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Intuitionist and Classical Dimensions of Hegel's Hybrid Logic

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Does Hegel's *The Science of Logic* (Hegel 2010) have *any* relation to or relevance for what is now known as "*the science of logic*"? Here a negative answer is as likely to be endorsed by many contemporary Hegel scholars as it is by mainstream analytic philosophers, but, of course, with different intent. The Hegel interpreters are likely to dismiss the relevance of the contemporary *science* of logic for philosophy, damning it as committed to what Hegel criticised as "the abstract understanding", *der bloss Verstand.* Mainstream analysts, in contrast, will use this to dismiss Hegel's philosophical claims, as analytic philosophy is widely seen as having revolutionised philosophy's method in the wake of the modernisation of the project of logic in the hands of Gottlob Frege and Bertrand Russell.

As this story is told from the analytic side, Gottlob Frege, a then relatively unknown German mathematician, had silently revolutionised logic in the last quarter of the nineteenth century, releasing it from it from the Aristotelian fetters within which it had been held for two thousand years. Frege's logic had the form of a universal language of concepts upon which the truths of arithmetic could be founded. In the hands of Russell and later analysts, the stakes involved in the elaboration of this "concept script" (*Begriffsschrift*) would be raised. Not only arithmetic but geometry would be shown to be grounded in it and, ultimately, the truths of all the empirical sciences and the claims of philosophy itself. Russell had learnt of Frege's work in 1900 at the *World Congress of Philosophy* and after this he had, with the help of his former mathematics teacher Alfred North Whitehead, thrown himself into a project of elaborating and promoting such a scientific logic. After ten years of labour, the first of the three volumes of *Principia Mathematica* would be born. In the telling of this story it is commonly noted that, prior to his conversion to Frege, Russell, like the majority of his philosophical Cambridge colleagues in the last decades of the nineteenth century, had practiced philosophy within a Kantian or Hegelian framework. Towards the end of the 1890s, however, under the influence of G. E. Moore, he was starting to find such an environment stifling and was becoming receptive to something new. This "new" would be the new logic, and from a combination of Frege's revolutionary logic and Moore's antagonism to idealism, "analytic philosophy" would come into existence.[1]

I suggest that it is wise to harbour suspicions about "revolutions", like this one, that hinge on years that end in two zeroes. In both mathematics and logic, for example, the nineteenth century had been a century of immense progress. In comparison, the mathematical advances on which Russell had drawn in elaborating Frege's *Begriffsschrift* had been comparatively narrow. The logic prized by "analytic" philosophy had, appropriately, grown out of the mathematical discipline of "analysis"—effectively a branch of mathematics that had developed in response to problems surrounding the *calculus* introduced in the

[1] I have sketched the relevance of this for the reception of Hegel in Redding 2007, Introduction.

seventeenth century by Newton and Leibniz. Cauchy, Weierstrass and others had sought to bypass difficulties with the notion of "infinitesimals" by giving rigorous definitions to intuitive notions like those of continuity and limit. Together with the eventual definition of the "real" numbers as well as Cantor's treatment of infinite sets, such notions had provided Russell with his mathematical starting point. However, analysis had been just *one* area of development in mathematics in the nineteenth century, with major developments having also taken place in both algebra and geometry, especially when combined in novel ways, as in disciplines such as projective geometry and linear algebra, which had both contributed to the framework within which physics would develop at the end of the century. Moreover, one of the first significant expressions of the characteristically "abstract" type of algebra in that century (Parshall 2008) had been the movement within mathematical logic that tends to be overlooked in the wake of the Frege–Russell "revolution"—the "algebra of logic" initiated in mid-century by the British mathematicians George Boole and Augustus de Morgan, in which algebra, now "disinterpreted" and so generalized beyond the domain of arithmetic, had been applied to the Aristotelian syllogistic.[2] While limited in its early forms, this tradition had been developed by others, especially Charles Sanders Peirce, in ways that, as has been argued by Hilary Putnam for example, paralleled the technical achievements of Frege (Putnam 1990).

Peirce, the son of a leading mathematician, had taken as relevant to logic advances in geometry and had opposed the movement of "arithmetization" central to developments in analysis informing the shape of the philosophical logic of analytic philosophy. Significantly, Peirce's approach to logical *categories* have been likened to Hegel's (Stern 2013) and Peirce himself had sensed parallels between Hegel's logical thought and the new geometry: "Most of what is true in Hegel is a darkling glimmer of a conception which the mathematicians had long before made pretty clear, and which recent researches have still further illustrated" (Peirce 1992, 296). Significantly, neither Whitehead nor Russell themselves had sprung from the earth as "analysts". Whitehead's first major book, *A Treatise on Universal Algebra* (Whitehead 1899), summed up the nineteenth-century developments of abstract algebra with a strong emphasis on geometry and the linear algebra or "vector analysis" of Hermann Grassmann. Russell, in his first work, *An Essay on the Foundations of Geometry* (Russell 1897), had drawn together Hegel's treatment of space in his *Philosophy of Nature* with the philosophical attitude Grassmann expressed towards these new forms of algebraic geometry.

