Gödel, Gentzen, Goodstein: The Magic Sound of a G-String[1]

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An Abstract Story of Numbers Series and Transfinite Ordinals

There can be beauty and depth and surprise in a mathematical result. At some rare occasions there is more: a touch of magic. Reuben Louis Goodstein (1912–1985) had his magic moment some time in 1941–42, amid a world in war. Here is the story of his result, *Goodstein's theorem*, in a concise and, it is hoped, suggestive formulation:

1. Take a number n, write it in *complete base 2*, as in $3 = 2^1 + 1$ and $4 = 2^2$ and $19 = 2^{2^2} + 2^1 + 1$.
2. Now change the base to 3 and subtract 1, as in $3^1 + 1 - 1 = 3$ and $3^3 - 1 = 26$ and $3^{3^3} + 3^1 + 1 - 1 = 7,625,597,484,990$
3. Change the base to 4 and subtract 1, to 5, to 6,…

The series for 3, denoted $G(3)$, has the value 0 when base 7 is reached. $G(4)$ has the value 0 when base $3 \times 2^{402653211} - 1$ is reached, so there is an incredible explosion from 4 on. Nevertheless:

Goodstein 1944: *All sequences $G(n)$ converge to* 0.

Brief sketch of proof: To each base n representation is associated a *transfinite ordinal*, with ω the ordinal of the natural numbers: $2^1 + 1$ becomes $\omega^1 + 1$, 2^2 becomes ω^ω, and $2^{2^2} + 2^1 + 1$ becomes $\omega^{\omega^\omega} + \omega^1 + 1$.

Crucial observation: Change of base of a number leaves the associated ordinal intact, subtracting one diminishes the associated ordinal.

The upper bound of the ordinals is $\varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$

There is no infinite descending sequence of ordinals below ε_0, so that the associated ordinals converge to 0. There is further a direct correspondence between the descending ordinals and descending natural numbers written in base n representation, by which the convergence to 0 of Goodstein sequences follows. Goodstein's article is "On the restricted ordinal theorem," *The Journal of Symbolic Logic*, vol. 9, pp. 33–41. He proved in fact a somewhat stronger result by which the successive bases can be taken from any nondecreasing sequence, say an exponentially growing one.

I claim: *No ordinary number theorist could have proved Goodstein's theorem*. To justify my claim, I just note that standard arithmetic is unable to prove the theorem: It is a "Gödel sentence" independent of the standard axioms of Peano arithmetic. One could of course maintain that the deeper parts of number theory are treated by analytic methods that go far beyond those of the elementary ones of Peano arithmetic. On the other hand, it is an empirical fact that no result of standard analytic number theory has surfaced that would have turned out to be a Gödel sentence. For example, even if Wiles's proof of Fermat's theorem is analytic, work is going on toward an elementary proof based on standard arithmetic induction.

How on earth was a result such as Goodstein's found and proved? This I will explain in what follows. Next, why did the result remain practically unknown for many decades? To that I have only partial answers.

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A First Pick of G

In September 1930, the perhaps sharpest mind of mathematics was hit by a surprise result that turned into one of the discoveries of a century: The attempts at giving to the foundations of mathematics an absolutely reliable justification, through a finitary proof of the consistency of mathematical reasoning, are unattainable. The result, Gödel's incompleteness theorem for arithmetic, shattered David Hilbert's foundational program. The sharpest mind in question was that of Johann von Neumann: Having listened to Gödel's apparently innocuous remarks at a conference in September 1930 in Hilbert's town of birth, Königsberg, he interrupted his lecture course on Hilbert's *Beweistheorie* (proof theory) in Berlin that Fall, to explain Gödel's result. Here is a contemporary account of the event by the well-known philosopher C. G. Hempel (2000, pp. 13–14):

> I took a course there with von Neumann which dealt with Hilbert's attempt to prove the consistency of classical mathematics by finitary means. I recall that in the middle of the course von Neumann came in one day and announced that he had just received a paper from … Kurt Gödel who showed that the objectives which Hilbert had in mind and on which I had heard Hilbert's course in Göttingen could not be achieved at all. Von Neumann, therefore, dropped the pursuit of this subject and devoted the rest of the course to the presentation of Gödel's results. The finding evoked an enormous excitement.

