, in which

case the fibre bundle defines what is known as a *principal fibre bundle*). The total space is composed by the *fibres F*, in correspondence with each point x of the base space. The simplest example of a fibre bundle is the so-called *product space* (E being equal to the product of B with F), which is a *trivial* fibre bundle; for example, a cylinder, with a circle as a base space and a line segment as the fibre. A classical example of a *non-trivial* fibre bundle, in contrast, is the *Möbius strip*, which resembles the product space locally, but not globally (it is 'twisted'). It is important to note first that each fibre is in itself an m-dimensional manifold. Thus, a fibre bundle is not simply a multi-dimensional space as any Riemannian or other topological spaces generally are, but a $(n+m)$-dimensional space. Second, and most importantly, the total space is in contact with the base, through the *bundle projection* (a continuous map) – the third basic component of the fibre bundle – which acts from the total space E to the base space B ($\pi : E \rightarrow B$). This means that the total space is *extrinsic*, in Lautman's sense, to the base space, but *not* external to it. In other words, the total space is formally distinct from, and irreducible to, the base space, but this does not imply that the total space need be conceived as a higher-dimensional ambient space in which the base space would be embedded. On the contrary, each fibre of the total space is an *internal* space, constituting the internal dimensions of the fibre bundle, although they are the *extra* dimensions *of* it. Finally, a *bundle connection* is an additional geometric structure that permits to compare infinitesimally neighbouring fibres and to describe the paths connecting them.

The topology of fibre bundles marks a theoretical renewal of Riemannian geometry. However, it must be noted that during the 1950s, when fibre bundle theory and gauge field theory had achieved their modern formulation, such theories were completely *independent* and unrelated to each other despite their common origins (Riemannian geometry, Einstein's general relativity and unified field theories). For this reasons, what is more important to stress here is that in the mid-1970s the physicists T. T. Wu and C. N. Yang were astonished to realise that gauge theories could be reframed and adequately formulated precisely through the mathematics of fibre bundle topology; we shall return to this in more detail. Moreover, this profound relationship between physical theories and topological–geometrical structures allowed for the understanding of a similarity between gauge theories and the theory of general relativity itself (see Cao 1997: 334). Ultimately, these extraordinary results have finally led to a new and more profound understanding of the *interdependence* between mathematics and physics.

VII. The Contemporary Physico-Mathematical Space and Deleuze's Plane of Immanence

Precisely these developments underlie Châtelet's interpretation and Deleuze's ultimate philosophical elaboration of Riemann's thought. The interrelation between mathematics and physics envisaged by Riemann's differential geometry and field–theoretical approach to physics, is by Châtelet understood in deep continuity with contemporary mathematical physics, as evidenced by fibre bundle topology, gauge field theories and modern interpretations of general relativity.[17]

In his paper *Le potentiel démoniaque. Aspects philosophiques et physiques de la théorie de Jauge*,[18] Châtelet proposes an interpretation of gauge theory in its fibre bundle formulation, showing how it definitively breaks the 'idealised' model of Euclid–Newton's space, time and matter, establishing, on the contrary, the 'effective' space of what he calls, evocatively, '*fibre-monads*' (also 'fibre-particles'). The revolutionary aspect of gauge theory, Châtelet suggests, lies in the definitive abandonment of the 'pure' (i.e. absolute) *representation* of classical (pre-Riemannian and pre-Einsteinian) geometry and physics, affirming instead the inseparable 'coupling' of mathematical structures and physical reality. In this way, Châtelet writes in a crucial passage, 'through the simultaneous dissolution of the categories of «purely geometrical» and «purely physical», it [the gauge theory] establishes a *proper physico-mathematical space*' (Châtelet 2010: 95; emphasis in original). How, then, does the gauge theory of elementary particle physics and fibre bundle topology attain this 'space'?

'Each monad [i.e., particle]', Châtelet writes, 'testing a path in the universe B [i.e., in the base space of the fibre bundle] establishes a kind of synthesis associated to each path' (Châtelet 2010: 100). This extremely succinct passage – which, as we shall see, will be taken up by Deleuze – condenses some key aspects of how the topological–geometrical structure of fibre bundles actually operates within gauge theory. To understand this, let us begin by briefly summarising some of the basic features of gauge field theory.[19]