The implicit idea of revolutionary discontinuity that accompanied the philosophical reception of *Principia Mathematica* quickly reduced the algebra of logic tradition to a historical footnote,[3] but as has been argued by others, the Frege–Russell tradition would become dependent on developments more continuous with the rival algebra of logic tradition (e.g., Brady 2000). In contrast, the idea that important progress had been made in the

[2] The idea of the "distinterpretation" of algebra to a purely syntactic calculus had been proposed by the Cambridge mathematician Duncan Gregory in 1840 (Ewald 1996, 443).
[3] For example, in the influential Kneale and Kneale 1962, Boole and his successors are afforded 30 pages in relation to over 300 devoted to Frege and his followers.

nineteenth century in a form of logic linked to *geometry* rather than arithmetic as favoured by the analysts had been asserted by Ernst Nagel (Nagel 1939). While the Hegelian/geometric framework sketched in Russell's early work was quickly abandoned, I want to examine the idea that Hegel's "logic" may indeed have had features that fitted these algebraic and geometric developments of the nineteenth century and that gives it features that bring it into relation with more recent logics, especially the logic of intuitionism, the mathematical movement started by the Dutch mathematician L. E. J. Brouwer, a critic of the logicism of Whitehead and Russell.[4] Hegel is, of course, known for his non-acknowledgement of the classical laws of logic, the law of non-contradiction and excluded middle. I will be arguing that, like intuitionistic logic, Hegel rejected the law of excluded middle as a type of logical *axiom* but allowed it to apply, along with the law of non-contradiction, in certain restricted contexts. These would reflect the somewhat "hybrid" nature of his logic that combined classical and intuitionistic features that would give to it a type of dynamism foreign to the Platonistic "classical" approach of Frege and Russell.[5]

1. Early Russell, Hegel and Geometry

Nicholas Griffin, in a study of early drafts of a paper published by Russell in 1901 (Griffin 2013), has noted that for a very short period after the encounter with Guiseppe Peano in 1900 when he learnt of the work of Frege, Russell had flirted with a logic based on the algebraic theory of *groups*. In the 1830s, group theory had arisen in relation to the theorisation of solutions to polynomial equations and in the 1870s Felix Klein had applied it to geometry so as to classify the various new forms of geometry that had developed during the nineteenth century. Griffin speculates:

> With the considerable advantages of hindsight, we can see now that group theory might have done as well as set theory. From the point of view of getting a clear conception of the basic ideas of mathematics, group theory has much to commend it, perhaps as much as set theory… [T]o someone coming to the project, as Russell was, from geometry, it could well seem that group theory had the advantage, since it had already considerable successes to its credit, especially in Klein's hands, whereas the effects of Cantorean set theory in geometry were problematic to say the least (Griffin 2013, 378).

Group theory would be replaced by modern set theory in Russell's project of seeking the logical *foundations* of mathematics, an occurrence Griffin puts down to various factors. One was the idea of *conceptual* analysis that Russell had come to adopt following Moore, in which analysis was understood as the "decomposition of complex terms (propositions) into

[4] Intuitionistic logic would be first developed by Brouwer's student Arend Heyting in the 1920s.
[5] The idea of a hybrid logic of this sort was suggested by Arthur Prior (Blackburn 2006. Blackburn emphasises Prior 2003, chs 11, 12, 16 and 19). Elsewhere (Redding 2017), I have drawn parallels between Prior's logic and Hegel's and the influence on Prior's approach by his teacher, the Hegelian John N. Findlay.

their simpler constituents" (Griffin 2013, 384). Group theory, especially applied to geometry, did not have this atomistic, decompositional nature.

Such a "decompositional" conception of analysis as described by the historian of analytic philosophy, Michael Beaney (Beaney 2007), had been starkly missing from Russell's first venture into foundational issues in mathematics in *An Essay on the Foundations of Geometry* in 1897. The form of geometry in question in this work was *projective* geometry, a type of geometry that had been proposed in the seventeenth century by Girard Desargues and a few others as a counter to the "analytic geometry" of Descartes (Andersen 2009, ch. 9). In analytic geometry, figures from Euclidean geometry could be reduced to variable-containing equations by the application of coordinates to the plane in which such figures were constructed. For example, plotted against the 'x' and 'y' coordinates, a circle of radius 1 unit with its centre at the origin of the co-ordinates could be represented by the equation $x^2 + y^2 = 1$. This meant that a line was essentially decomposable into a set of points—Descartes' "analytic" geometry clearly being prototypical of later forms of "decompositional" analysis (Beaney 2007, 2). In this way, Descartes had effectively reversed the approach found in Aristotle almost two thousand years before, in which a line was conceived as a *fundamentally* continuous magnitude, and a *point* was determined as the place where *two lines* intersected.

In contrast to Descartes' analytic geometry, Desargues' *projective* geometry was *non-metrical* and had developed from Renaissance theories of perspectival representation in painting (Andersen 2007, chs 1–5). Harold Coxeter has characterized projective geometry and its relation to this background as follows: "Plane projective geometry may be described as the study of geometrical properties that are unchanged by "central projection", which is essentially what happens when an artist draws a picture of a tiled floor on a vertical canvas. The square tiles cease to be square, as their sides and angles are distorted by foreshortening; but the lines remain straight, since they are sections (by the picture-plane) of the planes that join them to the artist's eye" (Coxeter 1987, 3). In short, for projective geometry, the "invariance" of crucial factors assumed within Euclidean geometry, such as the distance between two points, or the angle formed by two intersecting lines, no longer held. But if the geometer cannot rely on the invariance of, say, the right-angles defining a square or the uniform length of the radius of a circle, what *can* be relied upon? Something needs to remain invariant among projections for there to be geometrical *truths*. But while determinate distances or angles, considered as determinate *metrical relations* between a pair of points or lines respectively, were not preserved, certain "second-order" relations *among* such relations were. Crucial here was a peculiar double-ratio existing between two pairs of points on a line called the "cross-ratio". It turns out that if four *lines* are drawn through those points in a way such that those lines all meet at a point, O, lying outside the line, then that cross-ratio will be reproduced within *any* other line sectioning those four lines regardless of the *angle* at which it crosses them (see, e.g., Struik 2011, ch. 2-7).[6] That is, the cross-ratio is "projectively

[6] This is because the same cross-ratio relation holds among the *angles* formed by the lines intersecting at point O. This equivalence of the cross-ratio holding among four collinear points on the one hand and four concurrent lines on the other is an example of the duality between point and lines in projective geometry.

invariant". We will return to this phenomenon as it will turn out to have a peculiar significance for Hegel's logic.

