Neculai S. Teleman

From Differential Geometry to Non-commutative Geometry and Topology

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This book is a tribute to the memory of
Professor Enzo Martinelli,
with deep esteem and gratitude.
Neculai S. Teleman

Foreword

Neculai S. Teleman takes the reader on a fascinating expedition exploring the lands between the smooth and the continuous domains. For a very long time a continuous function was assumed to be differentiable perhaps with the exception of a finite or numerable but discrete set of points. The Weierstrass example (1872) of a continuous but nowhere differentiable function came as a great shock to mathematicians of the nineteenth century. A similar attitude was prevalent in the middle of the previous century when a topological manifold was considered to carry the unique smooth structure inducing the initial topology of the manifold. Then in 1956 John Milnor showed that on the 7-sphere there are several different smooth exotic structures. But in 1963 Michael F. Atiyah and Isadore M. Singer announced their index theorem which asserted that the index of the Laplace operator (associated with the smooth Riemannian metric) is equal to the topological index of the smooth manifold. Therefore, the index obtained from the data provided by the smooth structure is the same as the one provided by the data derived from the underlying topological structure. These two important and highly non-trivial results turned attention of many a fine mathematician of the 1960s to the regions where the smooth meets the continuous. A few years later (1965) S. P. Novikow published his famous result that the rational Pontrjagin classes of a smooth manifold are topological invariants. By the end of the decade, it was known that rational Pontrjagin classes can be defined for any topological manifold (R. Kirby and L.C. Siebelmann, 1969).

In 1971, I.M. Singer in *Future extensions of index theory and elliptic operators* wrote

> I was asked to speak on "possible future developments, give conjectures, and speculate about future advances." .... probably at this very moment a graduate student is busily at work on a theorem that might change present trends drastically. However, in a limited way, I will try and fulfill the charge – at least within the context of index theory for elliptic operators. My main theme is this: by and large this theory assigns an integer to an elliptic operator on a compact smooth manifold; I expect extensions of the present theory to new situations; i.e., non-smooth manifolds, non-manifolds of special type, and to a context where it is natural that integer be replaced by a real number.

A few years later Neculai S. Teleman arrived at MIT and in 1977 defended his Ph.D. thesis in which he extended the index theorem to combinatorial manifolds. Ever since he dedicated all his efforts to developing new tools and results which permit to extend the validity of the index theorem such as combinatorial Hodge theory on PL manifolds, (TAMS, 1979, Inv. Math., 1980), "elliptic machinery" and signature operators on Lipschitz manifolds (Publ. IHES, 1983). The following year he published a version of the index theorem for general topological manifolds using new tools, i.e. Kasparov's realisation of the analytic K-homology group $K_0(M)$ and Fredholm operators on non-compact manifolds. His quest to generalise various "smooth" constructions led him to use the existence of the quasiconformal structure on any topological manifold which finally turned his attention to the techniques and ideas of non-commutative geometry (1991).

The reader of the book is asked to embark together with the author on this fascinating journey which leads to the boundaries of our topological knowledge. Neculai S. Teleman introduces the reader step by step to the basic and then advanced tools of this fascinating theory: classical differential geometry, e.g. Riemannian metrics, Laplacian, Hodge theory and the smooth version of the index theorem. Then to go beyond that the basic elements of non-commutative geometry are presented as a method of delocalisation of our mathematical perception and natural extension of the classical geometrical tools. Having covered the known regions of geometry and topology of manifolds, the knowledgeable and experienced guide shows new fascinating vistas of non-commutative topology.

It is a very important volume which gathers and presents in an orderly manner the story of the index theorem. The tools and the results are scattered in numerous publications. But having them in one place is not the only advantage of this book. We get much more since the narrative is both objective and passionate at the same time as the narrator is one of the principal protagonists in this fascinating mathematical epic.

Kraków, Poland Robert Wolak
April 2019

Preface

This book combines some of the author's research contributions in index theory with elements of non-commutative geometry. The book ends with non-commutative *topology*.

This work shows that the index formula is a *topological statement*; on the other hand, this book gives a *re-formulation* of the index formula. The re-formulation should have important consequences.

From the non-commutative geometry side, this book explains how different results in index theory, obtained by classical geometry methods (Riemann–Roch, Hirzebruch [32], Atiyah and Singer [53], Teleman [83, 84]) and by non-commutative geometry (Connes [82], Connes and Moscovici [96], Connes et al. [101]), are connected. This has to do with the possibility, offered by different theories, to exhibit/describe the *topological index* of elliptic operators. The passage from classical differential geometry to non-commutative geometry, exemplified by the index formula done in this book, describes also the way how the author became interested in, and approached, non-commutative geometry. For this reason, this book does not intend to give a complete review of all its results or to give a complete list of its contributors. This book offers the reader a natural path starting with basic problems of differential geometry and leading to non-commutative geometry and topology. However, although this book does not intend to present the very significant developments of non-commutative geometry in *all directions*, it accompanies the reader towards present research problems in *index theory*.

This book has multiple goals:

1. The author did his Ph.D. with Prof. I. M. Singer (1977) working on extending the index theorem to combinatorial manifolds—a problem that was formulated by I. M. Singer in his programme article "Prospects of Mathematics" Symposium, March 16–18, Princeton [58]. One goal of this book is to *fill the existing gap* in the mathematical literature in reporting the developments which occurred in index theory after I. M. Singer presented his 1970-research programme. This report is certainly not complete.

2. It aims to provide a friendly introduction to non-commutative geometry. It studies index theory from a classical differential geometry perspective up to the point where classical differential geometry methods become insufficient.
3. Another goal is to present non-commutative geometry as a *natural continuation of classical differential geometry*. This is based on the following concepts:

   - Non-commutative geometry is *abstract index theory*.
   - Non-commutative geometry is a *de-localised theory*. In many cases the de-localisation occurs by replacing $\mathbf{C}^\infty$-homomorphisms by operators on L_2. This change replaces exact relations by L_2*-local averages*.
   - Non-commutative geometry is a *multi-distribution theory*. The *multi-distribution theory* occurs by means of two processes: (1) *polarisation* of algebraically higher order expressions; (2) each factor of the polarisation becomes a *mono-distribution theory*.
   - *Hochschild homology*—and its variants—replaces the de Rham homology.

4. Beginning with Riemann and up to the present times, index theory has passed through most chapters of mathematics: algebraic geometry, analytic geometry, differential geometry, combinatorial geometry, Lipschitz and quasi-conformal geometry as well as L_p-structures geometry. Another goal of this work is to show that while the objects of non-commutative geometry of a (classical) space X *live in all powers* X^p of X, classical differential geometry objects associated with X live on the *main diagonals of those powers*. The possibility of describing an object *classically* depends on whether the *non-commutative object* has a *classical limit*, i.e. whether its *restriction along the main diagonals* are defined; this depends on whether certain *products of distributions can be performed*.

   The book has the following main mathematical objectives:

   (i) it intends to provide a natural link between classical differential geometry and non-commutative geometry.
   (ii) it presents the basic algebraic structure and results which stay at the foundations of non-commutative differential geometry.
   (iii) it summarises the basic analytical structures on topological manifolds.
   (iv) it shows that the index theorem is a *topological statement*.
   (v) it summarises the results in index theory obtained with non-commutative geometry methods—local index theorems.
   (vi) it proposes *prospects in index theory*.
   (vii) it proposes *non-commutative topology*.

This book presents the mathematical facts in their chronological order most of the time. Both the de Rham isomorphism theorem and the Hodge decomposition theorem show that *real cohomology* may be extracted analytically from differentiable structures on topological manifolds. These results constitute two of the basic links between differentiable structures and topological structures. Given, however, that *not all* topological manifolds possess differentiable structures, and that the same topological structure might be compatible with *non-equivalent* differentiable

structures, there is a fundamental question to ask whether these relations could be derived purely in topological terms, on *all topological* manifolds. S. Novikov proved the fundamental result [41] stating that the rational Pontrjagin classes of *smooth* manifolds are *topological invariants*. Kirby-Siebenmann results [56] complete Novikov's theorem by stating that *all topological* manifolds possess rational Pontrjagin classes. Going along the same line of ideas, the following *fundamental question* arose: whether the Hodge decomposition theorem and index formula could be stated an *all topological* manifolds (see e.g., [58]).

This fundamental question was answered using Sullivan's fundamental result [75], which states that any topological manifold of dimension $\neq 4$ possesess a Lipschitz or quasi-conformal structure, *unique* up to Lipschitz/quasi-conformal isotopy. The first results on Hodge theory on combinatorial manifolds are due to Teleman [72, 76–78] and to Cheeger [79] on pseudo-manifolds. These results are successively extended using Sullivan's result on Lipschitz manifolds [83, 84]. Successive work due to Donaldson and Sullivan [92], Connes et al. [101] extends the index formula on quasi-conformal manifolds. This book summarises these results.

The first problem the author of this book undertook in his Ph.D. thesis was extending the Hodge decomposition theorem from smooth manifolds to pseudo-manifolds, see Teleman [72] and Singer [58]. At that time (1976) this subject was totally unexplored. In fact, I.M. Singer's programme [58], published in 1971, poses the following two questions too:

A "If M is a PL manifold, the L-polynomials are still well defined (Thom [25]), and one can define from them the rational Pontrjagin classes. The Hirzebruch signature theorem still holds. Is there an associated elliptic operator (as in the smooth case) whose index is the signature? On what spaces does it operate? Does it have a symbol and where does the symbol lie?"

B "... Are present analytic techniques strong enough to prove the Hodge theorem for smooth manifolds by approximations via combinatorial Laplacians associated with smooth triangulations? If so, this might shed some light on the PL-problem."

Looking retrospectively, it is important to remark that extending the Hodge isomorphism theorem and the index formula to spaces which are not *differentiable manifolds* was a daring task; the proof of the Hodge isomorphism theorem (see de Rham [86]) and the proof of the Atiyah–Singer index formula (see Palais seminar [42] and Atiyah and Singer [53]) heavily used the differentiable structure.

Combinatorial manifolds are the closest relatives of smooth manifolds. For this reason, the first problem addressed in trying to extend the index formula to more general contexts was to investigate whether Hodge theory could be extended from smooth manifolds to combinatorial manifolds.

Given that the Laplace operator is defined in terms of a chosen Riemannian metric on the manifold, it was crucial to understand in which respect the *special characteristics of the Riemannian metric* should be satisfied if a Hodge type result is desired. The de Rham [86] proof of the Hodge decomposition theorem showed that if one wishes to extend the Hodge theory to combinatorial manifolds in such a

way that *the main identities of the smooth Hodge theory be preserved*, see formula (1.198), the combinatorial Riemannian metric should satisfy the property that the local measure around each point x of the manifold (the total local solid measure of the simplices containing the point x) should be equal to the value of the solid angle about any point in the Euclidean space of the same dimension.

The following theorem answers this question.

Theorem 1 (Teleman [72, 76]) *Let M be a combinatorial manifold of dimension n. Let g be a combinatorial Riemannian metric on M with the property that for any point $x \in M$, the total local solid measure of the simplices containing the point x equals the value of the solid angle about any point in the Euclidean space of dimension n.*

Then the metric defines a smooth differentiable structure on M.

The following fundamental result was known.

Theorem 2 (Milnor [66]) *There are obstructions to the existence of smoothings of combinatorial manifolds, the Milnor [66] obstructions.*

These two results lead to the conclusion.

Corollary 1

(i) If one wishes to extend the Hodge theory to general structures, it is necessary to ignore *special requirements on the Riemannian metric.*
(ii) Riemannian metrics provide generalised smoothings.

The Laplace operator on smooth manifolds is defined exclusively in terms of a Riemannian metric on the manifold. On the other hand, the de Rham parametrix for the Laplacian [86] is expressed purely in terms of the *geodesic distance* function. Given that the signature operator plays a basic role in different proofs of the Atiyah–Singer index theorem (in differential geometry and in the Connes–Sullivan–Teleman and Hilsum index theorem) in non-commutative geometry it is important to understand the structure of the signature operator. In the smooth category the signature operator can be described completely in terms of the distance geodesic function; this is shown in Sect. 1.12.

This book is organised as follows.

Chapter 1 presents those tools of differential geometry which pass directly into non-commutative geometry. This is necessary to bring the reader from classical differential geometry topics to non-commutative geometry by providing a natural link between the two. This chapter ends with the geometry of the signature operator.

Chapter 2 introduces the first elements of non-commutative geometry. This chapter reconsiders the tools of differential geometry and finds the *minimal structure necessary* to make them work in non-commutative geometry. At this point we are at the boundary between differential geometry and non-commutative geometry. It shows that the *minimal structure* leads naturally to a new definition of differential forms, homology/cohomology and Chern–Weil theory. This is the way in which Hochschild homology and cyclic homology appear naturally. The

basic non-commutative geometry topics are presented here: universal algebra, non-commutative Chern–Weil theory, and Connes' long exact sequence.

Chapter 3 establishes the first relations of non-commutative differential geometry. It presents the basic results concerning Hochschild and cyclic homology/cohomology. It begins with the computation of the Hochschild homology of the algebra of smooth functions. The result has to be thought of as the link between differential geometry and non-commutative geometry. Its proof shows how combinatorics, topology and geometry interact towards the final result. Hochschild, cyclic and periodic cyclic homology are introduced and their principal properties are presented. The results due to Connes and Karoubi on the extension of Chern character to idempotents, extensions of associative algebras, K-theory and K-homology are presented. In the same section it is shown that the Chern character may be extracted from *direct connections*. Chapter 3 also presents the *periodic cyclic homology bi-complex* and homology.

Chapter 4 makes a *rapid panorama* of the analytic structures on topological manifolds. These analytical structures classify the various fields of geometry and are significant in index theory: algebraic geometry, analytic geometry, differential geometry, combinatorial geometry, Lipschitz, quasi-conformal and L_p geometry. Research interconnecting these fields, however, is not the norm. Global analysis on combinatorial, Lipschitz, quasi-conformal and L_p manifolds represent a small part of research. The majority of research is done in the *most regular cases*. Differential geometry benefits from the theory of partial differential equations. In spite of this reality, understanding the roots of fundamental problems in mathematics is of maximum importance. Index theory is one of them. It began with the Riemann–Roch theorem on compact Riemann surfaces. It was later extended by Hirzebruch on analytic manifolds and then by Atiyah and Singer on differentiable manifolds. Teleman proved that the index of *abstract elliptic operators* is a topological invariant. Presenting the hierarchy of analytic structures on topological manifolds helps the reader to place correctly the various index theorems. Hence, even though this chapter is short, we have decided to keep it as a separate chapter to provide a conceptual framework for the reader.

Chapter 5 presents the index theorems which were obtained with *classical differential geometry methods*: the Riemann–Roch index theorem, the Thom index theorem, Hirzebruch index theorem, Atiyah–Singer index theorem and Teleman index theorem.

Chapter 6 discusses index theorems which were obtained via *non-commutative geometry methods* (local index theorem and applications) due to Connes–Moscovici, Donaldson–Sullivan, Connes–Sullivan–Teleman and Hilsum.

Chapters 7–9 are devoted to *prospects in index theory*. In this part a new formulation of index theory is proposed using the newly introduced *local* structures and a few conjectures are formulated. More specifically, index theory is defined on an arbitrary pair $(\mathscr{A}, \mathscr{J})$ where $\mathscr{J}$ is an ideal of the *localised* ring $\mathscr{A}$. A new definition of the *topological* and *analytic* indices is proposed. The new formulation is based on the $T_*^{loc}(\mathscr{A})$-groups which replace the classical algebraic K_*-theory groups.

Chapter 10 introduces non-commutative topology. In this part we define non-commutative spaces, the homology of the new field and the Chern character of idempotents. This chapter is short because it is entirely new research that has not been published before. Nonetheless, we feel this work will provide a useful starting point for future research.

One of the main themes of this book is *locality*. The analytic structures (algebraic, analytical and differentiable) are sufficiently rich to allow writing down a formula for the topological index theorems. This is not the case of more general structures (combinatorial, Lipschitz or quasi-conformal structures). The possibility to have a *local formula for the topological index* depends on whether the *non-commutative topological index has a classical limit*.

The book introduces or recalls the following recent or new structures introduced by the author: *micro-localisation of Hochschild homology* [107, 121], *direct connection* [116, 117], *local periodic cyclic homology* [125], *local index formula* [130], *localised algebras* [125, 129], *T-completion* comp. [129], *topological periodic cyclic homology* [122], $(\tilde{b}, d)$ *bi-complex* [122], *Alexander–Spanier co-homology of localised rings*, *topological and analytic index, reformulation of index theory*, *local periodic cyclic characteristic classes* and *non-commutative topology*.

The contents of this book are self-contained. That said, in order to keep the size of this book reasonable, a few chapters are compact. In these cases, readers are invited to deepen their understanding further by reading the cited references. We hope this book will serve as a useful resource to introduce these concepts in non-commutative geometry to a broad audience of readers in the fields of mathematics and physics.

Ancona, Italy Neculai S. Teleman
April 2019

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Part I
Spaces, Bundles and Characteristic Classes in Differential Geometry

Chapter 1
Spaces, Bundles and Characteristic Classes in Differential Geometry



Abstract Part II prepares the reader to see how some of the basic notions of differential geometry pass into non-commutative geometry. The basic notions presented in the first chapter are reconsidered in the second chapter from a non-commutative geometry view point. Differential geometry begins with the algebra $\mathscr{A} = C^\infty(M)$ of *smooth functions* and builds up by adding multiple structures; classical index theory uses most of these structures. Non-commutative geometry is *abstract index theory*; its axioms comprise many of these structures. While differential geometry is built by summing up different structures, non-commutative geometry reverses this process. In differential geometry the *commutativity* and *locality* assumptions are built in by means of the construction of *differential forms*. There are two *basic differences* which summarise the passage from differential geometry to non-commutative geometry: in differential geometry (1) the basic algebra $\mathscr{A} = C^\infty$ *is commutative, has true derivations (differential fields), and has a topology*—the Fréchet topology; in non-commutative geometry, the basic algebra $\mathscr{A}$ *is not required to be commutative nor to have a topology, nor to have derivations*, (2) in differential geometry, the basic algebra $\mathscr{A}$ is used to produce *local objects*; in non-commutative geometry the *locality* assumption is removed. Non-commutative geometry finds and uses the *minimal structure* which stays at the foundation of geometry: of differential forms, product of (some) distributions, bundles, characteristic classes, cohomology/homology and index theory. The consequences of this discovery are far reaching.

1.1 Differential Forms

In this section we suppose that M is a paracompact smooth manifold of dimension n. We denote by $C^\infty(M)$ the unital associative algebra of smooth functions on M. This is a *commutative* algebra. We denote by $T(M)$ the *tangent bundle* to M and by $T^*(M)$ its *co-tangent* bundle. We denote by $\Gamma(T(M))$ the space of smooth sections of the tangent bundle $T(M)$ and by $\Omega^1(M)$ the space of smooth sections in the co-tangent bundle $T^*(M)$. The following result describes *local objects* in differential geometry.

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Proposition 1.1 (Peetre [26]) *Let Q be a $C^\infty(M)$-module and let $F : Q \longrightarrow Q$ be a linear mapping over the algebra of $C^\infty(M)$.*

Then F is a differential operator *if and only if F* decreases supports. *We will say that F is a* local operator. *In other words, if*

$$F(f \,.\, \gamma) \;=\; f \,.\, F(\gamma), \tag{1.1}$$

for any $f \in C^\infty(M)$, and any $\gamma \in Q$, then for any point $p \in M$, one has

$$(F(\gamma))(p) \;=\; F(\gamma(p)) \tag{1.2}$$

where $\gamma(p)$ is a jet.

Theorem 1.1

(i) *$\Gamma(T(M))$ is the space of* derivations *of the associative algebra $C^\infty(T(M))$, i.e. any $X \in \Gamma(T(M))$ is an $\mathbf{R}$-linear mapping $X : C^\infty(M) \longrightarrow \mathbf{R}$ which satisfies the* Leibnitz identity

$$X(\, f_1 \,.\, f_2 \,) \;=\; X(f_1) \,.\, f_2 \;+\; f_1 \,.\, X(f_2). \tag{1.3}$$

(ii) *The* differential *df of the smooth function f is defined by*

$$((\, df \,)(\, X \,))(p) := (\, Xf \,)(p), \tag{1.4}$$

for any $X \in \Gamma(T(M))$ and $p \in M$.

(iii) *$\Gamma(T(M))$ and $\Omega^1(M)$ are $C^\infty(M)$-modules.*

(iv) *$d\,1 \;=\; 0$.*

(v) *Taylor's formula with reminder and Schwartz' lemma on the commutativity of second order mixed partial derivatives, applied onto the smooth function f, imply*

$$(\, df \,)(p) \in C^\infty(M)/I_p^2, \tag{1.5}$$

where $I_p \subset C^\infty(M)$ is the ideal consisting of all smooth functions *which vanish at p.*

Definition 1.1 A k-differential form $\omega \in \Omega^k(M)$ is an k-multilinear $C^\infty(M)$-function

$$\omega : \otimes^k_{C^\infty(M)} \Gamma(T(M)) \longrightarrow C^\infty(M), \tag{1.6}$$

which is *skew-symmetric*

$$\omega(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_k) = -\,\omega(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_k). \tag{1.7}$$

Skew-symmetry is a form of *graded commutativity*.

Definition 1.2 For any $\omega \in \Omega^p(M)$ and $\sigma \in \Omega^q(M)$ define the *exterior product*, or *wedge product* $\wedge$

$$(\omega \wedge \sigma)(X_1, \ldots . X_{p+q}) := \frac{p!q!}{(p+q)!} \sum_{(\omega,\tau)\in S_{p+q}} \omega(X_{\sigma(1)}, \ldots ., X_{\sigma(p)}) \cdot \tau(X_{\sigma(p+1)}, \ldots ., X_{\sigma(p+q)}). \tag{1.8}$$

Definition 1.3 The *exterior derivative* $d\omega \in \Omega^r(M) \in \Omega^{r+1}(M)$ is defined by

$$d\omega(X_1, \ldots, X_i, \ldots, X_j, \ldots ., X_{k+1}) := \sum_{r=1}^{r=k+1} (-1)^r X_r \; \omega(X_1, \ldots, \hat{X}_r, \ldots ., X_{k+1}) +$$

$$\sum_{1\le i<j\le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots ., \hat{X}_j \ldots . X_{k+1}). \tag{1.9}$$

The non-commutative geometry does not assume any of the two requirements

(I) the *locality* and
(II) *graded commutativity*.

In non-commutative geometry both the space M and the algebra of smooth functions $C^\infty(M)$ are *replaced* by an arbitrary associative, unital or non-unital, commutative or non-commutative, algebra $\mathscr{A}$.

(i) In non-commutative geometry it is not assumed that the algebra $\mathscr{A}$ has true derivations (for example, the algebra of continuous functions does not have non-trivial derivations),
(ii) I_p, $p \in M$, may not be used for any individual point p,
(iii) the point-wise Taylor formula may not be used,
(iv) the differential forms and their derivatives need a new definition,
(v) the product of differential forms may not follow the classical definition. It is not postulated that the product of differential forms is graded commutative.

However, in non-commutative geometry the following properties are preserved

(vi) the *collection* I of local ideals I_p, $p \in M$, may be recovered from

$$I := Ker\{\mu : \mathscr{A} \otimes_{\mathbb{R}} \mathscr{A} \longrightarrow \mathscr{A}\}, \tag{1.10}$$

where μ denotes the multiplication in $\mathscr{A}$; this is going to be used into the construction of non-commutative differential forms,
(vii) the elements of the algebra $\mathscr{A}$ constitute a system of *generators* of the non-commutative differential forms,
(viii) the non-commutative differentials d produce complexes, i.e. $d^2 = 0$,
(ix) the graded Leibnitz formula should hold.

In differential geometry the locality requirement is too restrictive to allow making products of distributions of any kind; this difficulty persists at all levels of regularity below the C^1-regularity level. In non-commutative geometry the locality requirement is removed; the problem of multiplying distributions has new possibilities to be accomplished. Although it is not true that the existing non-commutative geometry succeeds to deal with the problem of multiplying *arbitrary* distributions, it is important to state that it is sufficiently powerful to cover the basic questions raised by the index formula on topological manifolds, if one takes advantage of the fundamental result due to Sullivan [75], which affirms that any topological manifold of dimension $\neq 4$ admits an unique Lipschitz or quasi-conformal structure. Sullivan's theorem together with the non-commutative geometry techniques produce results below the classical differential geometry of C^1-structures level, see e.g. Connes et al. [101].

Given any associative unital algebra $\mathscr{A}$, we anticipate by saying that the idea to define a complex, whose chains satisfy the conditions (viii) and (ix) only, leads to the *universal complex* $\mathscr{U}^*(\mathscr{A})$; this will be discussed in Sect. 2.4.10, see [9, 15]. In non-commutative geometry any differential form associated to the algebra $\mathscr{A}$ is the equivalence class of an element of the quotient of a certain sub-algebra of the universal complex; in the case of topological algebras the *projective tensor product completion* will be used, see Sect. 2.4.8.

1.2 de Rham Cohomology

The vector spaces $\Omega^k(M)$ together with the co-boundary homomorphisms d form the de Rham complex $\Omega^*_{dR}(M)$ of the smooth manifold M, $\Omega^0(M) := C^\infty(M)$.

The de Rham complex is an example of elliptic complex, see Atiyah and Bott [51].

Theorem 1.2 (de Rham [86]) *The homology of the de Rham complex is isomorphic to the singular cohomology $H^*(M, \mathbf{R})$.*

Karoubi [89] extended de Rham homology in the non-commutative context; more details will be given in Sect. 2.4.13.

1.3 Alexander–Spanier Cohomology

Alexander–Spanier cohomology is a non-commutative geometry object.

We assume that the topological space M is metrisable; we choose a metric d on M and we use this metric to define t-neighbourhoods U^k_t $(0 < t)$ of the main

diagonal in M^{k+1}

$$U_t^k := \{(x_0, x_1, .., x_k) \mid d(x_0, x_1) \ + \ d(x_1, x_2) \ + \ldots + \ d(x_{k-1}, x_k) \ + \ d(x_k, x_0) < t \ \}, \tag{1.11}$$

for $x_i \in M$.

The system of neighbourhoods U_t^k form a projective net U with respect to t: if $t_1 < t_2$, we consider $U_{t_1}^k \succ U_{t_2}^k$ given by restriction to the smaller space is placed in the rth-position. Let G be an arbitrary Abelian group.

Let

$$C_{AS,\,t}^k(G) := \{f \mid f : U_t^k \longrightarrow G, \ \ with \ f \ arbitrary \ function.\} \tag{1.12}$$

For any $f \in C_{AS,\,t}^k(G)$ and $0 \leq r \leq k+1$ define $\delta_r f \in C_{AS,\,t}^{k+1}(G)$

$$(\delta_r f)(x_0, x_1, .., x_{k+1}) := f(x_0, .., \hat{x_r}, \ldots, x_{k+1}). \tag{1.13}$$

This function is well defined because, if $(x_0, x_1, .., x_{k+1}) \in U_t^{k+1}$, the triangle inequality implies $(x_0, .., \hat{x_r}, \ldots, x_{k+1}) \in U_t^{k+1}$.

Define $d : C_{AS,\,t}^k(G) \longrightarrow C_{AS,\,t}^{k+1}(G)$

$$d \ f := \sum_{0 \leq r \leq k+1} (-1)^r \ \delta_r f. \tag{1.14}$$

The homomorphism d passes to the projective limit and satisfies $d^2 \ = \ 0$. Let

$$C_{AS}^k(U, G) = ProjLim_t C_{AS,\,t}^k \ (G). \tag{1.15}$$

The cohomology complex $C_{AS}^*(M, G) := \ \{C_{AS}^*(U, G), \ d\}$ is called *Alexander–Spanier complex*.

Theorem 1.3 (*Alexander–Spanier*, See [45])

For any locally finite simplicial or CW complex M, the homology of the Alexander–Spanier complex is canonically isomorphic to the singular cohomology $H^*(M, G)$.

Remark 1.1 In the Alexander–Spanier complex any cohomology k-chain on M is represented by a *non-local* function defined on M^k; this is a feature of non-commutative geometry.

Definition 1.4 The Alexander–Spanier co-chain ϕ has *cyclic symmetry* provided

$$\phi(x_0, x_1, \ldots, x_q) \ = \ (-1)^q \phi(x_1, \ldots, x_q, x_0).$$

The Alexander–Spanier co-chain ϕ is *skew symmetric* provided

$$\phi(x_{\sigma(0)}, x_{\sigma(1)}, \ldots, x_{\sigma(q)}) = Sign(\sigma)\ \phi(x_0, x_1, \ldots, x_q), \quad for\ any\ permutation\ \sigma.$$

Proposition 1.2 (See e.g. [96]) *Let $C^{*,\omega}_{AS}(M, G)$ denote the sub-complex of the Alexander–Spanier complex consisting of all co-chains which have cyclic symmetry or are skew-symmetric. Then*

(i) the inclusion of $H^{*,\omega}(M, G)$ into $H^{*,\omega}(M, G)$ induces isomorphisms in homology.
(ii) If M is a smooth manifold and $H^{*,\omega,smooth}(M, G)$ denotes the sub-complex of $H^{*,\omega}(M, G)$ consisting of all smooth, skew-symmetric Alexander–Spanier co-chains, then the inclusion of $C^{*,\omega,smooth}(M, G)$ into $H^{*,\omega}(M, G)$ induces isomorphisms in homology.
(iii) The mapping

$$f(\ x_0, x_1, \ldots, x_k\ ) \mapsto \frac{\partial^k}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_k^{i_k}} f(\ x_0, x_1, \ldots, x_k\ ))_{|\ diag} dx_1^{i_1} \wedge dx_2^{i_2} \wedge \ldots \wedge dx_k^{i_k} \tag{1.16}$$

is a chain homomorphism ξ_* from the Alexander–Spanier complex to de Rham complex which induces isomorphisms in homology

$$\xi_{k,*} : H^k_{AS}(M, \mathbb{R}) \longrightarrow H^k_{dR}(M). \tag{1.17}$$

The next observation is useful in non-commutative geometry.

Remark 1.2 Suppose *Cat* is a category of analytic structures on topological manifolds M with the property that the corresponding associative algebra $\mathscr{A}$ of functions on M is a topological algebra with the property that $Cat(M^{k+1})$ is the projective completion of the algebraic tensor product algebra $\otimes^{k+1}_{\mathbf{C}} Cat(M)$. This means that for any $f \in Cat(M^{k+1})$, the function f is given by the sum of a series

$$f = \sum_{i=1}^{\infty} f_0^i \otimes_{\mathbf{C}} f_1^i \otimes_{\mathbf{C}} \cdots \otimes_{\mathbf{C}} f_k^i. \tag{1.18}$$

This condition is satisfied, in particular, if $Cat = Top$ or $Cat = C^\infty$.

Then

$$d_r(f_0 \otimes_{\mathbf{C}} f_1 \otimes_{\mathbf{C}}, \ldots, \otimes_{\mathbf{C}} f_k) = f_0 \otimes_{\mathbf{C}} f_1 \otimes_{\mathbf{C}}, \ldots, \otimes_{\mathbf{C}} 1 \otimes_{\mathbf{C}} \ldots \otimes_{\mathbf{C}} f_k, \tag{1.19}$$

where the factor 1 is placed into the rth-position.

1.4 Alexander–Spanier Homology

We assume that M is a locally finite simplicial complex. We define the vector spaces

$$C^{AS}_{k,\,t}(\mathbf{R}) := \{\mu \mid \mu \ \ is\ a\ measure\ on\ \ U^k_t\}. \tag{1.20}$$

If $t_1 < t_2$, let ρ be the restriction of continuous functions defined on $U^k_{t_2}$ to the sub-space $U^k_{t_1}$. Let $\mu \in C^{AS}_{k,\,t_1}(\mathbf{R})$. Then the restriction ρ induces the measure $\rho_*(\mu)$ over the space $U^k_{t_2}$. The homomorphisms

$$\rho_* : C^{AS}_{k,\,t_1}(\mathbb{R}) \longrightarrow C^{AS}_{k,\,t_2}(\mathbb{R}) \tag{1.21}$$

define an inductive system of vector spaces

$$C^{AS}_k(M, \mathbb{R}) := IndLim_t C^{AS}_{k,\,t}(\mathbb{R}). \tag{1.22}$$

By definition, the real *Alexander–Spanier homology complex* is

$$C^{AS}_*(M, \mathbb{R}) := \{C^{AS}_*(M, \mathbb{R}),\ \partial\}. \tag{1.23}$$

and its homology is denoted $H^{AS}_*(M,\ \mathbb{R})$. For more information on Alexander–Spanier homology see Massey [73].

Theorem 1.4

(i) The Alexander–Spanier homology is functorially isomorphic to the real singular homology of M.
(ii) The integration induces a pairing

$$\int\ H^k_{AS}(M,\ \mathbb{R}) \otimes_{\mathbb{R}} H^{AS}_k(M,\ \mathbb{R}) \longrightarrow \mathbb{R}. \tag{1.24}$$

(iii) The pairing $\int$ coincides with the canonical pairing $H^k(M,\ \mathbb{R}) \otimes_{\mathbb{R}} H_k(M,\ \mathbb{R}) \longrightarrow \mathbb{R}$ via the isomorphism between the Alexander–Spanier cohomology/homology and the singular counter-parts.

1.5 Vector Bundles and Connections

The purpose of this section is to discuss how the vector bundles, linear connections and curvature of classical differential geometry pass into non-commutative geometry.

A vector bundle $\eta = (E, \pi, B)$ of rank N over the real or complex numbers consists of the *total space* E, *projection* $\pi : E \longrightarrow B$ and *base* B such that each

fibre $\pi^{-1}(x)$, for any point $x \in B$, is a vector space isomorphic to $\mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$, or $\mathbb{C}$. It is required that the bundle η be *locally trivial*, i.e. there exists an open covering $U = \{U_i\}$ of the base space and homeomorphisms $h_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{K}^n$ which sends each fibre $\pi^{-1}(x)$ isomorphically onto the fibre $\{x\} \times \mathbb{K}^n$, for any $x \in U_i$.

If all spaces in the definition of η are topological spaces, respectively, Lipschitz/quasi-conformal manifolds, or smooth manifolds, and the mappings π, h_i are, respectively, in the same category, the bundle η will be called a continuous bundle, respectively, Lipschitz/quasi-conformal or smooth bundle. Denote by *Cat* any of these categories.

If the base space is paracompact simplicial or CW-complex of finite dimension, then the bundle η may be embedded into a product bundle $\mathbf{N} := B \times \mathbf{K}^N$, with N sufficiently large, see Milnor [66].

We may choose in the product bundle $\mathbf{N}$ the product Hermitian, or Euclidean, structure, depending on whether $\mathbb{K} = \mathbf{C}$, or $\mathbb{R}$.

Let A_{Cat} be the associative unital algebra

$$A_{Cat} := \{f \mid f : B \longrightarrow \mathbf{K}, \quad where\ f\ is\ a\ Cat\ function\} \tag{1.25}$$

Let $px : \{x\} \times \mathbb{K}^N \longrightarrow \pi^{-1}(x)$ be the orthogonal projection, for any $x \in B$. The collection of matrices $p(x)$ produces an $N \times N$ matrix p with entries in A_{Cat}.

Theorem 1.5

(i) For any embedded bundle η into a product bundle of rank N there corresponds an $N \times N$ matrix p with entries belonging to the algebra A_{Cat}.

(ii) The matrix p is an idempotent ($p^2 = p$); if p is produced by pointwise orthogonal projections, then p is self-adjoint ($p^* = p$) and is called a projector.

(iii) Vice versa, for any idempotent matrix p with entries in the algebra A_{Cat} there corresponds a bundle η such that, up to an isomorphism, its corresponding idempotent is p

(iv) $1 - p$ is also an idempotent; if p is a projector, then $1 - p$ is the orthogonal projection onto the orthogonal complement $\eta^{\perp}$ of the bundle η.

(v) The direct sum $p \oplus (1 - p)$ decomposes the $\mathscr{A}_{Cat}$-free module of rank N into a direct sum

$$(\mathscr{A}_{Cat})^N = \Gamma(\mathbf{N}) = \Gamma(\eta) \oplus \Gamma(\eta^{\perp}) \tag{1.26}$$

and therefore the space $\Gamma(\eta)$ of sections of the bundle η is a finite projective $\mathscr{A}_{Cat}$-module.

(vi) Vice versa, to any finite projective $\mathscr{A}_{Cat}$-module M there corresponds a bundle η such that, up to an isomorphism, its associated $\mathscr{A}_{Cat}$-module is the module M.

Proof Only (iii) and (vi) need a discussion; we leave this as an exercise for the reader.

Remark 1.3 In non-commutative geometry bundles over the associative algebra $\mathscr{A}$ are by definition idempotents or, equivalently, finite $\mathscr{A}$-projective modules (Serre–Swan theorem) see [43].

An *idempotent defines a bundle together with an embedding into a product bundle*. In the case of a projective module, although one knows that such an embedding *exists*, the embedding is *not provided*.

1.6 Characteristic Classes

Characteristic classes are co-homological invariants of bundles.

1.6.1 Chern Classes

Chern classes are integral cohomology classes (of the base space) associated to complex vector bundles over CW-complexes.

Let $\xi : E \xrightarrow{\pi} M$ be a complex vector bundle of rank n over the CW-complex M. The *Chern classes* $c_i(E) \in H^{2i}(M, \mathbf{Z})$ represent the *cohomology obstruction* to the existence of $n - i + 1$-linearly independent sections in the bundle ξ, see Milnor [66].

The *Chern classes* are characterised by the following properties

Theorem 1.6

(i) For any complex vector bundle ξ over the CW-complex M, of rank n, the Chern class $c_i(\xi) \in H^{2i}(M, \mathbf{Z})$, $0 \leq i \leq n$; $c_0(\xi) = 1$, for $n + 1 \leq i$, $c_i(\xi) = 0$,

(ii) for any bundle as in (i), $c_n(\xi)$ represents the obstruction to the existence of one non-vanishing section in ξ

(iii) the Chern classes are natural, *i.e. for any continuous mapping $f : N \longrightarrow M$ and for any complex vector bundle ξ over M*

$$c_i(f^*\xi) = f^* c_i(\xi); \tag{1.27}$$

(iv) for any two bundles ξ_1, ξ_2 over M

$$c_i(\xi_1 \oplus \xi_2) = \sum_{r+s=i} c_r(\xi_1) \cup c_s(\xi_2). \tag{1.28}$$

The Chern class $c_n(\xi)$ from (ii) is the *Euler class* of ξ. For complex line bundles ξ $c_1(\xi)$ has a simple description in terms of the transition functions of the bundle, see Weil [22].

Definition 1.5 The formal power series

$$c(\xi) := \sum_{0 \leq r} c_r(\xi)\, t^r \tag{1.29}$$

is called *total Chern class*.

The property (1.28) becomes

$$c(\xi_1 \oplus \xi_2) = c(\xi_1) \cup c(\xi_2). \tag{1.30}$$

Definition 1.6 Let ξ_1 and ξ_2 be complex vector bundles over the spaces X_1, X_2, respectively. Their *external product* is $\xi_1 \times \xi_2$ a bundle over the space $X_1 \times X_2$.

If $X_1 = X_2 = X$ and $\iota : X \longrightarrow X \times X$ denotes the inclusion onto the diagonal, then we obtain the *internal product* $\xi_1 \otimes \xi_2$.

Theorem 1.7 *For the external product the total Chern class satisfies the relation*

$$c(\xi_1 \times \xi_2) = c(\xi_1) \times c(\xi_2). \tag{1.31}$$

while for the internal product one has

$$c(\xi_1 \otimes \xi_2) = c(\xi_1) \cup c(\xi_2). \tag{1.32}$$

1.6.2 Pontrjagin Classes

Definition 1.7 For any *real* vector bundle ξ, the Pontrjagin classes $p_i(\xi)$ are by definition, up to a sign, the even order Chern classes of the *complexified bundle*

$$p_i(\xi) := (-1)^i c_{2i}(\xi \otimes_{\mathbb{R}} \mathbb{C}). \tag{1.33}$$

Remark 1.4 The Chern classes of $c_i(\ \xi \otimes_{\mathbf{R}} \mathbf{C})$, $i = odd$, are 2-torsion elements of the corresponding cohomology groups, see Milnor [66, p. 174].

Theorem 1.8 (Milnor [39])

Let M be a smooth manifold and let $T(M)$ denote its tangent bundle.

The integer Pontrjagin classes $p_i(T(M)) \in H^{4i}(M, \mathbf{Z})$ *depend on the differentiable structure; they are not topological invariants.*

However, the image of the integral Pontrjagin classes into the rational cohomology, i.e. the *rational* Pontrjagin classes $p_i \in H^{4i}(M, \mathbf{Q})$, behaves differently. The following result is of fundamental importance.

Theorem 1.9 (Novikov [41])
The rational Pontrjagin classes *of the tangent bundle to any differentiable manifold*

$$p_i(T(M)) \in H^{4i}(M, \mathbf{Q}) \tag{1.34}$$

are topological invariants.

Prior to the Milnor and Novikov results, the following two theorems were known.

Theorem 1.10 (Rohlin and Svark [18]) *The rational Pontrjagin classes of a triangulated smooth manifolds are combinatorial invariants.*

Theorem 1.11 (Thom [24]) *The rational Pontrjagin classes are combinatorial invariants.*

The solution of the *Hauptvermutung* problem given by Kirby and Siebenmann [56] has as corollary the following theorem.

Theorem 1.12 (Kirby and Siebenmann [56], See Also [69])
Topological manifolds possess rational Pontrjagin classes.

The Novikov theorem gives information about the rational Pontrjagin classes of *smooth* manifolds. The Kirby–Siebenmann result covers *all* topological manifolds.

1.6.3 Chern Character

The Chern character derives from the Chern classes. The Chern character establishes an isomorphism between the rational K-theory with bi-graded rational cohomology.

Definition 1.8 Let ξ be a complex vector bundle of rank n. Let's represent the total Chern class

$$c(\xi) = \sum_{0 \le r \le n} c_r t^r := \prod_{1 \le r \le n} (1 + \lambda_r(\xi)\, t), \tag{1.35}$$

where t and λ_r are commuting indeterminate. (This decomposition, which we would referred to as *Hirzebruch reduction procedure*, could seem formal, but it has a topological justification, see Hirzebruch [32, section 4.2] and; see also Milnor [66, §19. and p. 307]).

For the definition of the *Chern character*, $Ch(\xi)$, see also [32, §. 10.1].

$$Ch(\xi) = \sum_{0 \le r \le n} e^{\lambda_r(\xi)\, t} := \sum_{0 \le s} Ch_s(\xi)\, t^s, \tag{1.36}$$

where

$$e^{\lambda_r(\xi)\, t} = \sum_{0 \le s} \frac{\lambda_r(\xi)^s}{s!}\, t^s = \sum_{0 \le s} \frac{1}{s!} \Big(\sum_{1 \le r \le n} \lambda_r(\xi)^s \Big)\, t^s \tag{1.37}$$

is the exponential series.

Formula (1.35) describes the Chern classes c_r as the rth-fundamental symmetric function in the variables $\lambda_1, \lambda_2, \ldots, \lambda_n$

$$c_r(\xi) = \sum_{1 \le i_1 < i_2 <, \ldots, < i_r \le n} \lambda_{i_1} \lambda_{i_2}, \ldots, \lambda_{i_r}. \tag{1.38}$$

Definition 1.9 The coefficients of the formal series (1.36) are called *components of the Chern character*.

The definition implies the first part of the next theorem.

Theorem 1.13

(i) The components of the Chern character are symmetric functions in the indeterminate λ_r and therefore $Ch_s(\xi)$ are well defined universal polynomials *with rational coefficients of the basic symmetric functions, $c_r(\xi)$.*
(ii) Vice versa, the Chern character determines the rational Chern classes by means of (inverted) universal polynomials *with rational coefficients.*

Proof See [32, Chapter 3] and Milnor [66, p. 306].

Chern–Weil theory—discussed in the next section—allows one to express the characteristic classes of bundles in terms of their *curvature*. In particular, the Chern character is given by the next theorem.

Theorem 1.14 *Let $p \in \mathbf{M}_n(M)$ be an idempotent, let $\nabla := p\ d\ p$ be its Levi-Civita connection, and let $R_\nabla = (d\ p)(d\ p)\ p$ be its curvature.*
Then

$$Ch_*(p) = tr\ e^{\frac{1}{2\pi i} R_\nabla t} \tag{1.39}$$

is a formal sum of closed forms in the de Rham complex $\Omega_*(C^\infty(M))$; this form represents the Chern character of p.

Proof The reason why this form represents the Chern character of the bundle follows from two facts. Firstly, one verifies that the formula is correct for complex

line bundles; secondly, one uses the Hirzebruch reduction procedure to extend the result from line bundles to arbitrary bundles.

This theorem passes to non-commutative geometry after the homology theory framework (called *cyclic homology*) is defined, see *Connes Karoubi Chern character*, discussed in Sect. 2.4.11.

1.6.4 Chern–Weil Theory

Definition 1.10 Let $\mathbf{M}_n(\mathbb{C})$ denote the algebra of $n \times n$-matrices with entries in $\mathbf{C}$. The function $P : \mathbf{M}_n(\mathbf{C}) \longrightarrow \mathbb{C}$, depending continuously of its entries, is called *invariant* provided for any matrices $M,\ N \in \mathbf{M}_n(\mathbf{C})$

$$P(MN) = P(NM), \tag{1.40}$$

or equivalently

$$P(MNM^{-1}) = P(N), \tag{1.41}$$

for M invertible.

(Clearly, if the first relation holds and M is invertible, replace in it $N = ZM^{-1}$ to get the second relation. Vice versa, if the second relation holds, with M invertible, take $N = ZM$ to get the first relation. Any singular matrix is the limit value of a continuous path of non-singular matrices).

Theorem 1.15 (Chern–Weil Theorem, See Chern [6] and Bott [63]) *Let ξ be any complex vector bundle of rank n over the smooth manifold M. Let ∇ be any smooth linear connection in ξ and let R_∇ be its curvature. Let P be any invariant function over the algebra of $n \times n$ matrices.*

Then

(i) $P(R_\nabla)$ *is a well defined differential form on* M
(ii) the differential form $P(R_\nabla)$ *is closed in the de Rham complex, i.e.* $d\ P(R_\nabla) = 0$
(iii) the de Rham cohomology class of the cycle $P(R_\nabla)$ *does not depend on the chosen connection* ∇. *An analogue statement holds for principal bundles.*

The determinant is an invariant function.
One has the following:

Theorem 1.16 *Let ξ be any complex vector bundle of rank n over the smooth manifold M. Let ∇ be any smooth linear connection in ξ and let R_∇ be its curvature. Then*

$$det(I_n + \frac{1}{2\pi i} R_\nabla\, t) = 1 + c_1(\xi)\, t + c_2(\xi)\, t^2 + \ldots + c_r(\xi)\, t^r + \ldots. + c_n(\xi)\, t^n \tag{1.42}$$

or

$$det(I_n + R_\nabla\, t) = 1 + \frac{1}{2\pi i} c_1(\xi)\, t + \frac{1}{(2\pi i)^2} c_2(\xi)\, t^2 + \ldots + \frac{1}{(2\pi i)^r} c_r(\xi)\, t^r + \ldots. \tag{1.43}$$

For any real vector bundle of rank n, the Pontrjagin classes are given by

$$det(I_n + R_\nabla\, t) = 1 + \sigma_1\, t + (2\pi)^{-2} p_1(\xi)\, t^2 + \sigma_3 t^3 + \ldots. + (2\pi)^{-2k}\, p_k(\xi)\, t^{2k} + \sigma_{2k+1}\, t^{2k+1} + \ldots. \tag{1.44}$$

while σ_{2k+1} are de Rham co-boundaries.

For any complex vector bundle ξ the Chern character is

$$Ch(\xi) = trace\, e^{\frac{1}{2\pi i} R_\nabla\, t}. \tag{1.45}$$

Vice versa, the components of the Chern character determine the Chern classes of the bundle by means of universal polynomials with rational coefficients.

The proof of these results could be found in [66], see also [63].

Theorem 1.17 *The Chern character passes to K-theory*

$$Ch : K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \bigoplus_{q=0,1,2,\ldots} H^{2q}(X, \mathbb{Q}). \tag{1.46}$$

It is a ring isomorphism.

$$Ch(f^*\, \xi) \;=\; f^*\, Ch(\xi) \tag{1.47}$$

$$Ch(\xi_1 \oplus \xi_2) \;=\; Ch(\xi_1) \;\oplus\; Ch(\xi_2) \tag{1.48}$$

$$Ch(\xi_1 \;\otimes_{\mathbf{C}}\; \xi_2) \;=\; Ch(\xi_1) \;\cup\; Ch(\xi_2). \tag{1.49}$$

1.6.5 Geometric Construction of the Chern Character

1.6.5.1 The Cube Construction

For this sub-section we use Teleman [44].

Let M be a smooth Riemannian manifold and let ξ be a vector bundle over M endowed with a linear connection ∇. Let $(x_1, x_2, \ldots, x_n)$ be local system of coordinates about the point p of the manifold. Let $X_1, X_2, \ldots, X_q$ be linearly independent tangent vectors to M at the point p. We may assume that these vectors are ortho-normal. The affine structure of the local coordinate system allows one to speak about segments joining points and coordinate lines. Let P be the q-dimensional parallelepiped, with respect to the local coordinates, whose sides are tangent, respectively, to the vectors $X_1, X_2, \ldots, X_q$ in such a way that the vectors X_i are the unit basic tangent vectors $X_i = \partial/\partial x_i$. Suppose the point p is the origin of the coordinate system.

Let $\tau(y, x)$ denote the parallel transport along the *segment* joining the point x with the point y. For each element σ in the permutation group Π_q and for each small $t \in \mathbf{R}^+$, let $\lambda_\sigma(t)$ be the unique path in P consisting of linear segments which start at $p = (0, 0, \ldots., 0)$ and finish at the point $(t, t, \ldots, t, 0, 0 \ldots, 0)$, such that the coordinates, $x_1, x_2, \ldots, x_q$ change just ones from 0 to t, in the order prescribed by the permutation σ. Let $x_{\sigma(i),t}$ denote the ith point of this path.

Let η be any vector belonging to the fibre of the bundle over the point p. Consider the alternating sum of the vectors obtained by transporting the vector η along the path $\lambda_\sigma(t)$; let $\Phi_q(\eta, t)$ be the resulting vector.

The correspondence $\eta \mapsto \Phi_q(\eta, t)$ is linear. Consider the function $\Phi_q(t)$ obtained by taking the trace of the function $\Phi_q(-, t)$

Definition 1.11

$$\Phi_q(t) := tr \sum_{\sigma \in \Pi_q} Sign(\sigma)\ \tau(x_{\sigma(q),t}, x_{\sigma(q-1),t}), \ldots\ldots, \circ\tau(x_{\sigma(2),t}, x_{\sigma(1),t}) \circ \tau(x_{\sigma(1),t}, 0). \tag{1.50}$$

In the sequel this construction will be referred to as the *cube construction*.

Theorem 1.18 (Teleman [44])

(i) For an even number of tangent vectors, the cube construction gives

$$Lim_{t \searrow 0} \frac{1}{t^{2q}} \Phi_{2q}(t) := \frac{q!(2q)!(-2\pi i)^q}{2^{2q}}\ \Psi_q, \tag{1.51}$$

where Ψ_q denotes the qth-component of the Chern character of the bundle ξ.

(ii) For an odd number of ortho-normal tangent vectors $\{X_1, X_2, \ldots, X_{2q}, X_{2q+1}\}$ *the cube construction gives*

$$Lim_{t\searrow 0}\frac{1}{t^{2q}}\Phi_{2q+1}(t) = \frac{q!(2q)!(-2\pi i)^q}{2^{2q}}\sum_{i=1}^{i=2q+1}(-1)^i\Psi_q(X_1, X_2, \ldots, \hat{X^i}, \ldots, X_{2q}, X_{2q+1}). \tag{1.52}$$

Remark 1.5 The geometric construction of the Chern character provided by Theorem 1.18 shows that the Chern character is the *extension of the Riemann curvature tensor* from dimension 2 to even higher dimensions.

1.6.5.2 The Simplex Construction

The *simplex construction*, due to Teleman [52], introduces two changes in the cube construction [44]: (1) the affine segments (relative to local coordinate charts) are replaced by *arcs of geodesics* joining close points (relative to an additional Riemannian metric on M), (2) the parallel transport along the sides of the cube is replaced by parallel transport along the sides of the simplex, which begin and finish at the first vertex of the simplex; the permutation σ describes the order in which the vertices of the simplex are used to produce the path.

Remark 1.6 The formula (1.51), improved by the simplex construction, may be rewritten using the notion of *direct connection* defined in Sect. 1.7. In this form it becomes the Chern character of idempotents. Applied to topological manifolds, it leads to the *rational Pontrjagin classes*, see Sect. 10.5.

1.6.6 Levi-Civita and Linear Connections of Idempotents

If M is a smooth manifold embedded in the Euclidean space $\mathbb{R}^N$, then its tangent bundle is embedded in a product bundle of rank N, denoted N. The fibre-wise orthogonal projection of the product bundle N onto the embedded tangent bundle produces an idempotent p.

The space of smooth sections in N will be denoted $\Gamma(\mathrm{N})$. The vectors belonging to $\Gamma(\mathrm{N})$ will be represented as *column vectors*. These vectors have N components; each component is a smooth function over M.

Instead of considering a manifold M and its tangent bundle embedded in an Euclidean space, we may consider an arbitrary idempotent $p \in \mathbf{M}_N(C^\infty(M))$. Let $\Gamma(p)$ denote the range of the idempotent p. This space represents the vector space of smooth sections in the bundle represented by p; it is a sub-space of the space of sections of the product bundle $\mathbf{N}$.

Let $e_i \in \Gamma(\mathbf{N})$, $1 \leq i \leq N$, denote the vector having all components equal to zero, with the exception of the ith component equal to 1. As the algebra $C^\infty(M)$

is commutative, the left and right module structures coincide. We insist, however, that in the next considerations we are going to work exclusively with the right module structure of $\Gamma(\mathrm{N})$ over the algebra $C^\infty(M)$. Any vector $(f_1, \ldots, f_N)^{tr} \in \Gamma(\mathrm{N})$ decomposes uniquely as a linear combination of the vectors e_i with smooth coefficients

$$(f_1, \ldots, f_N)^{tr} = \sum_i e_i \ f_i. \tag{1.53}$$

The right module structure is given by

$$(f_1, \ldots, f_N)^{tr} \cdot f = (f_1 \cdot f, \ldots, f_N \cdot f)^{tr}. \tag{1.54}$$

Although the algebra $C^\infty(M)$ is commutative and the following identity

$$p \cdot f = f \cdot p, \quad \forall\, f \in C^\infty(M), \tag{1.55}$$

holds, we are not going to use this identity. With this precaution, in all analogues situations in non-commutative geometry, the algebra $C^\infty(M)$ could be replaced by *any* associative algebra.

Definition 1.12 A *linear connection* into the bundle p, over the field $\mathbf{K} = \mathbf{R}$, *or* $\mathbb{C}$, is a $\mathbf{K}$-mapping $\nabla : \Gamma(p) \longrightarrow \Gamma(p) \otimes_{C^\infty(M)} \Omega^1(M)$, satisfying the Leibnitz relation.

$$\nabla(\xi . f) = (\nabla\, \xi\,)\, f + \xi\, df, \quad \forall \xi \in \Gamma(p), \quad \forall f \in C^\infty(M). \tag{1.56}$$

Proposition 1.3 *For any linear connection* ∇ *in the bundle* p*, there exists an unique* connection extension

$$\nabla : \Gamma(p) \otimes_{C^\infty(M)} \Omega^i(M) \longrightarrow \Gamma \otimes_{C^\infty(M)} \Omega^{i+1}(M) \tag{1.57}$$

satisfying the Leibnitz relation

$$\nabla(\,\xi \cdot \omega) = (\nabla\, \xi\,) \cdot \omega + (-1)^i\, \nabla\, \xi\, .(d\, \omega), \tag{1.58}$$

for any $\xi \in \Gamma(p) \otimes_{C^\infty(M)} \Omega^i(M)$ *and* $\omega \in \Omega^j(M)$.

Proof We define the extension of the linear connection

$$\nabla(\xi \otimes_{C^\infty(M)} \omega_i) = \nabla(\xi) \otimes_{C^\infty(M)} \omega_i + (-1)^i\, \xi \otimes_{C^\infty(M)} d\omega_i\ , \tag{1.59}$$

for any $\xi \in \Gamma(p)$ and $\omega_i \in \Omega_i(M)$.

The extension satisfies the Leibnitz relation.

$\nabla(\xi)$ is called *covariant derivative* of the section ξ. The following properties of linear connections are familiar.

Proposition 1.4 *Any linear connection satisfies the properties*

1. $\nabla_- : \Gamma(p) \longrightarrow \Gamma(p) \otimes_{C^\infty(M)} \Omega^1(M)$,
2. $\nabla_X(p)$ *is a* $C^\infty(M)$*-homomorphism in the argument* X,
3. $\nabla_X(\xi)$ *is* $\mathbb{R}$*-linear in the argument* ξ *and*
4. $\nabla_X(\xi)$ *satisfies the Leibnitz identity in the variable* ξ

$$\nabla_X(\xi\ f) = \xi \otimes_{C^\infty(M)} df + (\nabla_-(\xi))\ f, \quad \forall f \in C^\infty(M). \tag{1.60}$$

Proof We leave to the reader to check this proposition.

Proposition 1.5 *For any linear connection* ∇ *in the bundle* p*, the homomorphism* $R = \nabla \circ \nabla$ *is a* $C^\infty(M)$*-homomorphism.*

Proof One has

$$\nabla \circ \nabla(\xi.f) = \nabla(\nabla\xi).f + \xi \otimes_{C^\infty(M)} df) =$$
$$(\nabla(\nabla\xi)).f - (\nabla\xi)).df + (\nabla\xi)).df + (\nabla(\nabla\xi)).d(df)) = (\nabla(\nabla\xi)).f. \tag{1.61}$$

Definition 1.13 The endomorphism $R = \nabla \circ \nabla$ is called the *curvature* of the connection.

Proposition 1.6 *Let* ∇_1 *and* ∇_0 *be two linear connections in the bundle defined by the idempotent* p.

Then

$$\nabla_1 - \nabla_0 : \Gamma(p) \longrightarrow \Gamma(p) \otimes_{C^\infty(M)} \Omega^1(M) \tag{1.62}$$

is a $C^\infty(M)$*-homomorphism.*

Proof The definition of linear connections gives, for any $\xi \in \Gamma(p)$ and $f \in C^\infty(M)$

$$(\nabla_1 - \nabla_0)(\xi . f) =$$
$$(\nabla_1(\xi).f + \xi \otimes_{C^\infty(M)} df) - (\nabla_0(\xi).f + \xi \otimes_{C^\infty(M)} df) = ((\nabla_1 - \nabla_0)\xi).f. \tag{1.63}$$

Remark 1.7 All $C^\infty(M)$-modules considered next are right modules.

The idempotent p defines a vector bundle together with an embedding of it into a product bundle $\mathbb{N}$.

The structure equations describe how the connection and curvature of the bundle are seen from the product bundle perspective.

In the product bundle the exterior derivative, acting on functions belonging to $C^\infty(M)$, is available.

Theorem 1.19 (Structure Equations)

(i) First structure equation. *For any linear connection ∇ in the bundle p, there exist well defined differential forms $\omega^0_{ij} \in \Omega^1(M)$ such that*

$$\nabla(p\ e_i)\ = \sum_j\ (p\ e_j) \otimes_{C^\infty(M)}\ \omega^0_{ji}. \tag{1.64}$$

(ii) Second structure equation. *The curvature of the connection is given by the equations*

$$R(p\ e_i) = \sum_{k,j} (p\ e_k) \otimes_{C^\infty(M)} (\ \omega^0_{kj} \wedge \omega^0_{ji}\ +\ d\omega^0_{ki}\ ). \tag{1.65}$$

(In non-commutative geometry the algebra $\Omega^*(M)$ is replaced by another graded differential algebra having good homological properties—universal algebra*. In this algebra the wedge product is denoted by a dot. It is not graded commutative).*

Remark 1.8 If we denote by ω_{ji} *any* solution satisfying the first structure equation, such a solution is not unique. In fact, if $p\ e_j\ =\ 0$, the coefficient ω_{ji} is arbitrary. The theorem states that there exists a well defined particular solution ω^0_{ji}.

Proof

(i) By definition ∇ takes values in $\Gamma(p) \otimes_{C^\infty(M)}\ \Omega^1(M)$. This means that there exist elements $\xi_\iota \in \Gamma(p)$ and differential forms $\varphi_\iota \in \Omega^1(M)$ such that

$$\nabla(pe_i)\ =\ \sum_\iota \xi_\iota \otimes_{C^\infty(M)}\ \varphi_\iota. \tag{1.66}$$

As the elements $p\ e_i$ constitute a system of generators of the right module $\Gamma(p)$, there exists a system of elements $f_{\iota,i} \in C^\infty(M)$ such that

$$\xi_\iota\ =\ \sum_j (\ p\ e_j\ ).f_{j,\iota}. \tag{1.67}$$

Then

$$\begin{aligned}\nabla(p\ e_i)\ =\ \sum_{j,\iota} (\ p\ e_j\ ).f_{j,\iota} \otimes_{C^\infty(M)}\ \varphi_\iota\ = \\ \sum_{j,\iota} (\ p\ e_j\ ) \otimes_{C^\infty(M)}\ f_{j,\iota}.\varphi_\iota.\end{aligned} \tag{1.68}$$

Denote $\sum_{j,\iota} f_{j,\iota} \cdot \varphi_\iota = \omega_{ij}$. Then

$$\nabla(pe_i) = \sum_{j,i} (\, p\, e_j\,) \otimes_{C^\infty(M)} \omega_{ji}, \tag{1.69}$$

where $\omega_{ji} = \sum_{j,\iota} f_{j,\iota}.\varphi_\iota$.

(ii) We use the first structure equation to compute the curvature $R = \nabla\,\nabla$

$$R(\,p\,e_i\,) = \nabla\,\nabla\,(\,p\,e_i\,) = \sum_j \nabla\,((p\,e_j) \otimes_{(C^\infty M)} \omega_{ji}\,) =$$

$$\sum_j (\,\nabla\,(p\,e_j) \otimes_{(C^\infty M)} \omega_{ji} + (p\,e_j).d(\omega_{ji}) = \sum_{jk} ((p\,e_k) \otimes_{(C^\infty M)} \omega_{kj}\omega_{ji} + (p\,e_j).d(\omega_{ji}) =$$

$$\sum_{jk} ((p\,e_k) \otimes_{(C^\infty M)} \omega_{kj}\omega_{ji} + \sum_j (p\,e_j).d(\omega_{ji}) = \sum_{jk} ((p\,e_k) \otimes_{(C^\infty M)} \omega_{kj}\omega_{ji} + \sum_k (p\,e_k).d(\omega_{ki}). \tag{1.70}$$

This completes the proof of (ii).

Definition 1.14

(i) The forms ω_{ji} are called *connection components*.

(ii) The second structure equation may be rewritten (summation convention used)

$$R(p\,e_i) = (p\,e_k) \otimes_{C^\infty(M)} \Omega_{ki}, \tag{1.71}$$

where

$$\Omega_{ki} = \omega_{kj}\omega_{ji} + d\omega_{ki}. \tag{1.72}$$

The differential forms Ω_{ki} are called *curvature components*.

Remark 1.9 The connection coefficients and the curvature components defined above do not coincide with the classical definitions, see [66, pp. 294, 297]. Classically, the connection coefficients and the curvature components are defined in a *local system of coordinates*, or *product bundles*. Our treatment differs with respect to the classical treatment. In fact, our treatment describes these quantities *as if* the manifold M were embedded in an Euclidean space; compare also with [89, §1.9] and proof of Theorem 1.19.

Although the quantities involved in the structure equations we present here do not coincide with the classical definitions, the equations remain the same.

Formally, the second structure equation may be written

$$R = \Gamma\,\Gamma + d\,\Gamma. \tag{1.73}$$

Comparing with the second structure equation presented in [66, pp. 294, 297] we see that there is a sign discrepancy. The discrepancy is due to a different definition

of the connection extension, compare [66, p. 292]. Looking at the components only, the second structure equation writes

$$\Omega_{ki} \;=\; d\;\Gamma_{ki} \;+\; \Gamma_{kj}\;\Gamma_{ji}. \tag{1.74}$$

From this relation we get

$$d\;\Omega_{ki} \;=\; d\;d\Gamma_{ki} \;+\; (d\;\Gamma_{kj})\;\Gamma_{ji} \;-\; \Gamma_{kj}\;(d\;\Gamma_{ji}) \;=\; [\;d\Gamma,\;\Gamma\;]_{ki}, \tag{1.75}$$

or

$$d\;\Omega \;=\; [\;d\Gamma,\;\Gamma\;], \tag{1.76}$$

known as *Bianchi identity*.

1.6.7 Levi-Civita Connections

A particular case of a linear connection in the bundle p is the *Levi-Civita connection*.

Definition 1.15

$$\nabla_{-}(\xi) := \;p \circ d \circ p\;(\xi) \in \Gamma(p) \otimes_{C^\infty(M)} \Omega^1(M) \tag{1.77}$$

where d is the direct sum of exterior derivatives d

$$d\;(\;f_1,\;f_2,..,\;f_N\;)^{tr} := \;(d\;f_1,\;d\;f_2,..,d\;f_N)^{tr} \in \Gamma(p) \otimes_{C^\infty(M)} \Omega^1(M) \tag{1.78}$$

acting onto the sections $\xi = (\;f_1, f_2, \ldots, f_N\;)^{tr}, \quad f_i \in C^\infty(M)$, of the product bundle $M \times \mathbb{R}^N$.

The above formula has to be understood in the following manner. Of course, if ξ is a section of the bundle p, then $p(\xi) = \xi$. On the other side, $(d\;f_1, d\;f_2, .., d\;f_N)^{tr}(X) = (X\;f_1, X\;f_2, .., X\;f_N)^{tr}$ is a section in the product bundle $M \times \mathbb{R}^N$.

Let $p \;=\; (p_{ij})$. The product of matrices is given by

$$(p\;.\;p)_{ki} \;=\; \sum_j\; p_{kj}\;p_{ji} \;=\; p_{ki}. \tag{1.79}$$

The matrix p acts on the vector e_i by the formula

$$(p\;e_i) = \sum_j\; e_j\;p_{ji}. \tag{1.80}$$

Definition 1.16 Define the *Levi-Civita connection*

$$\nabla(p\ e_i)\ =\ p\ d\ (p\ e_i). \tag{1.81}$$

In this formula the operator d acts onto the components of vectors belonging to $\Gamma(\mathbf{N})$; these are smooth functions.

The Levi-Civita connection satisfies

$$\nabla(p\,e_i) = \nabla(p\,p\,e_i) = \nabla p\,(\sum_j e_j\ p_{ji}\ ) = \sum_j \nabla((p\,e_j)\,.\,p_{ji}) = \sum_j [\,\nabla(p\,e_j)\,p_{ji}\ +\ (p\,e_j)\,d\,p_{ji}\,] \tag{1.82}$$

Proposition 1.7 *For any idempotent p the following identities hold*

$$p\ (dp) = (dp)\ (1-p) \tag{1.83}$$

$$(1-p)\ (dp) = (dp)\ p \tag{1.84}$$

$$p\ (dp)\ (dp) = (dp)\ (dp)\ p. \tag{1.85}$$

Proof As the idempotent p satisfies the identity $p^2\ =\ p$, the first two identities follow by differentiation

$$(d\ p)\ p\ +\ p\ (d\ p)\ = d\ p^2\ =\ d\ p. \tag{1.86}$$

The third identity is a consequence of the first two.

Proposition 1.8 *The Levi-Civita connection is a linear connection.*

Proof We leave to the reader to check this result.

Remark 1.10 This definition of Levi-Civita covariant derivatives ∇_k coincides with the extension of the Levi-Civita connection [89, Proposition 1.10].

Theorem 1.20 *The Levi-Civita connection $\nabla = \{\nabla_k\}_k$ associated to the idempotent p satisfies the relation*

$$(\nabla_{k+1} \circ \nabla_k)(\ \omega_1, \omega_2, \ldots, \omega_N\ )^{tr}\ =\ p\ (d\ p) \wedge (d\ p) \wedge (\ \omega_1, \omega_2, \ldots, \omega_N\ )^{tr} \tag{1.87}$$

for any section $\xi\ =\ (\ \omega_1, \omega_2, \ldots, \omega_N\ )^{tr}$ in the bundle p_{Ω^k}.

Proposition 1.9 *The* curvature of the Levi-Civita connection *is*

$$R_\nabla\ :=\ p\ (d\ p) \wedge (d\ p). \tag{1.88}$$

Proof

$$
\begin{aligned}
(\nabla_{k+1} \circ \nabla_k)(\omega_1, \omega_2, \ldots, \omega_N)^{tr} &= (p\ d\ p)(p\ d\ p)(\omega_1, \omega_2, \ldots, \omega_N)^{tr} = \\
&= (p\ d\ p\ d\ p)(\omega_1, \omega_2, \ldots, \omega_N)^{tr} = \\
&= p\,[(d\ p) + p\ d] \wedge d\ p(\omega_1, \omega_2, \ldots, \omega_N)^{tr} = \\
&= p\ (d\ p) \wedge d\ p(\omega_1, \omega_2, \ldots, \omega_N)^{tr} = \\
&= p\ (d\ p) \wedge [(dp) \wedge (\omega_1, \omega_2, \ldots, \omega_N)^{tr} + p\ d(\omega_1, \omega_2, \ldots, \omega_N)^{tr}] = \\
&= p\ (d\ p)\,[(dp) \wedge (\omega_1, \omega_2, \ldots, \omega_N)^{tr} + p\ (d\ p)\ p\ d(\omega_1, \omega_2, \ldots, \omega_N)^{tr} = \\
&= p\ (d\ p) \wedge (dp) \wedge (\omega_1, \omega_2, \ldots, \omega_N)^{tr} + p\ (d\ p)\ p\ d(\omega_1, \omega_2, \ldots, \omega_N)^{tr} = \\
&= p\ (d\ p)\ (d\ p) \wedge (\omega_1, \omega_2, \ldots, \omega_N)^{tr} + p\ (d\ p)\ d\ (1-p)(\omega_1, \omega_2, \ldots, \omega_N)^{tr} =
\end{aligned}
\tag{1.89}
$$

(the section ξ belongs to the kernel of $1 - p$)

$$
= p\ (d\ p) \wedge (d\ p) \wedge (\omega_1, \omega_2, \ldots, \omega_N)^{tr}. \tag{1.90}
$$

Remark 1.11 The Levi-Civita connection suffices for the *definition* of the Chern character of idempotents in non-commutative geometry. However, linear connections are used to show that the Chern character may be defined in more general contexts.

The Chern character of idempotents will be discussed in Sect. 3.12.

1.6.8 Infinitesimal Connections

In this sub-section we will gradually extend the generality of our considerations until we will reach a point where the passage to non-commutative geometry becomes easily understood.

To begin with, we mention the following theorem.

Theorem 1.21 (Séminaire Sophus Lie, 1954–1955 [13, Ch. III, §. 9])
Lie algebra extensions

$$
0 \longrightarrow L' \longrightarrow L \longrightarrow L'' \longrightarrow 0, \tag{1.91}
$$

with L' a commutative Lie algebra, are classified by the elements of the group $H^2(L'', L')$.

We assume the reader is familiar with the notion of principal bundle. For more details, the reader may consult the treatise Kobayashi and Nomizu [34, Vol. I and II].

In 1957, Atiyah [17] associated with any *analytic* principal bundle P with structural group G over the manifold M an exact sequence of Lie algebras

$$0 \longrightarrow L(P) \longrightarrow Q \longrightarrow T \longrightarrow 0; \tag{1.92}$$

here Q denotes the Lie algebra of right-invariant tangent fields over the total space of the bundle, T denotes the Lie algebra of tangent fields on the base space and $L(P)$ denotes the Lie algebra of right-invariant tangent fields along the fibres of the bundle P. Atiyah uses this exact sequence to study analytic principal bundles.

We recall that for any vector bundle E of rank n over the field $\mathbf{K} = \mathbf{R}$, or $\mathbf{C}$, there is an *associated principal bundle* $P(E)$ with *structural group* $GL(n, \mathbf{K})$. The associated principal bundle $P(E)$ has as fibre over the point $x \in M$ the space of isomorphisms $ISO_{\mathbf{K}}(\mathbf{K}^n, E_x)$. The group $GL(n, \mathbf{K})$ acts to the right onto the total space of the bundle $P(E)$ while transforming its fibres in a simply transitively manner

$$ISO_{\mathbf{K}}(\mathbf{K}^n, E_x).GL(n, \mathbf{K}) \longrightarrow ISO_{\mathbf{K}}(\mathbf{K}^n, E_x). \tag{1.93}$$

The fibre of the associated principal bundle is diffeomorphic to the group $GL(n, \mathbf{K})$ and the group $GL(n, \mathbf{K})$ acts to the right.

A linear connection into the bundle E induces an *infinitesimal connection* into the associated principal bundle.

In this section we will be interested in *smooth* principal bundles P rather than analytic principal bundles. An *infinitesimal connection* in the principal bundle P is a $C^{\infty}(M)$-splitting $\rho : T \longrightarrow L(P)$ of the Lie algebra extension (1.93) of smooth tangent fields.

For any infinitesimal connection there is an associated *connection form* and *curvature*, see [34].

1.6.9 Characteristic Classes of Lie Algebra Extensions

For the next considerations we use Teleman [62]; this article gives an algebrisation of some of the basic geometric constructions in differential geometry and could be used as introduction to the theory of algebroids.

The purpose of the article [62] was multiple. On the one hand, its intent was to extend Theorem 1.21 to Lie algebra extensions by *non-commutative Lie algebras* $L^{'}$. On the other hand, the article gave a non-conventional definition of connection form and curvature; it made one *step further* towards the algebrisation of the theory of characteristic classes. The structure used in [62] is known as *Lie algebroid*, which was defined earlier, independently, by Pradines [48]. More information about algebroids may be found in [88, 105] and [118] by Mackenzie. The article [62] introduces *formal polynomial homotopies* and *formal integration*.

1.6.9.1 Connections and Curvature of Lie Algebra Extensions

We consider the following *algebroid structure* (1)–(9).

1. R is a commutative ring with $1 \in R$
2. F is a commutative R-algebra with $1 \in F$
3. $H,\ P,\ T$ are Lie R-algebras
4. the sequence

$$E: \quad 0 \longrightarrow H \xrightarrow{\iota} P \xrightarrow{\pi} T \longrightarrow 0$$

   is an exact sequence, where $\iota,\ \pi$ are Lie R-algebra Lie homomorphisms.
   We agree to identify $\iota(H)$ with H.
5. $H,\ P,\ T$ are F-modules, which induce the R-module structures (3).
6. $\iota,\ \pi$ are F-linear homomorphisms.
7. The exact sequence E is F-split, i.e. there exists an F-homomorphism $\nabla : T \longrightarrow P$ such that $\pi \circ \nabla = 1_T$.
   F is a T-Lie algebra module. Therefore F is a P-Lie algebra module by π.
8.

$$X(f_1.f_2) = (Xf_1).\ f_2 + f_1.\ X(f_2),$$

   for any $f_1,\ f_2 \in F$ and for any $X \in T$.
9.

$$[f\ X_1, X_2] = f\ [X_1, X_2] - (\pi(X_2))(f\ .\ X_1),$$

   for any $f \in F$ and any $X_1, X_2 \in P$.

Remark 1.12 The conditions (1)–(9) do not require *any* analytical structure, like smoothing or manifold structure; the ground ring $\mathbf{R}$ could be *any* commutative ring. Only at the end of the construction, in Sect. 1.6.9.3 where we are associating characteristic classes, we will need the ring $\mathbb{R}$ to contain the rational numbers.

Example 1.1 Let P be a principal bundle with structural Lie group G and base M in the smooth category. The bundle P provides an example of algebroid (1)–(9), where

1. R is the field $\mathbf{K} = \mathbb{R}$ or $\mathbb{C}$
2. F is th algebra of smooth functions on M with values in $\mathbf{K}$
3. H, resp. P, are the Lie algebras of right-invariant tangent fields on P which are tangent to the fibres, resp. of the total space of the bundle P, while T is the Lie algebra of tangent fields of the base space.
4. The projection π is the projection of a right-invariant tangent field on P onto the base space.

These algebras and modules satisfy all other requirements (1)–(9).

If the bundle P is an analytic bundle then there are *obstructions* to the existing of the splitting ∇, see Atiyah [17]. However, at the C^∞ level, the splitting exists [17] and provides a *classical infinitesimal connection* in P, see [34].

We come back to the general algebroid structure (1)–(9).

The Lie algebra H is an ideal in P. Therefore, H is a P-module given by the *adjoint representation* of P over H: for any $X \in P$ and $h \in H$

$$Xh := [X, h]; \tag{1.94}$$

the Lie module structure means

$$[X_1, X_2]h = X_1(X_2h) - X_2(X_1h); \tag{1.95}$$

this is the Jacobi identity, for any $X_1, X_2 \in P$ and $h \in H$.

The axioms imply that

$$[fX_1, X_2] = f[X_1, X_2], \quad \forall f \in F, \ \ \forall X_1 \in P \ \ and \ \ \forall X_2 \in H. \tag{1.96}$$

Definition 1.17 Define the *connection form*

$$\omega(X) := X - \nabla\pi(X), \qquad \forall X \in P. \tag{1.97}$$

Any element of H will be called *vertical*, while the elements ∇X, $\forall X \in T$, will be called *horizontal*.

It is easy to check that $\omega : P \longrightarrow H$. Given a commutative ring A and two A-modules, M and N, define

$$C^k_A(M, N) := \{f \mid f : M^k \longrightarrow N, \ \ with \ \ f \ \ A-multilinear \ \ and \ \ skew-symmetric\}. \tag{1.98}$$

In addition, if one supposes that M is Lie algebra over the ring A, define

$$d : C^k_A(M, N) \longrightarrow C^{k+1}_A(M, N) \tag{1.99}$$

by the familiar formula

$$\begin{aligned}(df)(X_1, X_2, \ldots, X_{k+1}) := &\sum_{1\le r\le k+1} (-1)^{r+1} X_r f(X_1, X_2, \ldots, \hat{X}_r, \ldots, X_{k+1}) \\ &+ \sum_{1\le r<s\le k+1} (-1)^{r+s} f([X_r, X_s], X_1, \ldots, \hat{X}_r, \ldots, \hat{X}_s, \ldots, X_{k+1}).\end{aligned} \tag{1.100}$$

Given that F is a T-module and H a P module, $\{C^*(T, F),\ d\}$ and $\{C^*(P, H),\ d\}$ are co-chain complexes.

Definition 1.18 The *basic curvature* is the form $\Omega_\nabla \in C_F^2(T, H)$ defined by

$$\Omega_\nabla(X_1, X_2) := \nabla[X_1, X_2] - [\nabla X_1, \nabla X_2] \tag{1.101}$$

for any $X_1, X_2 \in T$ while the *spatial curvature* is the form Ω_∇^* defined by

$$\Omega_\nabla^*(X_1, X_2) := [X_1, \omega(X_2)] - [X_2, \nabla(X_1)] - \omega([X_1, X_2]) - [\omega(X_1), \omega(X_2)] \tag{1.102}$$

for any $X_1, X_2 \in P$.

The basic curvature measures in which respect ∇ is not a Lie algebra homomorphism.

Remark 1.13 The reader should notice the difference between the definitions of *connection form* and *curvature* given above and the *classical definitions of connection form and curvature* of infinitesimal connections [34] and [17]. The definitions of connection form and curvature presented here are relevant to the theory of algebroids, compare Mackenzie [88, 105, 118]; they extend naturally to the case of short *exact sequences of associative algebras* and their Chern character: they pass easily into non-commutative geometry, see Sect. 2.3.3.

Proposition 1.10 *The spatial curvature and the basic curvature satisfy the relations*

$$\Omega_\nabla \in C_F^2(T, H) \tag{1.103}$$

$$\Omega_\nabla^* = \pi^* \Omega_\nabla, \quad therefore \quad \Omega_\nabla^* \in C_F^2(P, H) \tag{1.104}$$

$$\Omega_\nabla^*(X_1, X_2) = \nabla\pi[X_1, X_2] - [\nabla\pi X_1\ ,\ \nabla\pi X_2]. \tag{1.105}$$

Definition 1.19 We introduce the following operations

(i) $[\ \omega,\ \omega\ ] \in C_F^2(P,\ H)$ is defined by

$$[\ \omega,\ \omega\ ](X_1, X_2) := [\ \omega(X_1),\ \omega(X_2)\ ] \tag{1.106}$$

for any $X_1, X_2 \in P$.

(ii) $[\ \Omega_\nabla^*,\ \omega\ ] \in C_F^3(P, H)$

$$(X_1,\ X_2,\ X_3) :=$$
$$[\ \Omega_\nabla^*(X_1,\ X_2),\ \omega(X_3)] \ -\ [\ \Omega_\nabla^*(X_1,\ X_3),\ \omega(X_2)] \ +\ [\ \Omega_\nabla^*(X_2,\ X_3),\ \omega(X_1)], \tag{1.107}$$

for any $X_1, X_2, X_3 \in P$.

(iii) For any $\varphi \in C_F^k(T, E)$, with E an arbitrary T-module, the *covariant derivative* $D_\nabla(\varphi) \in C_F^{k+1}(T, E)$ is

$$D_\nabla(\varphi)(X_1, X_2, \ldots, X_{k+1}) := (d(\pi^* \varphi))(\nabla X_1, \nabla X_2, \ldots, \nabla X_{k+1}). \tag{1.108}$$

Proposition 1.11 (Structure Equation of Lie Algebra Extensions)
The spatial curvature satisfies the equation

$$d\,\omega = \Omega_\nabla^* + [\,\omega,\ \omega\,]. \tag{1.109}$$

Proposition 1.12 (Bianchi Identity of Lie Algebra Extensions)

(i) The exterior derivative of the spacial curvature Ω_∇^* satisfies the equation

$$d\Omega_\nabla^* = -[\Omega_\nabla^*,\ \omega], \tag{1.110}$$

while the covariant derivative of the basic curvature satisfies

$$D\Omega_\nabla = 0. \tag{1.111}$$

1.6.9.2 Constant Invariant Forms of Lie Algebra Extensions

We proceed with the study of short exact sequence of Lie algebras.

Definition 1.20 Let $\varphi : H \times H \times, \ldots, \times H \longrightarrow F$ be a k-multilinear F form.
The form φ is called *constant invariant form* provided it satisfies the condition

$$(\pi(X))\varphi(h_1, h_2, .., h_k) = \sum_{1 \le i \le k} \varphi(h_1, \ldots, [X, h_i], .., h_k), \tag{1.112}$$

for any $h_1, h_2, .., h_k \in H$ and $X \in P$.

To understand the above condition, we give a definition and theorem from the theory of invariant forms on Lie groups.

Definition 1.21 Let G be a Lie group. The k-differential form φ defined on G is called *invariant* differential form iff it is invariant at the adjoint action, i.e. for any element $g \in G$ and for any right-invariant tangent fields $X_1, X_2, \ldots, X_k$ on G, one has

$$\varphi(Ad_g^*(X_1), Ad_g^*(X_2), \ldots, Ad_g^*(X_k)) = \varphi(X_1, X_2, \ldots, X_k). \tag{1.113}$$

Theorem 1.22 *Let G be a connected compact Lie group. Then φ is an invariant k-differential form on G provided it satisfies the equation*

$$\sum_{1\leq i\leq k} \varphi(X_1, \ldots, [X_i, X], \ldots, X_k) \;=\; 0, \tag{1.114}$$

for any right-invariant tangent fields $X_1, X_2, \ldots, X_k$ and X on G.

We explain here the condition on φ of being constant invariant. For, we assume that the conditions (1)–(9) refer the algebroid associated to a principal G-bundle and we suppose that φ is a constant invariant form. Suppose $X \in H$. Equation (1.114) becomes

$$\sum_{1\leq i\leq k} \varphi(h_1, \ldots, [X, h_i], .., h_k) \;=\; 0, \tag{1.115}$$

which says that φ is an invariant differential form over the Lie algebra H, which represents the Lie algebra of right-invariant tangent fields along the fibres of the bundle. This gives an understanding of the *local* meaning of the condition (1.115). The fact that the equation should be satisfied for any $X \in P$ rather than $X \in H$, says that φ may not change when we pass from one local trivialisation of the principal bundle to another local trivialisation; this justifies calling such forms *constant* invariant forms.

1.6.9.3 Chern–Weil Theory for Lie Algebra Extensions

There is a Chern–Weil theory for principal bundles.

Theorem 1.23 (Chern–Weil)
Let ξ be a smooth *principal bundle with structural group G over M. Let ∇ be an infinitesimal connection in ξ. Let Ω_∇ be its curvature and let φ be any k-invariant form on the group G. Then*

(i)

$$\Phi_\nabla := \varphi(\Omega_\nabla, \Omega_\nabla, \ldots, \Omega_\nabla)$$

is a well defined $2k$-differential form on M.
(ii) The form Φ is closed.
(iii) The cohomology class of Ψ is independent of the connection ∇.

Theorem 1.24 (Teleman [62]) *Let (1)–(9) be an extension E of Lie algebras. Let ∇ be a connection in E and let Ω_∇ be its curvature form. Let φ be a k-invariant*

form of the extension. Then

1.

$$\Phi_\nabla := \varphi(\Omega_\nabla, \ldots, \Omega_\nabla) \in C^{2k}_\nabla(T, F)$$

is a well defined $2k$*-co-chain,*
2. *the form* Φ *is closed,*
3. *if the ground ring* R *contains the rational numbers, then the cohomology class* $[\ \Phi_\nabla\ ] \in H^{2k}(T, F)$ *does not depend on the connection* ∇.

Proof

1. is clear.
2. As in the classical case, the proof is based on the Bianchi identity Proposition 1.12.
3. The proof of the homotopy invariance of the characteristic classes [62, §5] uses *formal polynomial homotopies* between connections and a *formal integration* of powers of the corresponding curvatures. A *formal* Leibnitz–Newton type formula for *formal polynomial integration* is introduced; it requires rational numbers for its implementation.

For a complete proof see [62].

Theorem 1.24 passes into non-commutative geometry. The next remark will be used as the departure point for non-commutative considerations. For further reference, we summarise the result proven in this section.

Remark 1.14 For any short exact sequence of Lie algebras E (1)–(9) and any *constant invariant form* φ, $\Phi_\nabla \in C^{2k}(T, F)$ is a closed form; if the ground ring R contains the rational numbers, then the cohomology class of Φ_∇ is independent of the connection ∇.

Remark 1.15 The problem of defining characteristic classes of *short exact sequences of associative algebras* means going beyond the almost commutative case of Lie algebras; this will be discussed in a non-commutative context, in Sect. 2.3.3.

1.7 Direct Connections

The notion of *direct connection* was introduced by Teleman, originally using the term of *linear quasi-connection*, see [116], and later changed into *direct connections*, [121].

The genesis of this concept passed through a few stages. It started with the geometric interpretation of the Chern character [44]. This interpretation used cubes relative to the affine structure on local charts of coordinates and a parallel transport

defined on the bundle. The second stage was the article by Teleman [52]. This construction replaced cubes by simplices; such simplices were built using the parallel transport along *geodesics* of an additional Riemannian structure on the manifold. In the simplex construction the Riemannian metric was used as a *support* for the construction of simplices. The parallel transport along geodesics was used also in more recent articles, among which, in [96], for the study of the Novikov conjecture discussed in a non-commutative geometry context.

The next step derived from the study of the geodesic distance $d(x, y)$ between close points x, y belonging to a Riemannian smooth manifold [116, 121]. The purpose of the study was to extract the Pontrjagin classes of the manifold from the geodesic distance.

A further step was the extension of the procedure (used in [83, 121]) from the tangent bundle to an arbitrary smooth vector bundle; see also Kubarski and Teleman [117].

Direct connections *remove the need to introduce Riemannian metrics as support* for the parallel transport along geodesics.

We pass now to summarise the content of the articles [83, 121].

We recall that the geodesic distance $d(x, y)$ is a basic ingredient of the parametrix for the Laplacian, see Sect. 1.9.2. The parametrix uses the exterior powers of the double form on $M \times M$ derived from the geodesic distance

$$A(x, y) := -\frac{1}{2}\, d_x d_y d^2(x, y).$$

This double form was used in [83, 121] to produce the tensor $\tilde{A}$ on $M \times M$. The tensor $\tilde{A}$ is covariant in the first argument and contravariant in the second argument

$$\tilde{A}\,(\sum_i \xi_i \frac{\partial}{\partial x^i}) := \sum_i \xi_i \frac{\partial^2 d^2(x, y)}{\partial x^i \partial y^j} g^{jk} \frac{\partial}{\partial y^k}, \tag{1.116}$$

where g_{ij} are the local components of the Riemannian metric.

Theorem 1.25 *The tensor $\tilde{A}$ has the following properties:*

1.

$$\tilde{A} : T_x(M) \longrightarrow T_y(M) \tag{1.117}$$

is an isomorphism for x, y sufficiently close points and

2. $\tilde{A}(x, x) = Id$.

These properties are easily extended if we replace the tangent bundle by an arbitrary vector bundle.

Definition 1.22 (Teleman [116, 121]) A *direct connection* in the vector bundle ξ over M consists of a system of mappings $\tilde{A} : \xi_x \longrightarrow \xi_y$ from the fibre over the point x to the fibre over the point y such that

1.

$$\tilde{A} : \xi_x \longrightarrow \xi_y \tag{1.118}$$

is an isomorphism for x, y sufficiently close one to each other and
2. $\tilde{A}(x, x) = Id$.

Theorem 1.26 (Kubarski and Teleman [117]) *The* infinitesimal part of a smooth direct connection *in a smooth vector bundle is a linear connection in the bundle.*

Proof Let $A(x, y)$ be a smooth direct connection in the vector bundle ξ. We assume the bundle is locally trivialised, i.e. $A(x, y) : \mathbf{K}^n_x \longrightarrow \mathbf{K}^n_x$, $x, y \in coordinate\ chart$, is a vector valued function.

The infinitesimal part of the direct connection A is the second order partial derivative

$$\Gamma^{\beta}_{\alpha ij} := (\partial^2 A(x, y)/\partial x_i \partial y_j)_{|x=y}.$$

The functions $\Gamma^{\beta}_{\alpha ij}$ are the Christoffel coefficients of a linear connection.

For more details, please see [117].

Theorem 1.27 (Kubarski and Teleman [117]) *Smooth direct connections are more general than linear connections.*

We will see in the last chapter of this book that direct connections could be used on topological manifolds.

Proof Let Γ be a smooth linear connection in the vector bundle ξ. Let $\tau(y, x)_\gamma$ be the Γ-parallel transport of vectors along the curve γ, from the point x to the point y. Then Γ satisfies the condition

$$\tau(x, y)_\gamma \circ \tau(y, x)_\gamma = Id.$$

A direct connection does not have to satisfy this condition.

Definition 1.23 (Direct Connections in Algebraic Systems) M may be replaced by *any set* M, provided the notion of close points is defined; the fibres ξ_x may be replaced by *any algebraic structure*. (In particular, M could be replaced by any *simplicial complex* or even discrete space).

A *direct connection in the algebraic system* ξ is a system of mappings $\tilde{A} : \xi_x \longrightarrow \xi_y$ such that

1.

$$\tilde{A} : \xi_x \longrightarrow \xi_y \tag{1.119}$$

is an isomorphism for x, y sufficiently close one to each other
2. $\tilde{A}(x, x) = Id$.

Remark 1.16 A *particular case* of direct connection is the parallel transport along geodesics in any vector/principal bundle over a smooth manifold, see e.g. Teleman [52], Connes and Moscovici [96].

Definition 1.24 For any direct connection $\tilde{A}$ in the vector bundle ξ over the manifold M, define the function Φ_k over M^{k+1}

$$\Phi_k(x_0, x_1, \ldots, x_k) \ := \ tr \ \tilde{A}(x_0, x_1) \circ \tilde{A}(x_1, x_2) \circ, \ldots, \circ \tilde{A}(x_{k-1}, x_k) \circ \tilde{A}(x_k, x_0). \tag{1.120}$$

The function Φ_k reminds the cube [44] and simplex constructions [52] used in the geometric interpretations of the Chern character; compare with formula (1.51). In comparison with the cube and simplex constructions of the Chern character, the present formula eliminates both the need to consider arcs of geodesics—derived from a Riemannian metric on the base space—and the *skew-symmetrisation*. The elimination of skew-symmetrisation is one of the basic features of non-commutative geometry.

Theorem 1.28 (Extension of Chern–Weil Theory to Direct Connections [116, 117, 121]) *Let ξ be a complex/real bundle over a smooth manifold and $\tilde{A}$ a direct connection in ξ. For any natural even number $k = 2q$ consider the function Ψ_{2q}.*

The function Ψ_{2q} is a cyclic homology cycle. *(For cyclic homology see Sect. 3.21). Its* cyclic homology class *represents the Chern/Pontrjagin character of the bundle ξ (up to a multiplicative factor). The homological information contained by this class is concentrated along the diagonal of the product M^{2q+1}, see Sect. 3.21.*

Proof The proof is postponed to Sect. 3.24.

Remark 1.17 The interest for direct connections follows from the following facts.

1. Classical connections apply to vector/principal bundles over spaces with analytic structures, over manifolds. Direct connections do not require manifold structure. Direct connections work with *generic algebraic structures* over sets in which the notion of *neighbouring elements* is defined, as for example, in simplicial complexes.
2. Classical connections transport the fibres of vector/principal bundles along *smooth* curves. Direct connections define isomorphisms of the algebraic struc-

tures (not necessarily vector spaces or groups) from one point to a *neighbouring point*.
3. *Direct connections are more general than ordinary connections*.
4. Direct connections are suitable for many applications in non-commutative geometry, including the Chern–Weil theory.
5. Direct connections permit to obtain the *rational Pontrjsagin classes* (Sect. 10.5) on topological manifolds.

1.8 K^*-Theory of Banach Algebras

K-theory classifies *bundles* over associative algebras A. Isomorphism classes of vector bundles form a semi-group under direct-sum operation. The basic idea of K-theory consists of treating vector bundles as elements of a *commutative group*. In general, the procedure of passing from a semi-group to a group is called *Grothendieck completion*. Depending on the additional structures which are considered onto the algebra $\mathscr{A}$, different K-theories are obtained. If no topological structure is considered on the algebra $\mathscr{A}$, the corresponding K-theory is called *algebraic K-theory*. The K-theory for commutative algebras is closest to the original idea of classification of vector bundles. The definition of the *isomorphism equivalence relation* plays a fundamental role in the construction of classical K-theory and index theory. This is going to be reconsidered in the last chapters of this book. In that chapter new developments in algebraic K-theory are presented and the basis of a new *algebraic K-theory is proposed*.

The original K-theory was created for $C^0(X)$, the commutative algebra of continuous functions over CW-complexes, theory due to Grothendieck [16], Hirzebruch [32] and Atiyah [50]. From the very beginning K-theory played a fundamental role in index theory.

1.8.1 $K^*(C^0(X))$

Two continuous vector bundles ξ_1, ξ_2 over X will be called *C^0-isomorphic*, and we write $\xi_1 \sim \xi_2$, provided there exists a homeomorphism between their total spaces which induces isomorphisms between their fibres. If X were a smooth manifold, the homeomorphism could be required to be smooth, in which case the two bundles would be called *C^∞-isomorphic*. Over smooth manifolds, two vector bundles are C^0-isomorphic if and only if they are C^∞-isomorphic.