According to the Standard Model of particle physics, of which gauge theories represent the theoretical framework, elementary (quantum) particles are quanta indissolubly coupled to quantum fields. This means that all quantum particles are field quanta. Quantum fields are distinguished into two main types: (fermionic) *matter fields*, and their associated particles (fermions), such as electrons and quarks; (bosonic) *force fields*, and their associated particles (bosons), such as photons

and gluons. In gauge theory, force fields are known as *gauge fields*. Since these are the quantum fields that describe (non-gravitational) fundamental interactions, namely, electromagnetic, weak and strong forces, gauge fields are also defined as *interaction fields*. While fermions are actual quanta of matter (that is, the constituent of 'ordinary' matter), bosons are conceived as virtual quanta of force (interaction), and are called *gauge bosons*. Gauge bosons are virtual particles responsible for the transmission of interactions. For instance, photons are the gauge bosons that mediate electromagnetic interactions. In a nutshell, gauge theory can then be thought of as the quantum field theory of *matter* (matter fields) and *force* (force fields).

However, more important are the *interactions* between matter and force. Indeed, one of the key aspects revealed by quantum field theory is that in order for a quantum particle to exist, it must interact with some quantum field. In other words, a universe without interactions can not exist at all. Accordingly, since quantum particles are manifestations of interacting quantum fields, gauge theory is the theory that describes the dynamic processes of interaction between matter (matter fields) *and* force (force fields).

A quantum particle (as a quantum of matter field) interacting with a gauge field (force field) is affected by it. The resulting effect is observable as a change in the internal states of the particle (for example, a phase change in the wave function). What is the effective cause of this change? In classical field theories (Maxwell's theory of electromagnetism, for instance), changes were entirely ascribed to the influence of the (electromagnetic) field, or rather, its strength, that is, the intensity of the field. Mathematically, however, the problem is that for calculating the field strength one has to introduce some *potential* (for example, the electric scalar potential and the magnetic vector potential in the case of electromagnetism) in the form of potential functions, which allows to represent the 'tension' (to use Châtelet's expression in a previously quoted passage) of the field. The fundamental potentials in gauge theory are called *gauge potentials*. What is their more precise role and why are potentials problematic?

The point is that potentials have always had an ambiguous status within physics, beginning with their well-known 'arbitrariness'. The electric and magnetic potentials in Maxwell's equations, for instance, are invariant under a gauge transformation, that is, they leave the electric and magnetic fields completely unchanged. Thus, are the potentials actual physical entities, as fields are, or are they simply useful, but conventional and arbitrary mathematical devices permitting calculations

for deriving fields? For our purposes it is sufficient to highlight that during the last fifty years important experiments have in fact established the physical reality of potentials, stating the fact that they determine observable effects not entirely attributable to the field strength (the most important example here is the so-called Aharonov–Bohm effect).[20] In short, such experiments have demonstrated that the phase differences of quantum particles interacting with gauge fields are physically determined precisely by the gauge potentials. Therefore, they are not merely mathematical tools, but the field strength, or *gauge field strength*, is a quantity derived from them. In sum, gauge field potentials are the effective cause of changes in the internal states of matter fields.

What relationship can there be between such aspects of gauge field theory and fibre bundle topology? As previously mentioned, this is precisely what Wu and Yang proved in the mid-1970s. Fibre bundles provide the topological–geometrical structure that describes the dynamics of quantum field (matter fields and gauge fields) interactions. What Wu and Yang demonstrated was that *gauge fields* are *principal fibre bundles* (with a so-called Lie group as symmetry group), in which the base space is the space–time manifold, while the total space describes the dynamics of the internal states of interacting matter and force quantum fields (where force fields are principal fibre bundles and matter fields are so-called associated vector bundles). Each fibre at any space–time point thus corresponds to the internal (extra) dimension (the internal symmetry space) of quantum fields. In other words, the total space of the fibre bundle can be thought of as the 'space of phases' of a quantum particle moving through the base space (the space–time) and interacting with a quantum field (an electromagnetic field, for instance). As we have seen, phase differences are physically determined by the gauge potentials of the gauge field. In the fibre bundle formulation *gauge potentials* are indentified with the *bundle connection*, which describes the changes of the phase (amounting to a kind of 'twisting' of the fibres) along a path at different space–time points. The corresponding *gauge field strength* is then instead associated to the *curvature* of the connection. Because phase changes imply the curvature of the connection to be non-zero (that is, not 'flat'), the topology of the fibre bundle in gauge theories is, in general, 'twisted', which is to say, non-trivial.[21]