While Desargues' geometry had been eclipsed by the success of Descartes' analytic geometry in the eighteenth century, it had been rediscovered and developed in France in the first decades of the nineteenth (Gray 2007). It was later taken up by German mathematicians, and in the second half of the century, had become linked to group theory by Klein in his "Erlangen Program". From there it would feed into the complex forms of geometry utilized by Einstein, for example, in his general theory of relativity.

Within projective geometry, the relation of point and line came to be conceived in a new way that had elements of both Descartes' and Aristotle's approaches. In projective geometry parallel lines could be conceived of as *meeting* at some infinitely distant point, as in a point on the horizon in a perspectival painting, and so *any* two lines (now including parallel ones) could be conceived as determining a single point by their intersection, just as any two points could be conceived as determining the single line joining them. This resulted in the idea of a type of primitive equivalence and interchangeability as holding between points and lines, captured by the notion of "duality" (Gowers 2008) that would come to play an important part in subsequent mathematics as well as algebraic logic.[7] Hegel would come to work with a model of judgment which paired interchangeable logical forms for a single content in a similar way.

Russell's 1897 essay had been written during a period in which he adhered to some kind of mixture of Kantian and Hegelian philosophical ideas common at Cambridge at the time. Thus, in the Preface of the *Essay*, while acknowledging his "chief obligation" as being to Felix Klein (Russell 1897, v),[8] the idealist dimensions of his approach are signalled in his acknowledgement of the approach to *logic* of Bradley and Bosanquet and the dedication to McTaggart. While there is only one reference to Hegel (ibid., 138 n.2), it is clear that in terms of ideas concerning the relation of geometry to space, his approach is distinctly Hegelian. This comes out especially in his criticism of *Kant's* ultimately "subjectivist" approach to the domain of geometry as that of "pure intuition".

Calling upon the mathematician Hermann Grassmann's "profound philosophical introduction" to the book which introduced linear algebra in the 1840s, Russell notes that he had suggested that "Geometry, although improperly regarded as pure, was really a branch of applied mathematics, since it dealt with a subject-matter not created, like number, by the intellect, but given to it, and therefore not wholly subject to its laws alone" (Russell 1897, 134). This "given" was space itself, *qua* medium of extended objects arranged within it—a view Grassmann shared with Hegel. Nevertheless, Grassmann had believed it possible "to

[7] Thus, for any theorem in projective geometry that concerns some complex relation (or relation of relations) among some structured array of *points*, an equivalent theorem could be found that applied in the same way to an analogously structured array of *lines*.

[8] He also acknowledges Whitehead here for his own becoming aware of the "philosophical importance of projective Geometry" (ibid).

construct a branch of pure mathematics, a science, that is, in which our object should be wholly a creature of the intellect, which should yet deal, as Geometry does, with extension—extension as conceived, however, not as empirically perceived in sensation or intuition" (ibid.). Grassmann had called this "Extension Theory" (*Ausdehnungslehre*) and indeed others have believed that Grassmann, a student at the University of Berlin during Hegel's period there, had been deeply influenced by Hegel's *Science of Logic* (Wolff 1999, Lawvere 1991).[9] Russell does not mention Hegel at this point, but the Hegelian dimension of the move that this effects with respect to *Kant*'s conception of geometry is clear. Kant's "distinction between the subjective and the objective" (Russell 1897, 135) or intuition and concept had, with Grassmann, effectively been moved *within* the sphere of the intellect to enable a notion of pure extension to be explored entirely conceptually.[10] Hegel had responded to Kant's "subjective idealism" in much the same way, incorporating the relation between what Kant took to be *between* concepts and the non-conceptual form of "intuitive" cognition *into* the realm of concepts.

The point to be emphasised here, however, is the rejection of Descartes' analytic reduction of the geometric continuum into discrete points that results from Russell's embrace of the "qualitative" projective geometry. Externality, Russell points out, is an essentially relative conception: "nothing can be external to itself. To be external to something is to be another with some relation to that thing" (Russell 1897, 136) and hence "a position can have no intrinsic quality" (ibid., 137). In extensive magnitudes what are summed are not points but *relations*: "our form contains neither elements nor totality, but only endless relations—the terms of these relations being excluded by our abstraction from the matter which fills our form" (ibid.). But these are relations without prior existing *relata*. "One of the strangest properties" of the form of externality is that the summing of those infinite "would-be elements" of which it is made up involves the *summation of relations.* While "to speak of dividing or adding relations may well sound absurd" it is nevertheless difficult "to find an expression which shall be less improper. The fact seems to be, that externality is not so much a relation as bare relativity, or the bare possibility of a relation" (ibid. 138). At this point Russell refers in a footnote to a paragraph from Hegel's *Philosophy of Nature* describing space, qua "primary or immediate determination of nature" as "the abstract *universality of its self-externality,* its unmediated indifference … space is simply *continuous*, and is devoid of any determinate difference" (Hegel 1970, § 254). But this is only the *immediate* determination of space, and two paragraphs later Hegel notes how the concept of space must

[9] Grassmann seems to have been much more directly influenced by the idealist philosopher and hermeneutic theorist Friedrich Schleiermacher, and through him, by Friedrich Schlegel (Lewis 1977). Nevertheless, there clearly had been certain parallels in the approaches to logic common to Schleiermacher and Hegel, despite their differences.