Let $Vect(X)$ denote the set of C^0-isomorphism classes of vector bundles of finite rank over X.

Definition 1.25 Let $+$ denote the direct sum operation defined on $Vect(X)$.

Definition 1.26 Two vector bundles ξ_0, ξ_1 over the space X are called *homotopic* and we write $\xi_0 \sim_h \xi_1$ provided there exists a vector bundle ξ over the space $X \times [0, 1]$ such that $\xi_{|\, X\times\{i\}} \sim \xi_i,\ i = 0, 1$.

$\sim_h$ is an equivalence relation.

Theorem 1.29 *The functor*

$$Vect : Top \longrightarrow Vect(X)$$

is a contravariant functor.

This functor is a homotopic functor, *i.e. for any bundle ξ over the space $X \times [0, 1]$*

$$\xi \sim \xi_{|\, X\times\{0\}} \times [0, 1].$$

If $\xi_0 \sim_h \xi_1$, then their stabilisations are still isomorphic/homotopic.

Definition 1.27 The group $K^0(X)$ is by definition the Grothendieck completion of the semi-group $Vect(X)$.

The elements of the Grothendieck completion group are formal differences of elements $(\xi_1, \xi_2) \in Vect(X) \times Vect(X)$. Such a pair is called *virtual bundle of rank* $m - n$; here m and n represent the ranks of these bundles.

Definition 1.28 A vector bundle isomorphic to a product bundle is called *trivial* bundle.

The virtual rank of an element belonging to $K^0(X)$ is an invariant of the element.

Definition 1.29 Let

$$Rank : K^0(X) \longrightarrow \mathbb{Z} \tag{1.121}$$

denote the virtual rank function.

Definition 1.30

$$\tilde{K}^0(M) \;=\; Ker(Rank) \;=\; Ker\{K^0(X) \longrightarrow \mathbb{Z}\}. \tag{1.122}$$

In other words, the elements of $\tilde{K}^0(M)$ represent virtual bundles of rank zero. As the bundles ξ, $\xi \oplus \mathbf{N}$ are stably isomorphic, they represent the same element in $Vect/\sim_s$. Any element of $\tilde{K}$ may be represented by the difference $\xi - \ltimes \in K^0(M)$, which is a virtual bundle of rank 0, where n is the rank of the bundle ξ. Therefore, $\tilde{K}^0(X)$ could be defined also by

$$\tilde{K}^0(X) := Vect(X)/\sim_s, \tag{1.123}$$

with the additional information that its elements represent virtual bundles of rank zero.

$\tilde{K}^0(X)$ being by definition the kernel of a homomorphism between commutative groups, it is a commutative group. However, it is useful to understand directly how the opposite elements are produced in this group. In fact, any bundle ξ may be embedded as a sub-bundle of a product bundle $\mathbf{N}$. The bundle $\xi^\perp$, the orthogonal complement of the sub-bundle ξ, has the property that $\xi + \xi^\perp = \mathbf{N}$ and hence its class represents the opposite of the class of ξ in $\tilde{K}^0(X)$.

Definition 1.31 One defines $K^1(X) = \tilde{K}^0(\Sigma X)$, where Σ denotes *suspension*.

Recall that $\Sigma X = C_-(X) \cup_X C_+(X)$, where $C_-(X)$, $C_+(X)$ are cones over X and these cones are glued together along X, the base of the cones. Therefore, any vector bundle over ΣX is given by a pair of product bundles $\mathbf{N}$ over the cones $C_\pm(X)$ glued together by a *clutching function* $u \in \mathbf{GL}_N(\mathbf{C})$.

Definition 1.32 Let $u \in \mathbf{GL}_n(C^0(X))$. Then u and $u \oplus 1_N$ (in any order) are called *stably equivalent invertibles*.

For any associative algebra with unit $\mathscr{A}$, and for any invertible element $u \in \mathbf{GL}_N(\mathscr{A})$, the element $u \oplus 1_n \in \mathbf{GL}_{N+n}(A)$ is called *stabilised invertible* element of u and we write (in any order) $u \sim_s u \oplus 1_n$.

$\sim_s$ is an equivalence relation.

This gives us an equivalent definitions of $K^1(X)$

$$K^1(X) = \textit{stably homotopy classes of invertibles in } \mathbf{M}_n(C^0(X)). \tag{1.124}$$

Definition 1.33 The higher order K^i theory groups are defined by

$$K^i(X) = \tilde{K}(\Sigma^i X), \quad 1 \leq i. \tag{1.125}$$

For any closed sub-space $Y \subset X$, one define the *relative K^*-theory groups*

$$K^i(X, Y) := \tilde{K}^i(X/Y), \tag{1.126}$$

where X/Y denotes the quotient space.

Here we give the original definition of the relative K-theory groups.

Definition 1.34 Let (X, Y) be a pair of CW-complexes with $Y \subset X$ a closed sub-space. Define

$$\textit{Vect}(X, Y) := \{(\xi_1, \xi_0, \alpha) \mid \xi_1, \xi_0 \in \textit{Vect}(X),\ \textit{with } \alpha : \xi_{1|Y} \longrightarrow \xi_{0|Y} \textit{ isomorphism.}\} \tag{1.127}$$

Two triples (ξ_1, ξ_0, α), $(\varphi_1, \varphi_0, \beta)$ are *isomorphic* provided there exist two isomorphisms

$$h_1 : \xi_1 \longrightarrow \varphi_1 \tag{1.128}$$

$$h_0 : \xi_0 \longrightarrow \varphi_0 \tag{1.129}$$

such that $\beta \circ h_1 = h_0 \circ \alpha$.

The triples (ξ_1, ξ_0, α), $(\xi_1 + \mathbf{N}, \xi_0 + \mathbf{N}, \alpha + 1_N)$ (in any order) are called *stably equivalent*; we write

$$(\xi_1, \xi_0, \alpha) \sim_s (\xi_1 + \mathbf{N}, \xi_0 + \mathbf{N}, \alpha + 1_N). \tag{1.130}$$

The *sum of two triples* is defined by

$$(\xi_1, \xi_0, \alpha) + (\varphi_1, \varphi_0, \beta) := (\xi_1 + \varphi_1, \xi_0 + \varphi_0, \alpha \oplus \beta). \tag{1.131}$$

Theorem 1.30 *For any closed sub-space $Y \subset X$, define*

$$I\ (\xi_1, \xi_0, \alpha) := (\xi_1, \xi_0) \in K^0(X) \tag{1.132}$$

and

$$J\ (\xi_1, \xi_0) := (\xi_{1,\,|\,Y}, \xi_{0\,|\,Y}) \in K^0(Y) \tag{1.133}$$

Then, the sequence

$$K^0(X, Y) \xrightarrow{I} K^0(X) \xrightarrow{J} K^0(Y) \tag{1.134}$$

is exact.

This theorem states that the functor K^0 is *half exact*, see e.g. [68]. We already know that this is a homotopic functor. In this situation, the *Puppe sequence*, see [23], produces a *generalised co-homological functor*. Using the equivalent description of the relative K^* groups and the definition of higher order groups, the Puppe construction leads to the following

Theorem 1.31 *K^* is a* generalised co-homological functor *on the category of finite dimensional CW-complexes.*

In particular, for any paracompact finite dimensional CW-complex X and any closed sub-complex Y, and with δ denoting the connecting homomorphism*, the*

sequence

$$K^0(X,Y) \xrightarrow{I} K^0(X) \xrightarrow{J} K^0(Y) \xrightarrow{\delta} K^1(X,Y) \xrightarrow{\Sigma I} K^1(X) \xrightarrow{\Sigma J} K^1(Y) \xrightarrow{\Sigma\delta} \ldots.$$
$$\ldots \xrightarrow{\Sigma^{i-1}\delta} K^i(X,Y) \xrightarrow{I} K^i(X) \xrightarrow{J} K^i(Y) \xrightarrow{\Sigma^i\delta} \ldots.. \tag{1.135}$$

is exact.

Theorem 1.32 (Bott Periodicity, See [50, 57])

*The groups $K^i(C^0(X))$, $K^{i+2}(C^0(X))$ are isomorphic (*Bott periodicity*). The Bott periodicity isomorphism is functorial.*

Therefore, up to isomorphisms, there are only two K^i-theory groups: K^0 and K^1.

The Bott periodicity leads to the 6-term exact sequence

$$\begin{array}{ccccc} K^0(X/Y) & \to & K^0(X) & \to & K^0(Y) \\ \uparrow & & & & \downarrow \\ K^1(Y) & \leftarrow & K^1(X) & \leftarrow & K^1(X/Y) \end{array} . \tag{1.136}$$

$K^*(C^0(X))$ is used in index theory to follow up the contribution of the symbol of an elliptic operators on its index. Although the total space of the co-tangent bundle is not compact, only the restriction of the symbol to the unit co-sphere bundle in the index formula counts; this is a compact sub-space, if the base manifold is compact. For this reason, the K^*-theory construction has to be modified when it is used in index theory. The relevant K^*-theory in index theory is the K^*-theory *with compact supports*. We do not give more details on this notion; see [53] for more details.

The following result is of basic technical importance. We give it without proof.

Theorem 1.33

(i) The Chern character extends to the relative K^i-groups, $i = 0, 1$,

$$Ch : K^0(X,Y) \longrightarrow H^{even}(X,Y;\mathbf{R}) \tag{1.137}$$

$$Ch : K^1(X,Y) \longrightarrow H^{odd}(X,Y;\mathbf{R}). \tag{1.138}$$

(ii) $Ch : K^i(X,Y) \otimes_{\mathbb{Z}} H^i(X,Y;\mathbb{Q}), \quad i = ev, odd$ are isomorphisms.

(iii) The $K^*(C^0(X))$ has a multiplicative structure. The Chern character Ch is compatible with the multiplicative structures both onto $K^*(C^0)$ and $H^{even/odd}(-;\mathbf{Q})$.

1.8.2 K*-Theory of Banach Algebras

In the last part of this section we discuss briefly an extension of K^*-theory by replacing the algebra $C^0(X)$ by an arbitrary unital Banach algebra $\mathscr{A}$. In this case, vector bundles are replaced by *idempotents* of the algebra of matrices over $\mathscr{A}$.

Definition 1.35 Three equivalence relations can be defined onto the set if idempotents and invertibles over $\mathscr{A}$

1. *conjugation* $p_1, p_2 \in \mathbf{M}_n(\mathscr{A})$, idempotents are conjugated, and we write $p_1 \sim_c p_2$ provided there exists $u \in \mathbf{GL}_n(\mathscr{A})$ such that

$$p_1 = up_2u^{-1} \tag{1.139}$$

   $u_1, u_2 \in \mathbf{GL}_n(\mathscr{A})$, invertibles are conjugated, and we write $u_1 \sim_c u_2$ provided there exists $u \in \mathbf{GL}_n(\mathscr{A})$ such that

$$u_1 = uu_2u^{-1}. \tag{1.140}$$

2. *stabilisation* for idempotents: $p \sim_s p \oplus 0_n$
   and invertibles: $u \sim_s u \oplus 1_n$.
3. *homotopy*: for idempotents: $p_0, p_1 \in \mathbf{M}(\mathscr{A})$ are called *homotopic* and we write $p_0 \sim_h p_1$, provided there exists an idempotent $p \in \mathbf{M}(\mathscr{A}) \times [0, 1]$ such that its restrictions r_0, resp. r_1, to $\mathscr{A} \times \{0\}$, resp. $\mathscr{A} \times \{1\}$, is p_0, resp. p_1.
   By definition

$$\mathscr{A} \times [0, 1] := \{f \mid f : [0, 1] \longrightarrow \mathscr{A},\ f\ continuous\ function\}. \tag{1.141}$$

   r_0, r_1 are the restrictions of functions $f \in \mathscr{A} \times [0, 1]$ to $t = 0,\ 1$.
   Two homotopic invertible elements u_0, u_1 are defined analogously and we write $u_0 \sim_h u_1$.

$K^*(\mathscr{A})$-theory groups of the algebra $\mathscr{A}$ are defined

$$K^0(\mathscr{A}) := \{\ [p] - [q] \mid p^2 = p \in \mathbf{M}_m(\mathscr{A}),\ \ q^2 = q \in \mathbf{M}_n(\mathscr{A}),\ \ m, n \in \mathbf{N}\ \} \tag{1.142}$$

where [ ] means homotopy class

$$K^1(\mathscr{A}) := \pi_0\ (\ Lim_{n\to\infty}\mathbf{GL}_n(\mathscr{A})\ ). \tag{1.143}$$

To the $(X,\ Y)$ pair of compact non-empty topological spaces there corresponds the exact sequence of algebras of continuous functions

$$0 \to C^0(X,\ Y) \to C^0(X) \to C^0(Y) \to 0, \tag{1.144}$$

where

$$C^0(X,\ Y)\ =\ \{f\ \in\ C^0(X)\mid f(Y)\ =\ 0\}. \tag{1.145}$$

More generally, one may consider an arbitrary exact sequence of Banach algebras

$$0\to J\to A\to B\to 0. \tag{1.146}$$

Let $\tilde{J}:=J\oplus C.1$ be the algebra J with the unit $1\in C$ adjoined and let $\epsilon:\tilde{J}\to C$ be the augmentation mapping. By definition, for $i=0,1$

$$K^i(J)\ =\ Ker\ \epsilon_i \tag{1.147}$$

where $\epsilon_i:K^i(\tilde{J})\to K^i(C)$.

By construction, $K^i(C(X))\ =\ K^i(X)$, $i=0,1$.

The analogue of Bott periodicity holds for K-theory groups, see Wood [47]; hence, for any short exact sequence of Banach algebras and continuous mappings as above, the 6-terms K exact sequence of K-theory groups holds.

$$\begin{array}{ccccc} K^1(J) & \to & K^1(A) & \to & K^1(B) \\ \uparrow & & & & \downarrow \\ K^0(B) & \leftarrow & K^0(A) & \leftarrow & K^0(J) \end{array}. \tag{1.148}$$

Theorem 1.34 *For any paracompact CW-complex X, the Chern character establishes an isomorphism*

$$Ch_*:K^0(X)\otimes_{\mathbb{Z}}\mathbb{Q}\longrightarrow H^{ev}(X,\mathbb{Q}). \tag{1.149}$$

Definition 1.36 *Algebraic K_* theory* is a more general K-theory; it uses exclusively the *algebraic* structure of the algebra. It will be discussed in Sect. 7.2.1.

1.9 Metric Defined Operators

There are a few reasons why we have chosen to devote to the Hodge theory a special section. Smooth Hodge theory, through the Laplace operator, is one of the basic bridges connecting the local geometry (Riemannian metrics) with global invariants (real cohomology). The Laplacian is also a model for the class of elliptic operators. The Laplacian has an almost inverse, called *parametrix*. The parametrix belongs both to the class of *singular integral operators* and *pseudo-differential operators*.

Extending Hodge theory to more general structures was one of the basic questions posed by Singer in his programme [58].

The Laplacian is not interesting from the point of view of the index formula because its index is equal to zero.

Smooth Hodge theory was extended to more general analytic structures: on *pseudo-manifolds* by Cheeger [79], on combinatorial and Lipshitz manifolds by Teleman [77, 78, 83], by Donaldson and Sullivan on *quasi-conformal manifolds* [92] and by Connes et al. [101] on even dimensional quasi-conformal manifolds of dimensions $\neq 4$.

The *signature* operator will be discussed later in this chapter. It is an elliptic operator related to the Laplacian but much more interesting from the point of view of the index theory. It extends to all topological manifolds in dimensions $\neq 4$, see [78, 92] and [101]. Many proofs of the index theorem use the information provided by the signature operator. It is going to be discussed later in this book.

1.9.1 Laplace Operator

Any paracompact smooth manifold M possesses Riemannian metrics. We assume that M is also compact and oriented. We choose a Riemannian metric on M.

The orientation of $T^*(M)$ allows one to define the Poincaré pairing

$$P : \omega^k \otimes_{\mathbf{R}} \sigma^{n-k} \longrightarrow \int_M \omega \wedge \sigma \in \mathbf{R}. \tag{1.150}$$

The Poincaré pairing is non-degenerate. Stokes' formula implies that the Poincaré pairing passes to de Rham cohomology.

$$P^* : H^k_{dR}(M^n) \otimes_{\mathbf{R}} H_{dR}(M)^{n-k} \longrightarrow \int_M [\omega] \wedge [\sigma] \in \mathbf{R}. \tag{1.151}$$

The Riemannian metric induces a Euclidean structure on each exterior power k of the co-tangent bundle. Indeed, if $\omega_1, \omega_2 \ldots \omega_n$ is an ortho-normal basis in $T^*_p M^n)$, then $\{\omega_{i_1} \wedge \omega_{i_2} \wedge \ldots \wedge \omega_{i_k}\}$, $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ is an ortho-normal basis in $\wedge^k T^*_p(M^n)$.

The orientation of the co-tangent space $T^*_p(M)$ induces an orientation of this ortho-normal basis. Let

$$*_k : \wedge^k T^*_p M^n) \longrightarrow \wedge^{n-k} T^*_p M^n) \tag{1.152}$$

be the corresponding oriented isomorphism.

The star operators $*_k$ satisfy the relations

$$*_{n-k} \circ *_k \; = \; (-1)^{k(n-k)}. \tag{1.153}$$

The star operators are used to define a Hermitian scalar product on Ω^k

$$(\phi, \sigma) := \int_{M^n} \phi \wedge *\bar{\sigma}; \tag{1.154}$$

here we have assumed the differential forms take complex values. The star operators are used to produce the first order differential operators

$$\delta_k := (-1)^{n(k+1)+1} * d * : \Omega^k(M^n) \longrightarrow \Omega^{k-1}(M^n). \tag{1.155}$$

The operators δ_k are formal adjoints of the operators d, i.e.

$$(d\phi, \sigma) = (\phi, \delta\sigma), \quad (\delta\phi, \sigma) = (\phi, d\sigma). \tag{1.156}$$

The Laplace operator is the second order differential operator

$$\Delta_k := d\delta + \delta d : \Omega^k(M) \longrightarrow \Omega^k(M). \tag{1.157}$$

The solutions of the Laplace equation $\Delta(\omega) = 0$ are called *harmonic forms*.

The above identities imply that a harmonic form ω is characterised by the equations

$$d\omega = 0, \tag{1.158}$$

$$\delta\omega = 0. \tag{1.159}$$

1.9.2 Smooth Hodge Theory

Theorem 1.35 (Smooth Hodge Decomposition Theorem, See [86]) *Let M be any oriented closed smooth manifold. Then one has the orthogonal direct sum*

$$\Omega^k(M) = Ker\ \Delta_k(M) \oplus Im\ \Delta_k(M). \tag{1.160}$$

$$\Omega^k(M) = H^k_{dR}(M) \oplus d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M). \tag{1.161}$$

Hodge theorem states that any de Rham cohomology class contains one and only one harmonic representative.

The proof of the Hodge decomposition theorem is based on an explicit operator almost inverse of the Laplacian. A parametrix for the Laplace operator, and more generally, for smooth elliptic operators, may be produced by using the calculus of *pseudo-differential* operators, see, e.g. [53].

The Laplace operator Δ_k is defined in terms of the Riemannian metric. Its parametrix Ω_k may be expressed explicitly in terms of the metric by means of a singular integral operator (here we use the notation from de Rham [86])

$$\Omega_k(\varphi)(x) := \int_{M(y)} \omega_k(x, y) \wedge *\varphi, \tag{1.162}$$

where ω_k is the double form

$$\omega_k(x, y) \ = \ \frac{\sigma(x, y)\ \alpha_k(x, y)}{(n-2)s_n\ d(x, y)^{n-2}} \tag{1.163}$$

for $2 < n$, where $d(x, y)$ is the geodesic distance from the point x to the point y, σ is a cut-off smooth function on $M \times M$, identically 1 on a small neighbourhood of the diagonal; s_n is the area of the unit sphere in the Euclidean space of dimension n and

$$\alpha_k = \frac{1}{k!}(\ d_x d_y A(x, y)\ )^k, \quad for \ \ A(x, y) = -\frac{1}{2}d^2(x, y). \tag{1.164}$$

For $n = 2$ the parametrix is given by the singular integral operator

$$\Omega_2(\varphi)(x) := \ -\int_{M(y)} \omega_2(x, y) \wedge *\varphi(y), \tag{1.165}$$

$$\omega_2(x, y) = \ -\ \frac{\sigma(x, y)\ \alpha_k(x, y)}{s_2}\ ln\ d(x, y). \tag{1.166}$$

The double form $\omega_k(x, y)$ is symmetric in x and y.

Theorem 1.36 *The parametrix Ω_k satisfies the equations*

$$\Omega\,\Delta_k \ = \ Id._k - Q' \tag{1.167}$$

and

$$\Delta_k\Omega \ = \ Id._k - Q \tag{1.168}$$

where Q' Q are integral operators

$$(Q'\varphi)(x) = \int_{M(y)} q'(x, y) \wedge *\varphi(y) \tag{1.169}$$

and

$$(Q\varphi)(x) = \int_{M(y)} q(x, y) \wedge *\varphi(y) \tag{1.170}$$

whose kernels are

$$q(x, y) = -\Delta_k(x)\ \omega_k(x, y) \tag{1.171}$$

and

$$q'(x, y) = -\Delta_k(y)\ \omega_k(x, y) \tag{1.172}$$

Proof For the proof of this result see de Rham [86, §28]. The de Rham's proof is significant. In fact, after certain algebraic manipulations using Green's identities, in an ϵ-neighbourhood of the point x, the proof ends with the following computation

$$\begin{aligned} &(\Omega_k \nabla\varphi)(x) + Q^{'}\varphi(x) \ = \\ &= \ lim_{\epsilon \searrow 0} \int_{M(y)} [\delta(\ d(x, y)^n\ d\omega\ ) + d(\ d(x, y)^n\ \delta\ \omega\ )]\ \wedge_y\ *\varphi(y) \ = \\ &= \ lim_{\epsilon \searrow 0} \int_{M(y)} \int_{M(y)} \frac{n}{s_n} \varphi(x) *_y 1 + O(\ d(x, y)\ ) \ = \ \varphi(x). \end{aligned} \tag{1.173}$$

Definition 1.37 Let T be an integral operator T

$$(T\varphi)(x) := \int_{M(y)} t(x, y)\ \wedge\ \varphi(y), \tag{1.174}$$

where the kernel $t(x, y) \ = \ O(\ \frac{1}{d(x,y)^{n-r}})$, where $0 \leq r \leq n$; then the operator T is called *singular operator of order* r.

It is important to check the order of singularity of the above operators. The following result is proven in [86].

Theorem 1.37 *In spite of the fact that the operator* Ω_k *is of order* $n - 1$*, the operators* Q_k*,* $Q^{'}$ *are singular operators of order* $n - 2$*.*

This result is important. It shows that the parametric Ω_k (defined in terms of the geodesic distance) are good approximations of Δ_k^{-1}, as two orders of regularity are gained. This property will be used in Sect. 1.13 where we will describe the signature operator in geometric terms.

1.10 Analytic Preliminaries on Operators

1.10.1 Symbol

The symbol of an operator captures part of the information contained by the operator. Depending on the nature of the operator there are various types of symbols.

1.10.1.1 Symbol of Differential Operators

The *symbol* $\sigma(A)$ *of a differential operator* A registers the top order partial derivatives of the operator.

Let A be a differential operator of order r from the bundle φ_0 to the bundle φ_1 over the smooth manifold M. The symbol $\sigma(A)$ may be extracted from the operator by means of the formula

$$\sigma(A)(\xi,\ e) := A(f^r(x)\ \tilde{e}(x))(x), \tag{1.175}$$

where f is a smooth function defined on a neighbourhood of the point $x \in M$ such that f vanishes at x and $(df)(x) = \xi \in T^*(M)_x$, $\tilde{e}$ is a smooth section in the bundle φ_0 and $\tilde{e}(x) = e \in \varphi_{0,x}$.

The symbol

$$\sigma(A)(\ -\ ,\ -\ )(x) : T^*(M)_x \times \varphi_{0,\,x} \longrightarrow \varphi_{1,\,x} \tag{1.176}$$

is a homogeneous function of order r on the co-tangent space $T^*(M)_x$ and a linear function between the fibres of the bundles $\sigma(A)(\ \xi\ ,\ -\ )(x) : \varphi_{0,\,x} \longrightarrow \varphi_{1,\,x}$. For more details on differential operators see [42, Chapter IV].

1.10.1.2 Sobolev Spaces

Choose a non-negative integer s. On the algebra of smooth functions on the smooth manifold M, consider the family of semi-norms

$$||f||_{K,r} := Sup_{x\in K}|\partial^r f(x)\ /\ \partial^{r_1}x^{i_1}\ldots\partial^{r_h}x^{i_h}|, \tag{1.177}$$

where K is any compact set in M, $r \leq s$ is an arbitrary non-negative integer and $\sum r^i = r$.

Proposition 1.13 *The topology defined on the algebra* $C^\infty M)$ *by the family of semi-norms* $||\ -\ ||_{K,r}$ *with respect to all compact sets* $K \subset M$ *and non-negative integers* r *defines a* Fréché algebra.

Definition 1.38 Choose a locally finite covering of M by compact sets K. On the subset $C_0^\infty(M) \subset C^\infty(M)$, consisting of smooth functions on M having compact support, define the norm

$$|| f ||_s^2 := \sum_{K,\, 0 \le r \le s} || f ||_{K,\, r}^2, \tag{1.178}$$

where the sum is extended over all compacts K of the chosen covering.

The completion of $C_0^\infty(M)$ with respect to this norm is a Hilbert space, called *Sobolev space of order s*, denoted $H_{s,loc}(M)$.

If M is compact, then $H_{s,loc}(M)$ is denoted $H_s(M)$.

Although the Sobolev norm $|| \ - \ ||_s$ depends on the choice of the covering, the resulting topology is independent of the choice.

The definition of $H_{s,loc}(M)$ extends naturally to the space of sections in an arbitrary vector bundle ξ on M; the Hilbert space produced in this manner is denoted $H_{s,loc}(\xi)$.

Theorem 1.38 *Any differential operator A of order r defines a continuous operator*

$$A : H_{s,loc}(M) \longrightarrow H_{s-r,loc}(M), \quad for\ any\ r \le s. \tag{1.179}$$

For any $s < t$, the inclusion $H_t(M) \longrightarrow H_s(M)$ is compact.

1.10.1.3 Pseudo-Differential Operators and Their Symbol

The *Fourier transform* of the function f, denoted $\hat{f}$, or $\mathscr{F} f$, is defined on $L_2(\mathbf{R}^n)$ by the formula

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-i < x, \xi >} f(x)\, d\, x. \tag{1.180}$$

In this formula ξ is seen as a co-tangent vector to $\mathbf{R}^n$ and $< \ , \ >$ denotes the standard Euclidean scalar product.

Theorem 1.39 *The Fourier transform*

$$\mathscr{F} : L_2(\mathbf{R}^n) \longrightarrow L_2(\mathbf{R}^n) \tag{1.181}$$

is an isometry.

The inverse of this isometry is

$$(\mathscr{F}^{-1} f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{i < x, \xi >} f(\xi)\, d\, \xi. \tag{1.182}$$

The Sobolev spaces may be defined in $\mathbf{R}$ in an alternative way.

Theorem 1.40

$$H_s(\mathbf{R}^n) := H_{s,\,loc}(\mathbf{R}^n) = \{f \mid f \in L_2(\mathbf{R}^n) \ such\ that (1+|\xi|^2)^{s/2}\ \hat{f}(\xi) \in L_2(\mathbf{R}^n)\} \tag{1.183}$$

It is important to note that this definition of Sobolev spaces holds for *any real number s*. Although this definition uses the whole Euclidean space $\mathbf{R}^n$, it produces *local Sobolev spaces*; the local Sobolev spaces may be transferred on manifolds by mean of coordinate charts.

For more information about Fourier transform and Sobolev spaces see [42, Chapter X].

Definition 1.39 Let φ_0, φ_1 be two smooth vector bundles over the smooth manifold M. Let $\sigma(x,\ \xi)$, $x \in M$, $\xi \in T_x^*$, be a smooth function having the properties

1. with regard to the argument x, it is defined away from a small neighbourhood $\mathscr{U}$ of the zero section of the co-tangent spaces $T_x^*(M)$; with respect to the argument ξ, it is homogeneous of order r along the fibres of $T_x^*(M)$
2. for each pair $(x,\ \xi) \in T_x^* - \mathscr{U}$, $\sigma(\ x\ ,\ \xi\ ) : \varphi_{1,\,x} \longrightarrow \varphi_{2,\,x}$ is a linear mapping. If this linear mapping is an isomorphism for all such pairs, then the symbol is called *elliptic*.

$\sigma(\ -\ ,\ -\ )$ is called a *symbol of order r* from the bundle φ_0 to the bundle φ_1.

Let $Symb_r(\varphi_0,\ \varphi_1)$ denote the set of all symbols of order r between the two bundles.

Definition 1.40 Let $\sigma \in Symb_r(\varphi_0,\ \varphi_0)$. With the symbol σ one associates the operator

$$Op_\sigma f(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} \sigma(x,\xi)\hat{f}(\xi) d\ \xi. \tag{1.184}$$

The operator Op_σ is called *pseudo-differential operator* with symbol *pseudo-differential operator* σ.

Patching together pseudo-differential operators defined locally, one defines *pseudo-differential operators on smooth manifolds*.

Definition 1.41 A pseudo-differential operator Op_σ is called *elliptic* if its symbol is elliptic.

Theorem 1.41 *For any symbol of order r*

$$Op_\sigma : H_s(\xi_1,\ \xi_2) \longrightarrow H_{s-r}(\xi_1,\ \xi_2) \tag{1.185}$$

is continuous for any $s \in \mathbf{R}$.

Theorem 1.42 *On compact manifolds, the symbol of a pseudo-differential operators is a ring homomorphism; in particular,*

$$\sigma(A.B) = \sigma(A) \circ \sigma(B). \tag{1.186}$$

The following theorem is a fundamental result in the theory of elliptic operators.

Theorem 1.43 *If A is an elliptic pseudo-differential operator of order r and u is a solution of the equation*

$$A(u) = f, \quad where \quad f \in H_s(M), \tag{1.187}$$

then $u \in H_{s+r}(M)$.

1.10.1.4 Abstract Symbol

Definition 1.42 Let $B(T, S)$ denote the space of bounded operators from the Banach space S to the Banach space T. Let $K(T, S) \subset B(T, S)$ denote the sub-space of *compact operators*.

Lemma 1.1

$$K(T, S) \circ B(S, S) = B(T, T) \circ K(T, S) = K(T, S). \tag{1.188}$$

Definition 1.43 The *abstract symbol* $Symb\ (A)$ is the class of the operator A modulo the sub-space of compact operators $K(S, T)$

$$Symb\ (A) \in B(S, T)/K(S, T). \tag{1.189}$$

If $S = T$, $K(S, S)$ is an ideal in the algebra $B(S, S)$. The quotient algebra $B(S, S)/K(S, S)$ is called *Calkin algebra*.

1.10.2 Fredholm Operators

Definition 1.44 Let $A : S \longrightarrow T$ be a bounded operator between two Banach spaces.

Recall the operator A is called *Fredholm operator* provided

$$dim\ Ker\ A = finite, \quad A(S)\ is\ closed\ and\ dim : Coker\ A = finite. \tag{1.190}$$

Equivalently, A is a Fredholm operator iff there exists an operator $B \in B(T, S)$ such that

$$S_0 := 1 - BA \in K(S, S), \qquad S_1 := 1 - AB \in K(T, T). \tag{1.191}$$

We will say that B is an *almost inverse* of A, or *parametrix* for A.

Proposition 1.14

(i) Let $A_1 \in B(S, T)$ and $A_2 \in B(T, U)$. Then

$$Symb\ (A_2\ .\ A_1) = Symb\ (A_2) \circ Symb\ (A_1). \tag{1.192}$$

(ii) The operator A is a Fredholm operator if and only if $Symb(A)$ is invertible.

Theorem 1.44 *A pseudo-differential operator is a Fredholm operator if and only if it is an elliptic operator, or equivalently, if and only if its symbol is elliptic.*

1.10.3 Analytical Index

Definition 1.45 The *analytical index $a - Ind()$ of the Fredholm operator A* is

$$a - Ind(A) = dim\ Ker\ A - dim\ Coker\ A.$$

Theorem 1.45 (Calderon [49] and Hörmander [61], See Also [82, Proposition 6])
Let B be a parametrix for the operator A such that the operators

$$S_0 := 1 - BA, \qquad S_1 := 1 - AB. \tag{1.193}$$

be trace class *operators. Then*

$$a - Ind(A) = Tr\ S_0 - Tr\ S_1. \tag{1.194}$$

If S_0 and S_1 belong to the Schatten class L^n, see Sect. 1.12.4, then

$$a - Ind(A) = Tr\ (S_0)^n - Tr\ (S_1)^n, \tag{1.195}$$

Proposition 1.15 *The operators S_0, S_1 satisfy the identities*

$$A\ S_0 = S_1\ A, \qquad B\ S_1 = S_0\ B.$$

Proposition 1.16 *Suppose the operators S_0, S_1 belong to the Schatten class L^n. Let $\tilde{B}$ be the operator*

$$\tilde{B} := (1 + S_0 + \ldots + S_0^{n-1})\, B.$$

Then,

$$\tilde{S}_0 := 1 - \tilde{B}.A = S_0^n, \qquad \tilde{S}_1 := 1 - A.\tilde{B} = S_1^n.$$

Both operators $\tilde{S}_0$ and $\tilde{S}_1$ are trace class; therefore

$$a - Ind(A)\tilde{S}_0 = TrS_0^n - TrS_1^n.$$

Proof One has

$$\tilde{S}_0 = 1 - \tilde{B}A = 1 - (1 + S_0 + \cdots + S_0^{n-1})\, B\, .\, A = 1 - B\,(1 + S_1 + \cdots + S_1^{n-1}).\, A =$$
$$1 - B\, A\, (1 + S_0 + \cdots + S_0^{n-1}) = 1 - (1 - S_0)\,(1 + S_1 + \cdots + S_1^{n-1}) = S_0^n.$$

Analogously we find

$$\tilde{S}_1 = 1 - A\,\tilde{B} = 1 - A\,(1 + S_0 + \ldots + S_0^{n-1})\, B = S_1^n.$$

The procedure presented above is not *symmetric* on the pair of operators A, B. The next construction has *symmetry*, see [101, pg. 674].

Proposition 1.17 *Let $q(t)$ be the unique polynomial of degree $2l$, $n \leq 2l$ such that*

$$(1 - t)^{-\frac{1}{2}} = q(t) + O(2l + 1) \quad for\ t\ \ small.$$

Let $p(t)$ be the polynomial

$$p(t) = (1 - t)q^2(t) - 1 = O(2l + 1).$$

Define

$$\tilde{A} = q(S_1)\, A, \qquad \tilde{B} = q(S_0)\, B.$$

Then

$$a - Ind(A) = Tr\ p(S_1) - Tr\ p(S_0).$$

Proof One has

$$\tilde{S_0} := 1 - \tilde{B}\ \tilde{A} = 1 - q(S_0)\ B.q(S_1)\ A = 1 - q(S_0).q(S_0)\ B\ A =$$
$$1 - q(S_0).q(S_0)\ (1 - S_0) = -p(S_0) \in L^1 \tilde{S_1} := 1 - \tilde{A}\tilde{B} = -p(S_0) \in L^1.$$

Therefore

$$a - Ind(A) = Tr\ \tilde{S_0} - Tr\ \tilde{S_1} = Tr\ p(S_1) - Tr\ p(S_0).$$

1.10.4 Dirac Operators

We do not discuss Dirac operators in this book. Dirac operators exist on Riemannian smooth manifolds which admit a *spin structure*. Dirac operators are fundamental for certain differential geometry and physics phenomena, but the necessary spin structure *reduces their existence* on *all* topological manifolds; for this reason we do not study them here.

1.10.5 Signature Operators

In this section we use the notation of Sect. 1.9.

We assume here that M has even dimension $n = 2l$. Then

$$*_{n-k} \circ *_k = (-1)^k. \tag{1.196}$$

The star operator is modified

$$\tau_k := i^{k(k-1)+l} *_k . \tag{1.197}$$

The operator $\tau := \oplus_k \tau_k$ is an *involution*, $(\tau)^2 = I$, on $\Omega^* := \oplus_{0 \le k \le n} \Omega^k$. The bundles $\wedge^k \oplus \wedge^{n-k}$ are τ-invariant.

Definition 1.46 An element of the external power of the co-tangent bundle (not necessarily homogeneous), ξ, is called *positive*, resp. *negative*, provided $\tau(\xi) = \xi$, resp. $\tau(\xi) = -\xi$.

The *total wedge bundle* $\wedge := \oplus_{0 \le k \le n} \wedge^k$ decomposes into the orthogonal direct sum of the *positive and negative bundles* $\wedge^{\pm}$

$$\wedge = \wedge^+ \oplus \wedge^-. \tag{1.198}$$

Accordingly, a differential form (not necessarily homogeneous), ω, is called *positive*, resp. *negative*, if $\tau\ (\omega) = \omega$, resp. $\tau\ (\omega) = -\omega$.

Definition 1.47 The space Ω^* decomposes in the orthogonal direct sum of *positive and negative differential forms*

$$\Omega^* = \Omega^+ \ \oplus \ \Omega^-. \tag{1.199}$$

Due to the dimension n assumption, the expression of the operator δ becomes

$$\delta_k = - * \ d \ * . \tag{1.200}$$

Definition 1.48 Define the operator

$$D := \ d \ + \ \delta : \Omega^* \longrightarrow \Omega^*. \tag{1.201}$$

Proposition 1.18 *The operator D has the properties*

1. *it is self-adjoint,*
2. *it anti-commutes with* τ

$$D \ \tau \ = \ -\tau \ D \tag{1.202}$$

3. *it interchanges the spaces* $\Omega^{\pm}$

$$D : \Omega^{\pm} \longrightarrow \Omega^{\mp}. \tag{1.203}$$

Definition 1.49 We define the *signature operator* D^+ as the restriction of the operator D on the sub-space of positive differential forms. Its co-domain consists of negative forms. Likewise, D^- is the restriction on the sub-space of negative differential forms.

Proposition 1.19 *The operators* τ *and D, expressed with respect to the orthogonal direct sum decomposition* $\Omega \ = \ \Omega^+ \oplus \Omega^-$ *are*

$$\tau \ = \ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.204}$$

$$D \ = \ \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}. \tag{1.205}$$

The operator D is self-adjoint and it anti-commutes with τ*. One has*

$$D^{\pm *} \ = \ D^{\mp} \tag{1.206}$$

$$D^{\pm} \ \tau \ = \ - \ \tau \ \ D^{\pm}. \tag{1.207}$$

1.10.5.1 Signature Operators and Their Analytical Index

Signature operators play a fundamental role in the index theory. We will study signature operators both in the smooth and Lipschitz/quasi-conformal categories.

We begin by discussing signature operators in the smooth category.

Definition 1.50 If M is a closed, oriented topological manifold of dimension $4k$, then the *signature* M, denoted $Sig\ (M)$, is difference between the dimension of the $+1$ and -1-eigenspaces of the Poincaré pairing on the $H^{2k}(M, \mathbb{R})$.

Theorem 1.46 *The signature operator is an elliptic first order differential operator. Its analytical index is the signature of the manifold*

$$a - Ind(D^+) = Sig\ (M). \tag{1.208}$$

Proof The first part is clear. The proof of the second part is based on the following facts:

1. the adjoint of the signature operator is D^- and therefore
2. $dim\ Coker D^+ = dim\ Ker D^-$
3. the regularity of solutions of elliptic equations, Theorem 1.43 gives that $Ker\ D^+$ and $Coker\ D^+$ consist of smooth sections
4. the Hodge isomorphism Theorem 1.46 (1.208), combined with the fact that (i) the kernel and co-kernel of the signature operator, which originate from $\Omega^k \oplus \Omega^{n-k}$, $k \neq n/2$, cancel out; (ii) the remaining contribution to $a-Ind\ (D^+)$ concentrates to the middle dimension, where the properties of the involution τ and of the Poincaré pairing complete the argument.

The rest of the proof is left to the reader.

We will see that signature operators exist on any topological manifold endowed with a Lipschitz/quasi-conformal structure.

1.11 Statement of the Atiyah–Singer Index Theorem

The intent of this section is not to present a detailed discussion of the Atiyah–Singer index theorem. We state the Atiyah–Singer index theorem here to give us the possibility to indicate, in a chronological order, some of the developments which occurred in connection with it. We will return to the Atiyah–Singer index theorem in Sect. 5.6

Theorem 1.47 (Atiyah and Singer [42, 53, 54]) *Let A be an elliptic pseudo-differential operator on the smooth manifold M and let $\sigma(A)$ be its symbol. Then*

$$a - Ind(A) = \int_M Ch\ \sigma(A) \cup Td^{\mathbf{C}}(T^*(M)). \tag{1.209}$$

The Todd class is a basic ingredient of Hirzebruch's [32] formula both for the *topological index* of the signature operator and Hirzebruch's index formula for analytic bundles (1953); the Todd class is a basic ingredient of the expression for the topological index in all categories under consideration, going down up to topological manifolds. For more information about the Todd class the reader should refer to [32, 60] and [54]. It is also very interesting to read how Hirzebruch was lead to discover his index formulae [60].

Definition 1.51 (Todd Class, Todd [2] and Hirzebruch [32], See Also [60]) For any complex bundle η of rank n over the CW-complex X, its *Todd class* is defined by

$$Td(\eta) := \prod_{1\leq i\leq n} \frac{y_i}{1 - e^{-y_i}}. \tag{1.210}$$

where y_i are the *formal Chern split symbols*

$$c(\eta,\ t) := 1 + c_1(\eta)\, t + c_2(\eta)\, t^2 + \ldots + c_n(\eta)\, t^n = (1 + x_1\, t)\,(1 + x_2\, t)\,\ldots\,(1 + x_n\, t). \tag{1.211}$$

For any *real* vector bundle η of rank N, its *Todd class*, $Td^{\mathbb{C}}(\eta)$, is the Todd class of the bundle complexification

$$Td^{\mathbb{C}}(\eta) := Td(\eta \otimes_{\mathbb{R}} \mathbb{C}). \tag{1.212}$$

For any smooth manifold of dimension n, the *Todd class* of M is $Td^{\mathbb{C}}(T^*(M))$.

Remark 1.18 For any real vector bundle η the total Chern class $c_*(\eta, t) \in H^{ev}(X, \mathbf{Q})$ (torsion killed) is an alternating sum of Chern classes of degree $4k$, i.e. of Pontrjagin classes; its *Chern factorisation* may be expressed in terms of y_i, the basic characters of the standard torus of $O(n)$

$$\begin{gathered} c(\eta, t) := 1 + c_2(\eta)t^2 + \ldots + c_{2k}(\eta)t^{2k} + \ldots + c_{2[n/2]}(\eta)t^{2[n/2]} = \\ 1 - p_1(\eta)t^2 + p_2(\eta)t^4 + \cdots + (-1)^{[n/2]}p_{[n/2]}(\eta)t^{2[n/2]} = \\ \prod_i (1 + y_i\, t)(1 - y_i\, t) = \prod_i (1 - y_i^2\, t^2). \end{gathered} \tag{1.213}$$

Then

$$Td^{\mathbf{C}}(T^*(M)) = \prod_{1\leq i\leq [n/2]} \frac{-y_i}{1 - e^{y_i}} \frac{y_i}{1 - e^{-y_i}}. \tag{1.214}$$

These show that the Todd class $Td^{\mathbf{C}}(T^*(M))$ is a symmetric polynomial in the variables y_i^2. Therefore, the Todd class of $T^*(M)$ is given by an universal polynomial of the Pontrjagin classes $p_i(T^*(M))$. See [32, 54] for more details.

More generally, the *Hirzebruch genus associated to the power series*

$$\Phi(y) = 1 + a_1 y + a_2 y^2 + \ldots + a_n y^n + \ldots$$

is the cohomology class

$$\Phi(p_1, p_2, \ldots, p_{n/2}) = \prod_{1 \leq i \leq n/2} \Phi(y_i)$$

where the Pontrjagin classes of M are seen as symmetric functions of $y_1, y_2, \ldots, y_n$ (1.213).

In particular, the co-homological class associated to the power series $\mathscr{L}(y) = \sqrt{y}/tanh\sqrt{y}$ is the *Thom genus*, see Sect. 5.2.

Theorem 1.48 (Atiyah and Singer [54, p. 553])
Let V be a complex vector bundle over the compact CW-complex X. Let

$$\varphi_V : K(X) \longrightarrow K(V) \tag{1.215}$$

$$\psi_V : H^*(X, \mathbf{Q}) \longrightarrow H^*(V, \mathbf{Q}) \tag{1.216}$$

be the Thom isomorphisms in K-theory and cohomology.

Then, the Todd class $T(V)$ is the unique rational cohomology class $T(V) \in H^{ev}(X, \mathbb{Q})$ such that

$$\psi_V^{-1}\,(Ch\,(\varphi_V\,u)) = (Ch\,u)\,.\,T(V), \tag{1.217}$$

for any $u \in K(X)$.

For Thom isomorphism ψ_V in cohomology, the reader could refer to Milnor [66]. For Thom isomorphism in K-theory refer to Atiyah [50].

1.12 K-Homology

1.12.1 Clifford Algebras

Let $Q_{p,q}$ be the quadratic form defined on the real vector space $E = \mathbf{R}^{p+q}$

$$Q(x_1, \ldots, x_p, y_1, \ldots, y_q) = -(x_1^2 + \ldots + x_p^2) + (y_1^2 + \ldots + y_q^2). \tag{1.218}$$

Let $T(E)$ be the *tensor algebra*

$$T(E) := \mathbb{R} \oplus E \oplus (\otimes^2_{\mathbb{R}} E) \oplus \ldots . \oplus (\otimes^k_{\mathbb{R}} E) \oplus \ldots . \tag{1.219}$$

Let $I \subset T(E)$ be the bi-lateral ideal of $T(E)$ generated by the elements $x \otimes_{\mathbf{R}} x - Q(x) \,.\, 1$, for any $x \in E$.

Definition 1.52 Let $C_{p,q}$ be the quotient algebra

$$C_{p,q} := T(E)/I. \tag{1.220}$$

The algebra $C_{p,q}$ is called *Clifford algebra*.

Let $e_1, \ldots e_p, \epsilon_1, \ldots, \epsilon_q$ be the unit vectors in E corresponding to the components $x_1, \ldots, x_p, y_1, \ldots, y_q$.

Theorem 1.49 *The Clifford algebra $C_{p,q}$ is the free algebra generated by* 1 *and the elements e_i, $1 \leq i \leq p$, ϵ_j, $1 \leq j \leq p$ subject to the relations*

$$e_i^2 = -1, \quad for \;\; 1 \leq i \leq p; \quad e_i\, e_j = -e_j\, e_i \quad for \;\; 1 \leq i \neq j \leq p \tag{1.221}$$

$$\epsilon_j^2 = 1, \quad for \;\; 1 \leq j \leq q; \quad \epsilon_i\, \epsilon_j = -\epsilon_j\, \epsilon_i \quad for \;\; 1 \leq i \neq j \leq p \tag{1.222}$$

$$e_i\, \epsilon_j = -\epsilon_j\, e_i \quad for \;\; 1 \leq i \leq p, \; 1 \leq j \leq q. \tag{1.223}$$

The algebra $C_{p,q}$ is a Banach algebra with involution $*$*; the involution acts on the generators according to the relations*

$$e_i^* = -e_i, \quad for \;\; 1 \leq i \leq p; \quad \epsilon_j^* = \epsilon_j, \quad for \;\; 1 \leq j \leq q. \tag{1.224}$$

Definition 1.53 In the definition of the Clifford algebra $C_{p,q}$ we replace the space $\mathbf{R}^{p+q}$ by $\mathbf{C}^{p+q}$ and we keep the quadratic form Q unchanged. The corresponding algebra will be called *complex Clifford algebra*; it will be denoted by $C^{\mathbf{C}}_{p,q}$.

Theorem 1.50 *The complex Clifford algebras $C^{\mathbf{C}}_{p+1,q}$, $C^{\mathbf{C}}_{p,q+1}$ are isomorphic.*

Proof The function $f : C^{\mathbf{C}}_{p+1,q} \longrightarrow C^{\mathbf{C}}_{p,q+1}$, defined on the generators of the two algebras by the formulas

$$f(e_k) = e_k, \quad for \;\; 1 \leq k \leq p, \quad f(e_{p+1}) = i\, \epsilon_1$$

$$f(\epsilon_l) = \epsilon_l, \quad for \;\; 1 \leq l \leq q,$$

induces an algebra isomorphism.

For more details on Clifford algebras and modules see [38].

1.12.2 Kasparov K-Homology

Whitehead [31] gave a formal homological definition of *K-homology* (1962). Kasparov [67], using Atiyah's [50] definition of *abstract elliptic operator*, found an explicit realisation of this functor. Another description of the K-homology functor was given by Brown, Douglas and Fillmore [70] in terms of extensions of algebras. Here we describe the Kasparov construction. We use the following notation

1. $\mathscr{A}$ is a finitely generated Banach algebra with involution *.
2. H is a separable complex Hilbert space.
3. $L(H)$ is the Banach algebra of bounded operators on H. The adjoint mapping * is an involution on this algebra.
4. $K(H)$ is the algebra of *compact operators* on H;
   $A(H) := L(H)/K(H)$ is the *Calkin algebra*.
5. $U(H)$ is the group of *unitary operators* on H.
6. $C^{\mathbf{C}}_{p,q}$ is the complex Clifford algebra.

Definition 1.54 Let $E(\mathscr{A})$ consist of all triples $\{\varphi,\ \psi,\ F\}$ such that

1. $\varphi : A \longrightarrow L(H)$ is an involutive algebra homomorphism with $\varphi(1) = 1$,
2. $\psi : C^{\mathbf{C}}_{p,q+1} \longrightarrow L(H)$ is an involutive homomorphism which commutes with φ,
3. F is a Fredholm operator on H such that
4. F anti-commutes with the elements $\psi(e_1), \ldots, \psi(e_p), \psi(\epsilon_1, \ldots, \psi(\epsilon_{q+1})$ satisfying the properties

$$\varphi(f)\ F - F\ \varphi(f)\ =\ compact\ operator,\ \ for\ any\ f \in \mathscr{A}. \tag{1.225}$$

Definition 1.55 Define the set of *degenerate elements* of $E(\mathscr{A})$

$$D(\mathscr{A}) \subset E(\mathscr{A}) \tag{1.226}$$

consists of those elements $(\varphi,\ \psi,\ F)$ for which

$$\varphi(f)\ F\ -\ F\ \varphi(f)\ =\ 0,\ \ for\ any\ f \in \mathscr{A}. \tag{1.227}$$

There are two equivalence relations defined in $E(\mathscr{A})$

1. *homotopy*, denoted $\sim_h$. Two triples $(\varphi_0,\ \psi_0,\ F_0)$, $(\varphi_1,\ \psi_1,\ F_1)$ belonging to $E(\mathscr{A})$, are called *homotopic* provided there exists a family of triples $(\varphi_t,\ \psi_t,\ F_t) \in E(\mathscr{A})$, depending continuously on $t \in [0, 1]$, which begins and ends with the two triples.
2. *conjugation*, denoted $\sim_c$. Two triples $(\varphi_0,\ \psi_0,\ F_0)$, $(\varphi_1,\ \psi_1,\ F_1)$ are called *conjugated* provided there exists an auto-morphism u of the Hilbert space such that

$$(\varphi_0,\ \psi_0,\ F_0)\ =\ (u\varphi_1 u^{-1},\ u\psi_1 u^{-1},\ uF_1 u^{-1}). \tag{1.228}$$

Let $\tilde{E}(\mathscr{A})$ be the quotient of $E(\mathscr{A})$ modulo the two equivalence relations.
$E(\mathscr{A})$ becomes a semi-group under the direct sum operation

$$(\varphi_0,\ \psi_0,\ F_0)\ +\ (\varphi_1,\ \psi_1,\ F_1) := (\varphi_0 \oplus \varphi_1,\ \psi_0 \oplus \psi_1,\ F_0 \oplus F_1). \tag{1.229}$$

The direct sum becomes commutative after factorisation; one obtains the commutative semi-group $\tilde{E}(\mathscr{A})$.

The image of $D(\mathscr{A})$ in $\tilde{E}(A)$ is denoted $\tilde{D}(\mathscr{A})$.

Definition 1.56

$$K_{p,q}(A) := \tilde{E}(\mathscr{A})/\tilde{D}(\mathscr{A}). \tag{1.230}$$

Theorem 1.51

1. *(Kasparov [67, Proposition 1]): $K_{p,q}(\mathscr{A})$ is a group.*
2. *(Kasparov [67, Proposition 2]): In the definition of $E(\mathscr{A})$ the condition $F^* + F = \text{compact operator}$, or stronger condition $F^* + F = 0$, could be added; in this case, in the definition of $D(\mathscr{A})$, the condition $F^* + F = 0$ should be added*
3. *the condition $F^2 + 1 = \text{compact operator}$ could be added; in this case the condition $F^2 + 1 = 0$ should be added in the definition of $D(\mathscr{A})$.*
4. *(Kasparov [67, Theorem 2]): The groups $K_{p,q}(\mathscr{A})$ $K_{p+1,q+1}(A)$ are isomorphic.*

In non-commutative geometry the structure involved in Theorem 1.51 is called *spectral triple*.

Definition 1.57 (Spectral Triple) A *spectral triple* is a quadruple $(\mathscr{A}, H, \rho, F)$ where $\mathscr{A}$ is an associative algebra, H is Hilbert space, ρ is a representation of the algebra $\mathscr{A}$ in the Hilbert space H and F is Fredholm operator on H. In the next chapter we will not specify F, even if it is understood.

Spectral sequences are *non-commutative spaces*.

Definition 1.58 The complex *K_n-homology groups*, $n \in \mathbb{Z}$, is by definition

$$K_n(\mathscr{A}) = \begin{cases} K_{n,0}(A) & \text{for } 0 \le n \\ K_{0,n}(A) & \text{for } n \le 0. \end{cases} \tag{1.231}$$

Recalling the algebra isomorphism $f : C^{\mathbf{C}}_{p+1,q} \longrightarrow C^{\mathbf{C}}_{p+1,q}$ given by Theorem 1.50, we get

Theorem 1.52 *For any* finitely generated Banach algebra $\mathscr{A}$, *its complex K-homology groups are* 2*-periodic*

$$K_n(\mathscr{A})\ \textit{is isomorphic to}\ K_{n+2}(\mathscr{A}),\quad \textit{for any}\ n \in \mathbb{Z}. \tag{1.232}$$

Therefore, up to isomorphism, there are only two K-homology groups, $K_0(\mathscr{A})$ and $K_1(\mathscr{A})$.

*For any compact CW-complex X, the groups $K_i(C^0(X))$ are isomorphic to the K-*homology groups *of the space $K_*(X)$ defined by G. W. Whitehead [31].*

Theorem 1.53 *In view of Theorem 1.51*

1. *we can use $p = 1,\ q = 0$ to define $K_0(\mathscr{A})$*
2. *and $p = 0,\ q = 1$ to define $K_1(\mathscr{A})$.*

$K_*(C^0(\ -\ ))$ is a generalised homology functor *on the category of paracompact CW-complexes.*

$K_*(\ -\ )$ is a generalised homology functor *on the category of finitely generated Banach algebras.*

1.12.3 Even and Odd Spectral Triples

Spectral triples were introduced by Connes [82]. The definition of spectral triples uses part of Kasparov's [67] data which define K-homology. In particular, no assumption is made concerning a possible topology on the basic algebra. Spectral triples play in non-commutative geometry the role differential manifolds play in differential geometry.

Definition 1.59 (Even Spectral Triples) Let $\mathscr{A}$ be an associative algebra. Let H be a complex separable Hilbert space of infinite dimension.

Let $\rho : \mathscr{A} \longrightarrow L(H)$ be a representation of the algebra $\mathscr{A}$ on the Hilbert space H, where $\mathscr{L}(H)$ is the algebra of bounded operators on H.

Let $\gamma : L(H) \longrightarrow L(H)$ be an *involution*, $\gamma^2 = 1$ which defines a Z_2-grading of the Hilbert space H. We suppose that $\gamma isself - adjoint\ \gamma^* = \gamma$. We assume the representation ρ preserves the grading ($\rho\ \gamma = \gamma\ \rho$). Let $H^{\pm}$ be the ± 1-eigenspaces of the grading γ acting on H.

Let $D \in L(H)$ be a bounded operator satisfying the properties: $D^2 = 1,\ D^* = F,\ D\ \gamma = -\gamma\ D$.

Then $(\mathscr{A}, \rho, D)$ is called an *even spectral triple*.

If $[D, \rho(a)] \in L^p$, for any $a \in \mathscr{A}$, where L^p is the *Schatten class ideal* (see Sect. 1.12.4), then $(\mathscr{A}, \rho, F)$ is called *p-summable* spectral triple.

Let us explain the meaning of some requirements which are used in the definition of an even spectral triple.

The condition $D\ \gamma = -\gamma$ says that D exchanges the spaces $H_{\pm}$

$$D =: \begin{pmatrix} 0 & -G \\ F & 0 \end{pmatrix} : H_+ \oplus H_- \longrightarrow H_- \oplus H_+.$$

The condition $D^2 = -1 + \mathit{compact\ operator}$ says that $S_0 = 1 - GH$ and $S_1 = 1 - HG$ are *compact operators*. In this case, G and H are *parametrices* one to each other. If $D^2 = -1$, then $S_0 = 0$ and $S_1 = 0$.

The condition

$$\gamma\, \rho = \rho\, \gamma$$

says that the representation ρ keeps the spaces $H_\pm$ invariant. This means that for $\rho = \rho_+ \oplus \rho_-$, $\rho : H_\pm \longrightarrow H_\pm$.

If D were a pseudo-differential operator on the n-dimensional smooth manifold and f were a *continuous function* on M, then

$$\rho\, f\ \ D - D\, \rho\, f = \ \mathit{compact\ operator}.$$

However, this property could be sharpened if f were a *smooth function*

$$\rho\, f\ \ D - D\, \rho\, f\ \in L^{n+\epsilon},\ \ \forall \epsilon > 0.$$

Proposition 1.20 *Let F be an pseudo-differential operator on the smooth manifold M^n. Then there exists a parametrix G for F so that the corresponding operators S_0 and S_1 would be Schatten class operators in $L^{n+\epsilon}$, for any $0 < \epsilon$. In this case $(\mathscr{A}, \rho, D)$ is an $(n+\epsilon)$-summable spectral triple.*

Therefore, n-summability of spectral triples codifies the dimension of the underlying space.

More information about Schatten class operators is given in the next section (Sect. 1.12.4).

Remark 1.19 ϵ in the above formula could be removed if instead of ordinary trace we used the *Dixmier trace*, see Dixmier [40] and Connes [99]. We do not treat Dixmier trace in this book.

Definition 1.60 Suppose $\mathscr{E} = \{\mathscr{A}, \rho, F\}$ is a triple satisfying all conditions above, with one exception: $F^2 \neq 1$. The triple $\mathscr{E}$ is called a *pre-Fredholm module*.

Proposition 1.21 *Let $\mathscr{E} = \{\mathscr{A}, \rho, F\}$ be a pre-Hilbert module.*

There is a canonical procedure which associates with the pre-Hilbert module $\mathscr{E}$ a Fredholm module $\tilde{\mathscr{E}}$.

Proof For the proof we use [82, Appendix 2].

Assume F is given by the matrix

$$F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}.$$

operating on the Hilbert space $H = H^+ \oplus H^-$. By hypothesis

$$S_0 := 1 - Q\,P$$
$$S_1 := 1 - P\,Q$$

are compact operators.

Consider $H_{\mathbf{C}}$ to be the $\mathbb{Z}_2$-graded Hilbert space with $H^{\pm}_{\mathbb{C}} = \mathbb{C}$. Let $\tilde{H}$ be the graded tensor product $H \hat{\otimes} H_{\mathbf{C}}$. One gets

$$(H \hat{\otimes} H_{\mathbf{C}})^+ = H^+ \oplus H^-$$
$$(H \hat{\otimes} H_{\mathbf{C}})^- = H^- \oplus H^+$$
$$\tilde{H} = H^+ \oplus H^- \oplus H^- \oplus H^+.$$

Define

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : H_{\mathbf{C}} \longrightarrow H_{\mathbf{C}}.$$

$\tilde{H}$ becomes a left $\mathscr{A}$-module by means of the representation $\tilde{\rho}$

$$\tilde{\rho}(f)(\xi \tilde{\otimes} \eta) := (\rho(f)\xi) \tilde{\otimes} (e\ \eta)$$

Let $\tilde{F} : \tilde{H} \longrightarrow \tilde{H}$ be operator defined by the matrix

$$\tilde{F} = \begin{pmatrix} 0 & \tilde{Q} \\ \tilde{P} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & (1+S_0)Q & S_0 \\ 0 & 0 & S_1 & -P \\ P & S_1 & 0 & 0 \\ S_0 & -(1+S_0)Q & 0 & 0 \end{pmatrix}. \tag{1.233}$$

It is easy to check that $(\ \tilde{F}\ )^2 = 1$. The operators $\tilde{P}$ and $\tilde{Q}$ are *compact perturbations of the operators* P and Q.

Then $\tilde{\mathscr{E}} = \{\mathscr{A}, \tilde{\rho}, \tilde{F}\}$ is a Fredholm module.

Definition 1.61 (Odd Spectral Triples) Let $\mathscr{A}$ be an associative algebra. Let H be a complex separable Hilbert space of infinite dimension.

Let $\rho : \mathscr{A} \longrightarrow L(H)$ be a representation of the algebra $\mathscr{A}$ on the Hilbert space H, where $L(H)$ is the algebra of bounded operators on H.

Let $F \in L(H)$ be a bounded operator satisfying the properties: $F^2 = 1$, $F^* = F$, $F\gamma = -\gamma F$.

Then $(\mathscr{A}, \rho, F)$ is called an *odd spectral triple*.

If $[F, \rho(a)] \in \mathscr{L}^p$, for any $a \in \mathscr{A}$, then the spectral triple is called *p-summable*.

1.12.4 Schatten Class Operators

The operators **k** belonging to the Schatten class L^n are characterised by the speed their *characteristic values* λ_n converge at infinity. For any compact operator **k** the eigenvalues of the operator $\mathbf{k}^*\mathbf{k}$ are non-negative real numbers, discretely distributed. We put these numbers in increasing order and denote by s_n the nth such number. The number $\lambda_n = \sqrt{s_n}$ is by definition the nth *characteristic value* of the operator **k**.

Definition 1.62 A compact operator **k** defined on the Hilbert space H belongs to the *Schatten class* $L^p(H)$ provided its characteristic values λ_n satisfy the condition

$$||\mathbf{k}||_p := \Big(\sum_{1 \le n < \infty} \lambda_n^p \Big)^{1/p} < \infty. \tag{1.234}$$

$||-||_p$ is the norm in the space L_p.

Proposition 1.22

(i) Schatten class operators L^n are bi-lateral ideals of the space of compact operators $K(H)$.

(ii) Schatten class operators satisfy the Hölder-type relation

$$L^p \,.\, L^q \subset L^r, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 1 \le p, q < \infty. \tag{1.235}$$

(iii) The Schatten norms $||-||$ satisfy the Young-type inequality

$$||ST||_r \le ||S||_p \,.\, ||T||_q. \tag{1.236}$$

(iv) The Schatten class L^1 consists of trace class operators. *The operator trace Tr exists only for operators belonging to this class.*

(v) If $S \in L_p$, then

$$||S||_p = (Tr\, |S|^p)^{1/p}, \tag{1.237}$$

where $|S| := (S.S^*)^{1/2}$.

(vi) If $A_i \in L^n$, $1 \le i \le n+1$, then the operator trace $Tr(\, A_1 \,.\, A_2 \,.\, \ldots \,.\, A_n\,)$ *exists.*

More information about Schatten class operators could be found in Appendix, Part I [82].

Property (vi) is the main reason why n-summable Fredholm modules (defined next) are introduced in non-commutative geometry.

1.12.5 n-Summable Fredholm Modules

Definition 1.63 Assume that $\mathbb{K} = \mathbb{R}, \mathbb{C}$. A triple $\{\mathscr{A}, D, \rho\}$ consisting of the associative algebra $\mathscr{A}$, a Fredholm operator D and a representation ρ of $\mathscr{A}$ on the Hilbert space H, satisfying the Schatten condition $D \in L_n$ (1.234), is called an *n-summable Fredholm module* provided that the conditions discussed in Connes [82] Part I, Section 1, hold.

1.13 Geometry of the Signature Operator on Differentiable Manifolds

The next Theorem 1.57 provides a *non-commutative Thom–Hirzebruch formula*. The understanding of this section would become clear after reading Sects. 6.1 and 6.2 below. We present this topic here because it is formulated and uses exclusively the *smooth manifolds* structure.

The signature operators are at the root of the index formula on topological manifolds [83, 84, 101].

Remark 1.20 In this section we present the signature operators in the category of *smooth manifolds*. We will show in Chap. 6 that signature operators live on topological manifolds with *Lipschitz structure* and on *quasi-conformal manifolds*. We anticipate by saying that the study of the *index formula of signature operators index formula of signature operators* reflects the entire philosophy of this book. In this context the *topological index* is identified as a *cohomology class* [83]. The same phenomenon occurs in the quasi-conformal case [101]. The topological index of signature operators may be identified as a *differential form*, provided the *quasi-local index formula* [101] converges towards a differential form while approaching the diagonal; the limit is the Hirzebruch–Thom formula.

The interest in the present approach consists also of the fact that while the original Hirzebruch proof [32, 60] of the *Hirzebruch–Thom class* formula was based on Thom's results on *cobordism* [11], the present treatment is realised in the *non-commutative geometry* of the *manifold itself*. Involving cobordism technique to prove a certain property $\mathscr{P}$ on a manifold M means treating the property $\mathscr{P}$ on *all* manifolds which are *cobordant* with M.

In this section we produce a *non-commutative version of the signature operator* on smooth manifolds. At the end of this section we show how the *signature* of a smooth manifolds builds-up from the geodesic distance and give formulae aimed at finding the classical limit of the non-commutative signature operator.

This section reports joint work with Lescure and Teleman [124], unpublished (2007).

Definition 1.64 Let (M, g) be a Riemannian manifold of dimension $n = 2l$. Let us note by $r(x, y)$ the *geodesic distance* between x and y. One fixes a smooth cut-

off function λ positive, supported on $[-\rho, \rho]$ with ρ the injectivity radius of M, g. Assume moreover, that $1 - \lambda(r^2)$ is at least $O(r^4)$. Around the diagonal of $M \times M$ the function $A(x, y) = -\frac{r^2}{2}$ is smooth and we define the following bi-forms :

$$\alpha_p = \frac{1}{p!}\left(d_x d_y A(x, y)\right)^{\wedge^p}; \alpha = \sum_{p=0}^{n} \alpha_p$$

$$\omega = \frac{r^{2-n}\lambda(r^2)}{(n-2)s_n}\alpha.$$

Then, a parametrix Ω of the Laplacian Δ acting on forms is given by the following formula (de Rham's book [86]) :

$$\Omega\varphi(x) = \int \omega(x, y) \wedge *\varphi(y).$$

Associated with Ω, we introduce the following bounded operators acting on l-forms:

$$S_{\pm} = d\Omega\delta \pm \gamma d\Omega\delta\gamma$$

(with $\gamma = i^l *$, $\gamma^2 = 1$.)

Let $\wedge^{\pm}$ be the ± 1-eigenspaces of γ.

Definition 1.65 Let A be the *r-signature operator*

$$A = S_- \frac{1+\gamma}{2} : \wedge^+ \longrightarrow \wedge^-. \tag{1.238}$$

The operator A is called *r-signature operator* to distinguish it from the classical signature operator considered in Sect. 5.2. The discussion made in Sect. 5.2 shows that the analytical index of the signature operators depends on the *middle dimension* only. The domain of the r-signature operator is the space of differential forms of middle dimension. The signature operator and the r-signature operator are homotopically equivalent.

Proposition 1.23 *Let dimension of M be $n = 2l$, $l = even$. Then*

$$a - Ind(A) = Sig(M). \tag{1.239}$$

Proof Let $u \in \wedge^+$ and $A(u) = (S_- \frac{1+\gamma}{2})(u) = 0$. It is easy to see that u is an harmonic form. An argument similar to that done in Sect. 5.2 completes the proof.

Proposition 1.24

(i) The operator S_+ satisfies

$$S_+ \;=\; 1 \;+\; K,$$

where K is the integral operator

$$(K\varphi)(x) \;=\; \int_M (1 + *_x *_y)(d_x d_y \omega_{l-1}) \wedge *_y \varphi,$$

(ii) the (metric) kernel of K is

$$k_1(x, y) \;=\; (1 + *_x *_y)(d_x d_y \omega_{l-1}) \;=\; O(r^{2-n}).$$

Remark 1.21 The proposition above says that the operator S_+ is not interesting from the homological view point. However, we will see later that this operator may be used to regularise the r-signature operator, see Remark 1.22 and Sect. 1.13.1.

Proof Let φ be an l-form.

$$\begin{aligned}
S_+\varphi(x) \;&=\; (d\Omega\delta \;+\; (-1)^l * d\Omega\delta *)(\varphi) \\
&=\; d_x\Big(\int \omega_{l-1} \wedge *_y \delta_y \varphi\Big) + (-1)^l *_x d_x \int \omega_{l-1} \wedge *_y \delta_y *_y \varphi \\
&=\; (-1)^l \left(\int d_x \omega_{l-1} \wedge d_y *_y \varphi + \int *_x d_x \omega_{l-1} \wedge d_y \varphi \right) \quad \text{since} d_x \omega \;=\; O(r^{1-n}) \\
&=\; -\lim_{\varepsilon \to 0} \int_{r \geq \varepsilon} \big(d_y (d_x \omega_{l-1} \wedge *_y \varphi + *_x d_x \omega_{l-1} \wedge \varphi) - (d_x d_y \omega_{l-1}) \wedge *_y \varphi - d_y *_x d_x \omega_{l-1} \wedge \varphi \big) \\
&=\; \lim_{\varepsilon \to 0} \{ \int_{r=\varepsilon} d_x \omega_{l-1} \wedge *_y \varphi + *_x d_x \omega_{l-1} \wedge \varphi + \int_{r \geq \varepsilon} d_x d_y \omega_{l-1} \wedge *_y \varphi + *_x d_x d_y \omega_{l-1} \wedge \varphi \} \\
&=\; \lim_{\varepsilon \to 0} L_\varepsilon \varphi(x) + G_\varepsilon \varphi(x).
\end{aligned}$$

Let us examine the first integral depending on ε of the previous formula:

$$\begin{aligned}
L_\varepsilon \varphi(x) \;&=\; \int_{r=\varepsilon} d_x \omega_{l-1} \wedge *_y \varphi \;+\; *_x d_x \omega_{l-1} \wedge \varphi \\
&=\; \varepsilon^{-n} \int_{r=\varepsilon} r^n d_x \omega_{l-1} \wedge *_y \varphi \;+\; *_x r^n d_x \omega_{l-1} \wedge \varphi \\
&=\; \varepsilon^{-n} \int_{r \leq \varepsilon} d_y \{ r^n d_x \omega_{l-1} \wedge *_y \varphi \;+\; *_x r^n d_x \omega_{l-1} \wedge \varphi \} \\
&=\; \varepsilon^{-n} \int_{r \leq \varepsilon} d_y (r^n d_x \omega_{l-1}) \wedge *_y \varphi \;+\; d_y (*_x r^n d_x \omega_{l-1}) \wedge \varphi + O(\varepsilon) \\
&=\; \varepsilon^{-n} \int_{r \leq \varepsilon} (1 \;+\; *_x *_y)(d_y r^n d_x \omega_{l-1}) \wedge *_y \varphi \;+\; O(\varepsilon).
\end{aligned}$$

But

$$\begin{aligned} d_y(r^n d_x \omega_{l-1}) &= \frac{1}{s_n} d_y(r^n(-r^{1-n}\lambda(r^2)d_x(r)) \wedge \alpha_{l-1} + O(r^4)) \\ &= \frac{1}{s_n} d_y(\lambda(r^2)d_x(A) \wedge \alpha_{l-1}) + O(r^3) \\ &= \frac{1}{s_n}\lambda(r^2)d_x d_y(A) \wedge \alpha_{l-1} + O(r^3) \\ &= \frac{l}{s_n}\lambda(r^2)\alpha_l + O(r^3). \end{aligned}$$

Hence

$$L_\varepsilon \varphi(x) = \frac{l\varepsilon^{-n}}{s_n} \int_{r\leq\varepsilon} (1 + *_x *_y)(\lambda(r^2)\alpha_l) \wedge *_y \varphi + O(\varepsilon). \tag{1.240}$$

Now let us use normal coordinates centered at the point x. Since

$$-\frac{1}{2}\frac{\partial^2 r^2}{\partial x^i \partial y^j} = \delta_{ij} dx^i \otimes dy^j + O(r^3),$$

we get

$$\alpha_l = \sum_{|I|=l} dx^I \otimes dy^I + O(r^3) = \widetilde{\alpha}_l + O(r^3), \tag{1.241}$$

where the sum is taken over all multi-indices $I = (i_1, i_2, \ldots, i_l)$ with $i_1 < i_2 < \ldots < i_l$ and $\widetilde{\alpha}$ denotes the Euclidean analogue of α in this coordinates system. If $\overline{I}$ denotes the complementary multi-index of I and $\sigma(I)$ the signature of the permutation $(I, \overline{I})$, we check

$$*_x(dx^I) = \sigma(I)dx^{\overline{I}},$$

and

$$*_y(dy^I) = \sqrt{|g(y)|}\left(\sigma(I)g^{I,I}(y)dy^{\overline{I}} + \sum_{k\geq 1}\sum_{J,|J\cap I|=l-k} \sigma(J)g^{I,J}(y)dy^{\overline{J}}\right), \tag{1.242}$$

where, for multiindices I, J are of same length

$$g^{I,J} = \begin{vmatrix} g^{i_1,j_1} & g^{i_1,i_2} & \dots & g^{i_1,j_l} \\ g^{i_2,j_1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ g^{i_l,j_1} & \dots & & g^{i_l,j_l}. \end{vmatrix}$$

Hence, (1.242) becomes

$$*_y\,(dy^I) = \sigma(I)dy^{\overline{I}} + O(r^2), \tag{1.243}$$

Then (1.243) implies that $(1 + *_x *_y)(\alpha_l) = 2\widetilde{\alpha}_l + O(r^2)$ and using this in (1.240), we get

$$L_\varepsilon\varphi(x) = \frac{l\varepsilon^{-n}}{s_n}\int_{r\le\varepsilon}(\lambda(r^2)2\widetilde{\alpha}_l \wedge *_y\varphi + O(\varepsilon). \tag{1.244}$$

But

$$\widetilde{\alpha}(x, y) \wedge *_y\varphi(y) = \varphi(x) *_y 1 + O(r),$$

hence

$$\begin{aligned} L_\varepsilon\varphi(x) &= \frac{n\varepsilon^{-n}}{s_n}\varphi(x)\int_{r\le\varepsilon}\lambda(r^2) *_y 1 + O(\varepsilon) \\ &= \varphi(x)\left(\frac{n\varepsilon^{-n}}{s_n}\int_{r\le\varepsilon} *_y 1\right) + O(\varepsilon) \\ &= \varphi(x)\left(n\varepsilon^{-n}\int_{r\le\varepsilon} r^{n-1}dr\right) + O(\varepsilon) \\ &= \varphi(x) + O(\varepsilon). \end{aligned}$$

Coming back to S_+, we can now write

$$(S_+ - I)\varphi(x) = \lim_{\varepsilon\to 0} G_\varepsilon\varphi(x),$$

Let us investigate the right-hand side of this equality

$$G_\varepsilon\varphi(x) = \int_{r\ge\varepsilon}(1 + *_x *_y)(d_x d_y\omega_{l-1}) \wedge *_y\varphi.$$

Since $\omega = O(r^{2-n})$, the bi-form $d_x d_y \omega$ is $O(r^{-n})$ but we are going to check that the operator $1 + *_x *_y$ cancels its leading part.

Since α is closed, we have

$$\begin{aligned} d_x d_y(\omega_{l-1}) &= d_x d_y\Big(\frac{r^{2-n}\lambda(r^2)}{(n-2)s_n}\Big) \wedge \alpha_{l-1} \\ &= \frac{\lambda(r^2)}{(n-2)s_n} d_x d_y(r^{2-n}) \wedge \alpha_{l-1} + O(r^{4-n}) \\ &= \frac{r^{-n}\lambda(r^2)}{s_n}\Big\{\frac{n}{4r^2} d_x r^2 \otimes d_y r^2 - \frac{1}{2} d_x d_y r^2\Big\} \wedge \alpha_{l-1} \; + \; O(r^{4-n}). \end{aligned}$$

Let us denote $\widetilde{r}$ the Euclidean distance function in this coordinates system. Using (1.241) and the Donnelly formula, see [96, pg. 347]

$$r^2(x,y) \; = \; \widetilde{r}^2(x,y) \; - \; \frac{1}{3} R_{ikjl}(x_B) x^k x^l (y^i - x^i)(y^j - x^j) + O((x,y)^5) \text{ (Donnelly Formula)}, \tag{1.245}$$

we get

$$d_x d_y(\omega_{l-1}) \; = \; \frac{r^{-n}\lambda(r^2)}{s_n}\Big\{\frac{n}{4r^2} d_x \widetilde{r}^2 \otimes d_y \widetilde{r}^2 - \frac{1}{2} d_x d_y \widetilde{r}^2\Big\} \wedge \widetilde{\alpha}_{l-1} + O(r^{3-n}). \tag{1.246}$$

It remains to prove

$$(1 + \widetilde{*_x *_y})\Big(\{\frac{n}{4r^2} d_x \widetilde{r}^2 \otimes d_y \widetilde{r}^2 - \frac{1}{2} d_x d_y \widetilde{r}^2\} \wedge \widetilde{\alpha}_{l-1}\Big) = 0. \tag{1.247}$$

Firstly,

$$\begin{aligned} \frac{n}{4r^2} d_x \widetilde{r}^2 \otimes d_y \widetilde{r}^2 - \frac{1}{2} d_x d_y \widetilde{r}^2 &= -\frac{n}{r^2} \sum_{i,j} (x^i - y^i)(x^j - y^j) dx^i \otimes dy^j + \sum_i dx^i \otimes dy^i \\ &= \sum_i (1 - \frac{n}{r^2}(x^i - y^i)^2) dx^i \otimes dy^i + \sum_{i \neq j} (x^i - y^i)(x^j - y^j) dx^i \otimes dy^j. \end{aligned}$$

Set $\alpha_{ij} \; = \; (x^i - y^i)(x^j - y^j)$,

$$\begin{aligned} &\{\frac{n}{4r^2} d_x \widetilde{r}^2 \otimes d_y \widetilde{r}^2 - \frac{1}{2} d_x d_y \widetilde{r}^2\} \wedge \widetilde{\alpha}_{l-1} \\ &= \sum_{\substack{i,I \\ |I|=l-1}} (1 - \frac{n}{r^2}\alpha_{ii}) dx^i \wedge dx^I \otimes dy^i \wedge dy^I + \sum_{i \neq j} \alpha_{ij} \sum_{\substack{|I|=l-1 \\ i \neq j}} dx^i \wedge dx^I \otimes dy^i \wedge dy^I \\ &= \sum_{|I|=l} (l - \frac{n}{r^2} \sum_{k \in I} \alpha_{kk}) dx^I \otimes dy^I + \sum_{\substack{|I|=l-1 \\ i \neq j}} \alpha_{ij} dx^i \wedge dx^I \otimes dy^i \wedge dy^I. \end{aligned}$$

But

$$
\begin{aligned}
&(1+\widetilde{*_x *_y})\sum_{|I|=l}(l-\frac{n}{r^2}\sum_{k\in I}\alpha_{kk})dx^I\otimes dy^I\\
&=\sum_{|I|=l}(l-\frac{n}{r^2}\sum_{k\in I}\alpha_{kk})dx^I\otimes dy^I+\sum_{|I|=l}(l-\frac{n}{r^2}\sum_{k\in I}\alpha_{kk})dx^{\overline{I}}\otimes dy^{\overline{I}}\\
&=\sum_{|I|=l}(l-\frac{n}{r^2}\sum_{k\in I}\alpha_{kk})dx^I\otimes dy^I+\sum_{|I|=l}(l-\frac{n}{r^2}\sum_{k\in \overline{I}}\alpha_{kk})dx^I\otimes dy^I\\
&=\sum_{|I|=l}(2l-\frac{n}{r^2}\sum_{k=1}^{n}\alpha_{kk})dx^I\otimes dy^I=\sum_{|I|=l}(2l-\frac{n}{r^2}r^2)dx^I\otimes dy^I=0.
\end{aligned}
$$

On the other hand, we check that for all multi-index I of length $l-1$, not containing the indices i, j

$$
\widetilde{*_x *_y}(dx^i\wedge dx^I\otimes dy^j\wedge dy^I)=-dx^j\wedge dx^{\overline{I}\setminus\{i,j\}}\otimes dy^i\wedge dx^{\overline{I}\setminus\{i,j\}}. \tag{1.248}
$$

Indeed

$$
\widetilde{*_x}(dx^i\wedge dx^I)\;=\;\sigma(i,I)dx^{\overline{I}\setminus\{i\}}=\sigma(i,I)\varepsilon_j^{\overline{I}}\frac{i-j}{|i-j|}dx^j\wedge dx^{\overline{I}\setminus\{i,j\}},
$$

where $\varepsilon_j^{\overline{I}}$ is the signature of the permutation sending $\overline{I}$ to $(j,\overline{I}\setminus\{j\})$. Using the corresponding formula for $\widetilde{*_y}(dy^j\wedge dy^I)$, and the relation

$$
\sigma(i,I)\sigma(j,I)\varepsilon_j^{\overline{I}}\varepsilon_i^{\overline{I}}=(\sigma(I))^2=1.
$$

we get (1.248). Now

$$
\begin{aligned}
&(1+\widetilde{*_x *_y})\sum_{\substack{|I|=l-1\\ i\neq j}}\alpha_{ij}dx^i\wedge dx^I\otimes dy^i\wedge dy^I\\
&=\sum_{\substack{|I|=l-1\\ i\neq j}}\alpha_{ij}dx^i\wedge dx^I\otimes dy^i\wedge dy^I-\sum_{\substack{|I|=l-1\\ i\neq j}}\alpha_{ij}dx^j\wedge dx^{\overline{I}\setminus\{i,j\}}\otimes dy^i\wedge dx^{\overline{I}\setminus\{i,j\}}\\
&=\sum_{\substack{|I|=l-1\\ i\neq j}}\alpha_{ij}dx^i\wedge dx^I\otimes dy^i\wedge dy^I-\sum_{\substack{|I|=l-1\\ i\neq j}}\alpha_{ij}dx^j\wedge dx^I\otimes dy^i\wedge dx^I\\
&=\sum_{\substack{|I|=l-1\\ i\neq j}}\alpha_{ij}dx^i\wedge dx^I\otimes dy^i\wedge dy^I-\sum_{\substack{|I|=l-1\\ i\neq j}}\alpha_{ji}dx^i\wedge dx^I\otimes dy^j\wedge dx^I\\
&=0 \text{ since } \alpha_{ij}=\alpha_{ji}.
\end{aligned}
$$

This proves (1.247). Next

$$(1+*_x*_y)(d_xd_y\omega_{l-1}) = \frac{r^{-n}\lambda(r^2)}{s_n}(1+*_x*_y)\left(\left\{\frac{n}{4r^2}d_x\widetilde{r^2}\otimes d_y\widetilde{r^2}-\frac{1}{2}d_xd_y\widetilde{r^2}\right\}\wedge\widetilde{\alpha}_{l-1}\right)+O(r^{3-n})$$
$$= \frac{r^{-n}\lambda(r^2)}{s_n}(*_x*_y-\widetilde{*_x*_y})\left(\left\{\frac{n}{4r^2}d_x\widetilde{r^2}\otimes d_y\widetilde{r^2}-\frac{1}{2}d_xd_y\widetilde{r^2}\right\}\wedge\widetilde{\alpha}_{l-1}\right)+O(r^{3-n}).$$

It only remains to prove that $*_x*_y-\widetilde{*_x*_y}=O(r^2)$. To do this, we introduce some notations.

Let $\mathscr{R}$ be the Riemannian curvature of M, g :

$$\mathscr{R}(X,Y)Z = \nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z.$$

and set in local coordinates system, with local frame $e_1,\ldots,e_n$

$$R_{ijkl} = (\mathscr{R}(e_k,e_l)e_j,e_i)_g.$$

We know that in normal coordinates centered at a point x_B :

$$g_{ij}(y) = \delta_{ij}-\frac{1}{3}R_{ikjl}(x_B)y^ky^l+O(r^3)$$

Setting $R_{ij}(y) = R_{ikjl}(x_B)y^ky^l$, we get a symmetric matrix $R(y) = (R_{ij}(y))_{1\le i,j\le n}$ and

$$g(y) = I-\frac{1}{3}R(y)+O(r^3),$$

so

$$g^{-1}(y) = I+\frac{1}{3}R(y)+O(r^3).$$

If we note $R_{I,J}(y) = (R_{ij}(y))_{i\in I,\ j\in J}$ then

$$g^{I,I}(y) = 1+\frac{1}{3}Tr[R_{I,I}(y)]+O(r^3),$$

$$\sqrt{|g(y)|} = 1-\frac{1}{6}Tr[R(y)]+O(r^3),$$

$$g^{I,J}(y) = O(r^{2k}) \text{ if } |I\cap J| = |I|-k.$$

These data and (1.242) give

$$*_y(dy^I) = (1-\frac{1}{6}Tr[R(y)])\left(\sigma(I)\left(1+\frac{1}{3}Tr[R_{I,I}(y)]\right)dy^{\overline{I}} + \sum_{|J\cap I|=l-1}\sigma(J)g^{I,J}(y)dy^{\overline{J}}\right)+O(r^3),$$

$$= \widetilde{*_y}(dy^I) + \frac{1}{3}\sigma(I)\left(Tr[R_{I,I}(y)]-\frac{1}{2}Tr[R(y)]\right)dy^{\overline{I}}+\sum_{|J\cap I|=l-1}\sigma(J)g^{I,J}(y)dy^{\overline{J}}+O(r^3).$$

Since $R(y)=O(r^2)$; it follows that $*_x *_y - \widetilde{*_x *_y}$ is $O(r^2)$.

This ends the proof of Proposition 1.24

Corollary 1.1

$$S_-^2 = (d\Omega\delta)^2 + \gamma(d\Omega\delta)^2\gamma = S_+^2 = (1+K)^2.$$

We intend to implement the local index procedure [96] onto the operators $S_\pm$. We define

$$\theta := 1 - S_\pm^2 = -(2K+K^2).$$

Recall the following elementary result concerning the composition of week singular integral operators, see e.g. [86, Sect. 27].

Proposition 1.25 *Let L_1, resp. L_2, be a singular integral operator with kernel $L_1(x,y)=0(r^{e_1-n})$, resp. $L_2(x,y)=0(r^{e_2-n})$, with $e_1,e_2>0$, continuous away from the diagonal, with compact support. Let $L=L_1\circ L_2$ and let $L(x,y)$ be its kernel.*

(i) If $e_1+e_2<n$, then L is a singular integral operator with kernel $L(x,y)=0(r^{e_1+e_2-n})$

(ii) If $e_1+e_2>n$, then L is an integral operator with continuous kernel $L(x,y)=0(r^0)$.

From Proposition 1.24, Corollary 1.1 and Proposition 1.25 we get

Proposition 1.26 *The operator θ has the property that $\theta = -(2K+K^2)$ where K, resp. K^2, are integral operators with (metric) kernel $k_1(x,y) = O(r^{2-n})$, resp. $k_2(x,y) = O(r^{4-n})$.*

Let $\wedge^\pm$ denote the ±1-eigenspaces of the operator γ acting on the Hilbert space $\wedge$ of L_2- forms of middle degree l on M. Let Imd, $Im\delta$ denote the closed sub-spaces of all exact, co-exact forms of degree l. We recall that the Hodge decomposition theorem states

$$\wedge = Imd \oplus Im\delta \oplus \mathscr{H}^l,$$

where $\mathscr{H}^l$ denotes l-harmonic forms.

Proposition 1.27 *The operators $S_\pm$ have the properties*

(i) $\gamma S_\pm = \pm S_\pm \gamma$
(ii) $S_+(\wedge^\pm) \subset \wedge^\pm, \qquad S_-(\wedge^\pm) \subset \wedge^\mp$
(iii) $S_\pm(Imd) \subset Imd, \qquad S_\pm(Im\delta) \subset Im\delta$
(iv) $S_\pm^* = S_\pm$
(v) $\theta := 1 - S_\pm^2$ *commutes with* γ *and* $S_\pm$
(vi) θ *keeps invariant the spaces* $\wedge^\pm$, Imd *and* $Im\delta$.

Propositions 1.26 and 1.27 imply the following result.

Theorem 1.54 *The operator S_- satisfies the properties*

(i) S_- *is* quasi local *(its support may be chosen arbitrarily small around the diagonal)*
(ii) $(S_-)_{|Imd} = 1 + K'$
(iii) $(S_-)_{|Im\delta} = -1 + K''$,
where K', K'' are integral operators with metric kernel of type $O(r^{2-n})$.

S_- *is a* local Hodge decomposition*, as it is called in Connes et al. [101].*

We intend to apply the localisation procedure [96, 101] onto the *r-signature operator*

$$A \;=\; S_- \,\frac{1 \,+\, \gamma}{2} : \wedge^+ \longrightarrow \wedge^-. \tag{1.249}$$

For, we consider the parametrix

$$B : \wedge^- \longrightarrow \wedge^+$$

given by

$$B \;=\; \Sigma_N(\theta)\; S_- \,\frac{1 \,-\, \gamma}{2},$$

where

$$\Sigma_N(\theta) \;=\; \sum_{j=0}^{N} \theta^j.$$

and N is a sufficiently large number, which will be specified later.

The corresponding operators S_0, S_1 are

$$S_0 := 1-BA = 1 - \sum_{j=0}^{N} \theta^j . S_-^2 \frac{1+\gamma}{2} = 1 - \sum_{j=0}^{N} \theta^j . S_+^2 \frac{1+\gamma}{2} = \theta^{N+1} \frac{1+\gamma}{2}$$

$$S_1 := 1 - AB = 1 - \sum_{j=0}^{N} \theta^j . S_-^2 \frac{1-\gamma}{2} = 1 - \sum_{j=0}^{N} \theta^j . S_+^2 \frac{1-\gamma}{2} = \theta^{N+1} . \frac{1-\gamma}{2}.$$

The associated operator

$$R =: \wedge^+ \oplus \wedge^- \longrightarrow \wedge^+ \oplus \wedge^- \tag{1.250}$$

is given by the matrix

$$R = P - e = \begin{pmatrix} (S_0)^2 & S_0(1+S_0)B \\ S_1^+ A & -(S_1)^2 \end{pmatrix} =$$

$$= \begin{pmatrix} \theta^{2(N+1)} \frac{1+\gamma}{2} & \theta^{N+1}(1+\theta^{N+1}) \Sigma_N(\theta) S_- \frac{1-\gamma}{2} \\ \theta^{N+1} S_- \frac{1+\gamma}{2} & -\theta^{2(N+1)} \frac{1-\gamma}{2} \end{pmatrix} =$$

$$\begin{pmatrix} \theta^{2(N+1)} \frac{1+\gamma}{2} & \theta^{N+1} \Sigma_{2N+1}(\theta) S_- \frac{1-\gamma}{2} \\ \theta^{N+1} S_- \frac{1+\gamma}{2} & -\theta^{2(N+1)} \frac{1-\gamma}{2} \end{pmatrix}$$

and

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

General considerations [96] show that P is an idempotent ($P^2 = P$).
The operator P, seen as an operator on $\wedge$, is

$$P = \gamma \theta^{2(N+1)} + \theta^{N+1} [\frac{1-\gamma}{2} + \frac{1+\gamma}{2} \Sigma_{2N+1}(\theta)] \, S_- + \frac{1-\gamma}{2}.$$

We have proved the following result.

Theorem 1.55

$$Sig(M) = a-Ind(A) = Tr\ (S_0)^2 - Tr\ (S_1)^2 = Tr\ \theta^{2(N+1)} \frac{1+\gamma}{2} - Tr\ \theta^{2(N+1)} \frac{1-\gamma}{2} = Tr\ \gamma \theta^{2(N+1)}.$$

1.13.1 1-Parameter Family of Hodge Decomposition Operators

Remark 1.22 In the remaining part of this section we are going to make preparations for the localisation of the operator R. The r-signature operator is already local. We need to localise the operators depending on θ. Recall that $\theta = 1 - S_{\pm}^2 = -(2K + K^2)$, where $K = S_+ - 1$, cf. Proposition 1.24

We introduce the 1-parameter family of parametrices for the Laplacian Ω_t whose supports concentrate about the diagonal while $t \downarrow 0$

$$\Omega_t \varphi(x) = \int \omega_t(x, y) \wedge *\varphi(y),$$

where

$$\omega_t = \frac{r^{2-n} \lambda(\frac{r^2}{t^2})}{(n-2)s_n} \alpha.$$

and α is defined as above

$$\alpha_p = \frac{1}{p!} \left(d_x d_y A(x, y)\right)^{\wedge^p}; \alpha = \sum_{p=0}^{n} \alpha_p.$$

Associated to Ω_t we introduce the operators

$$S_{\pm,t} = d\Omega_t \delta \pm \gamma d\Omega_t \delta \gamma$$

acting on $l - forms$. Supposing the support of the function λ is in the interval $[-1, 1]$, the kernels of the operators $S_{\pm,t}$ have supports in the t-neighbourhood of the diagonal.

The corresponding operator K_t introduced in Proposition 1.24.

$$K_t = S_{+,t} - 1$$

is given by

$$(K_t \varphi)(x) = \int_M (1 + *_x *_y)(d_x d_y \omega_{l-1,t}) \wedge *_y \varphi;$$

its (metric) kernel is

$$k_{1,t}(x, y) = (1 + *_x *_y)(d_x d_y \omega_{l-1,t}).$$

We intend to study the kernel $k_{1,t}(x, y)$ while $t \downarrow 0$.