Von Neumann soon figured out a result he announced to Gödel in a letter of 30 November 1930: "I have been again concerned with logic recently, using the methods that you so successfully applied for the proof of undecidable properties. Here I came to a result that seems remarkable to me. Namely, I was able to prove that the consistency of mathematics is unprovable." (See Gödel's *Collected Works*,, vol. V.)

A Second Pick of G

Among von Neumann's audience in 1930 there had very likely been the young student Gerhard Gentzen (1909–1945) who in a letter of 13 December 1932 to his first university professor Hellmuth Kneser wrote:

> I have set as my specific task to find a proof of the consistency of logical deduction in arithmetic… The task becomes a purely mathematical problem through the formalization of logical deduction. The proof of consistency has been so far carried out only for special cases, for example, the arithmetic of the integers without the rule of complete induction. I would like to proceed further at this point and to clear at least arithmetic with complete induction. I am working on this since almost a year and hope to finish soon, and would then present this work as my dissertation (with Prof. Bernays).

This task required as a preliminary a study of logical deduction itself, which Gentzen cleared by his development of the proof systems of natural deduction and sequent calculus by May 1933. It took another 2 years for him to figure out and finish a proof of consistency of standard elementary arithmetic (Peano arithmetic, PA in what follows). Proud with the result, he sent his long paper, about a hundred typewritten pages, to his former professor Paul Bernays. The latter had become Hilbert's assistant in 1918 and was working on the Hilbert program. He also wrote the 2-volume *Grundlagen der Mathematik* (Foundations of Mathematics) that appeared under the name of Hilbert and Bernays in 1934 and 1939. In April 1933, right before Gentzen finished his doctoral work on logic, he was dismissed as "non-Aryan" from his professorship, and he lived in Zurich after 1934.

The way Gentzen first proved the consistency of PA is not easy to explain, but let's try to in some way at least: Assume some consequence C to have been concluded from a collection of assumptions denoted by Γ, and let C contain some overt or hidden falsity. Say, if C is a false numerical equality such as $0 = 1$, the case is clear, and if C is, say, a universal formula $\forall x A(x)$, let it have a false instance $A(n)$. We may not be able to find such an instance, but we may try to protect ourselves against a "worst possible case" in which such turns out. Our task is to show that in each such case, we are able to produce some hidden falsity in the assumptions Γ. Then, turning this to a contrapositive, if there are no hidden falsities in the assumptions Γ, no false consequence is hidden in C, either. The condition certainly holds if Γ is empty, so that any theorem C of PA is free from contradiction and PA is therefore consistent.

Gentzen set up a precisely defined *reduction procedure* by which we can answer to any proposed attempt at drawing a false conclusion from C, through the assumptions on which C depends. The crucial point of the proof is to show that the reduction always terminates in some finite number of steps. The matter is not at all clear, because each time there is a universal consequence, an infinity of choices of instances is possible, and this situation may repeat itself over and over again. For the termination proof, Gentzen used a principle he did not spell out, but that is known from L. E. J. Brouwer's intuitionistic mathematics as *bar induction*. A more sober terminology is *induction on well-founded trees*.

Bernays sailed in September 1935 in the company of Gödel across the Atlantic, to the Princeton institute. There they and probably also von Neumann studied Gentzen's manuscript and sent back letters with criticisms and suggestions. Only the responses of Gentzen have been preserved. It appears from these, as well as from later recollections of Bernays and the shorthand notes in which Gödel wrote up his work, that the main criticism was Gentzen's use of bar induction.