[10] Perception "presents us with various things, with discriminated and differentiate contents" but "what we wish to study here … is the bare possibility of such diversity" which can be considered the "form of externality". Russell then asks after the *properties* that such a form, when studied in abstraction, [must] necessarily possess" (ibid., 136). He answers that such a form must be essentially *relational*: "In the first place, externality is an essentially relative conception—nothing can be external to itself. … Hence, when we abstract a form of externality from all material content, and study it in isolation, position will appear, of necessity, as purely relative—a position can have no intrinsic quality" (ibid., 136–137).

be differentiated so to be "determinate and quantitative". While the point is "the *negation* of the immediate and *undifferentiated* self-externality of space itself", the determining self-sublating of the point is *the line* (ibid., §256 and Remark). Russell is quite explicit to the effect that this is the way he will enlarge upon the subject in the final chapter of his book (Russell 1897, 138).

2. Hegel, Geometry and Logic

The projective geometry discussed by Russell had clearly post-dated Hegel, but it is far from obvious that Hegel was unfamiliar with at least the general features of this movement. For example, the Swiss mathematician Johann Heinrich Lambert had pursued similar ideas in the eighteenth century (Andersen 2005, ch. 12), about which Hegel clearly had known (Hegel 2010, 544). Moreover, Hegel had in his library what was apparently the first book to reintroduce projective geometry and the notion of the cross-ratio in the nineteenth century. This was a 1803 German translation of *De la Corrélation des Figures de Géométrie* by Lazare Carnot published two years before (Carnot 1801). Carnot, a mathematician and military engineer, had been a hero of the French Revolution and the subsequent revolutionary wars. While projective geometry had only become widely known in the 1820s and 30s with figures such as Jean-Victor Poncelet, Carnot's 1801 book had re-introduced a specific and essentially prototypical *instance* of the cross-ratio called the *harmonic* cross-ratio as found earlier in Desargues and in late antiquity in the work of the Greek geometer Pappus of Alexandria. While Hegel never mentions Carnot's book on geometry, in *The Science of Logic* he relies heavily on an earlier work when discussing the problem of infinitesimals in the calculus of Leibniz and Newton. This was Carnot's *Reflexions sur la Metaphysique du Calcul Infinitesimal* published in 1797 that Hegel also possessed. Carnot himself had regarded his geometric work as a development of his solution to the problem of infinitesimals (Schubring 2005, 325).

Carnot's approach to the problem of *infinitesimals* in *Reflexions* was much like that of Russell in *Foundations* to the analogous problem of *points*. Infinitesimal quantities are "introduced in the calculus solely to facilitate the expression of the conditions proposed. It is clear that it is absolutely necessary to eliminate them from the calculation to obtain the desired result, that is, the ratios or relations (*rapports*) sought" (Carnot 1797, 30). These relations sought are what is symbolized in Leibniz's notation by '$\frac{dy}{dx}$' that Carnot interprets as a ratio or relation more fundamental than the supposedly infinitesimally small magnitudes being related. This is the lesson that Hegel had taken from that work and it links in turn to his treatment of the *ratios* as magnitudes in *The Science of Logic* (Hegel 2010, Bk. I, Sect. I, Ch. 3). Rejecting the idea that a ratio can be made determinate by its constitutive numbers, Hegel embraces the idea of a ratio being made determinate by its relation to *another ratio*—that was effectively the solution to the invariances under projection found in projective geometry as introduced by Carnot.

The *ratio of ratios* introduced by Carnot in his 1801 work shows an uncanny resemblance to the way Hegel, in Book III of *The Science of Logic*, conceives of the relation between two contrasting *qualitative* and *quantitative* forms of judgments he calls judgments of *inherence* (or judgments of *existence*) and judgments of *subsumption* (or judgments of *reflection*). The authority in logic at the *Tübingen Stift* while Hegel was there, Gottfried Ploucquet, had been an important eighteenth-century follower of Leibniz in his attempt to modernize the Aristotelian syllogism by the use of algebra, an approach independently discovered in the nineteenth century by Boole. Both Leibniz and Boole had mixed elements from Aristotle's *term logic* with a more *propositional* form of logic more typical of the ancient Stoics, and both struggled to unify these different approaches to judgment. Hegel had inherited this dual analysis of judgment structure. Judgments of inherence, with an Aristotelian *term-logical* structure could be differentiated from judgments of subsumption, with more "modern" propositional content, by the fact that each handled negation in a different way.