Since α is closed, both with respect to x and y, we have

$$d_x d_y(\omega_{l-1,t}) = d_x d_y\Big(\frac{r^{2-n}\lambda(\frac{r^2}{t^2})}{(n-2)s_n}\Big)\wedge\alpha_{l-1}$$

$$= \frac{1}{(n-2)s_n}\{\lambda(\frac{r^2}{t^2})d_x d_y(r^{2-n}) \quad + \quad d_x[\lambda(\frac{r^2}{t^2})]\otimes[d_y(r^{2-n})]$$

$$+ \quad [d_x(r^{2-n})]\otimes d_y[\lambda(\frac{r^2}{t^2})] \quad + \quad [d_x d_y\lambda(\frac{r^2}{t^2})][(r^{2-n})]\}\wedge\alpha_{l-1} \quad =$$

$$\frac{1}{(n-2)s_n}\{\lambda(\frac{r^2}{t^2})d_x d_y(r^{2-n}) \quad + \quad (2-n)\lambda'(\frac{r^2}{t^2})[d_x(\frac{r^2}{t^2})]\otimes[d_y r]r^{1-n}$$

$$+(2-n)\lambda'(\frac{r^2}{t^2})[d_x r]\otimes[d_y(\frac{r^2}{t^2})]r^{1-n} \quad + \quad [d_x d_y\lambda(\frac{r^2}{t^2})](r^{2-n})\}\wedge\alpha_{l-1} \quad =$$

$$\frac{1}{(n-2)s_n}\{\lambda(\frac{r^2}{t^2})d_x d_y(r^{2-n}) \quad + \quad (2-n)\lambda'(\frac{r^2}{t^2})[d_x(\frac{r^2}{t^2})]\otimes[\frac{d_y r^2}{2r}]r^{1-n}$$

$$+ (2-n)\lambda'(\frac{r^2}{t^2})[\frac{d_x r^2}{2r}]\otimes[d_y(\frac{r^2}{t^2})]r^{1-n} \quad + \quad d_x[\lambda'(\frac{r^2}{t^2})d_y(\frac{r^2}{t^2})](r^{2-n})\}\wedge\alpha_{l-1} \quad =$$

$$\frac{1}{(n-2)s_n}\{\lambda(\frac{r^2}{t^2})d_x d_y(r^{2-n}) \quad +$$

$$r^{-n}(2-n)\frac{r^2}{t^2}\lambda'(\frac{r^2}{t^2})\frac{1}{r^2}[d_x r^2]\otimes[d_y r^2] \quad +$$

$$+ \quad d_x[\lambda'(\frac{r^2}{t^2})d_y(\frac{r^2}{t^2})](r^{2-n})\}\wedge\alpha_{l-1} \quad =$$

$$\frac{1}{(n-2)s_n}\{\lambda(\frac{r^2}{t^2})d_x d_y(r^{2-n}) \quad +$$

$$r^{-n}(2-n)\frac{r^2}{t^2}\lambda'(\frac{r^2}{t^2})\frac{1}{r^2}[d_x r^2]\otimes[d_y r^2] \quad +$$

$$\lambda'(\frac{r^2}{t^2})[d_x d_y(\frac{r^2}{t^2})](r^{2-n}) \quad + \quad \lambda''(\frac{r^2}{t^2})[d_x(\frac{r^2}{t^2})]\otimes[d_y(\frac{r^2}{t^2})](r^{2-n})\}\wedge\alpha_{l-1} \quad =$$

$$\frac{1}{(n-2)s_n}\{\lambda(\frac{r^2}{t^2})d_xd_y(r^{2-n}) \quad +$$

$$r^{-n}\frac{r^2}{t^2}\lambda'(\frac{r^2}{t^2})[(2-n)\frac{1}{r^2}(d_xr^2)\otimes(d_yr^2)+(d_xd_yr^2)] \quad +$$

$$r^{-n}\frac{r^4}{t^4}\lambda''(\frac{r^2}{t^2})[\frac{1}{r^2}(d_xr^2)\otimes(d_yr^2)\}\wedge\alpha_{l-1} \quad =$$

$$\frac{r^{-n}}{(n-2)s_n}\{r^n\lambda(\frac{r^2}{t^2})d_xd_y(r^{2-n}) \quad +$$

$$\frac{r^2}{t^2}\lambda'(\frac{r^2}{t^2})[(2-n)\frac{1}{r^2}(d_xr^2)\otimes(d_yr^2)+(d_xd_yr^2)] \quad +$$

$$\frac{r^4}{t^4}\lambda''(\frac{r^2}{t^2})[\frac{1}{r^2}(d_xr^2)\otimes(d_yr^2)]\}\wedge\alpha_{l-1}.$$

On the other hand,

$$\lambda(\frac{r^2}{t^2})d_xd_y(r^{2-n}) =$$

$$\lambda(\frac{r^2}{t^2})d_xd_y((r^2)^{1-l}) =$$

$$\lambda(\frac{r^2}{t^2})d_x[(1-l)(r^2)^{-l}d_yr^2] =$$

$$\lambda(\frac{r^2}{t^2})(1-l)[(-l)(r^2)^{-1-l}(d_xr^2)\otimes(d_yr^2)+(r^2)^{-l}d_xd_yr^2] =$$

$$\lambda(\frac{r^2}{t^2})\frac{2-n}{2}[(-\frac{n}{2})(r^{-2-n})(d_xr^2)\otimes(d_yr^2)+(r^{-n})d_xd_yr^2] =$$

$$r^{-n}\{\lambda(\frac{r^2}{t^2})\frac{2-n}{2}[(-\frac{n}{2})\frac{1}{r^2}(d_xr^2)\otimes(d_yr^2)+d_xd_yr^2]\} =$$

$$r^{-n}(n-2)\lambda(\frac{r^2}{t^2})\{\frac{n}{4r^2}(d_xr^2)\otimes(d_yr^2)-\frac{1}{2}d_xd_yr^2\} =$$

Putting together the above results, we get

$$
\begin{aligned}
& k_{1,t}(x,y) \quad = \\
\frac{r^{-n}}{(n-2)s_n}(1+*_x*_y)\,\{\,[\;(n-2)\;\;\lambda(\tfrac{r^2}{t^2})\;\;(\frac{n}{4r^2}(d_xr^2)\otimes(d_yr^2)-\frac{1}{2}d_xd_yr^2) \\
& \quad - \quad (\frac{r}{t})^2\;\lambda'(\tfrac{r^2}{t^2})\;\;((n-2)\frac{1}{r^2}(d_xr^2)\otimes(d_yr^2)-(d_xd_yr^2)) \\
& \quad + \quad (\frac{r}{t})^4\;\lambda''(\tfrac{r^2}{t^2})\;(\frac{1}{r^2}(d_xr^2)\otimes(d_yr^2))\;]\;\wedge\;\alpha_{l-1}\,\}.
\end{aligned}
$$

1.13.2 The Cut-Off Function

We see that certain terms of the kernel $k_{1,t}(x,y)$ group together. We intend to use this feature in order to simplify its expression. For this purpose we use a special cut-off function

$$
\lambda = \begin{cases} 1+(l-1)x^l - lx^{l-1} & \text{for } 0 \le x \le 1 \\ 0 & \text{for } x > 1, \end{cases} \tag{1.251}
$$

where dimension of the manifold is $n = 2l$.

Proposition 1.28 *The function* λ *has the properties*

(i) $\lambda(0) = 1$, $\lambda(1) = 0$

(ii) $\lambda'(x)$ *is continuous on* $\mathbf{R}$, $\lambda'(x) = 0(x^{l-2})$, $\lambda'(1) = 0$

$$
\lambda'(x) = \begin{cases} (l-1)lx^{l-1} - l(l-1)x^{l-2} & \textit{for } x \le 1 \\ 0 & \textit{for } x > 1 \end{cases}
$$

(iii) $\lambda''(x)$ *is discontinuous in* 1 *and*

$$
\lambda''(x) = \begin{cases} (l-1)^2lx^{l-2} - l(l-1)(l-2)x^{l-3} & \textit{for } 0 \le x \le 1 \\ 0 & \textit{for } x > 1 \end{cases}
$$

(iv) λ *satisfies the differential equation*

$$
x^2\lambda''(x) - (n-2)x\lambda'(x) + \frac{n(n-2)}{4}\lambda(x) = 1.
$$

(v) for $4 \leq n$, λ *satisfies*

$$1 - \lambda(r^2) = 0(r^4).$$

We return to the formula for the kernel $k_{1,t}$, which may be rewritten

$$k_{1,t}(x, y) = \frac{r^{-n}}{(n-2)s_n}(1 + *_x *_y) \cdot$$

$$\{ [(\frac{n(n-2)}{4}\lambda(\frac{r^2}{t^2}) - (n-2)\frac{r^2}{t^2}\lambda'(\frac{r^2}{t^2}) + \frac{r^4}{t^4}\lambda''(\frac{r^2}{t^2})) \frac{1}{r^2}(d_x r^2) \otimes (d_y r^2)$$

$$- [(\frac{n-2}{2}\lambda(\frac{r^2}{t^2}) - \frac{r^2}{t^2}\lambda'(\frac{r^2}{t^2})) (d_x d_y r^2)] \wedge \alpha_{l-1} \}.$$

We use the cut-off function λ from Proposition 1.28 with $x = \frac{r^2}{t^2}$; by virtue of property (iv) the formula for $k_{1,t}(x, y)$ simplifies

$$k_{1,t}(x, y) = \frac{r^{-n}}{(n-2)s_n}(1 + *_x *_y)$$

$$\{\frac{n(n-2)}{4} \frac{1}{r^2}(d_x r^2) \otimes (d_y r^2) - [\frac{n-2}{2}\lambda(\frac{r^2}{t^2}) - \frac{r^2}{t^2}\lambda'(\frac{r^2}{t^2})] (d_x d_y r^2) \wedge \alpha_{l-1} \}$$

$$= \frac{r^{-n}}{2(l-1)s_n}(1 + *_x *_y)$$

$$\{l(l-1) \frac{1}{r^2}(d_x r^2) \otimes (d_y r^2) - [(l-1)\lambda(\frac{r^2}{t^2}) - \frac{r^2}{t^2}\lambda'(\frac{r^2}{t^2})] (d_x d_y r^2) \wedge \alpha_{l-1} \}$$

$$= \frac{r^{-n}}{2(l-1)s_n}(1 + *_x *_y)$$

$$\{l(l-1) \frac{1}{r^2}(d_x r^2) \otimes (d_y r^2) - [(l-1)(1 - \frac{r^n}{t^n})] (d_x d_y r^2) \wedge \alpha_{l-1} \}$$

$$= \frac{r^{-n}}{2s_n}(1 + *_x *_y)\{l \frac{1}{r^2}(d_x r^2) \otimes (d_y r^2) - (1 - \frac{r^n}{t^n}) (d_x d_y r^2) \wedge \alpha_{l-1} \}$$

$$= \frac{r^{-n}}{2s_n}(1 + *_x *_y) \{ [l \frac{1}{r^2}(d_x r^2) \otimes (d_y r^2) - (d_x d_y r^2)] \wedge \alpha_{l-1} \} - \frac{1}{t^n}\frac{l}{s_n}(1 + *_x *_y) \alpha_l.$$

The formula (1.247) along with the fact that $\star_x \star_y - \tilde{\star_x}\tilde{\star_y} = 0(r^2)$ implies that

$$(1 + *_x *_y) \{ [l \frac{1}{r^2}(d_x r^2) \otimes (d_y r^2) - (d_x d_y r^2)] \wedge \alpha_{l-1} \} = 0(r^2).$$

Theorem 1.56 *If the cut-off function* λ *(Proposition 1.28) is used, the kernel* $k_{1,t}(x, y)$ *of the operator*

$$S_{\pm,t} = d\Omega_t \delta \pm \gamma d\Omega_t \delta \gamma$$

simplifies to

$$k_{1,t}(x, y) = -\frac{1}{t^n}\frac{l}{s_n}(1 + *_x *_y) \ \alpha_l + 0(r^{n-2}). \tag{1.252}$$

The formula for the homological local index class is (compare with (1.51))

$$R(x_0, y_0) \otimes_\Lambda R(x_1, y_1) \otimes_\Lambda \ldots \otimes_\Lambda R(x_{2q}, y_{2q}) \otimes_\Lambda .$$

Theorem 1.57 *Let* M *be closed, oriented smooth manifold. Then the signature of* M *is*

$$Sig(M) = Trace\ R. \tag{1.253}$$

Part II
Non-commutative Differential Geometry

Chapter 2
Spaces, Bundles, Homology/Cohomology and Characteristic Classes in Non-commutative Geometry

Abstract In Chap. 1 we recalled some of the basic notions and results which are commonly used in differential geometry. We had presented them with the intent of showing how they pass into non-commutative geometry. By definition, a *non-commutative space* is a *spectral triple* $\{\mathscr{A}, \rho, F\}$ consisting of an associative, not necessarily commutative or topological algebra $\mathscr{A}$, a Fredholm operator F acting on a separable Hilbert space H and $\rho : \mathscr{A} \longrightarrow L(H)$ a representation of the algebra $\mathscr{A}$ onto the Hilbert space H, subject to additional conditions. Such a structure codifies an *abstract elliptic operator* defined by Atiyah (K-Theory, Benjamin, 1967). While in differential geometry *elliptic operators* are defined after multiple structures are summed up, in non-commutative geometry this process is reversed. The study of non-commutative geometry consists of finding the *hidden mathematical structures* codified by a spectral triple. In Chap. 2 we show how the notions of space, bundles, homology/cohomology and characteristic classes can be extracted from spectral triples. We stress that non-commutative geometry objects are defined in such a way that (1) the *locality* and (2) the *commutativity* assumptions, used in the differentiable geometry counterparts, are not postulated.

2.1 Basic Algebra $\mathscr{A}$ in Non-commutative Geometry

In differential geometry the basic algebra is the algebra of smooth functions $C^\infty(M)$ where M is a smooth manifold. This is a commutative associative algebra. It has many *derivations*: any tangent field is a derivation. The algebra $C^\infty(M)$ is a topological algebra too. For any relatively compact open subset K of M and finite order partial derivatives D^α

$$Sup_{x\in K} \mid D^\alpha f(x) \mid, \qquad \forall f \in C^\infty(M) \tag{2.1}$$

form a system of *semi-norms*. With respect to this system of semi-norms $C^\infty(M)$ becomes a Fréchét algebra.

In non-commutative geometry, the space M is replaced by an *associative algebra* $\mathscr{A}$. In non-commutative geometry the algebra $\mathscr{A}$ is not required either to be

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commutative or to have a topology. It is not required $\mathscr{A}$ to have a unit or not. It is not required $\mathscr{A}$ to have true derivations.

The motivation behind this choice is based on fundamental results which connect algebras with spaces. We list only few of them.

Theorem 2.1 (Gelfand and Naimark) *Any commutative C^*-algebra with unit is $*$-isomorphic to the $C^*(X)$-algebra of continuous functions on a compact topological space X.*

Any commutative C^*-algebra without unit is $*$-isomorphic to the algebra $C^*(X)$ of continuous functions on a locally compact topological space X which vanish at infinity.

Theorem 2.2 (Gelfand and Naimark [3]) *Any complex C^*-algebra is $*$-isomorphic to a $C^*(\mathscr{L}(X))$-algebra of bounded operators on a separable Hilbert space X.*

Theorem 2.3 (Gelfand and Mazur) *Any complex Banach algebra with unit in which all non-zero elements are invertible is isometrically isomorphic to the algebra of complex numbers.*

More information about these and other topics regarding topological algebras may be found in the books by Rickart [28], Dixmier [40], and Arveson [80].

A *space in non-commutative geometry* is given by a *spectral triple*. Spectral triples were defined in Sect. 1.12.3. A spectral triple is a system $\{\mathscr{A}, \rho, F\}$ where F is a Fredholm operator acting on a separable Hilbert space H and $\rho : \mathscr{A} \longrightarrow L(H)$ is a ring homomorphism into the algebra of bounded operators $\mathscr{L}(H)$; these entities have to verify additional conditions.

The information contained by the algebra $\mathscr{A}$ is very limited. This is the reason why extra structure is added in the definition of a spectral triple.

2.2 Bundles in Non-commutative Geometry

In non-commutative geometry, see Connes [82, 99], bundles appear in at least these modalities: in an explicit way, as finite projective modules, or idempotents in the matrix algebra $\mathbf{M}_*(\mathscr{A})$, where $\mathscr{A}$ is an associative algebra or, in an implicit way, either under the form of the Hilbert space H of spectral triples, or as equivalence classes of bundles, as elements of the K-theory groups.

2.3 Non-commutative Chern–Weil Theory

We saw in Sects. 1.6.6 and 1.6.7 that the *commutativity* of the ground algebra is not used in the definition of linear connections, curvature as well as in the construction of the structure equations. For this reason, the algebra of smooth functions may be replaced by an arbitrary associative algebra $\mathscr{A}$ over the ring $\mathbf{K}$.

We are going to see that the *minimal structure necessary to implement the Chern–Weil construction* may be decomposed in three steps, see [91]:

1. *first step*: the connection and curvature should be elements of a graded differential algebra,
2. *second step*: the homological information has to be collected in an interesting homology theory,
3. *third step*: a trace-type operator should be available.

The discussion below will hold at least in the case $\mathbf{K} = \mathbf{R},\ or\ \mathbf{C}$. If not differently specified, all tensor products are over the field $\mathbf{K}$.

2.3.1 Preparatory Material: Graded Differential Algebra $\tilde{\mathbf{B}}$

We insert this section with the only purpose of facilitating the reading of the topics related to characteristic classes; some of their proofs are omitted. With the present notation the degrees of chains and homology are increased by 1 with respect to the final setting. These notions and the degree shift will be discussed again in Sect. 2.4.

Definition 2.1 Let $\mathscr{A}$ be an associative algebra over the ring K.
We define

(i) the space of k-chains

$$\mathbf{B}_k = \otimes_K^k \mathscr{A} \tag{2.2}$$

(ii) the boundary $b' : \mathbf{B}_k \longrightarrow \mathbf{B}_{k-1}$

$$b'(a_1 \otimes \cdots \otimes a_k) = \sum_{0 \le r \le k-1} (-1)^{r-1} a_1 \otimes \ldots (a_r . a_{r+1}) \otimes \ldots a_k. \tag{2.3}$$

(iii) Let V be a K-algebra. Define the space of *co-chains on A with values in V*: $(\tilde{\mathbf{B}}, V)_p := Hom_K(\mathbf{B}_p(A),\ V)$.
Let δ be the transpose of b', i.e. $\delta_{p+1} : (\tilde{\mathbf{B}}, V)_{p+1} \longrightarrow (\tilde{\mathbf{B}}, V)_p$

$$(\delta_{p+1}\phi)(a_1 \otimes \cdots \otimes a_{p+1}) := \phi(b'(a_1 \otimes \cdots \otimes a_{p+1})).$$

(iv) On the space of co-chains $(\tilde{\mathbf{B}}, V)_*$ define the *product*, see Quillen [91]

$$\times : (\tilde{\mathbf{B}}, V)_p \times (\tilde{\mathbf{B}}, V)_q \longrightarrow (\tilde{\mathbf{B}}, V)_{p+q}$$

$$(\alpha_p \times \beta_q)\ (a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q) = \alpha_p(a_1 \otimes \cdots \otimes a_p)\ .\ \beta_q(b_1 \otimes \cdots \otimes b_q). \tag{2.4}$$

Proposition 2.1

(i) $\mathbf{B}_k$ *is an* $\mathscr{A}$ *bi-module given by the left and right multiplication of tensor product factors with elements of* $\mathscr{A}$,

(ii) $b^{'}\, b^{'} = 0$

(iii) $(\tilde{\mathbf{B}}, V)_*$ *is a graded differential algebra*

$$\delta(\alpha_p \times \beta_q) = (\delta\alpha_p) \times \beta_q + (-1)^p \alpha_p \times (\delta\beta_q) \tag{2.5}$$

(iv) if the algebra $\mathscr{A}$ *has an algebra homomorphism, called* augmentation, $\rho : \mathscr{A} \longrightarrow K$, *then the mapping* $s : \mathbf{B}_p \longrightarrow \mathbf{B}_{p+1}$

$$s(a_1 \otimes \cdots \otimes a_p) := (-1)^p a_1 \otimes \cdots \otimes a_p \otimes 1 \tag{2.6}$$

satisfies the identity

$$s\, b^{'} + b^{'}\, s = 1 \tag{2.7}$$

and therefore the bar complex $\mathbf{B}_*$ *is* acyclic.

Remark 2.1 The graded differential algebra structure on $(\tilde{\mathbf{B}}, V)_*$ derives from the *co-algebra* structure of $\mathbf{B}_*$, see [91]

$$\Delta(a_1 \otimes \cdots \otimes a_p) = \sum_r (a_1 \otimes \cdots \otimes a_r) \times (a_{r+1} \otimes \cdots \otimes a_p).$$

Proof

(iii) The expression of the boundary splits up in two parts

$$\begin{aligned} b^{'}\ (\, a_1 \otimes \cdots \otimes a_{p+q+1}\,) = \\ \sum_{1 \le r \le p} [\ (a_1 \otimes \ldots a_r . a_{r+1} \cdots \otimes a_{p+1}\ ] \otimes [\ a_{p+2} \otimes \cdots \otimes a_{p+q+1}\ ] + \\ (-1)^{p-1} [a_1 \otimes \cdots \otimes a_p]\ \otimes [\sum_{p+1 \le r \le p+q} (-1)^{r-1} a_{p+1} \otimes \cdots \otimes a_r a_{r+1} \otimes \cdots \otimes a_{p+q+1}]. \end{aligned} \tag{2.8}$$

We leave the remainder of the proof to the reader.

Definition 2.2

(i) Define the homomorphisms $T,\ N : B_k \longrightarrow B_k$ by the formulae

$$T(a_1 \otimes \cdots \otimes a_k) := (-1)^{k-2} a_2 \otimes a_3 \otimes \cdots \otimes a_k \otimes a_1 \tag{2.9}$$

$$N = \sum_{0 \le r \le k-2} T^r \tag{2.10}$$

(ii) A chain $\gamma \in B_k(\mathscr{A})$ is said to have *graded cyclic symmetry* and called *cyclic* provided $T(\gamma) = \gamma$, see [82]. The set of all cyclic chains in $B_k(\mathscr{A})$ is denoted $B_k^{\lambda}(\mathscr{A})$.

Proposition 2.2

(i) $b'(\mathbf{B}_k^{\lambda}(\mathscr{A})) \subset B_{k-1}^{\lambda}(\mathscr{A})$.
(ii) If the ring K contains the rational numbers, then $Ker(1-T) = ImN =$ space of cyclic elements.

Definition 2.3 The homology of the *cyclic complex* $\{(B_k^{\lambda}(\mathscr{A})),\ b'\}$ is called *cyclic homology of the algebra* $\mathscr{A}$, denoted $H^{\lambda}(\mathscr{A})$, see [82].

2.3.2 Associative Algebra Extensions and Their Geometry

In this section we extend the Chern–Weil theory to exact sequences

$$E \equiv \quad 0 \longrightarrow H \xrightarrow{\iota} P \xrightarrow{\pi} T \longrightarrow 0 \tag{2.11}$$

of associative algebras. In Sect. 1.6.9 we presented Chern–Weil theory on extensions of Lie algebras, see Teleman [62]. For the present discussion we use Quillen's article [91] with certain modifications. The invariants we obtain belong to the *cyclic cohomology* of the algebra $\mathscr{A}$, results due to Connes [82].

We consider an associative algebra $\mathscr{A}$. We assume the algebras P are unital and that $\pi(1_P) = 1_T$.

As said before, the problem is split in three steps.

The first step involves connections, curvature and the analogues of the structure equations. This is going to be done inside the graded differential algebra b'*-bar complex*, denoted $\mathbf{B}$, of the algebra $\mathscr{A}$, see Quillen [91]. The algebra $\mathbf{B}$ provides the differential calculus which replaces the Lie algebra context of [62]. The *second step* deals with the choice of the homological framework in which the minimal structure produces useful results. The third step deals with the *choice of trace*.

We anticipate by saying that the trace is characterised by its basic property of being invariant under *cyclic permutations* of the zero-degree factors. The *graded cyclic symmetry*, imposed on chains of any degree, is compatible with the ordinary trace and leads to *cyclic homology*, due to Connes [82].

We begin by discussing the *first step*.

Definition 2.4 Let $u : \mathscr{A} \longrightarrow T$ be an $\mathscr{A}$-homomorphism with $u(1_A) = 1_P$.

We assume there exists a K-homomorphism $\varphi : \mathscr{A} \longrightarrow P$, such that $u = \pi \circ \varphi$. $\varphi : \mathscr{A} = \mathbf{B}_1(\mathscr{A}) \longrightarrow P$ is called *connection*.

We associate with the *connection* φ its *curvature* $\omega : \mathbf{B}_2(\mathscr{A}) \longrightarrow P$

$$\omega(a_1 \otimes a_2) := \varphi(a_1.a_2) - (\varphi\, a_1).(\varphi\, a_2). \tag{2.12}$$

The curvature measures in which respect the connection is not multiplicative. As $\pi\ \omega = 0$, it follows that ω takes values in the ideal H. Taking into account the present structure, the objects introduced by Definition 2.1will be defined more specifically.

Definition 2.5

1. Let δ be the transposed $b^{'}$, acting on $Hom_K(\mathbf{B}(\mathscr{A}),\ P)$, resp. $Hom_K(\mathbf{B}(\mathscr{A}),\ H)$.

   For any $\phi \in Hom_K(\mathbf{B}_k(\mathscr{A}),\ P)$, resp. $\phi \in Hom_K(\mathbf{B}_k(\mathscr{A}),\ H)$, we define

$$(\delta\phi)(a_1 \otimes \cdots \otimes a_{k+1}) = \phi(\ b^{'}\ (a_1 \otimes \cdots \otimes a_{k+1})\ ).$$

2. On $Hom_K(\mathbf{B}(\mathscr{A}),\ P)$, resp. $Hom_K(\mathbf{B}(\mathscr{A}),\ H)$, we define the *product*: if

$$\alpha_p \in Hom_K(\mathbf{B}_p(\mathscr{A}),\ P)$$

   and

$$\beta_q \in Hom_K(\mathbf{B}_q(\mathscr{A}),\ P)$$

   one defines $\alpha_p.\beta_q \in Hom_K(\mathbf{B}_{p+q}(\mathscr{A}),\ P)$, resp. $Hom_K(\mathbf{B}_{p+q}(\mathscr{A}),\ H)$,

$$(\alpha_p.\beta_q)(a_1 \otimes \cdots \otimes a_{p+q}) := \alpha\ (a_1 \otimes \cdots \otimes a_p)\ .\ \beta\ (a_{p+1} \otimes \cdots \otimes a_{p+q}), \tag{2.13}$$

   where the last product is performed in P, resp. H.

Proposition 2.3 *$Hom_K(\mathbf{B}_*(\mathscr{A}),\ P)$, resp. $Hom_K(\mathbf{B}_*(\mathscr{A}),\ H)$, are graded differential algebras with respect to the boundary operator δ and the multiplication defined by (2.13).*

$$\delta(\alpha_p.\beta_q) = (\delta\alpha_p).\beta_q + (-1)^p\ \alpha_p.(\delta\beta_q). \tag{2.14}$$

Proof The property follows from the co-algebra structure Δ on $\mathbf{B}_*(\mathscr{A})$ [91, §1.2].

$$\Delta(a_1 \otimes \cdots \otimes a_k) = \oplus_{1\le r\le k}(a_1 \otimes \cdots \otimes a_k) \otimes (a_{r+1} \otimes \cdots \otimes a_k).$$

We have

$$\delta(\alpha_p.\beta_q)\ (a_1 \otimes \cdots \otimes a_{p+q+1}) = (\alpha_p.\beta_q)\ b^{'}(a_1 \otimes \cdots \otimes a_{p+q+1}) =$$

$$\alpha_p(\ b^{'}(a_1 \otimes \cdots \otimes a_{p+1})\ )\ .\ \beta_q\ (a_{p+2} \otimes \cdots \otimes a_{p+q+1}) +$$

$$(-1)^p\ \alpha_p\ (a_1 \otimes \cdots \otimes a_p)\ .\ \beta_q\ b^{'}(a_{p+1} \otimes \cdots \otimes a_{p+q}).$$

Proposition 2.4 (Second Structure Equation) *The curvature and the connection satisfy the equation*

$$\omega \;=\; \delta\varphi \;-\; \varphi \,.\, \varphi. \tag{2.15}$$

Proof Indeed

$$(\delta\,\varphi)(a_1 \otimes a_2) \;=\; \varphi(b^{'}\,(a_1 \otimes a_2)) = \varphi(a_1.a_2), \tag{2.16}$$

$$(\varphi \times \varphi)(a_1 \otimes a_2) = \varphi(a_1)\,.\,\varphi(a_2). \tag{2.17}$$

2.3.3 Connes Chern Character of Algebra Extensions

Proposition 2.5 (Bianchi Identity)

1. *Let φ be any* 1*-chain in the graded differential algebra and let ω be* defined *by the equation*

$$\omega := \;\delta\,\varphi \;-\; \varphi \,.\, \varphi. \tag{2.18}$$

Then φ and ω satisfy the generalised Bianchi identity

$$\delta\,\omega^q \;=\; -\,[\,\omega^q,\;\varphi\,]. \tag{2.19}$$

This formula for $q = 1$ is the Bianchi identity.
2. *The curvature ω and the connection form φ satisfy Eq. (2.19).*

Proof

1. For $q = 1$ the relation follows easily

$$\begin{aligned}\delta\,\omega \;&=\; \delta\,(\,\delta\,\varphi \;-\; \varphi\,\varphi\,) \;=\; -\,(\delta\,\varphi)\,\varphi \;+\; \varphi\,(\delta\,\varphi) \;=\\ -\,(\,(\delta\,\varphi) \;-\; \varphi\,\varphi)\,\varphi \;&+\; \varphi\,(\delta\,\varphi \;-\; \varphi\,\varphi) \;=\; -[\,\omega,\;\varphi\,].\end{aligned} \tag{2.20}$$

The general formula follows by induction on q

$$\begin{gathered}\delta\omega^q = \delta(\omega^{q-1}.\omega) =\\ (-1)[\,\omega^{q-1},\varphi\,].\omega \;+\; (-1)\omega^{q-1}.[\,\omega,\varphi\,] =\\ (-1)(\omega^{q-1}\varphi - \varphi\omega^{q-1}).\omega \;+\; (-1)\omega^{q-1}(\omega.\varphi - \varphi.\omega) =\\ (-1)\omega^q.\varphi - \varphi.\omega^q \;+\; (-1)(\,\omega^{q-1}.\varphi.\omega - \omega^{q-1}.\varphi.\omega\,) \;=\; -[\,\omega^q,\;\varphi\,].\end{gathered} \tag{2.21}$$

2. The connection φ and the curvature ω satisfy the *second structure equation* (2.15).

We introduce the chain

$$\Phi_q \ := \ \omega^q. \tag{2.22}$$

Formula (2.21) says that

$$\delta\Phi_q \ = \ [\ \Phi_q,\ \varphi\ ] \ = \ \Phi_q\ .\varphi \ - \ \varphi\ .\Phi_q. \tag{2.23}$$

We pass to the *second step*. In the sequel the bar complex $\mathbf{B}_*$ is understood to be associated to the algebra $\mathscr{A}$. We intend also that Hom means $Hom_{\mathbf{K}}$. We know that ω takes values in the *ideal* H. Therefore, Φ_q and $\delta\Phi_q$ belong to $Hom(\mathbf{B}(\mathscr{A}),\ H)$.

The analogue of the *constant invariant forms* of [62] is the *trace*. Recall that H is the first term of the exact sequence E. Let V a $\mathbf{K}$ bi-module and let $\tau : H \longrightarrow V$ a $\mathbf{K}$-homomorphism. The composition with the function τ produces a function

$$\tau : Hom(\mathbf{B}_k,\ H) \longrightarrow Hom(\mathbf{B}_k,\ V) \tag{2.24}$$

The following definition deals with the *third step*.

Definition 2.6 Let V be a K bi-module. A K-homomorphism $\tau : H \longrightarrow V$ is called a *trace* on H provided it vanishes on the commutators $[H, P]$. In other words, τ is a function

$$\tau : H/[H, P] \longrightarrow V. \tag{2.25}$$

Definition 2.7 Let τ be a trace on H. Define

$$\Psi_q \ := \ \tau \circ \omega^q. \tag{2.26}$$

For the definition of Ψ_q it would be sufficient to require that τ be defined on the qth power of H and that it would vanish on the commutators $[H^{q-1}, P]$. The *generalised Bianchi identity* combined with the trace property prove the next result.

Proposition 2.6 *Let τ be a trace on H.*
Then $\Psi_q \ := \tau \circ \omega^q \circ N$ is a cycle *in the complex* $\{\ Hom(\mathbf{B}_*,\ V), \delta\ \}$.

Remark 2.2 Proposition 2.6 tells that Ψ_q is a *cycle* in the complex $\{\ Hom(\mathbf{B}_*,\ V), \delta\ \}$. However, the homology of this complex does not provide always the kind of information we look for—see Sect. 1.6.3. Ψ_q does satisfy the trace conditions. This property is not used to the maximum of its possibilities: the cycle is not supposed to live in a complex in which *all* chains have *cyclic symmetry*. Connes [82] defined *graded cyclic symmetry* condition and defined the *cyclic complex* $\{\ B^{\lambda}_*(A), b'\ \}$ consisting of Hochschild chains which have *cyclic symmetry*. The homology of this complex is the *cyclic homology*. For the purpose of the Chern character, *Hochschild homology* is not an interesting homology theory for characteristic classes, see

Sect. 1.6.3. A refinement of it, called *cyclic homology*, or better *periodic cyclic homology*, discussed in Sect. 3.21 is the right homology theory.

Theorem 2.4 (Connes [82, Theorem 5], See Quillen [91, Theorem 1]) *Let E be an extension of associative algebras. Let φ be a connection in E and let ω be its corresponding curvature. Let $\tau : H^q \longrightarrow V$ be a trace vanishing on $[H^{q-1}, P]$. Then*

(i) the chain

$$\Psi_q := \tau \circ \omega^q \circ N \tag{2.27}$$

is a cycle in the cyclic complex $\tau \circ Hom(\mathbf{B}^{\lambda}_{2q}, H)$.

(ii) If the ring $\mathbf{K}$ is commutative and contains the rational numbers, then the homology class of the cycle Ψ_q is independent of the connection.

Proof The chain Ψ_q has graded cyclic symmetry because it belongs to the image of N and $Im\ N = Ker\ (1-T)$. $Ker\ (1-T)$ consists of cyclic elements and the boundary b' carries graded cyclic symmetric chains in graded cyclic symmetric chains, see Proposition 2.2

(i) Ψ_q is a cycle co-cycle because

$$\delta\Psi_q = \delta\tau \circ \omega^q \circ N = \tau \circ b'\omega^q \circ N = -\tau \circ [\,\omega^q, \varphi\,] \circ N = 0. \tag{2.28}$$

We pass to the proof of (ii). It remains to prove that the homology class of Ψ_q does not depend on the choice of the connection.

The present proof uses a homotopy argument realised on the product of the "base space", given by the algebra $\mathscr{A}$, with the interval $[0,\ 1]$, see Quillen [91, §2.1], compare [62, §5] and [89].

To this aim we keep both the algebra $\mathscr{A}$ and the homomorphism u unchanged and we replace the exact sequence $\mathscr{E}$ by the exact sequence

$$E \times K \equiv \quad 0 \longrightarrow H \otimes K \xrightarrow{\iota \times 1} P \otimes K \xrightarrow{\pi \times 1} T \otimes K \longrightarrow 0. \tag{2.29}$$

K_* is the de Rham complex, with polynomial coefficients, on the line. More precisely, the graded differential algebra K_* is

$$K := \mathbf{K}[\,t, dt\,], \quad dt.dt = 0; \tag{2.30}$$

the chains of K_0 are *polynomials* in t with coefficients in the ring K. The polynomials commute with dt; $K_1 := K_0.dt$.

Definition 2.8

(i) In the graded differential algebra $Hom(\mathbf{B}_*, V) \otimes K_*$ the boundary operator is $D = \delta + d_t$. The boundary δ operates on the first factor $Hom(\mathbf{B}_*, V)$ while

the *exterior derivative operator* d_t operates on the second factor K_*

$$d_t\,\phi \;:=\; dt\,.\,\frac{d}{dt}\phi \;:=\; dt\,.\,\dot{\phi}. \tag{2.31}$$

(ii) If $f \in Hom(\,\mathbf{B}_p,\;P) \otimes K_q$, the *total degree* of f is $|f| := p+q$.
(iii) If $f \in Hom(\,\mathbf{B}_p,\;P) \otimes K_q\,)$ and $f \in Hom(\,\mathbf{B}_r,\;P) \otimes K_s$, the *product* of the two co-chains is given by

$$(f \times g)(a_1 \ldots a_{p+r}) := (-1)^{p.s}\; f(a_1 \ldots a_p)\;.\; g(a_{p+1} \ldots a_{p+r}). \tag{2.32}$$

(iv) If $f \in Hom(\,\mathbf{B}_p,\;P) \otimes K_q$, the *total differential* of f is

$$Df \;:=\; \delta f \;+\; (-1)^p d_t f. \tag{2.33}$$

Lemma 2.1 *If* $f \in Hom(\,\mathbf{B}_p,\;P) \otimes K_q$ *and* $g \in Hom(\,\mathbf{B}_r,\;P) \otimes K_s$ *, then*

$$D(f \times g) \;=\; (Df) \times g \;+\; (-1)^{|\,f\,|} f \times (Dg). \tag{2.34}$$

Let φ_0 and φ_1 be two connections in the original exact sequence E. We introduce in $E \otimes K_*$ the connection

$$\varphi_t := \varphi_0.(1-t) \;+\; \varphi_1.t \;\in Hom(\,\mathbf{B}_1,\;P) \otimes K_0. \tag{2.35}$$

Definition 2.9 Define the *total curvature*

$$\omega_t := D\varphi_t \;-\; \varphi_t.\varphi_t \;=\; (\delta \;+\; d_t)\varphi_t \;-\; \varphi_t.\varphi_t \;=\; \tilde{\omega}_t \;+\; \dot{\varphi}_t.dt, \tag{2.36}$$

where

$$\tilde{\omega}_t := \delta\varphi_t \;-\; \varphi_t.\varphi_t \tag{2.37}$$

The total curvature is the curvature of the connection ω_t.

Using formula (2.31) we get

$$d_t\omega_t \;=\; d_t(\omega_1 \;+\; (\omega_1 \;-\; \omega_0).t) \;=\; (\omega_1 \;-\; \omega_0)\;.\;dt. \tag{2.38}$$

Word of Caution The form $\tilde{\omega}_t := \;\delta\omega_t \;-\; \omega_t\;.\;\omega_t$ *is not the curvature of the connection* ω_t. However, $\tilde{\omega}_t$ is a 2 co-chain associated to the form ω_t, with respect to the boundary map δ, in accordance to Proposition 2.5(1).

Proposition 2.5(1) says that

$$\delta\tilde{\omega}_t^q \;=\; -\,[\tilde{\omega}_t^q,\;\tilde{\omega}_t]. \tag{2.39}$$

Note that the boundary δ acts on the bar complex. Therefore, δ commutes with τ. Putting together these two facts, we get

$$\delta\tau\tilde{\omega}_t^q \;=\; 0. \tag{2.40}$$

We get

$$\omega_t^m \;=\; (\tilde{\omega}_t \;+\; (\omega_1 \;-\; \omega_0)\,.\,dt)^m \;=\; \tilde{\omega}_t^m \;+\; \sum_{1\le r\le m} \omega_t^{r-1}\,.\,\dot{\varphi}_t\,.\,\omega_t^{m-r}\,.\,dt. \tag{2.41}$$

The graded commutativity of the differential algebra K was definitely used in this formula, in particular $dt\,.\,dt \;=\; 0$.

Proposition 2.4, in which we replace the operator δ by D and the connection φ by the connection φ_t, gives

$$D\;\tau\;{\omega_t}^m\;N\;=0. \tag{2.42}$$

We get then

$$0 \;=\; D\;(\tau\circ\tilde{\omega}^m)\;N\;= \tag{2.43}$$

$$(\delta \;+\; d_t)\;\tau\;(\;\tilde{\omega}_t^m N \;+\; (\delta \;+\; d_t)\sum_{1\le r\le m} \tilde{\omega}_t^{r-1}\,.\,(\varphi_1 \;-\; \varphi_0)\,.\,\omega_t^{m-r}\;).dt\;N. \tag{2.44}$$

Formula (2.43) combined with (2.40) and the fact that the second term of (2.44) already contains the factor dt, imply

$$0 \;=\; d_t\;\tau\;\tilde{\omega}_t^m N \;+\; \delta\sum_{1\le r\le m} \tilde{\omega}_t^{r-1}\,.\,(\varphi_1-\varphi_0)\,.\,\omega_t^{m-r}\,.\,dt\,.\,N\,. \tag{2.45}$$

From this formula we find that

$$\frac{d}{dt}(\tau\tilde{\omega}_t^m)N.dt = -\delta\sum_{1\le r\le m} \tilde{\omega}_t^{r-1}\,.\,(\varphi_1-\varphi_0)\,.\,\omega_t^{m-r}\,.\,N\,.\,dt \tag{2.46}$$

or

$$\frac{d}{dt}(\tau\tilde{\omega}_t^m)N = -\delta\sum_{1\le r\le m} \tilde{\omega}_t^{r-1}\,.\,(\varphi_1 \;-\; \varphi_0)\,.\,\omega_t^{m-r}\,.\,N. \tag{2.47}$$

Let $V_i(\tau\tilde{\omega}_t^m N)$, $i = 0$ or 1 denote the value of $\tau\tilde{\omega}_t^m N$ at i. We integrate this last equation on the interval $[0,\ 1]$ to get

$$V_1(\tau\tilde{\omega}_t^m N) - V_1(\tau\tilde{\omega}_t^m N) = -\int_0^1 \delta \sum_{1\leq r\leq m} \tilde{\omega}_t^{r-1} \,.\, (\varphi_1 - \varphi_0) \,.\, \omega_t^{m-r} \,.\, N \,.\, dt. \tag{2.48}$$

But

$$V_i(\tau\tilde{\omega}_t^m N) \ = \ \tau\omega_i^m N, \quad i = 0,\ 1. \tag{2.49}$$

This completes the proof of Theorem 2.4.

Proposition 2.7 *Taking the trace* $\tau : P/H^{q+1} \longrightarrow V$ *instead, one obtain the Chern-Simons classes, see [91, §2.2].*

Remark 2.3 The proof of Theorem 2.4(ii), we have just presented, uses an opportunity left between the first and the second structure equations. To be more specific, the connection $\varphi_t = \varphi_0(1-t) + \varphi_1 t$ could be used in two different manners: either to consider the total curvature ω_t of the structure $E \times [0,\ 1]$ (associated to the boundary operator $D = \delta + d_t$), or to consider the curvature $\tilde{\omega}_t$ of the structure E (with respect to the boundary δ), for each value of the parameter t. The validity of the proof is based on the fact that the curvature $\tilde{\omega}_a$, for each value of the parameter $t = a$, may be obtained from the total curvature ω_t by replacing in its expression $t = a$.

Remark 2.4 The proof of Theorem 2.4 is based on the following facts:

1. the connection φ (of degree 1) and the curvature $\omega := \delta\,\varphi - \varphi \,.\, \varphi$ (of degree 2)—related by the *first and second structure equations*—live in the same graded differential complex algebra
2. there is a trace operator τ defined on the space where the curvature takes values
3. the homotopy between the two connections is a *polynomial homotopy*
4. there is a compatible graded operator N which carries the bar co-chains into cyclic co-chains $\mathbf{B} \circ N$; the homology of the cyclic complex provides a good homological realisation for the Chern–Weil construction.

This proof scheme may be used in all situations in which the above conditions are met.

For a *formal integration* procedure applied onto a polynomial homotopy see also [62].

2.3.4 Karoubi Chern Character of Idempotents

The extension of the Chern–Weil theory to idempotents/finite projective modules over arbitrary associative algebras is due to Karoubi [89] in the homological context and to Connes [82] in the cohomological case.

Here we present the non-commutative Chern–Weil theory using exclusively idempotent matrices.

To this aim, we consider an associative algebra $\mathscr{A}$, possibly non-commutative and we consider an idempotent p over this algebra. We keep in mind Remark 2.2 and we recall that the Chern–Weil procedure may be split in three steps. We begin with the *first step*.

For the first part of the construction, the homological support will be an arbitrary graded differential algebra $\Omega = \{\Omega_*, \ d\}$ in which $\Omega_0 = \mathscr{A}$. The algebra Ω replaces the de Rham complex used in the classical differential geometry. At the end of the construction, this algebra has to be defined further. Each Ω_k is an $\mathscr{A}$ bi-module. All $\mathscr{A}$-modules used in this section are right modules.

We consider the next two definitions and propositions, and we proceed as in Sects. 1.6.6 and 1.6.7.

Definition 2.10 Let $p \in \mathbf{M}_N(\mathscr{A})$ be an arbitrary idempotent and let $\Gamma(p)$ denote the space of its sections (i.e. elements of its range).

$\nabla : \Gamma(p) \longrightarrow \Gamma(p) \otimes_A \Omega_1$ is called *linear connection in the bundle* p provided it satisfies the condition

$$\nabla(\xi.f) \ = \ \nabla\xi.f \ + \ \xi \otimes_{\mathscr{A}} df, \ \ \forall\xi \in \Gamma(p) \ \ and \ \ \forall f \in A. \tag{2.50}$$

When not otherwise specified, the tensor product $\otimes$ is over $\mathbb{K}$.

Definition 2.11 Let $p \in \mathbf{M}_N(A)$ be an arbitrary idempotent.

The *Levi-Civita connection* is defined by the formula

$$\nabla \ = \ p \ d \ p. \tag{2.51}$$

In this formula the boundary d is acting onto the components of the elements of A^N, which are elements of the algebra $A = \Omega_0$.

Proposition 2.8 *Let p be an idempotent. Then*

(i) the Levi-Civita connection in p is a linear connection,
(ii) any idempotent p has linear connections.

Definition 2.12 For any linear connection ∇ its *connection extension* is defined by

$$\nabla : \Gamma(p) \otimes \Omega_k \longrightarrow \Gamma(p) \otimes \Omega_{k+1}. \tag{2.52}$$

Definition 2.13 The *curvature of the linear connection* ∇ is defined by

$$R \;=\; \nabla \circ \nabla. \tag{2.53}$$

Proposition 2.9 (Non-commutative Structure Equations) *For any linear connection* ∇ *in the bundle* p *one has the*

(i) first structure equation

(i.1) there exists a well defined *matrix solution* $\omega^0 \;=\; (\omega^0_{ji})$*, whose entries* ω^0_{ji} *are elements of* Ω^1*, particular solution of the equation in* $\omega \;=\; (\omega_{ij})$

$$\nabla(p\; e_i) = (p\; e_j) \;\otimes\; \omega_{ji}, \tag{2.54}$$

where

$$\omega^0_{ij} \;=\; \alpha_{ji} \;+\; dp_{ji}, \quad with \quad \nabla(p\; e_i) \;=\; \sum_{1\leq k\leq N} e_k \otimes \alpha_{ji} \tag{2.55}$$

(i.2) for the Levi-Civita connection, the matrix ω^0 *is*

$$\omega^0 \;=\; p\; dp. \tag{2.56}$$

(ii) for any pair of solutions satisfying Eq. (2.55), ω^1_{ji} *and* ω^2_{ji}*, the matrix*

$$\omega^t_{ji} \;=\; (1-t)\;\omega^1_{ji} \;+\; t\;\omega^2_{ji}, \quad \forall t \in [0,1] \tag{2.57}$$

satisfies the first structure equation

(iii) to any connection ∇ *and* any *solution* $\omega = \{\omega_{ji}\}$ *of the first structure equation, the curvature* $R \;=\; \nabla\;\nabla$ *is given by the matrix*

$$R(p\; e_i) \;=\; \sum_{1\leq k\leq N} (p\; e_k) \otimes_{\mathscr{A}} \Omega_{ki}. \tag{2.58}$$

The matrix $\Omega \;=\; (\Omega_{ki})$ *satisfies the* second structure equation

$$\Omega_{ki} \;:=\; \omega_{kj}\;.\;\omega_{ji} \;+\; d\;\omega_{ki}, \tag{2.59}$$

or

$$\Omega \;=\; \omega\;.\;\omega \;+\; d\;\omega. \tag{2.60}$$

Remark 2.5

(i) Notice that (i.1) associates a matrix; compare with the Karoubi [89] treatment where a matrix is associated to a linear connection in a free projective module.

(ii) In the sequel, although we could continue working with any matrix solution ω of the first structure equation, we will use the particular solution ω^0 given by (2.56).

Proof

(i.1) Consider the matrix of the idempotent p. Its entries are the components of the element $p\ e_i$

$$p\ e_i\ =\ e_1.p_{1i}\ +\ \ldots\ +\ e_N.p_{Ni}\ \in\ Im\ p,\quad p=(p_{ji}),\quad with\ \ p_{ji}\in\mathscr{A}. \tag{2.61}$$

p being an idempotent it satisfies

$$p\ e_i\ =\ p\ (p\ e_i)\ =\ (p\ e_1).p_{1i}\ +\ \ldots\ +\ (p\ e_N).p_{Ni}. \tag{2.62}$$

Given that ∇ is a linear connection, it satisfies

$$\nabla(p\ e_i):=\nabla(\ (p\ e_1).p_{1i}\ +\ \cdots+(p\ e_N).p_{Ni}\ )\ =$$
$$\nabla(p\ e_1)\ .\ p_{1i}\ +\ \ldots\ +\ \nabla\ (p\ e_N)\ .\ p_{Ni}\ +\ p\ e_1\otimes(dp_{1i})\ +\ \ldots\ +\ p\ e_N\otimes(dp_{Ni}). \tag{2.63}$$

$\nabla(p\ e_i)$ decomposes as a linear combination of the elements e_j, $1\le j\le N$, with uniquely defined coefficients $\alpha_{ji}\in\Omega^1$

$$\nabla(p\ e_i)=\sum_{1\le k\le N}e_k\otimes\alpha_{ki}. \tag{2.64}$$

We know however that $\nabla(p\ \ e_i)$ belongs to $(Im\ \ p)\otimes\Omega^1(A)$. Therefore, $\nabla(p\ e_i)\ =\ p(\nabla(p\ e_i))$, or

$$\nabla(p\ e_i)\ =\ p(\sum_{1\le j\le N}e_k)\otimes\alpha_{ji}\ =\ \sum_{1\le j\le N}(p\ e_j)\otimes\alpha_{ji}. \tag{2.65}$$

Therefore

$$\nabla(p\ e_i)\ =\ \sum_{1\le j\le N}(p\ e_j)\otimes(\ \alpha_{ji}+d\ p_{ji}\ ). \tag{2.66}$$

We have found well defined elements

$$\omega^0_{ji}\ =\ \alpha_{ji}+d\ p_{ji}, \tag{2.67}$$

which completes the proof of (i.1).

(i.2) For the Levi-Civita connection one has

$$\nabla(p\,e_i) \;=\; p\,d\,p\,(p\,e_i) \;=\; p\,d\,(p\,e_i) \;=\; p\,d\sum_{1\le j\le N} e_j.p_{ji} \;= \tag{2.68}$$

$$p\sum_{1\le j\le N} e_j\otimes dp_{ji} \;=\; \sum_{1\le j\le N}(p\,e_j)\otimes dp_{ji}. \tag{2.69}$$

Therefore, for the Levi-Civita connection one has $\omega^0 \;=\; p\otimes dp$.

(ii) For any two choices of the connection forms ω_{ji} there is an affine homotopy connecting them: if $\nabla(pe_i)=\sum_{1\le j\le N}(pe_j)\,\omega^0_{ji} \;=\; \sum_{1\le j\le N}(pe_j)\,\omega^1_{ji}$ then $\nabla_t(pe_i)=\sum_{1\le j\le N}(pe_j)\,(\,(1-t)\,\omega^0_{ji} \;+\; t\,\omega^1_{ji}\,)$ is a connection. It is important to notice that under the hypothesis, $\mathbf{R}\subset\mathscr{A}$, the affine homotopy is produced *inside* the algebra A: $\nabla^t \;=\; (1-t)\,\nabla^0 \;+\; t\,\nabla^1$; its matrix is $\omega^t_{ji} \;=\; (1-t)\,\omega^0_{ji} \;+\; t\,\omega^1_{ji}$.

(iii) The second structure equation follows from the definition of linear connections and the first structure equation

$$\begin{aligned} R(pe_i) \;&=\; \nabla\,\nabla\,(pe_i) \;=\; \nabla\,(\sum_{1\le j\le N}(p\,e_j)\,\otimes\,\omega_{ji}\,) \;=\\ (\,\nabla\sum_{1\le j\le N}(p\,e_j)\,)\,.\,\omega_{ji} \;&+\; \sum_{1\le j\le N}(p\,e_j)\,\otimes\,(d\,\omega_{ji}\,) \;=\\ \sum_{1\le k\le N}(p\,e_k)\,\otimes\,\omega_{kj}.\omega_{ji} \;&+\; \sum_{1\le k\le N}(p\,e_k)\,\otimes\,(d\,\omega_{ki}\,) \;=\\ \sum_{1\le k\le N}(p\,e_k)\,\otimes\,(\,\omega_{kj}.\omega_{ji} \;&+\; d\,\omega_{ki}\,). \end{aligned} \tag{2.70}$$

Proposition 2.10 *The second structure equation gives the* generalised Bianchi identity

$$d\,\Omega^m \;=\; \Omega^m\,.\,\omega \;-\; \omega\,.\,\Omega^m \;=\; [\,\Omega^m,\,\omega\,],\quad 1\le m. \tag{2.71}$$

In particular, for $m=1$, *one gets the* Bianchi identity

$$d\,\Omega \;=\; \Omega\,.\,\omega \;-\; \omega\,.\,\Omega \;=\; [\,\Omega,\,\omega\,]. \tag{2.72}$$

Proof For the proof of this proposition we use the proof of Proposition 2.4, with minor changes due to a sign change.

This ends the first step of the non-commutative Chern–Weil procedure. We pass to the *second step*.

2.3.4.1 Karoubi Chern Character

The hypotheses remain the same: $\mathscr{A}$ is an arbitrary unital associative algebra over the ring K and $p \in \mathbf{M}_n(\mathscr{A})$ is an idempotent over this algebra. ∇ is an arbitrary linear connection in the bundle defined by p and $R = \nabla \circ \nabla$ is its curvature. We assume the first step of the non-commutative Chern–Weil procedure is performed in the graded differential algebra $\Omega_* = \{\Omega_*,\ d\}$. The first and second structure forms are available and they satisfy the generalised Bianchi identities.

For the third step, one has to ensure that on the graded differential algebra Ω_* a *trace* is defined. The ordinary *trace* satisfies the property $tr\ (\alpha\ .\ \beta) = tr\ (\beta.\alpha)$; for graded chains, the *graded trace* should satisfy $tr\ (\alpha_r\ .\ \beta_s) = (-1)^{r.s} tr\ (\beta_s\ .\ \alpha_r)$. (If one of the factors had an inverse, the trace would turn out to be invariant under conjugation $tr\ (\beta\ .\ \alpha\ .\ \beta^{-1}) = tr\ \alpha$.)

Definition 2.14 The graded differential algebra $\bar{\Omega} = \{\ \bar{\Omega}_*,\ d\ \}$ defined by

$$\bar{\Omega}_k := \Omega_k\ /\ \sum_{r+s=k} [\Omega_r, \Omega_s], \tag{2.73}$$

where $[\Omega_r, \Omega_s]$ is the subset generated by all graded commutators

$$[\ \alpha_r,\ \beta_s\ ] = \alpha_r\ .\ \beta_s - (-1)^{r.s}\ \beta_s\ .\ \alpha_r \tag{2.74}$$

with $\alpha_r \in \Omega_r$ and $\beta_s \in \Omega_s$.

The boundary operator d satisfies the relation

$$d[\ \alpha_r,\ \beta_s\ ] = [\ d\ \alpha_r,\ \beta_s\ ] + (-1)^r[\ \alpha_r,\ d\ \beta_s\ ] \tag{2.75}$$

i.e. the boundary d passes to the quotient spaces. We denote the boundary map in the quotient spaces by the same symbol d.

Definition 2.15 Let $\tau : \mathbf{B}_k \longrightarrow \bar{\mathbf{B}}_k$ to be the projection mapping; we call it *graded trace*. This takes care of the *third step*.

Proposition 2.11 *The graded trace τ satisfies the relation*

$$\tau(\ \alpha_r\ .\ \beta_s\ ) = (-1)^{r.s}\tau(\ \beta_s\ .\ \alpha_r). \tag{2.76}$$

This relation holds for classical differential forms.

The function τ commutes with the boundary operator d.

Definition 2.16 For any idempotent p over the associative algebra $\mathscr{A}$, any graded differential algebra Ω_*, $\Omega_0 = \mathscr{A}$, and any positive integer q define the *Karoubi Chern character*

$$Ch(p)_q := \frac{1}{q!}\tau R^q \in \bar{\Omega}_{2q}. \tag{2.77}$$

Theorem 2.5 (Karoubi Chern Character [89, Theorems 1.19 and 1.22]) *Let $p \in \mathbf{M}_N(\mathscr{A})$ be an idempotent and let Ω_* be any graded differential algebra with $\Omega_0 = \mathscr{A}$. Let q be any natural number. Then*

(i) the chain $Ch(p)_q \in \bar{\Omega}_{2q}$ is a cycle.
(ii) its homology class $[Ch(p)_q] \in H_{2q}(\bar{\Omega})$ is independent of the connection.

Proof

(i) There is a direct argument due to Karoubi, see [89]—proof Theorem 1.19, which shows that $Ch_k(p)$ are cycles. To this purpose one introduces the operator $J := 2p - 1$. This operator satisfies $J^2 = 1$ and

$$\begin{aligned} Jdp &= (2p-1)\,dp = 2\,(dp)\,(1-p) - (dp) = \\ &(dp)\,(2-2p-1) = (dp)\,(1-2p) = -(dp)\,J. \end{aligned} \tag{2.78}$$

Proposition 2.10 says that τ commutes with d; we have then

$$\begin{aligned} d\,\tau\,[(p(dp)(dp)]^q = d\,tr\{p([(dp)(dp)]^q\} = d\,\tau\,\{p(dp)^{2q}\} = \\ \tau\,(dp)^{2q+1} = \tau\,J^2(dp)^{2q+1} = -\tau\,J(dp)^{2q+1}J = \\ -\tau\{J(dp)^{2q+1}J^{-1} = -\tau\,(dp)^{2q+1} \end{aligned} \tag{2.79}$$

which gives

$$\tau\,(dp)^{2q+1} = -\tau\,(dp)^{2q+1}. \tag{2.80}$$

If 2 is invertible in the ground ring $\mathscr{K}$ we get

$$tr\,[(dp)\,(dp)^{2q}] = 0. \tag{2.81}$$

For an arbitrary linear connection ∇ the generalised Bianchi identity (2.78) for its curvature form, in the graded differential algebra $\bar{\Omega}_*$, becomes

$$d[\Omega^m] = [\Omega^m.\omega - (-1)^{2m.1}\omega.\Omega_m] = 0. \tag{2.82}$$

These considerations complete the proof of (i).

(ii) The second part (ii) may be proven either by using Remark 2.2, or by using a homotopy operator. As for the proof of Theorem we use the de Rham complex on the line, with polynomial coefficients K_*. We replace the graded differential algebra Ω_* by the graded differential algebra given by the graded tensor product $\Omega_*\hat{\otimes}_K K$. The differential in this algebra is given by

$$D(\gamma \otimes_K \sigma) := (d\gamma) \otimes_K \sigma + (-1)^{deg\,\gamma}\,\gamma \otimes_K (d\sigma).$$

Let ∇_0 and ∇_1 are two linear connections in the bundle defined by the idempotent p. Then $\nabla_t := \nabla_0(1-t) + \nabla t$ is a linear connection in p, in the new algebra.

The connection form is

$$\omega_t = \omega_0.(1-t) + \omega_1.t. \tag{2.83}$$

and $d_t\omega_t = (\omega_1 - \omega_0).dt$. The second structure equation (2.60) gives the *total curvature*.

$$\begin{aligned}\Omega_t = (d + d_t)\omega_t + \omega_t.\omega_t := \tilde{\Omega}_t + d_t\omega_t = \\ \tilde{\Omega}_t + (\omega_1 - \omega_0).d_t.\end{aligned} \tag{2.84}$$

From this one gets, as in the proof of Theorem 2.4(ii)

$$\Omega_t^m = \tilde{\Omega}_t^m + \sum_{p+q=m-1} \tilde{\Omega}_t^p(\omega_1 - \omega_0)\tilde{\Omega}_t^q.dt. \tag{2.85}$$

Using the fact that the differential algebra K is *graded commutative*, it is easy to show that

$$\bar{\Omega}_* \otimes K_* = \Omega_* \otimes K_*. \tag{2.86}$$

From this we get that the graded traces of the two algebras $\tau_{\Omega\otimes K}$, τ_Ω are related by the formula

$$\tau_{\Omega\hat{\otimes}K} = \tau_\Omega \otimes Id_K. \tag{2.87}$$

Remark 2.3 may be used to complete the proof.

Theorem 2.5(ii) could be proved also by using a homotopy argument, see [89, §1.20]. The success of the procedure resides in the construction of a polynomial homotopy connecting any two connections.

Definition 2.17 Define the operator $h : K \longrightarrow \mathbf{K}$

$$h(\lambda) := \begin{cases} 0 & if \quad \lambda \in K_0 \\ \int_0^1 \mu\, dt & if \quad \lambda = \mu.dt \in K_1. \end{cases} \tag{2.88}$$

Proposition 2.12 *Denote by $V_i(\lambda)$ the value of the function λ at the point $i = 0$, or 1.*

The homotopy operator h satisfies

$$(d_t h + h d_t)(\lambda) = \begin{cases} V_1\lambda - V_0\lambda, & \forall \lambda \in K_0 \\ 0 & \forall \lambda.dt \in K_1. \end{cases} \tag{2.89}$$

For any graded algebra $\{\Omega_*, \delta\}$, the homotopy operator h extends naturally to an operator $h_\Omega : \Omega_* \hat{\otimes}_{\mathbf{K}} K \longrightarrow \Omega_*$ by the formula

$$h_\Omega\ (\omega \otimes_{\mathbf{K}} \lambda) := \omega \otimes_{\mathbf{K}} h(\lambda). \tag{2.90}$$

Proposition 2.13 *The homotopy operator h_Ω satisfies the identity*

$$((\delta\ +\ d_t)h_\Omega\ +\ h_\Omega(\delta\ +\ d_t)(\omega \otimes_{\mathbf{K}} \lambda)\ =\ \omega \otimes_{\mathbf{K}} \begin{cases} V_1\lambda\ -\ V_0\lambda, & \forall \lambda \in K_0 \\ 0 & \forall \lambda \in K_1. \end{cases} \tag{2.91}$$

The operator h_Ω passes to the quotient algebra

$$\Omega_* \hat{\otimes} K / [\Omega_* \hat{\otimes} K, \Omega_* \hat{\otimes} K] = \bar{\Omega}_* \hat{\otimes} K; \tag{2.92}$$

denote it by $\bar{h}_\Omega$.

The operator $\bar{h}_\Omega$ satisfies the relation

$$((\delta\ +\ d_t)\hat{h}_\Omega\ +\ \hat{h}_\Omega(\delta\ +\ d_t))(\bar{\omega} \otimes_{\mathbf{K}} \lambda) = \bar{\omega} \otimes_{\mathbf{K}} \begin{cases} V_1\lambda\ -\ V_0\lambda, & \forall \lambda \in K_0 \\ 0 & \forall \lambda \in K_1. \end{cases} \tag{2.93}$$

Proposition 2.14 (See [89, Ch. 1.20]) *Let γ be any $\delta\ +\ d_t$-cycle in the complex $\Omega_* \bar{\otimes} K$.*

Then $V_1(\gamma)\ -\ V_0(\gamma)$ is a δ-boundary in the complex $\bar{\Omega}_*$.

Proof Use the relation (2.93).

To complete the proof of Theorem 2.5, we apply Proposition 2.13 onto the chain $\tau\Omega_t^m$.

2.3.5 Connes Chern Character of Spectral Triples

In this section we construct Connes Chern character of spectral triples following Quillen's [91] setting: This means that we associate with any spectral triple $\mathscr{T}$ an extension of associative algebras $\mathscr{E}$. To do this we define a homomorphism u of the basic algebra into the quotient algebra of the extension and we consider a *splitting*, called *connection* φ; let ω be its corresponding *curvature*. The homological information contained by the spectral triple is extracted from the powers of the curvature.

Here are the details. We consider an associative algebra $\mathscr{A}$ and a representation $\rho : \mathscr{A} \longrightarrow L(H)$, where $L(H)$ is the algebra of bounded operators on the separable, infinite dimensional Hilbert space H.

2.3.5.1 Connes Chern Character of Odd Spectral Triples

Let $\mathscr{T} := (\mathscr{A}, \rho, F)$ be an odd spectral triple, see Sect. 1.12.3. We associate the short exact sequence of associative algebras

$$\mathscr{E} \equiv \quad 0 \longrightarrow \mathscr{L}^n(H) \xrightarrow{\iota} L(H) \xrightarrow{\pi} L(H) \,/\, \mathscr{L}^n \longrightarrow 0, \tag{2.94}$$

where $\mathscr{L}^n \longrightarrow$ is the ideal of Schatten class operators. As $\mathscr{L}^n(H)$ is an ideal in $L(H)$, π is an algebra homomorphism. We define the algebra homomorphism $u : \mathscr{A} \longrightarrow L(H)/\mathscr{L}^n(H)$, $u = \pi \circ \rho$.

We associate with F its 1-spectral component $p := 1/2(F + 1)$; it is an idempotent.

The idempotent $p \in L(H)$ has the property that $[p, \ \rho(a)]$ belongs to the Schatten class $\mathscr{L}^n$, for any $a \in \mathscr{A}$.

Define $u : \mathscr{A} \longrightarrow L(H) \,/\, \mathscr{L}^n$ to be $u(a_1) = \pi \ p \ \rho(a_1)$. It is clear that u is an algebra homomorphism

$$\begin{gathered} u(a_1.a_2) \ = \ \pi \ p \ \rho(a_1.a_2) \ = \ \pi \ p \ \rho(a_1).\rho(a_2) \ = \ \pi \ p^2 \ \rho(a_1).\rho(a_2) \ = \\ \pi \ p \ ([p, \rho(a_1)] \ + \ \rho(a_1)p).\rho(a_2) \equiv \pi \ (p \ \rho(a_1)).(p.\rho(a_2)) \ = \\ \pi \ (p \ \rho(a_1)).\pi \ (p.\rho(a_2)) \ = \ u(a_1).u(a_2). \end{gathered} \tag{2.95}$$

We define $\varphi : \mathscr{A} \longrightarrow L(H)$, $\varphi(a_1) := p \ \rho(a_1)$. φ is a *connection* of the extension $\mathscr{E}$; indeed, the relation $\pi \circ \varphi(a_1) = u(a_1)$ holds.

Let $\{ \ \mathbb{B}(\mathscr{A}), \ b^{'} \ \}$ be the bar complex. Consider the induced complex, with $b^{'}$ acting on the argument

$$Hom(\mathbb{B}, \ L(H)) := \{ \ Hom_{\mathbf{K}}(\mathbb{B}_*, \ L(H)), \ b^{'} \}. \tag{2.96}$$

The corresponding *curvature* is (see Sect. 2.3.3)

$$\omega(a_1 \otimes a_2) \ = \ (\delta\varphi \ - \ \varphi.\varphi)(a_1 \otimes a_2) \ = \ p \ a_1 \ a_2 \ - \ (pa_1).(pa_2). \tag{2.97}$$

The curvature belongs to $\mathscr{L}^n(H)$. The Hölder inequality implies that the power $n + 1$ of the curvature, ω^{n+1}, is trace class.

The next result is due to Connes [82, §1, 2] and [99].

Theorem 2.6 *Let* $\mathscr{T} = (\mathscr{A}, \rho, F)$ *be an n-summable* odd spectral triple. *Let* $p = 1/2(F + 1)$. *Let* $u : \mathscr{A} \longrightarrow L(H)/\mathscr{L}^n(H)$ *be defined by* $u(a) :=$

$\pi p\ .\ \rho(a)$ and the connection $\varphi(a) = p\ \rho(a)$. Let $\omega = \delta\ \varphi - \varphi\ .\ \varphi$ be the corresponding curvature.

(i) Then, for any $1 \leq m$ $Tr\ \omega^{n+m}$, $1 \leq$ is a Hochschild co-cycle of the algebra $\mathscr{A}$.

(ii) Let $\Psi_{n+m} := Tr\ \omega^{n+m} N, \quad 1 \leq m$. Then

$$\Psi_{n+m}\ N\ (a_1 \otimes a_2 \otimes \cdots \otimes a_{2(n+m)-1} \otimes a_{2(n+m)}) = \tag{2.98}$$

$$Tr\ F\ [F,\ \rho(-)]\ [F,\ \rho(-)] \ldots [F,\ \rho(-)]\ [F,\ \rho(-)]\ N\ (a_1 \otimes \cdots \otimes a_{2(n+m)})$$

is an odd degree cyclic co-cycle *of the algebra $\mathscr{A}$.*

(iii) If the ring $\mathbf{K}$ contains the rational numbers, then the Hochschild, resp. cyclic, cohomology class of the cycle $Tr\ \omega^{n+m}$, resp. Ψ_{n+m} is independent of the connection φ.

Proof To begin with, we compute

$$p\ [p, \rho(a_1)]\ [p, \rho(a_2)] = p\ (p\rho(a_1) - \rho(a_1)p)(p\rho(a_2) - \rho(a_2)p) =$$
$$p\ (p\rho(a_1)p\rho(a_2) - p\rho(a_1)\rho(a_2)p - \rho(a_1)pp\rho(a_2) + \rho(a_1)p\rho(a_2)p)) =$$
$$p\ \rho(a_1)p\rho(a_2) - p\rho(a_1).\rho(a_2)p - p\rho(a_1)p\rho(a_2) + p\rho(a_1)p\rho(a_2)p =$$
$$-p\ \rho(a_1)\ \rho(a_2)\ p + p\ \rho(a_1)\ p\ \rho(a_2)\ p = -\omega(a_1 \otimes a_2)\ p.$$

By iterating this formula, we obtain

$$p[p, \rho(a_1)][p, \rho(a_2)] \ldots [p, \rho(a_{2(n+m)-1})][p, \rho(a_{2(n+m)})] =$$
$$(-1)^{n+m}\omega^{n+m}(a_1 \otimes \cdots \otimes a_{2(n+m)})\ p.$$

Then, as $p\omega = \omega$

$$Tr\ \omega^{n+m}(a_1 \otimes \cdots \otimes a_{2(n+m)}) =$$
$$Tr\ p\ \omega^{n+m}(a_1 \otimes \cdots \otimes a_{2(n+m)}) =$$
$$(-1)^{n+m} p[p, \rho(a_1)][p, \rho(a_2)] \ldots [p, \rho(a_{2(n+m)-1})][p, \rho(a_{2(n+m)})].$$

From this we get

$$Tr\ (-2\omega)^{n+m}(a_1 \otimes \cdots \otimes a_{2(n+m)}) = \tag{2.99}$$

$$(F + 1)[F,\ \rho(a_1)][F,\ \rho(a_2)] \ldots [F,\ \rho(a_{2(n+m)-1})][F,\ \rho(a_{2(n+m)})].$$

The cyclic symmetrisation N and the cyclic permutation operator T acting onto the argument $a_1 \otimes \cdots \otimes a_{2(n+m)}$ have the effect

$$[F,\ \rho(-)][F,\ \rho(-)]\ldots[F,\ \rho(-)][F,\ \rho(-)]\ T\ N\ (a_1 \otimes \cdots \otimes a_{2n+m}) \ =$$
$$-[F,\ \rho(-)][F,\ \rho(-)]\ldots[F,\ \rho(-)][F,\ \rho(-)]\ N\ (a_1 \otimes \cdots \otimes a_{2(n+m)}).$$

Therefore

$$Tr\ [F, \rho(-)][F, \rho(-)]\ldots[F, \rho(-)][F, \rho(-)]N(a_1 \otimes \cdots \otimes a_{2(n+m)}) \ = \ 0. \tag{2.100}$$

Using this relation into the formula (2.99) gives

$$Tr\ (-2\omega)^{n+m}\ N\ (a_1 \otimes \cdots \otimes a_{n+m}) \ = \tag{2.101}$$

$$F[F,\ \rho(-)][F,\ \rho(-)]\ldots[F,\ \rho(-)][F,\ \rho(-)]\ N\ (a_1 \otimes \cdots \otimes a_{2(n+m)}),$$

which finishes the proof of the theorem.

Remark 2.6 Theorem 2.6(ii) shows that $\Psi_{n+m}\ N$ contains $2(n+m)$ commutator factors $[F, a], a \in \mathscr{A}$. By hypothesis, each such factor belongs to the Schatten class $\mathscr{L}^n(H)$. Therefore, in order to deal with trace class operators, the total number of commutators could be halved and the exact sequence $\mathscr{E}$ could be replaced by the exact sequence

$$\tilde{\mathscr{E}} \equiv \quad 0 \longrightarrow \mathscr{L}^{n/2}(H) \xrightarrow{\iota} L(H) \xrightarrow{\pi} L(H)\ /\ \mathscr{L}^{n/2} \longrightarrow 0; \tag{2.102}$$

the same proof applies without other modifications.

Changing the scalar factor in the definition of $\Psi_{n+m}\ N$, one obtains the next definition.

Definition 2.18 The *Connes Chern character of the odd spectral triple* $\mathscr{T}$ is by definition

$$Ch^{2m-1}\ N\ (a_1 \otimes a_2 \otimes \cdots \otimes a_{2(n+m)-1} \otimes a_{2m}) \ =$$
$$(-1)^m 2(m - \frac{1}{2})\ldots\frac{1}{2}\ Tr\ F\ [F,\ \rho(-)]\ldots[F,\ \rho(-)]\ N\ (a_1 \otimes \cdots \otimes a_{2m}) \tag{2.103}$$

($2m$-commutators) for $n/2 \leq m$.

2.3.5.2 Connes Chern Character of Even Spectral Triples

Let $\mathcal{T} = (\mathcal{A}, \rho, F)$ be an *even spectral triple* over the Hilbert space H, see Sect. 1.12.3.

We associate with $\mathcal{T}$ an *odd spectral triple* $\tilde{T} = (\tilde{\mathcal{A}}, \tilde{\rho}, \tilde{F})$, over a new Hilbert space $\tilde{H}$, to be defined.

We assume that the algebra $\mathcal{A}$ is trivially graded, i.e. its elements have degree zero. We define $\tilde{\mathcal{A}} := \mathcal{A} \otimes C_{1,1}$, where $C_{1,1}$ is the Clifford algebra over $\mathbf{C}$ with generators e_1 and ϵ_1, $e_1^2 = 1$, $\epsilon_1^2 = -1$, see Sect. 1.12.1. We view this algebra as a Z_2-graded algebra by considering $C_{1,1}^{ev} = \mathbf{C}e_1$ and $C_{1,1}^{odd} = \mathbf{C}\epsilon_1$. This means that e_1 has degree zero and ϵ_1 has degree one.

By definition $\tilde{H} := H^{ev} \oplus H^{odd}$, where $H^{ev} = H^{odd} = H$.

We define also, for any $a, b \in \mathcal{A}$

$$\tilde{\rho}(ae_1 + b\epsilon_1) := \begin{pmatrix} a & b \\ b & a \end{pmatrix} \tag{2.104}$$

seen as a matrix acting onto the direct sum which defines $\tilde{H}$.

We associate the short exact sequence of associative algebras

$$\tilde{\mathcal{E}} \equiv \quad 0 \longrightarrow \mathcal{L}^n(\tilde{H}) \xrightarrow{\iota} L^n(\tilde{H}) \xrightarrow{\pi} L^n(\tilde{H})/\mathcal{L}^n(\tilde{H}) \longrightarrow 0. \tag{2.105}$$

π is an algebra homomorphism because $\mathcal{L}^n$ is an ideal.

Define $u : \tilde{\mathcal{A}} \longrightarrow L(\tilde{H})/\mathcal{L}^n(\tilde{H})$ to be $u = \pi\ \tilde{\rho}$; it is a ring homomorphism because π and $\tilde{\rho}$ are ring homomorphisms.

Define

$$\tilde{F} = i \begin{pmatrix} 0 & F \\ -F & 0 \end{pmatrix}. \tag{2.106}$$

In this formula each component F of the matrix has degree one. Then $\tilde{\mathcal{T}}$ is an n-summable odd spectral triple over the algebra $\tilde{\mathcal{A}}$. Indeed, $\tilde{F}^2 = 1$ and $\tilde{F}^* = \tilde{F}$; moreover, the graded commutator is

$$i\,[\tilde{F},\ ae_1 + b\epsilon_1] = \begin{pmatrix} -[F,b] & -[F,a] \\ [F,a] & [F,b] \end{pmatrix} \in \mathcal{L}^n(\tilde{H}). \tag{2.107}$$

Define $\varphi : \tilde{\mathcal{A}} \longrightarrow L(\tilde{H})$

$$\varphi(ae_1 + b\epsilon_1) := a\ Id. + b\ \tilde{F}. \tag{2.108}$$

$\varphi : \mathscr{A} \longrightarrow L(H), \varphi(a_1) := p\rho(a_1)$. Then φ is a *connection* of the extension $\mathscr{E}$ as the relation

$$\pi \circ \varphi(a_1) \ = \ u(a_1) \tag{2.109}$$

is a consequence of the definitions of u and φ.

Let $\mathbf{B}(\mathscr{A}), \ b^{'} \ \}$ be the bar complex. Consider the induced complex, with $b^{'}$ acting on the argument

$$Hom(\mathbf{B}, \ L(H)) := \ \{ \ Hom_{\mathbf{K}}(\mathbf{B}_*, \ L(H)), \ b^{'}\}. \tag{2.110}$$

Then the *curvature*, see Sect. 2.3.3, ω of the connection ρ, $\omega \ = \ \delta\varphi - \varphi.\varphi$ belongs to $\mathscr{L}^n(H)$. Therefore, ω^{n+1} is trace class.

The next result is due to Connes [82, §1, 2] and [99].

Theorem 2.7 *Let $\mathscr{T} \ = \ (\mathscr{A}, \rho, F)$ be an n-*summable even spectral triple.
Let

$$Ch^{2m}(\mathscr{T})(a_0, a_1, \ldots, a_{2m}) \ := \ \frac{(-1)^m m!}{2} \ Tr \ (\gamma F[F, a_0] \ldots [F, a_{2m}]. \tag{2.111}$$

Then

(i) $Ch^{2m}(\mathscr{T})$ is an even degree cyclic co-cycle of the algebra $\mathscr{A}$ for any $n < m$.
(ii) If the ring $\mathbf{K}$ contains the rational numbers, then the cyclic cohomology class of the co-cycle $Ch^{2m}(\mathscr{T})$ is independent of the connection ρ.

Proof We have shown that there is a precise procedure which associates with any even spectral triple $\mathscr{T}$ an odd spectral triple $\tilde{\mathscr{T}}$. The proof of the theorem will follow the proof of Theorem 2.6 concerning odd spectral triples. To complete the proof two more ingredients are needed. The spectral triple $\tilde{\mathscr{T}}$ is over the algebra $\tilde{\mathscr{A}} \ = \ \mathscr{A} \otimes C_{1,1}$.

Proposition 2.15 *For any two associative algebras $\mathscr{A}$ and $\mathscr{B}$,*

$$C^p_\lambda(\mathscr{A} \otimes_{\mathbf{K}} \mathscr{B}) \ = \ \oplus_{p=r+s} C^r_\lambda(\mathscr{A}) \otimes_{\mathbf{K}} C^s_\lambda(\mathscr{B}). \tag{2.112}$$

2.3.6 Pairing n-Summable Fredholm Modules with K-Theory

We assume that $\mathbf{K} \ = \ \mathbf{R}$, or $\mathbf{C}$ and $D^2 \ = \ -1$; Proposition 1.21 shows that this condition may be satisfied.

Let $\mathscr{E} \ = \ \{\mathscr{A}, D, \rho\}$ be an n-summable triple over the $\mathbf{Z}_2$-graded Hilbert space H, with grading homomorphism γ. This data defines an element of the Kasparov K-homology group $K_0(\mathscr{A})$.

Let $p = (p_{ij}) \in \mathbf{M}_N(\mathscr{A})$ be an idempotent over the algebra $\mathscr{A}$. It represents an element of the K-theory group $K^0(\mathscr{A})$.

We are going to define the *pairing* between $\mathscr{E}$ and p denoted $\mathscr{E} * p$.

Let

1. $H^N := \oplus^N H$,
2. $\gamma^N := \oplus^N \gamma$
3. $D^N := \oplus^N D : H^N \longrightarrow H^N$
4. $\rho^N :=$ diagonal action of $\mathscr{A}$ on H^N, i.e.

$$\rho^N(f)(h_1, h_2, \ldots, h_N) = (\rho(f)h_1, \rho(f)h_2, \ldots, \rho(f)h_N).$$

D^N is given by the diagonal matrix

$$D^N = \begin{pmatrix} D & 0 & 0 & \ldots & 0 \\ 0 & D & 0 & \ldots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & D \end{pmatrix}. \tag{2.113}$$

$N \times N$ matrices whose components are operators on the Hilbert space H may be multiplied (to the left and to the right) by $N \times N$ matrices with entries in $\rho(\mathscr{A})$, with the usual matrix multiplication rule. In particular, the product of D^N by p is well defined.

For any n-summable Hilbert module the property $[D, \rho(f)] \in \mathscr{L}^n(H^N)$ becomes

$$[\, D^N, \, M \,] \in \mathscr{L}^n(H^N) \tag{2.114}$$

for any $N \times N$ matrix M with entries in $\rho(\mathscr{A})$.

Let D_p be the operator

$$D_p := p \, D^N \, p : H^N \longrightarrow H^N. \tag{2.115}$$

D_p has as domain and co-domain the Hilbert space $H_p := pH^N$.

Proposition 2.16

(i) For any matrix $N \times N$ matrix M with entries in $\rho(\mathscr{A})$ one has

$$\gamma^N \, M = M \, \gamma^N, \quad \textit{in particular } \gamma^N \, p = p \, \gamma^N,$$
$$\gamma^N \, H_p \subset H_p.$$

Proof

(i) Let $M = (\rho(f_{ij}))$. Then $\gamma^N M = (\gamma\ \rho(f_{ij})) = (\rho(f_{ij})\ \gamma)$.
(ii) $\gamma^N H_p = \gamma^N p H^N = p \gamma^N H^N = p H^N$.

Proposition 2.17 *For any n-summable triple $\mathscr{E}$ and idempotent p the corresponding operator $D_p := p D^N p$ has the following properties:*

(i) D_p satisfies property

$$\rho^N D_p - D_p \rho^N \in L^n \tag{2.116}$$

(ii)

$$\gamma^N D_p = -D_p \gamma^N; \tag{2.117}$$

(iii) the operator D_p splits up into its positive and negative parts

$$D_p = D_p^+ \oplus D_p^- \tag{2.118}$$

$$D_p^\pm = \frac{1 \mp \gamma^N}{2} D_p \frac{1 \pm \gamma^N}{2} \tag{2.119}$$

(iv)

$$\theta := I_N + (D^N)^2 \text{ belongs to } L^n(H^N), \tag{2.120}$$

(v)

$$\Theta_p := pI_N + D_p^2 \text{ belongs to } L^n(pH^N); \tag{2.121}$$

here I_N denotes the identity operator on the Hilbert space H^N (pI_N is the identity operator on pH^N).

Proof

(i) The operator ρ commutes both with p and D^N modulo Schatten class operators.
(ii) Obvious.
(iii) Property (ii) implies (iii).
(iv) We have

$$\theta := I_N + (D^N)^2 = I_N + D^N D^N = I_N - I_N - \text{Schatten class operator in } L^n.$$

(v) With regard to Θ_p, one has

$$\Theta_p := I_N p + D_p^2 = I_N p + pD^N p.pD^N p = I_N p + pD^N pD^N p =$$
$$pI_N p + ([D^N, p] + pD^N)D^N p = p(I_N + D^N D^N)p + [D^N, p]D^N p.$$

This operator belongs to $L^n(pH^N)$ because the two operators in the last member of the equality belong to this class.

(v) Part (v) of Propositions 2.17 and 2.16 say that H_p splits up into its positive and negative parts

$$H_p = H_p^+ \oplus H_p^-, \quad \gamma^N \mid H_p^\pm = \pm I$$

and along with it, the operator D_p too $D_p = D_p^+ \oplus D_p^+$.

Proposition 2.18 *The operator D_p exchanges the positive/negative parts*

$$D_p : H_p^\pm \longrightarrow H_p^\mp. \tag{2.122}$$

Proof The property follows from the commutation relations

$$\gamma^N D_p = \gamma^N pD^N p = p\gamma^N D^N p = -pD^N \gamma^N p = -pD^N p\gamma^N. \tag{2.123}$$

For this last part we use [101, (34), (35)]. The relation (2.120) becomes $D_p^2 := I_N p - \Theta_p$.

Let $q(t)$ be the unique polynomial of degree n such that

$$q(t)^2 = 1 + t + \ldots + t^n + \textit{higher order terms}. \tag{2.124}$$

Define

$$\tilde{D}_p = D_p\, q(\theta)\, p. \tag{2.125}$$

Then

$$\tilde{D}_p.\tilde{D}_p = (I_N p - \theta_p)(I_N + \theta + \ldots\theta_p^n + \textit{higher order terms in } \theta_p)\; D_p.D_p =$$

$$= I_N p + \tilde{D}_p.\tilde{D}_p(\theta_p^n + \text{higher order terms}) = I_N p - D_p.D_p\mathscr{K}, \tag{2.126}$$

where $\mathscr{K}$ is a trace class operator. Denote

$$\Theta(\mathscr{E}, p) = I_N p + \tilde{D}_p.\tilde{D}_p = D_p.D_p\mathscr{K} \in L^1. \tag{2.127}$$

Definition 2.19

$$\mathscr{E} * p := \text{ a - Ind}(\ \gamma^N \Theta(\mathscr{E}, p)\ ) \ = \ Trace(\ \gamma^N\ \Theta(\mathscr{E}, p)\ ). \qquad (2.128)$$

Remark 2.7 Let us explain the reasoning behind the construction of the operator $\mathscr{E} * p$.

(i) Although p and ρ^N do not commute (if $\mathscr{A}$ is not commutative), the operator D_p commutes with elements of the algebra $\mathscr{A}$, modulo compact operators. The operator D_p is called the *E twisted operator D along the bundle p*. Usually, the operator D_p is produced by choosing a linear connection in the bundle p, see e.g. [42]. The construction of the operator D_p we have just presented uses the Levi-Civita connection in the bundle p in a hidden fashion; see for example [83] when D is the signature operator.

(ii) The K-homology class, represented by $\mathscr{E}$, and the K-theory class represented by the idempotent p, produce the operator D_p. The pairing involves the analytic index of D_N^+ only. If the operators $S_0(D_N)$, $S_1(D_N)$ were trace class operators, then

$$\text{a - Ind } D_N^+ \ = \ TrS_0(D_N) \ - \ TrS_1(D_N),$$

however, $S_0(D_N),\ \ S_1(D_N) \in L^n(H)$ are not trace class operators and hence we are not allowed to use this relation. We may replace the operator D_p by the operator $\tilde{D}_p$, which is a compact deformation of D_p. By doing this, the positive parts of the two operators differ by a compact operator and therefore have the same index. In addition, Θ contains, as a factor, a sufficiently high power of a Schatten class operator and hence it is a trace class operator. The analytic index of $\tilde{D}_p^+$ may be extracted from the operator Θ

$$\mathscr{E} * p := \text{ a - Ind}(D_N^+) \ = \ Tr(\ \gamma^N \Theta(\mathscr{E}, p)\ ). \qquad (2.129)$$

(iii) $\mathscr{E} * p$ may be expressed algebraically in terms of the Connes Chern character of $\mathscr{E}$ and the Chern character of p.

2.3.7 Pairing of K-Homology with K-Theory—Explicit

Theorem 2.8 *We assume that the operator D of the previous section satisfies* $D^2 = -1$*; [82] shows that this condition may be satisfied. We consider the idempotent p as above. Then the algebraic expression for* $\mathscr{E} * p$ *is*

$$\mathscr{E} * p \ = \ a\text{ - }Ind\ D_p^n \ = \ Tr(\gamma^N p[D, p]^{2n'+1}), \quad n \le n' \qquad (2.130)$$

Proof Define

$$D_p^{\pm} := \frac{1 \mp \gamma^N}{2}\ p\ D^N\ p\ \frac{1 \pm \gamma^N}{2}.$$

Define

$$S_0^p := p\ \frac{1+\gamma}{2}\ +\ D_p\ D_p\ \frac{1+\gamma}{2}\ \in\ L^{n'}$$
$$S_1^p := p\ \frac{1-\gamma}{2}\ +\ D_p\ D_p\ \frac{1-\gamma}{2};\ \in\ L^{n'}.$$

For the computation of the analytic index of the operator D_p^+, $Tr(S_0)^{\tilde{n}} - Tr(S_0)^{\tilde{n}}$, $n \leq \tilde{n}$, we use (see Hörmander [74]). We get

$$\text{a - Ind}\ D_p^n\ =\ Tr\ (S_0^p)^{\tilde{n}}\ -\ Tr\ (S_1^p)^{\tilde{n}}\ =\ Tr\ \gamma^N\ (S_0^n)^{\tilde{n}}\ =$$
$$Tr\gamma^N(p\ +\ D_p^- D_p^+)\ =\ Tr\gamma^N\ p\ [D^N\ p]^{\tilde{n}}.$$

This follows from the following identities:

$$p\ [D,p]^2\ =\ p\ (D\ p\ D\ p - D\ p\ p\ D - p\ D\ D\ p + p\ D\ p\ D)\ =$$
$$p\ D\ p\ D\ p - p\ D\ p\ D - p\ D\ D\ p + p\ D\ p\ D\ =$$
$$p\ D\ p\ D\ p - p\ D\ D\ p\ =\ p\ D\ p\ D\ p\ +\ p$$

because $D^2\ =\ -1$. Analogously, one gets

$$[D,p]^2\ p\ =\ pDpDp\ +\ p.$$

The two identities imply

$$p\ [D,p]^2\ =\ [D,p]^2\ p; \tag{2.131}$$

additionally, $p[D,p]^2\ =\ [D,p]^2 p$.

From this we obtain

$$(p\ +\ D_p D_p)\ =\ p\ [D,p]^2\ =\ [D,p]^2 p.$$

Therefore,

$$\text{a - Ind}\ D_p^n\ =\ Tr\ \gamma^N\ (p\ [D,p]^2)^{n'},\quad n \leq n'.$$

There is another expression for the pairing, see [82, §3]

$$\mathscr{E} * p := \frac{(-1)^m}{2} Tr\gamma^N D[D, p]^{2n'+1}, \quad n \leq n'. \tag{2.132}$$

Indeed

$$\begin{aligned} Tr\gamma^N D^N [D^N, p]^{2n'+1} &= Tr\gamma^N D^N [D^N, p][D^N, p]^{2n'} \\ &= Tr\gamma^N D^N (p[D^N, p] + [D^N, p]p)[D^N, p]^{2n'} \end{aligned}$$

(here we have used the identity $[D^N, p] = p[D^n, p] + [D^N, p]p$)

$$\begin{aligned} Tr\gamma^N D^N p[D^N, p][D^N, p]^{2n'} &= \\ = -TrD^N \gamma^N p[D^N, p]^{2n'+1} &= \end{aligned}$$

(here we have used the identity $D^N [D^N, p]^{2n'+1} = -[D^N, p]^{2n'+1} D^n$)

$$\begin{aligned} -Tr\gamma^N p[D^N, p]^{2n'+1} D^N &= Tr\gamma^N pD^N [D^N, p]^{2n'+1} \\ Tr\gamma^N D^N [D^N, p]^{2n'+1} &= 2Tr\gamma^N D^N [D^N, p] p[D^n, p]^{2n'} = \\ &2(-1)^m Tr\gamma^N (p - pD^N pD^N p)^{n'+1}. \end{aligned}$$

For more details see [82, §3].

2.4 Non-commutative Homology

In the previous sections we saw that the Chen–Weil theory can be extended to possibly non-commutative associative algebras. We saw that the non-commutative Chern characters belong to different homology spaces: *cyclic cohomology* (Theorem 2.208) and *non-commutative de Rham homology* (Theorem 2.5). Using these two results as *motivation* we intend to study these homologies more thoroughly. This is going to be done in the next sections. We begin by discussing Hochschild homology and then cyclic homology. Before going to study cyclic homology, the following remark will be useful:

Remark 2.8 The *cyclic symmetry* property corresponds to the property of classical differential forms of remaining invariant under cyclic permutations of tangent fields.

The basic feature of cyclic symmetry consists of the fact that it is compatible—on even degree elements—with the symmetry property of the *trace* ($tr(A\ B) = tr(B\ A)$).

Homology theories of general associative algebras appeared in the mathematical literature about the middle of the last century. Some of them are: Hochschild

homology [4], the homology of a module over an associative algebra, see Chevalley and Eilenberg [5] and Mac Lane [68]. We have seen in Sect. 1.4. that the *Alexander–Spanier homology* [45] may be used to compute the simplicial cohomology of a CW-complex by means of *discontinuous co-chains*.

The first step we intend to accomplish is to show that the *Alexander–Spanier complex* and the *Hochschild complex* may be used as foundations of a *general homology theory* in non-commutative geometry and topology.

We start by defining the Hochschild complex and its homology.

Let $\mathscr{A}$ be an associative algebra over the ring $\mathbf{K}$ and M an $\mathscr{A}$ bi-module.

One associates the system of spaces and morphisms (the tensor products are over the ring K)

$$C_k(\mathscr{A}, M) := \{ \sum_{finite} m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_k \mid m \in M,\ a_i \in A\} \tag{2.133}$$

and $b : C_k(A, M) \longrightarrow C_{k-1}(\mathscr{A}, M)$ defined by

$$b := \sum_{0 \leq r \leq k} (-1)^r b_r, \tag{2.134}$$

where the *Hochschild face boundary map* and b_r is defined by

$$b_r(m \otimes a_1 \otimes \ldots \otimes a_k) := m \otimes a_1 \otimes a_2 \otimes \ldots\ldots \otimes a_{r-1} \otimes_K (a_k.a_{k+1}) \otimes_K \cdots \otimes a_k,$$
$$for\ \ 0 \leq r \leq k-1,\ \ a_0 = m, \tag{2.135}$$

and

$$b_k(m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_k) := (a_k\ .\ m) \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_{k-1}. \tag{2.136}$$

It is easy to check that $b\ b = 0$ and therefore $\{C_*, b_*\}$ is a homology complex, called *Hochschild complex of the algebra $\mathscr{A}$ with values in the module M*. Its homology is denoted $H_*(\mathscr{A}, M)$. By definition, the elements of $C_k(\mathscr{A}, M)$ are chains of degree k; the degree of the Hochschild homology is defined accordingly.

The case $M = \mathscr{A}$ is of particular interest; the corresponding Hochschild homology is denoted $H_*(\mathscr{A})$. Loday [97] denotes it $HH_*(\mathscr{A})$.

2.4.1 The Bar Resolution

We suppose that M is an $\mathscr{A}$ bi-module. We associate with the pair $\mathscr{A}, M$ the system of spaces and morphisms (for any $0 \leq k$)

$$\bar{C}_k(A, M) := C_k(A, M) \tag{2.137}$$

and $b_k^{'} : C_k(A, M) \longrightarrow C_{k-1}(A, M)$ defined by

$$b^{'} := \sum_{0 \leq r \leq k-1} (-1)^r b_r. \tag{2.138}$$

The Hochschild boundary b and the bar boundary $b^{'}$, acting onto the k-chains, are related by the formula

$$b = b^{'} + (-1)^k b_k; \tag{2.139}$$

for the definition of b_k see formula (2.135) above.

Definition 2.20 The *Hochschild complex* is

$$C_*^H(A) := \{C_p, b\}_{0 \leq p}. \tag{2.140}$$

Definition 2.21 The *bar complex* of the module M over the algebra A is

$$\{C_*(\mathscr{A}, M), b^{'}\}. \tag{2.141}$$

Proposition 2.19 *Let M be a right module over the associative algebra $\mathscr{A}$. Suppose the algebra $\mathscr{A}$ is unitary. Then the bar complex of the module M is acyclic.*

If $M = \mathscr{A}$ and $\mathscr{A}$ is unitary, then the bar complex of the algebra $\mathscr{A}$ is acyclic.

Definition 2.22 If the algebra $\mathscr{A}$ is unitary, the complex $\{C_*(\mathscr{A}, M), b^{'}\}$, resp. $\{C_*(\mathscr{A}), b^{'}\}$, is called *bar resolution of the module M*, resp. algebra $\mathscr{A}$, see [68].