Gentzen's response to the criticisms was to develop what is ever since known as *ordinal proof theory*. The first germs of such a theory are found in his shorthand notes from the Spring of 1933:

> The idea is roughly: Each proof has a (transfinite) *value*. Consistency of a system of proofs can be shown only through a proof that has a higher value than all of these. Therefore the theorem of Gödel.

If arithmetic induction is used in the derivation of $\forall x A(x)$, a derivation of any instance $A(n)$ can be constructed without the inductive step: first take the derivation of the base case $A(0)$, then instantiate the inductive step from $A(x)$ to $A(x + 1)$ by $x = 0$, to get $A(1)$ from $A(0)$. Next instantiate by $x = 1$ to get $A(2)$ from $A(1)$, etc., until you come to $A(n)$. Gentzen saw that each such proof of an instance must have a lesser value

than an inductive proof of $\forall x A(x)$, and because n is arbitrary, this is possible only through the use of a transfinite ordering of proofs. Each inductive step must be allotted at least the weight ω, the first transfinite ordinal, that of the natural numbers, and iterated inductions lead to exponentiation of ordinals, as in ω^{ω}. At this time, Gentzen thought he could order all arithmetical proofs by ordinals less than $\omega^{\omega^{\omega}}$, but then he realized this would not work. The proof by bar induction was thus a new idea.

In the Fall of 1935, after the criticisms from Princeton, Gentzen turned back to his older idea of a transfinite assignment of values to formal arithmetical proofs, this time with success. The right ordinal turned out to be the limit of the ω-towers $\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}$, namely what is denoted by $\varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$. Toward the end of the 1930s, Gentzen was able to prove that this ordinal is optimal as a characterization of the *proof-theoretical strength* of PA: a principle of proof that transcends the arithmetical induction of PA by just the right amount. It was at the same time an answer to a question Gentzen had in his earlier notes put as: *Where is the Gödel-point hiding?*

There have been lots of misunderstandings about Gödel's theorems of incompleteness and unprovability of consistency, and about Gentzen's work with the latter. Gentzen's point was not that there should be some doubt about consistency: The arithmetical axioms and rules are obviously correct, and logical inference in a formal system of proof maintains this correctness. The lesson from Gentzen's work in pure logic, his magnificent "Investigations into Logical Inference" (Gentzen 1934–1935) that established the two mentioned main proof systems of *natural deduction* and *sequent calculus*, was the following: It is not possible to master the structure of arbitrary proofs, but the following more subtle analysis usually works: First, given an arbitrary proof, show that it can be brought into some standard form (*normal* derivations in natural deduction, *cut-free* derivations in sequent calculus). Such standard derivations have remarkable properties, such as the *subformula property* of logical proofs: in a normal derivation, all formulas are parts of the claim or its assumptions.

Gentzen gave two analyses of the structure of proofs in arithmetic. In 1936, he considered arbitrary derivations and showed consistency in the way described above: If a falsity is hidden in a claim, there is some falsity hidden in some assumption. In 1938, he considered derivations of a contradiction, and analyzed their structure in a way analogous to pure logic. On a more general level, Gentzen's accomplishments can be described as follows. Gödel had proved incompleteness through a clearly artificial arithmetical coding of a property that would not otherwise have arisen in mathematics. Gentzen found a natural mathematical principle, transfinite induction up to the number ε_0, that is independent of elementary Peano arithmetic, thereby showing that Gödel's incompleteness theorem is no "freak" phenomenon. Subsequently, such principles have been found elsewhere in infinitary combinatorics, as we shall see later.

A Third Pick of G

Goodstein had studied his Gödel and Gentzen and more: He had written in 1934 in Cambridge, under the guidance of J. E. Littlewood, a master's thesis on transfinite numbers, a topic on which G. H. Hardy and Littlewood had worked extensively. He had also attended the lectures of the philosopher Ludwig Wittgenstein and was greatly impressed by Wittgenstein's "finitist" approach. The latter had, especially, come to the insight that there are parts of arithmetic (primitive recursive arithmetic) in which proofs by induction can be replaced by the uniqueness of functions defined by primitive recursion. Say, properties such as the associativity and commutativity of addition are usually proved by induction in PA, but these properties can equally well be recovered from recursion equations.