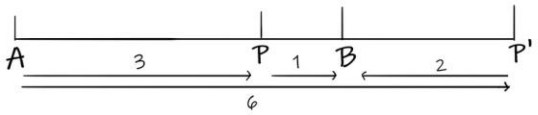
In negating a judgment of inherence, such as "the rose is red", the scope of negation extended only to the predicate: to deny that the rose is red is to imply that it is nevertheless a rose that has some *non-red* colour (Hegel 2010, 565). This leaves the negated judgment ("the rose is non-red") open to a further, "second" negation which extends to the *entire* content as with the "external negation" found in the Stoics (Bobzien 2020) as well as modern logics like that of Frege. Following Frege, Russell had *denied* that everyday judgments like Hegel's "the rose is red" were in fact *proper judgments*. Without complete propositional content, Russell came to treat such judgments as incapable of truth or falsity. Thus he treated in this way an example used by the early modal logician, Hugh MacColl, "Mrs Brown is not at home" (Russell 1906). Such an expression was really a type of predicate that would only be capable of truth or falsity if said of *a particular point in time*, as in "Mrs Brown is not at home at time t_1". Peirce, along with MacColl himself, had developed Boole's approach to logic and one aspect of this concerned the integration of propositional- and term-logical forms of judgment. This is the task in which, I suggest, the projective-geometrical dimension of *duality* in Hegel's logic is apparent, but to appreciate this we need to look closer to the nature of the harmonic cross-ratio, the paradigmatic *invariant* of projective geometry.

The harmonic cross-ratio posits an equivalence relation holding between *two* ratios, each being a ratio between the distances between two of four colinear points on a line, A, P, B and P', as in figure 1 below. In terms of the absolute values of the lengths involved, in the harmonic cross-ratio the ratio of the magnitudes AP to PB is the same as that of AP' to P'B. Here, P and P' are described as being "harmonic conjugates" with respect to the interval A and B, P said to divide AB *internally*, P' dividing it *externally*.[11] However, following an idea introduced by Kant in a paper in his precritical period of treating such line segments as *directed* (Kant 1992), an idea developed later by Grassmann, an interval such as AP can be read as extending *from* A *to* P or *from* P *to* A, such that one direction will be regarded as

[11] It turns out that A and B are similarly harmonic conjugates with respect to the interval PP'.

giving the magnitude a *positive* value and the other a *negative* value. Because in the double ratio below, P'B goes in the *reverse* direction to all the other line segments, one of the ratios of the double ratio will have a negative value in relation to the other, such that that $AP/PB = -(AP')/(P'B)$ (in the example below $\frac{3}{1} = -[\frac{6}{-2}]$).

Figure 1. Internal and external divisions of an interval



Here, while the point P divides the directed interval AB "internally", the point P' is said to divide it "externally". Because the direction of the line P'B is opposite to that of AP', it will be understood as having a negative value.

The cross-ratio and conjugates are called "harmonic" because the double ratio relates back to a sequence of four numbers found in the music theory of Pythagorean mathematicians around the time of Plato, a sequence called the "*harmonia*" or "musical tetraktys" (Barbera 1984). Late neo-Pythagorean authors had believed that this was the sequence of numbers referred to by Plato in his dialogue *Timaeus* as the "proportion" or "beautiful bond" unifying the parts of the "cosmic animal", a topic Hegel treats extensively in his *Lectures on the History of Philosophy*. According to Hegel, this structure had been appropriated by Aristotle and used to structure his syllogistic (Hegel 2006, 209–210).[12]

This equivalence of internal and external divisions of a *line*, I suggest, parallels the way in which Hegel describes the relation between two internally and externally divided *judgment forms* with opposed "inhering" and "subsuming" predicates, respectively. Indeed, the German term for judgment, "*Urteil*", already suggests such a *division*, "*Teilung*", into parts, *Teilen*. Aristotle had taken the terminology for his description of the syllogism from the music theory of contemporary Pythagorean mathematicians, who had sought ways of dividing an interval, the octave, by various "means" or "middle terms" so as to produce consonant smaller intervals (Einarson 1936). That a syllogism could be so "divided" comes naturally to this analogous application to logic. As for Hegel the syllogism provided the essential structure (the "truth") of the judgment, and so the judgment was similarly considered as divided in this way.

[12] I have presented the details of these links in Redding MS 2021, Part I.

These two judgment forms can be most readily distinguished by the different type of negation taken by each: in judgments of *inherence,* negation is "internal" in the sense that it is restricted in scope to the predicate. In judgments of *subsumption*, on the other hand, negation is "external", applying to both subject *and* predicate. But in each case, negation simply marks the difference between *what* is being asserted. In the former judgments, a predicate is asserted *of* a subject, as when some particular rose is said to be *red.* In the latter, the assertion applies to the complete sentence. In this example, what is asserted is "*that* the rose is red". The idea of internal or external *division* of the judgment thus seems to naturally apply.

George Boole would later treat the latter type of judgment as having a modified subject–predicate structure in which the *subject term* was now the complete sentence with the *predicate* predicating of the sentence that it was *true.* Thus he distinguished "primary propositions" as in "the sun shines", which classifies *an object* (the sun), within *a class of objects* (things that shine), from *secondary* propositions, such as "it is true that the sun shines", which classifies the *proposition*, that the sun shines, with the class of *true propositions* (Boole 1854, 38). This, I suggest, effectively coincides with Hegel's dualistic account of judgment, with the addition that for Hegel every judgment of inherence is able to be paired with a corresponding judgment of subsumption. In these cases, the *ultimate content* is the same while being expressed in *inverted* ways by each form.[13] This contrasts with the reductionistic accounts of both Boole and Frege. For Boole, secondary propositions presuppose and are constructed from primary propositions, while in the classical predicate calculus of Frege and Russell, only judgments with fully propositional contents are understood *as* judgments. From the latter perspective, Boole's primary judgments were regarded as incomplete (in Russell's terms, "*propositional functions*") that need to be converted into complete ones, just as MacColl's *tensed* judgment about Mrs Brown needed completion to an *untensed* form.[14]