Proof The mappings $s_{(k)} : \bar{C}_k(\mathscr{A}, M) \longrightarrow \bar{C}_{k+1}(\mathscr{A}, M)$, defined by

$$s_{(k)}(m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_k) := (-1)^k m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_k \otimes 1 \tag{2.142}$$

establish a homotopy between the identity and the zero homomorphism

$$s_{(k-1)} b^{'} + b^{'} s_{(k)} = 1. \tag{2.143}$$

Indeed, the identity follows from the relations

$$b^{'} s_{(k)}(m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_k) = (-1)^k b^{'}(m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_k) \otimes 1 + \tag{2.144}$$

$$(-1)^{2k} b^{'}(m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_k . 1$$

and

$$s_{(k-1)} b^{'}(m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_k) = (-1)^{k-1}(b^{'}(m \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_k)) \otimes 1.$$

2.4.2 Preliminaries: The Basic Structure

The cyclic group S_{n+1} acts on the tensor product C_n through its generator T_n whose action is

$$T_n(a_0 \otimes_K a_0 \otimes_K a_1 \ldots . \otimes_K a_n) := (-1)^n\, a_1 \otimes_K a_2 \otimes_K \otimes_K a_1 \otimes_K \ldots . \otimes_K a_n \otimes_K a_0. \tag{2.145}$$

The sign $(-1)^n$ is the sign of the cyclic permutation. This action is responsible for the basic properties of the cyclic homology, its refinements and derivatives; these will be discussed later.

Definition 2.23 A chain $\gamma \in C_n$ such that $T_n\gamma = \gamma$ is called *cyclic*.

The operator T_n satisfies

$$(T_n)^{n+1} = 1. \tag{2.146}$$

The *cyclic symmetrisation* operator N_n acting on C_n is defined by

$$N_n := \sum_{p=0}^{p=n} (T_n)^p. \tag{2.147}$$

If the ring $\mathbf{K}$ contains the rational numbers $\mathbb{Q}$, then the average J_n^λ over the *cyclic group* S_{n+1} is well defined; the space of cyclic elements is

$$J_n^\lambda := \frac{1}{n+1} N_n. \tag{2.148}$$

It is an idempotent. The image of the idempotent J_n^λ consists of all cyclic periodic elements.

Let $J_n^\perp := 1 - J_n^\lambda$ be the *complementary idempotent*. If no confusion could occur, the subscript n will be omitted.

Proposition 2.20

(i) The operators T and N satisfy

$$(1 - T)\, N = N\, (1 - T) = 0. \tag{2.149}$$

(ii) The complementary idempotent is

$$J_n^\perp = (1 - T)\tilde{J}^\perp = \tilde{J}^\perp\, (1 - T), \tag{2.150}$$

where

$$\tilde{J}^{\perp} := \sum_{p=0}^{p=n-1} \frac{n-p}{n+1} T^p. \tag{2.151}$$

(iii) The identity mapping of $C_n(\mathscr{A})$ decomposes

$$1 = \frac{1}{n+1} N + (1-T)\,\tilde{J}^{\perp} = \frac{1}{n+1} N + \tilde{J}^{\perp}(1-T). \tag{2.152}$$

Proof

(i) The identity (2.149) follows from (2.146)
(ii) One has

$$J_n^{\perp} = 1 - \frac{1}{n+1} N = 1 - \frac{1}{n+1} \sum_{p=0}^{p=n} T^p =$$

$$= \frac{1}{n+1} \sum_{p=0}^{p=n} (1 - T^p) = \frac{1}{n+1} \sum_{p=1}^{p=n} (1 - T)(1 + T^1 + \cdots + T^{p-1})$$

$$= (1 - T)\{ \sum_{p=0}^{p=n-1} \frac{n-p}{n+1} T^p \} = \{ \sum_{p=0}^{p=n-1} \frac{n-p}{n+1} T^p \}(1 - T). \tag{2.153}$$

(iii) It follows from (ii).

Corollary 2.1 *If the $\mathbb{K}$ contains the rationals, then*

(i)

$$Im J_n^{\lambda} = Ker J_n^{\perp} = Ker(1-T) = Im N, \tag{2.154}$$

(ii)

$$Ker J_n^{\lambda} = Im J_n^{\perp} = Im(1 - T) = Ker N. \tag{2.155}$$

Proof The first equalities (i) and (ii) follow from the projection properties.

(i) Relation (2.154) says that N and J^{λ} are scalar multiples of each other; therefore, they have the same kernel and range.

For the other equalities (i), the relation $N(1 - T) = 0$ implies $Im(1-T) \subseteq Ker N$. Vice versa, suppose that $x \in Ker N$. Then $x \in Ker J^{\lambda} = Im J^{\perp}$. But (2.155) tells that $Im J^{\perp} \subseteq Im(1 - T)$ and hence $x \in Im(1 - T)$.

(ii) The other equalities are proved analogously.

2.4.3 Relations Between N, $1-T$, b' and b

Here and in the next sections the associative algebra $\mathscr{A}$ is fixed; when no confusion could occur, $\mathscr{A}$ could be omitted.

Proposition 2.21

(i) The operators N, $1-T$ satisfy

$$(1-T)\,b' = b\,(1-T) \tag{2.156}$$

$$b'\,N = N\,b. \tag{2.157}$$

(ii) Therefore, the operators N, $1-T$ are degree preserving chain homomorphisms

$$1-T : \{C_*(A), b'\} \longrightarrow \{C_*(A), b\} \tag{2.158}$$

$$N : \{C_*(A), b\} \longrightarrow \{C_*(A), b'\}, \tag{2.159}$$

Proof The relations (i) are obtained by a straightforward computation, see [97, (2.1.5.2)] while (ii) follows from (i).

The next definition introduces the important $\{b, b^{'}\}$ bi-complex due to Loday and Quillen [85].

Definition 2.24 We denote by $\{b, b^{'}\}$ the bi-complex formed by the mappings $b,\ b^{'}, 1-T,\ N$

$$\begin{array}{ccccccccccc}
 & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
0 & \longleftarrow & \otimes^4\mathscr{A} & \xleftarrow{1-T} & \otimes^4\mathscr{A} & \xleftarrow{N} & \otimes^4\mathscr{A} & \xleftarrow{1-T} & \otimes^4\mathscr{A} & \xleftarrow{N} & \cdots \\
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
0 & \longleftarrow & \otimes^3\mathscr{A} & \xleftarrow{1-T} & \otimes^3\mathscr{A} & \xleftarrow{N} & \otimes^3\mathscr{A} & \xleftarrow{1-T} & \otimes^3\mathscr{A} & \xleftarrow{N} & \cdots \\
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
0 & \longleftarrow & \otimes^2\mathscr{A} & \xleftarrow{1-T} & \otimes^2\mathscr{A} & \xleftarrow{N} & \otimes^2\mathscr{A} & \xleftarrow{1-T} & \otimes^2\mathscr{A} & \xleftarrow{N} & \cdots \\
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
0 & \longleftarrow & \mathscr{A} & \xleftarrow{1-T} & \mathscr{A} & \xleftarrow{N} & \mathscr{A} & \xleftarrow{1-T} & \mathscr{A} & \xleftarrow{N} & \cdots
\end{array} \tag{2.160}$$

with general term $C_{p,q} := \otimes^{q+1}\mathscr{A}$.

The relations (2.156) and (2.157) allow us to consider two short exact sequences of complexes. The first exact sequence is

$$(ES_1) \equiv \quad 0 \longrightarrow \{\frac{C_*}{Ker(1-T)}, b'\} \stackrel{1-T}{\longrightarrow} \{C_*, b\} \stackrel{\Pi}{\longrightarrow} \{\frac{C_*}{Im(1-T)}, b\} \longrightarrow 0. \tag{2.161}$$

A second exact sequence of complexes will be needed

$$(ES_2) \equiv \quad 0 \longrightarrow \{Ker(1-T), b'\} \stackrel{\iota}{\longrightarrow} \{C_*, b'\} \stackrel{\Pi'}{\longrightarrow} \{\frac{C_*}{Ker(1-T)}, b'\} \longrightarrow 0. \tag{2.162}$$

where ι is the inclusion.

2.4.3.1 Study of the Exact Sequence (ES_1)

The first exact sequence leads to the exact sequence in homology

$$\ldots. \longrightarrow H_n(\frac{C_*}{Ker(1-T), b'}) \stackrel{(1-T)_*}{\longrightarrow} H_n(C_*, b) \stackrel{\Pi_*}{\longrightarrow} H_n(\frac{C_*}{Im(1-T)}, b) \stackrel{\delta_n}{\longrightarrow}$$

$$H_{n-1}(\frac{C_*}{Ker(1-T)}, b') \stackrel{(1-T)_*}{\longrightarrow} H_{n-1}(C_*, b) \stackrel{\Pi_*}{\longrightarrow} H_{n-1}(\frac{C_*}{Im(1-T)}, b) \stackrel{\delta_{n-1}}{\longrightarrow} \ldots\ldots \tag{2.163}$$

We need an explicit formula for connecting homomorphism δ_n. Let [ ] denote homology class.

Proposition 2.22

(i) For any cycle $x \in \{\frac{C_*}{Im(1-T)}, b\}$,

$$\delta_n([x]) = [\tilde{J}^{\perp} bx] \in H_{n-1}(\frac{C_*}{Ker(1-T)}, b') \tag{2.164}$$

(ii) $\delta_n(x) := \tilde{J}^{\perp} bx$ *is well defined at the level of chains*

$$\delta_n : \{\frac{C_*}{Im(1-T)}, b\}_n \longrightarrow \{\frac{C_*}{Ker(1-T)}, b'\}_{n-1} \tag{2.165}$$

and

(iii) δ_n *factories to the quotient spaces*

$$\delta_n : \frac{C_*}{Im(1-T)} \longrightarrow \frac{C_*}{Ker(1-T)}. \tag{2.166}$$

Proof

(i) Let $x \in C_n$ be such that $bx \in Im(1-T)$. Using Proposition 2.20, the chain bx may be decomposed

$$bx \;=\; \frac{1}{n+1}N(bx) \;+\; (1 \;-\; T)\tilde{J}^{\perp}(bx) \tag{2.167}$$

or, if we define $y = \tilde{J}^{\perp}(b\,x)$, as $N\;b\;x \;=\; 0$, one has

$$(1 \;-\; T)y \;=\; bx \tag{2.168}$$

which gives

$$\delta_n([x]) \;=\; [\tilde{J}^{\perp} b\;x]. \tag{2.169}$$

(ii) Although $\delta_n(x) := \tilde{J}^{\perp} b\;x$ is supposed to act at the level of homology classes, it makes sense at the level of chains

$$\delta_n : C_n \longrightarrow C_{n-1}. \tag{2.170}$$

(iii) Moreover, if $x \in Im(1 \;-\; T)$, i.e. $x \;=\; (1 \;-\; T)\bar{x}$, then

$$\begin{aligned} &\delta_n(x) = \delta_n((1 \;-\; T)\bar{x}) = \tilde{J}^{\perp} b((1 \;-\; T)\bar{x}) \;= \\ &\tilde{J}^{\perp}((1 \;-\; T)b')\bar{x}) \;=\; (1 - \frac{1}{n}N)b'\bar{x} \;= \end{aligned} \tag{2.171}$$

$$b'\bar{x} - \frac{1}{n}Nb'\bar{x} \;\cong b'\bar{x} \quad (mod.\; Ker(1 \;-\; T)). \tag{2.172}$$

2.4.3.2 Study of the Exact Sequence (ES_2)

The second exact sequence leads to the exact sequence in homology

$$\ldots \xrightarrow{\delta'_{n+1}} H_n\{Ker(1 \;-\; T), b'\} \xrightarrow{\iota_*} H_n\{C_*, b'\} \xrightarrow{\Pi'_*} H_n\{\frac{C_*}{Ker(1 \;-\; T)}, b'\} \xrightarrow{\delta'_n}$$

$$H_{n-1}\{Ker(1-T), b'\} \xrightarrow{i_*} H_{n-1}\{C_*, b'\} \xrightarrow{\Pi'_*} H_{n-1}\{\frac{C_*}{Ker(1-T)}, b'\} \xrightarrow{\delta'_{n-1}} \ldots \tag{2.173}$$

Proposition 2.23 *If the algebra A is unital, then the complex $\{\frac{C_*}{Ker(1-T)}, b'\}$ is acyclic.*

Proof Let

$$s_n : C_n \longrightarrow C_{n+1} \tag{2.174}$$

be defined by

$$s_n(a_0 \otimes_K a_1 \otimes_K \ldots .. \otimes_K a_n) := 1 \otimes_K a_0 \otimes_K a_1 \otimes_K \ldots .. \otimes_K a_n, \quad 0 \leq n, \tag{2.175}$$

and $s_{-1} = 0$. A simple computation shows that s establishes a homotopy of the identity to the zero homomorphism

$$b's_n + s_{n-1}b' = 1. \tag{2.176}$$

Remark 2.9 The acyclic complex $\{C_*, b'\}$ is known in the literature as the *bar resolution* of the algebra A, see MacLane [68, §10.2].

Proposition 2.24

(i) The connecting homomorphism

$$\delta'_n : H_n\{\frac{C_*}{Ker(1 - T)}, b'\} \longrightarrow H_{n-1}\{Ker(1 - T), b'\} \tag{2.177}$$

is an isomorphism. For any $x \in C_n$ such that $b'x \in Ker(1 - T)$,

$$\delta'_n[x] = [b'x]. \tag{2.178}$$

(ii) The inverse of the connecting homomorphism, $(\delta')_n^{-1}$, is

$$\delta_n^{'-1} : H_{n-1}\{Ker(1 - T), b'\} \longrightarrow H_n\{\frac{C_*}{Ker(1 - T), b'}\} \tag{2.179}$$

is given by the formula

$$\delta_n'^{-1}([y]) = [s_{n-1}y]. \tag{2.180}$$

(iii) The homology isomorphism $\delta_n'^{-1}$ extends to a homomorphism

$$\delta_n'^{-1} : Ker(1 - T)_{n-1} \longrightarrow C_n \tag{2.181}$$

given by

$$\delta_n'^{-1}(y) := s_{n-1}y. \tag{2.182}$$

Proof

(i) Let $x \in C_n$ such that $b'x \in Ker(1 - T)$. Then $\delta'_n([x]) = [b'x] \in H_{n-1}\{Ker(1 - T), b'\}$.

(ii) Suppose now $y \in Ker(1 - T)_{n-1}$ be such that $b'y = 0$. Let $x = s_{n-1}y$. Then $b'x = b's_{n-1}y = (1 - s_{n-2}b')y = y - s_{n-2}b'y = y$. Therefore, x is a cycle in $\{\frac{C_*}{Ker(1-T)}, b'\}$. From (i) we know that $\delta'[x] = [y]$. As δ' is an isomorphism, it follows that

$$(\delta'_n)^{-1}([y]) = [s_{n-1}y]. \tag{2.183}$$

2.4.4 Hochschild and Cyclic Homology

Definition 2.25 The homology of the complex $\{C_*(\mathscr{A}), b\}$ is the *Hochschild homology of the algebra* $\mathscr{A}$ with coefficients in $\mathscr{A}$, denoted $H_*(\mathscr{A})$. Other authors, see e.g. Loday [97], denote it by $HH_*(A)$.

Remark 2.10 Note that the elements of $C_n(\mathscr{A}) = \otimes_K^{n+1}$ are by definition chains of degree n, and hence there is a shift of one unit between the number of factors in the tensor product and the degree of the corresponding Hochschild homology classes.

Remark 2.11 If M is an $\mathscr{A}$-bi-module, then the elements of $M \otimes_K \otimes_K^n \mathscr{A}$ are called n-chains of the algebra $\mathscr{A}$ with coefficients in M.

More generally, one defines

$$C_n(\mathscr{A}, M) := M \otimes_A C_n(\mathscr{A}) = M \otimes_K C_{n-1}(\mathscr{A}), \tag{2.184}$$

endowed with the Hochschild boundary b

$$b(m \otimes_K a_1 \otimes_K a_2 \otimes_K \cdots \otimes_K a_n) := \tag{2.185}$$

$$ma_1 \otimes_K a_2 \otimes_K \cdots \otimes_K a_n - m \otimes_K b'(a_1 \otimes_K a_2 \otimes_K \cdots \otimes_K a_n) + (-1)^n a_n m \otimes_K a_1 \otimes_K a_2 \otimes_K \cdots \otimes_K a_{n-1}$$

Definition 2.26 (First Definition of Cyclic Homology) The homology of the complex $\{\frac{C_*(\mathscr{A})}{Im(1-T)}, b\}$ is the *cyclic homology* of the algebra $\mathscr{A}$, denoted $H_*^\lambda(\mathscr{A})$. The same degrees convention applies as in the Hochschild case.

The cyclic homology may be computed using a different complex, as specified by the next proposition.

Proposition 2.25

(i) The cyclic symmetrisation *induces an isomorphism of complexes*

$$N_* : \{\frac{C_*}{Im(1-T)}, b\} \longrightarrow \{Ker(1-T)_*, b'\}. \tag{2.186}$$

(ii) The mappings N_*^{-1} *defined by*

$$N_n^{-1}(x) \cong \frac{1}{n+1}x \cong \frac{1}{(n+1)^2}Nx \qquad (mod. \quad Im(1-T)). \tag{2.187}$$

induce the chain homomorphism

$$N_*^{-1} : \{Ker(1-T)_n, b'\} \longrightarrow \{\frac{C_n}{Im(1-T)_n}, b\}, \tag{2.188}$$

inverse to the chain isomorphism N_* *given by (2.186).*

This result justifies the next definition.

Definition 2.27 (Second Definition of Cyclic Homology)

$$H_n^{'\lambda}(A) := H_n\{\frac{C_*}{Ker(1-T)}, b'\}. \tag{2.189}$$

Remark 2.12 The first definition of cyclic homology is used in the literature more frequently than the second, see [82] and [97, 2.1.4, pg. 54].

In this section only, to distinguish the second definition of cyclic homology from the first one, we have added an accent in its notation.

Proof Relations (2.186) say that N_n is a chain homomorphism before factorisation. As $N(1-T) = 0$, N passes to the quotient of the domain and furthermore N induces a chain homomorphism given by (2.186).

Now we check that N_*^{-1} induce a chain homomorphism; the other statements are clear. Let $x \in Ker(1-T)_n$. We apply the decomposition (2.152) onto the element bx and we recall (2.157) $Nb = b'N$; we get

$$bx = \frac{1}{n}Nbx + (1-T)\tilde{J}_{n-1}^{\perp}bx \tag{2.190}$$

or

$$bx = \frac{1}{n}b'Nx + (1-T)\tilde{J}_{n-1}^{\perp}bx = \frac{n+1}{n}b'x + (1-T)\tilde{J}_{n-1}^{\perp}bx. \tag{2.191}$$

We recall that $b'(Ker(1 - T)) \subset Ker(1 - T)$. The relation (2.152) applied onto bx is

$$b\frac{1}{n+1}x = \frac{1}{n}b'x + (1 - T)\frac{1}{n+1}\tilde{J}^{\perp}_{n-1}b\,x \tag{2.192}$$

or

$$b\,N^{-1}x = N^{-1}b'\,x + (1 - T)\frac{1}{n+1}\tilde{J}^{\perp}_{n-1}b\,x, \tag{2.193}$$

which completes the proof.

Remark 2.13 It is important to note that b' transforms cyclic invariant chains into cyclic invariant chains while b does not. The defect to the cyclic invariance of b is provided by the above relation

$$b\,x - \frac{n+1}{n}b'\,x = (1 - T)\tilde{J}^{\perp}_{n-1}b\,x. \tag{2.194}$$

However,

$$b(\frac{C_n}{(1 - T)C_n}) \subset \frac{C_{n-1}}{(1 - T)C_{n-1}}. \tag{2.195}$$

2.4.5 Connes' Exact Sequence

We rewrite the exact sequence (ES_1) (2.163) using the first definition of cyclic homology; we use Definitions 2.25, 2.26 and (2.186) to get the exact sequence

$$\ldots \xrightarrow{\delta_{n+1}} H'_n(\frac{C_*}{Ker(1-T)}) \xrightarrow{(1-T)_*} H_n \xrightarrow{\Pi_*} H^{\lambda}_n \xrightarrow{\delta_n} H'_{n-1}(\frac{C_*}{Ker(1-T)}) \xrightarrow{(1-T)_*} H_{n-1} \xrightarrow{\Pi_*} \ldots. \tag{2.196}$$

Recalling that $\delta_n : H'_n(\frac{C_*}{Ker(1-T)}) \longrightarrow H'^{\lambda}_{n-1}$ are isomorphisms, see formula (2.177), the exact sequence above involves the Hochschild homology and the cyclic homology expressed in the H^{λ}_*-form or in the H'^{λ}_*-form. Depending on which description of the cyclic homology we choose, we obtain two different forms of the Connes' exact sequence (Ces), which we refer to as the $Ces - b$, $Ces - b'$, respectively

$$(Ces - b) \equiv \quad \cdots \xrightarrow{\delta_{n+1}} H^{\lambda}_{n-1} \xrightarrow{B_{n-1}} H_n \xrightarrow{I_n} H^{\lambda}_n \xrightarrow{S_n} H^{\lambda}_{n-2} \xrightarrow{B_{n-2}} H_{n-1} \xrightarrow{I_{n-1}} \ldots \tag{2.197}$$

and

$$(Ces - b') \equiv \quad \ldots \xrightarrow{\delta_{n+1}} H'^{\lambda}_{n-1} \xrightarrow{B'_{n-1}} H_n \xrightarrow{I'_n} H'^{\lambda}_n \xrightarrow{S'_n} H'^{\lambda}_{n-2} \xrightarrow{B'_{n-2}} H_{n-1} \xrightarrow{I'_{n-1}} \ldots \quad (2.198)$$

Keeping track of the morphisms involved in the definition of the homomorphisms B, I and S we are able to write down explicitly their expressions.

We study first (2.197). The composition of isomorphisms

$$H'_n(\frac{C_*}{Ker(1-T)}) \xrightarrow{\delta'_n} H'^{\lambda}_{n-1} \xrightarrow{N_{*,n-1}} H^{\lambda}_{n-1} \quad (2.199)$$

allows one to replace $H'_n(\frac{C_*}{Ker(1-T)})$ by H^{λ}_{n-1}. After this replacement, the homomorphism S_{n+1} becomes

$$S_{n+1} := N_{*,n-1} \circ \delta'_n \circ \delta_{n+1} : H^{\lambda}_{n+1} \longrightarrow H^{\lambda}_{n-1} \quad (2.200)$$

while the homomorphism $(1-T)_{*,n}$ becomes

$$B_{n-1} := (\delta'_n)^{-1} \circ N^{-1}_{*,n-1} : H^{\lambda}_{n-1} \longrightarrow H_n. \quad (2.201)$$

The replacement keeps the homomorphism $\Pi_{*,n}$ unchanged; this is going to be denoted by I_n.

Theorem 2.9 *In the Connes' exact sequence (2.197) the homomorphisms S, B and I are*

(i)

$$S_{n+1} := N_{*,n-1} \circ \delta'_n \circ \delta_{n+1} : H^{\lambda}_{n+1} \longrightarrow H^{\lambda}_{n-1} \quad (2.202)$$

(ii)

$$B_{n-1} := (1-T)_{*,n-1}(\delta'_n)^{-1} \circ N^{-1}_{*,n-1} : H^{\lambda}_{n-1} \longrightarrow H_n. \quad (2.203)$$

(iii)

$$I_n := \Pi_{*,n} : H_n \longrightarrow H^{\lambda}_n. \quad (2.204)$$

Although the homomorphisms S, B, I were expressed as homomorphisms relating various *homologies*, they are given by well defined formulas valid at the level of *chains*.

Theorem 2.10 *The homomorphisms S, B and I, defined at the chains level, are given by the following formulas:*

$$S(x) = N \circ b' \circ \tilde{J}^{\perp} b(x) \quad (see\ (2.164)) \tag{2.205}$$

$$\tilde{B}_{n-1}(x) = \frac{1}{n^2}(1 - T)\, s\, N\, x \tag{2.206}$$

$$I_n(x) = \Pi_N(x) : C_n \longrightarrow \frac{C_n}{Im(1-T)_n}. \tag{2.207}$$

If the coefficients $1/n^2$ did not belong to the ground ring, the operator $\tilde{B}$ could be replaced by $n^2\tilde{B}$

$$B_{n-1}(x) = (1 - T)\, s\, N\, x. \tag{2.208}$$

Theorem 2.11 *The corresponding $I^{'}$, $B^{'}$, $S^{'}$ operators are*

$$S^{'}_p = \delta^{'}_{p-1*}\delta_{p*}N^{-1}_{p*} \quad (\text{ where } N^{-1}_{p*} \text{ is given by } (2.188)\ ). \tag{2.209}$$

$$B^{'} = (1 - T)_*(\delta^{'}_{p-1*})^{-1} \tag{2.210}$$

$$I^{'}_p = N_{p*}\pi_{p*}. \tag{2.211}$$

These operators may be defined at the chain level as in Theorem 2.10.

Theorem 2.12 (Comparison Theorem)
*Let Φ_*, Ψ_* be homomorphisms between 3-*periodic exact sequences *which make the following diagram commutative:*

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{B} & HH_n(\mathscr{A}) & \xrightarrow{I} & H^{\lambda}_n(\mathscr{A}) & \xrightarrow{S} & H^{\lambda}_{n-2}(\mathscr{A}) & \xrightarrow{B} & HH_{n-1}(\mathscr{A}) & \xrightarrow{B} & \cdots \\
 & & \downarrow{\scriptstyle \Phi_n} & & \downarrow{\scriptstyle \Psi_n} & & \downarrow{\scriptstyle \Psi_{n-2}} & & \downarrow{\scriptstyle \Phi_{n-1}} & & \\
\cdots & \xrightarrow{B^{'}} & U_n & \xrightarrow{I^{'}} & V_n & \xrightarrow{S^{'}} & V_{n-2} & \xrightarrow{B^{'}} & U_{n-1} & \xrightarrow{I^{'}} & \cdots
\end{array} \tag{2.212}$$

Suppose $\Phi_n = \Psi_n = 0$ for $n < 0$.
Suppose also that Φ_p, $0 \leq p$, are isomorphisms.
Then Ψ_p are isomorphisms.

Proof Use five lemma.

2.4.6 Chern–Weil Characteristic Classes of Direct Connections

The geometrical construction of the Chern character presented in Sect. 1.6.5 may be reformulated using *direct connections*, see [117, 121, 122].

2.4.7 Non-localised Alexander–Spanier Complex

Definition 2.28 Let $\mathscr{A}$ be a unital algebra. The *non-localised Alexander–Spanier complex* associated with the algebra $\mathscr{A}$ is

$$C_*^{AS}(\mathscr{A}) \ := \ \{C_*(\mathscr{A}) \ , \ d \ \}, \tag{2.213}$$

where the co-boundary $d : C_k(\mathscr{A}) \longrightarrow C_k(A)$ is defined by

$$d \ = \ \sum_{0 \leq r \leq k+1} (-1)^r \ d_r, \tag{2.214}$$

where

$$d_r(\ a_0 \otimes_K a_1 \otimes_K \cdots \otimes_K a_k) \ := \ \ a_0 \otimes_K \ldots a_{r-1} \otimes_K 1 \otimes_K a_k \cdots \otimes_K a_k. \tag{2.215}$$

It is clear that $d^2 \ = \ 0$.

A Word of Caution The reader should distinguish between the non-localised Alexander–Spanier complex $C_{AS}^*(M, G)$, associated to a CW-complex M, defined in Sect. 1.3 and the Alexander–Spanier complex $C_*^{AS}(\mathscr{A})$, associated to an associative algebra $\mathscr{A}$, defined here. There are two basic differences between these two definitions. Supposing that G is an algebra, one may define $\mathscr{A} = \{f \mid f : M \longrightarrow G\}$. However, the chains in $C_*^{AS}(\mathscr{A})$ are finite sums of tensor products of functions in $\mathscr{A}$, while in $C_{AS}^*(M, G)$ all functions $f : M^k \longrightarrow G$ are allowed. However, the main difference between the two complexes consists of the fact that in the complex $C_*^{AS}(\mathscr{A})$ the *localisation* at the diagonal is not done.

Recall that $C_k(\mathscr{A})$ is an $\mathscr{A}$ bi-module through the left and right multiplications by α and β given by

$$\alpha(a_0 \otimes a_1 \otimes \cdots \otimes a_k)\beta := \ (\alpha a_0) \otimes a_1 \otimes \cdots \otimes (a_k \beta). \tag{2.216}$$

Definition 2.29 For any $a \in A$ define the *differential* of a

$$da \ = \ 1 \otimes_{\mathbf{K}} a - a \otimes_{\mathbf{K}} 1 \in A \otimes_K A. \tag{2.217}$$

Remark 2.14 If the elements of the algebra $\mathscr{A}$ were *functions* on some space X, df would represent the *difference function* $df(x, y) = f(y) \ - \ f(x)$, for any $x, y \in X$.

Proposition 2.26

(i) da is the AS boundary of $a \in C_0(\mathscr{A})$.
(ii) For any $a, \ b \in A$ one has $d(a.b) \ = \ (da).b \ + \ a.(db)$, where the products refer to the bi-module structure of the space of chains of degree 1.

(iii) *For any* $c \in \mathbf{K}$, $c.1_{\mathscr{A}}$ *could be seen as a* constant *of the algebra A; in fact,* $d(c.1_{\mathscr{A}}) = 0$. *In particular, for any* $c \in \mathbf{K}$ *and* $a \in \mathscr{A}$, *one has* $d(c.a) = c.(da)$ *and* $d(a.c) = (da).c$.

(iv) *For any* $a_1, a_2 \in \mathscr{A}$, *one has* $d(a_1.a_2) = (da_1).a_2 + a_1.(da_2)$.

Proof One has

$$d(c.1_{\mathscr{A}}) = 1_{\mathscr{A}} \otimes_{\mathbf{K}} c.1_A - c.1_{\mathscr{A}} \otimes_{\mathbf{K}} 1_{\mathscr{A}} = 1_{\mathscr{A}} \otimes_{\mathbf{K}} c.1_A - 1_A.c \otimes_{\mathbf{K}} 1_{\mathscr{A}} = 0. \tag{2.218}$$

We leave to the reader to check the other relations.

Definition 2.30 Define the *product in* $C_*^{AS}(\mathscr{A})$

$$\times : C_r(\mathscr{A}) \otimes C_s(\mathscr{A}) \to C_{r+s}(\mathscr{A}). \tag{2.219}$$

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_r) \times (b_0 \otimes b_1 \otimes \ldots \otimes b_s) := a_0 \otimes a_1 \otimes \cdots \otimes (a_r b_0) \otimes b_1 \otimes \ldots \otimes b_s. \tag{2.220}$$

This product coincides with the classical product of co-chains in the Alexander–Spanier complex, before the localisation at the diagonal.

Proposition 2.27 *Let A be any unital associative algebra.*

(i) *The product in* $C_*^{AS}(\mathscr{A})$ *is associative.*
The complex $\mathscr{C}_*(\mathscr{A}) := \{\sum_{r=0}^{\infty} C_r(\mathscr{A}), d\}$ *is a graded differential complex: for any* $\omega \in C_r(\mathscr{A})$ *and* $\sigma \in C_s(\mathscr{A})$ *one has*

$$d(\omega \times \sigma) = (d\omega) \times \sigma + (-1)^r \omega \times (d\sigma). \tag{2.221}$$

2.4.8 Augmentation

Definition 2.31 Let $\mathscr{A}$ a unital associative algebra over the unitary commutative ring K. Let $1_{\mathscr{A}}$, resp. $1_{\mathbf{K}}$, be their units. We say that the algebra $\mathscr{A}$ has an *augmentation* if a $\mathbf{K}$-algebra homomorphism $u : \mathscr{A} \longrightarrow \mathbf{K}$ is given such that $u(1_{\mathscr{A}}) = 1_{\mathbf{K}}$.

Proposition 2.28

(i) *Define* $\iota : \mathbf{K} \longrightarrow \mathscr{A}$ *by* $\iota(k) := k.1_{\mathscr{A}} = 1_{\mathscr{A}}.k$. *Then, the mapping* ι *is injective and henceforth* $\mathbf{K}$ *may be thought as a sub-algebra of the algebra* $\mathscr{A}$.

(ii) *For any* $a \in \mathscr{A}$, *one has* $a - \iota\, u(a) \in Ker\ u$.

Proof

(i) For any $k \in \mathbf{K}$ one has

$$u\,\iota(k) \;=\; u\,(1_A.k) \;=\; u\,(1_A).k \;=\; 1_{\mathbf{K}}k \;=\; k.$$

Proposition 2.29

(i) With respect to the additive group structure of the algebra $\mathscr{A}$, one has the direct sum decomposition

$$\mathscr{A} \;=\; \mathbf{K} \oplus A/\mathbf{K}. \tag{2.222}$$

We denote $\mathscr{A}/\mathbf{K} := \bar{\mathscr{A}}$.

(ii) The multiplicative structure of $\mathscr{A}$ is given by

$$(k_1,\; a_1).(k_2,\; a_2) \;=\; (k_1k_2,\; k_1a_2 \;+\; a_1k_2 \;+\; a_1.a_2) \tag{2.223}$$

for $k_1, k_2 \in \mathbf{K}$ and $a_1, a_2 \in \bar{\mathscr{A}}$.

A unit will be denoted by 1.

Suppose the algebra $\mathscr{A}$ has an augmentation $\rho \;:\; A \;\longrightarrow\; K$. Then the augmentation ρ provides a K-modules splitting of the exact sequence

$$0 \longrightarrow \mathbf{K} \longrightarrow \mathscr{A} \longrightarrow \mathscr{A}/\mathbf{K} \longrightarrow 0. \tag{2.224}$$

From this it follows that one has the direct sum of K-modules

$$\mathscr{A} \;=\; \mathbf{K} \oplus \mathscr{A}/\mathbf{K}.$$

We saw that for any $c \in K$, $dc \;=\; 0$. Therefore, the differential $da, a \in A$ depends only on the class of a in A/K. Viceversa, if $da := 1 \otimes_K a \;-\; a \otimes_K 1 \;=\; 0$, then $a \in K$.

Definition 2.32 The class of a in $\mathscr{A}/\mathbf{K}$ is denoted $\bar{a}$.

Remark 2.15 If the algebra $\mathscr{A}$ is unitary we will use the unit as discussed above. Many authors prefer to add an external unit $\bar{1}$ even if the algebra $\mathscr{A}$ has one. In the extended algebra $\tilde{\mathscr{A}} := \mathbf{K}\bar{1} \oplus \mathscr{A}$ the multiplication is defined by the same formula: for $(k_1, a_1), (k_2, a_2) \in \mathbf{K}\bar{1} \oplus \mathscr{A}$

$$(k_1\bar{1},\; a_1).(k_2\bar{1},\; a_2) \;=\; (k_1k_2\bar{1},\; k_1a_2 \;+\; a_1k_2 \;+\; a_1.a_2). \tag{2.225}$$

The element $\bar{1}$ becomes a unit of the algebra $\tilde{\mathscr{A}}$; if the algebra $\mathscr{A}$ had a unit 1, this element would no longer be a unit in the extended algebra.

Therefore, *if an algebra does not have a unit, we may always add a unit to it. However, in this book we are not going to replace a unit of the algebra by an external unit, unless specific needs would not require to do so.*

Proposition 2.30 *Let $\mathscr{A}$ be a unital associative algebra and let $\rho : \mathscr{A} \to K$ be an augmentation of A.*

Then, the AS complex $\{C_*^{AS}(\mathscr{A}), d\}$ is acyclic.

Proof Define $h : C_r^{AS}(\mathscr{A}) \to C_{r-1}^{AS}(\mathscr{A})$, by the formula

$$h(a_0 \otimes a_1 \otimes \cdots \otimes a_r) \; = \; \rho(a_0)a_1 \otimes \cdots \otimes a_r. \tag{2.226}$$

It satisfies the identity

$$dh \; + \; hd \; = \; 1. \tag{2.227}$$

Remark 2.16 In the classical Alexander–Spanier complex, the augmentation homomorphism ρ is the valuation of functions at one point.

Definition 2.33 If the algebra $\mathscr{A}$ has a locally convex topology, it is natural, see Connes [82], to replace the *algebraic tensor product* used in the definition of $\mathscr{C}_i(\mathscr{A})$ by a *topological tensor product* completion to produce $\hat{\mathscr{C}}_i(\mathscr{A})$. The elements of $\hat{\mathscr{C}}_r(\mathscr{A})$ are called *continuous Alexander–Spanier co-chains*. More often the *projective tensor products* are used. The *continuous Hochschild and cyclic homologies* are defined with the same topological completion agreement.

Remark 2.17 For smooth manifolds M, in the particular case of the algebra $\mathscr{A} = C^\infty(M)$, endowed with the Fréchét topology and for the *projective tensor product* completion, the *continuous Alexander–Spanier co-chains* consist of all smooth functions on various powers of M. The homology of this complex is still acyclic; the same homotopy operator h may be used. In addition, if the complex of *continuous Alexander–Spanier chains* is replaced by the *complex of germs* of such functions about the diagonals, the classical Alexander–Spanier theorem, see Spanier [45], states that its homology is canonically isomorphic to the de Rham cohomology.

2.4.9 The Operator σ

Definition 2.34 The *Karoubi operator* σ [89] (see also other references [104, 115]) connects the Hochschild complex and the Alexander–Spanier complex; it is given by the formula

$$db \; + \; bd := \; 1 \; - \; \sigma. \tag{2.228}$$

The way the operator σ is defined insures that it commutes both with d and b. In addition, σ is a chain homomorphism both in the Alexander–Spanier and

in the Hochschild complex. The images of the operator σ, and of its powers, are sub-complexes of both complexes. Additionally, as σ is homotopic to the identity with respect to the Alexander–Spanier and Hochschild boundaries, the inclusions of these sub-complexes into the Alexander–Spanier, resp. Hochschild, complexes induce isomorphisms between their respective homologies.

Proposition 2.31

1. The operators b' and d satisfy the identity

$$b'\,d\;+\;d\,b'\;=\;0. \tag{2.229}$$

2. The action of σ onto $(f_0\otimes f_1\otimes\cdots\otimes f_k)$ is given by

$$\sigma(f_0\otimes f_1\otimes\cdots\otimes f_k)\;=\;(-1)^{k+1}((df_k).f_0)\otimes f_1\otimes\cdots\otimes f_{k-1}. \tag{2.230}$$

3. The iterated kth power of σ on $C_k(A)$ is given by the expression

$$\sigma^k(f_0\otimes f_1\otimes\cdots\otimes f_k)\;=\;df_1.df_2\ldots df_k.f_0. \tag{2.231}$$

Proof

1. The definition of the Alexander–Spanier and Hochschild boundaries give us

$$d\,b'\;=\;\sum_{0\le i\le j\le k-1}(-1)^{i+j}b_{j+1}d_i\;+\;\sum_{0\le j<i\le k}(-1)^{i+j}b_jd_{i+1}, \tag{2.232}$$

where d_i and b_j are, respectively, the Alexander–Spanier and Hochschild boundary faces.

We have also on the other hand

$$\begin{aligned} b'\,d\;=\;&\sum_{0\le i<j+1\le k}(-1)^{i+(j+1)}b_{j+1}d_i\;+\;\sum_{0\le j<i\le k}(-1)^{(i+1)+j}b_jd_{i+1}\;+\\ &\sum_{0\le j\,=\,i\le k}(-1)^{(j+j)}b_jd_j\;+\;\sum_{0\le j\le k,i=j+1}(-1)^{(j+j)}b_jd_{j+1}\;=\\ &\sum_{0\le i<j+1\le k}(-1)^{i+(j+1)}b_{j+1}d_i\;+\;\sum_{0\le j<i\le k}(-1)^{(i+1)+j}b_jd_{i+1}\;=\;-\,d\,b'. \end{aligned} \tag{2.233}$$

2. To avoid ambiguity we use (k) to indicate the degree of chains. A direct computation gives

$$\begin{aligned} &(-1)^{k+1}(b_{k+1}\,d\;-\;d\,b_k)(f_0\otimes f_1\otimes\cdots\otimes f_k)\;=\\ &f_0\otimes f_1\otimes\cdots\otimes f_k\;+\;(-1)^k((df_k)\,f_0)\otimes f_1\otimes\cdots\otimes f_{k-1}. \end{aligned} \tag{2.234}$$

We get

$$(1 - \sigma)(f_0 \otimes f_1 \otimes \cdots \otimes f_k) = f_0 \otimes f_1 \otimes \cdots \otimes f_k + (-1)^k((df_k) f_0) \otimes f_1 \otimes \cdots \otimes f_{k-1},$$

or

$$\sigma(f_0 \otimes f_1 \otimes \cdots \otimes f_k) = (-1)^{k+1}((df_k).f_0) \otimes f_1 \otimes \cdots \otimes f_{k-1}. \quad (2.235)$$

This completes the proof of part 2.

We may compute the kth power of σ

$$\sigma^k (f_0 \otimes f_1 \otimes \cdots \otimes f_k) = df_1.df_2 \ldots df_k.f_0. \quad (2.236)$$

The proposition is proved.

Definition 2.35 Let Π_k be given by the operator

$$\Pi_{(k)} := (1 - b\, d)\, \sigma_{(k)}^k. \quad (2.237)$$

Theorem 2.13

1. *The $\Pi_{(k)}$ is given by the formula*

$$\Pi_{(k)} (f_0 \otimes f_1 \otimes \cdots \otimes f_k) = f_0\ df_1 \ldots . df_k. \quad (2.238)$$

2. *The operator $\Pi_{(k)}$ is an* idempotent on $C_k(\mathscr{A})$, *($(\Pi_{(k)})^2 = \Pi_{(k)}$).*
3. *The idempotent $\Pi_{(k)}$ commutes with the operators d and b; therefore, d and b keep the ranges of the operator $\Pi_{(k)}$ invariant*

$$d\ \Pi_{(k)} = \Pi_{(k+1)}\ d \quad (2.239)$$

$$b\ \Pi_{(k)} = \Pi_{(k-1)}\ b. \quad (2.240)$$

Remark 2.18 Distinguish $f_0\ df_1 \ldots . df_k$, defined here, from the same notation used by Loday [97, §1.3.11] to denote the image of this element through the anti-symmetrisation operator ϵ_k, or $f_0\, df_1. \wedge \cdots \wedge .df_k$. In both cases, the products are over the algebra $\mathscr{A}$.

Proof

1. The definition of the operator $\Pi_{(k)}$ (2.238) and the formula (2.228) give

$$\Pi_{(k)}(f_0 \otimes f_1 \otimes \cdots \otimes f_k) = (1 - b\, d)\, df_1 \ldots . df_k\ f_0 =$$

$$df_1 \ldots . df_k\ f_0 - (-1)^k b\ (df_1\ \ldots\ df_k\ df_0) =$$

$$df_1\ \ldots . df_k\ f_0 - (-1)^k b\ [df_1 \ldots . df_k\ (1 \otimes f_0 - f_0 \otimes 1)] = f_0\, df_1\ \ldots\ df_k;$$

here we have used the fact that $b_i(1 \otimes f_{i-1} - f_{i-1} \otimes 1) = 0$, for $0 \leq i \leq k-1$. Part 1 is proven.

2. The reader could verify easily that

$$\begin{aligned} \Delta(f_0 \otimes f_1 \otimes \cdots \otimes f_k) := (1 - \Pi_{(k)})(f_0 \otimes f_1 \otimes \ldots \otimes f_k) = \\ f_0 \otimes f_1 \otimes \ldots \otimes f_k - f_0\, df_1 \ldots df_k. \end{aligned} \tag{2.241}$$

Definition 2.36 Chains $f_0 \otimes f_1 \otimes \cdots \otimes f_k$ in which at least one of the factors f_i belongs to $\mathbf{K}$, $1 \leq i \leq k$, is called a *degenerate chain*.

3. The relation $d\ 1 = 0$ and the expression of $\Pi_{(k)}$ imply that $\Pi_{(k)}$ carries degenerate chain into zero. In other words

$$\Pi_{(k)}\, \Delta = \Pi_{(k)}\, (1 - \Pi_{(k)}) = 0. \tag{2.242}$$

This shows again that $\Pi_{(k)}$ is an idempotent. Additionally, this argument says that the range of the operator $1 - \Pi_{(k)}$ is a subset of the space of degenerate chains. On the other hand, any degenerate chain belongs to the kernel of $\Pi_{(k)}$, or to the range of $1 - \Pi_{(k)}$. This double inclusion implies that the set of degenerate chains coincides with the range of the complementary idempotent $1 - \Pi_{(k)}$.

4. The identity 1 already proven implies

$$\begin{aligned} d\ \Pi_{(k)}(f_0 \otimes f_1 \otimes \cdots \otimes f_k) = d(f_0.df_1 \ldots df_k) = \\ d\ f_0\ df_1\ \ldots\ df_k = 1.df_0\,.\,df_1\ \ldots\ df_k = \\ \Pi_{(k+1)}\, d_0\, (f_0 \otimes f_1 \otimes \cdots \otimes f_k) = \Pi_{(k+1)}\, d\, (f_0 \otimes f_1 \otimes \cdots \otimes f_k) \end{aligned} \tag{2.243}$$

because $\Pi_{(k+1)}\, d_i\ \ (f_0 \otimes f_1 \otimes \cdots \otimes f_k) = 0$ for any $1 \leq i \leq k+1$.

5. On the other hand

$$\begin{aligned} b\, \Pi_{(k)} = b\, (1 - bd)\sigma^k_{(k)} = b\ \ (1 - bd)(1 - bd - db)\sigma^{k-1}_{(k)} = b\ \ (1 - bd - db)\sigma^{k-1}_{(k)} = \\ b\ \ (1 - db)\sigma^{k-1}_{(k)} = (1 - bd)b\ \ \sigma^{k-1}_{(k)} = (1 - bd)\, \sigma^{k-1}_{(k-1)}\ \ b = \Pi_{(k-1)}\, b. \end{aligned} \tag{2.244}$$

Definition 2.37

1. By definition the *normalised Hochschild complex*, resp. *normalised Alexander–Spanier complex*, is

$$\begin{aligned} \tilde{C}_k(\mathscr{A}) \quad := \quad \Pi_{(k)}(C_k(\mathscr{A})) = \\ \{ \sum_{finite/series} f_0\ \ df_1.df_2.\ldots.df_k, \quad f_i \in \mathscr{A}/\mathbf{K},\ 1 \leq i \leq k \}; \end{aligned} \tag{2.245}$$

2. The range of the operator Π is the *universal algebra*.
3. The *complex of degenerate elements* of the Hochschild complex, resp. of the Alexander–Spanier complex, consists of the chains

$$\tilde{C}^0_k(\mathscr{A}) \quad := \quad \{finite\ sums/series\ of\ degenerate\ elements\ of\ degree\ k.\} \tag{2.246}$$

For any $a \in \mathscr{A}$, denote by $\bar{a}$ its image in the non-degenerated sub-complex.

Proposition 2.32

1. *The complexes* $\{C_*(\mathscr{A}), d\}$, $\{C_*(\mathscr{A}), b\}$ *decompose in direct sum of complexes*

$$\{C_*(\mathscr{A}), d\} \equiv \{\tilde{C}_*(\mathscr{A}), d\} \oplus \{C^0_*(\mathscr{A}), d\}. \tag{2.247}$$

$$\{C_*(\mathscr{A}), b\} \equiv \{\tilde{C}_*(\mathscr{A}), b\} \oplus \{C^0_*(\mathscr{A}), b\}. \tag{2.248}$$

2. *The homology of the sub-complex of* $\{\tilde{C}_*(\mathscr{A}), d\}$ *and the homology of the sub-complex of degenerate Alexander–Spanier chains* $\{C^0_*(\mathscr{A}), d\}$ *are trivial*

$$H_*\{\tilde{C}_*(\mathscr{A}), d\} = H_*\{C^0_*(\mathscr{A}), d\} = 0. \tag{2.249}$$

3. *The homology of the normalised Hochschild complex* $\{\tilde{C}_*(\mathscr{A}), b\}$ *and the homology of the complex of degenerate Hochschild chains,* $\{C^0_*(\mathscr{A}), b\}$, *are*

$$H_k\{\tilde{C}_*(\mathscr{A}), b\} = H_k(\mathscr{A}), \quad H_*\{C^0_*(\mathscr{A}), b\} = 0. \tag{2.250}$$

4. *Any chain of degree k of the normalised Hochschild complex* $\{\tilde{C}_*(\mathscr{A}), b\}$ *is a finite sum/series, of elements of the form*

$$a_0.da_1.\ldots.da_k, \tag{2.251}$$

where d is the Alexander–Spanier boundary operator and the product is performed in the same graded differential complex.
5. *On the sub-complex* $\tilde{C}_k(\mathscr{A})$ *one has the identity*

$$(1 - b\,d)\,\sigma^k_{(k)} = 1. \tag{2.252}$$

Proof The decomposition of Part 1 follows from the fact that Π is an idempotent. The two sub-complexes are the ranges of the idempotents Π and $1 - \Pi$.

Part 2 and Part 3 follow from the acyclicity of the non-local Alexander–Spanier complex, Proposition 2.30 along with Part 1. and the next lemma.

Lemma 2.2 *The operator* $h : C_k^0(\mathscr{A}) \longrightarrow C_{k+1}^0(\mathscr{A})$

$$h(a_0 \otimes a_1 \otimes \cdots \otimes a_k) \;=\; (-1)^i (a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_k), \qquad (2.253)$$

where $a_i \;=\; 1$, $1 \leq i \leq k$, *is the first factor of the tensor product equal to* 1, *satisfies the identity*

$$h.b \;+\; b.h \;=\; 1 \;-\; h. \qquad (2.254)$$

We leave to the reader the task to check the identity.

Part (v) follows from the fact that $(1 \;-\; bd)\sigma_{(k)}^k \;:=\; \Pi_k$ is an idempotent.

2.4.10 Universal Graded Differential Algebra

The next proposition summarises the basic properties of the idempotents Π_k.

Proposition 2.33 $\Pi_k : C_k(\mathscr{A}) \longrightarrow C_k(\mathscr{A})$ *has the following properties*

$$\Pi_k \;=\; (1 \;-\; b\,d)\;\sigma^k, \qquad (2.255)$$

where b = *Hochschild boundary and* d = *Alexander–Spanier boundary*

$$\Pi_k^2 \;=\; \Pi_k \qquad (2.256)$$

$$Im\;\Pi_k \;=\; space\;of\;non\;degenerate\;chains \qquad (2.257)$$

$$Im\;(1 \;-\; \Pi_k) \;=\; space\;of\;degenerate\;chains \qquad (2.258)$$

$$\Pi_k\;(a_0 \otimes a_1 \otimes \cdots \otimes a_k) \;=\; a_0.da_1.\ldots.da_k, \qquad (2.259)$$

where dots represent products relative to the $\mathscr{A}$*-bi-module structure and*

$$da_i \;=\; 1 \otimes a_i \;-\; a_i \otimes 1 \;=\; 1 \otimes \bar{a}_i \;-\; \bar{a}_i \otimes 1,$$

with $\bar{a}_i \;=\; [a_i] \in \mathscr{A}/\mathbf{K}$*; therefore,* $a_0.da_1.\ldots.da_k$ *belongs to the image of* Π_k *and, as* Π_k *is an idempotent, one has*

$$\Pi_k(\;a_0.da_1.\ldots.da_k\;) \;=\; a_0.da_1.\ldots.da_k, \qquad (2.260)$$

$$\Pi_{k-1}\;b \;=\; b\;\Pi_k \qquad (2.261)$$

$$\Pi_{k+1}\;d \;=\; d\;\Pi_k \qquad (2.262)$$

$$H_k\{Im\,\Pi_*,\ b\} \ = \ H_k\{C_*(\mathscr{A}),\ b\} \ = \ Hochschild\ homology \tag{2.263}$$

$$H_k\{Im\,\Pi_*,\ d\} \ = \ H_k\{C_*(\mathscr{A}),\ d\} \ = \ (non-localised)\ Alexander{-}Spanier\ homology \ = \ 0. \tag{2.264}$$

Definition 2.38 Let $\mathscr{U}_*(\mathscr{A}) := \oplus_{0\leq k}\ Im\,\Pi_k$.

Proposition 2.34

(i) For any $1 \leq k$ *and* $a_0, a_1, \ldots, a_{k+1} \in \mathscr{A}$ *one has*

$$da_0.da_1 \ldots da_k\ .\ a_{k+1} \ =$$
$$(-1)^{k+1} a_0.da_1.\ldots.da_{k+1} \ + \ \sum_{0\leq r\leq k} (-1)^{k-r} da_0.\ldots.da_{r-1}.d(a_r.a_{r+1}).\ldots.\ da_{k+1}. \tag{2.265}$$

(ii) $\mathscr{U}_*(\mathscr{A})$ *is a graded differential algebra with respect to the Alexander–Spanier boundary and the Alexander–Spanier product.*

Proof

(i) An easy computation shows

$$\sum_{0\leq r\leq k} (-1)^r da_0.\ldots.da_{r-1}.d(a_r.a_{r+1}).\ldots.da_{k+1} =$$
$$\sum_{0\leq r\leq k} (-1)^r da_0.\ldots.da_{r-1}.((da_r).a_{r+1} \ + \ a_r(da_{r+1})).\ldots.da_{k+1} \ =$$
$$a_0.da_1.\ldots.da_{k+1} \ + \ (-1)^k da_0.da_1.\ldots.da_k\ .\ a_{k+1}. \tag{2.266}$$

From this it follows that

$$da_0.da_1 \ldots da_k\ .\ a_{k+1} \ =$$
$$(-1)^{k+1} a_0.da_1 \ldots da_{k+1} \ + \ \sum_{0\leq r\leq k} (-1)^{r-k} da_0 \ldots da_{r-1} d(a_r.a_{r+1}) \ldots da_{k+1}. \tag{2.267}$$

(ii) The Alexander–Spanier product of the chains

$$a_0.da_1.\ldots.da_p \ \times \ b_0.db_1.\ldots.db_q \tag{2.268}$$

is

$$a_0.da_1.\ldots.da_p\ .\ b_0\ .\ db_1.\ldots.db_q. \tag{2.269}$$

Part (i) says that this is an element of $\mathscr{U}_{p+q}$. It is sufficient to bring the factor b_0 to the left by incorporating the same factor into a linear combination of terms which depend of $a_0, a_1 \ldots a_p, b_0$ and which terms belong to $\mathscr{U}_*(\mathscr{A})$.

The formulas (2.265) and (2.265) say two things: (1) the Alexander–Spanier right multiplication of elements of $\mathscr{U}_*(\mathscr{A})$ by elements of the algebra $\mathscr{A}$ give elements of $\mathscr{U}_*(\mathscr{A})$, (2) the formula provides a procedure to incorporate right factors like a_{k+1} into a linear combination of elements belonging to $\mathscr{U}_*(\mathscr{A})$.

More generally, one has the following result:

Proposition 2.35 *For any* $a_0\ ,\ b_0 \in \mathscr{A}$ *and* $a_1, \ldots, a_k,\ b_1, \ldots, b_k \in \mathscr{A}/\mathbf{K}$, *one has*

$$a_0\ .\ b_0\ .\ da_1\ .\ b_1\ .\ da_2\ .\ b_2 \ldots da_k\ .\ b_k \in \mathscr{U}_k(\mathscr{A}). \tag{2.270}$$

Proof We use Proposition 2.35 applied on the factor b_1: we incorporate it into linear combinations of terms which depend of $a_0, a_1 \ldots a_p, b_0$ and which terms belong to $\mathscr{U}_*(\mathscr{A})$. The same procedure will be applied then onto the factors $b_2, b_3, \ldots, b_q$.

Definition 2.39 The graded differential algebra $\mathscr{U}_*(\mathscr{A})$ is called *universal graded differential algebra* of $\mathscr{A}$.

Proposition 2.36 *The boundary operator d acts on an arbitrary chain of* $\mathscr{U}_*(\mathscr{A})$ *by the formula*

$$d\ (\ a_0.(da_1).(da_2).\ldots.(da_k)\ )\ =\ (da_0).(da_1).(da_2).\ldots.(da_k). \tag{2.271}$$

Proposition 2.37

1. *For any two chains* $\alpha \in C_p(\mathscr{A})$ *and* $\beta \in C_q(\mathscr{A})$

$$\Pi_{p+q}(\alpha.\beta) \neq \Pi_p(\alpha).\Pi_q(\beta).$$

2. *However, for* $\alpha, \beta \in \mathscr{U}_*(\mathscr{A})$ *one has*

$$\Pi_{p+q}(\alpha.\beta)\ =\ \Pi_p(\alpha).\Pi_q(\beta). \tag{2.272}$$

Proof 1. Let $\alpha\ =\ a_0 \otimes 1$, with $a_0 \neq 0$ and $\beta\ =\ b_0 \otimes b_1$, with b_0, b_1 not belonging to the sub-algebra $\mathbf{K}$. Then,

$$\Pi_2(\alpha.\beta)\ =\ \Pi_2(a_0 \otimes 1.b_0 \otimes b_1)\ =\ a_0 db_0.db_1 \neq 0, \quad and \tag{2.273}$$

$$\Pi_1(\alpha).\Pi_1(\beta)\ =\ (a_0.d1).(b_0.db_1)\ =\ 0.(b_0.db_1)\ =\ 0. \tag{2.274}$$

2. The hypothesis says that $\Pi_p(\alpha)\ =\ \alpha$, $\Pi_q(\beta)\ =\ \beta$ and $\Pi_{p+q}(\alpha.\beta)\ =\ \alpha.\beta$. Therefore, $\Pi_{p+q}(\alpha.\beta)\ =\ \alpha.\beta\ =\ \Pi_p(\alpha).\Pi_q(\beta)$.

Proposition 2.38 *Let* $\Omega^0_1 := Ker\ \mu$, *where* $\mu : \mathscr{A} \otimes_{\mathbf{K}} \mathscr{A} \longrightarrow \mathscr{A}$ *is the multiplication in the algebra* $\mathscr{A}$.

Then $\mathscr{U}_p(\mathscr{A}) \equiv \Omega^0_1 . \ldots . \Omega^0_1$ (p *factors*), *modulo degenerate chains.*

Proof Any element $\gamma \in Ker\ \mu$ is $\gamma = \sum_i a_i.b_i = 0$. Then

$$\sum_i a_i \otimes b_i = \sum_i a_i.(1 \otimes_{\mathbf{K}} b_i - b_i \otimes_{\mathbf{K}} 1) = \sum_i a_i.db_i \quad modulo\ degenerated\ chains.$$

For $\sum_i a_i\ .\ b_i \in Ker\ \mu$ define

$$i(\ (a_1 \otimes_{\mathbf{k}} b_1).{}_{\mathscr{A}} \ldots .{}_{\mathscr{A}}(a_k \otimes_{\mathbf{K}} b_k)\ ) := (a_1.db_1).{}_{\mathscr{A}} \ldots .{}_{\mathscr{A}}(a_k.db_k), \tag{2.275}$$

which shows that $(\Omega^0_1)^p$ is contained in $\mathscr{U}_p$, modulo degenerate chains.

Viceversa, any element of the form $a.db$ belongs to $Ker\ \mu$; then any element $a_0.da_1.\ldots.da_k \in \mathscr{U}_k$ may be written

$$a_0.(da_1)\ldots(da_k) := (a_0.da_1).{}_{\mathscr{A}} \ldots .{}_{\mathscr{A}}(da_k) \in (Ker\ \mu).{}_{\mathscr{A}} \ldots .{}_{\mathscr{A}} (Ker\ \mu), \tag{2.276}$$

which shows that $\mathscr{U}_k$ is a subset of $(\Omega^1_0)^k$, modulo degenerate chains.

Proposition 2.39 *If the algebra* $\mathscr{A}$ *has an augmentation* $\rho : A \longrightarrow \mathbf{K}$, *then the graded universal differential algebra* $\mathscr{U}_*(\mathscr{A})$ *is acyclic.*

Proof The complex $\mathscr{U}_*(\mathscr{A})$ has a homotopy operator

$$s\ (\ a_0.(da_1).(da_2).\ldots.(da_k)\ ) := (-1)^k a_0.(da_1).(da_2).\ldots.(da_{k-1})\ .\ \rho(a_k),\ with\ \rho(a_k) = \alpha \in \mathbf{K}. \tag{2.277}$$

with

$$s\ d\ +\ d\ s\ =\ 1 \quad in\ positive\ degrees \tag{2.278}$$

and

$$s\ (a_0) = 0\ \ in\ degree\ 0. \tag{2.279}$$

The *homotopy operator* is the valuation function at one point in the non-local Alexander–Spanier complex, see Sect. 2.4.7.

2.4.11 Non-commutative de Rham Homology and Cyclic Homology

For this topic we refer to Karoubi [89]; for more information see Loday [97].

Definition 2.40 Let $\mathscr{A}$ be an augmented associative algebra over the commutative ring $\mathbf{K}$. Let $\mathscr{U}_*(\mathscr{A})$ be its universal graded differential algebra. Define

$$\bar{\mathscr{U}}_*(\mathscr{A}) := \mathscr{U}_*(\mathscr{A}) \,/\, [\mathscr{U}_*(\mathscr{A}), \mathscr{U}_*(\mathscr{A})], \tag{2.280}$$

where $[-,-]$ denotes the *graded commutator*

$$[\alpha_p,\ \beta_q] := \alpha_p \,.\, \beta_q - (-1)^{p.q}\ \beta_q \,.\, \alpha_p. \tag{2.281}$$

Recall the projection mapping

$$\tau : \mathscr{U}_*(\mathscr{A}) \longrightarrow \bar{\mathscr{U}}_*(\mathscr{A}) \tag{2.282}$$

it was called *graded trace*.

For $k = 0$, *by definition* $[\ \mathscr{U}_*(\mathscr{A}), \mathscr{U}_*(\mathscr{A})\ ] := \mathbf{K} + [\ \mathscr{A}, \mathscr{A}\ ]$.

Define $H_k^{dR}(\mathscr{A}); = \ H_k\{\bar{\mathscr{U}}_*(\mathscr{A}),\ d\}$; it is called *reduced non-commutative de Rham homology* of the algebra $\mathscr{A}$.

2.4.12 Reduced Cyclic Homology

In this sub-section we introduce *reduced cyclic homology*, see Karoubi [89, 2.9] and Loday [97, §2.2.13].

Recall the definition of the cyclic complex

$$C_*^\lambda(\mathscr{A}) := \{C_p(\mathscr{A}) \,/\, (1 - T)C_p(\mathscr{A}),\ \ b\ \}_{0\le p}. \tag{2.283}$$

Definition 2.41 Let $\bar{C}_p^\lambda(\mathscr{A})$ be the quotient of the module $C_p^\lambda(\mathscr{A})$ by the sub-module generated by all chains which have at least one factor 1 placed in *any* position i, $0 \le i \le p$. The boundary operator b passes to the quotient.

The complex $\bar{C}_p^\lambda(\mathscr{A})$ with boundary b is called *reduced cyclic complex*; its homology is called *reduced cyclic homology*, denoted $\bar{H}_p^\lambda(\mathscr{A})$.

There is an isomorphism of complexes

$$a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_p \longrightarrow a_0 \,.\, d\bar{a}_1 \ldots d\bar{a}_p. \tag{2.284}$$

Proposition 2.40 *If the algebra $\mathscr{A}$ has an augmentation, then*

$$\bar{C}_p^\lambda(\mathscr{A}) \ = \ C_p^\lambda(\mathscr{A}/\mathbf{K}). \tag{2.285}$$

Proof Note that, according to the definition of $\bar{C}_p^{\lambda}(\mathscr{A})$, the elements $N(1 \otimes a_1 \otimes \cdots \otimes a_p)$ represent the zero element; recall that N is the cyclic symmetrisation operator.

Theorem 2.14 (Karoubi [89, Theorem 2.15]) *For any augmented associative algebra $\mathscr{A}$ the reduced non-commutative de Rham homology of the algebra is*

$$\bar{H}_p^{dR}(\mathscr{A}) \;=\; Ker\ B, \tag{2.286}$$

where $B : \bar{H}_p^{\lambda}(\mathscr{A}) \longrightarrow \bar{H}_p(\mathscr{A})$ is the Connes homomorphism (2.208).

The main difficulty in proving this result consists of the fact that the non-commutative de Rham homology is defined in a *cohomologic* complex while its counterpart, the cyclic homology, is defined in a *homologic* complex. The proof is based on the operator σ, which connects the homologic and cohomologic environments. We will show that the operator σ is closely related to the cyclic permutation operator T.

We pass to the proof of the theorem.

Proposition 2.41 *For any elements $a_0, a_1, \ldots a_k, a_{k+1} \in \mathscr{A}$ one has*

$$b\ (\ a_0\ .\ da_1 \ldots da_{k+1}\ )\ =\ (-1)^k\ [a_0\ .\ da_1 \ldots da_k\ ,\ a_{k+1}] \tag{2.287}$$

i.e.

$$Im\ b_{|\mathscr{U}_*} \subset [\mathscr{U}_*,\ \mathscr{U}_*]. \tag{2.288}$$

Proof One has

$$b\ (\ a_0\ .\ da_1 \ldots da_k.da_{k+1}\ )\ =\ b\ (\ a_0\ .\ da_1 \ldots da_k.(1 \otimes a_{k+1} - a_{k+1} \otimes 1)\ ) =$$
$$(-1)^{k+1}\ .\ (\ a_{k+1}\ .\ a_0\ .\ da_1\ \ldots\ da_k\ -\ a_0\ .\ da_1\ \ldots\ da_k\ .\ a_{k+1}\ ). \tag{2.289}$$

Recall the expression of the operator σ acting on an arbitrary chain

$$\sigma(\ a_0\ \otimes\ a_1 \otimes \cdots \otimes\ a_p\ )\ =\ (-1)^{p+1} da_p\ .\ a_0 \otimes a_1\ \otimes \cdots \otimes\ a_{p-1}. \tag{2.290}$$

The operator σ commutes with the operators b, d, Π_p and therefore it passes to the universal algebra $\mathscr{U}_*$.

Proposition 2.42 *For any $1 \leq k \leq p-1$, the powers of the operator σ^k are given by the formulae*

$$\sigma^k(a_0\ \otimes \cdots \otimes\ a_p)\ =\ (-1)^{k(p+1)} da_{p-k+1}\ .\ \ldots da_p\ .\ a_0\ \otimes\ a_1 \cdots \otimes\ a_{p-k}$$
$$\sigma^k(a_0\ .\ da_1.\ldots.da_p)\ =\ (-1)^{k.(p+1)}\ da_{p-k+1} \ldots da_p\ .\ a_0\ .\ da_1\ .\ \ldots da_{p-k}.$$

For $k = p$ one has

$$\sigma^p(\, a_0 \otimes a_1 \ldots\ldots \otimes a_p \,) \;=\; (-1)^{p.(p+1)}\, da_1 \,.\, da_2 \ldots\ldots da_p \,.\, a_0$$
$$\sigma^p(\, a_0 \,.\, da_1 \ldots\ldots da_p \,) \;=\; (-1)^{p(p+1)}\, da_1 \,.\, da_2 \ldots da_p \,.\, a_0.$$

For $k = p+1$ one has

$$\sigma^{p+1}\,(\, a_0 \,\otimes\, a_1 \ldots\ldots \otimes\, a_p \,) = (-1)^{p.(p+1)} da_p \,.\, a_0 \,.\, da_1 \ldots\ldots da_{p-1} + (-1)^p b\,(\, da_0 \,.\, da_1 \ldots\ldots da_p \,)$$
$$\sigma^{p+1}(\, a_0 \,.\, da_1 \ldots\ldots da_p \,) = (-1)^{p(p+1)}\, da_p \,.\, a_0 \,.\, da_1 \,.\, da_2 \ldots da_{p-1} + (-1)^p b\,(\, da_0 \,.\, da_1 \ldots\ldots da_p \,).$$

Proposition 2.43

1. For any chain in $C_p(\mathscr{A})$ one has

$$\Pi_p\,(\sigma \;-\; T)(\, a_0 \otimes a_1 \otimes \cdots \otimes a_p \,) = (-1)^{p+1} d(\, a_p \,.\, da_0 \,.\, da_1 \ldots\ldots da_{p-1} \,). \tag{2.291}$$

2. More generally, for any $1 \le k \le p$ one has

$$\Pi_p\,(\sigma^k \;-\; T^k)(\, a_0 \;\otimes\; a_1 \otimes \cdots \otimes a_p \,) = d \;-\; boundary. \tag{2.292}$$

Proof

1.

$$\Pi_p\,(\sigma - T)\,(\, a_0 \;\otimes\; a_1 \otimes \cdots \otimes\; a_p \,) =$$
$$\Pi_p\,(-1)^{p+1}\,(\, da_p \,.\, a_0 \otimes \cdots \otimes a_{p-1} + a_p \otimes a_0 \otimes \cdots \otimes a_{p-1} \,) =$$
$$(-1)^{p+1}\Pi_p(\,(1 \otimes a_p \;-\; a_p \otimes 1) \,.\, a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{p-1} \;+\; a_p \otimes a_0 \otimes \cdots \otimes a_{p-1} \,) =$$
$$(-1)^{p+1}\,((1 \otimes a_p.a_0 - a_p \otimes a_0) \otimes a_1 \cdots \otimes a_{p-1} \;+\; a_p \otimes a_0 \otimes a_1 \cdots \otimes a_{p-1}) =$$
$$(-1)^{p+1}\,\Pi_p\,(1 \otimes (a_p.a_0) \otimes a_1 \otimes \cdots \otimes a_{p-1}) =$$
$$(-1)^{p+1}\;1 \,.\, d(a_p \,.\, a_0) \,.\, da_1 \ldots\ldots da_{p-1} \;=\; (-1)^{p+1} d(\, a_p \,.\, a_0 \,) \,.\, da_1 \ldots\ldots da_{p-1}. \tag{2.293}$$

2. We intend to compute

$$\Pi_p\;\;(\sigma^k \;-\; T^k)\,(\, a_0 \otimes a_1 \otimes \cdots \otimes a_p \,) \;=$$
$$\Pi_p(\;(-1)^{k.(p+1)}\; da_{p-k+1} \,.\, da_{p-k+2} \;\ldots\; da_p \,.\, a_0 \otimes a_1 \cdots \otimes a_{p-k+1} \;-$$
$$(-1)^{k.p}\;\; a_{p-k+1} \otimes a_{p-k+2} \otimes \ldots a_p \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{p-k} \,). \tag{2.294}$$

We consider the term $\Theta := da_{p-k+1}.da_{p-k+2}\dots da_p.a_0\otimes a_1\cdots\otimes a_{p-k+1}$; we express it as a sum of tensor products. We group these terms in three categories: (1) terms, degenerated or not degenerated, which begin with the factor 1, (2) terms which do not begin with the factor 1 but which are manifestly degenerated (i.e. contain at least one factor 1) and (3) the remaining terms. Denote their sum, respectively, by Θ_1, Θ_2 *and* Θ_3. Then $\Pi_p\Theta_1$ is a d-boundary and $\Pi_p\Theta_2 = 0$.

The first group of terms is

$$\Theta_1 \;=\; 1\otimes a_{p-k+1}.da_{p-k+2}\dots da_p.a_0\otimes a_1\cdots\otimes a_{p-k+1}. \tag{2.295}$$

The remaining terms are

$$\Theta_2 \;+\; \Theta_3 \;=\; -a_{p-k+1}\otimes 1.da_{p-k+2}\dots da_p.a_0\otimes a_1\cdots\otimes a_{p-k+1} \tag{2.296}$$

and

$$\Theta_3 \;=\; (-1)^k a_{p-k+1}\otimes a_{p-k+2}\otimes\cdots\otimes a_p.a_0\otimes a_1\cdots\otimes a_{p-k+1}. \tag{2.297}$$

Π_p kills the degenerate terms Θ_2. Then

$$\Pi_p(\Theta) \;=\; \Pi_p(\, da_{p-k+1}.da_{p-k+2}\dots da_p.a_0\otimes a_1\cdots\otimes a_{p-k+1}\,) \;=$$
$$(d-boundary) \;+\; (-1)^k a_{p-k+1}.da_{p-k+2}.\cdots\otimes da_p.da_0.da_1.da_{p-k+1}. \tag{2.298}$$

We plug this relation into the formula (2.294) to complete to proof of part 2.

This completes to proof of Theorem 2.14.

This theorem shows that the operator σ is closely related to the operator T. We intend to better understand this relationship.

One has

$$\sigma\;\Pi_k\;(\,a_0\otimes a_1\otimes\cdots\otimes a_k\,) = \sigma(a_0\;da_1\;.\;da_2.\dots.da_k) \;=\; (-1)^{k+1}da_k\;.\;a_0\;.\;da_1.\dots.da_{k-1}. \tag{2.299}$$

Iterating this relation with respect to r, one obtains for $1\le r\le k$

$$\sigma^r\;\Pi_k(\,a_0\otimes a_1\otimes\cdots\otimes a_k\,) = (-1)^{r(k+1)}da_{k-r+1}\;.\;da_{k-r+2}\dots a_0\;.\;da_1.\dots.da_{k-r}. \tag{2.300}$$

The last relation obtained in this way is

$$\sigma^k\;\Pi_k\;(\,a_0\otimes a_1\otimes\cdots\otimes a_k\,) \;=\; (-1)^{k(k+1)}\;da_1\;.\;da_2\dots a_k\;.\;a_0. \tag{2.301}$$

Proposition 2.44

(i) On the quotient space $\mathscr{U}_p/Im\ b$ one has

$$\sigma^p = 1. \tag{2.302}$$

(ii) The action of σ on $\mathscr{U}_p$ is given by the formula

$$\sigma(a_0 \,.\, da_1 \ldots . da_p) = (-1)^{p-1}\, da_p \,.\, a_0 \,.\, da_1 \ldots . da_{p-1}. \tag{2.303}$$

Proof

(i) The operator σ commutes with b and d and hence it keeps each p-component $\mathscr{U}_p$ of the universal algebra invariant. Indeed

$$\sigma \Pi_p = \sigma(1 - b\,d)\sigma^p = (1 - b\,d)\sigma\sigma^p = (1 - b\,d)\sigma^p\sigma = \Pi_p\sigma. \tag{2.304}$$

We know also that $\Pi_p := (1 - b\,d)\sigma^p = 1$ on $\mathscr{U}_p$. Therefore

$$\sigma^p = 1 \quad on\ the\ quotient\ space\ \ \mathscr{U}_p\ /\ Im\ b. \tag{2.305}$$

(ii) For any $a_0 \,.\, da_1 \ldots . da_p \in \mathscr{U}_p$ one has

$$\begin{gathered}\sigma(\, a_0 \,.da_1 \ldots . da_p\,) = \\ \sigma(\, a_0 \,.\, da_1 \ldots . da_{p-1}(1 \otimes a_p - a_p \otimes 1)\,) = \\ (-1)^{p+1}(\, da_p.(a_0.da_1 \ldots . da_{p-1}) - (d1)(\,(a_0 \,.\, da_1 \ldots . da_{p-1}) \,.\, a_p\,) = \\ (-1)^{p+1}\, da_p \,.\, a_0 \,.\, da_1 \ldots . da_{p-1}.\end{gathered} \tag{2.306}$$

Proposition 2.45 *For any $1 \leq k \leq p-1$, the powers of the operator σ^k, acting on the universal algebra $\mathscr{U}_*$, are given by the formula*

$$\sigma^k(a_0 \,.\, da_1 \ldots . da_p) = (-1)^{k.(p+1)}\, da_{p-k+1} \ldots . da_p . a_0 \,.\, da_1 \ldots . da_{p-k}. \tag{2.307}$$

For $k = p$ one has

$$\sigma^p(a_0 \,.\, da_1 \ldots . da_p) = (-1)^{p.(p+1)}\, da_1 \,.\, da_2 \ldots . da_p . a_0. \tag{2.308}$$

and therefore

$$\sigma^p\ \Pi_p(\, a_0 \otimes a_1 \otimes \cdots \otimes a_p\,) = (-1)^{p(p+1)} da_1 . da_2 \ldots a_p . a_0. \tag{2.309}$$

From here we get

$$\sigma^{p+1}\,\Pi_p(\,a_0\otimes a_1\otimes\cdots\otimes a_p\,)=$$
$$(-1)^{(p+1)(p+1)}(\,d(a_p.a_0).da_1.da_2\ldots da_{p-1}-da_0.da_1.da_2\ldots da_{p-1}.a_p\,)\;=$$
$$(-1)^{p+1}(\,d(a_p).a_0\;+\;a_p(da_0)\,).da_1.da_2\ldots da_{p-1}-da_0.da_1.da_2\ldots da_{p-1}.a_p\,)=$$
$$(-1)^{p+1}(\,d(a_p).a_0.da_1.da_2\ldots da_{p-1}-[\,da_0.da_1.da_2\ldots da_{p-1},\;a_p\,]\,). \tag{2.310}$$

Recall that commutators $[\,da_0.da_1.da_2\ldots da_{p-1},\;a_p\,]$ belong to the image of b, Proposition 2.41.

The next result is a consequence of Proposition 2.45.

Proposition 2.46 *There is a natural action of the group* $\mathbf{Z}_p$ *onto* $\mathscr{U}_p/Im\ b$*; the operator* σ *acts as a generator of this group action.*

Proposition 2.47

(i) Let $\alpha_p\;=\;a_0\;.\;da_1.\ldots.a_p$*, with* $1\le p$ *and* $\beta_q\;=\;b_0\;.\;b_1.\ldots.b_q$ *be two chains of* $\mathscr{U}_*(\mathscr{A})$*. Let* σ *be the operator acting on chains of degree* $p+q$*.*

The graded commutator of the chains α,β *is*

$$[\,\alpha_p\;,\;\beta_q\,]\;=\;\alpha_p\;.\;\beta_q-(-1)^{p.q}\;\beta_q\;.\;\alpha_p=(1\;-\;\sigma^q)(\,\alpha_p\;.\;\beta_q\,). \tag{2.311}$$

The space of graded commutators $[\,\alpha_p,\;\beta_q\,]$*,* $1\le p$*, is the sub-module of* $\mathscr{U}_*$ *generated by* $Im(1\;-\;\sigma^k)$*,* $1\le k\le p$*.*

(ii) For $p=0$*, one has*

$$[a_0,\;b_0.db_1.\ldots.db_q]\;=\;(-1)^q\;b\;(b_0.db_1.\ldots.db_q\;.\;da_0). \tag{2.312}$$

Proof

(i) We use Proposition 2.45 formula (2.307) to get

$$(db_1.\ldots.db_q)(a_0.da_1.\ldots.da_p.b_0)\;=$$
$$(-1)^{q.(p+q+1)}\sigma^q(\,(a_0.da_1.\ldots.da_p).(b_0.db_1.\ldots.db_q)\,). \tag{2.313}$$

Then, the graded commutator of the chains α_p and β_q is

$$[\alpha_p,\beta_q]\;=\;\alpha_p.\beta_q\;-\;(-1)^{p.q}(-1)^{q.(p+q+1)}\sigma^q(\alpha_p.\beta_q)=(1\;-\;\sigma^q)(\alpha_p.\beta_q), \tag{2.314}$$

(ii) follows from Proposition 2.41.

Proposition 2.47 shows that the factorisation of $\mathscr{U}_p$ by all graded commutators is given by the formula

$$\bar{\mathscr{U}}_p \;=\; \mathscr{U}_p \;/\; (\; Im\, b \;+\; \sum_{1\leq k\leq p} Im(1 \;-\; \sigma^k)\;). \tag{2.315}$$

Proposition 2.48 *We assume* $\mathbf{Q} \subset \mathbf{K}$. *Then*

$$\bar{\mathscr{U}}_p \;=\; Ker(1 \;-\; \sigma) \;\; on \;\; \mathscr{U}_p / Im\, b, \tag{2.316}$$

i.e. the module of σ*-invariant elements of* $\mathscr{U}_p \;/\; Im\; b$.

Proof Use Proposition (2.47) and the fact that σ has order p on $\mathscr{U}_p \;/\; Im \;\; b$, Proposition 2.41.

Proposition 2.48 states who are the chains which belong to the complex $\{\; \bar{\mathscr{U}}_*,\;\; d\;\}$. It remains to compute its homology. We intend to relate the homology of this complex with the reduced cyclic homology of the algebra $\mathscr{A}$. For this reason it will be useful to represent the elements of the complex $\bar{\mathscr{U}}_*$ by elements of the reduced cyclic complex. The next proposition shows how this can be done. To do this step we remember that the elements of the complex $\bar{\mathscr{U}}_*$ have cyclic σ-symmetry modulo $Im\; b$.

Proposition 2.49 *The correspondence*

$$\Phi_p : a_0 \;.\; da_1 \ldots . da_p \mapsto a_0 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} a_p \tag{2.317}$$

(i) defines a bijection

$$\Phi_p : \bar{\mathscr{U}}_p \;/\; Im\; d \longrightarrow \bar{C}^{\lambda}_p(\mathscr{A}) \;/\; Im\; b \tag{2.318}$$

whose inverse is Π_p

$$\Pi_p : \bar{C}^{\lambda}_p(\mathscr{A}) \;/\; Im\; b \longrightarrow \tilde{\mathscr{U}}_p(\mathscr{A}) \;/\; Im\; d, \quad modulo\; factorisations. \tag{2.319}$$

(ii) Φ *interchanges* σ *and* T

$$\Phi_p\; \sigma \;=\; T\; \Phi_p. \tag{2.320}$$

Proof We prove that Φ_p is well defined

1.

$$\Phi(\; 1 \otimes da_0 \otimes da_1 \otimes \cdots \otimes da_p\;) \;=\; d(\; da_0 \otimes da_1 \otimes \cdots \otimes a_p\;) \;=\; da_0 \otimes\; da_1 \otimes \cdots \otimes da_p \;=\; 0 \in \tilde{\mathscr{U}}_p(\mathscr{A}) \;/\; Im\; d.$$

2.

$$\Phi\,(-1)^{p+1}(\,a_p\,da_1\ldots da_{p-1}\;-\;a_0\,da_1\ldots da_{p-1}\,a_p\,)\;=\;b\,(\,a_0\,da_1\ldots da_p\,)\;=\;0\in\bar{\mathscr{U}}_p(\mathscr{A})\;/\;Im\,d$$

3.

$$\Phi(\,a_0\,da_1.\ldots.da_p\;-\;(-1)^{p-1}\,da_p\,a_0\,da_1.\ldots.da_{p-1}\,)\;=$$

$$(-1)^p\,d(a_p\,.\,a_0).\ldots.da_{p-1}+a_0\,da_1.\ldots.da_p\;-\;(-1)^p\,a_p\,da_0\ldots da_{p-1}\;=\;0\in\bar{\mathscr{U}}_p(\mathscr{A})\;/\;Im\,d.$$

Therefore, Ψ is well defined.

Following the same kind of computations, one may verify that the inverse of Φ is Π_p, modulo factorisations

$$\Pi_p\,(\,a_0\,\otimes\,a_1\,\otimes\cdots\otimes a_p\,)\;=\;\Phi^{-1}\,(\,a_0\,\otimes\,a_1\,\otimes\cdots\otimes a_p\,)\;=\;a_0\,da_1\;\ldots\;da_p. \tag{2.321}$$

(ii) Proposition 2.48 says that $\bar{\mathscr{U}}_p$ consists of elements $a_0.da_1.\ldots.da_p\;\in\;\mathscr{U}_p\;/\;Im\,b$; these elements have σ-cyclic symmetry in $\mathscr{U}_p\;/\;Im\,b$. They correspond through the mapping Φ to elements $a_0\otimes a_1\otimes\cdots\otimes a_p$ which have cyclic symmetry modulo the image of b.

The bijection Φ interchanges T with σ.

Proposition 2.50

(i) Let Z_p denote the module of Hochschild cycles of degree k. Then the composition of mappings

$$\mathscr{H}_p:\quad Z_p(A)\xrightarrow{\Pi_p}\mathscr{U}_p(A)\xrightarrow{\tau}\bar{\mathscr{U}}_p(A) \tag{2.322}$$

induces an injection of the Hochschild homology $H_p(A)$ into $\bar{\mathscr{U}}_p(A)$.

(ii) Let π denote the canonical projection

$$\pi:\bar{\mathscr{U}}_p(A)\longrightarrow\bar{\mathscr{U}}_p(A)\;/\;Im\,b. \tag{2.323}$$

Then $\pi\circ\mathscr{H}:H_p(\mathscr{A})\longrightarrow\bar{\mathscr{U}}_p(A)\;/\;Im\,b$ is injective.

(iii) $H_p(\mathscr{A})$ is a sub-module of σ-invariant elements of $\bar{\mathscr{U}}_p(A)\;/\;Im\,b$.

Proof

(i) Consider an arbitrary Hochschild cycle γ of degree p. Proposition 2.32 says that the cycle $\Pi_p(\gamma)$ is co-homologous to the cycle γ. We have to show that the graded trace homomorphism $\tau:\mathscr{U}_p\longrightarrow\bar{\mathscr{U}}_p$ is injective on the set of b-cycles $\Pi_p(\gamma)$. The module $\bar{\mathscr{U}}_*$ is obtained from $\mathscr{U}_*$ through two factorisations: one with respect to the *Im b*, and the second with respect to all commutators of elements in $\mathscr{U}_*$. After the first factorisation is performed, the factorisation with

respect to all graded commutators is equivalent to the factorisation through the sub-module generated by $Im(1 - \sigma^k)$, $1 \leq k \leq p$.

The factorisation with respect to the image of the homomorphism b does not alter the homology class of $\Pi_p(\gamma)$. We have to show that the factorisation through the second set of commutators is injective on homology classes. We know that $\Pi_p(\gamma)$ is a b-cycle; we get from this

$$(1 - \sigma)\Pi_p(\gamma) = (bd + db)\Pi_p(\gamma) = b\, d\Pi_p(\gamma); \tag{2.324}$$

therefore, $[\sigma\Pi_p(\gamma)] = [\Pi_p(\gamma)]$, where $[\ ]$ is the homology class of a cycle in the homology of the complex $\mathscr{U}_p \,/\, Im\ b$.

Recall that $\Pi_p = (1 - b\ d)\sigma^p$, that Π_p is the identity on $\mathscr{U}_p$ and that $\sigma^p = 1$ on $\mathscr{U} \,/\, Im\ b$. Suppose now that $\Pi_p\gamma$ is a sum of commutators of the second type

$$\Pi_p\gamma = \sum_{1\leq k\leq p} (1 - \sigma^k)\gamma_k = (1 - \sigma)\theta, \tag{2.325}$$

where $\theta \in \mathscr{U}_p/Im\ b$. Then, as $\sigma^p = 1$ on this space

$$(1 + \sigma + \ldots \sigma^{p-1})\ \Pi_p\gamma = (1 + \sigma + \ldots \sigma^{p-1})(1 - \sigma)\theta = 0. \tag{2.326}$$

We assume that $\mathbf{Q} \subset \mathbf{K}$. We pass to the b-homology classes of the cycles in the formula above. We get

$$p\ [\Pi_p\gamma] = 0, \quad i.e.\ [\Pi_p\gamma] = 0. \tag{2.327}$$

For $p = 0$, $H_0^{dR}(\mathscr{A}) = \mathscr{A} \,/\, (\mathbf{K} + [\mathscr{A}, \mathscr{A}]) = \bar{\mathscr{U}}_0(\mathscr{A})$.

(ii) Factorisation through boundaries does not affect homology classes of cycles.

(iii) The operator σ commutes with b.

2.4.13 de Rham Homology vs. Reduced Cyclic Homology

Proposition 2.50(i) represents the chains of the complex $\bar{\mathscr{U}}_p/Im\ d$ as elements of the module $\bar{C}_p^\lambda \,/\, Im\ b$. We have to know how the boundary d of this complex is transferred under this representation.

Proposition 2.51

(i) The operator $B : \bar{C}_p^{\lambda} / Im\ b \longrightarrow H_{p+1}(\mathscr{A})/Im\ b$, *defined on the reduced complex becomes*

$$B(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \sum_{0 \le r \le p} (-1)^{rp} 1 \otimes a_r \otimes \ldots a_p \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}. \tag{2.328}$$

(ii) Assume that $\mathbf{Q} \subset \mathbf{K}$. *Then*

$$\frac{1}{p+1} \Pi_p\ B = d$$

and therefore one has the commutative diagram

$$\begin{array}{ccc} \mathscr{U}_p & \xrightarrow{B} & \mathscr{U}_{p+1} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ \bar{\mathscr{U}}_p & \xrightarrow{(p+1)d} & \bar{\mathscr{U}}_p \end{array} \tag{2.329}$$

(iii) B and b anticommute.

Proof

(i) Recall that $B = (1 - T)\ s\ N$, where N is the cyclic symmetrisation operator. In the reduced complex $T\ s = 0$. Therefore, the operator B becomes

$$B(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \sum_{0 \le r \le p} (-1)^{rp} 1 \otimes a_r \otimes \ldots a_p \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}. \tag{2.330}$$

(ii)

$$\Pi_{p+1}\ B(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = \sum_{0 \le r \le p} (-1)^{rp} 1.da_r \ldots . da_p.da_0.da_1 \ldots . da_{r-1} =$$

$$d \sum_{0 \le r \le p} (-1)^{rp} a_r.da_{r+1} \ldots da_p.da_0.da_1 \ldots . da_{r-1}. \tag{2.331}$$

Proposition 2.50 says that the Hochschild homology $H_p(\mathscr{A})$ is a sub-module of $\bar{\mathscr{U}}_p(\mathscr{A})/Im\ b$ and a fortiori a sub-module of $\mathscr{U}_p(\mathscr{A})/Im\ b$.

Proposition 2.52 *For any unital associative algebra $\mathscr{A}$ one has the commutative diagram*

$$\begin{array}{ccc} \bar{\mathscr{U}}/Im\ d \equiv \mathscr{U}_p^{\lambda}/Im\ b & \overset{B=(p+1)d}{\longrightarrow} & \bar{\mathscr{U}}_{p+1}/Imb \\ b\downarrow & & \downarrow b \\ \bar{\mathscr{U}}_{p-1}^{\lambda} & \overset{B}{\longrightarrow} & \bar{\mathscr{U}}_p \end{array} \tag{2.332}$$

For any $\gamma \in \bar{\mathscr{U}}_p/Im\ d$ one has $B(\gamma) \in H_{p+1}(\mathscr{A})$ if and only if $b\ B(\gamma) = 0$. But the operator B on the second line of the commutative diagram (2.332) is injective, see part (i) of Proposition 2.51. Therefore, $b(\gamma) = 0$. This shows that the pth homology group of the complex

$$\bar{\mathscr{U}}_0 \overset{d}{\longrightarrow} \ldots \overset{d}{\longrightarrow} \bar{\mathscr{U}}_p \longrightarrow \bar{\mathscr{U}}_{p+1}/H_{p+1}(\mathscr{A}) \tag{2.333}$$

is $\bar{H}_p^{\lambda}(\mathscr{A})$. This is true because, in spite of the fact that the formula consents the boundary $d\gamma$ to belong to $\bar{H}_{p+1}(\mathscr{A})$, the proof shows that necessarily it should be a cycle with respect to the boundary operator b. This completes the proof of Theorem 2.14.

2.5 Relation Between B and d

Proposition 2.53 *Suppose $x \in C_p^{\lambda}(\mathscr{A})$.*
Then

$$\Pi_{p+1}\ (B\ x) = d\ \Pi_p\ (x). \tag{2.334}$$

Proof We have, see (2.208)

$$\begin{gathered} \Pi_{p+1}\ B(\ a_0 \otimes a_1 \otimes \ldots . \otimes a_p) = \Pi_{p+1}\ (\ (1 - T)(1 \otimes N(\ a_0 \otimes a_1 \otimes \ldots . \otimes a_p))) = \\ \Pi_{p+1}\ (1 - T)(1 \otimes (\ a_0 \otimes a_1 \otimes \ldots . \otimes a_p)) = \\ \Pi_{p+1}\ (1 \otimes (\ a_0 \otimes a_1 \otimes \ldots . \otimes a_p)) = \Pi_{p+1}\ (\ 1 \otimes a_0 \otimes a_1 \otimes \ldots . \otimes a_p) = \\ 1.da_0 . \ldots . da_p = d\ (\ a_0 . da_1 . \ldots . da_p\ ). \end{gathered} \tag{2.335}$$

2.6 Non-commutative Cohomology

2.6.1 Hochschild $HH_*(\mathscr{A})$ and Cyclic $H^*_\lambda(\mathscr{A})$ Cohomologies

Hochschild cohomology was defined in [4]; *cyclic cohomology* was defined by Connes [82]; see also [99].

Although these notions appeared in previous sections in relation to non-commutative forms of Chern–Weil theory we discuss here *non-commutative cohomology* in a more organic fashion.

Definition 2.42 To begin with, we assume that $\mathbf{K}$ is a commutative ring and that $\mathscr{A}$ is an associative $\mathbf{K}$-algebra.

The *Hochschild cohomology complex* $\{C^*(\mathscr{A}), b^*\}_*$ of $\mathscr{A}$ over $\mathbf{K}$ is the dual of the *Hochschild homology complex*

$$C^p(\mathscr{A}) := Hom_{\mathbf{K}}(\otimes_{\mathbf{K}}^{p+1}(\mathscr{A}, \mathbf{K})) \tag{2.336}$$

with co-boundary

$$(b^p f)(\gamma) := f(b_p\gamma), \quad \forall \gamma \in Hom \otimes_{\mathbf{K}}^{p+1}(\mathscr{A}, \mathbf{K}). \tag{2.337}$$

The operators T^p and N^p are *transposed operators* of T_p, N_p defined on the Hochschild complex

$$(T^p f)(\gamma) := f(T_p\gamma), \quad (N^p f)(\gamma) := f(N_p\gamma), \quad \forall \gamma \in Hom_{\mathbf{K}}(\otimes_{\mathbf{K}}^{p+1}\mathscr{A}). \tag{2.338}$$

The *Hochschild cohomology* of the algebra $\mathscr{A}$ is the homology of the dual complex $C^*(\mathscr{A})_*$. We denote it by $HH^*(\mathscr{A})$.

A *cyclic co-chain* f is a chain of the dual complex $f \in C^p(\mathscr{A})$ having *cyclic symmetry* $T^p(f) = f$. The cyclic p co-chains form the *cyclic* $\mathbf{K}$-module $C^p_\lambda(\mathscr{A})$. We know that the Hochschild boundary b sends a cyclic chain in a cyclic chain; therefore the *cyclic cohomology complex* $\{C^*_\lambda(\mathscr{A}), b^*\}_*$ is well defined. The homology of this complex is the *cyclic cohomology* of the algebra $\mathscr{A}$. It is denoted $H^*_\lambda(\mathscr{A})$.

Theorem 2.15 (Connes [82]—Connes' Long Exact Sequence) *We assume* $\mathbf{K}$ *is a commutative ring which contains the rational numbers and* $\mathscr{A}$ *is a* $\mathbf{K}$*-algebra. Then, the following sequence*

$$\ldots \xrightarrow{S} H^p_\lambda(\mathscr{A}) \xrightarrow{I} HH^p(\mathscr{A}) \xrightarrow{B} H^{p-1}_\lambda(\mathscr{A}) \xrightarrow{S} H^{p+1}_\lambda(\mathscr{A}) \xrightarrow{I} HH^{p+1}(\mathscr{A}) \xrightarrow{B} \ldots \tag{2.339}$$

is exact.

Proof The proof is obtained by transposing Connes' long exact sequence in homology.