This is what encloses Châtelet's passage quoted above. The space–time paths of the 'monads' (particles) in the base space are correlated to the fibres in the internal (extra) dimensions of the total space (the 'space of phases'). This space, through the connection (the gauge potential) and its curvature (the gauge field strength), 'synthesises' the dynamics

of the fundamental interactions. The potential, associated with the connection, 'creates the quanta of the force field' (Châtelet 2010: 106), namely, the virtual particles (gauge bosons) of the interaction field (gauge field), while by the non-zero curvature of the connection 'the monad experiences the rebellion of space to every capture through a Cartesian grid (*quadrillage*)' (106), that is, through a trivial fibre bundle. The non-zero curvature indicates thus the non-trivial ('twisted') nature of space, or, as Châtelet stresses, the '*heterogeneity of space*' (105; emphasis in original). The fibre bundle formulation of gauge theory then describes, Châtelet suggests, the *open* space of the virtual 'potentialities' and 'intensities' of matter, where the 'monads' (particles) are necessarily coupled to the 'fibres' (quantum fields), which are themselves 'affected by the indomitable exteriority of the others' (100). In brief, the 'fibre-monads' are necessarily modified by interaction processes of matter and force. Ultimately, through the identification of the *connection* (a topological–geometrical notion) with the *potential* (finally an experimentally stated physical entity), the mathematical physics of fibre bundle topology and gauge field theory fulfils the condition of *opening* originally demanded by Riemann, and attains the *neo*-Leibnizian perspective of the *contemporary physico-mathematical space*.

This is exactly what Deleuze derives from Châtelet, condensing in a short but illuminating passage in *The Fold* what he defines as 'the new monadology in mathematics since Riemann' (Deleuze 1993: 154, n.16):

> Modern mathematics has been able to develop a fibered conception according to which 'monads' test the paths in the universe and enter in syntheses associated with each path. It is a world of captures instead of closures. (Deleuze 1993: 81)

As should be evident, this passage paraphrases that by Châtelet which we have analysed above, thus revealing which mathematical and physical implications it makes implicit reference to.[22] What is important for Deleuze here is that such aspects of contemporary science enable him to point out one of the major limits of Leibniz's metaphysics, that is, the condition of absolute closure which grounds his monadology, and therefore to show the difference with what he defines as the neo-Leibnizianism which characterises contemporary philosophy. Indeed, contrary to Leibniz's monadic closure, and to the requisites of convergence and compossibility, the contemporary universe, Deleuze argues, is a 'chaosmos', an *open* world in becoming, in which 'Beings are pushed apart, kept open through divergent series

and incompossible totalities that pull them outside' (Deleuze 1993: 81). However, this outside, as we shall see, is not an external, supplementary, dimension, but the intrinsic opening to the field of interactions, to 'changing captures' (81), in which 'the monads penetrate each other, are modified' (137).

As a consequence of this affirmation of openness and interaction in accordance with contemporary science, Leibniz's 'monadological' metaphysics, as Deleuze had underlined already in *A Thousand Plateaus*, must be replaced by a new 'nomadological' perspective (see also Plotnitsky 2009: 206):

> Although the 'monads' are no longer thought to be closed upon themselves, and are postulated to entertain direct, step-by-step (*de proche en proche*) local relations, the purely monadological point of view proves inadequate and should be superseded by a 'nomadology' (the ideality of striated space versus the realism of smooth space). (Deleuze and Guattari 1987: 574, n.27/616, n.26)

The classical 'ideality' of Euclid–Newton's space, time and matter, and *also* the conditions of closure, convergence and compossibility of Leibniz's monadology, are thus rejected in favour of the nomadological *realism* of the contemporary physico-mathematical space.

Deleuze's philosophical elaboration in his late works finds therefore a deep correlation with contemporary mathematical physics. Deleuze's theory of rhizomatic multiplicities and smooth spaces discloses the field of *interactions*, and thus the *connection* and *heterogeneity* of multiplicities (see Deleuze and Guattari 1987: 7). Deleuze's famous description of a Riemannian space as 'pure patchwork' finds here its original sense: an heterogeneous, 'amorphous collection' of infinitesimally neighbouring 'patches' of space, the connections of which are not predetermined, and 'do not imply an ambient space in which the multiplicity would be immersed' (493). Like the internal (extra) dimensions of fibred spaces in gauge theory that 'capture' fundamental interactions, Deleuze's field of interactions characterises the open space of 'extrinsic coexistence', the plane or 'milieu of exteriority' of connections, as 'extrinsic relations' (435, 353).