Frege had criticised Boole's system for the fact that its two term-logical and proposition-logical parts "run alongside one another, so that one is like the mirror image of the other, but for that very reason stands in no organic relation to it" (Frege 1979, 14). Well before Boole, however, Hegel, had effectively addressed Frege's complaint by the model of the *equivalence* of the inverse judgment forms (judgments of inherence and subsumption)

[13] In relation to this, Michael Wolff draws the parallel to Grassmann's idea of the absolute value of a directed line segment: "Hegel actually anticipates an important idea in the newer algebra and vector algebra [of] Hermann Grassmann … Hegel … introduces the concept of absolute value as the concept of a substrate of logical reflection" (Wolff 1999, 15).

[14] This is still the dominant attitude found in contemporary modal logic where modal expression, deemed incomplete because contextual, need to be converted into the forms of classical predicate calculus. This reduction, however, was resisted by Prior in his tense logic (Prior 2003). On parallels between Prior's approach to modal logic and that of Hegel, see Redding 2017.

paralleling the dual ratios formed by "internal" and "external" divisions of a directed line segment.[15]

It might be expected that the irreducibility of *internally asserted* sentences in the system will play havoc with the traditional laws of logic as usually conceived. In the Frege-Russell type of quantified predicate calculus there is only one form of negation, external negation, and that means that as a relation between propositions, negation takes the form of *contradiction.* Typically, the classical laws of non-contradiction (LNC) and excluded middle (LEM) are expressed in terms of propositional negation: LNC as the law ~(*p* and ~*p*) and LEM as *p* or ~*p.* But in both Aristotle and Hegel, judgments can, reflecting the role of internal negation in term logics, be opposed to each other in terms of *contrariety*. For example, the simple judgment of inherence, "the rose is red" will typically be denied by contrary assertions to the fact that the rose is *another colour* (Hegel 2010, 565). Moreover, one might suspect here that the meaning of "this rose is not red", understood as a negation of the assertion of the predicate of the subject ("this rose is *non*-red"), might itself be dependent on some more determinate predication, as in "this rose is yellow" or "this rose is pink" and so on. Here, logical relations cannot be simply abstracted from epistemological issues: it could *not* be known that a rose was not red without *somebody* knowing what colour it actually was.

This idea has received its most explicit expression in modern intuitionistic logic which is not committed to LEM and in the following section I will explore the possibility of this as a model for Hegel's way of dealing with these issues.

3. Intuitionistic Logic's Rejection of LEM as a model for Hegel's Logic.

Brouwer had conceived his intuitionism as a philosophical perspective on mathematics that was more in accord than its rivals—the logicism of Frege and Russell and the formalism of Hilbert and his school—with the implicit methodology of mathematics and the experience of the mathematician. As the name implies, "intuitionism" had connections to Kant's earlier philosophy of mathematics in which the truths of mathematics were seen as based upon "pure intuition", with intuition itself characterised as a as *non-conceptual* form of cognition. Brouwer, however, was working *downstream* of those nineteenth-century developments in geometry, such as projective geometry, that had challenged Kant's assumptions about the universality of *Euclidean* geometry and his reaction to this was distinctive.

Rather than abandon the concept–intuition distinction along the lines of a generally *conceptual* approach as reflected in Frege's break with Kantianism, Brouwer criticised the exclusivity of the "*logico-linguistic method*, which operated on words by means of logical

[15] Hegel's idea of equivalence here might be likened in spirit to the so-called "redundancy theory of truth", which relies on the intuition that the judgment "it is true that the sun shines" says no more than "the sun shines". That the predicate "is true" adds nothing to the content might be thought to testify to the unity of the "logical substate" involved, but it is *not* redundant in as much as it signals a change in logical *form*.

rules, sometimes without any guidance from experience and sometimes even starting from axioms framed independently of experience" (Brouwer 1981, 2).[16] Indeed, Whitehead and Russell might be regarded as exemplifying the "logico-linguistic" approach to mathematics to which Brouwer was opposed. Thus, Whitehead had stated in his *Treatise on Universal Algebra* that "Mathematics in its widest signification is the development of all types of formal, necessary, deductive reasoning … The sole concern of mathematics is the inference of proposition from proposition" (Whitehead 1898, vi) while Russell, in *The Principles of Mathematics*, would proclaim pure mathematics to be "the class of all propositions of the form '*p* implies *q*', where *p* and *q* are propositions containing one or more variables, the same in the two propositions, and neither *p* nor *q* contains any constants except logical constants" (Russell 1903, § 1). Before Brouwer, Hegel too had objected to such a conception of mathematics.

As Michael Detlefsen has summarized, Brouwer had stressed the need to account for "the seeming difference in the epistemic conditions of provers whose reasoning is based on genuine insight into the subject-matter being investigated, and would-be provers whose reasoning is not based on such insight". The new epistemology would require a new conception of inference, "for in order for a truth to be proven it requires that it be 'experienced' in a certain way" (Detlefsen 1990, 501). Classical epistemology allowed abstraction to some propositional content that may not be "experienceable" in *any* particular way and this was expressed in the abstract nature of the associated classical logic.