Definition 2.43 More generally, given a bi-module $\mathscr{M}$ over the algebra $\mathscr{A}$, the *Hochschild co-homology of $\mathscr{A}$ with values in M* is the homology of the cohomology complex $C^*(\mathscr{A},\mathscr{M}) = \{C^*(\mathscr{A},\mathscr{M}), b^*\}$ where

$$C^p(\mathscr{A},\mathscr{M}) = Hom_{\mathbf{K}}(\otimes^p_{\mathbf{K}}\mathscr{A},\mathscr{M}) \tag{2.340}$$

and co-boundary b^p

$$\begin{aligned}(b^p f)(a_1\otimes_{\mathbf{K}} a_1\otimes_{\mathbf{K}}\cdots\otimes_{\mathbf{K}} a_{p+1}) &= \\ a_1.f(a_2\otimes_{\mathbf{K}}\cdots\otimes_{\mathbf{K}} a_{p+1}) &+ \\ +\sum_{r=1}^{r=p}(-1)^r\ f(a_1\otimes_{\mathbf{K}}\cdots\otimes_{\mathbf{K}} a_r.a_{r+1}\otimes_{\mathbf{K}}\ldots a_{p+1}) &+ \\ (-1)^{p+1}f(a_1\otimes_{\mathbf{K}} a_2\otimes_{\mathbf{K}}\cdots\otimes_{\mathbf{K}} a_p).a_{p+1}.&\end{aligned} \tag{2.341}$$

(Recall that $\mathscr{M}$ is an $\mathscr{A}$-bi-module provided $\mathscr{M}$ is a $\mathbf{K}$-module and for any $a, b \in \mathscr{A}$ and $m \in \mathscr{M}$ one has $a(mb) = (am)b$.)

The *Hochschild complex of co-chains over $\mathscr{A}$ with values in $\mathscr{M}$*, $C^*(\mathscr{A},\mathscr{M}) = \{C^*(\mathscr{A},\mathscr{M}), b^*\}$ is by definition given by

$$C^p(\mathscr{A},\mathscr{M}) = Hom_{\mathbf{K}}(\otimes^p_{\mathbf{K}}\mathscr{A},\mathscr{M}) \tag{2.342}$$

with co-boundary b^p

$$\begin{aligned}(b^p f)(a_1\otimes_{\mathbf{K}} a_1\otimes_{\mathbf{K}}\cdots\otimes_{\mathbf{K}} a_{p+1}) &= \\ a_1.f(a_2\otimes_{\mathbf{K}}\cdots\otimes_{\mathbf{K}} a_{p+1}) &+ \\ +\sum_{r=1}^{r=p}(-1)^r f(a_1\otimes_{\mathbf{K}}\cdots\otimes_{\mathbf{K}} a_r.a_{r+1}\otimes_{\mathbf{K}}\ldots a_{p+1}) &+ \\ (-1)^{p+1}f(a_1\otimes_{\mathbf{K}} a_2\otimes_{\mathbf{K}}\cdots\otimes_{\mathbf{K}} a_p).a_{p+1}&\end{aligned} \tag{2.343}$$

The *Hochschild cohomology of the algebra $\mathscr{A}$ with values in the bi-module $\mathscr{M}$* is denoted $H^*(\mathscr{A},\mathscr{M})$.

There are two cases of bi-modules of interest (a) $\mathscr{M} = \mathscr{A}^*$, the dual of $\mathscr{A}$, and (b) $\mathscr{M} = \mathscr{A}$. The bi-module structure on $\mathscr{A}^*$ is defined by $(afb)(x) := f(bxa)$, where $f \in \mathscr{A}^*$ and $a, b, x \in \mathscr{A}$. The bi-module structure on $\mathscr{A}$ is obvious.

The correspondence

$$\Phi : Hom_{\mathbf{K}}(\otimes_{\mathbf{K}}^{p}, \mathscr{A}^*) \longrightarrow Hom(\otimes_{\mathbf{K}}^{p+1}, \mathbf{K}) \tag{2.344}$$

$$\Phi(f)(a_0 \otimes_{\mathbf{K}} a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} a_p) = (f(a_1, \ldots, a_p))(a_0) \tag{2.345}$$

transforms the complex $\{C^*(\mathscr{A}, \mathscr{A}^*, \ b^*)$ into the Connes Hochschild cohomology complex $\{C^*(\mathscr{A}), b^*\}_*$, i.e., the case (a) leads to the Connes' definition of Hochschild cohomology.

The case (b) $\mathscr{M} = \mathscr{A}$ was considered by Gerstenhaber [35]. It is relevant in *deformation theory*.

2.6.2 Quillen Non-commutative Cohomology, $H^*(\mathscr{A}, V)$

Remark 2.19 In most constructions of the Chern character of idempotents, see Connes [82, 99] and [89], the Chern character appears in two modalities: in *co-homological* and *homological* contexts. The homological Connes Chern character is obtained by considering tensor powers of the idempotent inside the *periodic cyclic homology complex*, see [99], see also Loday [97, §8.3.3]. This is going to be discussed in Sect. 3.12. The *homological Connes Chern character* manifests itself through its pairing with *cyclic cohomology* [82] or with the Alexander–Spanier cohomology, see Connes and Moscovici [96] and Connes et al. [101]. The trace is provided either by the *graded trace interpretation of the cyclic co-homology*, or by the *operator trace*.

In Sect. 2.3.2 we studied Quillen's [91] general scheme which allows one to extend the Chern–Weil theory to the non-commutative associative algebra category. In Sect. 2.7 we saw how this applies to Karoubi's [89] treatment of the topic. Karoubi's construction requires a co-homological graded algebra in which the powers and the boundaries both of the connection form and curvature form may be computed. The necessary multiplicative structure is realised inside the *universal algebra* of the associative algebra $\mathscr{A}$. The *trace* is provided by the factorisation of the universal algebra through graded commutators. We recall, the universal algebra of $\mathscr{A}$, factorised through graded commutators, produces a co-homological complex whose homology is called *non-commutative de Rham homology* of the algebra. The *Karoubi Chern character* belongs to the *non-commutative de Rham homology*.

Karoubi [89, §2.15] proves that the non-commutative reduced de Rham homology is the kernel of the Connes B operator $B : H_p^{\lambda}(\mathscr{A}) \longrightarrow H_{p+1}(\mathscr{A})$, see Theorem 2.14.

The main difficulty in proving this result consists of the fact that the *non-commutative de Rham homology* is defined inside a *cohomology* context, while the cyclic homology counterpart is defined in a *homology* theory.

Another way of producing non-commutative cohomology derives from Quillen's [91] method. To do this, one fixes a **K**-algebra V and one considers the *graded differential cohomology complex*

$$\{Hom_{\mathbf{K}}(\otimes^{p}_{\mathbf{K}}\mathscr{A}, V),\ b^{'}\}_{0\leq p}. \tag{2.346}$$

The case $V = \mathbf{K}$ leads to the Hochschild cohomology (and cyclic cohomology) which was discussed above.

The *multiplicative structure* in this cohomology complex is provided both by the *co-algebra* structure of the tensor algebra of $\mathscr{A}$ and the multiplication in V. The co-boundary operators are, respectively, the transpose of the Hochschild boundary b and the transpose of the bar operator $b^{'}$.

Definition 2.44 Let $\mathscr{A}$ be an associative algebra and V a **K**-module.

Define

$$C^{0}(\mathscr{A}, V) := \mathscr{A} \tag{2.347}$$

$$C^{p}(\mathscr{A}, V) := Hom_{\mathbf{K}}(\otimes^{p}_{\mathbf{K}}\mathscr{A}, V), \quad 1 \leq p. \tag{2.348}$$

Let $d^{'}$, resp. $d : C^{p}(\mathscr{A},\ V) \longrightarrow C^{p+1}(\mathscr{A},\ V)$, be the boundary operators b', resp. b transposed: they are given by the formulae

$$\begin{gathered}(\ d^{'}\ \phi\ )(a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} \otimes_{\mathbf{K}} a_{p+1}) := \\ \phi\ (b^{'}\ (a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} \otimes_{\mathbf{K}} a_{p+1})),\end{gathered} \tag{2.349}$$

for any $\phi \in C^{p}(\mathscr{A}, \mathbf{K}),\quad or\quad C^{p}(\mathscr{A}, \mathscr{A})$ where

$$\begin{gathered}b^{'}\ (a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} \otimes_{\mathbf{K}} a_p) = \\ \sum_{1\leq r\leq p-1} (-1)^{r-1} a_1 \otimes \cdots \otimes a_{r-1} \otimes a_r . a_{r+1} \otimes \cdots \otimes a_p.\end{gathered} \tag{2.350}$$

and

$$\begin{gathered}(\ d\ \phi\ )(a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} \otimes_{\mathbf{K}} a_{p+1}) := \\ (\ d^{'}\ \phi\ )(\ a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} \otimes_{\mathbf{K}} a_{p+1}\ ) + \\ (-1)^{p-1}\ a_{p+1}\ .\ \phi\ (a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} a_p\ ).\end{gathered} \tag{2.351}$$

Define the *graded cyclic permutation* acting onto the argument

$$(\ T_p\ \phi\ )(a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} a_p) := (-1)^{p-1}\ \phi\ (a_2 \otimes_{\mathbf{K}} a_3 \cdots \otimes_{\mathbf{K}} a_p \otimes_{\mathbf{K}} a_1) \tag{2.352}$$

and the *cyclic symmetrisation* operator

$$N_p \;=\; 1 \;+\; T_p \;+\; \ldots \;+\; T_p^{p-1}. \tag{2.353}$$

Definition 2.45 Let $\phi \in C^p(\mathscr{A},\ V)$ and $\psi \in C^q(\mathscr{A},\ V), V \;=\; \mathscr{A}$.
Define their *product* $\phi\ .\ \psi \in C^{p+q}(\mathscr{A},\ V)$

$$\begin{gathered}(\phi\ .\ \psi)\ (a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} \otimes_{\mathbf{K}} a_{p+q}) \;:=\\ \phi\ (a_1 \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} \otimes_{\mathbf{K}} a_p)\ \ .V\ \ \psi\ (a_{p+1} \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} \otimes_{\mathbf{K}} a_{p+q}).\end{gathered} \tag{2.354}$$

This product uses the *co-algebra structure of the tensor algebra* of $\mathscr{A}$ with respect to the argument; regarding the values, the product uses the multiplication in V.

The following property follows easily:

Proposition 2.54

(i)

$$d^{'}\ .\ d^{'} \;=\; 0, \quad d\ .\ d \;=\; 0. \tag{2.355}$$

(ii) For any $\phi \in C^p(\mathscr{A},\ V)$ *and* $\psi \in C^q(\mathscr{A}, V)$ *the Leibnitz identity holds*

$$d^{'}\ (\phi\ ._V\ \psi) \;:=\; (\ d^{'}\ \phi\ )\ .\ \psi \;+\; (-1)^{p-1}\ \phi\ .\ (d^{'}\ \psi\ ). \tag{2.356}$$

Proposition 2.55 *The operators* $d, d^{'}, T$ *and* N *satisfy the identities*

$$T_p^p \;=\; 1, \;\; N_p(1 \;-\; T_p) \;=\; (1 \;-\; T_p)N_p \;=\; 0 \tag{2.357}$$

$$d^{'}(1 - T) \;=\; (1 - T)d \tag{2.358}$$

$$dN = Nd^{'}. \tag{2.359}$$

Proof The operators T, N, d and d' are defined on the tensor algebra of the algebra $\mathscr{A}$. These relations are the transposed relations (2.156) and (2.157).

Definition 2.46 Let $C^*_\lambda(\mathscr{A},\ V) \subset C^*(\mathscr{A},\ V)$ denote the subset of co-chains having *cyclic symmetry*.
Define *Hochschild-type cohomology*

$$H^*(\mathscr{A},\ V) := Homology^*\{C^*(\mathscr{A}),\ V),\ d\}. \tag{2.360}$$

Define *cyclic-type cohomologies*

$$H^*_\lambda(\mathscr{A},\ V) := Homology^*\{C^*_\lambda(\mathscr{A}),\ V),\ d'\}. \tag{2.361}$$

Theorem 2.16 *Let $\mathscr{A}$ be a unital associative algebra over the commutative ring* **K**. *Suppose the ring* **K** *contains the rational numbers. Let V be a* **K***-module.*

Then, one has the Connes-type long exact sequence

$$\ldots \xrightarrow{\tilde{S}} \tilde{H}^p_\lambda(\mathscr{A},\ V) \xrightarrow{\tilde{I}} \tilde{H}^p(\mathscr{A},\ V) \xrightarrow{\tilde{B}} \tilde{H}^{p-1}(\mathscr{A},\ V) \xrightarrow{\tilde{S}} \tilde{H}^{p+1}_\lambda(\mathscr{A},\ V) \xrightarrow{\tilde{I}} \ldots \tag{2.362}$$

Proof This long exact sequence is a formal consequence of the transposed relations (2.156) and (2.157). For more details see the proof of Connes long exact sequence in homology.

2.6.3 Non-commutative de Rham Cohomology with Values in an Algebra, $H^*_{dR}(\mathscr{A}, V)$

In this section we *propose* a *non-commutative de Rham* ***co****-homology* obtained by combining Karoubi's [89] and Quillen's [91] methods.

We start with the cohomology complex $\{C^*(\mathscr{A},\ V),\ d^{'}\}$ considered by Quillen—see section above. This means that

$$\{C^*(\mathscr{A}, V), d^{'}\} = \{Hom_{\mathbf{K}}(\otimes^p_{\mathbf{K}}\mathscr{A}, V) \tag{2.363}$$

is a co-homological differential algebra with respect to the co-boundary $d^{'}$—the *transposed of the bar operator*. The multiplication is given by the *co-algebra* structure of the tensor algebra of $\mathscr{A}$ (of the argument) and the multiplication in V.

Recall the product of co-chains

$$(\phi\ .\ \psi)(a_1 \otimes \cdots \otimes a_1 \otimes a_{p+q}) := \phi(a_1 \otimes \cdots \otimes a_p)\ .\!\!_{\mathscr{A}}\ \psi(a_{p+1} \otimes \cdots \otimes \otimes a_{p+q}). \tag{2.364}$$

The Leibnitz identity holds with respect to the co-boundary $d^{'}$.

The Karoubi [89] method consists of passing to the module of *graded traces*.

Definition 2.47 The *graded commutator* of degree t is defined by

$$[C(\mathscr{A},\ V), C(\mathscr{A},\ V)]_t := \textit{module generated by } [C^p(\mathscr{A},\ V), C^p(\mathscr{A},\ V)],\quad p+q=t \tag{2.365}$$

$$[\phi_p,\ \psi_q] := \phi_p\ .\ \psi_q\ +\ (-1)^{p.q}\psi_p\ .\ \phi_q. \tag{2.366}$$

The *de Rham cohomology complex* of the algebra $\mathscr{A}$ with values in V consists of the module of *graded traces*

$$\{C^*_{dR}(\mathscr{A},\ V),\ d^{'}\} := \{C^t(\mathscr{A},\ V)/[C(\mathscr{A},\ V), C(\mathscr{A},\ V)]_t,\ d^{'}\}_{0\leq t}. \tag{2.367}$$

The homology of this complex will be called *non-commutative de Rham cohomology of the algebra $\mathscr{A}$ with values in V*; we will denote it by $H^p_{dR}(\mathscr{A},\ V)$.

Remark 2.20 In this book, when we refer to *non-commutative de Rham cohomology* we intend that the corresponding cohomology complex is produced through the factorisation by the sub-module of *graded commutators*.

Problem 2.1 Express $H^p_{dR}(\mathscr{A},\ V)$ in terms of the Hochschild/cyclic cohomology of $\mathscr{A}$ and elements of V.

2.6.4 Cyclic Co-cycles vs. Closed Graded Trace

We have seen in the previous sections that *graded trace* and *cyclic symmetry* are used in the constructions of non-commutative Chern–Weil characters and we found various relations between them. In this section we define the *character of a cyclic cycle* and we establish its relationship with cyclic cohomology. There are two modalities to present this argument. We start by presenting Connes method [82].

We recall that a *cyclic co-cycle* f is a Hochschild co-cycle, $b^*f = 0$, which has cyclic symmetry, $Tf = f$. We recall also that the universal algebra is the range of the idempotents Π_* introduced in Sect. 2.4.10. The elements of the universal algebra are precisely the *non-degenerate* elements of the Hochschild homology complex of the algebra $\mathscr{A}$. The product and co-boundaries in the universal differential algebra are given by the (non-localised) Alexander–Spanier operations (Sect. 2.4.7). Although the universal algebra can be described also as $\mathscr{A} \otimes^*_{\mathbf{K}} (\mathscr{A}/\mathbf{K})$, the reader should not confuse the elements of the universal algebra with the elements of the Hochschild homology complex.

Definition 2.48 Suppose $\mathscr{A}$ is an associative algebra with, or without unit. We associate with $\mathscr{A}$ a new algebra

$$\tilde{\mathscr{A}} := \{a\ +\ \lambda.1 \mid \forall a \in \mathscr{A},\ \ and\ \lambda \in \mathbf{K}\}. \tag{2.368}$$

The algebra structure in $\tilde{\mathscr{A}}$ is given by

$$(a\ +\ \lambda.1)(b\ +\ \mu.1)\ =\ (a.b\ +\ \lambda.b\ +\ a.\mu, \lambda.\mu.1).$$

$1 \in \tilde{\mathscr{A}}$ is a unit.

The algebra $\mathscr{A}$ is replaced with the algebra $\tilde{\mathscr{A}}$.

Proposition 2.56 *The universal algebra of* $\tilde{\mathscr{A}}$ *is*

$$\mathscr{U}^*(\tilde{\mathscr{A}}) = \tilde{\mathscr{A}} \otimes_{\mathbf{K}} \otimes^*_{\mathbf{K}} \mathscr{A}. \tag{2.369}$$

Definition 2.49 Given any *co-chain* $f \in C^p(\tilde{\mathscr{A}})$, one associates with f its *character* τ_f. It is a linear mapping defined over the p-component of the universal algebra $\mathscr{U}^p(\tilde{\mathscr{A}})$ with values in $\mathbf{K}$, see [82]

$$\tau_f(a_0 + \lambda.1).da_1.\ldots.da_p) := f(a_0.da_1.\ldots.da_p). \tag{2.370}$$

Remark 2.21 τ_f is d-closed. Indeed

$$\begin{aligned} \tau_f d(a_0 + \lambda.1).da_1.\ldots.da_{p-1}) &= \tau_f(da_0.da_1.\ldots.da_{p-1}) = \\ \tau_f((0+1)da_0da_1\ldots da_{p-1}) &= f(0 \otimes a_0 \ldots a_{p-1}). \end{aligned} \tag{2.371}$$

The character τ_f of the co-chain f registers the values f takes on all non-degenerate Hochschild chains.

Remark 2.22 The universal algebra is d-acyclic. However, after the factorisation by graded commutators one obtains the *non-commutative de Rham complex*. It is not acyclic. Therefore, factorisation by graded commutators has the effect of reducing the Alexander–Spanier co-chains to the diagonal—see Sect. 1.3.

Definition 2.50 The co-chain $\tau \in \mathscr{U}^p(\tilde{\mathscr{A}}) \longrightarrow \mathbf{K}$ is a *graded trace* provided it vanishes on **all** *graded commutators* of degree p

$$\tau \mid [C^p(\mathscr{A}, \mathscr{A}^*),\ C^p(\mathscr{A}, \mathscr{A}^*)]_p = 0. \tag{2.372}$$

Definition 2.51 The co-chain τ is called *closed* if it vanishes on all d-boundaries $\tau_{d\omega} = 0$.

Proposition 2.57 (Connes [82, 99]) *Let* $\mathscr{A}$ *be an associative algebra.*
The correspondence

$$\tau : \{ f | f = b - closed\ cyclic\ co - chain\ over\ \tilde{\mathscr{A}} \} \longrightarrow \tag{2.373}$$

$$\{ \tau_f \mid \tau_f = d - closed\ graded\ trace \} \tag{2.374}$$

is a $1-1$ *correspondence.*

Remark 2.23 The reader should not be surprised that the d-closeness condition is part of the statement of the proposition. In fact, in various constructions of the non-commutative Chern–Weil character the d-closeness of co-chains comes together with the graded trace condition.

Lemma 2.3 *Given that the universal algebra $\mathcal{U}_*(\mathcal{A})$ is generated both by the elements $a \in \mathcal{A}$ and differentials $da = 1 \otimes a - a \otimes 1$, τ is a graded trace if and only if τ vanishes on all graded commutators of type $\tau[\, a_0 da_1 \dots a_p,\ a_{p+1}\,]$ and $\tau[\, a_0 da_1 \dots da_p,\ da_{p+1}\,]$, for all $a_i \in \mathcal{A}$.*

Proof

1. Suppose $f \in C^p_\lambda(\tilde{\mathcal{A}})$ and $b\, f = 0$. We have defined

$$\tau_f((a_0 + \lambda.1)da_1 \dots da_p) := f(a_0 \otimes a_1 \otimes \cdots \otimes a_p). \tag{2.375}$$

We saw that τ_f is a d-closed graded trace.

We have

$$\begin{aligned} 0 = &\sum_{0\le r\le p} (-1)^r f(a_0 \otimes \cdots \otimes (a_r.a_{r+1}) \otimes \cdots \otimes a_{p+1}) + \\ &(-1)^{p+1} f((a_{p+1}.a_0) \otimes a_1 \otimes \cdots \otimes a_p) = \\ &\sum_{0\le r\le p} \tau_f(a_0 da_1 \dots d(a_r.a_{r+1}) \dots da_{p+1}) + \\ &(-1)^{p+1} \tau_f(a_{p+1}.a_0 da_1 \dots .da_p) = \\ &\tau_f(a_0 da_1 \dots d(a_r.a_{r+1}) \dots a_{p+1}) + \\ &(-1)^{p+1} \tau_f(a_{p+1}.a_0 da_1 \dots .da_p) = \\ &[\, \tau_f(a_0 da_1 \dots d(a_r.a_{r+1}) \dots a_p),\ a_{p+1})]. \end{aligned} \tag{2.376}$$

This proves that τ_f vanishes on all graded commutators of type

$$\tau[(a_0 + \lambda 1).da_1 \dots a_p,\ a_{p+1}].$$

Viceversa, the same computation read backwards shows that if τ_f vanishes on such commutators, then f is a Hochschild co-cycle.

Now we consider a commutator of the second type. Recall that in the universal algebra degenerate elements are ignored. Therefore, in the universal algebra

we have

$$\begin{gathered}
(da_p(a_0+\lambda.1).da_1\dots da_{p-1}) \;= \\
\Pi(da_p(a_0+\lambda.1)da_1\dots da_{p-2}(1\otimes a_{p-1}-a_{p-1}\otimes 1)) \;= \\
\Pi(da_p(a_0+\lambda.1)da_1\dots da_{p-2}(1\otimes a_{p-1})) \;= \\
\Pi(da_p(a_0+\lambda.1)da_1\dots da_{p-2}\otimes a_{p-1})) \;=\; \dots \;= \\
\Pi(da_p(a_0+\lambda.1)\otimes a_1\otimes a_{p-2}\otimes\dots\otimes a_{p-1}) \;= \\
\Pi((1\otimes a_p-a_p\otimes 1)(a_0+\lambda.1)\otimes a_1\otimes a_{p-2}\otimes\dots\otimes a_{p-1}) \;= \\
\Pi((1\otimes a_p\,-;a_p\otimes 1)(a_0\;+\;\lambda.1)\otimes a_1\otimes a_{p-2}\otimes\dots\otimes a_{p-1}) = \\
\Pi((1\otimes(a_p(a_0+\lambda.1)\otimes a_1\otimes a_{p-2}\otimes\dots\otimes a_{p-1}) = \\
-\Pi(a_p\otimes(a_0+\lambda.1)\otimes a_1\otimes a_{p-2}\otimes\dots\otimes a_{p-1}) = \\
d(a_p(a_0+\lambda.1)da_1da_{p-2}\dots da_{p-1}) \\
-a_pd(a_0+\lambda.1)da_1.da_{p-2}.\dots.da_{p-1}.
\end{gathered} \tag{2.377}$$

As the graded trace is d-closed, we get

$$\begin{gathered}
\tau_f(da_p(a_0\;+\;\lambda.1).da_1\dots da_{p-1}) \;=\; -\,\tau_f(\,a_p.d(a_0\;+\;\lambda.1).da_1.da_{p-2}.\dots.da_{p-1} \;= \\
-f(\,a_p\otimes(a_0\;+\;\lambda.1)\otimes a_1\otimes a_{p-2}.\dots\otimes a_{p-1}).
\end{gathered} \tag{2.378}$$

We know that f is a cyclic co-chain; therefore,

$$\begin{gathered}
\tau_f(da_p(a_0+\lambda.1).da_1\dots da_{p-1}) = \\
-\,f(\,a_p\otimes(a_0\;+\;\lambda.1)\otimes a_1\otimes a_{p-2}.\dots\otimes a_{p-1}) \;= \\
(-1)^{p+1}f(\,(a_0\;+\;\lambda.1)\otimes a_1\otimes a_{p-2}.\dots\otimes a_{p-1}\otimes a_p) \;= \\
(-1)^{p+1}\tau_f((a_0\;+\;\lambda.1).da_1\dots da_{p-1}da_p).
\end{gathered} \tag{2.379}$$

The same computation read backwards may be used to prove that if τ_f vanishes on commutators of the second type, then the co-chain f has cyclic symmetry.

Remark 2.24 The replacement of the algebra $\mathscr{A}$ with the algebra $\tilde{\mathscr{A}}$ allows one to produce the d-closed graded trace τ_f. This could be done in another way if we assume that the algebra $\mathscr{A}$ is unital and we consider only those cyclic co-chains f which vanish when at least one of the arguments $a_0, a_1, \dots a_p$ belongs to the commutative ring $\mathbf{K}$. This relates to Karoubi's description of de Rham homology—*reduced cyclic homology*—see Sect. 2.4.12.

Remark 2.25 In the next section we will see that the continuous Hochschild and cyclic homologies of the algebra of smooth functions are, essentially, the space

of de Rham forms and de Rham cohomology of the space. On the other hand, de Rham differential forms and de Rham cohomology could be obtained even from the *localised Alexander–Spanier complex*. The reader could be tempted to conclude that cyclic cohomology takes care both of the *vanishing on graded commutators* property and of the *localisation* involved in the definition of the Alexander–Spanier complex, see Sect. 1.3. This is not the case—*cyclic cohomology does not replace localisation*. It is known that the *continuous cyclic homology of any Banach algebra is trivial*, see [99]; on the other hand, we will see later that the **localised cyclic homology** of the Banach algebra of Hilbert–Schmidt operators is isomorphic to the ordinary cohomology.

2.6.5 *Chern Character with Values in* $H^{ev}_{\lambda}(\mathbf{M}_N(\mathscr{A}),\ \mathscr{A})$

We start with an idempotent $p \in \mathbf{M}_N(\mathscr{A})$. It is an element of the algebra of matrices of size N; its entries belong to the algebra $\mathscr{A}$. We associate with p the Levi-Civita connection form $\omega = ||\omega_{ij}||$. Its entries, ω_{ij}, belong to the space $Hom_{\mathbf{K}}(\otimes_{\mathbf{K}}\mathscr{A}^2,\ \mathscr{A})$—the space of 1-forms of the algebra $\mathscr{A}$ with values in $\mathscr{A}$, see Sect. 2.3.4. The curvature matrix $\Omega = ||\Omega_{ij}||$ has entries in the space of 2-forms, $Hom_{\mathbf{K}}(\otimes_{\mathbf{K}}\mathscr{A}^3, \mathscr{A})$. The power q of the curvature Ω belongs to the space of $2q$-forms, $Hom_{\mathbf{K}}(\otimes^2_{\mathbf{K}}\mathbf{M}_N(\mathscr{A}), \mathbf{M}_N(\mathscr{A}))$.

We consider the form

$$\Psi_q(p) := \frac{1}{q!} tr\ (\Omega(p)^q N) \in C^{2q}_{\lambda}(\mathbf{M}_N(\mathscr{A}), \mathscr{A}); \tag{2.380}$$

here the cyclic symmetrisation operates on the argument while the trace acts onto the value of the homomorphism. The result is an element of the cyclic complex $C^*_{\lambda}(\mathbf{M}_N(\mathscr{A}), \mathscr{A})$.

We *propose* the following definition of Chern character of idempotents:

Theorem 2.17 *Let $p \in \mathbf{M}_N(\mathscr{A})$ be an idempotent.*

Then

$$\Psi_q(p) := \frac{1}{q!}\ tr\ (\Omega(p)^q N) \tag{2.381}$$

is a closed form in the cyclic complex $C^*_{\lambda}(\mathbf{M}_N(\mathscr{A})$.

2.6.6 Chern Character of Idempotents with Value in Non-commutative de Rham Cohomology

In this section we define a Chern character of idempotents with values in the *non-commutative de Rham cohomology of the algebra of matrices with entries in* $\mathscr{A}$, $\tilde{H}_{dR}(\mathbf{M}_N(\mathscr{A}), \mathbf{M}_N(\mathscr{A}))$.

We use [91]; we take $V = \mathbf{M}_N(\mathscr{A})$, see Definition 2.1.

With this choice of V, the ordinary trace of matrices may be used as a trace function and hence there is no need to factorise through graded commutators or to produce a trace function, see [91, §1.3]. From this point on, the Chern–Weil construction proceeds without difficulties.

Indeed, start with an idempotent $p \in \mathbf{M}_N(\mathscr{A})$.

Consider the Levi-Civita connection, given by a matrix with entries in $\mathbf{M}_N(\mathscr{A})$

$$\omega = p \,.\, dp \,.\, p \in C^1(\mathbf{M}_N(\mathscr{A}), \mathbf{M}_N(\mathscr{A})). \tag{2.382}$$

We produce a *matrix* ω with entries in $\mathscr{A}$ which describes the connection form ω, as done in Proposition 2.9. The boundary $d^{'}$ (the transpose of $b^{'}$) and the product introduced by Definition 2.1, satisfy the Leibnitz identity.

We define the curvature

$$R = d^{'}\,\omega - \omega \,.\, \omega. \tag{2.383}$$

The curvature verifies the generalised Bianchi identities

$$d^{'} R^q = -[R^q,\ \omega]. \tag{2.384}$$

Define

$$\Psi_q = tr\ R^q \in C^{2q}(\mathbf{M}_N(\mathscr{A}), \mathbf{M}_N(\mathscr{A})). \tag{2.385}$$

Theorem 2.18

(i)

$$d^{'}\,\Psi_q = 0. \tag{2.386}$$

(ii) The homology class of $\Psi_q \in H^{2q}(\mathbf{M}_N(\mathscr{A}), \mathbf{M}_N(\mathscr{A})))$*, (with respect to the co-boundary* $d^{'}$*) depends only on the homotopy class of the idempotent* p*.*

Proof The proof uses the following structure:

1. the computations are performed inside a graded differential co-homological complex $\{\Omega_*, d\}$ with $\Omega_0 = \mathscr{A}$

2. the curvature is $R = d^{'} \omega - \omega . \omega$, where ω is the connection form of the idempotent p
3. any two connections are ω_0, ω_1 are affinelly related $\omega_t = \omega_0(1-t) + \omega_1.t$.

2.7 Connes Chern Character in K-Homology

Connes Chern character in K-homology is defined on spectral triples [82, 99]. For this topic see Sect. 2.3.5.

We recall that spectral triples are refinements of K-homology classes, see Sect. 1.12.3

2.8 Connes Karoubi Chern Character in K-Theory

The cycles of $K^0(\mathscr{A})$ are idempotents $p \in \mathbf{M}_N(\mathscr{A})$. The Chern character of idempotents was discussed in Sect. 2.3.4.

The next proposition produces the Connes Karoubi Chern character of an idempotent p by passing directly to cyclic homology, without passing through the connection—curvature step.

Proposition 2.58 *Let $p \in \mathbf{M}_N(\mathscr{A})$ be an idempotent.*
Then

$$\Psi_q(p) := (-1)^{p-1} \frac{2p!}{2(p)!} tr\ p(dp.dp)^p \in H^{\lambda}_{2p}(\mathbf{M}_N(\mathscr{A})) \equiv H^{\lambda}_{2p}(\mathscr{A}), \tag{2.387}$$

where the last equivalence is the Morita isomorphism*, discussed later, see Sect. 3.11.*

Here tr is the *non-commutative trace of matrices*

$$tr\ (a_0 \otimes a_1 \otimes \cdots \otimes a_r) := \sum_{i_0, i_1, \ldots i_r} a^{i_1}_{0,i_0} \otimes a^{i_2}_{1,i_1} \otimes \cdots \otimes a^{i_0}_{r,i_r} \tag{2.388}$$

and $dp = 1 \otimes p - p \otimes 1$.

The cycles of $K^0(\mathscr{A})$ are invertible matrices $u \in \mathbf{GL}(\mathscr{A})$. The *Connes Chern character for invertible matrices* is given by the following result:

Proposition 2.59 *Let $u \in \mathbb{GL}(\mathscr{A})$.*

Then

$$\Psi_p(u) :=$$

$$tr\,((u^{-1} - 1) \otimes (u - 1) \otimes (u^{-1} - 1) \otimes (u - 1) \otimes \cdots \otimes (u^{-1} - 1) \otimes (u - 1)) \in$$

$$\in H^{\lambda}_{2p-1}(\mathbf{M}_N(\mathscr{A})) \equiv H^{\lambda}_{2p-1}(\mathscr{A}), \quad 2p\ factors, \tag{2.389}$$

represents the Connes Chern character of the invertible u up to a multiplicative constant.

2.9 Pairing K-Theory with Alexander–Spanier Cohomology; Towards Local Index Theorem

For the topics discussed in this section we refer to Connes and Moscovici [96].

In this section we present only the introductory algebraic framework onto which the Connes–Moscovici *local index theorem* is based.

In Part IV we are going to show how these constructions apply to the case of the *index theory on quasi-conformal manifolds*. Next, in Part V, we will show how these arguments *change* in the case of the T-theory.

Here M is a *smooth manifold*, which for simplicity we assume to be compact. Let $\mathscr{A}$ be the algebra of complex valued continuous functions on M. Let $\rho : \mathscr{A} \longrightarrow \mathbf{B}(H)$ be the obvious representation, where $\mathbf{B}(H)$ is the algebra of bounded operators defined onto the L_2 *space of half densities*, $H = L_2(M, |\Lambda|^{1/2}M)$. (Here we assume that M is oriented. The *bundle of half densities* is a line bundle whose transition functions are square root of the determinant of the Jacobi matrices of coordinates changes).

$\mathscr{L}$ denotes the norm closure algebra of the space of pseudo-differential operators of order zero into the space $\mathbf{B}(H)$. Let $\mathscr{K}$ denote the algebra of compact operators on H. Let $C(S^*M)$ denote the unit co-sphere bundle of M. One gets the exact sequence of algebras

$$\mathscr{E} \equiv \quad 0 \longrightarrow \mathscr{K} \xrightarrow{\iota} \mathscr{L} \xrightarrow{\sigma} C(S^*M) \longrightarrow 0, \tag{2.390}$$

where σ is the *symbol* of operators.

Theorem 2.19 *There exists a canonical construction which produces a* pairing

$$K^1(C(S^*M) \otimes H^{even}_{AS}(M) \longrightarrow \mathbf{C}. \tag{2.391}$$

The construction of the pairing passes through the following steps:

1. Let $A \in \mathscr{L}$ be an elliptic *pseudo-differential operator* on the smooth manifold M. One considers the symbol $a \in C(S^*M)$ of the operator $A : H_0 \longrightarrow H_1$. Denote by $b = a^{-1}$ its inverse.
2. Let $B \in \mathscr{L}$ be a *lift of the symbol* b. The operators A and B could be chosen, modulo compact operators, so that as to have arbitrarily small supports about the diagonal of $M \times M$.
3. One considers the *defects of the parametrix* B: $S_0 := 1 - BA$, $S_1 = 1 - AB$. *A priori*, these operators are pseudo-differential operators of order -1. They may be modified so as to become *smoothing operators*. Let $\mathscr{L}^\infty$ denote the algebra of smoothing operators.
4. One constructs the invertible operator

$$L := \begin{pmatrix} S_0 & -(1+S_0)B \\ A & S_1 \end{pmatrix} \in \mathbf{M}_2(H_0 \oplus H_1). \tag{2.392}$$

The inverse of the operator L is

$$\mathbf{L}^{-1} = \begin{pmatrix} S_0 & (1+S_0)B \\ -A & S_1 \end{pmatrix}. \tag{2.393}$$

The operator L is a lift of the operator

$$\begin{pmatrix} 0 & -B \\ A & 0 \end{pmatrix} \in \mathbf{M}_2(H_0 \oplus H_1). \tag{2.394}$$

5. With the operator L one constructs the idempotent

$$P := L\, e_1\, L^{-1} \in \mathscr{L}^\infty + D^0 \tag{2.395}$$

and the smoothing operator

$$R := P - e_2 \in \mathscr{L}^\infty, \tag{2.396}$$

where

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in D^0 \subset \mathbf{M}_2(H_0 \oplus H_1), \tag{2.397}$$

(D^0 is the algebra of differential operators of order zero).

Let ∂ be the connected K-theory homomorphism associated to the exact sequence $\mathscr{E}$. Then

$$\partial(a) = [P] - [e_2]. \tag{2.398}$$

6. **Definition 2.52** Let $A_0, A_1, \ldots, A_p$ be bounded operators on the Hilbert space H with the property that $A_0, A_1 \ldots A_p$ is a trace class operator. Suppose also that $f_0, f_1, \ldots, f_p \in \mathscr{A}$. Define

$$Index_{f_0 \otimes f_1 \otimes \cdots \otimes f_p} (A_0 \otimes A_1 \otimes \cdots \otimes A_p) := Tr\ (A_0\ f_0\ A_1\ f_1 \ldots A_p\ f_p). \tag{2.399}$$

Proposition 2.60 *Let $A_0, A_1, \ldots, A_p$ be bounded operators on the Hilbert space H with the property that $A_0.A_1 \ldots A_p$ is a trace class operator. Suppose also that $f_0, f_1, \ldots, f_{p-1} \in \mathscr{A}$. Then*

$$Index_f\ b\ (A_0 \otimes A_1 \otimes \cdots \otimes A_p) = Index_{d\ f}\ (A_0 \otimes A_1 \otimes \cdots \otimes A_p). \tag{2.400}$$

where b is the Hochschild boundary and d is the Alexander–Spanier coboundary.

Proof The reader can check easily the proposition.

Proposition 2.61 *Let $P_t \in \mathscr{L}^\infty + D^0$ be a piecewise C^1-path of idempotents, $t \in [0, 1]$, where $f \in \mathscr{A}$ is a* skew-symmetric function. *Let q be an even number. Then*

$$\begin{aligned} &Index_f(P_1 \otimes P_1 \otimes \cdots \otimes P_1) - Index_f(P_0 \otimes P_0 \otimes \cdots \otimes P_0) = \\ &(q+1) \int_0^1 Index_{df}\ (T_t \otimes P_t \otimes \cdots \otimes P_t) dt, \end{aligned} \tag{2.401}$$

where $T_t = (1 - 2P_t)\frac{d}{dt}P_t$.

Proof It is easy to check that $\frac{d}{dt}P_t = [T_t, P_t]$. Then

$$\frac{d}{dt}\, Index_f\ (P_t \otimes \cdots \otimes P_t) = \sum_0^q Index_f\ (P_t \otimes \cdots \otimes \frac{d}{dt}P_t \otimes \cdots \otimes P_t) =$$
$$\sum_0^q\ Index_f\ (P_t \otimes \cdots \otimes [T_t, P_t] \otimes \cdots \otimes P_t) = (q+1) Index_f\ ([T_t, P_t] \otimes P_t \otimes \ldots P_t) =$$

(because f is skew-symmetric and henceforth it is stable under graded cyclic permutations)

$$= (q+1) Index_f\ b\ (T_t \otimes P_t \otimes P_t \otimes \ldots P_t) = (q+1) Index_{df}\ (T_t \otimes P_t \otimes P_t \otimes \cdots \otimes P_t)$$

(in view of Proposition 2.59).

7. **Definition 2.53** Let $f = f(x_0, x_1, \ldots, x_q) = \sum_{n=1}^\infty f_0^n \otimes f_1^n \otimes \cdots \otimes f_q^n$ be a smooth function, where $f_r^n \in \mathscr{A}$, $0 \leq r \leq p$. Define

$$Index_f\ (A_0 \otimes \cdots \otimes A_q) := \sum_n Index_{f_0^n \otimes \cdots \otimes f_q^n}\ (A_0 \otimes \cdots \otimes A_q). \tag{2.402}$$

8. **Proposition 2.62** *Let f be a skew-symmetric smooth function and q an even number.*

   Then Propositions 2.60, 2.61 hold in the case of

$$\tau_f^q(R) := Index_f\ (R \otimes R \otimes \cdots \otimes R).$$

9. The operators A, B have supports on an arbitrarily small neighbourhoods of the diagonal. Consequently, the operators S_0, S_1, P and R have arbitrarily small supports about the diagonal. For this reason, if f is a smooth Alexander–Spanier co-cycle of degree $q = even$, whose support is small about the diagonal and contained in the support of the operator R^{q+1}, then Propositions 2.60 and 2.61 show that $\tau_f^q(R)$ is an Alexander–Spanier co-cycle whose cohomology class depends only on the cohomology class of f. This information refers to just one lift L. We need to prove that $\tau_f^q(R)$ is independent of the local lift.

   Let A_i, resp. B_i, $i = 0,\ 1$, be two *local* lifts of the symbol a, resp. symbol b. By "local" we mean that both operators have support in a chosen, arbitrarily small neighbourhood $\mathscr{U}$ of the diagonal. Let $R_i = P_i - e_2$ be the operators R and P defined in Part 5 above.

   $A_t := (1 - t)A_0 + t\,A_0$, resp. $B_t := (1 - t)B_0 + tB_1$, $t \in [0,\ 1]$ is a local homotopy between A_0 and A_1, resp. B_0 and B_1.

   For any $t \in [0,\ 1]$ consider $S_{0,t} := 1 - B_tA_t$ and $S_{1,t} := 1 - A_tB_t$. With $S_{0,t}$ and $S_{1,t}$ one manufactures the path of idempotents P_t and the path of smoothing operators $R_t = P_t - e_2$. This argument, combined with Propositions 2.60, 2.61, and 2.62, show that the pairing (2.391) is independent of the local lifts of a and a^{-1}.

This completes the proof of Theorem 2.19.

Chapter 3
Hochschild, Cyclic and Periodic Cyclic Homology



Abstract Hochschild homology (along with cyclic and periodic cyclic homologies) plays in the non-commutative geometry the role which de Rham cohomology plays in the classical geometry. It is defined for any *associative algebra*. The *Hochschild chains* over the algebra $\mathscr{A}$ are not *localised* and the operations with the chains over the algebra $\mathscr{A}$ are not *commutative*. If the algebra were the algebra of differentiable functions over a topological manifold M, the corresponding Hochschild chains would be differentiable functions over M^N. Cyclic/periodic cyclic homology of the $\mathscr{A}$ were introduced to extend the Chern–Weil characteristic classes to idempotents over $\mathscr{A}$. Cyclic/periodic cyclic homology represents the minimal algebraic structure for which the Chern–Weil construction works. The cyclic/periodic cyclic homology of the algebra of differentiable functions constitutes the **link** between the classical differential geometry and non-commutative geometry.

3.1 Hochschild Homology of the Algebra of Smooth Functions, $HH_*(C^\infty(M, \mathbb{R}))$

In this section we prove that the *continuous Hochschild homology* in degree k of the *algebra of smooth functions* on a differentiable manifold M is the space of k-differential forms on M. This result was proved by Connes [82] for compact manifolds M and was extended to paracompact manifolds by Teleman [107, 114]. Connes proof uses a *projective resolution* of the algebra of smooth functions. Teleman's proof follows up a Hochschild p-cycle of the algebra, which *a priori* is supported on $M \times M \times \cdots \times M$ ($k+1$ factors) and shows that its homology class lives on the main diagonal. The proof shows that the cycle can be replaced by co-homologous chains having smaller and smaller supports. The procedure is decomposed into steps. The various steps show that the Hochschild homology class may be reduced gradually to arbitrary small neighbourhoods of the diagonal, then to *germs* of the chain all along the diagonal and finally to *infinite jets along the diagonal*. The infinite jets along the diagonal form a complex called *diagonal complex*. The last step computes the homology of the diagonal complex. It is shown that the homology of the diagonal complex depends only on the first order

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derivatives of the chain (in the direction of all factors, with the exception of the first) along the diagonal.

It is important to conclude that the Hochschild complex, in the case of the algebra of smooth functions, takes care of all ingredients involved in the construction of differential forms: derivations (tangent fields), *localisation* (multi-linearity over the algebra of smooth functions) and *skew symmetry* of differential forms.

This result is the **link between classical differential geometry (which turns up to live on the main diagonal of the various powers of the manifold) and non-commutative geometry (which lives on the various powers of the space)**.

The *result* of a classical differential geometry problem could be expressed as a *classical differential form* provided its *restriction to the diagonal* makes sense, i.e. whether the *product of distributions* involved in the problem may be performed with the classical differential geometry analysis. This conclusion becomes clearer when it is applied onto the various stages if the index formula, from Riemann curves, passing through smooth manifolds and extended to topological manifolds.

This result should not be *confused* with the Hochschild–Kostant–Rosenberg theorem [33] regarding the Hochschild homology of *smooth algebras*. *The algebra of smooth functions on a differentiable manifold is not a smooth algebra*.

Another procedure which allows to compute the Hochschild homology of function algebras is presented in the article by Brasselet et al. [113].

3.1.1 Preliminaries on the Algebra of Smooth Functions

In this section we compute the Hochschild homology of the algebra $\mathscr{A} = C^\infty(M, \mathbf{R})$, where M is a *paracompact* smooth manifold

$$C_k(C^\infty(M, \mathbf{R})) = \{ f \mid f : M^{k+1} \longrightarrow \mathbf{R}, \quad f = smooth. \}. \tag{3.1}$$

The *Hochschild boundary* $b = \sum_{0 \le r \le k} (-1)^r \ b_{k,r}$ is the alternating sum of the *face operators* $b_{k,r}$. The face operator $b_{k,r}$ acting onto the function f is the restriction of f to the *diagonal* $\iota_{k,r} : M^k \hookrightarrow M^{k+1}$ and $b_{k,r} f =: f \circ \iota_{k,r}$

$$(\iota_{k,r})(x_0, x_1, \ldots, x_{k-1}) = \begin{cases} (x_0, x_1, \ldots, x_{r-1}, \ x_r, x_r, \ , \ldots, x_{k-1}) & \text{if } 0 \le r \le k-1 \\ (x_0, x_1, \ldots, x_{k-1}, x_0) & \text{if } r = k. \end{cases}$$

Let Δ_k be the main diagonal in M_k, $\Delta_k = \{(x, \ x, \ldots, \ x) \mid x \in M\}$. Let g be a Riemannian on M and r a Riemannian metric on M. We chose a Riemannian metric d on M. We suppose the corresponding geodesic distance has the injectivity radius is at least 1.

Let $\lambda : [0, \infty) \longrightarrow [0, 1]$ be a decreasing function on M. We assume the support of λ is contained in the interval $[0, 1]$ and we assume that λ_t is identically 1 on the interval $[0, 1/2]$.

We denote by λ_t be a rescale of the function λ

$$\lambda_t(x) \;=\; \lambda \,.\, (\frac{x}{t}), \qquad for \quad 0 < t.$$

We denote by ρ_k the distance to the diagonal

$$\rho_k : M_{k+1} \longrightarrow [0, 1]$$

$$\rho_k(\, x_0,\; x_1, \ldots, x_k\,) \;=\; d^2(\, x_0, x_1\,) + d^2(\, x_1, x_2\,) + \ldots..+ d^2(\, x_{k-1},\; x_k\,) + d^2(\, x_k, x_0\,).$$

We denote by U_t any *t-neighbourhoods of the diagonal*

$$U_t := \{(\, x_0,\; x_1, \ldots,\; x_k\,) \mid \rho_k(\, x_0, x_1, \ldots, x_k\,) \;<\; t\}.$$

We denote by $\rho_{k,t}$ the function $\lambda_t \circ \rho_k$. The function $\rho_{*,\,t}$ is a chain homomorphism of the cyclic Hochschild complex, see Connes [82]. The function $\rho_{k,\,t}$ has support in the t-neighbourhood of the diagonal. It is identically 1 on the $U_{t/2}$-neighbourhood of the diagonal.

Theorem 3.1 (Localisation on the First Factor) *Let ξ be any smooth function on M. Then the multiplication of the elements in C_k by the function ξ_k, $0 \leq k$, is a chain homomorphism of the Hochschild complex.*

The proof is evident and left to the reader.

3.1.2 Localisation of Germs Along the Diagonal

We are going to show here that the class of any Hochschild cycle depends only on its *germ* about the main diagonal Δ.

To do so, we decompose any Hochschild cycle $f \in C_k$ into a sum of two cycles

$$f = \rho_{k,t} \,.\, f \;+\; (1 - \rho_{k,t}) \,.\, f.$$

The support of the first cycle is in U_t, while the support of the second term is identically zero on $U_{\frac{t}{2}}$.

Definition 3.1 Let C^t_* be the sub-complex of C_*. This sub-complex consists of those chains f which vanish on the t-neighbourhood of the diagonal.

Let us denote by C^0_* the inductive limit of C^t_* as t decreases to 0.

We will show that the complex C^0_* is acyclic. To do so, we construct a homotopy between the identity and the zero homomorphism in the complex C^0_*.

Definition 3.2 We define the *extension operators*

$$\mathscr{E}_t : C_k \longrightarrow C_{k+1}$$

$$\mathscr{E}_t(f)\,(\,x_0 \ldots, x_{k+1}\,) \;=\; \lambda_t(\,d^2(\,x_0, x_1\,)\,)\,.\,f(\,x_1 \ldots, x_{k+1}\,).$$

Proposition 3.1 *The extension operators satisfy the identities*

$$\mathscr{E}_t : C_k^s \longrightarrow C_{k+1}^{\frac{s}{4}}, \quad \forall\, 0 \leq s. \tag{3.2}$$

Proof Suppose $(x_0 \ldots, x_{k+1})$ is a point such that

$$d^2(\,x_0, x_1\,) \;+\; d^2(\,x_1, x_2\,) \;+\; \ldots \;+\; d^2(\,x_k, x_{k+1}\,) \;+\; d^2(\,x_{k+1}, x_0\,) \;<\; \frac{s}{4}.$$

For such a point one has

$$d^2(\,x_1, x_2\,) \;+\; d^2(\,x_2, x_3\,) \;+\; \ldots \;+\; d^2(\,x_k, x_{k+1}\,) \;<\; \frac{s}{4}$$

$$d^2(\,x_0, x_1\,) \;+\; d^2(\,x_{k+1}, x_0\,) \;<\; \frac{s}{4}, \quad d^2(\,x_0, x_1\,) \;<\; \frac{s}{4}, \quad d^2(\,x_{k+1}, x_0\,) \;<\; \frac{s}{4}.$$

From these inequalities we get

$$d^2(\,x_{k+1}, x_1\,) \leq (d(\,x_{k+1}, x_0\,) \;+\; d(\,x_0, x_1\,))^2 \;<\; \frac{3s}{4}$$

$$d^2(\,x_1, x_2\,) \;+\; d^2(\,x_2, x_3\,) \;+\; \ldots \;+\; d^2(\,x_k, x_{k+1}\,) \;+\; d^2(\,x_{k+1}, x_1\,) \;<\; s.$$

For such a point, if f vanishes on U_s, then $\mathscr{E}_t f$ vanishes on the $U_{\frac{s}{4}}$.

Definition 3.3 Let $\mathscr{N}_t$ be the function defined by the relation

$$\delta\,\mathscr{E}_t \;+\; \mathscr{E}_t\,\delta \;=\; 1 \;-\; \mathscr{N}_t.$$

Then

$$(\mathscr{N}_t\; f)(x_0, x_1, \ldots., x_k) \;=\; (-1)^k\;\lambda_t(\,d^2(x_0, x_1)\,)\,[\;f(x_1, x_2, .., x_k, x_0\,) \;-\; f(\,x_1, x_2, .., x_k, x_1)\,].$$

Proposition 3.2 *One has the relation*

$$[\;\mathscr{N}_t\;]^k \mid C_k^{(k+k^2)\,\cdot\, t} \;=\; 0.$$

Proof Notice, the function $\mathcal{N}$ is a linear combination of functions involving f, their restrictions to certain diagonals and permutations of the arguments

$$[\,\mathcal{N}_t\,]^k\ f(\,x_0, x_1, \ldots, x_k\,) \;=\; \prod_{i=0}^{k-1} \lambda_t(d^2(x_i, x_{i+1}))\cdot(\,\mathcal{F}\ f\,)\,(\,x_0, x_1, \ldots, x_k\,).$$

The first factor of this expression is not going to vanish at an arbitrary point $(\,x_0, x_1, \ldots, x_k\,)$ unless

$$d^2(\,x_i, x_{i+1}\,) \;<\; t, \quad for \quad 0 \le i \le k-1.$$

For such a point the triangle inequality gives

$$d(\,x_0, x_k\,) \;<\; k\,.\,t^{\frac{1}{2}}.$$

We have further

$$\sum_{i=0}^{k} d^2(\,x_i, x_{i+1}\,) \;<\; k\,t \;+\; k^2\,t.$$

If the function f vanishes on $U_{(k+k^2)\,.\,t}$ one has necessarily

$$[\mathcal{N}_t]^k\ \ f\ (\,x_0, x_1, \ldots, x_k\,) \;=\; 0.$$

Proposition 3.2 is proven.

Lemma 3.1 *$\mathcal{N}_t$ satisfy the relations*

$$\delta\ \mathcal{N}_t \;=\; \mathcal{N}_t\ \delta.$$

Proof One has

$$\delta\,\mathcal{N}_t \;=\; \delta\,(1 \,-\, [\delta\mathcal{E}_t \,+\, \mathcal{E}_t\,\delta]) \;=\; \delta - \delta\,\mathcal{E}_t\delta \;=\; (\,1 - [\delta\,\mathcal{E}_t \,+\, \mathcal{E}_t\,\delta]\,)\,\delta \;=\; \mathcal{N}_t\,\delta.$$

Definition 3.4 Let us introduce the operator

$$\mathcal{K}_t \,:=\, \mathcal{E}_t\,.\,\sum_{r=0}^{r=k-1} [\,\mathcal{N}_t\,]^r. \tag{3.3}$$

Proposition 3.3 *The functions $\mathcal{K}_t$ satisfy the identity*

$$\mathcal{K}_t : C_k^{(k\,+\,k^2)\,.\,t} \longrightarrow C_{k+1}^{(k\,+\,k^2)\,.\,4^{-(k+1)}\,.\,t}$$

and the homotopy identity

$$\delta \mathscr{K}_t \ + \ \mathscr{K}_t \, \delta \ = \ 1.$$

Proof The first part of the statement follows from relation (3.2) along with the observation that $\mathscr{K}_t$ is a linear combination of operators of the form

$$\mathscr{E}_t \ (\ \delta \ \mathscr{E}_t \ )^{\, r} \quad and \quad \mathscr{E}_t \ (\ \mathscr{E}_t \ \delta \ )^{\, r}, \quad with \quad 0 \leq r \leq k.$$

The second part (3.3) follows from Proposition 3.2. Indeed, it tells that $\mathscr{N}_t^k \ = \ 0$ on the space $C_k^{(k+k^2).t}$. In addition,

$$1 \ = \ 1 - \mathscr{N}_t^k \ = \ (\ 1 - \mathscr{N}_t \ ) \, . \ \sum_{r=0}^{r=k-1} [\mathscr{N}_t]^r \ = \ (\ \delta \mathscr{E}_t \ + \ \mathscr{E}_t \ \delta \ ) \, . \ \sum_{r=0}^{r=k-1} [\mathscr{N}_t]^r \ =$$

$$\delta \ \mathscr{E}_t \, . \ \sum_{r=0}^{r=k-1} [\mathscr{N}_t]^r \ + \ \mathscr{E}_t \, . \ \sum_{r=0}^{r=k-1} [\mathscr{N}_t]^r \ \delta.$$

Remark 3.1 It is easy to check that C_t^* is acyclic. To prove this it is sufficient to have a pair of operators $\mathscr{K}_{k-1,t}, \ \ \mathscr{K}_{k,t}$ which satisfy

$$(\delta \ \mathscr{K}_{k-1,t} \ + \ \mathscr{K}_{k,t} \ \delta)(f_k) \ = \ f_k.$$

Notice that it is not necessary to keep the same operators $\mathscr{K}_{k,t}$ for all k.

Theorem 3.2 *The complex of* germs C_*^0 *is acyclic.*

Proof Use induction limit.

Definition 3.5 (Fine Algebra) We suppose $\mathscr{A}$ be an algebra of functions on a metric space M. The algebra is called *fine* iff for any open covering $\mathscr{U}$ of M there exists a partition of unity with functions belonging to the algebra $\mathscr{A}$ whose supports belong in the sets of the covering $\mathscr{U}$.

We have proved the following result.

Theorem 3.3 *Let M be metric space and $\mathscr{A}$ a fine algebra of functions on M.*

The Hochschild homology of $\mathscr{A}$ depends only on the germs of functions on the main diagonals of the powers of M.

3.1.3 Micro-Localisation of Hochschild Homology

The micro-localisation of the Hochschild homology of the algebras of smooth functions is obtained by cutting progressively, towards the diagonal, the support of the cycle $f \in C^\infty(M)$. We show at the same time that the Hochschild homology class of the cycle f depends only on the ∞-jet of the germ of f along the diagonal. We show also that the homology class of f depends only on the skew-symmetrised 1-jet of f with respect to all variables with the exception of the first one.

In the previous sub-section we saw that any Hochschild cycle may be localised with respect to the first variable (Theorem 3.1). We have showed that its homology class depends only on its restriction to an arbitrary small neighbourhood of the diagonal (Theorem 3.19). Therefore we may suppose that $M = \mathbb{R}^n$.

Definition 3.6 Denote by J^∞_k the vector space of ∞-jets of $f \in C^\infty(M_{k+1}, \mathbb{R})$, with respect to all variables, with the exception of the first variable x_0, along the main diagonal.

Any element of the vector space J^∞_k is a formal power series in the variables $x^i_j - x^i_0$, with $1 \leq i \leq n$ and $1 \leq j \leq k$, with coefficients that are smooth functions in x_0. We have produced the short exact sequence of complexes

$$0 \longrightarrow J^0_* \longrightarrow C_* \longrightarrow J^\infty_* \longrightarrow 0. \tag{3.4}$$

Remark 3.2 The complex J^∞_* is isomorphic to the Hochschild complex

$$C_{C^\infty(M),*}(\, J^\infty(M),\ C^\infty(M)\,).$$

$C^\infty(M)$ is seen as a $J^\infty(M)$-module. The module action is given by the multiplication of jets by jets depending on the variables $x_0, x_1, \ldots, x_k$.

Theorem 3.4 *$J_{0,*}$, endowed by the Hochschild boundary, is complex.*
J^∞_* is acyclic.

Proof The function $\frac{d}{dt}\,\lambda_{C(k)\,.\,t}$ (where $C(k)$ are constants) vanishes on a neighbourhood of the diagonal. We define a new function $\frac{d}{dt}\,\{\,(\lambda_{C(k).t} \circ \rho_k)\,.\,f\}$. We introduce it into the formula (3.3). We obtain the relation

$$\frac{d}{dt}\{\,(\lambda_{C(k)\,.\,t} \circ \rho_k)\,.\,f\} \;=\; [\mathscr{K}_t\,\delta_k \;+\; \delta_{k+1}\mathscr{K}_t]\,(\,\frac{d}{dt}(\,\lambda_{C(k)\,.\,t} \circ \rho_k)\,.\,f).$$

Let us suppose that the support of f is contained in $U_{1/2C(k)}$; we integrate the above equality on the interval $[\epsilon, 1]$. We obtain

$$f - (\lambda_{C(k).\epsilon} \circ \rho_k) \cdot f = (\lambda_{C(k)} \circ \rho_k) \cdot f - (\lambda_{C(k)\,.\,\epsilon} \circ \rho_k) \cdot f = \int_\epsilon^1 \frac{d}{dt} (\lambda_{C(k)\,.\,t} \circ \rho_k) \cdot f \, dt =$$

$$\int_\epsilon^1 \mathscr{K}_t \, \delta_k \frac{d}{dt} (\lambda_{C(k)\,.\,t} \circ \rho_k) \cdot f \, dt + \int_\epsilon^1 \delta_{k+1} \, \mathscr{K}_t \, \frac{d}{dt} (\lambda_{C(k)\,.\,t} \circ \rho_k) \cdot f \, dt.$$

δ commutes with the multiplier $\frac{d}{dt}(\lambda_{C(k).t} \circ \rho_k)$. As a consequence we have

$$f - (\lambda_{C(k)\,.\,\epsilon} \circ \rho_k) f = (\delta_{k+1} \mathscr{K}(\epsilon) + \mathscr{K}(\epsilon) \delta_k) f. \tag{3.5}$$

In this formula

$$\mathscr{K}(\epsilon) f = \int_\epsilon^1 \mathscr{K}_t \frac{d}{dt} (\lambda_{C(k).t} \circ \rho_k) f \, dt. \tag{3.6}$$

Theorem 3.4 follows from the next lemma.

Lemma 3.2 *There is*

$$Lim_{t \to 0} (\lambda_{C(k)\,.\,t} \circ \rho_k) . f = 0, \quad \forall f \in J_k^0. \tag{3.7}$$

Define

$$\mathscr{K} f \doteq Lim_{\epsilon \to 0} \mathscr{K}(\epsilon) f. \in J_{k+1}^0$$

From this it follows

$$f = (\delta_{k+1} \mathscr{K} + \mathscr{K} \delta_k) f.$$

Proof While $t \to 0$ the integrands and their partial derivatives are of finite order $O(1/t^{r+1})$ at the main diagonal; this divergence is absorbed by the infinite order vanishing of f along the main diagonal

$$f(x_0, x_1, \ldots x_k) = 0(m)(x_1 - x_0, x_2 - x_0, \ldots, x_k - x_0),$$

for any non-negative number m.

The homotopy formula (3.5) implies the last part of the lemma.

Notice that the induced boundary operator in J_*^∞ acts pointwisely.

Theorem 3.5 *Let $\mathscr{A}$ be the algebra $C^\infty(M, \mathbb{R})$.*
Then the homomorphism

$$j^\infty : C_* \to J^\infty_* \tag{3.8}$$

induces isomorphism in homology.
Therefore

$$HH_k(C^\infty(M)) \;=\; H_k(J^\infty_*(M)). \tag{3.9}$$

3.1.4 Hochschild Homology of J^∞_*

Theorems 3.1 and 3.5 say that the Hochschild homology of the algebra $\mathscr{A}$ is local and that is concentrated along the each diagonal of $\mathbb{R}^n$. Therefore, it is sufficient to compute this homology in the case of $M \;=\; \mathbb{R}^n$.

Denote by $(\, x_0,\; x_1, \ldots,\; x_k\,) \in (\, \mathbb{R}^n\,)^{k+1}$ and by $(\, x^1_i, \ldots,\; x^n_i)$ the coordinates in the ith-factor $\mathbb{R}^n$ of this space.

Definition 3.7 Denote by J^∞_k the vector space of infinite jets of smooth functions on $(\mathbb{R}^n)^{k+1}$ along the main diagonal, in the direction of the factors $0 < i$.

Any element of J^∞_k is represented a formal power series in $x^j_i \;-\; x^j_0$, whose coefficients are smooth functions in x_0, see Sect. 3.1.3.

Theorem 3.6

1. *$J^\infty(M)$ is a $J^\infty(M)$-module given by the multiplication of functions with the 0th-order component of the jet.*
2. *One has*

$$H_k(\, J^\infty_*\,) \;=\; HH_{C^\infty(M),\,k}(\, J^\infty(M), C^\infty(M)\,).$$

3. *The homology is*

$$H_k(\, J^\infty_*\,) \;=\; \{\sum_{1\le j_1,\ldots,j_k\le n} a_{j_1,\,j_2,\ldots,\,j_k}(x_0)(\, x^{j_1}_1 \;-\; x^{j_1}_0\,) \wedge \cdots \wedge (\, x^{j_k}_k \;-\; x^{j_k}_0)\},$$

where $a_{\,j_1,\,j_2,\ldots,\,j_k}(x_0)$ are smooth functions and $\wedge$ indicates skew-symmetrisation of the variables $x_i \;-\; x_0\,,$ for $0 < i$.

Proof To prove the theorem we need the next five lemmas.

Definition 3.8

1. For any monomial

$$a_j^i(\ x_0\ )(\ x_{i_1}^{j_1}\ -\ x_0^{j_1})^{\alpha_1}\ldots.(\ x_{i_r}^{j_r}\ -\ x_0^{j_r})^{\alpha_k}, \tag{3.10}$$

   k is called of *homological* degree, while $\sum_i |\alpha_i|$ is called *algebraic degree*.
2. A *degenerated monomial* is a monomial which does not depend on one or more of the vector variables $x_1, \ldots, x_k$; such a monomial is called *lacunary*. We denote by $J_*^{\infty,L} \subset J_*^{\infty}$ the sub-complex of *lacunary* chains.
3. Denote by $J_*^{\infty,T}$ the sub-complex generated by all remaining monomials.

Lemma 3.3 *The complex J_*^{∞} decomposes in a direct sum of complexes*

$$J_*^{\infty}\ =\ J_*^{\infty,L} \oplus J_*^{\infty,T}.$$

Lemma 3.4 *The sub-complex of lacunary chains is acyclic.*

Proof Proposition 2.32(iii) says that the lacunary complex is acyclic. We give here a direct proof in this specific case.

Definition 3.9

1. A *lacuna* is a *maximal string* of consecutive missing vector variables in a lacunary monomial.
2. The *length* of a lacuna is the number of missing variables in the lacuna.
3. The *index* of a lacuna is the index of the first missing vector variable in the lacuna.
4. The *order* of a monomial with lacunae is the first missing variable $x_1, \ldots, x_k$, counted from the left.

For any lacuna, we agree to replace any missing vector variable by the factor 1; in this way any lacuna of length h will have h consecutive factors 1.

Let $k+1$ indicate the number of variables. Suppose the monomial contains the factor $(x_{i_k}^{j_k}\ -\ x_0^{j_k})$ to a positive power. Then the last Hochschild face operators are zero.

Lemma 3.5 *The complex $J^{\infty,T}$ is a direct sum of complexes*

$$J^{\infty,T} \simeq C^{\infty}(\mathbb{R}^n) \otimes_{\mathbb{R}} B_{\star}(J_0^{\infty}(\mathbb{R}^n).$$

In this formula $B_{\star}(J_0^{\infty}(\mathbb{R}^n))$ is the bar complex *of the algebra $J_0^{\infty}(\mathbb{R}^n)$ of jets vanishing at the origin.*

Proof We notice the first and the last Hochschild face operators acting on monomials of $J^{\infty,\,T}$ are zero.

The last face operator is zero; therefore, the Hochschild boundary reduces to the bar boundary operator $b^{'}$. The algebra $B_{\star}(J_0^{\infty}(\mathscr{R}^n))$ algebra does not have a unit. For this reason, its bar complex is not necessarily acyclic, comp. Proposition 2.19.

Lemma 3.6 *$B_\star$ decomposes in a direct sum*

$$B_\star(\ J_0^\infty(\mathbb{R}^n))\ )\ =\ \sum_r\ J_0^r.$$

In this formula J_0^r is the sub-complex of $B_\star(J_0^\infty(\ \mathbb{R}^n))\ )$ generated by all monomials of total degree r.

Proof The boundary $b^{'}$ preserves the algebraic degree of monomials. Let J_0^r be the sub-complex of $B_\star(J_0^\infty(\mathbb{R}^n)))$ generated by all monomials of total degree r.

The complex $B_\star(J_0^\infty(\mathbb{R}^n)))$ splits up in the direct sum of complexes

$$B_\star(J_0^\infty(\mathbb{R}^n)))\ =\ \sum_r J_0^r.$$

Lemma 3.7 *One has the decomposition by complexes*

$$H_r(\ J_0^r\ )\ =\ \wedge^r(T^\star(\ \mathbb{R}^n)\ ). \tag{3.11}$$

Proof The proof of the lemma uses a spectral sequence associated to a filtration of J_0^r and an induction on r.

For $r = 1$, the result is clear. We suppose the lemma has been proved for $r-1$. We introduce the following increasing filtration on the complex J_0^r: F_p^r be the sub-complex of J_0^r generated by the monomials

$$a^i_j(x^{j_1}_{i_1})^{\alpha_1}\dots(x^{j_r}_{i_r})^{\alpha_k},\quad \text{such that } \alpha_1 \text{ divides r-p.} \tag{3.12}$$

We denote by $F^r_{p,q}$ the subset of F^r_p of chains having homological degree $p+q$.

Let b^0 the boundary operator defined on the term E^0 of the spectral sequence; it is the bar boundary operator minus the first face operator. By the induction argument we know that the only non-zero terms of E^1 is

$$E^1_{p,1}\ =\ \mathbb{R}_{r-p}[\ x^1,\dots,x^n\ ]\otimes_{\mathbb{R}}\wedge^p T^\star(\mathbb{R}^n),\qquad for\qquad 1\le p\le r-1,$$

where we have denoted by $\mathbb{R}_k[\ x^1,\dots,x^n\ ]$ the space of homogeneous polynomials of degree k.

We think of the elements of $E^1_{p,1}$ as differential forms of degree p on $\mathbf{R}^n$ whose coefficients are homogeneous polynomials of degree $r-p$. The boundary operator b^1, acting on E^1, is the *interior product* i_X of forms with the tangent field

$$X=\sum_{k=1}^n x^k\frac{\partial}{\partial x^k}.$$

Recall that d is the differential of forms. We compute E^2. We define the operators

$$h_p := d : E^1_{p,1} \longrightarrow E^1_{p+1,1}, \qquad for\ 2 \leq p \leq r-1.$$

Recall any k-differential form

$$\omega = \sum_{\alpha} a_\alpha(x) dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_k}$$

satisfies the identity

$$(i_X \circ d + d \circ i_X)(\omega) = k\, \omega + \sum_{\alpha} (X\, a_\alpha(x)) dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_k}.$$

We put together this identity with the Euler identity and we find that $\omega \in E^1_{p,1}$ satisfies the homotopy formula

$$(i_X \circ d + d \circ i_X)(\omega) = (p + (r-p))\, \omega = r\, \omega.$$

This homotopy formula shows that if

$$d(E^1_{p,1}) \subset E^1_{p+1,1},$$

then $E^2_{p,1} = 0$.

We begin by considering $r \leq n$. The term $d(E^1_{r-1,1})$ is not contained in $E^1_{r,0}$ because the differential forms in $E^{p,1}$ should have as coefficients polynomials of degree at least one.

If $2 \leq r - p$, the operator $h := d$ is a well defined operator on $E^1_{p,1}$. In this case the homotopy formula (3.1.4) shows that $E^2_{p,1} = 0$. As $E^1_{r,1} = 0$, we get

$$E^2_{r-1,1} = Ker\{E^1_{r-1,1} \to E^1_{r-2,1}\}. \tag{3.13}$$

The homotopy formula (3.1.4) applied onto the term $\omega \in E^1_{r-1,1}$ gives

$$(b^1 \circ d + d \circ b^1)(\omega) = r\, \omega.$$

These formulas show ω is in $Ker\{E^1_{r-1,1} \to E^1_{r-2,1}\}$ if and only if $\omega \in (i_X \circ d)(\mathbb{R}_1 \otimes_{\mathbb{R}} \wedge^{r-1} T^\star(\mathbb{R}^n)$. But, on the other hand, we have

$$d(\mathbb{R}_1 \otimes_{\mathbb{R}} \wedge^{r-1} T^\star(\mathbb{R}^n) = \wedge^r T^\star(\mathbb{R}^n). \tag{3.14}$$

We obtain

$$\omega \in i_X(\wedge^r T^\star(\mathbb{R}^n)).$$

This corresponds via the previous identification

$$to\ jets \longleftrightarrow differential\ forms\ with\ polynomial\ coefficients$$

to r-skew-symmetric multilinear jets.

For $r > n$ the homotopy formula (3.1.4) implies $E^2_{p,q} = 0$ for any p, q.

Therefore, $E^2_{p,q} = E^\infty_{p,q}$.

This completes the proof of Theorem 3.6.

Theorem 3.7 (Teleman [107, 114], Connes [82])

1. *Let M be a smooth manifold M.*
   One has the isomorphism

$$J : HH_k(C^\infty(M)) = C^\infty(M, \wedge^k T^*M). \tag{3.15}$$

2. *The isomorphism J is induced by the mapping $J : C_k(C^\infty(M)) \longrightarrow \Omega^k(M)$*

$$J\ f(x_0, x_1, \ldots, x_k) := \left(\frac{\partial^k f(x_0, x_1, \ldots, x_k)}{\partial x_1^{i_1} \ldots \partial x_k^{i_k}}\right)_{diag} dx_1^{i_1} \wedge \cdots \wedge dx_k^{i_k}. \tag{3.16}$$

3.2 Computation of $HH_*(C^\infty(M))$ Using One Derivative

Remark 3.3 In the previous section we computed the Hochschild homology of the algebra of smooth functions showing how the *topology, the Fréchet structure and the local algebra interact*. In this section we compute the Hochschild homology of the algebra of smooth functions $\mathscr{A}$ using just one partial derivative.

The method we present here shows that *Schwarz' lemma* on mixed second order partial derivatives enters into the computation of the Hochschild homology. This is particularly significant in the case of functions with reduced regularity. *This is the* glue *which connects the L_2 (consisting of discontinuous functions) analysis with local topological data*. This connection enters, implicitly, into the picture by means of distribution theory.

Definition 3.10 Let $f \in \mathscr{A}$. Define

$$j^1_k(f) := \left(\frac{\partial^k f(x_0, x_1, \ldots, x_k)}{\partial x_1^{i_1} \ldots \partial x_k^{i_k}}\right)_{|x_0 = x_1 = \cdots = x_k} dx_1^{i_1} \otimes \cdots \otimes dx_k^{i_k}.$$

The Schwarz lemma tells that the order in which the partial derivatives appear in this formula does not matter. The RHS of this formula is independent of local coordinates.

Proposition 3.4 *The homomorphisms j^1_* satisfy the identities*

$$j^1_{k-1}(\,bf\,) \;=\; 0. \tag{3.17}$$

Proof

$$\begin{aligned}
j^1_{k-1}(\,bf\,) &:= \frac{\partial^{k-1}(bf)(x_0,x_1,\ldots,x_{k-1})}{\partial x_1^{i_1}\ldots\partial x_{k-1}^{i_{k-1}}}_{|x_0=x_1=\ldots=x_{k-1}} \times\, dx_1^{i_1}\otimes\cdots\otimes dx_{k-1}^{i_{k-1}} \;= \\
&(-1)^0\frac{\partial^{k-1}}{\partial x_2^{i_1}\ldots\partial x_k^{i_{k-1}}}\;\; f(x_0,x_0,\ldots,x_{k-1}) + \\
&\sum_{1\le r\le k-1}(-1)^r\Big(\frac{\partial^{k-1}}{\partial x_1^{i_1}\ldots\partial x_r^{i_r}\partial x_{r+2}^{i_{r+1}}\ldots\partial x_k^{i_{k-1}}}+\frac{\partial^{k-1}}{\partial x_1^{i_1}\ldots\partial x_{r-1}^{i_{r-1}}\partial x_{r+1}^{i_r}\ldots\partial x_k^{i_{k-1}}}\Big)\; f(x_0,\ldots,x_r,x_r,\ldots x_{k-1}) + \\
&(-1)^k\frac{\partial^{k-1}}{\partial x_1^{i_1}\ldots\partial x_{k-1}^{i_{k-1}}}\;\; f(x_0,\ldots,x_{k-1},x_0)\;_{|x_0=x_1=\ldots x_{k-1}=x_k}\;\times\, dx_1^{i_1}\otimes\cdots\otimes dx_{k-1}^{i_{k-1}} \;= \\
&\Big((-1)^0\frac{\partial^{k-1}}{\partial x_2^{i_1}\partial x_3^{i_2}\ldots\partial x_k^{i_{k-1}}}\big(\,f(x_0,x_0,x_1,x_2,\ldots,x_{k-1})\;-\;f(x_0,x_1,x_1,x_2,\ldots,x_{k-1})\,\big)_{|x_0=x_1=\ldots=x_k}+ \\
&(-1)^1\frac{\partial^{k-1}}{\partial x_1^{i_1}\partial x_3^{i_2}\ldots\partial x_k^{i_{k-1}}}\big(\,f(x_0,x_1,x_1,x_2,\ldots,x_{k-1})-f(x_0,x_1,x_2,x_2,\ldots,x_{k-1})\,\big)_{|x_0=x_1=\cdots=x_k}+ \\
&(-1)^2\frac{\partial^{k-1}}{\partial x_1^{i_1}\partial x_2^{i_2}\partial x_4^{i_3}\ldots\partial x_k^{i_{k-1}}}\big(\,f(x_0,x_1,x_2,x_2,\ldots,x_{k-1})-f(x_0,x_1,x_2,x_3,x_3,\ldots,x_{k-1})\,\big)_{|x_0=x_1=\ldots=x_k}+ \\
&\qquad\ldots+ \\
&(-1)^{k-1}\frac{\partial^{k-1}}{\partial x_1^{i_1}\partial x_2^{i_2}\partial x_3^{i_3}\ldots\partial x_{k-1}^{i_{k-1}}}\big(\,f(x_0,x_1,x_2,\ldots,x_{k-1},x_{k-1})\;-\;f(x_0,x_1,x_2,\ldots,x_{k-1},x_0)\,\big)_{|x_0=x_1=\ldots=x_k}\Big)\;\times \\
&\times dx_1^{i_1}\otimes\cdots\otimes dx_{k-1}^{i_{k-1}} \;=\; 0+0+0+\ldots+0=0.
\end{aligned} \tag{3.18}$$

Proposition 3.4 tells that j^1_* induces the zero boundary homomorphism on J^1_*. Clearly, $0\circ j^1_k \;=\; j^1_{k-1}\circ 0$. This means that j^1_* is a chain homomorphism.

Hence, $Ker\, j^1_*$ is a sub-complex of the complex $\{C_*(\mathscr{A})\}$.

Notice that $d\xi_1^{i_1}\otimes\cdots\otimes d\xi_k^{i_k}$ constitute a basis of $T^*_{x_0}(M)\otimes\cdots\otimes T^*_{x_0}(M)$. One gets the following result.

Proposition 3.5 *The space Ker j^1_k is generated by chains $f\in C_k(\mathscr{A})$ with the property that*

$$\Big(\frac{\partial^k f}{\partial x_1^{i_1}\ldots\partial x_k^{i_k}}\Big)_{|x_0=x_1=\ldots=x_k} \;=\; 0. \tag{3.19}$$

Definition 3.11 Let M be a smooth manifold. Let $\mathscr{A}:=C^\infty(\,M\,)$.

Let

$$C_k^0(\mathscr{A}) := Ker\ j_k^1. \tag{3.20}$$

Let b denote the boundary in the complex $C_*^0(\mathscr{A})$.

One has the short exact sequence of complexes

$$0 \to C_*^0 \xrightarrow{i_*} C_* \xrightarrow{j_1} J_*^1 \to 0. \tag{3.21}$$

Proposition 3.6 *The inclusion i_* induces the zero homomorphism in homology*

$$0 = i_{k,*} : H_k(C_*^0) \longrightarrow H_k(C_*). \tag{3.22}$$

Proof

(1) C_*^0 contains *all degenerate chains* C_*^{deg}. Indeed, any degenerate chain of degree k is a sum of chains γ_h which do not depend on at least one of the variables $x_1, \dots x_k$. For such a chain $j_1(\gamma_h) = 0$.
(2) The complex $\{ C_*^0 \}$ *splits up in a direct sum of complexes*

$$\{ C_*^0,\ b \} = \{ C_*^{deg},\ b \} \oplus \{ \tilde{C}_*^0,\ b \}. \tag{3.23}$$

Indeed, $1 - \Pi_k$ (restricted to C_k^0) is a projection onto C_k^{deg}; it commutes with b.
(3) From part (2) we get

$$i_{k,*} = i_{k,*}(H_k\{ C_*^{deg},\ b \}) + i_{k,*}(H_k\{ \tilde{C}_*^0,\ b \}). \tag{3.24}$$

(4) Degenerate chains do not contribute to the Hochschild homology; therefore, $i_{k,*}(H_k\{ C_*^{deg},\ b \}) = 0$.
(5) We are going to prove that $i_{k,*}(H_k\{ \tilde{C}_*^0,\ b \}) = 0$.

Let $f \in \tilde{C}_k^0$; we intend to analyse f about the diagonal. This is a local problem. We introduce a new system of coordinates $\xi = (\xi_1, \xi_2, \dots, \xi_k)$ for $x_0 = (x_0^1, \dots, x_0^k)$, $\xi_i := x_i - x_0$, $1 \le i \le k$.

The Taylor expansion of $f \in \tilde{C}_*^0$ at a point of the diagonal gives

$$f(x_0, x_1, \dots, x_k) = \sum_{i_1 \dots i_k} C_{i_1 \dots i_k} \frac{\partial^k f}{\partial \xi_1^{i_1} \dots \partial \xi_k^{i_k}} \xi_1^{i_1} \dots \xi_k^{i_k} + \text{higher order terms}. \tag{3.25}$$

Here we have specified the lowest order (non-vanishing terms) $C_{i_1,\dots,i_k}$.
Proposition 3.5 says that $(\partial^k f / \delta x_1^{i_1} \dots \delta x_k^{i_k})_{|\delta x_0 = x_1 = \dots = x_k} = 0$.

The next lemma completes the proof of Proposition 3.6.

Lemma 3.8 *For any $f \in C_k^0$ the Taylor expansion of f begins with terms of order $k+1$.*

Definition 3.12 Consider the operators $\mathscr{K}_k$ (3.3), $\mathscr{K}(\epsilon)$ and $\mathscr{K}$ restricted to $\tilde{C}_*^0$. Recall that these operators satisfy the identities (3.5) and (3.6):

$$f \ - \ (\lambda_{C(k),\epsilon} \circ \rho_k) \ . \ f = (\delta_{k+1} \ \mathscr{K}(\epsilon) \ + \ \mathscr{K}(\epsilon) \ \delta_k) \ f \tag{3.26}$$

$$\mathscr{K}(\epsilon) f \ = \ \int_\epsilon^1 \mathscr{K}_t \ \frac{d}{dt}(\lambda_{C(k),t} \circ \rho_k) \ f \ dt \tag{3.27}$$

Then we define $\mathscr{K} f$ as

$$\mathscr{K} f \ = \ lim_{t\to 0} \ \mathscr{K}_\epsilon \ f \ = \ lim_{t\to 0} \int_\epsilon^1 \mathscr{K}_t \ \frac{d}{dt}(\lambda_{C(k),t} \circ \rho_k) \ f \ dt \tag{3.28}$$

$$f \ = \ (\delta_{k+1} \ \mathscr{K} \ + \ \mathscr{K} \ \delta_k) \ f. \tag{3.29}$$

Let $f \in C_k^0$. The last identity (3.29) is valid provided the following two relations hold:

$$lim_{t\to 0} \ \mathscr{K}(\epsilon) \ f \in C_k \tag{3.30}$$

$$lim_{t\to 0} \ (\lambda_{C(k),\epsilon} \circ \rho_k) \ . \ f \ = \ 0. \tag{3.31}$$

Both relations follow from Lemma 3.8. This completes the proof of Proposition 3.6.

3.2.1 The Homology of $\tilde{C}_*$

Proposition 3.7 *One has the exact sequence of complexes*

$$0 \longrightarrow H_k(\tilde{C}_*^1) \xrightarrow{j_{k,*}} H_k(J^1) \xrightarrow{\delta_k} H_{k-1}(C_*^0) \longrightarrow 0, \tag{3.32}$$

where δ_* is the connecting homomorphism.

Proof The proposition follows from the exact sequence (3.21) and Lemma 3.8.

For any $1 \leq r \leq k-1$ consider

$$D \ = \ \frac{\partial^{s_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{s_{r-1}}}{\partial x_{r-1}^{i_{r-1}}} \left(\frac{\partial^{s_r}}{\partial x_r^{i_r}} \frac{\partial^{s_{r+1}}}{\partial x_r^{i_{r+1}}}\right) \frac{\partial^{s_{r+2}}}{\partial x_{r+1}^{i_{r+2}}} \cdots \frac{\partial^{s_k}}{\partial x_{k-1}^{i_k}}.$$

Lemma 3.9 *One has the identity*

$$(\ D\ b\ f\ )_{|x_0 = x_1 = ... = x_k}\ = \tag{3.33}$$

$$(-1)^r(\ \frac{\partial}{\partial x_1^{i_1}} \dots \frac{\partial}{\partial x_{r-1}^{i_{r-1}}}(\frac{\partial}{\partial x_r^{i_r}}.\frac{\partial}{\partial x_{r+1}^{i_{r+1}}} + \frac{\partial}{\partial x_r^{i_{r+1}}}.\frac{\partial}{\partial x_{r+1}^{i_r}}).\frac{\partial}{\partial x_{r+2}^{i_{r+2}}} \dots \frac{\partial}{\partial x_k^{i_k}} f\ )_{|x_0 = x_1 = ... = x_k}.$$

Proof We split $D\ b\ f$ in three parts

$$Dbf\ =\ I\ +\ II\ +\ III,$$

where

$$I :=\ (-1)^s\ D \sum_{0 \le s \le r-1} f(\ x_0, \dots, x_s, x_s, \dots x_{r-1}, x_r, \dots x_{k-1}\ )_{|x_0 = x_1 = ... = x_k}.$$

$$II :=\ (-1)^r\ D\ f(\ x_0, \dots, x_{r-1}, x_r, x_r, x_{r+1}, \dots x_{k-1}\ )_{|x_0 = x_1 = ... = x_k}.$$

$$III :=\ (-1)^s\ D \sum_{r+1 \le s \le k-1} f(\ x_0, \dots, x_r, \dots x_s, x_s, \dots, x_{k-1}\ )_{|x_0 = x_1 = ... = x_k}.$$

One has

$$I\ =$$

$$(-1)^{r-1}(\ \frac{\partial^{r-1}}{x_1^{i_1} \dots x_{r-1}^{i_{r-1}}} \frac{\partial^2}{\partial x_{r+1}^{i_r} \partial x_{r+1}^{i_{r+1}}} \frac{\partial^{k-r-1}}{\partial x_{r+2}^{i_{r+3}} \dots \partial x_k^{i_k}}\ )(\ f\ )_{|x_0 = x_1 = \dots = x_k}$$

$$II\ =$$

$$(-1)^r(\ \frac{\partial^{r-1}}{\partial x_1^{i_1} \dots \partial x_{r-1}^{i_{r-1}}}.(\frac{\partial^2}{\partial x_r^{i_r} \partial x_r^{i_{r+1}}} + \frac{\partial^2}{\partial x_r^{i_r} \partial x_{r+1}^{i_{r+1}}} + \frac{\partial^2}{\partial x_r^{i_{r+1}} \partial x_{r+1}^{i_r}} + \frac{\partial^2}{\partial x_{r+1}^{i_r} \partial x_{r+1}^{i_{r+1}}}).\frac{\partial^{k-r-1}}{\partial x_{r+2}^{i_{r+3}} \dots \partial x_k^{i_k}}\ )(\ f\ )_{|x_0 = x_1 = ... x_k}.$$

$$III\ =$$

$$(-1)^{r+1}(\ \frac{\partial^{r-1}}{\partial x_1^{i_1} \dots \partial x_{r-1}^{i_{r-1}}} \frac{\partial^2}{\partial x_r^{i_r} \partial x_r^{i_{r+1}}} \frac{\partial^{k-r-1}}{\partial x_{r+2}^{i_{r+3}} \dots \partial x_k^{i_k}})(\ f\ )_{|x_0 = x_1 = ... x_k}.$$

Summing up I, II and III proves Lemma 3.9.

Lemma 3.9 says that the image of j^1_{k*} consists of skew-symmetric chains.

Theorem 3.8

(i) The image of the homomorphism $j_{k,*}$ consists of skew-symmetric *chains*

$$j_{k,*}(H_k(\tilde{C}_*)) = \{ \sum_{1 \le j_1, \dots, j_k \le n} a_{j_1, j_2, \dots, j_k}(x_0)\, dx_1^{j_1} \wedge \dots \wedge \dots\, dx_k^{j_k} \};$$

here $a_{j_1, j_2, \dots, j_k}(x_0)$ are smooth functions.

(ii) $HH_k(C^\infty(M)) = \Omega^k(M) = k-$ differential forms.

Proof Let

$$\gamma = \sum_{i_1, \dots, i_k} a_{i_1, \dots, i_k}(x_0)\, dx_1^{i_1} \wedge \dots \wedge dx_k^{i_k} \in C_k(J^1) \tag{3.34}$$

be any skew-symmetric chain.

Associate with γ the locally defined skew-symmetric function

$$f = \sum_{i_1, \dots, i_k} a_{i_1, i_2, \dots, i_k}(x_0)\, (x_1^{i_1} - x_0^{i_1}) \wedge \dots \wedge \dots (x_k^{i_k} - x_0^{i_k}) \in C_k. \tag{3.35}$$

Then f is a *cycle* in the complex C_*; clearly, $j_1(f) = \gamma$. Therefore, any skew-symmetric chain $\gamma \in J^1_k$ is in the image of j_{k*}.

Vice versa, suppose f is a cycle in C_k. Lemma 3.9 gives

$$0 = (D\, b\, f)_{|\, x_0 = x_1 = \dots = x_k} = \tag{3.36}$$

$$(-1)^r \frac{\partial^{r-1}}{\delta x_1^{i_1} \dots \delta x_{r-1}^{i_{r-1}}} \left(\frac{\partial^2}{\delta x_r^{i_r} \delta x_{r+1}^{i_{r+1}}} + \frac{\partial^2}{\delta x_{r+1}^{i_r} \delta x_r^{i_{r+1}}} \right) \frac{\partial^{k-r}}{\delta x_{r+2}^{i_{r+2}} \dots \delta x_k^{i_k}})_{|\, x_0 = x_1 = \dots = x_k}.$$

This means that the components of j^1_k are skew-symmetric in the set of indices $i_1, \dots, i_k$. This completes the proof of (i).

(ii) The second part of the theorem follows from part (i) along with the fact that $j^1_{k,*}$ is a monomorphism.

3.3 Division and Hochschild Homology

For this section we use [113]. The arguments discussed here are extracted from correspondence with Lescure [124], unpublished.

Let $\mathscr{A}$ be an associative algebra $\mathscr{A}$. Let

$$\lambda \;=\; \sum_i \alpha_i \otimes \beta_i \;\in \mathscr{A} \otimes \mathscr{A}. \tag{3.37}$$

Let $h_j : C_k \longrightarrow C_{k+1}$, $0 \le j \le k$ be given by the formula

$$h_j\ (a_0 \otimes \cdots \otimes a_k) \;=\; a_0 \otimes \cdots \otimes a_j \lambda \otimes \cdots \otimes a_k, \qquad 0 \le j \le k, \tag{3.38}$$

and $\lambda_j \;=\; b_{j+1}\, h_j : C_k(\mathscr{A}) \longrightarrow C_k(\mathscr{A})$, i.e.

$$\lambda_j\ (a_0 \otimes \cdots \otimes a_k) \;=\; \begin{cases} a_0 \otimes \cdots \otimes a_j \lambda a_{j+1} \otimes \cdots \otimes a_k & \text{if } j < k \\ b_{k+1}\ (a_0 \otimes \cdots \otimes a_k\ .\ \lambda) & \text{if } j = k, \end{cases}$$

where b_{k+1} is the last Hochschild boundary face, see Sect. 2.4. and

$$a_j\ \lambda\ a_{j+1} \;=\; \sum_i (a_j \alpha_i) \otimes (\beta_i a_{j+1}) \in\ C_1.$$

Definition 3.13 Suppose $\lambda = \sum_i\ \alpha_i \otimes \beta_i \in \mathscr{A} \otimes \mathscr{A}$ and $\sum_i\ \alpha_i.\beta_i = 1 \in \mathscr{A}$. Let $h : C_k(\mathscr{A}) \longrightarrow C_k(\mathscr{A})$

$$h := \sum_{0 \le j \le k} (-1)^j h_j \tag{3.39}$$

Proposition 3.8

(i) One has

$$b\, h \;+\; h\, b \;=\; \sum_{0 \le j \le k} (\ 1 - \lambda_j\ ) := \ \mathscr{L}_k. \tag{3.40}$$

(ii) The operator $\mathscr{L}_*$ is a homomorphism of the Hochschild complex $\{C_*(\mathscr{A}),\ b\}$.

Remark 3.4 If $\lambda \;=\; 1 \otimes 1$, then the operator h_j is the degeneracy map s_j, see Loday [97, §1.6]. In this case $\lambda_j \;=\; 1$ and $\mathscr{L} \;=\; 0$.

Definition 3.14 Define

$$C_k^0(\mathscr{A}) \;:=\; h\ (\ C_k(\mathscr{A})\ ). \tag{3.41}$$

Proposition 3.9 *Suppose the sub-complex C_*^0 is h-stable. Suppose also the operator $\mathscr{L}$ is invertible on C_*^0.*

Then the sub-complex C_*^0 is acyclic.

Proof Let $\mathscr{L}^{-1}$ be the inverse of $\mathscr{L}$ on $C_k^0(\mathscr{A})$. Let $\mathscr{H} := h\,\mathscr{L}^{-1}$. Then

$$\begin{aligned} \mathscr{H}\,b + b\,\mathscr{H} &= (h\mathscr{L}^{-1})\,b + b\,(h\,\mathscr{L}^{-1}) = \\ h\,b\,\mathscr{L}^{-1} &+ b\,h\,\mathscr{L}^{-1} = \mathscr{L}\,\mathscr{L}^{-1} = 1. \end{aligned} \tag{3.42}$$

Proposition 3.9 can be used to produce a second proof of Theorem 3.7. With a slight modification of the considerations above, we can produce another proof of Proposition 3.6. In fact, let M be a smooth manifold and $\mathscr{A} = C^\infty(M)$. Let d be the geodesic distance associated to a Riemannian metric on M. Define $\lambda(x, y) := 1 + d^2(x, y)$.

Define the operator h by the same formula (3.39). The corresponding operator $\mathscr{L}$ is

$$(\mathscr{L} f)(x_0, \ldots, x_k) = f(x_0, \ldots, x_k) \,.\, (d^2(x_0, x_1) + \cdots + d^2(x_{k-1}, x_k) + d^2(x_k, x_0)).$$

Define $C_k^0 := \mathscr{L} C_k(\mathscr{A})$.

One has the exact sequence

$$0 \longrightarrow C_k^0 \longrightarrow C_k \longrightarrow C_k/C_k^0 \longrightarrow 0. \tag{3.43}$$

One has the analogue of Proposition 3.5.

Proposition 3.10 *The sub-complex C_*^0 has trivial homology in the complex C_*^0.*

Proof One has the identity on C_*^0

$$(h\,b + b\,h)\,f = f\,.\,\mathscr{L}. \tag{3.44}$$

Any element of C_*^0 is the $\mathscr{L}$-multiple of an element of C_*. Let $\mathscr{L}^{-1} : C_k^0 \longrightarrow C_k$ be mapping given by the division by $\mathscr{L}$. Then $h\mathscr{L}^{-1} : C_k^0 \longrightarrow C_{k+1}$ satisfies the identity

$$(h\,\mathscr{L}^{-1})\,b + b\,(h\,\mathscr{L}^{-1}) = 1 \in C_k. \tag{3.45}$$

More applications of the techniques presented in this section can be found in [113].