However, this extrinsic field of interactions and connections should not be confused with an external dimension that could encompass the multiplicities. On the contrary, just as the internal dimensions of a fibre bundle, as we have seen, are extrinsic without being external, Deleuze's multiplicities (as well as Riemannian manifolds) also have no supplementary dimensions of embedding (see Deleuze and

Guattari 1987: 9). For this reason, Deleuze, following Lautman, opposes the connection of Riemannian spaces ('accumulation'), or smooth spaces, with what he terms 'Euclidean conjunction' (486, 573, n.19). Indeed, the latter, Deleuze argues, consists in overlaying 'upon each point of smooth space a tangent Euclidean space endowed with a sufficient number of dimensions' and then in reintroducing 'parallelism between two vectors, treating multiplicity as though it were immersed in this homogeneous and striated space' (373, 556, n.39). The 'Euclidean conjunction' to which Deleuze refers corresponds therefore to nothing other than Levi-Civita's parallel transport of vectors on Riemannian manifolds (that is, an 'Euclidean connection' in Cartan's terminology), which, as we have seen, precisely implied an extrinsic and external viewpoint, opposed to Riemann's intrinsic differential geometry.[23] Thus, I would suggest, Deleuze's field of interactions (and connections) and the multiplicities might be then conceived of more adequately as a (non-trivial) fibre bundle, in which the base space would correspond to the multiplicities and the total space to the internal (extra) dimensions of interactions and connections, a virtual field of 'heterogeneous potentials and intensities' (Deleuze 1994: 50).

This entails that the condition of opening affirmed by Deleuze does not imply that the field of interactions (and connections) constitutes an external space with respect to the multiplicities, while defining a proper and irreducible 'milieu'. To better understand this point, let us return briefly to Deleuze's interpretation of Leibniz's monadic structure. Despite the condition of closure, Deleuze shows that even Leibniz's monad possesses 'a minimum of outside', however, as he states, 'a strictly complementary form of outside' (Deleuze 1993: 111). What is of interest to us is that, in elucidating this kind of complementarity, Deleuze refers to the topological property of 'unilaterality' of the monad, which implies 'a torsion of the world, an infinite fold, that can be unwrapped (*se déplier*) in conformity with the condition [of closure] only by recovering the other side, not as exterior to the monad, but as the exterior or outside *of* its own interiority' (111/149). The topology that Deleuze has in mind here is thus precisely that of a Möbius strip (see also Deleuze 1990: 20, n.10, 337), which, as we have seen, is itself a non-trivial fibre bundle. Indeed, in *The Logic of Sense* Deleuze, following Lautman, had already underlined that to grasp the one-sided character of the Möbius strip it is necessary to split it 'by unfolding it in its length (*en le dépliant dans sa longueur*), by untwisting it (*en le détordant*)' (Deleuze 1990: 20/31; translation modified). According to Lautman, this proves that the unilaterality is an *extrinsic* (or 'relational') property 'since, to

be realised, it is necessary to split the ring and untwist it, which implies a rotation around an axis exterior to the surface of the ring' (Lautman 2011: 117). For this reason, Deleuze maintains it is only by 'unfolding' the surface that 'the dimension of sense [i.e. the outside] appears of itself, in its irreducibility, and also in its genetic power' (Deleuze 1990: 20). Nevertheless, it can be shown that the unilaterality is reducible to the Möbius strip's *intrinsic* property of being non-orientable, that is, 'folded' and 'twisted' (non-trivial) in itself (see Lautman 2011: 117–8). Therefore, the outside is strictly complementary (not exterior, or rather, not external) to the monad because, as with a Möbius strip, *the extrinsic property of unilaterality is equivalent to the intrinsic property of non-orientability*. At the same time, however, this equivalence does not eliminate the difference and heterogeneity (irreducibility) of the outside with respect to the monad.

Deleuze conceives of the relationship between the field of interactions (and connections) and the multiplicities, in the same manner. If there is between them a reciprocal irreducibility, which assures an extrinsic (outside) dimension of openness, at the same time, they also possess a strict complementarity that prevents any kind of transcendence. The field of interactions (and connections) corresponds then to what Deleuze calls the 'plane of consistency' of multiplicities, which defines precisely their internal exteriority; that is, their pure *immanence*. Indeed, the plane of consistency constitutes the 'outside of all multiplicities', whose dimensions 'increase with the number of connections that are made on it' (Deleuze and Guattari 1987: 9), while remaining a single, pure *plane of immanence*: the 'not-external outside and the not-internal inside', or simply, the 'absolute outside' (Deleuze and Guattari 1994: 59–60). Ultimately, the topology of Deleuze's plane of immanence, as field of interactions and connections, and the multiplicities which fill it, appears to be that of a kind of folded and twisted (non-trivial) fibre bundle, which finds its correlate in the physico-mathematical space of contemporary science.