Within intuitionism, one particular expression of this attitude would be the rejection of indirect proofs in mathematical reasoning, such as found in traditional "*reductio ad absurdem*" arguments. Some geometric truth, for example, could not be simply established by showing that its negation was self-contradictory, because such a form of proof was not accompanied by any corresponding *mathematical* insight. This, in turn, put pressure on the Law of Excluded Middle which, stated in modern form, that if a proposition *p* was false then its contradictory ~*p* must be *true.*[17] Brouwer thus allowed a role for undecidable propositions—ones for which there existed neither *proofs* nor *refutations.* In turn, this was linked to the idea of the role of the experience of some concrete "witness" or "proof object" required for the truth of any *abstractly* propositional claim (Bridges and Palmgren 2018, section III).

A generalisation of this idea of the experience of concrete witnesses required for the determination of abstractly formulated judgments has recently been put forward by Robert Stalnaker in the context of metaphysical debates over the nature of possible worlds (Stalnaker 2012). In these debates, possible worlds are often discussed as sets of "maximal

[16] At his most extreme, Brouwer seemed to suggest that concepts were relevant only to the linguistic expression of mathematical truths and not the truths themselves.
[17] C.f.: "The belief in the universal validity of the principle of the excluded third in mathematics [i.e., LEM] is considered by the intuitionists as a phenomenon of the history of civilization of the same kind as the former belief in the rationality of π, or in the rotation of the firmament about the earth" (Brower 1981, 7).

propositions", a proposition being maximal "in the sense that for every (actual) proposition, either it or its contradictory is entailed by it". But Stalnaker then notes that "a proposition might be maximal in this sense while failing to be fully specific, where a proposition is fully specific only if for every existential proposition that it entails, it also entails a singular proposition that is a witness to that existential proposition" (Stalnaker 2012, 19). Stalnaker does not mention intuitionism, which is the home of the notion of a "witness", but his claim closely fits with the spirit of intuitionistic logic.

For his part, Detlefsen has pursued a suggestively similar "modal" interpretation of Brouwer's conception of mathematical knowledge: "Mathematical knowledge becomes more than simply knowledge of a mathematical proposition, and is distinguished by a certain mode or kind of cognitive state as well" (Detlefsen 1990, 505–506). From this perspective, "to mathematically know a truth was to 'experience' it in a certain way" (ibid.). In a similar way, Stalnaker claims that in relation to the determinateness of our claims about the *actual* world, in contrast to claims about merely possible alternatives to it, the semantics of our sentences must go beyond a consideration of truth-values of abstract propositions. Within the Frege–Russell logical system, an existential statement about a *red rose* would have the form that, positing some indefinite "x" declares it to be a rose and to be red. To be maximally *specific*, however, this abstract existential statement would need to be accompanied by a singular "witness statement" about some *specific* identifiable rose—"*this* rose"—that, on the basis of experience of it, it is thereby said to be *red.* In this analysis, Hegel's abstract judgments of *subsumption* in order to have determinate content, must in some way stand in relation to some equivalent judgment of *inherence.* This is the model I have described as the equivalence of inverses, and the idea of such a "witness" provides us with a tool for making explicit what Hegel had in mind about the role of simple judgments of inherence about specific objects.

In contrast to the Frege–Russell conception of a singular underlying judgment form, as we have seen, Hegel more closely approximates the post-Boolean "algebra of logic" tradition with its two contrasting judgment types. While the task for the post-Booleans had been to show how these two judgment forms which "ran alongside one another" could be understood as standing in some "organic" or systematic relation (Frege 1979, 14), I have suggested that we understand Hegel's idea of an equivalence holding between these opposed forms as supplying an answer. However, if such forms are equivalent, it could not be that the former form is about something in the world (the sun) and the latter about a *linguistic representation* of that worldly thing, as Boole's "secondary proposition" suggests. That is, the equivalence involved implies that we should not think of the distinction as between "de-re" and "de-dicto" judgments. This being the case, I suggest that for Hegel we use the idea of "*de-facto*" rather than "*de-dicto*" for Boole's secondary propositions. If primary judgments are about *things*, secondary judgments are more about "facts".

From an Hegelian point of view, of course, the problem with the notion of "fact" will be the *atomistic* connotations that the idea has in the modern world. As Wittgenstein

famously wrote, “The world is a totality of facts, not of things”, “The world divides into facts” and these facts are *atomic* in the sense that “any one can either be the case or not be the case, and everything else remain the same” (Wittgenstein 1922, §§ 1.1, 1.2, 1.21). There can be no sense for Hegel that the world is an assemblage of such “facts”, but the common assumption that Hegel is thereby committed to a type of *holism* as found in Bradley, however, in which the determination of elements of the world runs through a type of ontological pyramid from top-down rather than bottom-up is, I belief, false. The analysis of Hegel’s judgment form in terms of the principle of the equivalence of logical inverses helps show why.[18]