3.4 Combinatorics Behind Homology Theories

Here we present the combinatorics which stays at the foundations of some of the most familiar homology theories (non-localised Alexander–Spanier homology and cohomology [45], Hochschild [68, 82, 99] and modified Hochschild homology [119, 123]) (later on called *topological Hochschild homology*) in the cases in which the *product* is *pointwisely defined*, as in the case of smooth functions.

This discussion does not regard the case of the algebra of integral or pseudo-differential operators. The Hochschild and modified Hochschild homology are built on associative algebras, possibly endowed with a topological structure. The construction we present here extends the definition of Hochschild and modified Hochschild homology to a *mixed context* in which the role of the topological tensor products is replaced by a functor which associates with any power of a base space a set of functions with certain regularity properties, as it occurs in the case of smooth functions. This situation is related to the case discussed in Sect. 3.1. For more details the reader may consult [120]. More related information may be found in [119, 123].

3.4.1 Notation

Let X be an arbitrary set. Let X^{k+1} denote the direct product $X \times X \times \ldots . \times X$, $(k+1)$ factors, where k is an arbitrary non-negative integer.

Let $\iota_{k,r} : X^k \longrightarrow X^{k+1}, 0 \leq i \leq k+1$, be the *diagonal homomorphisms* defined by (3.2) and let Δ be the main diagonal.

Denote by $D_{k,r} \subset X^{k+1}$ the range of $\iota_{k,r}$ and let $D_k = \bigcup_{r=0}^{r=k} D_{k,r}$ the union of diagonals of X^{k+1}.

Recall the definition of *face homomorphisms* $\delta_{k,r} : X^{k+1} \longrightarrow X^k$, for $0 \leq r \leq k$,

$$\delta_{k,r}(x_0, x_1, \ldots, x_k) = (x_0, x_1, \ldots, \hat{x_i}, \ldots., x_k),$$

used in the definition of *simplicial homology*.

Let $\mathbb{K}$ be a commutative ring with unit 1. Let $C_k(X, \mathbb{K})$ the free $\mathbf{K}$-module with generators $(x_0, x_1, \ldots, x_k)$, with $x_i \in X$. Denote by $[x_0, x_1, \ldots, x_k]$ such generators.

Definition 3.15 $D_k(X, K) \subset C_k(X, \mathbf{K})$ is the sub-module generated by the elements of D_k.

Let $\iota_{k,r}$, resp. $\delta_{k,r}$, be the $\mathbf{K}$-linear extension of $\iota_{k,r}$, resp. $\delta_{k,r}$.

The next definition introduces boundary maps which are *duals* of the Alexander–Spanier and Hochschild boundary maps.

Definition 3.16 We define the (non-localised) *pointwise Alexander–Spanier* homomorphism

$$\delta_k : C_k(X, \mathbf{K}) \to C_{k-1}(X, \mathbf{K}), \qquad \delta_k := \sum_{r=0}^{r=k} (-1)^r \delta_{k,r}. \tag{3.46}$$

and the *pointwise Hochschild* homomorphisms

$$\iota_k : C_k(X, \mathbf{K}) \to C_{k+1}(X, \mathbf{K}), \qquad \chi_k := \sum_{r=0}^{r=k+1} (-1)^r \chi_{k(r)}. \tag{3.47}$$

It is not difficult to verify that $\delta_{k-1} \circ \delta_k = 0$ and $\chi_{k+1} \circ \chi_k = 0$.

We consider the cohomology complexes $\{C_*(M,K),\ \chi_*\}$, resp. $\{C_*(M,K),\ \delta_*\}$, called *pointwise Hochschild cohomology complex*, resp. (non-localised) *pointwise Alexander–Spanier homology complex*, of X.

3.4.2 Total Difference Operators

Definition 3.17 We consider the direct sum $C_*(X,\mathbf{K}) := \oplus_{k=0}^{k=\infty} C_k(X,\mathbf{K})$. On this space we introduce $\mathbf{K}$-homomorphism product defined on generators by

$$[x_0,x_1,\ldots,x_k]\,.\,[y_0,y_1,\ldots,y_l] = [x_0,x_1,\ldots,x_k,\ y_0,y_1,\ldots,y_l]. \quad (3.48)$$

Proposition 3.11 *For any $U \in C_k(X,\mathbf{K})$ and $V \in C_l(X,\mathbf{K})$ one has*

$$\delta(U\,.\,V) = (\delta U)\,.\,V + (-1)^{k+1}U\,.\,(\delta V). \quad (3.49)$$

Definition 3.18 The *pointwise total difference operator*

$$\pi_k : C_k(X,\mathbf{K}) \longrightarrow C_k(X,\mathbf{K}) \quad (3.50)$$

is defined by the formula

$$\pi_k\,([x_0,x_1,\ldots,x_k]) = [x_0]\,.\,([x_1]-[x_0])\,.\,([x_2]-[x_1])\ldots([x_k]-[x_{k-1}]). \quad (3.51)$$

Theorem 3.9 *Consider $[x_0,x_1,\ldots,x_k] \in C_k(X,\mathbf{K})$; one has*

$$\pi_k\,([x_0,x_1,\ldots,x_k]) = [x_0,x_1,\ldots,x_k] + \textstyle\sum_\alpha U_\alpha, \text{ where } U_\alpha \in D_k(X,\mathbf{K}) \quad (3.52)$$

$$\text{for any } \gamma \in D_k(X,\mathbf{K}),\quad \pi_k(\gamma) = 0, \quad (3.53)$$

$$\pi_k^2 = \pi_k, \quad (3.54)$$

$$\text{the range of the idempotent } 1-\pi_k \text{ is } D_k(M,K). \quad (3.55)$$

Proof We look for those terms of the RHS of (3.51) which might not belong to $D_k(X,\mathbf{K})$. We will do this by making partial consecutive products beginning with the first factor of the RHS of (3.51). For example, there is only such a term as a result of multiplication of the first two factors: $[x_0].[x_1]$. Continuing in this fashion we find that there is only one such term which might not belong to $D_k(X,\mathbf{K})$; this is $[x_0].[x_1]\ldots[x_k] = [x_0,x_1,\ldots,x_k]$. This proves the first part of the theorem.

It is clear that for any $\gamma \in D_k(X,\mathbf{K})$ one has $\pi_k(\gamma) = 0$.

The third and the fourth relations are consequences of the first two.

Definition 3.19 Define by

$$E_k(X, \mathbf{K}) := \textit{the range of the idempotent } \pi_k.$$

Corollary 3.1

(i) $E_k(X, \mathbf{K})$ *is generated by elements* $\pi_k[x_0, x_1, \ldots, x_k]$ *with the property that* $x_0 \neq x_1 \neq \cdots \neq x_k$.

(ii) The $\mathbf{K}$*-module* $C_k(X, \mathbf{K})$ *decomposes in a direct sum of sub-modules*

$$C_k(X, \mathbf{K}) \;=\; E_k(X, \mathbf{K}) \oplus D_k(X, \mathbf{K}), \tag{3.56}$$

Proposition 3.12

(i) One has

$$\chi_k \circ \pi_k\,[x_0, x_1, \ldots, x_k] \;= \tag{3.57}$$
$$(-1)^{k+1}[x_0]\,.\,([x_1] \;-\; [x_0]) \ldots ([x_k] \;-\; [x_{k-1}])\,.\,([x_0] \;-\; [x_k]).$$

(ii)

$$\pi_{k+1} \circ \chi_k\,[x_0, x_1, \ldots, x_k] \;=$$
$$= (-1)^{k+1}[x_0]\,.\,([x_1] \;-\; [x_0]) \ldots ([x_k] \;-\; [x_{k-1}])\,.\,([x_0] \;-\; [x_k]). \tag{3.58}$$

(iii) Therefore, $\chi_k \circ \pi_k \;=\; \pi_{k+1} \circ \chi_k$.

Proof

(i) We have

$$\chi_k \circ \pi_k\,[x_0, x_1, \ldots, x_k] \;=$$
$$\sum_{r=0}^{r=k} (-1)^r [x_0]\,.\,([x_1] \;-\; [x_0])\,.\,([x_2] \;-\; [x_1]) \ldots ([x_r]^2 \;-\; [x_{r-1}]^2)\,.\,[x_{r+1} \;-\; x_r] \ldots ([x_k] \;-\; [x_{k-1}]) \;=$$
$$(-1)^{k+1}[x_0]\,.\,([x_1] - [x_0]) \ldots ([x_k] \;-\; [x_{k-1}])\,.\,([x_0] \;-\; [x_k]). \tag{3.59}$$

To prove (ii), notice that $\pi_{k+1}\chi_{k,i}\;([x_0, x_1, \ldots, x_k]) \;=\; 0$ for $0 \leq i \leq k$. Therefore,

$$\pi_{k+1} \circ \chi_k\,[x_0, x_1, \ldots, x_k] \;=\; (-1)^{k+1}\pi_{k+1}[x_0, x_1, \ldots, x_{k-1}, x_k, x_0] \;= \tag{3.60}$$
$$= \;(-1)^{k+1}[x_0]\,.\,([x_1] \;-\; [x_0]).\ldots.([x_k] \;-\; [x_{k-1}])\,.\,([x_0] \;-\; [x_k]).$$

Property (iii) follows from (i) and (ii). This completes the proof of Proposition 3.12.

Proposition 3.13

(i) One has

$$(\delta_k \pi_k)\,[x_0, x_1, \ldots, x_k] = \tag{3.61}$$
$$([x_1] - [x_0])\ .;\ ([x_2] - [x_1]).\ldots.([x_i] - [x_{i-1}]).\ldots.([x_k] - [x_{k-1}]).$$

(ii)

$$(\pi_k \circ \delta_k)\,[x_0, x_1, \ldots, x_k] =$$
$$([x_1] - [x_0])\ .\ ([x_2] - [x_1]).\ldots.([x_i] - [x_{i-1}])\ .\ \ldots.\ ([x_k] - [x_{k-1}]). \tag{3.62}$$

(iii) Therefore, $\delta_k \circ \pi_k = \pi_{k-1} \circ \delta_k$.

Proof One has

$$(\delta_k \pi_k)\,[x_0, x_1, \ldots, x_k] = \delta_k\,([x_0]\,.\,([x_1] - [x_0])\,.\ldots.\,([x_k] - [x_{k-1}])) = \tag{3.63}$$
$$([x_1] - [x_0])\ldots([x_i] - [x_{i-1}]).\ldots.([x_k] - [x_{k-1}]) +$$
$$\sum_{i=1}^{i=k}(-1)^i[x_0]\,.\,([x_1] - [x_0])\,.\ldots.\,(\,(([\hat{x_i}] - [\hat{x_i-1}])\,)\,.\ldots.([x_k] - [x_{k-1}]) =$$
$$([x_1] - [x_0])\,.\,([x_2] - [x_1])\ldots([x_i] - [x_{i-1}]).\ldots.([x_k] - [x_{k-1}]).$$

Now we compute $\pi_k \circ \delta_k$. One has

$$(\,\pi_{k-1} \circ \sum_{r=0}^{r=k}(-1)^r \delta_{k,r}\,)\,[x_0, x_1, \ldots, x_k]\,) \tag{3.64}$$
$$= \pi_{k-1} \sum_{r=0}^{r=k}(-1)^r [x_0]\,.\,([x_1] - [x_0])\ldots([x_{r-1}] - [x_{r-2}])\,.\,([x_{r+1}] - [x_r])\ldots([x_k] - [x_{k-1}]))$$
$$= ([x_1] - [x_0])\,.\,([x_2] - [x_1])\ldots([x_i] - [x_{i-1}])\ldots([x_k] - [x_{k-1}]).$$

This completes the proof of Proposition 3.13.

Corollary 3.2

(i)

$$[x_0]\,[x_1] - [x_0])\,.\,([x_2] - [x_1])\ldots([x_k] - [x_{k-1}])\,([x_0] - [x_k]) \in E_{k+1}(M, \mathrm{K}) \tag{3.65}$$

$$([x_1] - [x_0])\,.\,([x_2] - [x_1])\ldots([x_k] - [x_{k-1}]) \in E_{k-1}(M, \mathrm{K}). \tag{3.66}$$

(ii) One has the direct sum decomposition by sub-complexes

$$\{C_k(M,K),\ \chi_*\} = \{E_k(M,K),\ \chi_*\} \oplus \{D_k(M,K),\ \chi_*\} \tag{3.67}$$

$$\{C_k(M,K),\ \delta_*\} = \{E_k(M,K),\ \delta_*\} \oplus \{D_k(M,K),\ \delta_*\} \tag{3.68}$$

Proof Part (i) and (ii) follow from Propositions 3.12 and 3.13.

For more details see [114].

3.5 The Karoubi-Type Operator

Definition 3.20 In analogy with the Definition 2.32 define *pointwise Karoubi operator* σ (comp. [89, 104, 115]) by the formula

$$\delta_{k+1} \circ \chi_k + \chi_{k-1} \circ \delta_k = 1 - \sigma_k. \tag{3.69}$$

The analogue of the bar operator b' is the operator χ' defined below.

Definition 3.21 The operator χ' is given by the formula

$$\chi'_k\,[x_0, x_1, \ldots, x_k] = \sum_{i=0}^{i=k} (-1)^i\,[x_0, x_1, \ldots, x_i, x_i \ldots .x_k]. \tag{3.70}$$

Proposition 3.14 *The operator* χ' *satisfies the relation*

$$\delta_{k+1} \circ \chi'_k + \chi'_{k-1} \circ \delta_k = 0. \tag{3.71}$$

Proof The first term of (3.71) is

$$\delta_{k+1} \circ \chi'_k\,[x_0, x_1, \ldots, x_k] = \delta_{k+1} \circ \sum_{i=0}^{i=k} (-1)^i [x_0, x_1, \ldots, x_i, x_i \ldots .x_k] =$$

$$\sum_{0 \le j < i} \sum_{i=0}^{i=k} (-1)^{i+j} [x_0, x_1, \ldots, \hat{x_j}, \ldots x_i, x_i \ldots .x_k] + \sum_{i < j \le k} \sum_{i=0}^{i=k} (-1)^{i+j+1} [x_0, x_1, \ldots, x_i, x_i \ldots \hat{x_j}, .., x_k].$$

The second term of (3.71) is

$$\chi'_{k-1} \circ \delta_k\,[x_0, x_1, \ldots, x_k] = \chi'_{k-1} \sum_{j=0}^{j=k} (-1)^j [x_0, x_1 \ldots \hat{x_j}, .., x_k] =$$

$$= \sum_{0\le i<j} \sum_{j=0}^{j=k} (-1)^j [x_0, x_1 \dots x_i, x_i \dots \hat{x_j}, .., x_k] \; +$$

$$+ \sum_{j<i\le k} \sum_{j=0}^{j=k} (-1)^{i+j+1} \; [x_0, x_1 \dots \hat{x_j} \dots x_i, x_i \dots x_k].$$

Summing up these results proves (3.71).

Proposition 3.15 *The operator σ satisfies the relations*

$$\sigma_k \; [x_0, x_1, \dots, x_k] \; = (-1)^{k+1} \; [x_1, \dots, x_k] \; . \; [x_1 \; - \; x_0] \tag{3.72}$$

$$\sigma_k^k \; [x_0, x_1, \dots, x_k] \; = [x_k] \; . \; [x_1 \; - \; x_0] . [x_2 \; - \; x_1] \dots [x_k \; - \; x_{k-1}]. \tag{3.73}$$

Proof We have

$$(\delta_{k+1} \circ \chi_k \; + \; \chi_{k-1} \circ \delta_k) \; [x_0, \; x_1, \dots, x_k] \; =$$

$$(-1)^{k+1} \delta_{k+1} \circ \chi_{k,k+1} \; [x_0, x_1, \dots, x_k] \; + \; (-1)^k \chi_{k-1,k} \circ \delta_k \; [x_0, x_1, \dots, x_k] \; =$$

$$(-1)^{k+1} \; (\sum_{i=0}^{i=k} (-1)^i \; [x_0, x_1, \dots, \hat{x}_i, \dots, x_k, x_0] \; + \; (-1)^{k+1} \; [x_0, x_1 \dots x_k] \;) \; +$$

$$(-1)^k \; \chi_{k-1,k} \; (\; [x_1, x_1, \dots, x_k] \; + \; \sum_{i=1}^{i=k} (-1)^i \; [x_0, x_1, \dots, \hat{x}_i \dots x_k] \;) \; =$$

$$\sum_{i=0}^{i=k} (-1)^{i+k+1} \; [x_0, x_1, \dots, \hat{x}_i \dots x_k, x_0] \; + \; [x_0, x_1, \dots, x_k] \; +$$

$$(-1)^k [x_1, x_2, \dots, x_k, x_1] + \sum_{i=1}^{i=k} (-1)^{i+k} \; [x_0, x_1, \dots, \hat{x}_i, \dots, x_k, x_0] \; =$$

$$(\; (-1)^{k+1} [x_1, x_2, \dots, x_k, x_0] \; + \; \sum_{i=1}^{i=k} (-1)^{i+k+1} [x_0, x_1, \dots, \hat{x}_i \dots x_k, x_0] \;) \; + \; [x_0, x_1 \dots x_k] \; +$$

$$(-1)^k [x_1, x_2, \dots, x_k, x_1] \; + \; \sum_{i=1}^{i=k} (-1)^{i+k} [x_0, x_1, \dots, \hat{x}_i, \dots, x_k, x_0] \;) \; =$$

$$[x_0, x_1, \dots, x_k] - (-1)^{k+1} \; (\; [x_1, x_2, \dots, x_k, x_1] - [x_1, x_2, \dots, x_k, x_0] \;).$$

This proves (3.72). The relation (3.73) is obtained from (3.72) by iteration.

Theorem 3.10 *The operator σ satisfies the identity*

$$\pi_k \; = \; \sigma_k^{\; k} \; (\; 1 \; - \; \delta_{k+1} \circ \chi_k \;). \tag{3.74}$$

Proof We compute the RHS of (3.74).

$$\sigma_k^{\,k}\,(\,1\,-\,\delta_{k+1}\circ\chi_k\,)\,[x_0,x_1,\ldots,x_k]\;=$$

$$\sigma_k^{\,k}\,[x_0,x_1\ldots x_k]-\sigma_k^{\,k}\,\circ\,\delta_{k+1}\,(\sum_{i=0}^{i=k}(-1)^i[x_0,x_1\ldots x_i,x_i\ldots x_k]\;+\;(-1)^{k+1}[x_0,x_1\ldots x_k,x_0]\,)\;=$$

$$[x_k].[x_1\;-\;x_0].[x_2\;-\;x_1]\ldots[x_k\;-\;x_{k-1}]$$

$$-\sigma_k^{\,k}(\;\sum_{0\le j<i}\;\sum_{i=0}^{i=k}(-1)^{i+j}[x_0,x_1\ldots\hat{x}_j\ldots x_i,x_i\ldots x_k]\;+\;\sum_{i<j\le k}\;\sum_{i=0}^{i=k}(-1)^{i+j+1}[x_0,x_1\ldots x_i,x_i\ldots\hat{x}_j\ldots x_k]\,)$$

$$-\;(-1)^{k+1}\;\sigma_k^{\,k}\;\delta_{k+1}\;[x_0,x_1\ldots x_k,x_0]\;=$$

$$[x_k].[x_1-x_0].[x_2\;-\;x_1]\ldots[x_k\;-\;x_{k-1}]-\;(-1)^{k+1}\;\delta_{k+1}\;\sigma_{k+1}^{\,k}\;[x_0,x_1\ldots x_k,x_0]\;=$$

$$[x_k].[x_1\;-\;x_0].[x_2\;-\;x_1]\ldots[x_k\;-\;x_{k-1}]$$

$$-(-1)^{k+1}(-1)^{k.(k+2)}\;\delta_{k+1}\;[x_k][x_0].[x_1\;-\;x_0].[x_2\;-\;x_1]\ldots[x_k\;-\;x_{k-1}]\;=$$

$$[x_k].[x_1\;-\;x_0].[x_2\;-\;x_1]\ldots[x_k\;-\;x_{k-1}]+\;(\,[x_0]\;-\;[x_k]\,).[x_1\;-\;x_0].[x_2\;-\;x_1]\ldots[x_k\;-\;x_{k-1}]\;=$$

$$[x_0].[x_1\;-\;x_0]\,.\,[x_2\;-\;x_1]\;\ldots\;[x_k\;-\;x_{k-1}]\;=\;\pi_k([x_0,x_1\ldots x_k]).$$

The following expression gives another formula for the operator π_k:

$$\pi_k\;[x_0,x_1,x_2,\ldots,x_k]\;\;=$$

$$=\delta_{k+1,k+1}\circ(\delta_{k+2,k}\;-\;\delta_{k+2,k+1})\circ\cdots\circ(\delta_{2k,3}\;-\;\delta_{2k,4})\circ(\delta_{2k+1,1}\;-\;\delta_{2k+1,2})\circ$$

$$\circ\chi_{2k,0}\circ\ldots\circ\chi_{k+2,k-2}\circ\chi_{k+1,k-1}\circ\chi_{k,k}\;\;[x_0,x_1,x_2,\ldots,x_k]\,.$$

3.6 Homological Consequences

Definition 3.22 A $(\delta\;,\;\chi)$-*structure* on a system of **K**-modules consists of a system $\{C_k\}_{k=0,1,2\ldots}$ of $\mathbb{K}$-homomorphisms

$$\delta_k : C_k \longrightarrow C_{k-1}, \qquad \delta_{k-1}\circ\delta_k = 0 \tag{3.75}$$

$$\chi_k : C_k \longrightarrow C_{k+1}, \qquad \chi_{k+1}\circ\chi_k = 0 \tag{3.76}$$

such that, with the *abstract* σ *operators* defined by the formulas

$$\delta_{k+1}\circ\chi_k\;+\;\chi_{k-1}\circ\delta_k\;=\;1-\sigma_k, \tag{3.77}$$

the operators

$$\pi_k\;:=\;\sigma_k^{\,k}\;(\;1\;-\;\delta_{k+1}\circ\chi_k\;) \tag{3.78}$$

are idempotents, $\pi_k \circ \pi_k = \pi_k$.

Remark 3.5 Abstract structures satisfying relations (3.75), (3.76) were defined by Burghelea [87]; later they were called *mixed complexes* by Kassel [90]. For more information on these topics the reader may consult Karoubi [89], Cuntz [115], Cuntz and Quillen [104], Loday [97].

Formula (3.78) is related to the formula (2.6.8.1-2) from Loday' book [97, page 85]. Loday's formula is stated on the *normalized* algebraic Hochschild complex associated to an *associative algebra*; a normalized complex is a *quotient* complex. Notice that our treatment does not require any *ground associative algebra* nor *algebraic tensor products* or *topological tensor products*, compare, e.g. Connes [82, 99] or Connes and Moscovici [96]. For more applications in this direction see [119, 123].

Definition 3.23 Given a (δ, χ)-structure on the system $\{C_k\}_{k=0,1,2...}$ of **K**-modules, define

$$E_k := Image\ (\pi_k) \tag{3.79}$$

$$D_k := Image\ (1 - \pi_k) \tag{3.80}$$

Section 3.6. provides an example of (δ, χ)-structure.

Theorem 3.11 *Let (δ, χ) be defined on the system of* **K***-modules $\{C_k\}_{k=0,1,2...}$. Then*

(i) the boundary operators δ_* and χ_* commute with the idempotents π_*, $1 - \pi_*$
(ii) one has the direct sum decomposition by sub-complexes

$$\{C_*, \delta\} = \{E_*, \delta\} \oplus \{D_*, \delta\} \tag{3.81}$$

$$\{C_*, \chi\} = \{E_*, \chi\} \oplus \{D_*, \chi\} \tag{3.82}$$

(iii) the sub-complexes $\{D_*, \delta\}$, $\{D_*, \chi\}$ are acyclic.

Proof

(i) One has

$$\delta_k\, \pi_k = \delta_k\, \sigma_k^k\, (1 - \delta_{k+1} \circ \chi_k) = \sigma_{k-1}^k\, \delta_k\, (1 - \delta_{k+1} \circ \chi_k) = \sigma_{k-1}^k\, \delta_k$$

and

$$\pi_{k-1}\, \delta_k = \sigma_{k-1}^{k-1}\, (1 - \delta_k \circ \chi_{k-1})\, \delta_k \sigma_{k-1}^{k-1}\, \delta_k\, (1 - \chi_{k-1} \circ \delta_k) =$$
$$\sigma_{k-1}^{k-1}\, \delta_k\, (\sigma_k + \delta_{k+1} \circ \chi_k) = \sigma_{k-1}^{k-1}\, (\sigma_{k-1}\delta_k + \delta_k\, \delta_{k+1} \circ \chi_k) =$$
$$\sigma_{k-1}^{k-1}\, \sigma_{k-1}\, \delta_k = \sigma_{k-1}^k\, \delta_k.$$

These two equalities give $\delta_k\ \pi_k\ =\ \pi_{k-1}\ \delta_k$.

To prove the second commutativity relation, we compute, on one side $\chi_k\ \pi_k$

$$\chi_k\ \pi_k\ =\ \chi_k\ \sigma_k^k\ (\ 1\ -\ \delta_{k+1}\ \chi_k\ )\ =\ \sigma_{k+1}^k\ \chi_k\ (\ 1\ -\ \delta_{k+1}\ \chi_k\ )\ =$$
$$\sigma_{k+1}^k\ (\ 1\ -\ \chi_k\ \delta_{k+1}\ )\ \chi_k\ =\ \sigma_{k+1}^k\ (\ \sigma_{k+1}\ +\ \delta_{k+2}\ \chi_{k+1}\ )\ \chi_k\ =\ \sigma_{k+1}^{k+1}\ \chi_k.$$

On the other hand

$$\pi_{k+1}\ \chi_k\ =\ \sigma_{k+1}^{k+1}\ (\ 1\ -\ \delta_{k+2}\ \chi_{k+1}\ )\ \chi_k\ =\ \sigma_{k+1}^{k+1}\ \chi_k.$$

These last two relations imply $\chi_k\ \pi_k\ =\ \pi_{k+1}\ \chi_k$.

(ii) Property (i) implies (ii).

(iii) Suppose $\omega\ \in\ D_k$ and $\delta\ \omega\ =\ 0$. Then $0\ =\ \pi_k(\omega)\ =\ \sigma_k^k\ (\ 1\ -\ \delta_{k+1}\ \chi_k\ )\ (\omega)$. Therefore, $\sigma_k^k\ (\omega)\ =\ \delta_{k+1}\ \chi_k\ (\omega)$. This shows that $\sigma_k^k\ (\omega)$ is homologous to zero in the homology of the complex $\{C_*, \delta\}$. However, σ_* and all its powers are homologous to the identity. Hence, there exists $\theta\ \in\ C_{k+1}$ such that $\omega = \delta_{k+1}\ (\theta)$. This property along with (i) give

$$\omega\ =\ (\ 1\ -\ \pi_k\ )(\omega)\ =\ (\ 1\ -\ \pi_k\ )\delta_{k+1}(\theta)\ =\ \delta_{k+1}(\ 1\ -\ \pi_k\ )(\theta).$$

Therefore, ω is homologous to zero in the complex $\{D_*, \delta\}$, i.e. the complex $\{D_*, \delta\}$ is acyclic.

To prove the acyclicity of the complex $\{D_*, \chi\}$, we assume that $\omega\ \in\ D_k$ and $\chi_k\ \omega\ =\ 0$. Then $0\ =\ \pi_k(\omega)\ =\ \sigma_k^k\ (\ 1\ -\ \delta_{k+1}\ \chi_k\ )\ (\omega)\ =\ \sigma_k^k\ (\omega)$. Therefore, $\sigma_k^k\ (\omega)\ =\ 0$. As $\sigma_k^k\ (\omega)\ =\ \chi_{k-1}(0)$, we get that $\sigma_k^k\ (\omega)$ is χ-homologous to zero in the complex $\{C_*, \chi\}$. This gives, as before, that ω is χ-homologous to zero in the complex $\{C_*, \chi\}$. This means that there exists $\theta\ \in\ C_{k-1}$ such that $\omega = \chi_{k-1}\ (\theta)$. This relation and -i) imply

$$\omega\ =\ (\ 1\ -\ \pi_k\ )(\omega)\ =\ (\ 1\ -\ \pi_k\ )\chi_{k-1}(\theta)\ =\ \chi_{k-1}(\ 1\ -\ \pi_k\ )(\theta);$$

this shows that the complex $\{D_*, \chi\}$ is acyclic.

3.7 Pointwise Alexander–Spanier and Hochschild Homology

Theorem 3.12 *The complex* $\{E_*, \delta\}$ *is acyclic.*

Proof We show that the total pointwise Alexander–Spanier complex $\{C_*(M,\ \mathrm{K}),\ \delta\}$ is acyclic. Once this result is established, the decomposition formula (3.81) along with Theorem 3.11(iii) imply that the sub-complex $\{E_*(M,\ \mathrm{K}),\ \delta\}$ is acyclic too.

To prove the first statement, let P be a fixed point in M. Define the operator $h_k^\delta : E_k \longrightarrow E_{k+1}$

$$h_k^\delta(\ [x_0]\ .\ [x_1]\dots[x_k]\ )\ ) := \ [P]\ .\ [x_0]\ .\ [x_1]\dots[x_k]. \tag{3.83}$$

The operators h_k^δ produce a homotopy between the identity and the zero homomorphism

$$(\ \delta_{k+1}\circ h_k^\delta + h_{k-1}^\delta\circ\delta_k)\ (\ [x_0].[x_1]\dots[x_k]\ ) =$$

$$[x_0]\ .\ [x_1]\dots[x_k] - [P]\ .\ \delta_k\ [x_0]\ .\ [x_1]\dots[x_k]\ \ +\ [P]\ .\ \delta_k(\ [x_0]\ .\ [x_1]\dots[x_k]\ )\ =\ [x_0]\ .\ [x_1]\dots[x_k].$$

This result is well known in the Alexander–Spanier homology theory; it appeared in different forms in Chaps. 1 and 2.

Theorem 3.13 *The homology of the complex* $\{E_*,\ \chi\}$ *is*

$$H_k(E_*,\ \chi) = 0,\quad 1 \le k.$$

Proof On $\{E_*(M, \mathbf{K}),\ \chi\}$ we define the operator

$$h_k^\chi(\ \ [x_0]\ .\ ([x_1] - [x_0])\dots([x_{k-1}] - [x_{k-2}])\ .\ ([x_k] - [x_{k-1}])\ \ ) :=$$
$$[\ x_0\ ]\ .\ ([x_1] - [x_0])\dots([x_{k-1}] - [x_{k-2}]).$$

We show that the operator h_k^χ is well defined.

Lemma 3.10 *Suppose*

$$\sum_{i_0,i_1,\dots,i_k} a_{i_0,i_1,\dots,i_k}[x_{i_0}]\ .\ ([x_{i_1}] - [x_{i_0}])\dots([x_{i_k}] - [x_{i_{k-1}}]) = \tag{3.84}$$

$$\sum_{i_0,i_1,\dots,i_k} b_{i_0,i_1,\dots,i_k}[x_{i_0}]\ .\ ([x_{i_1}] - [x_{i_0}])\dots([x_{i_k}] - [x_{i_{k-1}}]),$$

where

$$a_{i_0,i_1,\dots,i_k},\ b_{i_0,i_1,\dots,i_k}\ \in \mathbf{K},\quad \textit{and}\ \ x_0 \ne x_1 \ne \dots \ne x_k.$$

Then $a_{i_0,i_1,\dots,i_k} = b_{i_0,i_1,\dots,i_k}$.

Therefore, any element of $E_*(M, \mathbf{K})$ *has a unique representation (3.84).*

Proof We prove the lemma by induction on k. Lemma is clearly true for $k = 0$. Suppose the lemma is true for $k - 1$. Chose a point $x_0 \in X$. Identity (3.84) gives

$$\sum_{i_1,\dots,i_k} a_{i_0,i_1,\dots,i_k}[x_{i_0}]\ .\ ([x_{i_1}] - [x_{i_0}])\dots([x_{i_k}] - [x_{i_{k-1}}])\ =\ \sum_{i_1,\dots,i_k} b_{i_0,i_1,\dots,i_k}\ [x_{i_0}]\ .\ [([x_{i_1}] - [x_{i_0}])\dots([x_{i_k}] - [x_{i_{k-1}}])$$

or

$$\sum_{i_1,\ldots,i_k} a_{i_0,i_1,\ldots,i_k}([x_{i_1}]-[x_{i_0}])\ldots([x_{i_k}]-[x_{i_{k-1}}]) = \sum_{i_1,\ldots,i_k} b_{i_0,i_1,\ldots,i_k}([x_{i_1}]-[x_{i_0}])\ldots([x_{i_k}]-[x_{i_{k-1}}])$$

which implies

$$\sum_{i_1,\ldots,i_k} a_{i_0,i_1,\ldots,i_k}[x_{i_1}]\ldots([x_{i_k}]-[x_{i_{k-1}}]) = \sum_{i_1,\ldots,i_k} b_{i_0,i_1,\ldots,i_k}[x_{i_1}]\ldots([x_{i_k}]-[x_{i_{k-1}}]).$$

The induction hypothesis completes the proof of the lemma.

We come back to the proof of Theorem 3.13. We use Proposition 3.12 to get

$$\begin{aligned}(\,\chi_{k-1}\circ h_k^{\chi} + h_{k+1}^{\chi}\circ\chi_k\,)(\,[x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_k]-[x_{k-1}])\,) =\\ \chi_{k-1}([x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]) +\\ (-1)^{k+1}h_{k+1}^{\chi}(\,[x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_k]-[x_{k-1}]).([x_0]-[x_k])\,) =\end{aligned}$$

$$\begin{aligned}(-1)^k\,[x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_0]-[x_{k-1}]) +\\ (-1)^{k+1}\,[x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_k]-[x_{k-1}]).\end{aligned}$$

This proves that the identity of $\{E_*(M,\mathrm{K}),\ \chi\}$ is homotopic to the chain homomorphism U.

$$\begin{aligned}U(\,[x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_k]-[x_{k-1}])\,)\\ := [x_0].([x_1];\,[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_0]-[x_{k-1}]).\end{aligned}\tag{3.85}$$

Proposition 3.12, applied onto $[x_0, x_1, \ldots, x_{k-1}]$, allows us to write

$$\begin{aligned}U(\,[x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_k]-[x_{k-1}])\,)\\ := (-1)^k\chi_{k-1}\,(\,[x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}])\,).\end{aligned}\tag{3.86}$$

This last result shows that any chain in $\{E_*(M,\mathrm{K}),\ \chi\}$ is homotopic to a χ-boundary. This proves Theorem 3.13

Remark that the operator U is an idempotent. In fact, its range is the sub-module generated by the elements $[x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_0]-[x_{k-1}])$ and $U([x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_0]-[x_{k-1}]) = [x_0].([x_1]-[x_0])\ldots([x_{k-1}]-[x_{k-2}]).([x_0]-[x_{k-1}])$.

Definition 3.24 We introduce the *pointwise modified Hochschild boundary*

$$\chi_n' : C_n(M,\mathrm{K}) \longrightarrow C_{n+1}(M,\mathrm{K}) \tag{3.87}$$

by the formula

$$\chi'_n := \sum_{i=1}^{i=n} (-1)^{i-1}\, C_n^i\, (\chi_n\, \delta_{n+1})^{i-1} \chi_n. \tag{3.88}$$

Theorem 3.14 *The pointwise modified Hochschild operator commutes with the pointwise Alexander–Spanier operator*

$$\chi'_{n-1}\, \delta_n \;+\; \delta_{n+1}\, \chi'_n \;=\; 0. \tag{3.89}$$

Proof The result follows from a straightforward computation. The formula (3.88) is a dual form of the relation (5.85) from [123].

Modified Hochschild homology will be reconsidered in Sect. 10.2.

3.8 Derivations

In this section we suppose $\mathscr{A}$ to be an associative algebra over the ring $\mathbf{K}$ and $\mathscr{M}$ to be an $\mathscr{A}$-bi-module.

Definition 3.25 A $\mathbf{K}$-mapping $D : \mathscr{A} \longrightarrow \mathscr{A}$ is called *derivation* provided it satisfies the identity

$$D(a.b) \;=\; D(a)\,.\,b \;+\; a\,.\,D(b) \quad \forall a, b \in \mathscr{A}. \tag{3.90}$$

Definition 3.26 Let $u \in \mathscr{A}$. The mapping $ad(u) : \mathscr{M} \longrightarrow \mathscr{M}$ is defined by

$$ad(u)(a) := u\,.\,a \;-\; a\,.\,u \;=\; [u,\; a]. \tag{3.91}$$

It is called *inner derivation*.

Definition 3.27 Recall $Cp(\mathscr{A}, \mathscr{M}) := \mathscr{M} \otimes_{\mathbf{K}}^{p} \mathscr{A}$. The inner derivation $ad(u)$, $u \in \mathscr{A}$, extends to a mapping $Cp(\mathscr{A}, \mathscr{M}) \longrightarrow Cp(\mathscr{A}, \mathscr{M})$

$$ad(u)(a_0 \otimes a_1 \otimes \cdots \otimes a_p) := \sum_{0 \le r \le p} a_0 \otimes \cdots \otimes a_{r-1} \otimes [u, a_r] \otimes \cdots \otimes a_p. \tag{3.92}$$

Definition 3.28 Let $u \in \mathscr{A}$. Define $h(u) : Cp(\mathscr{A}, \mathscr{M}) \longrightarrow Cp(\mathscr{A}, \mathscr{M})$

$$h(u)(a_0 \otimes a_1 \otimes \cdots \otimes a_{p+1}) := \sum_{0 \le r \le p} (-1)^r\, a_0 \otimes \cdots \otimes a_{r-1} \otimes u \otimes \cdots \otimes a_p. \tag{3.93}$$

Proposition 3.16 *The mapping* $h(u)$

(i) commutes with the Hochschild boundary b,
(ii) satisfies the homotopy identity

$$b\, h(u) \;+\; h(u)\, b \;=\; -ad(u), \tag{3.94}$$

(iii) and hence induces the zero homomorphism onto the Hochschild homology.

Proof We leave the proof to the reader.

3.9 Anti-Symmetrisation Map

Let S_p denote the *group of permutations* of the set $\{1, 2, \ldots, p\}$, or *symmetric group*.

Definition 3.29 The *anti-symmetrisation map* ϵ_p

$$\epsilon_p : C_p(\mathscr{A}, \mathscr{M}) \longrightarrow C_p(\mathscr{A}, \mathscr{M}) \tag{3.95}$$

is acting onto generators by the formula

$$\epsilon_p(m \otimes a_1 \otimes \cdots \otimes a_p) := \tag{3.96}$$
$$\textstyle\sum_{\sigma \in S_p} sign(\sigma)\; m \otimes a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \cdots \otimes a_{\sigma^{-1}(p)} := \quad (\forall m \in \mathscr{M},\ a_r \in \mathscr{A})$$
$$m \otimes a_1 \wedge \cdots \wedge a_p.$$

Let $\Lambda_p(\mathscr{A}, \mathscr{M})$ denote the range of the anti-symmetrisation mapping.
Any element of $\Lambda_p(\mathscr{A}, \mathscr{M})$ will be called *anti-symmetric*.
Any element of $\mathscr{M} \otimes \wedge^p \mathscr{A}$ is already anti-symmetric.

It is natural to consider the anti-symmetrisation because it is one of the operations involved in the definition of the classical differential forms.

Definition 3.30 In the universal complex $\mathscr{U}$ of the algebra $\mathscr{A}$ the *anti-symmetrisation* mapping is defined by the formula

$$\epsilon_p\ (f_0.df_1 \ldots df_p) := \sum_{\sigma \in S_p} sign(\sigma)\ f_0.df_{\sigma^{-1}(1)}.\ldots.df_{\sigma^{-1}(p)}. \tag{3.97}$$

Proposition 3.17 *Anti-symmetrisation commutes with the projection onto the normalised sub-complex*

$$\epsilon_p\, \Pi_p \;=\; \Pi_p\, \epsilon_p. \tag{3.98}$$

Proof It follows from the formula

$$\Pi_p\ (f_0 \otimes f_1 \otimes \cdots \otimes f_p)\ =\ f_0.df_1.\ldots.df_p.$$

3.10 Chevalley–Eilenberg Map

Definition 3.31 The *Chevalley–Eilenberg map* $\delta : \Lambda_p(\mathscr{A}, \mathscr{M}) \longrightarrow \Lambda_{p-1}(\mathscr{A}, \mathscr{M})$ is given by the formula

$$\delta\ (m \otimes a_1 \wedge a_2 \wedge \cdots \wedge a_p)\ = \tag{3.99}$$

$$\sum_{1\leq r\leq p} (-1)^r\ [\ m,\ a_r\ ] \otimes a_1 \wedge \ldots \hat{a}_r \cdots \wedge a_p\ +\ \sum_{1\leq r<s\leq p} (-1)^{r+s-1}\ m \otimes [a_r,\ a_s] \wedge a_1 \wedge \ldots \hat{a}_r \cdots \wedge \hat{a}_s \wedge \ldots a_p.$$

Notice that the Chevalley–Eilenberg mapping uses exclusively *Lie module structure* of $\mathscr{M}$ over the underlying Lie algebra $\mathscr{A}$.

Theorem 3.15 *Let $\mathscr{A}$ be an associative algebra and $\mathscr{M}$ an $\mathscr{A}$-bi-module. One has the commutative diagram*

$$\begin{array}{ccc} \Lambda_p(\mathscr{A}, \mathscr{M}) & \xrightarrow{\epsilon_p} & C_p(\mathscr{A}, \mathscr{M}) \\ \delta \downarrow & & \downarrow b \\ \Lambda_{p-1}(\mathscr{A}, \mathscr{M}) & \xrightarrow{\epsilon_{p-1}} & C_{p-1}(\mathscr{A}, \mathscr{M}). \end{array} \tag{3.100}$$

In other words, the anti-symmetrisation transforms, essentially, the Hochschild boundary into the Chevalley–Eilenberg mapping.

Proof The proof, comp. [97], is done by induction on p. We suppose the theorem is true for p

$$\epsilon_{p-1}\ \delta\ =\ b\ \epsilon_p. \tag{3.101}$$

We denote $(a_1 \otimes a_2 \otimes, \ldots, \otimes a_p,\ a_{p+1})\ =\ (\bar{a},\ a_{p+1})$. The next relation reflects the action of the symmetric group

$$\epsilon_{p+1}\ (\bar{a},\ a_{p+1})\ =\ (-1)^p\ h(a_{p+1})(\epsilon_p(\bar{a})). \tag{3.102}$$

The following relations are consequences of (3.102), (3.94), (3.101), (3.102), respectively

$$b\ \epsilon_{p+1}\ (\bar{a},\ a_{p+1})\ =\ (-1)^p b\ h(a_{p+1})\ \epsilon_p(\bar{a})\ =$$
$$(-1)^p(\ -ad(a_{p+1}\ )-h(a_{p+1})\ b\ )\epsilon_p\ (\bar{a})\ =$$

$$(-1)^{p+1} ad(a_{p+1})\ \epsilon_p(\bar{a}) \ + \ (-1)^{p-1} h(a_{p+1})\ \epsilon_{p-1}\ \delta(\bar{a}) =$$
$$(-1)^{p+1} ad(a_{p+1})\ \epsilon_p(\ \bar{a}) \ + \ \epsilon_p(\ \delta(\bar{a}),\ a_{p+1}) \ = \ \epsilon_p\ \delta\ (\bar{a},\ a_{p+1}).$$

3.11 Morita Equivalence

Definition 3.32 Let $\mathscr{A}$ be a unital algebra over the ring $\mathbb{K}$.
Let $\iota'_{n,p} : C_p^{tr}(\mathbf{M}_n(\mathscr{A})) \longrightarrow C_p(\mathbf{M}_n(\mathscr{A}))$ be the inclusion of

$$C_p^{tr};(\mathbf{M}_n\ (\mathscr{A})) := \tag{3.103}$$
$$\{ \sum_{1\leq i_0,i_1,\ldots,i_n\leq n} E_{i_0,i_1}(a_0) \otimes_{\mathbf{K}} E_{i_1,i_2}(a_1) \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} E_{i_{p-1},i_p}(a_{p-1}) \otimes_{\mathbf{K}} E_{i_p,i_0}(a_p)\ \},\ \ \forall a_i \in \mathscr{A}$$

into the space of Hochschild chains over the matrix algebra over $\mathscr{A}$.
Recall, $E_{ij}(a)$ is the $n \times n$-matrix whose $(i,\ j)$-entry is a.
Denote by $H_*^{tr}(\mathbf{M}_n(\mathscr{A}))$ the homology of the complex $\{\ C_*^{tr}(\mathbf{M}_n(\mathscr{A})),\ b\ \}$.

Definition 3.33 The *non-commutative trace* is the mapping $tr : C_p^{tr}(\mathbf{M}_n\ (\mathscr{A})) \longrightarrow C_p(\mathscr{A})$

$$tr\ (\ \sum_{1\leq i_0,i_1,\ldots,i_n\leq n} E_{i_0,i_1}(a_0) \otimes_{\mathbb{K}} E_{i_1,i_2}(a_0) \otimes_{\mathbf{K}} \cdots \otimes_{\mathbb{K}} E_{i_{p-1},i_p}(a_{p-1}) \otimes_{\mathbb{K}} E_{i_p,i_0}(a_p)\ \ ) :=$$
$$\sum_{1\leq i_0,i_1,\ldots,i_n\leq n} a_0 \otimes_{\mathbb{K}} a_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} a_{p-1} \otimes_{\mathbb{K}} a_p \in C_p\ (\mathscr{A}) \tag{3.104}$$

Definition 3.34 There is a natural algebra monomorphism $\iota : \mathscr{A} \longrightarrow \mathbf{M}_n(\mathscr{A})$

$$\iota(a) := E_{11}(a). \tag{3.105}$$

Theorem 3.16 (Morita Equivalence) *The inclusions*

(i) $\iota_* : C_*(\mathscr{A}) \longrightarrow C_*^{tr}(\mathbf{M}_n((\mathscr{A}))$,
(ii) $\iota'_* : C_*^{tr}(\mathbf{M}_n(\mathscr{A})) \longrightarrow C_*(\mathbf{M}_n((\mathscr{A}))$,
(iii) the non-commutative trace $tr : C_*^{tr}(\mathbf{M}_n(\mathscr{A})) \longrightarrow C_*(\mathscr{A})$
induce isomorphisms in homology,
(iv) ι_* *and the non-commutative trace* tr *are inverse one to each other.*

Definition 3.35 The isomorphism $\iota'_* \circ \iota_* : HH_*(\mathscr{A}) \longrightarrow HH_*(\mathbf{M}_n(\mathscr{A}))$ is the *Morita isomorphism*.

Proof To prove the theorem we use twice the same kind of spectral sequence. We begin with (ii).

We introduce a filtration on the complex $\{\ C_*(\mathbf{M}_N(\mathscr{A})),\quad b\ \}$

$$F_r\ C_{r+q}(\mathbf{M}_N(\mathscr{A})) := \tag{3.106}$$

$$\{\ \sum_{i_0,\ldots i_r,|,i_{r+1},j_{r+1},\ldots i_{r+q},j_{r+q}} E_{i_0,i_1}(a_0)\otimes\cdots\otimes E_{i_{r-1},i_r}(a_{r-1})\otimes E_{i_r,j_{r+1}}(a_r)\otimes$$

$$\otimes E_{i_{r+1},j_{r+1}}(a_{r+1})\otimes E_{i_{r+2},j_{r+2}}(a_{r+2})\otimes\ldots E_{i_{r+q},j_{r+q}}(a_{r+q})\ \}.$$

In particular, $F_pC_p(\mathbf{M}_N(\mathscr{A}))$ is

$$\{\ \sum_{i_0,\ldots,i_p} E_{i_0,i_1}(a_0)\otimes\cdots\otimes E_{i_p,i_0}(a_r)\ \}. \tag{3.107}$$

This leads to a first quadrant spectral sequence. More specifically,

$$E^0_{r,q} = \begin{cases} C_q(\mathbf{M}_n(\mathscr{A})) & \text{for } r=0,\ 0\le q \\ C_r^{\,tr}(\mathbf{M}_n(\mathscr{A})) & \text{for } q=0,\ 0\le r \end{cases} \tag{3.108}$$

For the computation of the terms $E^1_{r,q}$ notice that

$$d^0_{r,q} = \begin{cases} b'\otimes 1_q & \text{for } 0<q,\ 0\le r \\ b & \text{for } q=0,\ 0\le r; \end{cases} \tag{3.109}$$

here, b is the Hochschild boundary of the complex $C^{tr}_*(\mathbf{M}_n(\mathscr{A}))$, b' is the b'-bar boundary in the complex $C_*^{\,tr}(\mathbf{M}_n(\mathscr{A}))$ and 1_q is the identity map of $C_{q-1}(\mathbf{M}_n(\mathscr{A}))$.

Lemma 3.11 *The complex* $\{\ C_*^{\,tr}(\mathbf{M}_n(\mathscr{A})),\ b'\}$ *is acyclic.*

Proof The operator s

$$s(\ E_{i_0i_1}(a_0)\otimes E_{i_1i_2}(a_1)\otimes\ldots E_{i_pi_0}(a_p)\ ) :=$$
$$E_{i_0i_0}(1)\otimes E_{i_0i_1}(a_0)\otimes E_{i_1i_2}(a_1)\otimes\ldots E_{i_pi_0}(a_p).$$

satisfies the homotopy identity $b'\ s + s\ b'\ =\ 1$.

Therefore

$$E^1_{r,q} = \begin{cases} 0 & \text{for } 0<q,\ 0\le r \\ H_r^{\,tr}(\mathbf{M}_n(\mathscr{A})) & \text{for } q=0,\ r\le 0. \end{cases} \tag{3.110}$$

The $E^2_{r,q}$ terms are

$$E^2_{p,q} = \begin{cases} 0 & \text{for } 0 < q,\ 0 \le r \\ H_r{}^{tr}(\mathbf{M}_n(\mathscr{A})) & \text{for } q = 0,\ 0 \le r. \end{cases} \tag{3.111}$$

The spectral sequence converges to $HH_*(\mathbf{M}_n(\mathscr{A}))$; it collapses at E^2. Part (ii) of Theorem 3.16 is proven.

To prove (i) we consider the following filtration of the complex $\{\ C_*{}^{tr}(\mathbf{M}_n(\mathscr{A})), b\ \}$

$$F_r\ C^{tr}_{r+q}(\mathbf{M}_N(\mathscr{A})) := \tag{3.112}$$

$$\{ \sum_{i_0=\cdots=i_r,|,i_{r+1},\ldots i_{r+q+1}} E_{i_{r+q+1},i_0}(a_0) \otimes \cdots \otimes E_{i_o,i_0}(a_{r-1}) \otimes E_{i_0,i_{r+1}}(a_r) \otimes$$

$$\otimes E_{i_{r+1},i_{r+2}}(a_{r+1}) \otimes E_{i_{r+2},i_{r+3}}(a_{r+2}) \otimes \ldots E_{i_{r+q},i_{r+q+1}}(a_{r+q})\ \}.$$

The spectral sequence terms E^0 are

$$E^0_{r,q} = \begin{cases} C^{tr}_q(\mathbf{M}_n(\mathscr{A})) & \text{for } r = 0,\ 0 \le q \\ C_r(E_{11}(\mathscr{A})) & \text{for } q = 0,\ 0 \le r. \end{cases} \tag{3.113}$$

$$E^1_{r,q} = \begin{cases} 0 & \text{for } 0 < q,\ 0 \le r \\ HH_r(\mathscr{A}) & \text{for } q = 0,\ r \le 0. \end{cases} \tag{3.114}$$

$$E^2_{p,q} = \begin{cases} 0 & \text{for } 0 < q,\ 0 \le r \\ HH_r(\mathscr{A}) & \text{for } q = 0,\ 0 \le r. \end{cases} \tag{3.115}$$

The spectral sequence collapses at E^2 and converges to $H_*{}^{tr}(\mathbf{M}_n(\mathscr{A}))$. Part (i) of Theorem 3.16 is proven.

(iii) Clearly

$$1 = tr \circ \iota_{p,*} : C_p(\mathscr{A}) \longrightarrow C_p(\mathscr{A}). \tag{3.116}$$

Part (i) says that $\iota_{p,*}$ induces an isomorphism in homology. Therefore, the graded trace tr is an isomorphism. Moreover, tr and ι_*, acting in homology, are inverse one to each other.

Theorem 3.16 is proven.

3.12 Connes–Chern Character of Idempotents

Theorem 3.17

(i) Let $e \in \mathscr{A}$ be an idempotent of the algebra $\mathscr{A}$ and q an even number. Then

$$Ch_q(e) := \frac{q}{2}! \otimes_{\mathbf{K}}^{q+1} e = \frac{q}{2}! \, e \otimes_{\mathbf{K}} e \otimes_{\mathbf{K}} e \ldots e \otimes_{\mathbf{K}} e \in H_q^{\lambda}(\mathscr{A}) \tag{3.117}$$

is a cyclic cycle.

(ii) If $e \in \mathbf{M}_n(\mathscr{A})$ then

$$\tilde{Ch}_q(e) := \frac{q}{2}! \, tr \, (\otimes_{\mathbf{K}}^{q+1} e) = \frac{q}{2}! \, tr \, (e \otimes_{\mathbf{K}} e \otimes_{\mathbf{K}} \cdots \otimes_{\mathbf{K}} e) \in H_q^{\lambda}(\mathscr{A}) \tag{3.118}$$

is a cyclic cycle.

(iii) Both cycles represent the Connes–Karoubi–Chern character of e up to a multiplicative factor.

(iv) $\tilde{Ch}_q(e)$ and $Ch_q(e)$ are equivalent under the Morita isomorphism.

Proof We use the second definition of cyclic homology (2.189). It is clear that both chains have cyclic symmetry; it is easy to see that both chains are b'-close. Theorem 3.16 along with Theorem 2.12 show that both homology classes represent well defined cyclic cohomology classes of the algebra $\mathscr{A}$.

3.13 $S\,(Ch_q\ e) = Ch_{q-2}\ e$

Theorem 3.18 *For any idempotent $e \in \mathscr{A}$ and any even number q one has*

$$S\,(Ch_q\ e) = Ch_{q-2}\ e. \tag{3.119}$$

Proof We use the first definition of cyclic homology. Recall the formula (2.205) for the operator $S : H_p^{\lambda}(\mathscr{A}) \longrightarrow H_{p-2}^{\lambda}(\mathscr{A})$

$$S\,(x) = N \circ b' \circ \tilde{J}^{\perp} \circ b\,(x).$$

As q is even, $b\,(\otimes^{q+1} e) = \otimes^{q} e$. Then

$$\tilde{J}^{\perp}\, b\,(\otimes^{q+1} e) =$$

$$\sum_{r=0}^{r=q-1} \frac{q-1-r}{q} T^r(\otimes^q e) = \sum_{r=0}^{r=q-1} \frac{q-1-r}{q}(-1)^r(\otimes^q e) =$$

$$\sum_{r=0}^{r=q-1}(-1)^r(1-\frac{1+r}{q})(\otimes^q\ e)\ =\ \sum_{r=0}^{r=q-1}(-1)^{r+1}\frac{1+r}{q}(\otimes^q\ e)\ =$$

$$\frac{-1}{q}\sum_{r=0}^{r=q-1}(-1)^r(1+r)(\otimes^q\ e)\ =\frac{-1}{q}(-1)\frac{q}{2}(\otimes^q\ e)\ =\frac{1}{2}\ (\otimes^q\ e)\ .$$

We get further

$$b^{'}\tilde{J}^{\perp}\ b\ (\otimes^{q+1}\ e)\ =\ b^{'}(\frac{1}{2}\ \otimes^q\ e)=\frac{1}{2}\ (\otimes^{q-1}\ e)$$

or

$$S\ (\otimes^{q+1}\ e)\ =\ \frac{1}{2}\ N\ (\otimes^{q-1}\ e)\ =\frac{1}{2}\ \sum_{r=0}^{r=q-1}\ (\otimes^{q-1}\ e)\ =\ \frac{q}{2}\ (\otimes^{q-1}\ e)\ =\ \frac{q}{2}\ (\otimes^{q+1}\ e),$$

which proves the theorem.

Remark 3.6 Theorems 3.17, 3.18 are significant. Recall that in the classical differential geometry the definition of the Chern character of vector bundles ($tr\ e^{\frac{R}{2\pi i}}$) exhibits, at the same time, all components of the Chern character. On the other hand, in non-commutative geometry, for each q, the cyclic homology class $[Ch_q\ \ e]\ \in\ H_q^{\lambda}(\mathscr{A})$ *captures part of the components of the Chern character* (make reference to the computation of the cyclic homology of the algebra of smooth functions). Theorem 3.18 shows that the operators S re-organise the totality of components of the Chern character. *Periodic cyclic homology* restores this feature of the classical differential geometry. Periodic cyclic homology is the right container for the Chern character. This is going to be discussed in the last chapter of this book, in the *noncommutative topology*.

3.14 Changing the Ground Ring $\mathbb{K}$

Definition 3.36 Let S be an algebra over the ring $\mathbf{K}$. Suppose $\mathscr{A}$ is an S-algebra. Define

$$C_p^S(\mathscr{A}):=\ \otimes_S^{p+1}\mathscr{A}\ /\ \sim \tag{3.120}$$

where

$$a_0\otimes_S\cdots\otimes_S a_p.s\ \sim\ s.a_0\otimes_S\cdots\otimes_S a_p$$

for any $s \in S$. The Hochschild boundary b passes to this quotient

$$b : C_p^S(\mathscr{A}) \longrightarrow C_{p-1}^S(\mathscr{A}). \tag{3.121}$$

Let

$$\pi : C_p(\mathscr{A}) \longrightarrow C_p^S(\mathscr{A}) \tag{3.122}$$

be the obvious epimorphism.

The homology of the complex $\{ C_*^S(\mathscr{A}),\ b \}$ is denoted by $HH_*^S(\mathscr{A})$.

3.15 Separable Algebras

Definition 3.37 Let S be an algebra over the ring $\mathbf{K}$. S^{op} is the additive group S in which the order of factors inside products is reversed. It is an algebra.

The unital algebra S over the ring $\mathbf{K}$ is called *separable* provided

1. there exists an element $e = \sum_{1 \leq i \leq n} u_i \otimes_{\mathbf{K}} v_i \in S \otimes_{\mathbf{K}} S^{op}$ such that

$$\sum_{1 \leq in} (u_i \ . \ v_i)_S = 1 \in S \tag{3.123}$$

(the index S indicates the algebra in which the product is performed)

2. for any $s \in S$ one has

$$s\, e = e\, s \in S \otimes_{\mathbf{K}} S^{op}. \tag{3.124}$$

It turns out that e is an idempotent. Indeed

$$e \ . \ e = (\sum_{1 \leq i \leq n} u_i \otimes_{\mathbb{K}} v_i).(\sum_{1 \leq j \leq n} u_j \otimes_{\mathbf{K}} v_j) =$$
$$\sum_{1 \leq i,j \leq n} u_i \otimes_{\mathbf{K}} (v_i v_j u_j)_{S^{op}} = \sum_{i,j} u_i \otimes_{\mathbf{K}} (u_j . v_j . v_i)_S =$$
$$\sum_i u_i \otimes_{\mathbf{K}} (\sum_j u_j . v_j) v_i = \sum_i u_i \otimes_{\mathbf{K}} v_i = e.$$

Theorem 3.19 *Let S a separable algebra over the ring $\mathbf{K}$. Assume $\mathscr{A}$ is an S-algebra. Let π be the canonical epimorphism (3.122) $\pi : C_p(\mathscr{A}) \longrightarrow C_p^S(\mathscr{A})$. Then*

$$\pi_{p*} : HH_p(\mathscr{A}) \longrightarrow HH_p^S(\mathscr{A}) \tag{3.125}$$

are isomorphisms for any p.

Proof We use the idempotent e to produce a splitting $\psi : C_p^S(\mathscr{A})$ for the epimorphism π (3.122)

$$\psi\ (a_0 \otimes_S a_1 \otimes_S \cdots \otimes_S a_p) := \tag{3.126}$$
$$\sum_{1\le i_0,i_1,\ldots,i_p\le n}\ v_{i_0}a_0u_{i_1} \otimes_{\mathbb{K}} v_{i_1}a_1u_{i_2} \otimes_{\mathbb{K}} \ldots v_{i_{p-1}}a_{p-1}u_{i_p} \otimes_{\mathbb{K}} v_{i_p}a_pu_{i_0}.$$

ψ is well defined. To show this we begin with the first tensor product. For any $s \in S$ and using (3.124) we get

$$\psi\ ((a_0.s) \otimes_S a_1 \otimes_S \cdots \otimes_S a_p) :=$$
$$\sum_{1\le i_0,i_1,\ldots,i_p\le n}\ v_{i_0}(a_0.s)u_{i_1} \otimes_{\mathbb{K}} v_{i_1}a_1u_{i_2} \otimes_{\mathbb{K}} \ldots v_{i_{p-1}}a_{p-1}u_{i_p} \otimes_{\mathbb{K}} v_{i_p}a_pu_{i_0} =$$
$$\sum_{1\le i_0,i_1,\ldots,i_p\le n}\ v_{i_0}a_0u_{i_1} \otimes_{\mathbb{K}} (v_{i_1}.s)a_1u_{i_2} \otimes_{\mathbb{K}} \ldots v_{i_{p-1}}a_{p-1}u_{i_p} \otimes_{\mathbb{K}} v_{i_p}a_pu_{i_0} =$$
$$\psi\ (a_0 \otimes_S (s.a_1) \otimes_S \cdots \otimes_S a_p).$$

The other tensor products are treated similarly.

To prove (3.125) we use the same kind of spectral sequence we used to prove Theorem 3.16(i) and (ii). For, we introduce a filtration of the complex $\{\ C_*(\mathscr{A}),\ b\ \}$

$$F_r\ C_{r+q}(\mathscr{A}) :=$$
$$\{\sum_{finite}\ \sum_{1\le i_0,\ldots,i_r\le n} a_0u_{i_1}\ \otimes_{\mathbb{K}}\ v_{i_1}a_1u_{i_2}\ \otimes_{\mathbb{K}}\ \cdots \otimes \otimes_{\mathbb{K}} v_{i_{r-1}}a_ru_{i_r} \otimes_{\mathbb{K}} a_{r+1}\ \otimes_{\mathbb{K}}\ \ldots\ \otimes_{\mathbb{K}}\ a_p\}$$

for $a_i \in \mathscr{A}$. In particular

$$F_r\ C_r(\mathscr{A}) :=$$
$$\{\sum_{finite}\ \sum_{1\le i_0,\ldots,i_r\le n} v_{i_0}a_0u_{i_1}\ \otimes_{\mathbb{K}}\ v_{i_1}a_1u_{i_2}\ \otimes_{\mathbb{K}}\ \cdots \otimes \otimes_{\mathbb{K}} v_{i_r}a_ru_{i_0}\ \}.$$

The E^2 terms are

$$E^2_{p,q}\ =\ \begin{cases} 0 & \text{for } 0 < q,\ 0 \le r \\ HH_r^S(\mathscr{A}) & \text{for } q = 0,\ 0 \le r. \end{cases} \tag{3.127}$$

The spectral sequence collapses at E^2; it converges to $HH_*(\mathscr{A})$. This completes the proof of the theorem.

For more information about separable algebras see [68, Ch. VII, §5].

3.16 Excision

Definition 3.38 (Wodzicki [94]) The algebra $\mathscr{I}$ is called *H-unital* provided the complex $\{ \otimes_{\mathbf{K}}^{p} \mathscr{I} \otimes_{\mathbf{K}}^{p} V, \quad b^{'} \}_{0 \leq p}$ is acyclic for any $\mathbf{K}$-module V.

If the algebra $\mathscr{I}$ had a unit, it would have acyclic homology; this explains the term *H-unital*.

Theorem 3.20 (Wodzicki [94, Theorem 3.1]) *Let $\mathscr{A}$ be a unital associative algebra and $\mathscr{I} \subset \mathscr{A}$ a bi-lateral ideal. Suppose $\mathscr{I}$ is H-*unital*. Consider the exact sequence of associative algebras*

$$\mathscr{E}: \quad 0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{A} \longrightarrow \mathscr{A}/\mathscr{I} \longrightarrow 0. \tag{3.128}$$

Then the long exact sequence in Hochschild and cyclic homology

$$\ldots \longrightarrow HH_k(\mathscr{I}) \longrightarrow HH_k(\mathscr{A}) \longrightarrow HH_k(\mathscr{A}/\mathscr{I}) \longrightarrow \ldots \tag{3.129}$$

$$\ldots \longrightarrow H_k^{\lambda}(\mathscr{I}) \longrightarrow H_k^{\lambda}(\mathscr{A}) \longrightarrow H_k^{\lambda}(\mathscr{A}/\mathscr{I}) \longrightarrow \ldots \tag{3.130}$$

are exact.

Proof For the proof see [97, §1.4] or [98].

3.16.1 Shuffle Product

The *shuffle product* mimics the *triangulation of the direct product of two simplices*.

Definition 3.39 Let $\mathscr{A}$ and $\mathscr{B}$ be two unital associative algebras.
Let $\sigma = a_0 \otimes a_1 \otimes \cdots \otimes a_p \in C_p(\mathscr{A})$ and $\tau = b_0 \otimes b_1 \otimes \cdots \otimes b_q \in C_q(\mathscr{B})$.
The *shuffle product*, see [68, pg. 243], [97, §4.2.1],

$$sh(\sigma, \tau) = \sigma \times \tau \in C_{p+q}(\mathscr{A} \otimes \mathscr{B})$$

is given by

$$\sigma \times \tau := \tag{3.131}$$

$$\sum_{\pi \in \tilde{S}_{p+q}} \epsilon(\pi)\, \pi\, (\, (a_0 \otimes b_0) \otimes (a_1 \otimes 1) \otimes \cdots \otimes (a_p \otimes 1) \otimes (1 \otimes b_1) \otimes \cdots \otimes (1 \otimes b_q)\,),$$

where $\tilde{S}_{p+q}$ consists of all permutations of $p+q$ objects with the property that
$1 \leq \pi_1 < \pi_2 < \cdots < \pi_p \leq p+q$ and $1 \leq \pi_{p+1} < \pi_{p+2} < \cdots < \pi_{p+q} \leq p+q$;
$\epsilon(\pi)$ is the sign of π.

To imagine this formula, think of σ as the simplex $[a_0, a_1, a_2, \ldots, a_p]$ and of τ as the simplex $[b_0, b_1, b_2, \ldots, b_q]$. Their direct product is decomposed as an union of simplices given by formula (3.131).

Proposition 3.18 *The shuffle product satisfies*

$$b(\sigma_p \times \tau_q) \;=\; b(\sigma_p) \times \tau_q \;+\; (-1)^p \sigma_p \times b(\tau_q). \tag{3.132}$$

Proof We leave the proof to the reader.

3.16.2 $HH_n(\mathscr{A} \otimes \mathscr{B}) = \sum_{p+q=n} HH_p(\mathscr{A}) \otimes HH_q(\mathscr{B})$

Theorem 3.21 (Eilenberg–Zilber Theorem for Hochschild Homology) *Let $\mathscr{A}$ and $\mathscr{B}$ be unital algebras over the field $\mathbb{K}$.*

Then the sum of the shuffle mappings

$$sh_* : \sum_{p+q=n} HH_n(\mathscr{A}) \otimes_{\mathbf{K}} HH_q(\mathscr{B}) \longrightarrow HH_n(\mathscr{A} \otimes_{\mathbf{K}} \mathscr{B}) \tag{3.133}$$

produce isomorphisms for any n.

Proof (Schetch). Proposition 3.18 and Theorem 3.22 imply that the sum of the shuffle mappings induce morphisms sh_* in homology. Call this homomorphism f. The inverse of f is constructed using acyclic model argument, see [68, Chapter VIII]; see also [97, Theorem 4.2.5. pg. 123].

3.17 Tensor Product of Complexes

Definition 3.40 Let $\mathscr{A} = \{A_p, \partial\}$, resp. $\mathscr{B} = \{B_q, \partial'\}$, be a complex of $\mathbb{K}$ right, resp. left, modules over the ring $\mathbb{K}$. Their *tensor product* is $\mathscr{A} \otimes_{\mathbb{K}} \mathscr{B}$, where

$$(\mathscr{A} \otimes_{\mathbb{K}} \mathscr{B})_n := \sum_{p+q=n} A_p \otimes_{\mathbb{K}} B_q \tag{3.134}$$

with boundary

$$\partial'' \gamma_p \otimes \sigma_q := \partial(\gamma) \otimes_{\mathbf{K}} \sigma \;+\; (-1)^p \gamma \otimes_{\mathbb{K}} \partial'(\sigma). \tag{3.135}$$

Theorem 3.22 *Let $\mathscr{A} \otimes_{\mathbf{K}} \mathscr{B}$ be a product of complexes. Suppose that B_q and $H_q(\mathscr{B})$ are* flat, *or that* **K** *is a* field. *Then the homology of $\mathscr{A} \otimes \mathscr{B}$ is*

$$H_n(\mathscr{A} \otimes \mathscr{B})_n \;=\; \sum_{p+q} H_p(\mathscr{A}) \otimes H_q(\mathscr{B}). \tag{3.136}$$

Proof The proof is based on a spectral sequence associated to a filtration of the tensor product of complexes, see, e.g., [68, §9].

3.18 The Cyclic Bi-complex $\{b,\ b^{'}\}$ and Cyclic Homology

Definition 3.41 The *cyclic bi-complex* $\{b, b^{'}\}$ is the bi-complex introduced by Definition 2.24 and Sect. 2.4.3.

The cyclic bi-complex was introduced by Loday and Quillen [85]. Recall that this bi-complex was used to derive Connes' exact sequence.

In this section we show that *cyclic homology* of this complex is the *homology of the total complex* $Tot\{b,\ b'\}$ associated with the bi-complex $\{b,\ b'\}$.

As an application, we will use this bi-complex to compute the cyclic homology of the algebra of smooth functions. We will show that the cyclic homology of the algebra $C^{\infty}(M)$ is, essentially, the de Rham cohomology of the space M. This result is important because it represents the *link between (co)homology in classical differential geometry and non-commutative geometry*. The same result exhibits a *defect of cyclic homology*. *Periodic cyclic homology* corrects this defect. These features will be discussed later.

Definition 3.42 We associate with the bi-complex $\{b, b^{'}\}$ (2.178) the complex $Tot\{b,\ b'\}$

$$Tot\{b,\ b'\} := \{\ Tot\{b,\ b'\}_*,\ \partial\ \}, \tag{3.137}$$

$$Tot\{b,\ b'\}_n := \sum_{p+q=n} C_{p,q} = \sum_{0\leq q\leq n} \otimes^{q+1}\mathscr{A}, \qquad \partial\ := b - b^{'}.$$

Theorem 3.23 *Let $\mathscr{A}$ be an associative algebra, not necessarily unital.*
Then the homology of the complex $Tot\{b,\ b'\}$ is the cyclic homology

$$H_n((Tot\{b,\ b'\})\ =\ H_n^{\lambda}(\mathscr{A}). \tag{3.138}$$

Proof We give the proof of the theorem for unital algebras. For the general case see [97, §2.1.3].

The proof is based on a spectral sequence argument. For, we consider the spectral sequence associated to the filtration of $Tot\{b,\ b'\}$ with respect to the second index q

$$F^{''}_{p,q} := \oplus_{0\leq r\leq q} C_{r,p}(\mathscr{A}). \tag{3.139}$$

We get a first quadrant spectral sequence, $E^0_{p,q}\ =\ C_{p,q}$.

$$E^1_{p,q}\ =\ \begin{cases} \otimes^{q+1}(\mathscr{A})/(1-T)\otimes^{q+1}(\mathscr{A}) & \text{if}\quad p=0 \\ 0 & \text{if}\quad p>0; \end{cases} \tag{3.140}$$

$d^1 = b(\,mod.\ 1-T)$, or

$$
\begin{array}{ccccccccc}
\cdots & & \cdots & \cdots & \cdots & \cdots \\
\downarrow b & & \downarrow b' & \downarrow b & \downarrow b' \\
\otimes^4\mathscr{A}/(1-T)\otimes^4\mathscr{A} & \xleftarrow{1-T} & 0 & \xleftarrow{N} & 0 & \xleftarrow{1-T} & 0 & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow b' & \downarrow b & \downarrow b' \\
\otimes^3\mathscr{A}/(1-T)\otimes^3\mathscr{A} & \xleftarrow{1-T} & 0 & \xleftarrow{N} & 0 & \xleftarrow{1-T} & 0 & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow b' & \downarrow b & \downarrow b' \\
\otimes^2\mathscr{A}/(1-T)\otimes^2\mathscr{A} & \xleftarrow{1-T} & 0 & \xleftarrow{N} & 0 & \xleftarrow{1-T} & 0 & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow b' & \downarrow b & \downarrow b' \\
\mathscr{A}/(1-T)\mathscr{A} & \xleftarrow{1-T} & 0 & \xleftarrow{N} & 0 & \xleftarrow{1-T} & 0 & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow b' & \downarrow b & \downarrow b' \\
0 & & 0 & 0 & 0 & \cdots
\end{array}
\tag{3.141}
$$

Therefore, using the first definition of cyclic homology, we get

$$
E^2_{p,q} = \begin{cases} H^{\lambda}_q(\mathscr{A}) & \text{if} \quad p=0 \\ 0 & \text{if} \quad p>0. \end{cases}
\tag{3.142}
$$

The spectral sequence collapses at E^2; it converges to the homology of the total complex $Tot\{b,\ b'\}$. The theorem is proven.

We consider the filtration of the bi-complex $\{b,\ B\}$ with respect to the first index p

$$
F^{'}_p := \sum_{0\le r\le p} C_{r,q} = \sum_{0\le r\le p} \otimes^{q+1}\mathscr{A}.
\tag{3.143}
$$

Then $E^0_{p,q} = C_{p,q}$ with the boundary map $d^0_{p,q\,*} = b,\ or\ b^{'}_{*}$.

The term E^1 is

$$\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
0 & \longleftarrow HH_3(\mathscr{A}) & \xleftarrow{1-T} & 0 & \xleftarrow{N} & HH_3(\mathscr{A}) & \xleftarrow{1-T} & 0 & \xleftarrow{N} \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
0 & \longleftarrow HH_2(\mathscr{A}) & \xleftarrow{1-T} & 0 & \xleftarrow{N} & HH_2(\mathscr{A}) & \xleftarrow{1-T} & 0 & \xleftarrow{N} \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
0 & \longleftarrow HH_1(\mathscr{A}) & \xleftarrow{1-T} & 0 & \xleftarrow{N} & HH_1(\mathscr{A}) & \xleftarrow{1-T} & 0 & \xleftarrow{N} \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
0 & \longleftarrow HH_0(\mathscr{A}) & \xleftarrow{1-T} & 0 & \xleftarrow{N} & HH_0(\mathscr{A}) & \xleftarrow{1-T} & 0 & \xleftarrow{N} \cdots
\end{array} \tag{3.144}$$

with boundary map $d^1_{p,q} = 0$ for any p and q. Indeed, if p is odd/even, the domain/co-domain of $d^1_{p,q}$ is zero. For the same reason all boundary maps $d^{2r+1}_{p,q}$ vanish. Therefore,

$$E^{2r}_{p,q} = H_*(E^{2r-1}) = E^{2r-1} \quad for\ any\ 1 \leq r.$$

Proposition 3.19 *The boundary map $d^2_{p,q} : E^2_{p,q} \longrightarrow E^2_{p-2,q+1}$, $p = even$, is the operator B (2.208)*

$$B : HH_p(\mathscr{A}) \longrightarrow HH_{p+1}(\mathscr{A}).$$

Proof In the spectral sequence under consideration $d^0_{2p,q} = b_*$ and $d^0_{2p+1,q} = b'_*$. For which concerns d^1 one has $d^1_{2p,q} = N_*$ and $d^1_{2p+1,q} = (1-T)_*$.

Let $[x] \in E^2_{2p,q} = H_*(E^1)$ be the homology class of a d^1-cycle $x \in E^1$, (i.e. $d^1(x) = 0$, or $N_*(x) = 0$). The element x is the homology class of a d^0-cycle $\bar{x} \in C_{2p,q}$, (i.e. $b_*(\bar{x}) = 0 \in E^0$).

The d^1-cycle condition $d^1(x) = N_*(x) = 0$ means that $d^1(x)$ is the 0-homology class belonging to the sub-complex F_0, or $N_*(\bar{x}) = b'(y)$, for some $y \in C_{2p-1,q+1}$. Given that the b'-complex is acyclic, we may use the homotopy s to take $y = s\,(N\,\bar{x}) = 1 \otimes N\bar{x}$, i.e. $N_*(\bar{x}) = b'(1 \otimes N\bar{x}) \in C_{2p-1,q}$.

From these relations we get

$$b\,(1-T)\,(y) = -(1-T)\,b'\,(y) = 0$$

or

$$b\,(1-T)\,(1 \otimes N\,\bar{x}) = -(1-T)\,b'\,(1 \otimes N\,\bar{x}) = 0.$$

This says that $z := (1-T)\,(1\otimes N\,\bar{x})$ is a b-cycle in $C_{2p-2,q+1}$; it determines a class $[z] = [(1-T)\,(1\otimes N\,\bar{x})] \in E^2_{2p-2,q+1}$. Therefore

$$d^2([x]) = [z] = [(1-T)\,(1\otimes N\bar{x})] = B[x],$$

see (2.208).

3.19 Computation of $H_*^\lambda(C^\infty(M))$

Now we are in the position to compute the *continuous cyclic homology* $H_*^\lambda(C^\infty(M)$. We know from Sect. 3.1 that the continuous Hochschild homology $HH_q(C^\infty(M))$ is the space of q-differential forms. On the other hand, for any associative algebra $\mathscr{A}$, we considered in Sect. 3.14 a spectral sequence which converged to the cyclic homology of the algebra $\mathscr{A}$. This spectral sequence may be used even in the case of *continuous cyclic homology* if finite sums are replaced by series. We showed also that the d^2-boundary map is the Connes' operator B.

Lemma 3.12 *For any associative algebra $\mathscr{A}$ the operators b and B anti-commute*

$$B\,b + b\,B = 0. \tag{3.145}$$

Proof The relation (3.145) follows from (2.156) and (2.157). In fact, we get

$$\begin{aligned} B\,b\gamma &= (1-T)\,(1\otimes(N\,b\,\gamma)) = (1-T)\,(1\otimes b'\,N\,\gamma)\\ b\,B\gamma &= b(1-T)\,(1\otimes(N\gamma)) = (1-T)b'(1\otimes(N\gamma)) =\\ &(1-T)\,(N\gamma - 1\otimes b'N\,\gamma) = -(1-T)\,(1\otimes b'N\,\gamma). \end{aligned}$$

Lemma 3.13 *One has*

$$Image\ E^2_{2p,q} = Image\ B_q = d\,\Omega^q(M) = all\ exact\ (q+1)-differential\ forms.$$

Proof We already know that $HH^q(C^\infty(M)) = \Omega^q(M) = q-differentialforms$. We know also that, if $f \in C_q(C^\infty(M))$ is a Hochschild cycle, its Hochschild homology class is the differential form

$$[f] = J_q(f) := \frac{\partial^q}{\partial x_1^{i_1}\dots\partial x_q^{i_q}}_{|diag}\ dx_1^{i_1}\wedge\dots\wedge dx_q^{i_q}. \tag{3.146}$$

Lemma 3.12 tells that B induces a well defined homomorphism $B : HH^q(C^\infty(M)) \longrightarrow HH^{q+1}(C^\infty(M))$. The (anti)commutativity between B and b gives

$$B[f] = -J_{q+1}\left((1-T)(1\otimes N\, J_q(f))\right) = J_{q+1}(1\otimes N\, J_q f) = d\, J_q\, N\, f \quad (3.147)$$

because $J_{q+1}T(1\otimes\omega) = 0$ for any differential from ω. Moreover, the set $d\, J_q\, N\, f$, for arbitrary cycle f, is the set of *all* exact (q+1)-differential forms.

Lemma 3.14

(i) $F_{2p+3,q}/F_{2p,q}$ is independent of p.
(ii) The spectral sequence collapses at E^3: $E^\infty_{p,q} = E^3_{p,q}$.

Proof

(i) The quotient space is

$$F_{2p+3,q}/F_{2p,q} = \sum_{2p+1\le r\le 2p+3} C_r(C^\infty(M)).$$

The boundary maps involved in the definition of $E^r_{p,q}$ are $N = \sum_{0\le r\le q} T^r$ and $(1-T)$. The operator $\frac{1}{q+1}N$ is an idempotent and its complement contains the factor $1-T$, see Sect. 2.4.2. Therefore, the kernel and the image of any composition of these maps may not produce new d^r-cycles and boundaries beyond those produced by d^2.

(ii) Part (i) says that $F^3 = F^\infty$.

A general property of spectral sequences (see [45, page 471] or [97, Theorem D.4 page 418]) states that E^∞ provides a filtration of the homology of *Tot* by means of sub-groups $G_q \subset H^\lambda(C^\infty(M))$ and $F^{\infty,q} = G_q/G_{q-1}$. Therefore

Lemma 3.15

$$H^\lambda_n(C^\infty(M)) = \oplus_{2p+q=n}\, E^3_{2p,q}.$$

Theorem 3.24 *For any paracompact smooth manifold M*

$$H^\lambda_n(C^\infty(M)) = \Omega_n(M)/d\Omega_{n-1}(M) \oplus_{1\le 2r\le n} H^{n-2r}_{dR}(M). \quad (3.148)$$

Proof Based on general properties of spectral sequences, we get $H^\lambda_n(C^\infty(M)) = \oplus_{2p+q=n}\, E^3_{2p,q}$. The top term ($q = n-2p$) is

$$E^3_{2p,n-2p} = Ker\, d^2_{2p,n-2p}/Image\, d^2_{2p,n-2p-1} = \Omega_n/d\Omega_{n-1}. \quad (3.149)$$

The other terms are $Ker\, d^2_{2p,n-2r}/Image\, d^2_{2p,n-2r-1} = H^{n-2r}_{dR}(M)$. This completes the proof of the theorem.

Remark 3.7 The top term ($q = n - 2p$) and the other terms E^3 have different structures. While all other terms represent de Rham cohomologies, the numerator of the top term is the Ω_n instead of $Ker\ d^2$. This is due to the *configuration of the spectral sequence*, which derives the fact that the bi-complex $\{b, b^{'}\}$ is limited to the left (being a first quadrant bi-complex). This is the *defect of cyclic homology* we referred to above. It was exhibited on the case of the algebra of smooth functions. *Periodic cyclic homology* repairs this defect.

3.20 $H^{\lambda}_{*}(\mathscr{A})$

Theorem 3.23 says that the total homology of the bi-complex $\{b,\ b^{'}\}$ is the cyclic homology.

Theorem 3.25 *There exists a homomorphism*

$$H^{\lambda}_{n}(\mathscr{A}) \longrightarrow (\ HH_n(\mathscr{A})/B\ (HH_{n-1}(\mathscr{A}))\ ) \oplus_{2\leq 2r\leq n}\ (Ker\ B_{n-2r}/Im\ B_{n-2r-1}). \tag{3.150}$$

This homomorphism is an isomorphism for $\mathscr{A} = C^{\infty}(M)$.

Proof Consider the spectral sequence associated to the filtration $F^{'}$ of the bi-complex $\{b,\ b^{'}\}$ (3.143).

The term E^2 is the operator B, Proposition 3.19

$$E^2_{2p+1,q} = 0, \quad E^2_{2p,q} = HH_q(\mathscr{A}) \tag{3.151}$$

$$B : E^2_{2p,q} \longrightarrow E^2_{2p-2,q+1}. \tag{3.152}$$

and

$$E^3 = E^2. \tag{3.153}$$

The spectral sequence collapses at the term E^3 Lemma 3.14(ii).

The boundary map d^2 is $d^2_{2p,q} = B : HH_q \longrightarrow HH_{q+1}$.

3.21 Periodic Cyclic Homology

Periodic cyclic homology may be introduced in different ways. The original definition is [99, p. 301] (see Remarks 3.6 and 3.7)

Definition 3.43

$$H^{\lambda,per}(\mathscr{A})\ :=\ Lim_{\rightarrow} S : H^{\lambda}_{n}(\mathscr{A}) \longrightarrow H^{\lambda}_{n-2}(\mathscr{A}). \tag{3.154}$$

Therefore, the operator S passes to periodic cyclic homology. It fits in the Connes' long exact sequence

$$(Ces - b)^{per} \equiv \tag{3.155}$$
$$\dots \xrightarrow{\delta_{n+1}} H^{\lambda}_{n-1} \xrightarrow{B_{n-1}} H_n \xrightarrow{I_n} H^{\lambda,per}_n \xrightarrow{S_n} H^{\lambda,per}_{n-2} \xrightarrow{B_{n-2}} H_{n-1} \xrightarrow{I_{n-1}} \dots$$

Periodic cyclic homology may be introduced using the $\{B,\ b\}$ bi-complex, to be defined next.

Recall $B\,b\ +\ b\,B\ =\ 0$, Lemma 3.12.

Definition 3.44 By definition the {B, b} bi-complex is given by the first quarter diagram

$$\begin{array}{ccccccccc}
 & \dots & & \dots & & \dots & & \dots & \dots \\
 & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & \\
0 & \longleftarrow C_3(\mathscr{A}) & \xleftarrow{B} & C_2(\mathscr{A}) & \xleftarrow{B} & C_1(\mathscr{A}) & \xleftarrow{B} & C_0(\mathscr{A}) & \longleftarrow \dots \\
 & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow & \\
0 & \longleftarrow C_2(\mathscr{A}) & \xleftarrow{B} & C_1(\mathscr{A}) & \xleftarrow{B} & C_0(\mathscr{A}) & \longleftarrow & 0 & \longleftarrow \dots \\
 & \downarrow b & & \downarrow b & & \downarrow & & \downarrow & \\
0 & \longleftarrow C_1(\mathscr{A}) & \xleftarrow{B} & C_0(\mathscr{A}) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \dots \\
 & \downarrow b & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longleftarrow C_0(\mathscr{A}) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow \dots
\end{array} \tag{3.156}$$

The general term is $C_{p,q} = C_{q-p}(\mathscr{A})$ for $0 \le p \le q$; otherwise, it is zero.

Theorem 3.26 *The homology of the total complex of the* $\{B, b\}$ *bi-complex is the cyclic homology*

$$H_p(Tot\{B, b\}(\mathscr{A})) \ = \ H^{\lambda}_p(\mathscr{A}). \tag{3.157}$$

Proof We consider the spectral sequence associated to the filtration with respect to the first index

$$F_{p,q} := \oplus_{0\le r\le p}\, C_{r,r+q}.$$

The first spectral sequence terms are

$$E^0_{p,q} \ = \ C_{p,q} \tag{3.158}$$
$$E^1_{p,q} \ = \ HH_{q-p}(\mathscr{A}) \tag{3.159}$$

$$E^2_{p,q} \;=\; Ker\; B_q/Im\; B_{q-1}, \quad B_q : HH_q(\mathscr{A}) \to HH_{q+1}(\mathscr{A}). \tag{3.160}$$

Excepting index shifts, the terms E^2 coincide with the terms E^2 (3.151).

Proposition 3.20 *The complex $\{B, b\}$ is obtained from the bi-complex $\{b, b^{'}\}$ by eliminating the acyclic columns $b^{'}$, see [97, §2.1.7, 2.5.10].*

Proof See [97, §2.5.10].

We said at the beginning of this section that *periodic cyclic homology* repairs a defect of cyclic homology, see Remark 3.7. The repair is done by extending the $\{B, b\}$ bi-complex to the left, above the bisectrice of the first and third quadrants.

Definition 3.45 The *periodic cyclic bi-complex* $\{B, b\}^{per}$ is the diagram

$$\begin{array}{ccccccccccc}
 & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & \\
 & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & & & \\
\cdots & \longleftarrow C_3(\mathscr{A}) & \xleftarrow{B} & C_2(\mathscr{A}) & \xleftarrow{B} & C_1(\mathscr{A}) & \xleftarrow{B} & C_0(\mathscr{A}) & \longleftarrow & \cdots & \\
 & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow & & & \\
\cdots & \longleftarrow C_2(\mathscr{A}) & \xleftarrow{B} & C_1(\mathscr{A}) & \xleftarrow{B} & C_0(\mathscr{A}) & \longleftarrow & 0 & \longleftarrow & \cdots & \\
 & \downarrow b & & \downarrow b & & \downarrow & & \downarrow & & & \\
\cdots & \longleftarrow C_1(\mathscr{A}) & \xleftarrow{B} & C_0(\mathscr{A}) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \\
 & \downarrow b & & \downarrow & & \downarrow & & \downarrow & & & \\
\cdots & \longleftarrow C_0(\mathscr{A}) & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\cdots & \longleftarrow 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
 & \cdots & & \cdots & & \cdots & & \cdots & & \cdots &
\end{array} \tag{3.161}$$

In the sequel we will use .

Definition 3.46 The *total product complex associated to the bi-complex* $\{B, b\}^{per}(\mathscr{A})$ is called *periodic cyclic complex*; it consists of two terms $C^{\lambda,per}_{ev/odd}(\mathscr{A})$

$$\begin{aligned} C^{\lambda,per}_{ev}(\mathscr{A}) &:= \prod_{0\le p} C_{2p}(\mathscr{A}) \\ C^{\lambda,per}_{odd}(\mathscr{A}) &:= \prod_{0\le p} C_{2p+1}(\mathscr{A}) \end{aligned} \tag{3.162}$$

and boundary maps

$$B - b : C^{\lambda,per}_{ev/odd}(\mathscr{A}) \longrightarrow C^{\lambda,per}_{odd/ev}(\mathscr{A}). \tag{3.163}$$

The boundary map $B - b$ preserves the parity of chains and not their degree. The homology of the total product complex associated to the bi-complex $\{B, b\}^{per}(\mathscr{A})$ is the *periodic cyclic homology* of the algebra $\mathscr{A}$. Periodic cyclic homology is Z_2-graded; it is denoted $H^{\lambda,per}_{ev/odd}(\mathscr{A})$.

Definition 3.47

(i) Let $(a_{p,q}) \in \{B, b\}$ be a chain. Let $\{B, b\}^{const} \subset \{B, b\}$ be the sub bi-complex consisting of those chains for which $a_{p,q} = a_{q-p} \in C_{q-p}(\mathscr{A})$.
(ii) $Tot\{B, b\}^{const}$ is a sub-complex of $Tot\{B, b\}$.

3.22 $H^{\lambda,per}_{ev/odd}(C^\infty(M)) = H^{ev/odd}_{dR}(M)$

Definition 3.48 For any $f \in C_k(C^\infty(M)$ define the *partially skew-symmetrised function*

$$f^\wedge(x_0, x_1, \ldots, x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} f(x_0, x_{\sigma(1)}, \ldots, x_{\sigma(k)}). \tag{3.164}$$

Lemma 3.16 *The homomorphism J, defined for any element $f \in C_k(C^\infty(M)$, see Theorem 3.7(ii) and (3.146), satisfies*

(i)

$$J\ (1 \otimes f_0 \otimes \cdots \otimes f_k) = d\ J\ (f_0 \otimes \cdots \otimes f_k) \tag{3.165}$$

(ii)

$$J\ T\ (1 \otimes f_0 \otimes \cdots \otimes u_k) = 0 \tag{3.166}$$

(iii)

$$J\ b\ (f_0 \otimes \cdots \otimes f_k) = 0. \tag{3.167}$$

(iv)

$$J\ B\ (f_0 \otimes \cdots \otimes f_k) = (k+1)\ d\ J\ (f_0 \otimes \cdots \otimes f_k). \tag{3.168}$$

(v) J induces a bi-complex chain homomorphism to the bi-complex of differential forms

$$J : \{B, b\} \longrightarrow \{d, 0\}.$$

Proof The operator J is defined locally. A simple computation in local coordinates proves (i) and (ii).

(iii) $b(f_0 \otimes \cdots \otimes f_k)$ is a Hochschild cycle whose homology class is zero. We know from Theorem 3.7(ii) that its homology class is given by the differential form $J\ b\ f(x_0 \otimes \cdots \otimes f_k)$ (J establishes an isomorphism between Hochschild homology and differential forms).

(iv) One has

$$J\ B\ (f_0 \otimes \cdots \otimes f_k) = J\ (1 - T)\ (1 \otimes N\ f_0 \otimes \cdots \otimes f_k) = \tag{3.169}$$
$$J\ (1 \otimes N\ (f_0 \otimes \cdots \otimes f_k)) = d\ J\ N(f_0 \otimes \cdots \otimes f_k) = (k+1)\ d\ J\ (f_0 \otimes \cdots \otimes f_k).$$

(v) Follows from (iii) and (iv).

Theorem 3.27 *For any paracompact smooth manifold M, the chain homomorphism J induces an isomorphism from the Z_2-graded periodic cyclic homology of the algebra of smooth functions on M to the Z_2-graded de Rham cohomology of M*

$$H^{\lambda,per}_{ev}(\mathscr{A}) = H^{ev}_{dR}(M)$$

$$H^{\lambda,per}_{odd}(\mathscr{A}) = H^{odd}_{dR}(M).$$

Proof The proof is based on the comparison of the spectral sequences (corresponding to the first filtration) associated to the bi-complex $\{B, b\}$ and the de Rham bi-complex $\{d, 0\}$. To this end, we define for the first bi-complex the filtration

$$F^{per}_{ev,q} := \prod_{\infty \le p \le +\infty} C_{2p+q}(C^\infty(M)), \quad q \le 0$$
$$F_{odd,q} := \prod_{\infty \le p \le +\infty} C_{(2p+1)+q}(C^\infty(M)), \quad q \le 0$$
$$F_{ev/odd,q} = 0 \ for\ q < 0;$$

and for the de Rham bi-complex, the filtration

$$F^{dR}_{ev,q} := \prod_{\infty \le p \le +\infty} \Omega^{2p+q}(C^\infty(M)), \quad q \le 0$$
$$F_{odd,q} := \prod_{\infty \le p \le +\infty} \Omega^{(2p+1)+q}(C^\infty(M)), \quad q \le 0$$
$$F_{ev/odd,q} = 0 \ for\ q < 0.$$

The corresponding E^0 and E^1 terms are

$$E^{0,per}_{p,q} = C_{p,q}, \quad E^{0,dR}_{p,q} = \Omega^{p,q},$$

$$E^{1,per}_{q=ev/odd} = \prod_{q=ev/odd} HH_q(\mathscr{A}), \quad E^{1,dR}_{q=ev/odd} = H^{q=ev/odd}_{dR}(M).$$

Lemma 3.16 states that J is a bi-complex homomorphism, while Theorem 3.7 states that J_* is an isomorphism at the level of E^1-terms. Therefore, J induces an isomorphism between the homologies of the associated total complexes.

On the other hand, periodic cyclic homology is, by definition, the homology of the total product complex associated to the bi-complex $\{B, b\}$. The boundary map in the periodic cyclic complex is $B - b$.

3.23 Proof of Theorem 3.27 Explicit

It is instructive to illustrate how periodic cyclic homology of the algebra of smooth functions relates to the de Rham cohomology.

Theorem 3.28 *The bi-complex homomorphism* $J : \{B, b\}^{const}(C^\infty(M)) \longrightarrow \{d, 0\}(\Omega_* M)$ *induces isomorphisms in homology.*

To prove the theorem we show separately that J induces an epimorphism/monomorphism in homology.

Proposition 3.21 J_* *is surjective.*

Proof We have to prove that for any closed de Rham form ω_r, $(d\omega_r = 0)$, of degree k there exists a sequence of functions $0, \ldots, 0, f_k, f_{k+2}, f_{k+4}, \ldots, f_{k+2r}, \ldots$ with the following properties ($f_p \in C_p(C^\infty M)$):

(1) $f_{k-2r} = 0$ for any r, $0 < r$
(2) $J\ f_k = \omega_r$
(3) $J\ f_{k+2r} = 0$ for any r, $0 < r$
(4) $B\ f_{k+2r} = b\ f_{k+2r+2}$ for any integer r.