Goodstein's first mathematical work of substance was a long manuscript with the title "Function Theory in an Axiom-Free Equation Calculus," of which he writes in a letter of 29 July 1940 to Bernays: "The manuscript which accompanies this letter gives some account of a new formal calculus for the Foundations of Mathematics on which I have been working for the past six years." Bernays responded with a 10-page tightly typewritten letter that displayed his complete mastery of the calculus of Goodstein. He also suggested that Goodstein tone down some of his claims:

> Generally my meaning is that your attempt could be quite as well, and perhaps even better appreciated, if you could deliver it from the polemics against the usual mathematical logics which seem to me somewhat attackable, in particular as regards your arguments on the avoidability of quantifiers. Of course in your calculus, like in the recursive number theory, quantifiers are not needed. But with respect to the "current works on mathematical philosophy" the thesis that "the apparent need for the sign '(x)' [the universal quantifier] arose from a confusion of the two different uses. . . of variable signs" can hardly be maintained.

Goodstein's original manuscript has been lost and nothing of the aforementioned is found in the printed version of 1945, but the samples of polemics cited by Bernays are rather of the type in which Wittgenstein was involved in the mid-1930s.

Goodstein had sent his only copy of the manuscript to Bernays and was asking to get it back, but there were difficulties: In a letter of 25 March 1941, he writes: "The manuscript you sent off on January 24th has not yet reached me but I think 10 weeks is not an undue delay. In any case, during the past few months I have succeeding in reconstructing a good deal of the paper from memory."

A paper of Goodstein's of 1958 in a Bernays-Festschrift contains the explanation of the delays:

> Doubts concerning the validity of the use of quantifiers and of indirect proof led me, in 1938, to the construction of the equation calculus, a free variable system in which recursive arithmetic and recursive analysis may be codified without postulating any logical axioms. Finding no appreciation of this system in England or the United States, I sent an account of it to Professor Bernays just before the outbreak of war and our subsequent correspondence travelled to and fro across Europe, no doubt unnecessarily worrying the Censorship of three countries. . . The encouragement which he then offered me gave me the

strength of purpose to continue working during the dark days of the war.

Nothing of these sentiments is seen in the correspondence. In Great Britain, at least Max Newman had to ascertain that the formulas in the letters were not any secret war code, and in Switzerland, the civil defense of the ETH-Zurich led by Prof. M. Plancherel had a similar task (I owe the latter to Prof. Erwin Engeler). The third country may have been the United States or Portugal, because letters between Great Britain and Switzerland had to be sent via Lisbon.

The last letters about the equation calculus, from the Spring of 1941, provide already an idea of the genesis of Goodstein's theorem: "I have as a matter of fact proved the theorem that a decreasing sequence of ordinals is finite, without using transfinite induction, to ordinals up to and including $\omega^{\omega^{\omega}}$ but this is not in the paper I am sending you." A second paper came perhaps a year later.

Here is the likely reasoning that led to Goodstein's magic moment: Gentzen had shown that the proof-theoretic ordinal of Peano arithmetic is $\varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$, the upper limit of ordinals of the form $\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}$. Each of these corresponds in an obvious way to a finite tower of exponentials $n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}$ that show the amazing growth rates illustrated earlier. On the other hand, the fundamental property of ordinals is that they are *well-founded:* Each descending sequence of ordinals is finite. Therefore, starting with some $\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}$ and passing to lesser ordinals, as in the Goodstein proof, will eventually lower the tower of ω's until a finite ordinal is reached, and eventually 0. The ordering of the ordinals is not the same as that of natural numbers, but there is a correspondence: Given two ordinals $\alpha < \beta$, if the corresponding base n numbers are denoted by $\alpha(n)$ and $\beta(n)$, then from some n on, $\alpha(n) < \beta(n)$, where the latter is the order of finite natural numbers (see Stillwell 2010 for a nice presentation).