Following Stalnaker, we might consider abstract propositional contents as perfectly suited for thought about alternative possibilities to what is found in the actual world. In one of Stalnaker’s examples, it could have possibly been the case that Saul Kripke had seven sons and, hence, a *seventh* son. In the language of possible-worlds semantics, there exists a possible state of the world in which Kripke *does* have a seventh son. But Stalnaker points to differences in the logical features of propositions about an actual person, like Kripke himself, on the one hand, and those about some merely possible person, his possible seventh son, on the other. In relation to Kripke, we had speculated about an alternate possible world in which he has different properties to the ones he actually has—that of possessing a seventh son. But Stalnaker suggests that we cannot simply transfer this type of status to a merely *possible* object, like Kripke’s possible seventh son. For every true *generic* statement about Kripke, say that he has blue eyes, there will be a corresponding *specific* statement, such as, when one points to a colour on a colour chart and says that Kripke’s eyes are *this* shade of blue. But let’s say that Kripke’s possible seventh son has brown eyes. Here generic propositions about his eyes will not be accompanied by the type of specific propositions (the witness statements) that accompany generic propositions about Kripke himself. In Hegel’s terminology, there will be judgments of subsumption that are true of Kripke’s possible seventh son, but no corresponding judgments of *inherence*. As with the intuitionist’s knowledge of objects of the mathematical world, on Stalnaker’s account, knowledge of objects of *the actual world* will, in Detlefsen’s terms, be “distinguished by a certain mode or kind of cognitive state as well” (Detlefsen 1990, 505–506).

4. Conclusion: Hegel’s Predicative Duality and the Dynamics of Inquiry

Hegel’s predicative duality clearly situates his logic more within the Boolean “algebra of logic” tradition than within the classical logic of Frege and Russell, but within such approaches I have suggested that his position on the classical logical laws puts him closer to *intuitionistic* variants of that logic. Thus, rather than Boolean logic, which conforms to the laws of non-contradiction and excluded middle, Hegel’s might be better compared to the type

[18] In this respect, the 1897 Russell had not fully appreciated the relevance of the parallels between Hegel and projective geometry that he helped unearth. When he turned against “Hegel” his target was Bradley’s conceptual holism rather than the real thing.

of mathematical logic proposed in the 1920s by Arend Heyting, the student of Brouwer who developed an algebraic form of logic which, while similar to the approach of Boole, departed from it in important ways.

Crucially, *negation* does not function within the Heyting algebra underlying his intuitionistic logic as it does in that underlying Boolean logic. For Heyting, "true" will effectively have to be understood as indicating *proven* by the experience of a witness and "false" as similarly indicating *refuted* by some experience of a contrary witness. In Hegel, this will be reflected by the idea that a negative judgment, such as the "rose is not red", presupposes an experience of the rose as having *some other colour.* For intuitionistic logic, this has the consequence that "true" and "false" are no longer *complementary* notions. In short, "*not p*" is *not* defined as the contradictory of "*p*", such that "not *p*" is false when "*p*" is true and true when "*p*" is false. Rather, Heyting's alternative was to define negation by utilizing a consequence that had followed from Russell's original definition of material implication—the fact that a *false* proposition implies *any other* proposition.[19] But if implication is to be used in the definition of negation, clearly the relation of implication cannot *itself* be defined in ways that employ the negation operator, as is done in classical logic. The relation of implication must be defined in a different way and the implication that results in a Heyting algebra is *weaker* than that found in classical logic.

This does not mean that intuitionistic logic is simply different to or incommensurable with classical logic, however. Rather, it means that the Heyting algebra underlying inituitionistic logic is able to be strengthened by the addition of certain other axioms that, in specific contexts, convert it to a Boolean one. This allows the judgments to feed into different aspects of inquiry resulting in an overall *dynamic* system, but this type of dynamism might also, I suggest, be seen as implicit within Hegel's logic as well.

Consider, for example, singular judgments of inherence, considered as functioning like Stalnaker's singular "witness statements", and so as providing intuitionist-type *proofs* for corresponding general claims. In verifying a theory, one will want to call upon specific witness statements rather than their abstract equivalents. But in other contexts, one will rely on the equivalent abstractly *propositional* claims to understand *how* such claims relate to *other* claims *via* the formal implication relations among their contents. Only abstract judgments will be needed, for example, in considering logical relations holding within some hypothetical model of reality that is yet to be subject to empirical verification.

Such an idea of logic does not conform to the requirement of constructing some ultimate logical language—a Leibnizian "*characteristica universalis*" or Fregean "*Begriffsschrift*"—but Hegel was critical of the conception of any such "standard language in which each concept is presented as a connection of other concepts" in such a way that "a content would still retain the same determinations that it has when fixed in isolation" (Hegel 2010, 608). A

[19] In one of Brouwer's examples, $\sim p \supset (p \supset 1 = 2)$.

similar rejection of any idea of a universal language would later be expressed by the Cambridge logician W. E. Johnson, who continued to hold to the algebraic approach of the Booleans after the appearance of *Principia Mathematica* (Johnson 1921–1924). Johnson treated logic as a mathematical device useful in specific contexts to aid human reasoning, but which itself *depended* upon human reasoning and human language for its interpretation. As he had put it in the early 1890s, a logical calculus "aims at exhibiting, in a non-intelligent form, those same intelligent principles that are actually required for working it" (Johnson 1892, 3). That is, in order for these "disinterpreted" strings of symbols to be grasped as *logical*, they must once more be *reinterpreted* which suggests two things: they must rely upon the resources of everyday language and thought for their reinterpretation, and that *reinterpretation* could in turn result in *new meanings* being given to those elements.[20]

The projects of Leibniz and Frege might be seen as historically continuous with a conception of logic going back to William of Ockham, for whom logic was essentially an attempt to find the language implicit in the thoughts of an omniscient god. The conceptions of logic found in Hegel, Peirce, Heyting and Johnson are more ones conceived as fit for humans who can reflect on and so correct their fallible reasoning processes.

[20] On the resistance of the logic of Frege and Russell to reinterpretation, see Goldfarb 1982, 694.

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