As J induces isomorphism on the Hochschild homology (Theorem 3.7), given the differential form ω_k there exists a Hochschild cycle f_k so that $J\ f_k = \omega_k$, and, of course, $b\ f_k = 0$.

Bf_k is a b-cycle because $b\ B\ f_k = -B\ b\ f_k = -B(0) = 0$. Being a b-cycle, its Hochschild homology class is determined by the form $J\,Bf_k = kd\,J f_k = kd\omega_k = 0$. Therefore, Bf_k is a b-boundary. Therefore, there exists $\tilde{f}_{k+2}$ so that $Bf_k = b\,\tilde{f}_{k+2}$.

We would like the relation $J\,\tilde{f}_{k+2} = 0$ to be satisfied. As this might not be true, we will be forced to modify $\tilde{f}_{k+2}$. For, consider $\sigma_{k+2} = J\,\tilde{f}_{k+2}$. For the differential

form σ_{k+2} there exists a b-cycle h_{k+2} so that $J h_{k+2} = \sigma_{k+2}$. Replace $\tilde{f}_{k+2}$ by $f_{k+2} = \tilde{f}_{k+2} - h_{k+2}$; then $B\ f_k = b\ \tilde{f}_{k+2} = b(\tilde{f}_{k+2} - h_{k+2}) = b\ f_{k+2}$ and $J\ f_{k+2} = 0$.

Lemma 3.17 (Induction Lemma for Surjectivity)
Let $f_r \in C_r(\mathscr{A})$ and $f_{r+2} \in C_{r+2}(\mathscr{A})$ be given so that

(i) $J\ f_{k+2} = 0$ *and*
(ii) $b\ f_{k+2} = B\ f_k$.

Then there exists $f_{r+4} \in C_{r+4}(\mathscr{A})$ such that

(i) $J\ f_{k+4} = 0$ *and*
(ii) $b\ f_{k+4} = B\ f_{k+2}$.

Proof of the Induction Lemma for Surjectivity.

Consider $B\ f_{k+2}$. It is a b-cycle because $b\ B\ f_{k+2} = -B\ b\ f_{k+2} = -B\ B\ f_k = 0$. Its Hochschild homology class is measured by $J\ B\ f_{k+2} = (k+2)\ d\ J\ f_{k+2} = (k+2)\ d\ 0 = 0$. Therefore, there exists a function $\tilde{f}_{k+4}$ so that $B\ f_{k+2} = b\ \tilde{f}_{k+4}$. Again, we would like that $J\tilde{f}_{k+4} = 0$, which might not be satisfied. We consider $J\tilde{f}_{k+4} := \sigma_{k+4}$. There exists a b-cycle h_{k+4} so that $J\ h_{k+4} = \sigma_{k+4}$. We replace $\tilde{f}_{k+4}$ by $f_{k+4} := \tilde{f}_{k+4} - h_{k+4}$. Then $B\ f_{k+2} = b\tilde{f}_{k+4} = b\ (\tilde{f}_{k+4} - h_{k+4}) = bf_{k+4}$ and $J\ f_{k+4} = 0$. This completes the proof of the lemma.

The induction lemma completes the argument and therefore $J_{even/odd}$ are epimorphisms.

Proposition 3.22 *J_* is injective.*

Proof Assume that $\{f_\epsilon, f_{\epsilon+2}, f_{\epsilon+4}, \ldots..\}$, ($\epsilon = 0$, or 1) is a $(B-b)$-cycle and that its image in the de Rham cohomology is zero

$$J\ f_{\epsilon+2r} = d\ \omega_{\epsilon+2r-1}$$

for each positive integer r.

Treat first the case of J_{ev}. We need to find a sequence of functions $\{g_1, g_3, g_5, \ldots..\}$ so that

$$B\ g_{2r-1} - b\ g_{2r+1} = f_{2r}.$$

The functions g_i will be constructed by induction beginning from the left. The initial data is

$$b\ f_0 = 0, \qquad J\ f_0 = 0$$

$$b\ f_2 = B\ f_0.$$

The first equation to solve is

$$-b\, g_1 \;=\; f_0. \tag{3.170}$$

We know that f_0 is a b-cycle because any chain of degree zero is a b-cycle; we have to prove that it is a b-boundary. J gives its Hochschild homology class. By hypothesis, we know that

$$J\, f_0 \;=\; 0 \in H^0_{dR}(M).$$

But $J\ f_0 \;=\; f_0$ for any chain of degree zero. Therefore $f_0 = 0$. Therefore, an arbitrary b-cycle g_1 solves the equation for g_1. We will see later that g_1 has to be further determined. For this reason this function will be temporarily denoted by $\tilde{g}_1$.

We pass now to the determination of g_3.

The initial data is

$$B\, f_0 - b\, f_2 \;=\; 0$$

$$J\, f_2 \;=\; d\, \omega_1$$

$$-b\, \tilde{g}_1 \;=\; 0.$$

The function g_3 has to satisfy the equation

$$B\, \tilde{g}_1 \;-\; b\, g_3 \;=\; f_2. \tag{3.171}$$

For it to exist, it is necessary that $f_2 \;-\; B\tilde{g}_1$ be a b-boundary. For, we check first whether it is a b-cycle

$$b\,(f_2 \;-\; B\,\tilde{g}_1) \;=\; b\, f_2 \;-\; b\, B\tilde{g}_1 \;=\; b\, f_2 \;+\; B\, b\tilde{g}_1 \;=\; b\, f_2 \;-\; Bf_0 \;=\; 0.$$

Next, we check its Hochschild homology class

$$J\,(f_2 \;-\; B\,\tilde{g}_1) \;=\; d\,\omega_1 \;-\; d\,J\,\tilde{g}_1 \;=\; d\,(\omega_1 \;-\; J\,\tilde{g}_1).$$

At this point we impose on the b-cycle $\tilde{g}_1$ the supplementary condition that

$$J\,\tilde{g}_1 \;=\; \omega_1$$

and then g_3 exists so as to satisfy Eq. (3.170).

This is the basis of induction for the determination of the functions g_{2r+1}.

Now we consider J_{odd}. To prove that J_{odd} is a monomorphism, we assume that the sequence of functions $\{f_1, f_3, f_5, \ldots..\}$ is given so that

$$B\, f_{2r-1} \;=\; b\, f_{2r+1}, \quad for\ \ any\ \ 1 \leq r$$

and that

$$J\ f_{2r-1}\ =\ d\ \omega_{2r-2}.$$

We have to prove that there exist functions $\{\ g_0, g_2, g_4, \ldots..\}$ so that

$$B\ g_{2r-2}\ -\ bg_{2r}\ =\ f_{2r-1}.$$

As in the case of J_{odd} we proceed by increasing induction on r.

We state first the basis of induction. The first equation solve is

$$B\ g_0\ -\ bg_2\ =\ f_1. \tag{3.172}$$

To determine g_0 we apply J to both sides of Eq. (3.172). We get

$$J\ (B\ g_0\ -\ bg_2)\ =\ J\ f_1,$$

which becomes

$$d\ J\ g_0\ =\ J\ f_1$$

or, as $J\ g_0\ =\ g_0$

$$d\ g_0\ =\ d\omega_0.$$

Therefore, $g_0\ =\ \omega_0\ +\ C$, where C is a constant.

We replace g_0 in the above equation, which becomes

$$B\ (\omega_0\ +\ C)\ -\ b\ g_2\ =\ f_1$$

or, as $B\ C = 0$,

$$B\ \omega_0\ -\ b\ g_2\ =\ f_1. \tag{3.173}$$

Notice that $f_1\ -\ B\ \omega_0$ has degree one and hence it is a b-cycle. We compute its Hochschild homology class

$$J\ (f_1\ -\ B\ \omega_0)\ =\ d\ \omega_0\ -\ d\ \omega_0\ =\ 0.$$

Therefore there exists g_2, determined modulo a b-cycle. There exist g_0 and g_2 so that Eq. (3.172) be satisfied.

With the basis of induction assured by the fulfilment of Eqs. (3.170), (3.171) and (3.172), the induction argument is provided by the next lemma.

Lemma 3.18 (Induction Lemma for Injectivity) *We assume the following data is given*

(i) $f_{r-1} \in C_{r-1}(\mathscr{A})$ *and* $f_{r+1} \in C_{r+1}(\mathscr{A})$ *satisfy the* $(B-b)$*-boundary condition*

$$B\ f_{r-1}\ =\ b\ f_{r+1} \tag{3.174}$$

(ii) so that $J_*\ (f_{r+1})\ =\ 0 \in H^{r+1}_{dR}(M)$*, i.e.*

$$J\ f_{r+1}\ =\ d\ \omega_r, \tag{3.175}$$

with $\omega_r \in \Omega^r(M)$

(iii) $g_{r-2} \in C_{r-2}(\mathscr{A})$ *and* $\tilde{g}_r \in C_r(\mathscr{A})$ *such that*

$$B\ g_{r-2}\ -\ b\ \tilde{g}_r\ =\ f_{r-1}. \tag{3.176}$$

Let $h_r \in C_r(\mathscr{A})$ *be a* b*-cycle* $h_r \in C_r(\mathscr{A})$ *such that*

$$J\ h_r\ =\ \frac{1}{r}\omega_r\ -\ J\tilde{g}_r. \tag{3.177}$$

Then there exists $\tilde{g}_{r+2} \in C_{r+2}(\mathscr{A})$ *so that*

$$B\ (\ \tilde{g}_r\ +\ h_r\ )\ -\ b\tilde{g}_{r+2}\ =\ f_{r+1}. \tag{3.178}$$

Proof In order to satisfy Eq. (3.178) it is necessary that

$$C := f_{r+1}\ -\ B\ (\ \tilde{g}_r\ +\ h_r\ ) \tag{3.179}$$

be a b-boundary. For, we prove first that it is a b-cycle. Indeed

$$\begin{gathered} b\ C\ =\ b\ (\ f_{r+1}\ -\ B(\ \tilde{g}_r\ +\ h_r\ )\ )\ = \\ b\ f_{r+1}\ +\ B\ b(\ \tilde{g}_r\ +\ h_r\ )\ = \\ b\ f_{r+1}\ +\ B(\ B\ g_{r-2}\ -\ f_{r-1}\ )\ +\ B\ b\ h_r\ = \\ (\ b\ f_{r+1}\ -\ B\ f_{r-1}\ )\ +\ B\ b\ h_r\ =\ 0. \end{gathered} \tag{3.180}$$

The last equality uses the co-cycle condition (3.174) and the hypothesis that h_r is a b-cycle.

We determine the Hochschild homology class of the cycle C. We have

$$\begin{gathered} J\ C\ =\ J\ (\ f_{r+1} - B\ (\ \tilde{g}_r\ +\ h_r\ )\ ) = \\ =\ d\ \omega_r\ -\ r\ d\ (\ J\tilde{g}_r\ +\ Jh_r\ ) = \\ =\ d\ (\ \omega_r\ -\ r\ J\tilde{g}_r\ -\ r\ (\ \frac{1}{r}\omega_r\ -\ J\tilde{g}_r)\ )\ =\ 0. \end{gathered} \tag{3.181}$$

Therefore, there exists $\tilde{g}_{r+2}$ so that Eq. (3.178) be satisfied. To complete the induction argument, we define $g_r := \tilde{g}_r + h_r$ and replace in Eq. (3.174) $\tilde{g}_r$ by g_r. This equation is not affected by this replacement because h_r is a b-cycle. This ends the proof of the injectivity.

Corollary 3.3 *Let $C = \{f_\epsilon, f_{\epsilon+2}, f_{\epsilon+4}, \ldots..\}$, ($\epsilon = 0$, or 1) be a $(B-b)$-cycle. Then*

(i)

$$J(C) = \{ Jf_\epsilon, Jf_{\epsilon+2}, Jf_{\epsilon+4}, \ldots..\}$$

is a sequence of closed differential forms in the Z_2-graded de Rham complex

(ii) $J_*(C) \in H_{dR}^{ev/odd}(M)$ is given by the sequence of de Rham cohomology classes

$$J_*(C) = \{ [Jf_\epsilon], [Jf_{\epsilon+2}], [Jf_{\epsilon+4}], \ldots..\}.$$

It is easy to prove part (i) although it is a direct consequence of Lemma 3.16

$$d\ J\ f_{\epsilon+2r} = \frac{1}{\epsilon+2r}\ J\ B\ f_{\epsilon+2r} = \frac{1}{\epsilon+2r}\ J\ b\ f_{\epsilon+2r+2} = 0.$$

Corollary 3.4 *In order for the de Rham cohomology class $[\omega_k] \in H_{dR}^k(M)$ to be realised in the periodic cyclic complex it is necessary to use a $(B-b)$ cycle involving an infinite sequence of chains (functions) $0, \ldots, 0, f_k, f_{k+2}, f_{k+4}, \ldots, f_{k+2r}, \ldots$.*

Proof This is a direct consequence of the proof of Lemma 3.17.

3.24 Periodic Cyclic Connes–Chern Character of Idempotents

Theorem 3.29 *Let e be an idempotent over the algebra $\mathscr{A}$. Then*

$$U_{2q} := -\frac{1}{2}4^q q!(1-T)(1 \otimes^{2q} e) + 4^q q!(\otimes^{2q+1} e) \tag{3.182}$$

is an even periodic cyclic homology cycle *of the algebra $\mathscr{A}$.*

It represents the Connes–Chern character of the idempotent e, up to a multiplicative factor.

Proof We consider the following chain in the $\{B, b\}^{per}$ bi-complex

$$U_{2q} := \alpha_{2q}(1-T)(1 \otimes^{2q} e) + \beta_{2q} \otimes^{2q+1} e. \tag{3.183}$$

Then

$$B\,U_{2q-2} := 0 + \beta_{2q-2}\,B\,(\otimes^{2q-1} e) = \tag{3.184}$$
$$\beta_{2q-2}\,(1-T)(\,1\otimes N(\otimes^{2q-1} e)) = 2q.\,\beta_{2q-2}\,(1-T)(\,1\otimes(\otimes^{2q-1} e)).$$

On the other hand,

$$b\,U_{2q} := b\,(\,\alpha_{2q}\,(1-T)(1\otimes^{2q} e) + \beta_{2q}\,(\otimes^{2q+1} e)\,) = \tag{3.185}$$
$$\alpha_{2q}\,b\,(1-T)(1\otimes^{2q} e) + \beta_{2q}\,b\,(\otimes^{2q+1} e) =$$
$$\alpha_{2q}\,(1-T)\,b'(1\otimes^{2q} e) + \beta_{2q}\,(\otimes^{2q} e) =$$
$$\alpha_{2q}\,(1-T)\,((\otimes^{2q} e) - 1\otimes b'(\otimes^{2q} e)) + \beta_{2q}\,(\otimes^{2q} e) =$$
$$\alpha_{2q}\,[((\otimes^{2q} e) - T(\otimes^{2q} e)) - (1-T)\,(1\otimes(\otimes^{2q-1} e)] + \beta_{2q}\,(\otimes^{2q} e) =$$
$$\alpha_{2q}\,[((\otimes^{2q} e) + (\otimes^{2q} e)) - (1-T)\,(1\otimes(\otimes^{2q-1} e)] + \beta_{2q}\,(\otimes^{2q} e) =$$
$$(2\alpha_{2q} + \beta_{2q})(\otimes^{2q} e) - \alpha_{2q}\,(1-T)(1\otimes(\otimes^{2q-1} e))].$$

We intend to determine α_{2q} and β_{2q} so that the chain $\{U_{2q}\} \in \{B, b\}^{per}$ be a cycle for the boundary map $B-b$. This condition imposes on the coefficients the relations

$$2q\beta_{2q-2} + \alpha_{2q} = 0 \;\; and \;\; 2\alpha_{2q} + \beta_{2q} = 0.$$

We find $\alpha_{2q} = -\frac{1}{2}\beta_{2q}$ and therefore $\beta_{2q} = 4q\;\beta_{2(q-1)}$. If we take $\beta_0 = 1$, we get $\beta_{2q} = 4^q q!$.

Therefore

$$U_{2q} := -\frac{1}{2}\,4^q\,q!\,(1-T)\,(1\otimes^{2q} e) + 4^q q!(\otimes^{2q+1} e) \tag{3.186}$$

is a cycle for the boundary map $B - b$.

3.25 Direct Connections and Chern Character of Idempotents

In this section we discuss *direct connections*, introduced in Sect. 1.7, from the noncommutative geometry view point and we prove Theorem 1.28.

Theorem 3.30 *Let M be a smooth manifold and let $e \in \mathbf{M}_N(C^\infty(M))$ be an idempotent.*

(i) Then $A(y,x) := e(y)\;e(x) : Image\;e(x) \longrightarrow Image\;e(y)$ is a direct connection in the bundle Image e.

(ii) We associate with the direct connection A the function (1.120)

$$\Phi_k(x_0, x_1, \ldots, x_k) := tr\ A(x_0, x_1) \circ A(x_1, x_2) \circ, \ldots, \circ A(x_{k-1}, x_k) \circ A(x_k, x_0). \tag{3.187}$$

Then function Φ_k coincides with the function

$$\Psi_k(e)(x_0, x_1, \ldots, x_k) := \sum_{i_0, i_1, \ldots, i_k} e^{i_1}_{i_0}(x_0) \otimes e^{i_2}_{i_1}(x_1) \otimes \cdots \otimes e^{i_0}_{i_k}(x_k).$$

(iii) The function Ψ_k is a periodic cyclic homology cycle which represents (up to a multiplicative factor) the Chern/Pontrjagin character of the bundle Image e.

(iv) For any direct connection $\bar{A}$ in the bundle Image e, the corresponding function (1.120) represents (up to a multiplicative factor) the Pontrjagin/Chern character of the idempotent e.

Proof

(i) We use the fact that e is an idempotent.

(ii) We evaluate the function Φ_k

$$\Phi_k(\, x_0, x_1, \ldots, x_k\,) :=$$
$$tr\ \tilde{A}(x_0, x_1) \circ \tilde{A}(x_1, x_2) \circ, \ldots, \circ \tilde{A}(x_{k-1}, x_k) \circ \tilde{A}(x_k, x_0) =$$
$$tr\ (e(x_0)e(x_1)) \circ (e(x_1)e(x_2)) \circ \cdots \circ (e(x_k)e(x_0)) =$$

and because e is an idempotent, we have further

$$\Phi_k(\, x_0, x_1, \ldots, x_k\,) = = tr\ e(x_0) \circ e(x_1) \circ e(x_2)) \cdots \circ e(x_k) = \Psi_k(e)(\, x_0, x_1, \ldots,\ x_k),$$

(iii) The functions Φ_k, Ψ_k are periodic cyclic homology cycles of the algebra of smooth functions. Theorem 1.28 says that the periodic cyclic homology class of the function Ψ is, via Morita equivalence, the Chern character of the bundle *Image e*

(iv) Whatever the direct connection in the bundle *Image e* would be, its periodic cyclic homology class depends only on its infinitesimal part, represented by the operator J (3.146), along the diagonal. We know from Theorem 1.26 that the infinitesimal part of any direct connection is a classical linear connection. On the other hand, we know that the classical Chern character is independent of the linear connection.

This completes both the proof of the theorem and the proof of Theorem 1.28.

3.26 Connes–Chern Character $Ch : K^i(\mathscr{A}) \longrightarrow H_i^{per,\lambda}(\mathscr{A})$

In this section we construct the Connes–Chern character on K^i, $i = 0,\ 1$, with values in the periodic cyclic homology. For this presentation we use Gracia-Bondia et al. [112, §10.2, Pg. 446–447].

In this section we assume that $\mathscr{A}$ is a unital ring.

We recall the definition of the $(b, B)(\mathscr{A})$ bi-complex. Its general term is

$$\begin{aligned} C_{p,q} &= \otimes^{p-q}\mathscr{A}, \quad q \leq p \\ C_{p,q} &= 0, \quad p < q \\ b &: C_{p,q} \longrightarrow C_{p,q-1} \\ B &: C_{p,q} \longrightarrow C_{p+1,q}. \end{aligned}$$

Proposition 3.23 *Let $e \in \mathbb{M}_n(\mathscr{A})$ be an idempotent. Then e satisfies the relations*

$$\begin{aligned} b(e\ (de)^{2p}) &= e(de)^{2p-1} & b\ (de)^{2p} &= (2e-1)(de)^{2p-1} \\ B(e\ (de)^{2p}) &= (2p+1)(de)^{2p+1} & B\ (de)^{2p} &= 0 \\ b(e\ (de)^{2p+1}) &= 0 & b\ (de)^{2p+1} &= 0 \\ B(e\ (de)^{2p+1}) &= 0 & B\ (de)^{2p+1} &= 0. \end{aligned}$$

3.26.1 Chern Character $Ch : K^0(\mathscr{A}) \longrightarrow H_{ev}^{per,\lambda}(\mathscr{A})$

Proposition 3.24 (Definition)
Let $e \in \mathbb{M}_n(\mathscr{A})$ be an idempotent. Let

$$Ch\ (e) = Tr\ e + \sum_{p=1}^{\infty} (-1)^p \frac{(2p)!}{p!}\ (e - \frac{1}{2})(de)^{2p}. \tag{3.188}$$

Then $Ch_{ev}\ (e)$ is an even cycle in the $(b, B)(\mathscr{A})$ cyclic complex. Its homology class is the periodic cyclic Connes–Chern character of *e.*

Proof Use Proposition 3.23.

3.26.2 Chern Character $Ch : K^1(\mathscr{A}) \longrightarrow H_{odd}^{per,\lambda}(\mathscr{A})$

Proposition 3.25 (Definition) *Let $u \in \mathbb{M}_n(\mathscr{A})$ be an invertible matrix. Let*

$$Ch_{odd}\ (u)\ =\ \sum_p (-1)^{2p+1}(2p+1)!\ Tr\ (u^{-1}du).(du^{-1}du).\ldots.(du^{-1}du). \tag{3.189}$$

Then $Ch_{odd}\ (u)$ *is an odd cycle in the* $(b, B)(\mathscr{A})$ *cyclic complex. Its homology class is the* periodic cyclic Connes–Chern character of u.

Proof Use Proposition 3.23.

Part III
Index Theorems

Chapter 4
Analytic Structures on Topological Manifolds



Abstract The aim of this part is to show the reader the classification of *analytic* structures on topological manifolds. This discussion should show that passing from an analytic structure $\mathscr{C}$ to a more regular one $\mathscr{C}'$ implies two things: (1) the passage is *not ever possible*, i.e. there are *obstructions*, (2) even if the passage is *possible*, the resulting structures $\mathscr{C}'$ *might be not* $\mathscr{C}'$*-equivalent*. The reader is invited to appreciate Sullivan result (Sullivan and Sullivan, Geometric topology. Localisation. Periodicity and Galois Symmetry. The 1970 MIT Notes. Ed. A. Ranicki) which states that in dimension $\neq 4$ the passage from the category of topological manifolds to quasi-conformal/Lipschitz manifolds is *always possible, in a unique way*.

4.1 Topological Manifolds

Definition 4.1 A topological manifold M^n of dimension n is a Hausdorf topological space which is locally homeomorphic to $\mathbb{R}^n$.

From the definition it follows that on a topological manifold M^n there exists an *atlas of local charts*, i.e. homeomorphisms $f_\alpha : U_\alpha \longrightarrow V_\alpha$, where U_α is an open set in $\mathbb{R}^n$ and V_α is an open set in M^n. On V_α there are *local coordinates* $(x_1, x_2, \ldots, x_n)$, the image of the point $(x_1, x_2, \ldots x_n) \in U_\alpha$

The *transition functions* $f_{\beta,\alpha} : V_\alpha \cap V_\beta \longrightarrow V_\alpha \cap V_\beta$, where $f_{\beta,\alpha} = f_\beta \circ f_\alpha$, are *homeomorphisms*.

We are going to mention some fundamental results concerning topological manifolds which are related to topics treated in this book.

Theorem 4.1 *Any closed oriented topological manifold satisfies the* Poincaré duality.

Theorem 4.2 *There are Poincaré duality spaces which are not homotopy equivalent to any topological manifold.*

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Definition 4.2 A topological manifold is a *stable manifold* if and only if there exists an atlas of local coordinates $f_{\beta\alpha} = h_1 \circ h_2 \circ \cdots \circ h_n$, where h_i are homeomorphisms with the property that for each factor there exists a non-vanishing open set in the domain of h_i on which $h_i = identity$, with $1 \leq i \leq n$.

Theorem 4.3 Stable homeomorphism theorem *(Brown and Gluck [36], Kirby [55])* *Any orientation preserving homeomorphism of* $\mathbb{R}^n$ *is stable.*

Theorem 4.4 *Any topological manifold is decomposable in* handles*, see [56].*

Theorem 4.5 (*(Annulus Theorem*) (Rado [1], Quinn [81], Kirby [55])) *For any* homeomorphism $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *which maps the unit ball* $\mathbf{B}$ *into its interior, the set* $\mathbf{B} - h(interior(\mathbf{B}))$ *is homeomorphic to the annulus* $S^{n-1} \times [0, 1]$.

Theorem 4.6 (Hauptvermutung (Kirby and Siebenmann [56])) *Let* $f : M_0^n \longrightarrow M_1^n$ *be a* homeomorphism *between two* combinatorial manifolds.

Then, in general, there does not exist a combinatorial homeomorphism between them.

The negative answer is due to Milnor [66].

Given a homeomorphism h between two combinatorial manifolds of dimension $\geqslant 5$, *h is* isotopic *(resp. if the manifolds are simply connected then the homeomorphism h is* homotopic*) to a* combinatorial homeomeorphism *if and only if the* obstruction $\mathbf{k}(h) \in H^3(M, \mathbb{Z}/2\mathbb{Z})$ *is equal to zero.*

If M^m *is a* compact topological manifold*,* $m \geqslant 5$*, then* M^m has a combinatorial structure *if and only if the* Kirby–Siebenmann obstruction $\mathbf{k}(M) \in H^4(M, \mathbb{Z}/2\mathbb{Z})$ *is equal to zero.*

The PL-structures on M^m *are in one to one correspondence with elements of* $H^3(M, \mathbb{Z}/2\mathbb{Z})$.

Theorem 4.7 (Kirby and Siebenmann [56]) *Any topological manifold has* rational Pontrjagin classes.

In the next section we give a classification of analytical structures on topological manifolds.

4.2 Quasi-Conformal and Lipschitz Manifold Structures

Definition 4.3

(i) Let $h : U_1 \longrightarrow U_2$ be a homeomorphism between two open sets U_1 U_2 in $\mathbb{R}^n$. h is called *quasi-conformal homeomorphism* provided the following condition is satisfied:

$$0 < k < \bar{Lim}_{r \searrow 0} \frac{max|h(x) - h(y)|, \quad |x - y| = r}{min|h(x) - h(y)|, \quad |x - y| = r} < K < \infty. \tag{4.1}$$

(ii) h is called a *Lipschitz homeomorphism* provided there exist two positive constants k, K such that

$$k\,|x-y| \leq\ |h(x)-h(y)| \leq\ K\,|x-y|, \quad for\ any\ x, y \in U_1. \tag{4.2}$$

Definition 4.4 Let M^n be a topological manifold. Let $\mathscr{A} = \{f_{\beta\alpha}\}$ be a complete atlas of local coordinates on M^n. If each transition function $f_{\beta\alpha}$ is a quasi-conformal, resp. Lipschitz, homeomorphism, then the manifold M^n is called a *quasi-conformal*, resp. *Lipschitz*, manifold.

Theorem 4.8 (Sullivan [75])

(i) Any closed topological manifold M^n of dimension $n \neq 4$ has a quasi-conformal, resp. Lipschitz, structure.
(ii) For any two quasi-conformal, resp. Lipschitz, atlases $\mathscr{A}_0$, $\mathscr{A}_1$ on M^n, there exists an isotopy*, i.e. a continuous family of quasi-conformal, resp. Lipschitz, homeomorphisms $h_t : (M^n, \mathscr{A}_0) \longrightarrow (M^n, \mathscr{A}_1)$ with $Lim_{t\searrow 0} h_t = Identity$.*

In dimension 4 there is a totally different picture.

Theorem 4.9

(i) There exist topological manifolds of dimension 4 *which do not possess any Lipschitz or quasi-conformal structure.*
(ii) Even if such structures exist, they are not necessarily isotopic.

4.3 Combinatorial and PL Manifold Structures

All simplices of a triangulation are supposed to be closed.

Definition 4.5 Let M^n be a simplicial complex which is also a topological manifold. M^n is called a *combinatorial manifold* if the triangulation satisfies the following condition: for any simplex σ^p, $dim\sigma = p$, of the triangulation, the *link of the simplex* σ^p, denoted $Lk(\sigma^p)$, is equivalent to the sphere of dimension $n-p-1$.

The *star of the simplex* σ^p, denoted $Sr(\sigma^p)$, consists of all simplices of the triangulation which contain σ^p as a face.

The link of σ^p is the union of all simplices of the $Sr(\sigma^p)$ which are disjoint of σ^p.

A simplicial complex which consists of maximal simplices of dimension n such that any simplex of dimension $n-1$ is a face of exactly two n-dimensional simplices, is called *pseudo-manifold* of dimension n.

Remark 4.1 There exist pseudo-manifolds which are not combinatorial manifolds.

Theorem 4.10 *Let M^n be a topological manifold. For $5 \leq n$, M^n admits a combinatorial manifold structure if and only if the* Kirby–Siebenmann *obstruction vanishes. The Kirby–Siebenmann obstruction $\kappa(M^n) \in H^4(M^n, \mathbb{Z}_2)$.*

For $5 \leq n$, if the Kirby–Siebenmann obstruction vanishes, the corresponding combinatorial manifold structures are parameterised by the elements of the group $H^3(M; \mathbb{Z}_2)$.

Definition 4.6 A PL-manifold is a combinatorial manifold given by a complete atlas of *piecewise linear* transition functions.

4.4 Differentiable Manifolds

There are *obstructions* to existence of *differentiable structures* on *combinatorial manifolds*.

Suppose M^n is a *combinatorial* manifold $\{\mathscr{S}(M^n)\}$ endowed with a *combinatorial Riemannian metric* g. Therefore, each closed simplex $\alpha \in \mathscr{S}(M^n)$ is endowed with a *smooth* Riemannian metric g_α; at any common face τ of two such simplices, the Riemannian metrics g_α of the neighbouring simplices α agree.

At any point $x_0 \in \alpha$ the *solid angle* $\mathscr{W}(\alpha, x_0)$ of the simplex α, at the point x_0, is well defined. The Riemannian metric g defines already a smooth manifold structure on the complement of the $n-2$-skeleton.

Theorem 4.11 (Teleman [76]) *We intend to define a differentiable structure $\mathscr{D}$ on M^n such that the Riemannian metric g would become a smooth Riemannian metric on the smooth manifold $(M^n, \mathscr{D})$.*

This is possible if and only if g satisfies the following condition: for each simplex τ which contains the point x_0 and for all neighbouring simplices α, the following condition has to be satisfied:

$$\sum_{\alpha} \mathscr{W}(\alpha, x_0) \;=\; \textit{solid angle about a point in the Euclidean space } \mathbf{E}^n. \tag{4.3}$$

Proof The condition is necessary.

We pass to the second part. The proof shows that if the condition (4.3) is satisfied, then, at the point x_0, the solid angles $\mathscr{W}(\alpha, x_0)$ fit together as the pieces of a patchwork; they will create the tangent space of the smooth manifold M^n at the point x_0.

The construction of the tangent space at the point x_0 is done by induction on n. Supposing the tangent spaces have already been constructed in dimension $n-1$, one passes to dimension n. To perform the construction in dimension n, one starts with an isometric affine embedding of any n-dimensional simplex α_0 in the Euclidean space $\mathbb{E}^n$. One continues the construction by attaching isometrically any n-simplex of M^n which has in common with the simplex α a simplex of dimension $n-1$, and so

on. If the condition (4.3) is satisfied, the construction produces a unique realisation of the star of the point x_0 in $\mathbb{E}^n$.

The realisation of the star of the point x_0 in the Euclidean space is completely determined once the first embedding α_0 has been chosen. This construction is unique, modulo an element of the orthogonal group $O(n)$. Let $\mathscr{D}_{x_0}$ be the space of all combinatorial matrices which satisfy the condition (4.3), factorised by the action of the group $O(n)$.

Let $\mathscr{D}(\mathscr{S}(\ M^n)\ )$ be the filtration with base M^n and fibres $\mathscr{D}_{x_0}$. Therefore, the differentiable structure exists if and only if the vibration $\mathscr{D}(\mathscr{S}(\ M^n)\ )$ has a global section. There is an obstruction to the existence of a global section.

Choose a combinatorial metric g. Let $\Gamma_{x_0} \in \pi_{\mathscr{D}_{x_0}}$ be the local obstructions. These obstructions belong to $H^k(M^n, \Gamma_{k-1})$, where k is the dimension of the simplex α containing the point x_0 as an interior point.

These obstructions depend on the particular combinatorial structure chosen on the PL-manifold M^n. To obtain an answer independent of the chosen combinatorial structure, we have to sub-divide the simplices of the manifold $\{\mathscr{S}(\ M^n)\ \}$. We obtain the result.

Theorem 4.12 (Teleman [72]) *If M^n is a PL-manifold of dimension n there are* obstructions to the existence of a differentiable structure *on M^n. The obstructions belong to*

$$H^k(\tilde{M}^n,\ \pi_{k-1}(PL_k/O_k)), \quad 1 \leqq k \leqq n.$$

The problem of determining conditions under which there exist differentiable structures on combinatorial manifolds was clarified by the work of Munkres [29, 30] and Hirsch and Mazur [65].

4.5 Analytic Manifolds

Let V^n be a *complex algebraic manifold*. This means that V is covered with a complete system of *smooth coordinate charts* U_α such that the *changes of coordinates* are given by *holomorphic functions*

$$f_{\beta\alpha}, \quad f_{\beta\alpha} = holomorphic\ functions.$$

On the manifold V^n one has *two systems of local coodinate charts* ($1 \leq p \leq n$)

$$z = x_p + i\ y_p$$
$$\bar{z} = x_p - i\ y_p,$$

and differentials

$$dz = dx_p + i\, dy_p$$
$$d\bar{z} = dx_p - i\, dy_p.$$

In an analogous fashion on treats the space of sections of *holomorphic complex vector bundles over* V^p.

Definition 4.7 Let W^q be an analytic vector bundle with fibre $\mathbb{C}^q$ over the analytic manifold V^p.

Let $h_{j,i}$ be the holomorphic transition functions of W^q. Let $\bar{h}_{j,i}$ be the *conjugate* of $h_{j,i}$ and let $\bar{W}$ be the obtained bundle.

Let

$$\mathscr{U}^{m,n}(V^p, W^q) = \{ \text{ smooth functions of the bundle } V^p \otimes_{\mathbb{R}} \bar{W}^q \tag{4.4}$$

which depend on z with respect to the bundle $V^p$and of $\bar{z}$ with respect to W^q }. (4.5)

Theorem 4.13 (Dolbeault [10]) *Let V^n be a complex analytic manifold.*
Let $\mathscr{U}^p(V^n)$ be the sheaf of smooth functions of V^n.
Then the cohomology of the Dolbeault complex

$$0 \longrightarrow \mathscr{U}^0(V^n) \longrightarrow \mathscr{U}^1(V^n) \longrightarrow \ldots \longrightarrow \mathscr{U}^m(V^n) \longrightarrow \mathscr{U}^{m+1}(V^n) \longrightarrow \ldots \longrightarrow \mathscr{U}^p(V^p) \longrightarrow 0 \tag{4.6}$$

is finite dimensional.

Definition 4.8

(i) Let ψ and ψ^{-1} be the *anti-isomorphisms* $\psi : W \longrightarrow W^*$ and $\psi^{-1} : W^* \longrightarrow W$.

(ii) Since the bundles W and W^* are analytic and (χ = anti-symmetrisation) define

$$\# = \psi \otimes (\chi *) : \mathscr{U}^{m,n}(V, W) \longrightarrow \mathscr{U}^{p-m,q-n}(V, W^*) \tag{4.7}$$

$$\tilde{\#} = \psi^{-1} \otimes (\chi *) : \tilde{\mathscr{U}}^{m,n}(V, W^*) \longrightarrow \mathscr{U}^{p-m,q-n}(V, W). \tag{4.8}$$

(iii) Define the complex Laplacian $\vartheta = -\tilde{\#}\tilde{\partial}\#$.

$$\Box^{m,n} = \tilde{\vartheta}\, \delta^{m,n} + \delta\, \vartheta^{m,n}. \tag{4.9}$$

(iv) The complex Laplacian is an elliptic operator. Then the solutions of $\Box^{m,n} = 0$ are finite dimensional spaces; any solution of this equation is called *harmonic form*.

Theorem 4.14 (Kodaira [8]) *Let W^q be a complex analytic vector bundle over the complex analytic manifold V^p.*

Then the space of harmonic forms is finite dimensional. It is isomorphic to the space $H^{m,n}$.

Theorem 4.15 (Serre [12]) *Let W^q be a complex analytic vector bundle over the complex analytic manifold V^p.*

The spaces $H^{m,n}(V, W)$, $H^{p-m,q-n}(V, W^*)$ are isomorphic.

Definition 4.9 Let V be a complex analytic manifold.

An *hermitian metric* on V is

$$ds^2 = 2\sum g_{\alpha,\beta}(z^\alpha, \bar{z}^\beta);\ dz^\alpha \wedge d\bar{z}^\beta, \qquad g_{\bar{\alpha},\beta} = g_{\beta\alpha}. \tag{4.10}$$

To each hermitian metric there is associated an exterior differential form

$$\omega = i \sum g(z^\alpha, z^{\bar{\beta}})\, dz_\alpha \wedge d\bar{z}_\beta. \tag{4.11}$$

The manifold V is called a *Kähler manifold* provided

$$d\,\omega = 0. \tag{4.12}$$

Definition 4.10 A *Hodge manifold* is a Kähler manifold with a canonical differential 2-form that is integral.

Theorem 4.16 (Kodaira [8]) *A compact complex manifold is algebraic if and only if it is a Hodge manifold.*

Chapter 5
Index Theorems in Differential Geometry



Abstract The Riemann–Roch theorem counts the zeroes and poles of a meromorphic function over a Riemann surface. The theorem was extended over complex analytic manifolds by Hirzebruch. Atiyah–Singer formula, valid on differentiable manifolds, explains that the formula holds because it is related to *elliptic operators*. The index formulas were extended to topological manifolds by N. Teleman. The Teleman formula produces the topological index as a *cohomology class*. It is not represented by a *cohomology form* because the Chern–Weil construction involves products of the curvature which could not be performed within classical differential geometry. This problem is re-considered within *non-commutative geometry* in the next chapter.

5.1 Riemann–Roch Index Theorem

In this section we present the statement of the Riemann–Roch theorem. This section does not give the proof of it. It gives only the main idea of the proof.

The Riemann–Roch theorem is a statement regarding the number of zeroes and poles of meromorphic functions on *Riemann surfaces*, i.e. complex algebraic closed manifolds of complex dimension 1. Topologically, such spaces are oriented closed surfaces.

Any meromorphic function f on a Riemann surface has a finite number of zeroes z_i and poles p_j. The zeroes and poles produce a *divisor* $D(f)$

$$D(f) \;=\; \sum_i a_i z_i \;-\; \sum_j b_j p_j \tag{5.1}$$

where a_i, resp. b_j, are the orders of the zeroes z_i, resp. poles p_j. A divisor associated with a meromorphic function is called a *principal divisor*. Two divisors are called *linearly equivalent* provided their difference is a principal divisor.

Analogously, on any Riemann surface one may define meromorphic 1-forms $\omega = f(z)dz$. To each non-vanishing meromorphic 1-form, one associates its divisor in the same manner. The divisors associated with two non-vanishing meromorphic

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1-forms are linearly equivalent. Therefore, any such divisor K is called a *canonical divisor*.

For any divisor D, $\mathscr{L}(D)$ is by definition the $\mathbb{C}$-dimension of the vector space of meromorphic functions f such that all coefficients of $D(f) + D$ are non-negative.

The *genus* of the Riemann surface S is $g(S) := \frac{1}{2}\, dim_{\mathbb{Q}} H^1(S, \mathbb{Q})$.

For any divisor $D = \sum_i n_i x_i$ the number $deg\ (D) = \sum_i n_i$ is called the *degree*.

Theorem 5.1 (Riemann–Roch Theorem) *Let S be a Riemann surface and D a divisor on S. Then*

$$\mathscr{L}(D) - \mathscr{L}(K - D) = deg(D) + (1 - g(S)), \tag{5.2}$$

The Riemann–Roch index theorem was extended by F. Hirzebruch [32]. For an algebraic proof see [71, §IV.1]; see also [100, pg. 127]. The Riemann–Roch theorem corresponds, in the case of the Atiyah–Singer index theorem, to the elliptic operator $\frac{\partial}{\partial z}$ defined on *Riemann spaces*. For more information see [54] and [42, §XIX.4].

5.2 Thom Index Formula

Theorem 5.2 *There exists a construction which produces the $\mathscr{L}_*$-character of combinatorial manifolds.*

In the remaining part of this section we refer to [66, §18 and §20] to exhibit this construction.

Definition 5.1 Let $\mathscr{C}$ denote the class of all finite Abelian groups. A homomorphism $f : A \longrightarrow B$ between Abelian groups is called $\mathscr{C}$ *isomorphism* iff both the kernel and the co-kernel of f belong to the class $\mathscr{C}$.

Theorem 5.3 *Let X be a finite complex which is $(k-1)$-connected, $2 \leq k$. If $r < 2k + 1$, the Hurewicz homomorphism*

$$\pi_r(X) \longrightarrow H_r(X, \mathbb{Z}) \tag{5.3}$$

is a $\mathscr{C}$-isomorphism.

In the particular case $X = S^k$ this result becomes

Theorem 5.4 *The Hurewicz homomorphism*

$$\pi_r(S^k) \longrightarrow H_r(S^k, \mathbb{Z}) \tag{5.4}$$

is a $\mathscr{C}$-isomorphism for $r < 2k + 1$.

Transversality holds in the combinatorial category.

Let M^n be a combinatorial manifold of dimension n and $f : M^n \longrightarrow S^{n-4i}$ a combinatorial map. Let x_0 be a point in S^k. The mapping f may be assumed to be *transversal to the point* x_0. This means

1. $N^{4i} := f^{-1}(x_0)$ is a closed oriented combinatorial manifold of dimension $4i$,
2. N^{4i} has a *combinatorial normal bundle* in M^n, with total space ξ and fibres ξ_x, $x \in N^{4i}$,
3. such that f carries each fibre (ξ_x, x) homeomorphically onto (D, x_0).

Let g_D be a generator of the group $H^{n-4i}(S^{n-4i}, \mathbb{Z})$ having support in D. Let $g_{N^{4i}}$ be a generator of $H^{4i}(N^{4i}, \mathbb{Z})$. Then $g_{N^{4i}} \cup f^*(g_D)$ is a fundamental class of M^n. One has

$$(g_{N^{4i}} \cup f^*(g_D))[M^n] = (g_{N^{n-k}})[N^{n-k}] = Signature\ (N^{4i}); \tag{5.5}$$

the last equality is a consequence of the Thom isomorphism in the bundle ξ and the signature Theorem 1.46.

Definition 5.2 If $n < 2k - 1$ the set of all homotopy classes of maps $f : M^n \longrightarrow S^k$ is an Abelian group $\pi^k(M^n)$ called *k-th cohomotopy group of* M^n.

A result by Serre says that $f \mapsto f^*(u)$ is a $\mathscr{C}$ isomorphism

$$\pi^k(M^n) \longrightarrow H^k(M^n, \mathbb{Z}). \tag{5.6}$$

The images $f^*(u)$ generate a subgroup of finite index in $H^k(M^n, \mathbb{Z})$. We take $k = n - 4i$ so that the inequality $n < 2k - 1$ holds. The Poincaré duality implies that $\mathscr{L}_i(T(M^n))$ is determined by all pairings $< \mathscr{L}_i \cup f^u, \mu_n >$, where $\mu_n \in H^n(M^n, \mathbb{Z})$ is a generator.

5.3 Thom–Hirzebruch Signature and Index Theorem

The reader is advised to read Hirzebruch's article "The Signature Theorem: Reminiscenses and Recreation" published in "Prospects of Mathematics", Annals of Mathematics Study Nr. 70, 1971, [14]. It describes the process which led him to formulate and then prove the generalisation of the Riemann–Roch theorem in the analytic category. This was a work involving J. P. Serre, K. Kodaira, D. C. Spencer, R. Thom, J. Milnor. Thom's results on cobordism [7, 24] played a decisive role. Hirzebruch' s formula contains already all ingredients of the Atiyah–Singer index formula, except the *elliptic operators*.

Before formulating Hirzebruch's signature formula we make a short review on *bordism groups* in the category of oriented smooth manifolds.

Two closed, oriented smooth manifolds M_0, M_1 of dimension n are called *bordant* provided there exists a compact, oriented smooth manifold W of dimension

$n+1$ whose boundary is diffeomorphic to $M_1 - M_0$, ($-M_0$ is the manifold M_0 with reversed orientation).

Let Ω_n denote the set of all equivalence classes of bordant manifolds of dimension n. The *disjoint union* of oriented manifolds makes Ω_n a commutative group. Ω_n is called *bordism group* of order n.

Let Ω_* be the union, with respect to n, of all bordism groups Ω_n.

The direct product of elements of Ω_* makes Ω_* a graded commutative ring (the orientation is responsible for the property $[M^m] \times [N^n] = (-1)^{m.n} [N^n] \times [M^m]$).

Theorem 5.5 (Thom [7, 24]) *The groups Ω_n are finite for $n \neq 4k$.*

The group Ω_{4k} is isomorphic to a direct sum

$$\Omega_{4k} \equiv \ Finite\ group \oplus \mathbf{Z}^{\pi(k)}, \tag{5.7}$$

where π_k is the number of distinct partitions of k.

The sum of products of complex projective spaces of even complex dimension

$$\sum_{k_1 + \ldots + k_l = k} P_{2k_1}(\mathbb{C}) \times P_{2k_2}(\mathbb{C}) \times \cdots \times P_{2k_l}(\mathbb{C}) \tag{5.8}$$

constitute a set of free group generators of $\Omega_k \otimes_{\mathbb{Z}} \mathbb{Q}$.

This result produces a group homomorphism

$$G : \Omega_k \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q} \tag{5.9}$$

which, on such generators, is given by the formula

$$G\ (P_{2k_1}(\mathbb{C}) \times P_{2k_2}(\mathbb{C}) \times \cdots \times P_{2k_l}(\mathbb{C})) := \tag{5.10}$$

$$p_{k_1}(P_{2k_1}(\mathbb{C}))\ \cup\ p_{k_2}(P_{2k_2}(\mathbb{C}))\ \cup\ p_{k_l}(P_{2k_l}(\mathbb{C}))\ [M].$$

For more details on Thom's theorem, the reader could refer to [19].

Theorem 5.6 (Thom–Hirzebruch Signature Theorem) *Let M be a closed, oriented smooth manifold of dimension $4k$.*

Then

$$Sig\ M = L_k(p_1, p_2, \ldots, p_k)\ [M], \tag{5.11}$$

where $p_1, p_2, \ldots, p_k$ are the Pontrjagin classes of M and L_k is an universal polynomial associated to the power series

$$Td(M) = \frac{\sqrt{z}}{tanh\ \sqrt{z}} \tag{5.12}$$

through Hirzebruch's splitting procedure (see Definition 1.7).

Definition 5.3 The class $Td(M)$ is called *Todd class*.

Proof The *Poincaré symmetric bilinear form* is given by the evaluation of the cup product of $2k$-singular cohomology classes on the fundamental class of M. It is non-degenerate. The difference between the number of positive squares of this form and the number of negative squares is *signature of the manifold*, denoted $Sig\ M$.

The signature satisfies the properties

1. $Sig\ (M_1 + M_2) = Sig\ (M_1) + Sig\ (M_2)$
2. $Sig\ (-M_1) = -Sig\ (M_1)$
3. $Sig\ (M_1 \times M_2) = Sig\ (M_1) \times Sig\ (M_2)$
4. if the manifold M^{4k} is an oriented boundary, $M^{4k} = \partial W^{4k+1}$, then all *Pontrjagin numbers* vanish; recall that for any closed oriented manifold M^{4k} its Pontrjagin numbers are

$$p_{k_1}(M) \cup p_{k_2}(M) \cup \cdots \cup p_{k_{1l}}(M)\,[M],\quad k_1 + k_2 + \ldots + k_l = k.$$

Properties 1–4 show that the signature defines a homomorphism

$$Sig : \Omega_{4k} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

The signature homomorphism may be compared with the homomorphism G. To this aim we use the fact that the *total Pontrjagin class* of the complex projective space is given by

$$p_*(P_k(\mathbf{C})) = (1 - c_1^2)^{n+1},$$

where $c_1 \in H^2(P_k(\mathbb{C}), \mathbb{Z})$ is the first Chern class of $P_k(\mathbb{C})$, see Milnor [66, §14.10 and Example 15.6]. These show that the signature homomorphism and the homomorphism G agree on generators (5.8).

This completes the proof.

5.4 Hirzebruch Index Theorem

Theorem 5.7 (Hirzebruch [32, pg. 155] (The Riemann–Roch Theorem for Complex Analytic Vector Bundles Over Analytic Manifolds)) *Let V be an algebraic manifold of dimension n and let W be a complex analytic vector bundle over V.*

Then the Euler–Poincaré characteristic *of the cohomology of V with coefficients in the sheaf of holomorphic sections in W is given by the formula*

$$\sum_{i=0}^{i=n} (-1)^i dim\ H^{2i}(V, W) =$$

$$\sum_{i=0}^{i=n} (-1)^i dim\ H^{2i}(V, W) - \sum_{i=0}^{i=n} (-1)^i dim\ H^{2i+1}(V, W) = Analytic\ Index\ \partial =$$

$$(Ch\ W\ .\ TdV)\ [V]. \tag{5.13}$$

Proof The proof is based on the theorems due to A. Borel, K. Kodaira, J.-P. Serre, D. C. Spencer, R. Thom, presented in Sect. 4.5.

5.5 Grothendieck Index Theorem

In this section we present the statement of the Grothendieck theorem. This section does not give the proof of it. It gives only the main idea of the proof.

In which follows we present the *Grothendieck–Riemann–Roch theorem* [16].

The Grothendieck theorem extends the Hirzebruch result.

In this book we use the A. Borel, J.-P. Serre article [21]. For the notions which are not specified hereby the reader is invited to look at [21, Sects. 1–16]. More detailed proofs could be found in Ryan [132].

We begin by recalling the notation for $f^!$, $f_!$ see [21, §5], c and d.

Definition 5.4 Let $f : X \longrightarrow Y$ be a morphism.

If E is a fibre bundle with base X we define $f^* : K(X) \longrightarrow K(Y)$ as the *induced fibre bundle* with base Y, and $f_* : K(Y) \longrightarrow K(X)$ as the *induced morphism* in K-theory.

By definition K^0 the space of bundles modulo the relation $[E] = [E^{'}] + [E^{"}]$, where

$$\mathscr{E} \equiv \quad 0 \longrightarrow E^{'} \longrightarrow E \longrightarrow E^{"} \longrightarrow 0,$$

where $\mathscr{E}$ is an exact sequence.

In this section we modify the definition of K^0.

One has a homomorphism $K^0 \longrightarrow K_0$; therefore, K_0 is K^0-module.

On the *Grothendieck group of coherent sheaves* one defines the *multiplication* $[E_1]\ .\ [E_2].\ := [E_1] \otimes [E_2]$.

Definition 5.5 If $f : X \longrightarrow Y$ is a morphism, define

$$\alpha \in K_0(Y),\ and\ \beta \in K_0(X)\ then\ f_* f^*(\alpha)\ .\ \beta = \alpha\ (f_*\beta).$$

Proposition 5.1 *If X is non-singular, then the map $K^X \longrightarrow K_X$ an isomorphism.*

For the proof of this proposition see Hartshorne [71].

Theorem 5.8 (Grothendieck–Riemann–Roch Theorem [16, 21]) *Let X and Y be two quasi-projectives, irreducible non-singular algebraic manifolds. Let $f : Y \longrightarrow X$ a* proper morphism. *Let $Td(X)$, resp. $Td(X)$, be the Todd class of X, resp. Y, and let $y \in K(Y)$.*

Then

$$f_*(Ch\ y\ .\ Td\ Y) = Ch(f_!(y)\ .\ Td\ X). \tag{5.14}$$

The remaining part of this section is devoted to its proof.

Definition 5.6 ($K^0(X) = K_0(X)$)

This relation allows to extend the definition of *Chern classes* to *coherent sheaf*.

c_k represents the *Chern class of the coherent sheaf*. This is an homogeneous class of dimension $2k$.

Lemma 5.1 *Let*

$$Z \xrightarrow{g} Y \xrightarrow{f} X \tag{5.15}$$

be proper morphisms. Let $z \in K(Z)$ and $y = g^!(z)$.

Then

a. *if Riemann–Roch is true pour* $(\mathbf{g}, \mathbf{z})$ *and for* $(\mathbf{f}, \mathbf{y})$, *then it is true for* $(\mathbf{f}\,\mathbf{g}, \mathbf{z})$ (5.16)

b. *if Riemann–Roch is true pour* $(\mathbf{f}\,\mathbf{g}, \mathbf{z})$ *and for* $(\mathbf{f}, \mathbf{y})$, *and if* $\mathbf{f}_*$ *is injective, then it is true for* (g, z). (5.17)

This lemma shows that to prove the Riemann–Roch theorem it is enough to verify it on *projections* and *monomorphisms*.

Lemma 5.2 *Let*

$$f : Y \longrightarrow X$$

and

$$f' : Y' \longrightarrow X'$$

be proper morphisms; let $y \in K(Y)$ and $y' \in K(Y')$.

Then, if the Riemann–Roch theorem is true for (f, y) and for (f', y'), then it is true for $(f \times f', y \otimes y')$.

Lemma 5.1 shows that it is sufficient to prove the Riemann–Roch theorem in the following situations:

a. $Y = X \times P$, where P is projective space and $f : X \times P \longrightarrow X$ is the projection on the first factor.

b. $f : Y \longrightarrow X$ is a monomorphism of Y on a closed sub-manifold of X.

According to Lemma 5.2, it follows that

a′. The homomorphism $K(X) \otimes K(P) \longrightarrow K(X \times P)$ is onto.

a″. Is true for l'application of P onto the point.

This proves that the Hirzebruch–Riemann–Roch is true for P.

Proposition 5.2 *Let X be an arbitrary algebraic manifold. Let Y be an* affine space *of finite dimension.*
Then

$$K(X) = K(X \times Y). \tag{5.18}$$

Proposition 5.3 *Let X be an algebraic manifold and P a projective space.*
Then the homomorphism

$$K(X) \otimes K(P) = K(X \times P) \tag{5.19}$$

is surjective.

Proposition 5.4 *Let $\mathscr{F}$ be a coherent sheaf on the projective space P.*
Then the Hirzebruch–Riemann–Roch theorem

$$\varkappa\,(Ch(\mathscr{F})\,.\,Td(P)) = \chi(P, \mathscr{F}) \tag{5.20}$$

is true.

These take care of $\mathbf{a}'$. and $\mathbf{a}''$.

Proposition 5.5 *The Riemann–Roch theorem is true for $X = Y \times P$ where*

$$i : Y \longrightarrow Y \times P \tag{5.21}$$

P is a projective space and i is the application

$$i(y) = (y, p_0), \quad y \in Y, \quad p_0 \in P. \tag{5.22}$$

We have the commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ \downarrow g & & \downarrow f \\ Y & \xrightarrow{i} & X. \end{array} \tag{5.23}$$

Lemma 5.3 *Let G be a vector bundle of rank k on the algebraic variety X. Then one has*

$$Ch(\lambda_{-1} G) = C_k(G^*) \cdot Td(G^*)^{-1}. \tag{5.24}$$

Lemma 5.4 *We use the notation of Lemma 5.3. Then*

1. $f_*(1) = 1$; *therefore* $f_* f^* = 1$.
2. $g_*(c_{p-1}(F)) = 1$.
3. $f^! i_! = j_!(g^!(y) \cdot \lambda_{-1} F^*)$, $y \in K(Y)$.
4. $\lambda_{-1} F^* \equiv 0$ *modulo* $(1 - L^*)$, *if* $p \geqq dim Y + 2$.

The proof of this lemma will complete the check of the Grothendieck–Riemann–Roch theorem.

5.6 Atiyah–Singer Index Formula

In this section we combine the first proof of the Atiyah–Singer index formula [42], (Seminaire Palais, Annals of Mathematics Studies, Princeton, 1965) with Teleman [83, 84]. The Palais seminaire [42] contains the basic analysis and topology which enter into the index formula and its indices. One of the basic parts of the original proof [42] is the *cobordism invariance* of the analytical index, presented in Sect. 5.9.1. It requires a delicate analysis of *boundary value of elliptic opertators*. In this section we replace the Palais cobordism treatment with another technique, see Teleman [83, 84], based on the interpretation of elliptic operators as representing K-homology classes. This technique extends to *Lipschitz and topological manifolds*, to be discussed in Sects. 5.10 and 5.11.

5.7 The Theorem

Let $D : \xi_0 \longrightarrow \xi_1$ be an elliptic pseudo-differential operator of order $k \in \mathbb{Z}$ from the bundle ξ_0 to the bundle ξ_0 over the manifold M. The following indices are introduced:

$$Analytic\ index\ of\ D := the\ index\ of\ the\ operator\ D := i_a(D). \tag{5.25}$$

$$Topological\ index\ of\ D := Ch(Symb_k(D)) \cup Td(M)[M] := i_t(D). \tag{5.26}$$

Theorem 5.9 (Atiyah and Singer [53]) *Let M be a closed* smooth *manifold and $D : \xi_0 \longrightarrow \xi_1$ an* elliptic pseudo-differential *operator of order k from the bundle ξ_0 to the bundle ξ_1 on M.*

Then

$$i_a(D) = i_t(D). \tag{5.27}$$

5.8 Idea of the Proof

Proof The operator D and its indices live on the total space of the co-tangent bundle T^*M. The *first step* of the proof consists of showing that both indices may be defined on M. More specifically, one shows that both indices are determined by the K-theory class of the symbol of the operator; then, by means of the Chern character one passes to the de Rham cohomology of the total space, and finally, using Thom isomorphism, one passes to the rational de Rham cohomology of the base space. At that point we will see that (*second step* of the proof) if the index formula holds on twisted signature operators, then it holds on any elliptic operator. We explain next the first step.

The *operator* D is determined by its symbol up to compact operators. On the other hand, two elliptic operators which differ by a compact operator have the same index. Therefore, the analytical index is determined by the K-theory of the symbol of the operator.

The topological index is defined in terms of characteristic classes. It depends only on the stable homotopic class of the symbol. The characteristic classes are compatible with the equivalence relations which define K-theory. Therefore, the topological index depends on the K-theory class of the symbol.

Given that both indices take value in the torsion free group $\mathbb{Q}$, there is no loss of information if we ignore torsion. We know that the *rational* K-theory is isomorphic to the *rational* de Rham cohomology

$$Ch : K^i(M \times M, M \times M - \nabla_M) \otimes \mathbb{Q} \longrightarrow \tag{5.28}$$

$$H_{AS}^{i\ mod 2}(M \times M, M \times M - \nabla_M, \mathbb{Q}) \equiv H_{AS}^{i\ mod 2}(M, \mathbf{Q}); \tag{5.29}$$

the last isomorphism is the Thom isomorphism.

Definition 5.7 The *twisted signature operator*, or *signature operator* D_ξ *with values in vector bundles* ξ over M, is by definition

$$D_\xi := p\,(D_M^+ \oplus \cdots \oplus D_M^+)\,p, \quad n\ terms\ D_M^+, \tag{5.30}$$

where the *idempotent* $p \in \mathbf{M}_n(C^\infty(M))$ represents the bundle ξ and D_M^+ is the signature operator.

Rellich lemma implies that D_ξ^+ produces an operator whose indice depend solely on the K-theory class of the idempotent p. At the level of K-theory, one has

$$Ch(D_\xi^+) = Ch(D_M^+) \otimes_{\mathbb{C}} Ch(\xi). \tag{5.31}$$

Definition 5.8 Define $\tilde{i}_a(D_\xi^+)$, $\tilde{i}_t(D_\xi^+)$, $i(\xi) : K_0(M) \longrightarrow \mathbb{Q}$: for any $\xi \in K_0(M)$

$$\tilde{i}_a(\xi) := i_a(D_\xi^+) \tag{5.32}$$

$$\tilde{i}_t(\xi) := i_t(D_\xi^+) \tag{5.33}$$

$$i(\xi) := \tilde{i}_a(\xi) - \tilde{i}_t(\xi). \tag{5.34}$$

Proof We do not know (yet) a more explicit expression of the analytical index. We know, however, that $i_(\xi)$ depends solely on the K^0-cohomology class of the bundle ξ. The bundles ξ may be used to produce twisted signature operators D_ξ^+. It is sufficient to prove the next lemma.

Lemma 5.5 *Suppose* $dim\ M = even$. *At the level of sets*

$$\{Ch\,(D_\xi^+) \mid for\ all\ \xi \in K^0(M) \otimes \mathbb{Q}\} = \{Ch\,(\xi) \mid for\ all\ \xi \in K^0(M) \otimes \mathbb{Q}\}. \tag{5.35}$$

To prove the lemma it is sufficient to know the Chern character of D_ξ^+. One has, see [42, Pg. 224]

$$Ch\ D_M^+ = 2^m - x, \quad x \in H_{comp}^{ev-\{0\}}(T^*, \mathbb{Q}), \quad m = dim\ M/2 = l.$$

As

$$(1 - x)(1 + x + \cdots + x^m) = 1 - x^{m+1} = 1,$$

it follows that ChD_M^+ is a unit in de Rham cohomology $H_{dR,comp}^{ev}(T^*(M), \mathbb{Q})$. Given that $Ch\ (D_\xi^+) = Ch\ (D_M^+) \otimes_{\mathbb{Q}} Ch\ ([\xi])$, its image in $H_{dR,comp}^{ev}(M, \mathbb{Q})$, through the Thom isomorphism is also a unit. Therefore, the set of the Chern

characters of all elements $[\xi] \in K^0_{comp}(M) \otimes \mathbb{Q}$ exhausts the even rational co-homology. As the Chern character establishes an isomorphism with the even rational co-homology of M, we have proved the next proposition.

Proposition 5.6 *Suppose we ignore torsion. The knowledge of the homomorphism i on twisted signature operators determines i on any elliptic operator.*

The section below provides the second step of the proof of Atiyah–Singer index formula.

5.9 The Atiyah–Singer Index Formula Holds on Twisted Signature Operators

Theorem 5.10 *The Atiyah–Singer index formula Theorem 5.9 holds for twisted signature operators.*

Therefore, Theorem 5.9 holds for all elliptic operators.

5.9.1 Cobordism Invariance of the Analytical Index

The next material was extracted, with permission, from [83].

Theorem 5.11 *Suppose M^{2l} is the boundary of a smooth oriented compact manifold W^{2l+1}. Suppose the complex vector bundle ξ, over the base M^{2l}, is the restriction of a complex vector bundle $\tilde{\xi}$ over W^{2l+1}.*

Then the analytic index of the twisted elliptic operator D^+_ξ vanishes:

$$i_a(D^+_\xi) = 0. \tag{5.36}$$

Remark 5.1 The reader will notice that this proof is homological in nature; it uses exclusively the underlying *Lipschitz* structure. For this reason, this material will be used in the next section (Sect. 5.10) regarding Lipschitz manifolds.

We choose a smooth Riemannian metric on M. We work with the de Rham cohomology.

In the quasi-conformal/Lipschitz case the L_2-cohomology replaces de Rham cohomology: see Teleman [83], Donaldson and Sullivan [92] and Connes et al. [101].

5.9.2 Hodge Theory and the Analytic Index of Twisted Signature Operators

We assume M is oriented and closed. Under these conditions we have Hodge theory (Sect. 1.9.2, Theorem 1.35), and signature operators, see Sect. 1.10.5, Theorem 1.46.

The next theorem is the main result of this section: it extends Theorem 1.46 to twisted signature operators.

Definition 5.9 The *Thom genus* is the characteristic class obtained through the Hirzebruch genus procedure (Sect. 1.11) applied onto the power series

$$\mathscr{L}(z) = \frac{\sqrt{z}}{tanh\sqrt{z}} \tag{5.37}$$

Theorem 5.12 *Let M be a closed, oriented, smooth manifold of dimension $2l$ and ξ a smooth vector bundle over M.*

Then

$$i_a(D_\xi^+) = \mathscr{L}(M)[M] \tag{5.38}$$

where $\mathscr{L}(M)$ is the Thom genus.

5.9.3 Excision for Twisted Signature Operators

The data defining twisted signature operators is *local*. Elliptic operators define K_0-theory homology classes (Sect. 1.11); their elements are defined *locally*. Based on these two facts we will think of twisted signature operators as leaving, *locally*, the same kind of information as simplicial homology classes do leave on the space. The treatment below realises this idea.

Definition 5.10 Let M be a Riemannian, oriented, closed smooth manifold. Let M_1, M_2 be closed, oriented Riemannian, smooth manifolds of dimension $2l$. Let ξ, ξ_1, resp. ξ_2, be complex vector bundles over M, M_1, resp. M_2, endowed with Hermitian connections Γ, Γ_1, resp. Γ_2.

Suppose $i_\alpha : (M, \Gamma) \longrightarrow (M_\alpha, \Gamma_\alpha)$, $\alpha = 1, 2$, are smooth, isometric embeddings, compatible with the Hermitian connections.

Under these conditions we will say that the triples $(M_\alpha, \Gamma_\alpha, M)$, $\alpha = 1, 2$, are *excisive*.

The next lemma contains the basic analytic information related to exclusivity .

Lemma 5.6 *Let $(M_\alpha, \Gamma_\alpha, M)$, $\alpha = 1, 2$, be two excisive triples, where Γ_α is a Hermitian bundle over M_α, $\alpha = 1, 2$.*

Then, one has the continuous family, depending on the parameter $t \in [0, \pi/2]$, *of Fredholm operators*

$$\begin{pmatrix} D^+_{\xi_1} & 0 \\ 0 & -D^+_{\xi_2} \end{pmatrix} \equiv (D^+_{\xi_\alpha}\ f_\alpha)^*(D^+_{\xi_\alpha}\ f_\alpha)\ +\ g^*_\alpha . g_\alpha, \tag{5.39}$$

with

$$f_\alpha\ =\ \varphi^2_\alpha\ cost\ +\ \psi^2_\alpha$$

$$g_\alpha\ =\ \varphi^2_\alpha\ sint,$$

and

$$\varphi_\alpha\ =\ sin\ \lambda_\alpha \tag{5.40}$$

$$\psi_\alpha\ =\ cos\ \lambda_\alpha \tag{5.41}$$

where λ_α *is a smooth function with the properties*

$$\lambda_\alpha : M_\alpha \longrightarrow [0, \pi/2] \tag{5.42}$$

$$\lambda_\alpha\ is\ identically\ 0\ on\ M_\alpha\ -\ \bar{U}. \tag{5.43}$$

Proof We construct two smooth functions λ_α satisfying the conditions of the lemma.

For the proof of the formula (5.39) see [83, §12].

Theorem 5.13 (Teleman [83, Theorem 12.1, Pg. 66]) *Let* M_α, $\alpha = 1, 2$, *be closed, oriented smooth Riemannian manifolds with metrics* Γ_α. *Let* ξ_α, θ_α *be smooth vector bundles over* M_α *with Hermitian connections* ∇_α, ∇'_α.

Suppose U *is an open sub-manifold in* M_1 *and* M_2 *and that* $V \subset U$ *is an open subset with* $\bar{V} \subset U$.

These pairs are required to satisfy the following hypotheses:

1. $\{(\nabla_1, \Gamma_1), (\nabla_2, \Gamma_2),\ U\}$ *is an excisive triple with identifying isomorphism*

$$\xi_1 \mid U \xrightarrow{\Lambda_{21}} \xi_2 \mid U;$$

2. $\{(\nabla'_1, \Gamma_1), (\nabla'_2, \Gamma_2),\ U\}$ *is an excisive triple with identifying isomorphism*

$$\theta_1 \mid U \xrightarrow{\Lambda'_{21}} \theta_2 \mid U;$$

3. $\{(\nabla_\alpha, \Gamma_\alpha), (\nabla'_\alpha, \Gamma_\alpha),\ M_\alpha - \bar{V}\}$ *is an excisive triple with identifying isomorphism*

$$\xi_\alpha \mid (M_\alpha\ -\ \bar{V}) \xrightarrow{\Sigma_\alpha} \theta_\alpha \mid (M_\alpha - \bar{V});$$

4. the identifying isomorphisms are compatible, i.e. the diagram

$$\begin{array}{ccc} \xi_1 \mid (U - \bar{V}) & \xrightarrow{\Lambda_{21}} & \xi_2 \mid (U - \bar{V}) \\ \downarrow \Sigma_1 & & \downarrow \Sigma_2 \\ \theta_1 \mid (U - \bar{V}) & \xrightarrow{\Lambda'_{21}} & \theta_2 \mid (U - \bar{V}) \end{array} \tag{5.44}$$

is commutative.

Then

$$i_a(D^+_{\xi_1}) \; - \; i_a(D^+_{\xi_2}) \; = \; i_a(D^+_{\theta_1}) \; - \; i_a(D^+_{\theta_2}). \tag{5.45}$$

(Compare with the Excision Axiom of Atiyah and Singer [53]).

Proof We use Lemma 5.6 three times, two times with respect to the excisive triples (1) ed (2), $\{(\nabla_1, \Gamma_1), (\nabla_2, \Gamma_2), \; U\}$, resp. $\{(\nabla'_1, \Gamma_1), (\nabla'_2, \Gamma_2), \; U\}$, and once more with respect to the excisive triple (3) $\{(\nabla_1, \Gamma_1), (\nabla'_2, \Gamma_2), \; U - \tilde{V}\}$.

As a result of this procedure we obtain an *invertible scalar* Fredholm operator; its analytical index, clearly, vanishes. This property derives from the formula (5.39); the *scalar* part derives from the scalar part of the formula (5.39).

See [83] for more details.

5.9.4 Cobordism Invariance

Definition 5.11

1. Let M^n_1, $M^n)_2$ be two smooth, closed, oriented manifolds of dimension n. The two manifolds are called *orientation cobordant* provided there exists a smooth, oriented, closed manifold W^{n+1} of dimension $n + 1$, such that the oriented boundary of W^{n+1} consists of M^n_1 and M^n_2.

   Denote by Ω^{Diff}_n the Abelian group of all diffeomorphic classes of smooth, oriented closed cobordant manifolds of dimension n and $\Omega^{Diff} = \sum_{n \in \mathbb{N} \cup 0}$. The sum in Ω^{Diff} is given by disjoint union; the opposite is given by changing the orientation.

   Ω^{Diff} is a *ring*. The product in Ω^{Diff} is given by the direct product.
2. Suppose W^{n+1} establishes an oriented cobordism between the manifolds M^n_1, M^n_2. Let ξ be a complex vector bundle over W^{n+1} and let ξ_1, resp. ξ_2, be its restriction over M_1, resp. M_2. Let ξ_1, resp. ξ_2, be the restriction of ξ to M^n_1, resp. M^n_2. Then the bundles ξ_1, ξ_2 are called *orientation cobordant*.

Denote by $\Omega_n^{Diff}(\mathbb{BGL})$ the group of all orientation cobordant complex smooth bundles over $M \in \Omega_n^{Diff}$ and

$$\Omega^{Diff}(\mathbb{BGL}) = \sum_{n\in\mathbb{N}\cup 0} \Omega_n^{Diff}(\mathbb{BGL}). \tag{5.46}$$

Theorem 5.14 (Thom [24]) *$\Omega^{Diff} \otimes \mathbb{Q}$ is a free Abelian ring generated by complex projective spaces of even complex dimension.*

We state the next result even if it is not necessary for the proof of the Atiyah–Singer index formula.

Theorem 5.15 (Wall [27]) *Two smooth manifolds are oriented cobordant if and only if they have the same Pontrjagin and Stieffel–Whitney classes.*

For the proof of the Atiyah–Singer index formula for twisted signature operators we need to know when two bundles are cobordant. Given that any vector bundle is classified by the homotopy class of a mapping into $\mathbb{BGL}$, the next result gives the answer.

Theorem 5.16 *The Abelian group $\Omega^{Diff}(\mathbb{BGL}) \otimes \mathbb{Q}$ is generated by*

1. *products of even dimensional complex projective spaces*

$$P_{\mathbb{C}}^{2n_{i_1}} \times P_{\mathbb{C}}^{2n_{i_2}} \cdots \times P_{\mathbb{C}}^{2n_{i_p}}, \quad p\ arbitrary, \tag{5.47}$$

2. *products of generators of $K^0(S^{even})$ over even dimensional spheres.*

5.9.5 The Analytic and Topological Indexes of Twisted Signature Operators

Theorem 5.17 *The analytical index $\tilde{i}_a$ and the topological index $\tilde{i}_a$ have the properties*

1. *they are additive with respect to direct sums of operators.*
2. *they are multiplicative with respect to direct products of operators.*
3. *they are cobordism invariant.*

Proof For (1) and (2) see [42, Theorem 4, Ch. XV].
Theorem 5.18 below states (3).

Definition 5.12 Let M^n be a CAT-manifold (CAT = smooth or Lipschitz). Let $i : S^p \times B^{n-p} \longrightarrow M$ be a CAT-embedding.

Then i defines an *elementary surgery* through the following construction: Consider the manifold $M^n \times [0,1]$ to which the *handle* $B^{p+1} \times B^{n-p}$ is added at the level $M^n \times \{1\}$

$$W^{n+1} \;=\; M^n \times [0,1] \cup_i B^{p+1} \times B^{n-p}. \tag{5.48}$$

The manifold $(M^n - i(S^p \times B^{n-p})) \cup (B^{p+1} \times S^{n-p})$ is said to be obtained from the manifold M^n through the surgery i.

Theorem 5.18 *Let W^{2l+1} be a smooth, oriented closed manifold with boundary $M_1 \cup M_2$ (disjoint union). Let ξ be a complex bundle over W^{2l+1} whose restrictions to M_1, resp. M_2, are ξ_1, resp. ξ_2. Let Γ be a Hermitian metric in ξ whose restrictions to ξ_1, resp. ξ_2, are Γ_1, resp. Γ_2.*

Then

$$\tilde{i}_a(D^+_{\xi_1}) \;=\; \tilde{i}_t(D^+_{\xi_2}). \tag{5.49}$$

Proof The cobordism W^{2l+1} may be decomposed in *elementary surgeries*, see [64].

We are going to prove that elementary surgery does not modify analytical indices. Consider then an elementary cobordism 5.12 and a Hermitian bundle ξ over W^{n+1} of rank N. Let

$$M_1 := \; M^n \times \{0\} \tag{5.50}$$

$$M_2 := \; (M^n - i(S^p \times B^{n-p})) \cup (B^{p+1} \times S^{n-p}). \tag{5.51}$$

On M_1 we consider the bundles

$$\xi_1 \;=\; restriction\ of\ \xi\ at\ M_1^n \tag{5.52}$$

$$\xi_2 \;=\; product\ bundle\ of\ Rank\ N\ over\ M_1^n. \tag{5.53}$$

On M_2 we consider the bundles

$$\theta_1 \;=\; restriction\ of\ \xi\ at\ M_2^n \tag{5.54}$$

$$\theta_2 \;=\; product\ bundle\ of\ Rank\ N\ over\ M_2^n. \tag{5.55}$$

The restriction of the bundle ξ at the sub-space $i(S^p \times \mathbf{B}^{n-p})$ is trivial because this sub-space is part of the contractible sub-space $i(S^p \times \mathbf{B}^{n-p}) \cup \mathbf{B}^{p+1} \times \mathbf{B}^{n-p}$. We may choose a trivialisation of it. This shows that the triples (M_1^n, ξ_1, Γ_1), (M_1^n, ξ_1, Γ_1) are excisive. This observation realises the condition (1) of Theorem 5.13.

We proceed in the same manner with the triples (M_2^n, ξ_2, Γ_2); this realises the condition (2) of Theorem 5.13.

The condition (3) of the Theorem 5.13 is provided by the set $V = i(\mathbf{S}^p \times \mathbf{B}^{n-p})$ which is present both in M_1^n and M_2^n. Therefore, Eq. (5.45) holds.

But

$$i_a(D^+_{\theta_1}) \;=\; Signature(M^n_1)\;.\;Rank\;\theta_1 \tag{5.56}$$

$$i_a(D^+_{\theta_2}) \;=\; Signature(M^n_2)\;.\;Rank\;\theta_2. \tag{5.57}$$

The two manifolds M^n_1, M^n_2 are cobordant. Therefore they have the same signature, see Sect. 1.8.2. Equation (5.45) implies then $i_a(D^+_{\xi_1}) \;=\; i_a(D^+_{\xi_2})$.

It is not difficult to see that the topological index behaves correctly with respect to direct sums, direct products and cobordisms; see Chap. 1 and [42].

5.9.6 Excisive Triples

Definition 5.13 Let $M_\alpha, \alpha \;=\; 1,\; 2$, be two smooth manifolds. Let $U,\; V$ be two open sub-manifolds in M_α; $U \subset M_\alpha$ and $\bar{V} \subset U$.

Let φ and θ be complex vector bundles on $M_\alpha, \alpha \;=\; 0,\; 1$.

We suppose also that

$$(\varphi \mid U\;) \text{ be an excisive triple with identifying isomorphism} \tag{5.58}$$

$$(\varphi \mid U\;) \xrightarrow{\Lambda_{12}} (\varphi \mid U\;) \tag{5.59}$$

$$(\theta \mid U\;) \text{ be an excisive triple with identifying isomorphism} \tag{5.60}$$

$$(\theta \mid U\;) \xrightarrow{\Lambda_{12}} (\;\theta \mid U\;). \tag{5.61}$$

Suppose the next diagram is commutative

$$\varphi_\alpha \mid (\;M_\alpha - \bar{V}\;) \xrightarrow{\Lambda_\alpha} \theta_\alpha \mid (\;M_\alpha - \bar{V}\;). \tag{5.62}$$

Suppose also the next diagram is commutative

$$\begin{array}{ccc} \varphi_1 \mid (\;U - \tilde{V}\;) & \xrightarrow{\Lambda_{21}} & \varphi_2 \mid (\;U - \tilde{V}\;) \\ \Big\downarrow \Sigma_1 & & \Big\downarrow \Sigma_2 \\ \theta_1 \mid (\;U - \tilde{V}\;) & \xrightarrow{\Lambda'_{21}} & \theta_2 \mid (\;U - \tilde{V}\;) \end{array} \;. \tag{5.63}$$

Then the two structures are called *excisive triples*.

Compare it with Excision Axiom of Atiyah and Singer [53].

Theorem 5.19 *Suppose all preceding hypotheses of the excessive triples are in place. Then*

$$Index\;D^+_{\varphi_1} \;=\; Index\;D^+_{\varphi_2}. \tag{5.64}$$

Proof The theorem follows from the following relation:

$$\begin{gathered} Index\ D^{+}_{\varphi_1} - Index\ D^{+}_{\varphi_2} = Index\ D^{+}_{\theta_1} - Index\ D^{+}_{\theta_2} = \\ Rank\ \theta_1\ .\ Sig\ M_1 - Rank\ \theta_2\ .\ Sig\ M_2 = 0. \end{gathered} \tag{5.65}$$

Proof Let J_α be connections in the two bundles. We are going to show that

$$\begin{gathered} Index\ [\ (D_{\varphi_1},\ J_{\varphi_1}) - (D_{\varphi_2},\ J_{\varphi_2})\ ] - \\ [\ (D_{\theta_1},\ J_{\theta_1})\ ] - [\ (D_{\theta_2},\ J_{\theta_2})\ ] = 0. \end{gathered} \tag{5.66}$$

Finally in order to prove these relations we shall perform homotopies of the operators in the two brackets and ultimately a homotopy of their difference.

Let

$$\varphi,\ \ \psi :\ M_\alpha \longrightarrow [\ 0, 1\ ],\quad \alpha = 1,\ ;2 \tag{5.67}$$

be two smooth functions with the property

$$Supp\ \varphi_1(x) = Supp\ \psi_2(x)\ \ for\ any\ x \in U. \tag{5.68}$$

We assume also $x \in U$

$$\varphi_\alpha(x) = \psi_\alpha(x). \tag{5.69}$$

Such functions satisfy

$$\text{any x} \in M_\alpha,\quad \varphi^2(x) + \psi^2(x) = 1,\quad Supp\ \psi_\alpha \subset (M_\alpha - \tilde{V}),\quad \alpha = 1, 2. \tag{5.70}$$

Such functions may be constructed as follows: Let

$$\lambda :\ \bar{U} \longrightarrow [\ 0, \frac{\pi}{2}\ ] \tag{5.71}$$

be a smooth function with the property

$$\lambda \mid \partial\bar{U} = 0,\quad and\quad \lambda \mid V = \frac{\pi}{2}. \tag{5.72}$$

Let λ be a smooth function obtained by extending by zero on $M_\alpha - \bar{U}$. We get a smooth function $\lambda_\alpha, \alpha = 1,\ 2.$

$$\varphi_\alpha = sin\ \lambda \tag{5.73}$$

$$\psi_\alpha = cos\ \lambda,\quad \alpha = 1,\ 2. \tag{5.74}$$

These functions have the required properties.

First Step—Here we deal with the first bracket of (5.66). We split out D_{φ_α} and D_{ψ_α} by means of the functions φ and ψ, in a sum of two operators, one with support in U and the other with support in $M_\alpha - V$. We will focus on the operators D_{φ_α}. Consider the case with

$$M_1 = M_2, \quad \varphi_1 = \psi_2. \tag{5.75}$$

It follows that

$$\begin{pmatrix} D_{\varphi_1} & 0 \\ 0 & -D_{\varphi_2} \end{pmatrix} \equiv \begin{pmatrix} \phi_1\ D_{\varphi_1}\ \phi_1 & 0 \\ 0 & -\phi_2\ D_{\varphi_2}\ \phi_2 \end{pmatrix} + \begin{pmatrix} \psi_1\ D_{\varphi_1}\ \psi_1 & 0 \\ 0 & -\psi_2\ D_{\varphi_2}\ \psi_2 \end{pmatrix}. \tag{5.76}$$

We have the homotopy

$$\Delta_t\ (\ \varphi_1, \varphi_2\ ) = \begin{pmatrix} \varphi_1\ D_{\varphi_1}\ cos\ t & -\Lambda_{12}\ \varphi_2^2\ sin\ t \\ \Lambda_{21}\ \varphi_1^2\ sin\ t\ \phi_2\ D_{\varphi_2}\ \phi_2\ cos\ t \end{pmatrix} + \begin{pmatrix} \psi_1\ D_{\varphi_1}\ \psi_1 & 0 \\ 0 & \psi_2\ D_{\varphi_1}\ \psi_2 \end{pmatrix} \tag{5.77}$$

$$t \in [\ 0, \frac{\pi}{2}\ ], \tag{5.78}$$

where

$$\Delta\ (\varphi_1, \xi_2)\ :\ H_{\xi_1} \oplus H_{\xi_2} \longrightarrow H_{\xi_1} \oplus H_{\xi_2}. \tag{5.79}$$

It is not difficult to see that $\Delta_t\ (\ \varphi_1\ ,\ \varphi_2,\ J_{\varphi_1} \oplus -\ J_{\varphi_2}\ )$ satisfies the conditions

$$J \circ D_t = -D_t \circ J \tag{5.80}$$

$$D^* = -D. \tag{5.81}$$

This leads us to a 4×4-matrix. We have

$$\Delta_t^2(\ \varphi_1, \varphi_2\ ) = \begin{pmatrix} L_{11}\ L_{12} \\ L_{21}\ L_{22} \end{pmatrix} \tag{5.82}$$

where we have the relations

$$\begin{aligned} L_{11} &= (\varphi_1 D_{\varphi_1} \varphi_1 \cos t + \psi_1 D_{\varphi_1} \psi_1)^2 - \varphi_1^2 \varphi_2^2 \sin^2 t \\ L_{12} &= -(\varphi_1 D_{\varphi_1} \varphi_1 \cos t + \psi_1 D_{\varphi_1} \psi_1) \Lambda_{12} \varphi_2^2 \sin t + \\ &\quad \Lambda_{12} \varphi_2^2 \sin t \ (\varphi_2 D_{\varphi_2} \varphi_2 \cos t + \psi_2 D_{\varphi_2} \psi_2) \\ L_{21} &= \Lambda_{21} \varphi_1^2 \sin t (\varphi_1 D_{\varphi_1} \varphi_1 \cos t + \psi_1 D_{\varphi_1} \psi_1) - \\ &\quad (\varphi_2 D_{\varphi_2} \varphi_2 \cos t + \psi_2 D_{\varphi_2} \psi_2) \Lambda_{21} \varphi_1^2 \sin t \\ L_{22} &= (\varphi_2 D_{\varphi_2} \varphi_2 \cos t + \psi_2 D_{\varphi_2} \psi_2) - \varphi_1^2 \varphi_2^2 \sin^2 t. \end{aligned} \tag{5.83}$$

We have got the matrix

$$\begin{pmatrix} (\varphi_1^2 \cos t + \psi_1^2) D_{\varphi_1}^2 - \varphi_1^2 \varphi_2^2 \sin^2 t & 0 \\ 0 & (\varphi_2^2 \cos t + \psi_2^2)^2 D_{\varphi_2} - \varphi_1^2 \varphi_2^2 \sin^2 t \end{pmatrix}. \tag{5.84}$$

Let

$$f_\alpha = \varphi_\alpha^2 \cos t + \psi_\alpha^2 \tag{5.85}$$

$$g_\alpha = \varphi_\alpha^2 \sin t, \qquad \alpha = 1, 2. \tag{5.86}$$

It is not difficult to see that for any $x \in U$ we have

$$\varphi_1(x) = \varphi_2(x) \tag{5.87}$$

$$\varphi_1(x) \psi_1(x) = \varphi_1(x) \psi_1(x) = \varphi_2(x) \psi_1(x) = \varphi_2(x) \psi_2(x). \tag{5.88}$$

We also note that the matrix (5.84) is

$$\begin{pmatrix} (D_{\varphi_1} f_1)^* (D_{\varphi_1} f_1) + g_1^* g_1 & 0 \\ 0 & (D_{\varphi_2} f_2)^* (D_{\varphi_2} f_2) + g_2^* g_2 \end{pmatrix} \tag{5.89}$$

This leads us to the relation

$$Index\ D_{\varphi_1}^+ - Index\ D_{\varphi_2}^+ = Index\ (\Delta_{\frac{\pi}{2}} (\varphi_1, \varphi_2) J_{\varphi_1} \oplus - J_{\varphi_2}), \tag{5.90}$$

where

$$\Delta_{\frac{\pi}{2}} (\varphi_1, \varphi_2) = \begin{pmatrix} \psi_1 D_{\varphi_1} \psi_1 & -\Lambda_{12} \varphi_2^2 \\ \Lambda_{21} \varphi_1^2 & \psi_2 D_{\xi_2} \psi_2. \end{pmatrix} \tag{5.91}$$

Second Step—We deal here with the second bracket in the same way. We get

$$Index\ D_1^+ \ -\ Index\ D_2^+ \ =\ Index\ \Delta_{\frac{\pi}{2}}(\theta_1,\theta_2),\ J_{\theta_1}\oplus -J_{\theta_2}). \tag{5.92}$$

Here we have

$$\Delta_{\frac{\pi}{2}}(\theta_1,\theta_2)\ =\begin{pmatrix}\phi_1 D_{\theta_1}\psi_2^2 & -\Lambda_{12}^{'}\varphi_2^2\\ \Lambda_{21}^{'}\varphi_1^2 & -\phi_2 D_{\theta_2}\psi_2\end{pmatrix} \tag{5.93}$$

Third Step—We pass now to the proof of Theorem 5.18. For this we consider the homotopy operator

$$(\ \Delta_{\frac{\pi}{2}},\varphi_1,\varphi_2\ )\oplus -\ (\ \Delta_{\frac{\pi}{2}},\theta_1,\theta_2\ )\ =\ (\ \Delta_t(\ \varphi_1,\varphi_2,\theta_1,\theta_2\ )\ )\ =$$

$$\begin{pmatrix}A_{11},A_{12},A_{13},A_{14}\\ A_{21},A_{22},A_{23},A_{24}\\ A_{31},A_{32},A_{33},A_{34}\\ A_{41},A_{42},A_{43},A_{44}\end{pmatrix} \tag{5.94}$$

where we have

$$\begin{aligned}
A_{11} &= \psi_1\ D_{\varphi_1}\ \psi_1\ cos\ t\\
A_{12} &= -\Lambda_{12}\ \varphi_2^2\\
A_{13} &= -\Lambda_1^{-1}\ \psi_1^2\ sin\ t\\
A_{14} &= 0\\
A_{21} &= \Lambda_{21}\ \varphi_1^2\\
A_{22} &= \psi_2\ D_{\varphi_2}\ \psi_2\ cos\ t\\
A_{23} &= 0\\
A_{24} &= -\Lambda_2^{-1}\ \psi_2^2\ sin\ t\\
A_{31} &= \Lambda_1\ \psi_1^2\ sin\ t\\
A_{32} &= 0\\
A_{33} &= -\psi_1\ D_{\theta_1}\ \psi_1\ cos\ t\\
A_{34} &= \Lambda_{12}^{'}\ \varphi_2^2\\
A_{41} &= 0\\
A_{42} &= \Lambda_2\ \psi_2^2\ sin\ t\\
A_{43} &= -\Lambda_{21}^{'}\ \varphi_1^2\\
A_{44} &= \psi_2\ D_{\theta_2}\ \psi_2\ cos\ t
\end{aligned} \tag{5.95}$$

This maps $H_{\varphi_1} \oplus H_{\varphi_2} \oplus H_{\theta_1} \oplus H_{\theta_2}$ into itself for $t \in [0, \frac{\pi}{2}]$. Notice that this homotopy acts effectively $H_{\varphi_1} \oplus H_{\theta_1}$ and on $H_{\varphi_2} \oplus H_{\theta_2}$.

We verify now that $\Delta_t(\ \varphi_1, \varphi_2, \theta_1, \theta_2\ )$ is a Fredholm operator—see Appendix of [83]. We have

$$\Delta(\ \varphi_1, \varphi_2, \theta_1, \theta_2\ ) \ = \ \begin{pmatrix} Fredholm\ oper. & 0 & 0 & 0 \\ 0 & Fredholm\ oper. & 0 & 0 \\ 0 & 0 & Fredholm\ oper. & 0 \\ 0 & 0 & 0 & Fredholm\ oper. \end{pmatrix}. \tag{5.96}$$

We have

$$\begin{aligned} &\Delta(\varphi_1, \varphi_2, \theta_1, \theta_2) \ \circ \ (J_{\xi_1} \oplus -J_{\xi_2} \oplus -J_{\xi_1} \oplus J_{\xi_2}) \ = \\ &-(J_{\xi_1} \oplus -J_{\xi_2} \oplus -J_{\xi_1} \oplus J_{\xi_2}) \ \circ \ \Delta(\xi_1, \xi_2, \theta_1, \theta_2). \end{aligned} \tag{5.97}$$

We have

$$Index(\ \Delta_{\frac{\pi}{2}}(\ \varphi_1, \varphi_2, \theta_1, \theta_2\ ), \quad J_{\varphi_1} \oplus -J_{\varphi_2} \oplus J_{\theta_1} \oplus -J_{\theta_2}\ ) \ = \ 0. \tag{5.98}$$

Then proves the relation.

This proposition implies

$$\Delta_{\frac{\pi}{2}}(\xi_1, \xi_2, \theta_1, \theta_2) \ = \tag{5.99}$$

$$\begin{pmatrix} 0 & -(\Lambda_{21}\varphi_1^2)^* & -(\Sigma_1\psi_1^2)^* & 0 \\ \Lambda_{21}\varphi_1^2 & 0 & 0 & -\Sigma_2^{-1}\psi_2^2 \\ \Sigma_1\psi_1^2 & 0 & 0 & \Lambda 12^{'}\varphi_2^2 \\ 0 & (\Sigma_2^{-1}\psi_2^2) & -(\Lambda_{12}^{'}\varphi_2^2)^* & 0 \end{pmatrix} \tag{5.100}$$

The operator Δ anti-commutes with the operator

$$J \ = \ J_{\varphi_1} \ \oplus \ -J_{\varphi_2} \ \oplus \ -J_{\theta_1} \ \oplus \ J_{\theta_2}. \tag{5.101}$$

To prove this it to show

$$(J_{\varphi_\alpha})^* \ = \ J_{\xi_\alpha}, \quad (J_{\theta_\alpha})^* \ = \ J_{\theta_\alpha}, \quad \alpha = 1,\ 2, \tag{5.102}$$

see together with the commutation relation, leads us to

$$\Lambda_{21}\ \varphi_1^2\ J_{\xi_1} \ = \ J_{\xi_2}\ \Lambda_{21}\ \varphi_1^2 \tag{5.103}$$

$$\Lambda_{12}^{'}\varphi_2^2\ J_{\theta_2} \ = \ J_{\theta_1}\ \Lambda_{12}^{'}\ \varphi_2^2 \tag{5.104}$$

$$\Sigma_1\ \psi_1^2\ J_{\xi_1} \ = \ J_{\theta_1}\ \Sigma_1\ \psi_1^2 \tag{5.105}$$

$$\Sigma_2^{-1}\ \psi_2^2\ J_{\theta_2} \ = \ J_{\varphi_2}\ \Sigma_2^{-1}\psi_2^2. \tag{5.106}$$

We take into account the commutation relations, after considering the next equations, we get

$$(I) \quad J_{\varphi_1}\ (\Lambda_{21}\varphi_1^2\ )^* \ = \ (\Lambda_{21}\varphi_1^2)^* J_{\varphi_2} \tag{5.107}$$

$$J_{\theta_2}(\Lambda_{12}^{'}\varphi_2^2)^* \ = \ (\Lambda_{12}^{'}\varphi_2^2)^2)^* J_{\theta_1} \tag{5.108}$$

$$(II) \quad J_{\varphi_1}(\Sigma_1\psi_1^2)^* \ = \ (\Sigma_1\psi_1^2)^* J_{\theta_1} \tag{5.109}$$

$$J_{\theta_2}(\Lambda_2^{-1}\psi_2^2)^* \ = \ (\Lambda_2^{-1}\psi_1^2)^2)^* J_{\theta_2}. \tag{5.110}$$

Considering the equation tells us

$$Index\ (\Delta_{\frac{\pi}{2}}(\varphi_1, -\varphi_2, -\theta_1, \theta_2)) \ = \ Index\ (\Delta,\ J_{\varphi_1} \oplus -J_{\varphi_2} \oplus -J_{\theta_1} \oplus J_{\theta_2}). \tag{5.111}$$

The operator Δ is skew-adjoint and hence

$$Index(\ \Delta, J\ ) \ = \ dim\ Ker^{+}\Delta \ - \ dim\ Ker^{-}\Delta. \tag{5.112}$$

We show now that $Ker\Delta \ = \ 0$; then, a fortiori, it follows that $(\Delta,\ J) \ = \ 0$. This formula implies the next relations.