Notes

1. The most important studies on Deleuze and Riemann are Plotnitsky 2006 and 2009. In general, for Deleuze's engagement with mathematics and science, see DeLanda 2002; Duffy 2006; Marks 2006.
2. All page references to Deleuze's works are made distinct by the use of the slash, referring first to the English version and secondly to the French. This is true in general of all authors quoted.

3. The French edition of 1922, as well as the English edition of 1952, correspond to the 1921 fourth enlarged German edition of Weyl's book, originally published in 1918. For a detailed analysis of Weyl's masterwork see Scholz and Coleman 2001.
4. In general, see Laugwitz 2008; specifically, Boi 1992; Scholz 1992; Monastyrsky 2008.
5. The example of colour, as a continuous manifold, is already present in Riemann's lecture (see Riemann 2007: 653). Deleuze's other source for this point (as in general for Riemann's theory) is certainly Jules Vuillemin, which refers to Helmholtz (see Vuillemin 1962: 409, n.2).
6. Deleuze's source for the differential quadratic form of a Riemannian space is Lautman (2011: 97–8), which depends on Cartan and Weyl. See Weyl 1952: 90.
7. For discussion on Riemann's distinction see, for example, Laugwitz 2008: 305–8, and in relation to Deleuze, Plotnitsky 2006.
8. The French term '*amorphe*' would become a typical Deleuzian characterisation of Riemannian manifolds, as well as of smooth spaces (see Deleuze and Guattari 1987: 476, 477, 485, 488). See also Lautman 2011: 98, and for an analysis, Plotnitsky 2006.
9. On Riemann and Leibniz see Laugwitz 2008, but also, in relation to Deleuze, see Plotnitsky 2009.
10. This correlation established by Weyl between Riemannian geometry and Faraday–Maxwell field theory is also analysed by Charles Alunni (2006), showing its relevance for Bachelard.
11. Originally published in 1979, this paper is now included in the new edition of Châtelet's works (see Châtelet 2010: 85–94). All quotations of Châtelet are my translations.
12. Note that the phrase by Riemann to which the 'Proustian' title of Châtelet's paper refers is precisely that which we encountered in Weyl, and that Deleuze synthesises in *Bergsonism*.
13. For an introduction to fibre bundles and bundle connections, see, for example, Isham 2001: ch. 5, 6; Penrose 2005: ch. 15. For an historical account see Bourguignon 1992.
14. For all these aspects see Coleman and Korté 2001. For an historical reconstruction of Weyl's theory in the framework of the attempts of unified field theories in the early twentieth century, see the excellent study of P. V. Vizgin 2011.
15. Note, *en passant*, that Ehresmann, the first great scholar of Cartan, was a close friend of Lautman. On the relevance of this friendship for Lautman's philosophy of mathematics, see the challenging remarks laid out by Fernando Zalamea in his introduction to Lautman's works (Lautman 2011: xxxv, 267, n.14).
16. For the historical developments of fibre bundle topology from the strict mathematical (algebraic and topological) viewpoint see Dieudonné 2009.
17. As Châtelet himself observes, 'modern presentations of general relativity by the mathematical physicists utilize always more the language of "fibred spaces" ("*fibrés*")' (Châtelet 2010: 290, n.15).
18. For indications about the various versions and published forms of this paper see Châtelet 2010: 302.
19. It is clearly beyond the scope of the present paper to provide a detailed discussion of gauge theory. For a detailed account see Zeidler 2011, and for philosophical analyses see Brading and Castellani 2003; Healey 2007.
20. Note that, in another important paper, Châtelet too, discusses the Aharonov–Bohm effect, considering it as a compelling example of the new

physico-mathematical space introduced by gauge theory (see Châtelet 2010: 109–31).

21. The fibre bundle is trivial in the Abelian case of quantum electrodynamics (in the absence of magnetic monopoles), while it is non-trivial in the more general non-Abelian Yang–Mills' theories for the weak and strong interactions.
22. Note that Deleuze always refers to Châtelet's paper on Riemann, although the passage cited here from *The Fold* is clearly extrapolated from the second paper by Châtelet that we have considered. This proves, given their relationship as colleagues and friends, that Deleuze must have read Châtelet's paper on gauge theory, at least in one of its versions.
23. Note Deleuze's subtle conceptual displacement between 'Euclidean connection' and 'Euclidean conjunction' for indicating Levi-Civita's parallelism, and the opposition between the latter and the proper 'connection' of Riemannian spaces ('accumulation'); see Deleuze and Guattari 1987: 486, 510.

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