The finite analogy to Gentzen's tower of ω's gave the incredibly growing Goodstein series, and Gentzen's induction the principle of proof by which these series converge to 0. The result was far beyond everything else that had been seen in number theory before, but it took a long time to bring home this fact.

How Do You Publish Your Results During a World War?

Now we come to the concrete side of Goodstein's discovery: having found the result, he wrote a paper, about 10 pages, with the title "A Note on Gentzen's Theorem." He then sent it to Princeton where the noted logician Alonzo Church was editing at the time the only journal devoted to logic, *The Journal of Symbolic Logic*. Church in turn sent the paper to Bernays for refereeing, although none of the anonymous kind we are used to: Bernays wrote directly to Goodstein on 1 September 1942, telling him that Church had asked him to act as a referee. The letter consists of 4 pages of single-spaced typewritten comments and suggestions. Something went amiss perhaps, for Bernays sent afterward a cable to Goodstein to which the latter responded that he "has gladly accepted all your suggestions." In the end, Goodstein had received the suggestions of Bernays, perhaps after a delay with the censors. Nevertheless, this first round was not enough, for on 24 February 1943, Bernays sent another very long letter to Goodstein, with the words that he "first hoped, as you did, that this manuscript could be sent directly to Prof. Church. But at regarding it nearer I found several points to require still a modification."

One of the crucial items in the exchange concerned the independence of Goodstein's theorem, denoted P in the paper, from Gentzen's formalization of number theory, denoted Z^*. Bernays makes a page of comments and then suggests that "perhaps on account of these circumstances it will be the best to cancel the paragraph on the undecidability of P in Z^*." In his reply of 23 April 1943, Goodstein writes that he has rewritten again the manuscript, "gladly accepting all the corrections and suggestions which you so kindly made," further adding that he has "omitted the paragraph on the Gentzen system Z^*, for the reasons you gave."

Goodstein had no proof of the independence of his theorem from PA, but it is clear to a careful reader of his paper that he indeed was convinced of the said independence. Even contemporary reviewers saw this, as in J. McKinsey's account in *Mathematical Reviews*, 1947:

> By the "restricted ordinal theorem," the author means the proposition that a descending sequence of ordinals, each less than ε (the first ordinal to satisfy $\varepsilon = \omega^{\varepsilon}$), is necessarily finite. This theorem is of importance because of the crucial role it plays in Gentzen's proof of the consistency of his "reine Zahlentheorie." In view of the results of Gödel, the restricted ordinal theorem cannot be proved in Gentzen's system; for, if it could, Gentzen's proof of consistency could be formalized within the system. The author formulates an interesting purely number-theoretic proposition which is equivalent to the restricted ordinal theorem.

Goodstein's ordeals with the manuscript were not finished with the second revision, for Bernays wrote a third letter dated 29 September 1943, with further suggestions as long as the paper itself. He writes that he "felt obliged to send you the manuscript once again, for final revision." In the end, concluding his own part, he writes: "I am afraid you will be somewhat disappointed by my demand... but I think you will find, after all, my criticism to be justified, and I hope, that you will not have so much trouble by the modifications to be applied, as I had with the revision."

Now, finally, Goodstein could send the paper for publication.

Where Was Gentzen?

In September 1939, an almost total darkness fell on Europe, and contacts were cut. Gentzen had been corresponding with Bernays until the outbreak of the war, but was then called on military duty. Gentzen's mother was an American of German descent and his English was excellent, and his task was to listen to the radio traffic of the British, later Allied airplanes, not anything one with a mere formal schooling in English could easily accomplish. A nervous breakdown led to an end of this period.

In late 1939, Gentzen met Gödel who gave a guest lecture in Göttingen, requesting the presence of Gentzen, and in 1942, he was able to send for publication what remained his last great work, with a title that amounts to "Provable and Unprovable Initial Cases of Transfinite Induction in Number Theory." This paper was the worthy successor of the proof of consistency of Peano arithmetic. It contains a direct proof of the unprovability of ε_0-induction in PA. Therefore, because anything is provable in an inconsistent system, PA is consistent.

Amid the silence and isolation, there arrived in 1943 in Zurich an issue of what had been the leading mathematics journal worldwide, the *Mathematische Annalen*, with Gentzen's paper on the initial cases of transfinite induction in it. It was the next-to-last issue to come out for a very long time, printed on low-quality paper that has turned yellow and brittle with the years. For Bernays, if not for Goodstein, it was at least a sign that Gentzen was alive and working, and Bernays dutifully produced a review of the article, published in 1944 in the USA while the war was still going on. Of Goodstein's theorem, the first number-theoretic application of Gentzen's induction principle, the latter cannot have known anything. Gentzen was positioned after his convalescence since 1943 in the German university of Prague; in May 1945, he was interned with the remaining German professors, and in August 1945, he died of starvation in a prison cell, merely 35 years old.

The Afterlife of Goodstein's Theorem

Why did the sensational application of transfinite ordinals to number theory not become generally known until decades later? Part of the reason is, in my opinion, that Gentzen was so far ahead of all others that his line of work was continued only from the 1950s on. Part of the reason is also in the title suggested by Bernays, for many people would have checked what "a note on Gentzen's theorem" has to say, when instead "the restricted ordinal theorem" doesn't say a thing. Thus, Gödel, for example, seems not to have even noticed this remarkable follow-up to his incompleteness results.

Another part of the reason for the early neglect of Goodstein's theorem lies in *a false modesty*. The numerical explosion is witnessed by one single number 4^{256} and no comment whatsoever, which perhaps doesn't strike the reader's eye amid a cramped text. The same attitude is well displayed by Goodstein's subservient acceptance of each and every comment and criticism of Bernays. The latter persuaded him to suppress the claims to independence from the final version, and one can only speculate what effect a clear-cut conjecture of independence could have had on future research. It is difficult to understand why Bernays downplayed the matter in such a way, requesting categorically the deletion of any reference to a possible independence, when the matter was clear to Goodstein and to some of those who read his article.

Gödel's original undecidable sentences were obtained through an arithmetical coding of the *provability predicate*, which led to the construction of an arithmetical statement that expressed, through coding, its own unprovability. As mentioned, it has often been felt that the creation was artificial and would not concern "ordinary mathematics." When the topic was revived in the latter part of the 1970s, the stress was on the mathematical naturalness of the undecidable propositions, such as the combinatorial *Ramsey principle* of Paris and Harrington in 1977. The independence of Goodstein's theorem from the axioms of PA was proved by Kirby and Paris in 1982. There was much publicity about these results at the time, as they were the first examples of natural mathematical incompleteness results. In these discussions, Gentzen's priority in having found a natural mathematical principle unprovable in PA, as shown by Gentzen's direct argument in 1943, was forgotten. *Summa summarum:* To everyone his right share: to Gentzen for having found a natural principle of pure mathematics that is expressible but not provable in PA, to Goodstein for having found a purely arithmetical, astonishing application of Gentzen's principle, and to the logicians of the latter days for having proved that things indeed stand as Goodstein had thought.

Unpublished Sources

The letters between Bernays and Goodstein are found in the Bernays collection of ETH-Zurich. Gentzen's manuscripts are in two slim folders of stenographic writing that the present author is preparing for publication in English translation. Gödel's work on bar induction in Gentzen's consistency proof is found in his *Arbeitshefte*, written in the Gabelsberger stenography and available in microfilm form, in particular *Arbeitsheft* 4, pp. 39–50 and *Arbeitsheft* 11, p. 28. For details on these sources, see von Plato (2014).

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