We take care now of $(\ x_1, x_2, y_1, y_2\ ) \ \in \ Ker\ \Delta$.

$$\Lambda_{21}\varphi_1^2\ x_1 \ - \ \Sigma_2^{-1}\psi_2^2\ y_2 \ = \ 0 \tag{5.113}$$

$$\Sigma_1\psi_1^2\ x_1 \ + \ \Lambda_{12}^{'}\varphi_2^2\ y_2 \ = \ 0 \tag{5.114}$$

$$-(\Lambda_{21}\varphi_1^2)^*\ x_2 \ - \ (\Sigma_1\psi_1^2)^*\ y_1 \ = \ 0 \tag{5.115}$$

$$(\Sigma_2^{-1}\psi_2^2)^*\ x_2 \ - \ (\Lambda^{'}\varphi_2^2)\ y_1 \ = \ 0. \tag{5.116}$$

Now we solve the systems (I) and (II).

To solve the system (I), we multiply the first equation by $\Lambda_{12}\ \varphi_2^2$ and the second equation by $\Sigma_1^{-1}\ \psi_1^2$. These will eliminate y_2. This works because the commutativity of Λ and Σ is equivalent to the relation

$$(\ \Lambda_{12}\ \varphi_2^2)\ (\Sigma_2^{-1}\psi_2^2) \ = \ (\Sigma_1^{-1}\ \psi_1^2)\ (\Lambda_{12}^{'}\ \varphi_2^2). \tag{5.117}$$

x_1 has to satisfy the equation

$$[(\Lambda_{12}\ \varphi_2^2)\ (\Lambda_{21}\ \varphi_1^2) \ + \ (\Sigma_1^{-1}\ \psi_1^2)\ (\Sigma_1\psi_1^2)]\ x_1 \ = 0. \tag{5.118}$$

This implies $x_1 = 0$ (recall (5.70))

To solve the system (II) we use the same procedure, being aware of the extra complication due to the presence of the adjoints.

To proceed, we multiply the first equation (II) by $(\Sigma_1^{-1}\ \psi_1^2)$ and the second by $(\ \Lambda_{21}^2\varphi_1^2\ )^*$. The equality

$$(\ \Sigma_1^{-1}\psi_1^2\ )^*\ (\ \Lambda_{21}\varphi_1^2\ )^* = (\ \Lambda_{21}^{'}\varphi_1^2\ )^*\ (\ \Sigma_2^{-1}\psi_2^2\ )^* \tag{5.119}$$

allows us to eliminate x_2. Then y_1 has to belong to the kernel of the operator

$$(\Sigma_1^{-1}\ \psi_1^2)^*(\Sigma_1\ \psi_1^2)^* + (\Lambda_{21}^{'}\ \varphi_1^2)^*\ (\Lambda_{12}^{'}\ \varphi_2^2)^* \tag{5.120}$$

which turn us to invertible; therefore $y_1 = 0$.

This formula follows from the commutativity of the diagram by passing to adjoints. The operator is invertible because its adjoints are invertible.

To eliminate y_1 we multiply the first equation (II) by $(\Lambda_{12}\varphi_2^2\ )^*$ and the second by $(\ \Sigma_2\ \psi_2^2\ )^*$; invoking the same kind of arguments, we get $x_2 = 0$.

The techniques presented in this chapter are valid also in the next section devoted to the Lipschitz manifolds.

5.10 Teleman Index Formula: The Index Formula Is a Topological Statement

In the proceeding section we presented a proof of the Atiyah–Singer index formula. The proof we have presented used Palais [42] and Teleman [83]. That proof uses very little analytical structure. It is valid on Lipschitz manifolds as well as on any manifold structure which is a specialisation of the Lipschitz structure. For this reason we present here solely those topics which need a specific Lipschitz treatment. These are: (1) L_2-cohomology, (2) Hodge theory, (3) K-homology; see Teleman [83, 84].

5.11 Teleman Index Formula on Lipschitz Manifolds

5.11.1 Riemannian Structures: L_2-Cohomology

Definition 5.14 (Riemannian Metrics) Let M^n be a Lipschitz manifold. Cover M^n with a set of relatively compact local charts $\iota_\alpha : U_\alpha \longrightarrow V_\alpha$, $U_\alpha \subset \mathbb{R}^n$, $V_\alpha \subset M^n$, $\alpha \in I$.

On each set U_α we choose a *measurable* Riemannian metric $\bar{g}_\alpha$. We choose two constants $0 < k < K$ such that

$$k\ \bar{g}_0 \leq \bar{g}_\alpha \leq K\ \bar{g}_0, \quad any\ \alpha \in I,$$

where $\bar{g}_0$ is the standard Euclidean metric. We transfer the Riemannian metrics $\bar{g}_\alpha$ onto M^n through the coordinate charts ι_α. The metric $\bar{g}_\alpha$ induces onto the set of smooth manifolds with differential forms $\Omega^k(U_\alpha)$ that have relatively compact supports, the norm

$$||\omega_\alpha||^2_\alpha \ = \ \int_{U_\alpha} \omega_\alpha \wedge_\alpha *\omega_\alpha,$$

where the star operator $*_\alpha$ is associated to the Riemannian metric $\bar{g}_\alpha$.

The norms $|| - ||_\alpha$ are transferred into M^n by declaring each of the ι_α to be an isometry. For any $\omega \ = \{\iota_*^{-1}\omega_\alpha\}$ we declare

$$||\omega||^2_{M^n} \ = \ \sum_{\alpha \in I} ||\omega_\alpha||^2$$

The metric $g \ = \{\iota_{\alpha,*}(\bar{g}_\alpha)\}_{\alpha \in I}$ will be called *Lipschitz Riemannian metric on* M^n.

Definition 5.15 The completion of the space of *flat* i-differential forms with respect to the norms $|| - ||_{M^n}$ will be called the space of L_2-*differential forms*, denoted $L^i_2(M^n)$.

Theorem 5.20 (Whitney) *Let f be Lipschitz function defined on $U \subset \mathbb{R}^n$.*

Then the partial derivatives $\partial f/\partial x^i$ exist almost everywhere and belong to the space $L_\infty(U)$.

Definition 5.16 Let M be a Lipschitz manifold. Then $Td\ (M)$ is called *Todd genus* of M.

Proposition 5.7 *The spaces $L_2(M^n)$ are Hilbert spaces; the Hilbert space norms corresponding to two choices of the local measurable Riemannian metrics $\bar{g}_\alpha$ are equivalent.*

Although on a Lipschitz manifold there are no *smooth* differential forms, we may however consider such forms on any local coordinate chart V_α.

Definition 5.17 Let $\sigma \in \Omega^p$ be a p-differential form with measurable components with respect to the local coordinates in V_α. The differential form with measurable components $d\ \sigma \ \in \ \Omega^{p+1}$ is called the *distributional exterior derivative* of σ provided the $d\sigma$ satisfies the relation

$$\int_{V_\alpha} \sigma \wedge d\varphi \ = \ (-1)^{p+1} \int_{V_\alpha} d\sigma \wedge \varphi$$

for any smooth differential form $\varphi \in \Omega^{n-p-1}(V_\alpha)$.

Definition 5.18 Let $\sigma \in \Omega^p(V_\alpha)$ be a differential form with *bounded coefficients* with the property that the distributional exterior differential $d\sigma$ has bounded components.

Then, the differential form σ is called a *flat form*, see [20],

Definition 5.19 A *flat form on* M^n is by definition a form which satisfies the condition of the Definition 5.19 locally, for any $\alpha \in I$.

Flat forms may, in fact, be defined on any simplicial complex.

Theorem 5.21 (Whitney [20]) *The homology of the co-homology complex of flat forms is isomorphic to the real simplicial co-homology of the simplicial space.*

The proof is based on the generalised de Rham theorem.

Proposition 5.8 *Flat p-forms are dense in* $L_2^p(M^n)$.
The Hilbert space $L_2^p(M^n)$ *is the completion of the space of p-flat forms.*

Proof See [83].

Theorem 5.22 *The inclusion of the co-homology complex of flat differential forms into the co-homology complex* $\{L_2^*,\ d\ \}$ *induces an isomorphism in co-homology.*

The proof is based on the comparison of the spectral sequences associated with both complexes.

5.11.2 Strong and Weak Convergence: The Space W_1^p

Definition 5.20 Let a_n be a sequence in $L_2^p(M^n)$; suppose $lim_{n\to\infty} a_n = a \in L_2^p(M^n)$. Suppose there exists an element $da \in L_2^{p+1}(M^n)$ such that $lim_{n\to\infty} d\,a_n = da \in L_2^{p+1}(M^n)$.

We will say that $da \in L_2^{p+1}(M^n)$ is the *strong limit* of the sequence $\{a_n\}_{n\in\mathbb{N}}$ and we will write

$$lim_{n\to\infty} a_n = a \in L_2^p(M^n), \quad lim^s_{n\to\infty} d\,a_n = d\,a \in L_2^{p+1}(M^n). \tag{5.121}$$

On the other hand, suppose $\{a_n\}$ is a sequence in $L_2^p(M^n)$ with $lim_{n\to\infty} a_n = a$. Suppose there exists $da \in L_2^{p+1}(M^n)$ such that

$$lim_{n\to\infty} \int_{M^n} a \wedge d\varphi = (-1)^{p+1} \lim_{n\to\infty} \int_{V_\alpha} da \wedge \varphi$$

for any $\varphi \in L_2^{n-p-1}(M^n)$.

Then we will say that $da \in L_2^{p+1}(M^n)$ is the *week limit of* the sequence $\{a_n\}_{n\in\mathbb{N}}$. We will write $lim_{n\to\infty} a_n = a, lim^w_{n\to\infty} da_n = da.$ and $lim^s = lim^w = lim.$

Proposition 5.9 *The sequence* $\{da_n\}_{n\in\mathbb{N}} \in L_2^{p+1}(M^n)$ *converges strongly to* $da \in L_2^{p+1}(M^n)$ *if and only if* $\{da_n\}_{n\in\mathbb{N}}$ *converges weekly to* da.

Definition 5.21 By definition $W_1^p := \{a \in L_2^p(M^n)\ with\ da \in L_2^{p+1}(M^n)\ and\ d*a \in L_2^{n-p+1}(M^n)\}$.

Definition 5.22 We introduce on W_1^p the norm

$$||a||_1^2 := ||a||^2 + ||da||^2 + ||d*a||^2.$$

Proposition 5.10 $\{\ W_1^p,\ ||-||_1\ \}$ *is a Hilbert space.*

The spaces W_1^p are the Lipschitz analogue of the Sobolev spaces H_1.

5.11.3 Rellich Lemma on Lipschitz Manifolds

Theorem 5.23 *The continuous inclusion* $\{\ W_1^p,\ ||-||_1\ \} \longrightarrow \{\ L_2^p,\ ||-||\ \}$ *is compact.*

The statement is local; it may be verified on coordinate charts. The proof is based on two facts.

1. For any Lipschitz Riemannian metric $\bar{g}_\alpha$ the ranges of the operators d and $d*$ remain *transversal*, as in the case of the corresponding ranges associated to the standard Euclidean metric. This allows to reduce the problem to the Euclidean metric case.
2. In the case of the Euclidean case, the Fourier transform $\mathscr{F}$ applied onto exact differentials $d\sigma$ controls the *radial component* of $\mathscr{F}\sigma$; on the other hand, $\mathscr{F}$ applied onto $d*\sigma$ controls the transversal component of $\mathscr{F}\sigma$.

5.11.4 Lipschitz Hodge Theory: The Signature Operator

For this section see Teleman [77, 78, 83].

Definition 5.23 Any differential form $h \in W_1^p$ having the properties (i) $dh = 0$, (ii) $d*h = 0$ is called a *harmonic form.*

Let $\mathscr{H}^p$ denote the space of harmonic forms of degree p.

Theorem 5.24 (Lipschitz Hodge Theorem [78, 83])

1. *The space of harmonic forms $\mathscr{H}^p$ is finite dimensional. It is isomorphic to the real simplicial cohomology of the Lipschitz manifold.*
2. *The ranges of the operators d and $d*$ acting on W_1^p are closed.*
3. *The sub-spaces $\mathscr{H}^p$, dW_1^{p-1}, $d*W_1^{n-p+1}$ are mutually orthogonal. One has the orthogonal decomposition*

$$W_1^p = \mathscr{H}^p \oplus dW_1^{p-1} \oplus d*W_1^{n-p+1}. \tag{5.122}$$

5.11.5 The Index of Twisted Signature Operators Is a Lipschitz Invariant

The *signature operators* D^+ and *twisted signature operators* D_ξ^+ are defined as in the smooth case, see Sect. 1.13. Here we mention solely the specific Lipschitz phenomena.

5.11.6 The Analytic Index of D_ξ^+ Is a Lipschitz Invariant

The twisted signature operator D_ξ^+ depends on the choice of the idempotent p which represents the bundle ξ. The Lipschitz Rellich Lemma (Theorem 5.23) shows that modifying the idempotent p does modify the operator D_ξ^+ by adding a compact operator. This change does not modify the analytical index.

The Lipschitz Rellich lemma (Theorem 5.23) shows also that modifying the Hermitian metric Γ in the bundle ξ does modify the Fredholm operator by adding a compact term; this modification preserves the analytical index.

In the smooth case a change of the *smooth* Riemannian metric does not change the Sobolev spaces H_1. In the Lipschitz case, the change of the Lipschitz Riemannian metric *does modify the spaces* W_1^p.

Theorem 5.25 *The analytical index of the twisted signature operator D_ξ^+ is independent of the Lipschitz Riemannian metric.*

Proof We show that the analytical index does not change if the Lipschitz Riemannian metric is modified on a small subset V of the manifold. We may assume that V is contractible and hence that the bundle ξ is trivial over V. Theorem 5.13 of Sect. 5.9 holds in the Lipschitz category. We apply this theorem by making the following choices:

1. $M_1 = M_2 = M$
2. $\xi_1 = \xi_2 = \xi$, *both bundles trivial over* V, *with the same Hermitian metric*
3. $\theta_1 = \theta_2 =$ *product bundles*.

One has

$$i_a(D^+_{\theta_1}) = i_a(D^+_{\theta_2}) = (Signature\ M).(Rank\ \theta_\alpha), \quad \alpha = 1, 2. \tag{5.123}$$

Theorem 5.13 implies $i_a(D^+_{\xi_1})$ does not change if the Lipschitz Riemannian metric is modified.

5.11.7 The Index Formula Is a Topological Statement

At this point we know that the analytical index of twisted signature operators is a *Lipschitz invariant*. Moreover, the next result shows that the *analytical index* of twisted elliptic operators is a *topological invariant*.

Theorem 5.26 (Sullivan [75]) *For any closed even dimensional topological manifold M^n of dimension $n \neq 4$ there exists a* unique *Lipschitz, resp. quasi-conformal, structure, isotopic to the identity.*

The topological index of signature operators remains invariant under Lipschitz/quasi-conformal homeomorphisms. The isometry to the identity keeps the equivalence class of any bundle unchanged. Therefore, the *topological index* of twisted signature operators is a *topological invariant*.

It remains only to show that the analytical index and the topological index of twisted signature operators agree. This will follow from the following four theorems:

Theorem 5.27 (Kirby and Siebenmann [56, Essay III, Theorem 2.1]) *For any compact topological manifold W of dimension $m + 1 \leq 6$ and for any clean $m + 1$-sub-manifold $W_0 \subset W$ there exists a handlebody decomposition, i.e. a finite filtration*

$$W_0 \subset W_1 \subset \ldots W_r = W \tag{5.124}$$

by closed clean $m+1$-sub-manifolds such that if H_{k+1} denotes the closure of $W_{k+1} - W_k$, then,

$$(H_{k+1}, H_{k+1} \cap W_k) \tag{5.125}$$

is homeomorphic to $(\mathbf{B}^p, \partial\mathbf{B}^p) \times \mathbf{B}^q$, $p + q = n + 1$. Here $\mathbf{B}^p$ is the closed p-dimensional unit disc.

Theorem 5.18 and its proof hold in the Lipschitz category. We have then the following result:

Theorem 5.28 *The analytical index of twisted signature operators is invariant under cobordisms.*

Theorem 5.29 (Kirby and Siebenmann [56, Annex C, Proposition 10.3]) *The rational oriented* smooth *cobordism group of the classifying space of complex vector bundles and the rational oriented* topological *cobordism group of the classifying space of complex vector bundles are isomorphic*

$$\Omega_*^{Diff}(\mathbb{BGL}) \otimes \mathbb{Q} \equiv \Omega_*^{STOP}(\mathbb{BGL}) \otimes \mathbb{Q}, \ \textit{see [42] Ch. XVIII.} \tag{5.126}$$

Given that both indices are invariant under topological cobordisms, Theorem 5.29 allows us to compare the two indices in the smooth category. The Atiyah–Singer index theorem in the smooth category states that the two indices agree. We have proved the following result:

Theorem 5.30 *The analytic index and the topological index of twisted signature operators on even dimensional Lipschitz manifolds are equal.*

This completes the proof of the index formula for twisted signature operators on Lipschitz manifolds of even dimension.

We intend to extend the index formula to a larger class of Fredholm operators on Lipschitz manifolds. This will be discussed in the section below.

5.11.8 Teleman Index Formula on Topological Manifolds

The elements of the K_0-homology, in the smooth category, contain as cycles all elliptic pseudo-differential elliptic operators of order zero. We intend to extend the index formula from twisted elliptic operators on Lipschitz manifolds to cycles in K_0, see Sect. 2.8. This may be done, but there is a difficulty. The Kasparov K_*-homology is defined over *finitely generated Banach algebras*. Unfortunately, the algebra of Lipschitz functions on a Lipschitz manifold is not finitely generated. The solution [84] consists of replacing the algebra of Lipschitz functions on the manifold M with the finitely generated Banach algebra $C^1(U(M))$ of functions of class C^1 defined on a tubular neighbourhood $U(M)$, of the embedding of M into some $\mathbb{R}^N$, see [59, Corollary 4.6]; see also [102]. This leads us to the next result.

Theorem 5.31 ([84]) *For any compact Lipschitz manifold of even dimension one has*

1. $K_*(C^1(U(M)))$ *is isomorphic to* $K_*(C^0(M))$.
2. $K_*(C^1(UM))$ *contains twisted signature operators as cycles.*
3. *The index formula holds for* $K_*(C^1(UM))$*; it is a* topological statement.
4. *If M were a smooth manifold,* $K_*(C^1(U(M)))$ *would contain as cycles all elliptic pseudo-differential operators.*

Chapter 6
Index Theorems in Non-commutative Geometry



Abstract The index formula on topological manifolds may not be expressed by *de Rham-type forms*, because this leads to making products of distributions which in classical geometry may not be performed. The solution to this problem consists of replacing the de Rham forms by *quasi-local objects*, i.e. objects living on a neighbourhood of the diagonal in the different powers of the space. This may be done in non-commutative geometry; one obtains the *local index theorem of Connes, Moscovici*. In the specific case of topological manifolds, this leads to the Connes, Hilsum, Sullivan, Teleman index formulae.

6.1 Connes Moscovici Local Index Theorem

The local Index Theorem presented in this section is [96, Theorem (3.9) Pg. 368]. A generalisation of this result is presented in [103].

The reader should realise that the adjective *local* is used with two different meanings in this book. On the one hand, the *Connes–Moscovici local index theorem* [96] is based on the following construction: (1) inside classical algebraic K_i-theory, (2) of an *almost idempotent* $R(A)$ associated with an elliptic *pseudo-differential* operator A, (3) the index theorem is extracted by *pairing* $R(A)$ with *skew symmetric Alexander–Spanier cohomology classes*. The result uses cyclic homology of *idempotents*. *Skew symmetry* allows to apply the Connes-Karoubi-Chern character onto the almost idempotent $R(A)$. The Connes, Moscovici theorem produces the components of the Thom–Hirzebruch character, one at a time, one for each Alexander–Spanier class.

On the other hand, see Sect. 7.2.4, we produce a *localised homology theory* involving (1) a *local algebraic* T_i^{loc}-theory; (2) the definition of a Chern character of *almost idempotents*, involving the replacement of the basic algebra with *separable algebras*; and (3) The obtained information is treated inside the *periodic cyclic homology* (Sect. 3.45).

The index theorem we obtain produces the *whole* Thom–Hirzebruch character.

These ideas lead in Parts IV and V to further extensions of classical results.

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The Connes–Moscovici local index theorem will be discussed in Sect. 6.1. The next discussion touches Chern character of *almost idempotents*, the *Chern character of almost idempotents*. The *local periodic cyclic homology* will be treated in Sect. 7.2. The *local algebraic* T_*^{loc}-theory will make the object of Sect. 7.3.2.

Let $A : L_2(E) \to L_2(F)$ be an elliptic pseudo-differential operator of order zero from the vector bundle E to the vector bundle F on the compact smooth manifold M. Let $\sigma_{pr}(A) = \mathbf{a}$ be its *principal symbol*. This symbol is a continuous isomorphism from the bundle π^*E to the bundle π^*F over the unit co-sphere bundle $S(T^*M)$, where π is the co-tangent bundle projection. Let B be a pseudo-differential *parametrix* for the operator A; its principal symbol is $\sigma_{pr}(B) = a^{-1}$. The operators A and B may be supposed to have small supports about the diagonal in $M \times M$. The corresponding operators $S_0 = 1 - BA$ and $S_1 = 1 - AB$ may be supposed to be *s*moothing operators. Their distributional support is concentrated on an arbitrarily small neighbourhood of the diagonal.

Onto the operators A, B, S_0, S_1, one applies the construction of the connecting operator ∂, Definition 7.6.

The smoothing *residue* operator

$$\mathbf{R} := \mathbf{P} - \mathbf{P}_2 = \begin{pmatrix} S_0^2 & S_0(1+S_0)B \\ S_1 A & -S_1^2 \end{pmatrix} \tag{6.1}$$

is a difference of idempotents (of the space $L_2(E) \oplus L_2(F)$) with small support about the diagonal. The operator $\mathbb{R}$ is a *trace class* operator. Additionally, it belongs to the space $\mathscr{K}_M$ of *compact operators* on $L_2(M)$. This leads to the ring exact sequence

$$0 \to \mathscr{K}_M \to \mathscr{L}_M \to C(S^*M) \to 0, \tag{6.2}$$

where $\mathscr{L}_M$ is the norm closure in the algebra of bounded operators of the algebra of pseudo-differential operators of order zero and $C(S^*M)$ is the algebra of continuous functions on the unit co-sphere bundle to M.

After embedding the bundles E and F into the trivial bundle of rank N, the operators L, $\mathbf{P}$, $\mathbf{P}_2$, resp. R, may be seen as elements of the matrix rings $\mathbb{M}_N(\mathscr{L}_M)$, resp. $\mathbb{M}_N(\mathscr{K}_M)$;

$$\partial_1([\mathbf{a}]) = [\mathbf{P}] - [\mathbf{P}_2] \in K^0(\mathscr{K}_M). \tag{6.3}$$

Definition 6.1 $\mathbb{R}(x_0, x_1)$ be the *distributional kernel* of the residue operator $\mathbb{R}$. This is a smooth function on $M \times M$ with small support about the diagonal.

Definition 6.2 Let $A_0, A_1, \ldots, A_q$ be linear operators on M and let $A_i(x, y)$ be their distributional kernels; suppose at least one operator A_i is trace class. Let $\phi \in C^q(M)$ be a (local) Alexander–Spanier chain. Define the distribution on M^{q+1}

$$\tau(A_0, A_1, \ldots, A_q)(\phi) := (-1)^q \int_{M^{q+1}} trace\ A_0(x_0, x_q) \ldots . A_q(x_q, x_0)\ \phi(x_0, x_1 \ldots x_q). \tag{6.4}$$

Proposition 6.1 *Let $A_0, A_1, \ldots, A_q$ be linear operators satisfying Definition 6.2. Suppose also that $\phi \in C^{q,\lambda}(M)$ be a (local)* cyclic *Alexander–Spanier co-chain $(\phi(x_q, x_0, x_1 \ldots x_{q-1}) = (-1)^q \phi(x_0, x_1, x_2 \ldots x_q))$. Then*

(i) $\tau\ (A_0, A_1, \ldots, A_q)(\phi)$ is a cyclic chain (with respect to $A_0, A_1, \ldots, A_q$)
(ii)

$$\tau(\ b(A_0, A_1, \ldots, A_{q+1}))(\phi) \ := \ \tau(A_0, A_1, \ldots, A_q)\ (d\phi), \tag{6.5}$$

where b is the Hochschild boundary and d is the Alexander–Spanier co-boundary.

Proof By direct verification.

Proposition 6.2 *Let P_t be a C^1-path of idempotents in $\mathbb{M}_n(L_2(M))$, for $t \in [0, 1]$. Let $T_t := (1 - 2P_t)\frac{d}{dt}P_t$. Then*

$$\tau(\phi)(P_1, P_1, \ldots, P_1) - \tau(\phi)(P_0, P_0, \ldots, P_0)dt \ = \ (q+1) \int_0^1 \tau(d\phi)(T_t, P_t, \ldots, P_t)dt. \tag{6.6}$$

Let $C^{q,sk}(M)$ denote the space of Alexander–Spanier co-chains of degree q on M consisting of all smooth, *skew symmetric* (Definition 1.4) real valued functions ϕ defined on M^{q+1}, which have support on a sufficiently small tubular neighbourhood of the diagonal.

For any $[\mathbf{a}] \in K^1(C(S^*M))$ and for any even number q, define the linear functional

$$\tau^q_{\mathbf{a}} : C^q(M) \longrightarrow \mathbb{C} \tag{6.7}$$

given by the formula

$$\tau^q_{\mathbf{a}}(\phi) \ = \ \int_{M^{q+1}} \mathbf{R}(x_0, x_1)\mathbf{R}(x_1, x_2) \ldots \mathbf{R}(x_q, x_0)\ \phi(x_0, x_1, \ldots, x_q)\ dx_0.dx_1 \ldots .dx_q. \tag{6.8}$$

Define the *index class homomorphism*

$$Ind : K^1(C(S^*M)) \otimes_{\mathbb{C}} H^{ev,sk}_{AS}(M) \longrightarrow \mathbb{C} \tag{6.9}$$

$$Ind(\mathbf{a} \otimes_{\mathbb{C}} \phi) := \ \tau^q_{\mathbf{a}}\ (\phi), \tag{6.10}$$

where $H^{ev}_{AS}(M)$ denotes Alexander–Spanier cohomology given by *skew symmetric* (Definition 1.4) co-chains ϕ.

Propositions 6.1 and 6.2 show that the index class homomorphism is well defined.

Theorem 6.1 (, Connes and Moscovici [96, Theorem 3.9. Pg. 368]) *Let A be an elliptic pseudo-differential operator on M and let $[\phi] \in H^{2q,sk}_{comp}(M)$. Then*

$$Ind_{[\phi]}A = \frac{1}{(2\pi i)^q}\frac{q!}{(2q)!}(-1)^{dimM} < Ch\sigma(A)\tau(M)[\phi], [T^*M] > \tag{6.11}$$

where $\tau(M) = Todd(TM) \otimes \mathbb{C}$ and $H^*(T^*M)$ is seen as a module over $H^{*,sk}_{comp}(M)$

Proof See [96].

6.2 Donaldson, Sullivan Index Theorem

Sections 6.2–6.3 present the most general analytic structures on topological manifolds for which the index formula holds.

The Donaldson–Sullivan article [92] begins with the analysis of 4-dimensional quasi-conformal manifolds. It produces Hodge theory on such manifolds and computes the index of *signature* operators. The signature operator is defined on $L_{2+\epsilon}$ differential forms of degree 2.

The main results of [92] are

Theorem 6.2 *There are topological manifolds of dimension* 4 *which do not admit any quasi-conformal structure.*

Theorem 6.3 *There are quasi-conformal manifolds of dimension* 4 *which are homeomorphic but not quasi-conformally equivalent.*

6.3 Connes, Sullivan, Teleman Index Formula

For this section the reader is advised to consult [101]. On Lipschitz manifolds all differential forms with coefficients in L_p, for any $1 \leq p \leq \infty$, are well defined. On quasi-conformal manifolds the list of differentiable forms restricts drastically. On a quasi-conformal manifolds of dimension n, for any $0 \leq p \leq n$, there exist only p-differential forms which have components, locally, in $L_{n/p+\epsilon}$, $0 < \epsilon$.

In particular, on closed quasi-conformal manifolds of dimension $n = 2l$ there exist l-differential forms with components in $L_{2+\epsilon}$. On this space there exists the operator S

$$S(\omega) = \omega, \quad \omega \, is \, in \, the \, domain \, d \tag{6.12}$$

$$S(\omega) = -\omega, \quad \omega \, is \, in \, the \, domain \, \delta. \tag{6.13}$$

Theorem 6.4 *The homological information carried by the operator S is the signature.*

If the operator S is *localised* around the diagonal, then the information carried by this operator becomes much richer.

Theorem 6.5 *The homological information carried by the localised operator S is the Thom–Hirzebruch character.*

The localised S operator is made of open pieces of the operator S, defined on the sphere S^{2l}; obviously, the local quasi-conformal structure on S^{2l} coincides with the local quasi-conformal structure on M^{2l} on each piece. The operator S defined on the sphere S^{2l}, associated to the quasi-conformal structure μ, is

$$S_\mu = (1 + \mu)^{-1} (S_0 + \mu) S_0 (S_0 + \mu)^{-1} (1 + \mu), \tag{6.14}$$

where $||\mu|| < 1$ and S_0 is the flat conformal structure on the standard sphere.

The paper [101] makes some improvements with respect to the [96] paper. They are:

1. it proves the index theorem on the most general analytical structures on topological manifolds
2. it exhibits the basic feature of the signature operator: it shows that the basic index phenomena related to the signature operator reside on differential forms of *middle degree*
3. it *localises* the construction. The localisation is based on the system of neighbourhoods of the diagonal;
4. it produces directly the *homology index character* without the need to pair it with the Alexander–Spanier co-cycle. However, the idempotent e of the residue operator $R(A) = P - e$ is killed by *skew-symmetrising* the index character $\wedge^q R(A) := R(A) \otimes R(A) \otimes \cdots \otimes R(A)$. The construction produces the *components* of the index class $\wedge^q R(A)$, one for each even power q.

The same problem is discussed in Sect. 7.2.6. In that section we will define a *Chern character of differences of idempotents* $R(A) = P - e$, which would avoid the skew-symmetrisation of the index class.

6.3.1 Hilsum Index Formula

Hilsum [109] constructs, for any n-dimensional topological manifold endowed with an atlas of homeomorphisms possessing first order partial derivatives in L_p, $\frac{n(n+1)}{2} < p$, an unbounded Fredholm module which defines a class in the K-homology, whose Chern character is the Thom–Hirzebruch polynomial in the Pontrjagin classes of the manifold.

6.4 Index Theorem on Combinatorial Manifolds

This section continues the topics treated in Sect. 6.3.1. It illustrates the specific characteristics of the combinatorial structure. Specifically, one constructs a family of signature operators S_t depending on a parameter t and one shows that their index class converges, while $t \searrow 0$, to the *components of the Thom–Hirzebruch class*.

Remark 6.1 The construction uses exclusively the combinatorial structure.

Let M^{2l} be a closed oriented combinatorial manifold. We assume g to be a Riemannian metric on M which is affine on each simplex of M^{2l}. We assume these metrics agree on common faces of M^{2l}.

Choose an interior point p of a simplex σ of the triangulation and let $T_p(\sigma)$ be its tangent space. Extend the metric on $T_p(\sigma)$ by translation; denote this flat conformal structure μ_σ. Let S_σ be the corresponding signature operator defined on $T_p(\sigma)$, see (6.14).

We construct a *partition of unity* λ_t^σ having the following properties, where t is a parameter:

1. λ_t^σ is defined in terms of the metric μ_σ; the support of λ_t^σ is contained in a t-neighbourhood on the simplex $\sigma \in M^{2l}$
2. $Support\ (Lim_{t\searrow 0}\ \lambda_t^\sigma) = \sigma$
3. For points sufficiently far away from the lower dimensional boundary components of the simplex σ, the function λ_t^σ is invariant under translations of $T_p(\sigma)$ along σ.

We consider the operator

$$S_t = \sum_{\sigma \in M} \lambda_t^\sigma\ S_\sigma\ \lambda_t^\sigma. \tag{6.15}$$

The operator S_t, defined on M^{2l}, is a localisation of the signature operator S; $S^2 = 1 + finite\ rank\ operator$.

The operator K_t, whose construction is presented below, contains the desired M^{2l}-homological information. To define K_t one considers the operator $\theta_t = S_t^2 - 1$. θ_t belongs to the Lorentz space $\mathscr{L}^{2l,\infty}$, [101]. Let $q(x)$ be the unique polynomial of degree n such that

$$(1+x)^{-1/2} = q(x) + O(x^{n+1}),\quad t\ small.$$

Let $p(x)$ be a polynomial such that

$$p(x) \doteq (1+x)q(x)^2 - 1 = O(x^{n+1}).$$

The operator

$$H_t = \gamma p(\theta_t) + [1 - \frac{1+\gamma}{2} p(\theta_t)] q(\theta_t) S_t.$$

($\gamma = i^l *$) is the regularised operator S_t.

Theorem 6.6 *The operator*

$$K_t^\gamma \doteq H_t \gamma H_t + \gamma$$

is trace class; therefore, its kernel $k_t^\gamma(x, y)$ is integrable.

With the operator K_t^γ one associates the measure, as defined in [101, Theorem 2]

$$\tau_t^d(\gamma) \doteq trace(\wedge^{d+1} k_t(\gamma)).$$

This measure is skew-symmetric with respect to the orientation.

We construct a new measure which is independent of the orientation and which contains the same homology information as $\tau_t^d(\gamma)$

$$\tau_t^d \doteq \frac{1}{2}(\tau_t^d(\gamma) - \tau_t^d(-\gamma)).$$

Theorem 6.7

$$Lim_{t \searrow 0} trace(\wedge^{d+1} \tau_t^d) = L_d,$$

where L_d is simplicial chain of degree d on M. The convergence is in the sense of distributions, that is continuous functionals on the $(d+1)$-fold projective tensor product of the algebra A of piecewise differentiable functions on M^{2l}.

The homology class of the cycle L_d is $\lambda_d(L^{2l-d} \cap [M])$, where L^{2l-d} is the Hirzebruch-Thom L-class and where $\lambda_d = 2^{2m+1}(2\pi i)^{-m} \frac{m!}{(2m)!}$, where $2m = d$.

Lemma 6.1 *The measures $\tau_t^d(\mathcal{O})$ have the following properties:*

1. *its support is contained in a small tubular neighbourhood of the diagonal of M^{d+1}*
2. *for a small neighbourhood of the point $p \in \sigma$, $\tau_t^d(\mathcal{O})$ depends only on the geometric data around p*
3. *$\tau_t^d(-\mathcal{O}) = -\tau_t^d(\mathcal{O})$*
4. *one has*

$$\tau_t^d(\mathcal{O})(x_0, x_1, \ldots, x_d) = \frac{1}{t^{(d+1).2l}} \tau_t^d(\mathcal{O})(\frac{1}{t}.x_0, \frac{1}{t}.x_1, \ldots, \frac{1}{t}.x_d); \qquad (6.16)$$

here we have assumed $p = 0$.

The proof of the lemma is based on the notion of *Hodge equivalence*.

Definition 6.3 Let

$$h : (M_1,\ \mu_1,\ g_1, \mathscr{O}_1) \longrightarrow (M_2, \mu_2,\ g_2, \mathscr{O}_2)$$

be a combinatorial homeomorphism from the manifold M_1 to the manifold M_2, each equipped with structures as before. If h preserves these structures, we will say that h is a *Hodge equivalence*.

With the exception of the cut-off function, the construction of the measure $\tau_t^d(\mathscr{O})$ uses exclusively the conformal structure. For this reason the lemma could be proven following general arguments.

The next step of the proof consists of understanding, for each degree d, the role of the degree of the simplex σ with respect to d. For this part we use the next result.

6.4.1 Review of Orthogonal Invariants

Definition 6.4 Let V^q be an Euclidean vector space of dimension q. Let $O(V)$ be the group of its orthogonal transformations in V^q. The *diagonal action* of the group $O(q)$ on the direct sum $\oplus^d V$, of d copies of V, is given the action of the same orthogonal transformation on each summand of the direct sum.

A real polynomial function φ defined on $\oplus^d$ is called *odd invariant* over the orthogonal group iff for any $T \in O(V)$

$$\varphi(Tx_1, Tx_2, \ldots.., Tx_d) = |T|.\varphi(x_1, x_2, \ldots.., x_d),$$

where $|T| = \pm 1$, depending on whether T preserves or changes the orientation.

Theorem 6.8

1. *If $d < q$, then the only odd invariant polynomial is the zero polynomial.*
2. *If $d \geq q$, then any odd, invariant polynomial belongs to the ideal generated by the polynomials*

$$determinant\ [x_{i_1}, x_{i_2}, \ldots., x_{i_q}],$$

where $1 \leq i_1 < i_2 < i_3 < \ldots. < i_q \leq d$, see Weyl [46].

The measure $trace \wedge^{d+1} \tau_t^\sigma(\mathscr{O})$ is invariant at the diagonal action of the group of rotations in σ. This action is odd invariant, Definition 6.4. Let $dim(\sigma) = q$. Let F be a test function on $(M^{2l})^{d+1}$. A consequence of Lemma 6.1 is

1. if $d < q$ then $\int_M \ trace \wedge^{d+1} \tau_t^\sigma(\mathscr{O}) = o(t^\infty)$
2. if $d = q$ then

$$Lim_{t\searrow 0} \int_{M^{2l}} \ trace \wedge^{d+1} \tau_t^\sigma(\mathscr{O}) F(x)|dx| = L(\sigma) \int_{M^{2l}} dx_1 dx_2 \ldots dx_d F(x_0, x_1, \ldots, x_d)$$

3. if $d > q$ then $\quad Lim_{t\searrow 0} \int_{M^{2l}} \ trace \wedge^{d+1} d\tau_t^\sigma(\mathscr{O}) F(x)|dx| = 0.$

Once it has been proved that the limit exists, its value is determined as in the article [101].

Part IV
Prospects in Index Theory

Chapter 7
Algebraic Structures



Abstract In this chapter we introduce a new K-theory, called T_*-theory, and we reformulate the index theory.

7.1 Algebraic K_*-Theory

For a more expanded version of the topics presented here see [125]. These results lead in Chaps. 8 and 9 to further extensions of classical results.

The results of Sect. 1.8.1 may be extended (with limitations) to *arbitrary* rings. The corresponding K_*-theory is *algebraic K-theory*. For a complete treatment, the reader may consult the treaties by Milnor [66], Karoubi [89], Rosenberg [100] and Weibel [126].

Let $\mathscr{A}$ be a unital ring. The ring $\mathbb{M}_n(\mathscr{A})$ of $n \times n$ matrices with entries in $\mathscr{A}$, *idempotents*, invertible elements and stabilisation are defined as in Sect. 1.8.1.

Two matrices $m_1, m_2 \in \mathbb{M}_n(\mathscr{A})$ are called *conjugated* and we write $m_1 \sim_c m_2$, provided there exists an invertible matrix $u \in \mathbb{M}_n(\mathscr{A})$ such that $m_1 = u\, m_2\, u^{-1}$.

Definition 7.1 $K_0(\mathscr{A})$ is the *Grothendieck completion* of the set of equivalence classes of stabilised, conjugated idempotents over the algebra $(\mathscr{A})$.

$K_1(\mathscr{A})$ is the set of stabilised invertible elements of the ring $\mathscr{A}/[\mathscr{A}, \mathscr{A}]$, where $[\mathscr{A}, \mathscr{A}]$ is the multiplicative commutator sub-group. $K_1(\mathscr{A})$ is an Abelian group.

Definition 7.2 Let $\mathscr{A}$ be a unital ring and $\mathscr{J} \subset \mathscr{A}$ a bi-lateral ideal.

There corresponds the short exact sequence

$$0 \longrightarrow \mathscr{J} \xrightarrow{\iota} \mathscr{A} \xrightarrow{\pi} \mathscr{A}/\mathscr{J} \longrightarrow 0, \tag{7.1}$$

where ι is the inclusion and π the canonical mapping.

Theorem 7.1 *Let $\mathscr{A}$ be a unital ring and let $\mathscr{J} \subset \mathscr{A}$ a bi-lateral ideal.*

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Then there corresponds an exact sequence

$$\begin{array}{ccccc} K^1(J^+) & \to & K^1(A) & \to & K^1(B) \\ & & & & \partial \downarrow \\ K^0(B) & \leftarrow & K^0(A) & \leftarrow & K^0(J^+), \end{array} \tag{7.2}$$

where $\mathscr{J}^+$ is the unitised ring and where ∂ is defined below. Let $u \in \mathscr{A}/\mathscr{J}$ be an invertible element.

We define here the *connecting homomorphism* ∂. Let $u \in (M)_n(\mathscr{A})$ be an invertible element. Let u^{-1} be its inverse.

Let $A, B \in \mathbb{M}_n(\mathscr{A})$ be such that $\pi(A) = u$ and $\pi(B) = u^{-1}$.

We define $S_0 = 1 - BA$ and $S_1 = 1 - AB$. They belong to $\mathbb{M}_n(\mathscr{J})$. With the matrices A, B, S_0, S_1 one associates the invertible matrix [96]

$$L = \begin{pmatrix} S_0 & -(1+S_0)B \\ A & S_1 \end{pmatrix} \in \mathbb{GL}_{2n}(\Lambda_{\mathscr{A}}). \tag{7.3}$$

The inverse of the matrix L is

$$L^{-1} = \begin{pmatrix} S_0 & (1+S_0)B \\ -A & S_1 \end{pmatrix} \in \mathbb{GL}_{2n}(\mathscr{A}). \tag{7.4}$$

Let e_1, e_2 be the idempotents

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_{2n}(\mathscr{A}), \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{M}_{2n}(\mathscr{A}). \tag{7.5}$$

The invertible matrix L is used to produce the idempotent

$$P := L\, e_1\, L^{-1} = \begin{pmatrix} S_0^2 & S_0(1+S_0)B \\ S_1 A & 1 - S_1^2 \end{pmatrix} \in \mathbb{M}_{2n}(\mathscr{A}). \tag{7.6}$$

Definition 7.3 *Connecting homomorphism* for *K-theory* see [96].

For any $[u] \in K_1^{loc}(\mathscr{A})$ define the *connecting homomorphism* $\partial : K_1(\mathscr{A}) \longrightarrow K_0(\mathscr{A})$ by

$$\partial[u] := [P] - [(e_2] \in K_0^{loc}(\mathscr{A}). \tag{7.7}$$

7.2 *Local* Algebraic Structures

The index formula is a *local* statement made on global and local data. This is the reason why we introduce *local Alexander–Spanier cohomology*, *local periodic cyclic homology*, *local Chern character* and *local T_**-theory. Index theory should be

done: **Case 1**: for *arbitrary rings*, **Case 2**: for *rings of functions over topological manifolds*. Case 1 produces general index theorems, as for example, over *pseudo-manifolds*. Case 2 gives a general treatment of classical and non-commutative index theorems. All existing index theorems belong to the second category. The *tools of the theory* we propose would contain: *local* T_*-theory, *local periodic cyclic homology*, *local Chern character*. With regard to the tools used to express the index formula, there are *three stages*: *Stage I* done in T_i^{loc}-theory, *Stage II* is done in the local periodic cyclic homology and *Stage III* involves products of distributions, i.e. restriction to the diagonal. For each stage there corresponds a *topological index* and an *analytical index*. The *non-commutative topology* is mentioned in Part VI, Sect. 9.4.

7.2.1 K_*-Theory

The results of Sect. 1.8.1 may be extended (with limitations) to *arbitrary* rings. The corresponding K_*-theory is *algebraic K-theory*. For a complete treatment, the reader may consult the treaties by Milnor [66], Karoubi [89], Rosenberg [100] and Weibel [126].

Let $\mathscr{A}$ be a unital ring. The ring $\mathbb{M}_n(\mathscr{A})$ of $n \times n$ matrices with entries in $\mathscr{A}$, *idempotents*, invertible elements and stabilisation are defined as in Sect. 1.8.1.

Two matrices $m_1, m_2 \in \mathbb{M}_n(\mathscr{A})$ are called *conjugated* and we write $m_1 \sim_c m_2$, provided there exists an invertible matrix $u \in \mathbb{M}_n(\mathscr{A})$ such that $m_1 = u\, m_2\, u^{-1}$.

Definition 7.4 $K_0(\mathscr{A})$ is the *Grothendieck completion* of the set of equivalence classes of stabilised, conjugated idempotents over the algebra $(\mathscr{A})$.

$K_1(\mathscr{A})$ is the set of stabilised invertible elements of the ring $\mathscr{A}/[\mathscr{A}, \mathscr{A}]$, where $[\mathscr{A}, \mathscr{A}]$ is the multiplicative commutator sub-group. $K_1(\mathscr{A})$ is an Abelian group.

Definition 7.5 Let $\mathscr{A}$ be a unital ring and $\mathscr{J} \subset \mathscr{A}$ a bi-lateral ideal. There corresponds the short exact sequence

$$0 \longrightarrow \mathscr{J} \xrightarrow{\iota} \mathscr{A} \xrightarrow{\pi} \mathscr{A}/\mathscr{J} \longrightarrow 0, \tag{7.8}$$

where ι is the inclusion and π the canonical mapping.

Theorem 7.2 *Let $\mathscr{A}$ be a unital ring and let $\mathscr{J} \subset \mathscr{A}$ a bi-lateral ideal.*
Then there corresponds an exact sequence

$$\begin{array}{ccccc} K^1(J^+) & \to & K^1(A) & \to & K^1(B) \\ & & & & \partial \downarrow \\ K^0(B) & \leftarrow & K^0(A) & \leftarrow & K^0(J^+). \end{array} \tag{7.9}$$

where $\mathscr{J}^+$ is the unitised ring and where ∂ is defined below. Let $u \in \mathscr{A}/\mathscr{J}$ be an invertible element.

We define here the *connecting homomorphism* ∂. Let $u \in (M)_n(\mathscr{A})$ be an invertible element. Let u^{-1} be its inverse.

Let $A, B \in \mathbb{M}_n(\mathscr{A})$ be such that $\pi(A) = u$ and $\pi(B) = u^{-1}$.

We define $S_0 = 1 - BA$ and $S_1 = 1 - AB$. They belong to $\mathbb{M}_n(\mathscr{J})$. With the matrices A, B, S_0, S_1 one associates the invertible matrix [96]

$$L = \begin{pmatrix} S_0 & -(1+S_0)B \\ A & S_1 \end{pmatrix} \in \mathbb{GL}_{2n}(\Lambda_{\mathscr{A}}). \tag{7.10}$$

The inverse of the matrix L is

$$L^{-1} = \begin{pmatrix} S_0 & (1+S_0)B \\ -A & S_1 \end{pmatrix} \in \mathbb{GL}_{2n}(\mathscr{A}). \tag{7.11}$$

Let e_1, e_2 be the idempotents

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_{2n}(\mathscr{A}), \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{M}_{2n}(\mathscr{A}). \tag{7.12}$$

The invertible matrix L is used to produce the idempotent

$$P := Le_1L^{-1} = \begin{pmatrix} S_0^2 & S_0(1+S_0)B \\ S_1A & 1 - S_1^2 \end{pmatrix} \in \mathbb{M}_{2n}(\mathscr{A}). \tag{7.13}$$

Definition 7.6 *Connecting homomorphism* for *K-theory* see [96].

For any $[u] \in K_1^{loc}(\mathscr{A})$ define the *connecting homomorphism* $\partial : K_1(\mathscr{A}) \longrightarrow K_0(\mathscr{A})$ by

$$\partial[u] := [P] - [(e_2] \in K_0^{loc}(\mathscr{A}). \tag{7.14}$$

7.2.2 Localised Rings

Definition 7.7 (Localised Rings) Let $\mathscr{A}$ be a unital associative ring. The ring $\mathscr{A}$ is called *localised ring* provided it is endowed with an additional structure satisfying the axioms (1)–(3) below.

Axiom 1 The underlying space $\mathscr{A}$ has a decreasing filtration by sub-spaces $\{\mathscr{A}_\mu\}_{n\in\mathbb{N}} \subset \mathscr{A}$.

Axiom 2 $\mathbb{C}.1 \subset \mathscr{A}_\mu$, *for any* $\mu \in \mathbb{N}$

Axiom 3 For any $\mu,\ \mu' \in \mathbb{N}^+,\ \mathscr{A}_\mu \,.\, \mathscr{A}_{\mu'} \subset \mathscr{A}_{Min(\mu,\mu')-1}, \quad (\mathscr{A}_0 \,.\, \mathscr{A}_0 \subset \mathscr{A}_0)$.

Remark 7.1 A ring $\mathscr{A}$ could have different localisations.

7.2.3 Local *Alexander–Spanier Cohomology*

Definition 7.8 Let $\mathscr{A} = \mathscr{A}_\mu$ be a localised unital ring and G an Abelian group. Define

$$C^p_{AS}(\mathscr{A}_\mu,\ G) = \{ \sum_i g_i\ a^i_0 \otimes a^i_1 \otimes \cdots \otimes a^i_p, \quad g_i \in G, \quad a_i \in \mathscr{A}_\mu \}_{p=0,1,\ldots\infty}. \tag{7.15}$$

The boundary map d is defined as in the classical definition of Alexander–Spanier cohomology

$$d : C^p_{AS}(\mathscr{A}_\mu,\ G) \longrightarrow C^{p+1}_{AS}(\mathscr{A}_\mu,\ G) \tag{7.16}$$

$$d(g\ a_0 \otimes a^i_1 \otimes \cdots \otimes a_p) =$$
$$g\ [1 \otimes a_0 \otimes a^i_1 \otimes \cdots \otimes a_p - a_0 \otimes 1 \otimes a^i_1 \otimes \cdots \otimes a_p\ +\ \ldots\ +\ (-1)^{p+1} a_0 \otimes a^i_1 \otimes \cdots \otimes a_p \otimes 1]. \tag{7.17}$$

The *local* Alexander–Spanier cohomology is

$$H^{loc,p}_{AS}(\mathscr{A}) = ProjLim_\mu\ H_p(C^*_{AS}(\mathscr{A}_\mu,\ G),\ d). \tag{7.18}$$

7.2.4 Local *Periodic Cyclic Homology: Long Exact Sequence*

We assume that $\mathscr{A}_\mu$ is a localised unital ring.

The operators T, N, B, I, S presented in Sect. 2.4 pass to localised rings. Therefore *local cyclic homology*, *local periodic cyclic homology* may be extended to localised rings.

Theorem 7.3 *The* local Hochschild and cyclic homology *are well defined for a localised ring.*

One has the exact sequence (analogue of Connes' exact sequence, compare Sect. 2.4.5.)

$$\ldots \xrightarrow{B} H^{loc}_p(\mathscr{A}) \xrightarrow{I} H^{loc,\lambda}_p(\mathscr{A}) \xrightarrow{S} H^{loc}_{p-2}(\mathscr{A}) \xrightarrow{B} H^{loc}_{p-1}(\mathscr{A}) \xrightarrow{S} \ldots \tag{7.19}$$

Theorem 7.4 *For* $\mathscr{A} = C^\infty(M)$, *one has*

$$H^{loc}_p(C^{\infty(M)}) = H^{loc}_p(C^{\infty(M)}). \tag{7.20}$$

The *local bi-complex* $(b, B)(\mathscr{A}_\mu)$ is well defined for localised rings too. The general term of the $(b, B)(\mathscr{A}_\mu)$ bi-complex is

$$C_{p,q} = \otimes^{p-q+1}\mathscr{A}_\mu, \quad q \leq p \tag{7.21}$$

$$b : C_{p,q} \longrightarrow C_{p,q-1} \tag{7.22}$$

$$B : C_{p,q} \longrightarrow C_{p+1,q}. \tag{7.23}$$

Definition 7.9 The homology of the bi-complex $(b, B)(\mathscr{A}_\mu)$ has two components: the *even*, resp. *odd*, component corresponding to the $p - q = even\ number$, resp. $p - q = odd\ number$.

The *local periodic cyclic homology* of the localised ring $\mathscr{A}$ is

$$H^{loc,per,\lambda}_{ev),odd}(\mathscr{A}) := Lim Proj_\mu\ H_{ev,odd}H_*(\mathscr{A}_\mu). \tag{7.24}$$

7.2.5 Local *Chern Character* $T^{loc}_i \longrightarrow H^{loc,per,\lambda}_i$

In this section we localise the periodic even/odd Chern character. All operations involved in the construction (Sect. 3.26) may be localised; however, we should use the $(b, B)(\mathscr{A}_\mu)$ only onto *non degenerate* elements, i.e. onto the range of the idempotent Π, see Sect. 3.26.

Definition 7.10 (Local Periodic Cyclic Chern Character)
Let $\mathscr{A}_\mu$ be a localised ring.

1. For any idempotent $e \in \mathbb{M}_n(\mathscr{A}_\mu)$ define

$$Ch_{ev,\mathscr{A}_\mu}(e) = Tr\ e + \sum_{p=1}^{\infty}(-1)^p\frac{(2p)!}{p!}(e - \frac{1}{2})(de)^{2p}. \tag{7.25}$$

$Ch_{ev,\mathscr{A}_\mu}(e)$ is an even cycle in the $(b, B)(\mathscr{A}_\mu)$ cyclic complex. Its homology class is the *periodic cyclic Connes Chern character of* e.

Let $e \in T^{loc}_0(\mathscr{A})$; define

$$Ch_{ev}(e) = Proj Lim_\mu\ Ch_{ev,\mathscr{A}_\mu}(e) \in H^{loc,per,\lambda}_{ev}.(\mathscr{A}). \tag{7.26}$$

2. For any invertible $u \in \mathbb{M}_n(\mathscr{A}_\mu)$, define

$$Ch_{odd}(u) = \sum_p(-1)^{2p+1}(2p+1)!\ Tr\ (u^{-1}du)\ .\ (du^{-1}du)\ .\ \ldots(du^{-1}du). \tag{7.27}$$

Let $u \in T_1^{loc}(\mathscr{A})$; define

$$Ch_{odd}\ (u)\ =\ ProjLim_{\mu}\ Ch_{odd,\mathscr{A}_{\mu}}\ (u) \in H_{odd}^{loc,per,\lambda}(\mathscr{A}). \tag{7.28}$$

7.2.6 Local Index Theorem

In this section we consider an elliptic pseudo-differential operator, see Sect. 1.10.1. We are going to define the *Chern character of differences of idempotents*, without skew-symmetrisation. This solves the problem formulated in Sect. 6.3. For more details about this topic see Teleman [125, 128].

Lemma 7.1 *Let* $R(A) := P - e$, *where* P *and* e *are idempotents. Then* $R(A)$ *satisfies the identity*

$$R(A)^2\ =\ R(A)\ -\ (e\ .\ R(A)\ +\ R(A)\ .\ e). \tag{7.29}$$

Definition 7.11 Let $R(A) := P - e$, where P and e are idempotents. Let $\Lambda = \mathbb{C} + e\ .\ \mathbb{C}$. Then,

$$C_q\ (R(A)) := Tr\ \otimes_{\Lambda}^{2q+1}\ R(A) \tag{7.30}$$

is the *Chern character* of $R(A)$.

Lemma 7.2 Lemma 7.1 $C_q\ (R(A))$ *is a cycle in the complex* $\{\otimes_{\Lambda}^{*}\mathbb{M}_N(\mathscr{A})\}_*$, *such that*

$$\lambda\ .\ a_0 \otimes_{\Lambda} a_1 \otimes_{\Lambda} \cdots \otimes_{\Lambda} a_k\ =\ a_0 \otimes_{\Lambda} a_1 \otimes_{\Lambda} \cdots \otimes_{\Lambda} a_k\ .\ \lambda \tag{7.31}$$

for any $\lambda \in \Lambda$.

Proof One has

$$\begin{aligned} b'\ C_q\ (R(A))\ &=\ Tr\ [\ (R(A)\ -\ (e\ .\ R(A)\ +\ R(A)\ .\ e)) \otimes_{\Lambda} \cdots \otimes_{\Lambda} R(A) \ldots \\ &-R(A) \otimes_{\Lambda} R(A) \otimes_{\Lambda} \cdots \otimes_{\Lambda} (R(A)\ -\ (e\ .\ R(A)\ +\ R(A)\ .\ e))\ ]\ = \\ &Tr\ ;[\ -(e\ .\ R(A)\ +\ R(A)\ .\ e)) \otimes_{\Lambda} \cdots \otimes_{\Lambda} R(A) \ldots \\ &-R(A) \otimes_{\Lambda} R(A) \otimes_{\Lambda} \cdots \otimes_{\Lambda} (-(e\ .\ R(A)\ +\ R(A)\ .\ e)\ ] = \\ -Tr\ [\ e\ .\ R(A) \otimes_{\Lambda} \cdots \otimes_{\Lambda} R(A)\ &-\ R(A) \otimes_{\Lambda} \cdots \otimes_{\Lambda} R(A)\ .\ e\ ]\ =\ 0. \end{aligned} \tag{7.32}$$

$C_q\ (R(A))$ is a cyclic cycle. It represents a cyclic homology class in $H_*^{\lambda}(\otimes^{2q+1}(\mathscr{A}_{\mu}))$.

The ring Λ is separable, see definition (Sect. 3.15). Indeed the mapping

$$\mu : \Lambda \otimes \Lambda^{op} \longrightarrow \Lambda, \quad \mu(1 \otimes 1) = 1, \ \mu(1 \otimes e) = \mu(e \otimes 1) = e, \ \ \mu(e \otimes e) = e$$

has the Λ-bi-module splitting

$$s : \Lambda \longrightarrow \Lambda \otimes \Lambda^{op}, \quad s(1) \ = \ e \otimes e \ + \ (1 \ - \ e) \otimes (1 \ - \ e), \quad s(e) \ = \ e \otimes e.$$

Theorems 3.7, 3.19 states that the homology of the Hochschild complex is isomorphic to the homology of the complex $C^S(\mathscr{A})$. Using the long exact sequence in homology associated to the complex $H_*^\lambda(\otimes^{2q+1}(\mathscr{A}_\mu)$ and the Connes exact sequence, we find that the cyclic homology of $H_*^\lambda(\otimes^{2q+1}(\mathscr{A}_\mu)$ is isomorphic to the cyclic homology of the algebra $\mathscr{A}$. Here we have used the localisation of the algebra $\mathbb{M}_N(\mathscr{A})$ defined by the supports of the operators.

It remains to solve the problem of producing the *whole index class*. This could be obtained by the formula

$$Ch_{ev} \ = \ R(A) \ - \ Ch_{ev}\, e_{II}, \tag{7.33}$$

where $R(A)$ is the signature operator discussed in Sect. 1.13.

7.3 *Local* Algebraic T_i: Theory

This section deals with the program presented in [129]. For any associative ring $\mathscr{A}$ we define the *commutative groups* $T_i(\mathscr{A})$, $i \ = \ 0, 1$. We introduce the notion of *localised rings*, see Definition 7.7, $\mathscr{A} \ = \ \{\mathscr{A}_\mu\}$, given by a *linear filtration* of the algebra $\mathscr{A}$ and we associate the *commutative groups* $T_i^{loc}(\mathscr{A})$. Although we define solely $T_i^{loc}(\mathscr{A})$ for $i \ = \ 0, 1$, we expect our construction could be extended in higher degrees. *We stress that our construction of* $T_i(\mathscr{A})$ and $T_i^{loc}(\mathscr{A})$, i = 0, 1, *uses exclusively matrices*. The projective modules are totally avoided. The role of *equivalence of projective modules*, used in the classical construction of the algebraic K-theory, is played by *conjugation*. *The commutative group* $T_0(\mathscr{A})$ is by definition the *Grothendieck completion of the space of idempotent matrices factorised through the equivalence relations*: (1) *stabilisation* $\sim_s$, (2) *conjugation* $\sim_c$, and (3) for *localised groups*, $T_0^{loc}(\mathscr{A})$, *projective limit with respect to the filtration*, denoted $\sim_p$. The *non-localised* group $T_0(\mathscr{A})$ coincides with the classical algebraic K_0-theory group. The groups $T_1(\mathscr{A})$ are the quotient of $\mathbb{GL}(\mathscr{A})$ through the equivalence relation generated by (1) stabilisation $\sim_s$, (2) *conjugation* $\sim_c$ and (3) $\sim_{\mathbb{O}(\mathscr{A})}$, where $\mathbf{O}(\mathscr{A})$ is the *sub-group generated by elements of the form* $u \oplus u^{-1}$, *for any* $u \in \mathbb{GL}(\mathscr{A})$. For any $u_1, u_2 \in \mathbf{GL}(\mathscr{A}) \ / \ (\sim_s \cup \sim_c)$, one defines $u_1 \sim_{\mathbf{O}(\mathscr{A})} u_2$ provided there exist $\xi_1, \xi_2 \in \mathbb{O}(\mathscr{A})$ such that $u_1 \ + \ \xi_1 \ = \ u_2 \ + \ \xi_2$. The third relation is a particular case of a new completion procedure which we call *T*-

completion, see Sect. 7.3.8. The operation

$$\mathbb{GL}(\mathscr{A}) \,/\, (\sim_s \cup \sim_c) \longrightarrow \mathbb{GL}(\mathscr{A}) \,/\, (\sim_s \cup \sim_c \cup \sim_{\mathbf{O}(\mathscr{A})})$$

transforms the commutative *semi-group* $\mathbb{GL}(\mathscr{A}) \,/\, (\sim_s \cup \sim_c)$ in the commutative *group* $\mathbb{GL}(\mathscr{A}) \,/\, (\sim_s \cup \sim_c \cup \sim_{\mathbf{O}(\mathscr{A})})$. The groups $T_i^{loc}(\mathscr{A})$ follow the same construction as that of $T_i(\mathscr{A})$, provided the supports of the elements belong to $\mathscr{A}_\mu$. *Our definition of* $T_1(\mathscr{A})$ and $T_1^{loc}(\mathscr{A})$ *does not use the commutator sub-group* $[\mathbf{GL}(\mathscr{A}), \mathbf{GL}(\mathscr{A})]$ *or elementary matrices in its construction*. We define *short exact sequences of localised rings* and we get the corresponding open six terms exact sequence (Theorem 7.12). We stress that one has to take the tensor product of the expected six terms exact sequence by $\mathbb{Z}[\frac{1}{2}]$ in order to get the open six terms exact sequence. We expect the factor $\mathbb{Z}[\frac{1}{2}]$ to have important implications, among them, existence of a generator of the K-homology fundamental class and Kirby–Siebenmann obstruction class. This work shows that the basic relations which define T_1 and T_1^{loc} reside in the *additive* sub-group generated by elements of the form $u \oplus u^{-1}$, $u \in \mathbf{GL}(\mathscr{A})$, rather than in the *multiplicative* commutator sub-group $[\mathbf{GL}(\mathscr{A}), \mathbf{GL}(\mathscr{A})]$. Even for trivialy filtered algebras, $\mathscr{A} = \{\mathscr{A}_\mu\}$, for all $\mu \in \mathbb{N}$, the groups $T_1^{loc}(\mathscr{A})$ provides more information than the classical group $T_1(\mathscr{A})$. For the computation of the groups $T_i^{loc}(\mathbb{C})$ see Sect. 8.5 and [131].

7.3.1 Introductory Considerations

For the next considerations we have in mind the index formula applied onto the algebra of integral operators and pseudo-differential operators. Recall, the index formula is a *global statement* whose ingredients *may be computed by local data*. Our *leading idea* is to *localise* K-theory and periodic cyclic homology along the lines of the Alexander–Spanier cohomology, see [129].

We define *localised rings* $\mathscr{A}$ and we define and their *local* T-theory, $T_i^{loc}(\mathscr{A})$, $i = 0, 1$. Both the topological and analytic indices of an elliptic operator are stable under cutting the operators about the diagonal. This fact leads us naturally to the notion of *localised rings*. Based on this notion we pass to the problem of finding a natural K-theory able to control these *local entities*. These are the *local algebraic* T-theory groups, $T_i^{loc}(\mathscr{A})$, $i = 0, 1$ and *local cyclic homology*.

For the construction of the groups K_i the reader may refer to the books [66, 89, 100, 126], in the Banach algebras or C^*-algebras category, [108, 110].

Our construction of T_0^{loc} uses exclusively *idempotent matrices*. We avoid projective modules for the following reasons:

1. projective modules, in comparison with idempotent matrices, contain more ambiguity and
2. matrices are more suitable for controlling the *ring filtration data*; matrices are more prone to make calculations.

No reference to projective modules is used in our constructions.

$$K_1(\mathscr{A}) := \mathbf{GL}(\mathscr{A})/[\mathbf{GL}(\mathscr{A}), \mathbb{GL}(\mathscr{A})], \tag{7.34}$$

where $[\ \mathbb{GL}(\mathscr{A}), \mathbb{GL}(\mathscr{A})\ ]$ is the commutator normal sub-group of the group of invertible matrices $\mathbb{GL}(\mathscr{A})$. Our definition of *local* T-theory groups needs to keep track of the *number of multiplications* performed inside the algebra $\mathscr{A}$. In our case it is necessary to involve a *bounded* number of multiplications. *It is important to state that, in general, the number of multiplications needed to generate the whole commutator sub-group is un-bounded.*

It is known that the commutator sub-group is also generated by the *elementary* matrices. $T_0^{loc}(\mathscr{A})$ is by definition the Grothendieck completion of the semi-group of idempotent matrices in $\mathbb{M}_n(\mathscr{A}_\mu)$ modulo three equivalence relations: (i) stabilisation $\sim_s$, (ii) local conjugation $\sim_c$ by invertible elements $u \in \mathbb{GL}_n(\mathscr{A}_\mu)$ and (iii) projective limits with respect to $\mu \in \mathbb{N}$.

To understand the relationship between our definition of the group T_1^{loc} and K_1, recall that the commutator sub-group $[\mathbb{GL}(\mathscr{A}, \mathbb{GL}(\mathscr{A}]$ is given by arbitrary *products of multiplicative commutators*

$$[A, B] := A\ B\ A^{-1}\ B^{-1}, \quad for\ any\ A, B \in \mathbb{GL}_n(\mathscr{A}).$$

Supposing that A and B are conjugated, i.e. $A = U\ B\ U^{-1}$, we have

$$A = U\ B\ U^{-1} = U\ B\ U^{-1}\ .\ B^{-1}\ B = [U, B]\ .\ B.$$

This shows that if A and B are conjugated, they differ, multiplicatively, by a commutator. To complete this remark, we say that A and B are *locally conjugated* provided A, B and U belong to some $\mathbb{GL}_n(\mathscr{A}_\mu)$; here, $\mathscr{A}_\mu$ denote the terms of the filtration of $\mathscr{A}$.

We note that in the particular case $\mathscr{A} = \mathbb{C}$ the quotient of $\mathbb{GL}(\mathbb{M}(\mathbb{C}))$ through the commutator sub-group $\mathbb{GL}(\mathbb{M}(\mathbb{C}))$, $\mathbb{GL}(\mathbb{M}(\mathbb{C}))]$ gives much less information than $T_1^{loc}(\mathbb{M}(\mathbb{C}))$.

7.3.1.1 Factorisation of $u \otimes u^{-1}$

Proposition 7.1 *In the algebra of matrices over $\mathscr{A}$ one has the identity*

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.35}$$

This formula will play an important role in the construction of T_*-theory.

7.3.2 Algebraic T_i and T_i^{loc}-Theory

7.3.2.1 Introductory Considerations

Let $\mathscr{A}$ be a ring, with or without unit. If the unit will be needed, the unit will be adjoined.

Definition 7.12 Given the ring $\mathscr{A}$ we denote by $\mathbb{M}_n(\mathscr{A})$ the space of $n \times n$ matrices with entries in $\mathscr{A}$. $\mathbb{M}_n(\mathscr{A})$ is a bi-lateral $\mathscr{A}$-module.

Let $\mathbb{I}demp_n \subset \mathbb{M}_n(\mathscr{A})$ be the subset of idempotents p ($p^2 = p$) of size n with entries in $\mathscr{A}$.

Suppose $\mathscr{A}$ has a unit. We denote by $\mathbb{GL}_n(\mathscr{A})$ the sub-space of matrices M of size n, with entries in $\mathscr{A}$ which are invertible, i.e. there exists the matrix $M^{-1} \in \mathbb{M}_n(\mathscr{A})$ such that $M\ M^{-1} = M^{-1}\ M = 1$. $\mathbb{GL}_n(\mathscr{A})$ is a non-commutative group under multiplication.

Definition 7.13 The inclusions

(i) $\mathbb{I}demp_n(\mathscr{A}) \longrightarrow \mathbb{I}demp_{n+1}(\mathscr{A})$

$$p \mapsto p' = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \tag{7.36}$$

(ii) $\mathbb{GL}_n(\mathscr{A}) \longrightarrow \mathbb{GL}_{n+1}(\mathscr{A})$

$$u \mapsto u' = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \tag{7.37}$$

are called *stabilisations*.

(i) Stabilisation of idempotents and invertible matrices define two direct systems with respect to $n \in \mathbb{N}$.

(ii) If p or p' are idempotents and one of them is an iterated stabilisation of the other we write $p \sim_s p'$.

If u or u' are invertible matrices and one of them is an iterated stabilisation of the other we write $u \sim_s u'$.

7.3.3 Localised Rings

We recall the definition of *localised rings*.

Definition 7.14 Let $\mathscr{A}$ be a unital associative ring. The ring $\mathscr{A}$ is called *localised ring* provided it is endowed with an additional structure satisfying the axioms (1)–(4) below.

Axiom 1 The underlying space $\mathscr{A}$ has a decreasing filtration by sub-spaces $\{\mathscr{A}_\mu\}_{n\in\mathbb{N}} \subset \mathscr{A}$.
Axiom 2 $\mathbb{C}.1 \subset \mathscr{A}_\mu$, *for any* $\mu \in \mathbb{N}$.
Axiom 3 For any $\mu,\ \mu' \in \mathbb{N}^+,\ \mathscr{A}_\mu \,.\, \mathscr{A}_{\mu'} \subset \mathscr{A}_{Min(\mu,\mu')-1}, \quad (\mathscr{A}_0 \,.\, \mathscr{A}_0 \subset \mathscr{A}_0)$.

Definition 7.15 *Homomorphisms of localised rings. Induced homomorphism*

(i) A homomorphism from the localised ring $\mathscr{A} = \{\mathscr{A}_\mu\}_{\mu\in\mathbb{N}}$ to the localised ring $\mathscr{B} = \{\mathscr{B}_\mu\}_{\mu\in\mathbb{N}}$ is any ring homomorphism $\phi : \mathscr{A} \longrightarrow \mathscr{B}$ such that $\phi : \mathscr{A}_\mu \longrightarrow \mathscr{B}_\mu$, for any $\mu \in \mathbb{N}$.
(ii) Let $f : \mathscr{A} \longrightarrow \mathscr{B}$ be a localised ring homomorphism.

Let $f_* : \mathbb{M}_n(\mathscr{A}_\mu) \longrightarrow \mathbb{M}_n(\mathscr{B}_\mu)$ be the induced homomorphism which replaces any component a_{ij} of the matrix M with the component $f(a_{ij})$ of the matrix $f_*(M)$.

Remark 7.2 The notion of localised ring is different from the notion of *m-algebras* defined by Cuntz [115]. The sub-spaces $\mathscr{A}_n$ are not required to be *algebras*. In these cases the corresponding *local* T-theory could be not trivial.

Remark 7.3 In this section any localisation of the algebra of bounded operators on the Hilbert space $H := L_2(M \times M)$ derives from a *decreasing filtration* $\{\mathscr{U}_\mu\}$ towards the diagonal. In such a case, for $\mu^{'} > \mu$, one has the close inclusions

$$L_2(\mathscr{U}_{\mu'}) \subset L_2(\mathscr{U}_\mu). \tag{7.38}$$

For *any internal auto-morphism* of the algebra $\mathbb{B}(\mathscr{U}_{\mu'})$ induces an internal automorphism of the algebra $\mathbb{B}(\mathscr{U}_\mu)$ (any automorphism of the algebra $L_2(\mathscr{U}_\mu)$ will be extended by the identity on the complement of the Hilbert space $L_2(\mathscr{U}_{\mu'})$).

Remark 7.4 The immediate application of these considerations regards pseudo-differential operators on compact manifolds. The pseudo-differential operators on a compact smooth manifold form a *localised Banach algebra*. The filtration is defined in terms of the support of the operators; the bigger the filtration order is, the smaller the supports of the operators towards the diagonal are.

7.3.4 Local Mayer–Vietoris Diagrams

In this section we *adapt* Milnor's [66] description of the first two algebraic K-theory groups *to the case of localised rings*.

Let Λ, Λ_1, Λ_2 and Λ' be rings with unit 1 and let

$$\begin{array}{ccc} \Lambda & \xrightarrow{i_1} & \Lambda_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\ \Lambda_2 & \xrightarrow{j_2} & \Lambda' \end{array} \tag{7.39}$$

be a commutative diagram of homomorphisms of ring homomorphisms. All ring homomorphisms f are assumed to satisfy $f(1) = 1$. Any module in this book is a left module.

We assume the diagram satisfies the three conditions below.

Hypothesis 1 All rings and ring homomorphisms are localised, see Definition 7.7.

Hypothesis 2 Λ is a *local product* of Λ_1 and Λ_2, i.e. for any pair of elements $\lambda_1 \in \Lambda_{1,\mu}$ and $\lambda_2 \in \Lambda_{2,\mu}$ such that $j_1(\lambda_{1,\mu}) = j_2(\lambda_{2,\mu}) = \lambda' \in \Lambda'_\mu$, there exists only one element $\lambda_n \in \Lambda_\mu$ such that $i_1(\lambda_\mu) = \lambda_{1,\mu}$ and $i_2(\lambda_\mu) = \lambda_{2,\mu}$.

The ring structure in Λ is defined by

$$(\lambda_1, \lambda_2) + (\lambda'_1, \lambda'_2) := (\lambda_1 + \lambda'_1\ ,\ \lambda_2 + \lambda'_2),\quad (\lambda_1, \lambda_2)\ .\ (\lambda'_1, \lambda'_2) := (\lambda_1\ .\ \lambda'_1\ ,\ \lambda_2\ .\ \lambda'_2), \tag{7.40}$$

i.e. the ring operations in Λ are performed component-wise.

Hypothesis 3 At least one of the homomorphisms j_1 and j_2 is surjective.

Remark 7.5

(i) Let $M \in \mathbb{M}_n(\Lambda)$ be a matrix. It consists of a pair of matrices $(M_1, M_2) \in \mathbb{M}_n(\Lambda_1) \times \mathbb{M}_n(\Lambda_2)$ subject to the condition $j_{1,*}M_1 = j_{2,*}M_2$.
Any matrix $M \in \mathbb{M}_n(\Lambda)$ is called *double matrix*.

(ii) if (M_1, M_2), (N_1, N_2) are double matrices, belonging to (**resp.** $\mathbb{M}(\Lambda_1) \times \mathbb{M}(\Lambda_2)$) and $(\lambda_1, \lambda_2) \in \Lambda$, (**resp.** $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$) then relations (21.6) induce onto the space of double matrices, (**resp.** the space $\mathbb{M}(\Lambda_1) \times \mathbb{M}(\Lambda_2)$) the following operations:

$$(\lambda_1, \lambda_2)\ (M_1, M_2) = (\lambda_1\ M_1, \lambda_2\ M_2)$$

$$(M_1, M_2) + (N_1, N_2) = (M_1 + N_1,\ M_2 + N_2)$$

$$(M_1, M_2)\ .\ (N_1, N_2) = (M_1\ .\ N_1,\ M_2\ .\ N_2).$$

Definition 7.16 Any commutative diagram satisfying Hypotheses 1–3 will be called *local Mayer–Vietoris diagram.*

Standing Hypothesis In the remaining part of this chapter we assume that the ring $\mathscr{A}$ is localised. We assume also that $\Lambda_1 = \Lambda_2 = \mathscr{A}$ and that $J \subset \mathscr{A}$ is a bi-lateral ideal. Define $\Lambda := \{(\lambda_1, \lambda_2) \in \mathscr{A} \oplus \mathscr{A},\ such\ that\ \lambda_1 - \lambda_2 \in J\}$. Denote $i_\alpha(\lambda_1, \lambda_2) = \lambda_\alpha$, $\alpha = 1, or\ 2$. Denote also $\Lambda' := \mathscr{A}/J$ and $j_\alpha(\lambda_1, \lambda_2) := \lambda_\alpha\ mod.\ J$. One has the ring short exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{(i_1, i_2)} \Lambda_1 \oplus \Lambda_2 \xrightarrow{j_1 - j_2} \Lambda' \longrightarrow 0.$$

We suppose that the above scheme is a localised Mayer–Vietoris diagram.

7.3.5 Preparing the Definition of $T_0^{loc}(\mathscr{A})$ and $T_1^{loc}(\mathscr{A})$

Definition 7.17 We assume the ring $\mathscr{A}$ is localised.

We consider the space of *matrices with entries in* $\mathscr{A}_\mu$ and we denote it by $\mathbb{M}_n(\mathscr{A}_\mu)$.

We denote by $\mathbb{I}demp_n(\mathscr{A}_\mu)$ the space of idempotent matrices of size n with entries in $\mathscr{A}_\mu$.

We denote by $\mathbb{GL}_n(\mathscr{A}_\mu)$ the space of invertible matrices M of size n with the property that the entries of both M and M^{-1} belong to $\mathscr{A}_\mu$.

We denote by $\mathbb{I}demp(\mathscr{A}_\mu)$ the space of $\operatorname{inj\,lim}_n \mathbb{I}demp_n(\mathscr{A}_\mu)$.

We denote by $\mathbb{GL}(\mathscr{A}_\mu)$ the space of $\operatorname{inj\,lim}_n \mathbb{GL}_n(\mathscr{A}_\mu)$.

Definition 7.18

(i) Two matrices $s,\ t \in \mathbb{M}_n(\mathscr{A})$ are called *conjugated*, $s \sim_c t$, there exists $u,\ u^{-1} \in \mathbb{GL}_n(\mathscr{A})$ such that $s = u\ t\ u^{-1}$.

(ii) Two matrices are called $s,\ t \in \mathbb{M}_n(\mathscr{A}_\mu)$ will be called *locally conjugated* and we write $s \sim_{lc} t$, provided they are related through a *local* inner auto-morphism defined by $u,\ u^{-1} \in \mathbb{GL}_n(\mathscr{A}_\mu)$ such that $s = u\ t\ u^{-1}$.

In particular

(ii.1) two *idempotents* $p,\ q \in \mathbb{I}demp_n(\mathscr{A}_\mu)$ are *locally conjugated* and we write $p \sim_{lc} q$ provided there exists $u, u^{-1} \in \mathbb{GL}_n(\mathscr{A}_\mu)$ such that $q = u\ p\ u^{-1}$ and

(ii.2) two *invertible matrices* $s,\ t \in \mathbb{GL}_n(\mathscr{A}_\mu)$ are *locally conjugated* and we write $s \sim_{lc} t$ provided there exists $u, u^{-1} \in \mathbb{GL}_n(\mathscr{A}_\mu)$ such that $s = u\ t\ u^{-1}$.

If it is clear from the context, $\sim_{lc}$ will be simply denoted by $\sim_c$.

Proposition 7.2

(i) the spaces $\mathbb{I}demp_n(\mathscr{A}_\mu)$ *and* $\mathbb{GL}_n(\mathscr{A}_\mu)$ *are semi-groups with respect to the direct sum*

$$A + B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{7.41}$$

(ii) The stabilisation, conjugation and local conjugation of idempotents, resp. invertible matrices, commute.

The spaces $\mathbb{I}demp_n(\mathscr{A})$, $\mathbb{GL}_n(\mathscr{A}_\mu)$ are stable under stabilisations.

It is clear that the direct sum addition of idempotents, resp. invertibles, is compatible with the local conjugation equivalence relation. In fact, if s_1, s_2 are conjugated through an inner auto-morphism defined by the element u_1 ($s_1 \sim_l s_2$) and t_1, t_2 are conjugated through an inner auto-morphism defined by the element u_2 ($t_1 \sim_l t_2$), then $(s_1 + t_1) \sim_l (s_2 + t_2)$ are conjugated through the inner auto-morphism $u_1 \oplus u_2$.

Based on this observation, the associativity of the addition is now immediate.

From this it follows that $\mathbb{I}demp(\mathscr{A}_\mu)\ /\ \sim_{lc}$, resp. $\mathbb{GL}(\mathscr{A}_\mu)\ /\ \sim_l$, are associative semi-groups.

Proposition 7.3 *It follows that the semi-groups* $\mathbb{I}demp(\mathscr{A})\ /\ \sim_c$, $\mathbb{GL}(\mathscr{A}_\mu)\ /\ \sim_{lc}$ *are commutative.*

Proof One has the identity $A,\ B \in \mathbb{M}_n(\mathscr{A}_\mu)$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}, \tag{7.42}$$

which tells that $(A\ +\ B) \sim_c (B\ +\ A)$, resp. $(A\ +\ B) \sim_{lc} (B\ +\ A)$.

For more information on the classical algebraic K-theory and the *local* T-theory, see Sect. 7.3.1.

7.3.6 Definition of $T_0(\mathscr{A}_\mu)$ and $T_0^{loc}(\mathscr{A})$

We suppose the stabilisation is involved without being specified.

Definition 7.19 We assume $\mathscr{A}$ is a localised unital associative ring. We define

(i) $T_0(\mathscr{A}_\mu)$ to be $\mathbb{G}\ (\mathbb{I}dem p_n\ (\mathscr{A}_\mu)\ /\ \sim_c)$.
(ii) and $T_0(\mathscr{A})$ to be $ProjLim_{\mu\in\mathbb{N}}\ T_0(\mathscr{A}_\mu)$, where **G** means *Grothendieck completion*.

Proposition 7.4 *Any local inner auto-morphism induces the identity on* $T_0^{loc}(\mathscr{A})$.

For any unital associative algebra $\mathscr{A}$*, trivially localised* $\mathscr{A}_\mu\ =\ \mathscr{A}$*), one has* $T_0(\mathscr{A})\ =\ K_0(\mathscr{A})$.

Proof See Rosenberg [100, Lemma 1.2.1 and Theorem 1.2.3].

7.3.7 $T_1(\mathscr{A}_\mu)$ and $T_1^{loc}(\mathscr{A})$

As said in the previous sub-section, we assume the stabilisation is involved without being specified.

The equivalence class of the invertible element $u \in \mathbb{GL}(\mathscr{A})$ modulo conjugation, $[u]_{\sim_c}$, will be called the *abstract Jordan canonical form* of u. The group $T_1(\mathscr{A}_\mu)$ we are going to define preserves much of the information provided by the abstract Jordan form. The classical definition of K_1 extracts a minimal part of the abstract Jordan form. As the addition in the semi-group $\mathbb{GL}(\mathscr{A})$ is given by the direct sum and the Jordan canonical form $J(u)$ (in the classical case of the algebra $\mathbb{GL}_n(\mathbb{R})$) J behaves additively $(J(u \oplus v)\ =\ J(u) \oplus J(V)$ modulo permutations of the

Jordan blocks), given an arbitrary element $u \in \mathbb{GL}(\mathscr{A})$, it is not reasonable to expect existence of an element $\tilde{u}$ such that $[u+\tilde{u}]_{\sim_c} = [1_{2n}]_{\sim_c}$. Given that we want $T_1(\mathscr{A}_\mu)$ to be a group, we introduce the group structure (opposite elements) forcibly. In the case of the classical K_1-theory, the class of the element u^{-1} represents the opposite class, $-[u] \in K_1(\mathscr{A})$. In our case, the opposite elements will be introduced by means of the *T-completion* technique, to be explained next.

7.3.8 T-Completion

Proposition 7.5 *Let $\mathscr{S}$ be an additive commutative semi-group with zero element* 0. *Let $I : \mathscr{S} \longrightarrow \mathscr{S}$ be an* additive involution, *($I^2 = Id$), such that $I(0) = 0$.*

Define the equivalence relation $\sim_{\mathscr{O}}$ in $\mathscr{S}$: $u \sim_{\mathscr{O}} v$ iff these exist two elements $u_0, u_1 \in \mathscr{S}$, such that the elements u, v, $\xi_0 = u_0 + I(u_0)$ and $\xi_1 = u_1 + I(u_1)$, satisfy

$$u + \xi_0 = v + \xi_1. \tag{7.43}$$

(i) The equivalence relation $\sim_{\mathscr{O}}$ is compatible with the addition in $\mathscr{S}$.
(ii) $\mathscr{S}/\sim_{\mathscr{O}}$ is a group.
(iii) Let $[u]$ denote the $\sim_{\mathscr{O}}$-equivalence class of u. Then

$$-[u] = [\, I(u)\,]. \tag{7.44}$$

Definition 7.20 Define

$$\mathscr{O}(u) := u + I(u) \quad for\ any\ u \in \mathscr{S} \tag{7.45}$$

$$O(\mathscr{S}) := \{\mathscr{O}(u) \mid u \in \mathscr{S}\}. \tag{7.46}$$

Proof

(i) Obvious.
(ii) $\sim$ is clearly reflexive and symmetric. If $u_1 + \xi_1 = u_2 + \xi_2$ and $u_2 + \xi_3 = u_3 + \xi_4$, $u_1, u_2, u_3, u_4 \in O(\mathscr{S})$, then $u_1 + (\xi_1 + \xi_3) = u_2 + (\xi_2 + \xi_3) = u_3 + (\xi_2 + \xi_4)$, which shows that $\sim$ is transitive. Therefore $\sim$ is an equivalence relation.
(iii) One the other side, for any element $u \in \mathscr{S}$, one has $[u] = [u + 0] = [u] + [0] = [0] + [u]$, which shows that the class of [0] is a the zero element of $\mathscr{S}/\sim_{\mathscr{O}}$.

We have also $[u] + [I(u)] = [\mathscr{O}(u)] = [0 + \mathscr{O}(u)] = [0]$ because $0 \sim (0 + \mathscr{O}(u))$; therefore, $-[u] = [I(u)]$ exists. From this we get further $[\mathscr{O}(u)] = [u + I(u)] = [u] + [I(u)] = 0$.

7.3.9 *Definition of* $T_1(\mathscr{A}_\mu)$ *and* $T_1^{loc}(\mathscr{A})$

The construction of the groups $T_1(\mathscr{A}_\mu)$ and $T_1^{loc}(\mathscr{A})$ involves the T-completion (Sect. 7.3.8). We take

- $\mathscr{S} = \mathbb{GL}(\mathscr{A}_\mu) \sim_c$
- the involution $I : \mathbb{GL}(\mathscr{A}_\mu) \sim_{cl} \longrightarrow \mathbb{GL}(\mathscr{A}_\mu) \sim_{cl}$ is given by $I(u) = u^{-1}$
- the identity element $I \in \mathbb{GL}(\mathscr{A}_\mu)/\sim_c$ becomes the zero element of $\mathscr{S}$.

Here $\sim_l$ indicates that the invertible elements and their conjugation occurs locally. Note that if $u_1 \sim_{cl} u_2$, then $u_1^{-1} \sim_{cl} u_2^{-1}$.

The next proposition summarises the properties of $\mathscr{O}(\mathscr{A}_\mu)$.

Proposition 7.6

(i) The space $\mathscr{O}(\mathscr{A}_\mu)$ is a commutative group; the zero element is the class if the identity.

(ii) The mapping

$$\mathscr{O} : \mathbb{GL}(\mathscr{A}_\mu) \longrightarrow \mathscr{O}(\mathscr{A}_\mu) \tag{7.47}$$

is additive and commutes with local *conjugation*

$$\mathscr{O}(u_1 + u_2) = \mathscr{O}(u_1) + \mathscr{O}(u_2) \tag{7.48}$$

$$\mathscr{O}(\lambda\, u\, \lambda^{-1}) = \lambda\, \mathscr{O}(u)\, \lambda^{-1}, \quad \lambda \in \mathbb{GL}(\mathscr{A}_\mu), \tag{7.49}$$

i.e. if $u_1 \sim_{sl} u_2$ then $\mathscr{O}(u_1) \sim_{sl} \mathscr{O}(u_2)$

(iii)

$$\mathscr{O}(u^{-1}) \sim_c \mathscr{O}(u). \tag{7.50}$$

(iv)

$$\mathscr{O}(u_1 u_2) \neq \mathscr{O}(u_1)\mathscr{O}(u_2). \tag{7.51}$$

Definition 7.21

$$T(\mathscr{A}_\mu) := T - completion\ of\ \mathbb{GL}(\mathscr{A}_\mu)\ /\ \sim_{lc} \tag{7.52}$$

and

$$T^{loc}(\mathscr{A}) := \operatorname*{proj\,lim}_{\mu \in \mathbb{N}} T_1(\mathscr{A}_\mu). \tag{7.53}$$

Proposition 7.7

(i) $T_1(\mathscr{A}_\mu)$ *and* $T_1^{loc}(\mathscr{A})$ *are commutative groups.*
(ii) The inverse element of $[u]$ *is* $-[u] = [u^{-1}]$.

Proof (i) and (ii) follow from the properties of the T-completion, see Sects. 9.8 and 7.3.8.

7.3.10 Induced Homomorphisms

Definition 7.22 Let $f : \mathscr{A} \longrightarrow \mathscr{B}$ be a localised ring homomorphism.
Then the ring homomorphism f induces homomorphisms

$$f_* : T_0(\mathscr{A}_\mu) \longrightarrow T_0(\mathscr{B}_\mu), \qquad f_* : T_0^{loc}(\mathscr{A}) \longrightarrow T_0^{loc}(\mathscr{B}) \tag{7.54}$$

and

$$f_* : T_0(\mathscr{A}_\mu) \longrightarrow T_0(\mathscr{B}_\mu), \qquad f_* : T_1^{loc}(\mathscr{A}) \longrightarrow T_1^{loc}(\mathscr{B}). \tag{7.55}$$

7.3.11 Constructing Idempotents and Invertible Matrices Over Λ_μ

We returned to the situation presented in Sect. 7.3.4. Here we produce *local idempotents and invertibles* of the ring Λ. These results will be used in the proof of the *six terms exact sequence Theorem* 7.12.

Theorem 7.5 (Definition) *Let* $p_1 \in \mathbb{I}dem p_n(\Lambda_{1,\mu})$ *be an idempotent matrix, resp.* $p_2 \in \mathbb{I}dem p_n(\Lambda_{2,\mu})$, *with the property*

$$j_{1*}(p_1) = u\ j_{2*}(p_2)\ u^{-1}, \tag{7.56}$$

where $u \in \mathbb{GL}_n(\Lambda'_\mu)$ *is an invertible matrix.*

(i) Under these conditions there exists an idempotent double matrix $p \in \mathbb{I}dem p(\Lambda_\mu)$ *such that*

$$i_{1*}(p) = p_1 \oplus 0_n \ \text{ and } \ i_{2*}(p) = \tilde{p}. \tag{7.57}$$

The idempotent $\tilde{p}_2 \in \mathbb{M}_{2n}(\Lambda_{2,\mu})$ is conjugated to $p_2 \oplus 0_n$ through an invertible matrix $\tilde{U}(u) \in \mathbb{GL}_{2n}(\Lambda_{2,\mu})$, that is

$$\tilde{p}_2 \ = \ \tilde{U}(\ u\ )(\ p_2 \oplus 0_n\ )\tilde{U}(u)^{-1} \tag{7.58}$$

$$j_{2,*}(\ \tilde{p}_2\ ) \ = \ (\ u\ j_{1,*}(p_1)\ u^{-1}\ ) \oplus 0_n \ = \ j_{1,*}(\ p_1 \oplus 0_n\ ). \tag{7.59}$$

(ii) The double matrix idempotent is denoted $p \ = \ (\ p_1,\ p_2,\ \tilde{U}(u)\ )$.

Definition 7.23 Let $\tilde{U}(u)$ be the lifting of $u \otimes u^{-1}$ in $\Lambda_{2,\mu}$ (7.35).

Let $p \ = \ (\ p_1,\ p_2,\ \tilde{U}(u)\ )$ the idempotent over Λ_μ produced by Theorem 7.5.

The idempotent p doubles in size.

Condition says that $[\ j_{1*}\ p_1\ ] \ = \ [\ j_*\ p_2\ ] \in T_0(\Lambda'_\mu)$ (7.57). The first part (i) says that the pair $([\ p_1 \oplus 1_n\ ], [\ \tilde{p}_2]\ ) \in T_0(\Lambda_1) \oplus T_0(\Lambda_2)$ belongs to the image of $(\ i_{1*}, i_{2*}\ )$.

Proof We will operate onto objects related to $\Lambda_{2,\mu}$.

The next lemma is needed.

Lemma 7.3 *Let $p_1 \ = \ (a_{ij}) \in \mathbb{I}dem p_n(\Lambda_{1,\mu})$ and $p_2 \ = \ (b_{ij}) \in \mathbb{I}dem p_n(\Lambda_{2,\mu})$ be idempotents.*

We suppose the idempotents $j_{1*}(p_1)$, $j_{2*}(p_2)$ are conjugated through an inner auto-morphism defined by $u \in \mathbb{GL}_n(\Lambda'_\mu)$, i.e.

$$j_{1*}(p_1) \ = \ u\ j_{2*}(p_2)\ u^{-1}. \tag{7.60}$$

Assume, that the invertible element u lifts to an invertible element $\tilde{u} \in \mathbb{GL}_n(\Lambda_{2,\mu})$ (i.e. $j_{2*}\tilde{u} \ = \ u$).

Then $p \ = \ (p_1, p'_2, u) \in \mathbb{I}dem p_n(\Lambda_\mu)$ is an idempotent given by the double matrix

$$p \ = \ ((a_{ij}, c_{ij})), \tag{7.61}$$

where

$$(a_{ij}) \ = \ p_1 \in \mathbf{I}dem p_n(\Lambda_{1,\mu}) \ \ and \ \ p'_2 \ = \ (c_{ij}) := \ \tilde{u}\ p_2\ \tilde{u}^{-1} \in \mathbb{I}dem p_n(\Lambda_{2,\mu}). \tag{7.62}$$

Remark that in this lemma the size of the double matrix p does not change.

Proof of Lemma 7.2 We use Remark 7.5(ii). It is clear that the matrix p given by (8.30) is an idempotent. In fact, to evaluate p^2 amounts to compute separately the square of the first and second component matrices of the matrix p, i.e. the squares

of (a_{ij}) and (c_{ij}). These are

$$(a_{ij})^2 \;=\; (a_{ij}) \;\; and \;\; (c_{ij})^2 \;=\; (\,\tilde{u}\; p_2\; \tilde{u}^{-1}\,)^2 \;=\; \tilde{u}\; p_2^2\; \tilde{u}^{-1} \;=\; \tilde{u}\; p_2\; \tilde{u}^{-1} \;=\; (c_{ij}). \tag{7.63}$$

It remains to verify that $p \in \mathbb{M}_n(\Lambda_\mu)$, i.e. $j_{1*}(a_{ij}) \;=\; j_{2*}(c_{ij})$. This follows from (8.29) combined with (8.31)

$$j_{1*}(p_1) \;=\; u\; j_{2*}(p_2)\; u^{-1} \;=\; j_{2*}(\tilde{u}\; p_2\; \tilde{u}^{-1}) \;=\; j_{2*}(\tilde{p}_2). \tag{7.64}$$

This ends the proof of Lemma 7.3.

Lemma 7.4 *Suppose* $p_1 \;=\; (a_{ij}) \in \mathbb{I}demp_n(\Lambda_{1,\mu})$ *and* $p_2 \;=\; (b_{ij}) \in \mathbb{I}demp_n(\Lambda_{2,\mu})$ *are idempotents.*

Suppose additionally that the idempotents $j_{1*}(p_1)$, $j_{2*}(p_2)$ *are conjugated through an inner auto-morphism defined by* $u \in \mathbb{GL}_n(\Lambda_\mu^{'})$, *i.e.*

$$j_{1*}(p_1) \;=\; u\; j_{2*}(p_2)\; u^{-1}. \tag{7.65}$$

Then it follows that

(i) $j_{1*}(p_1 \oplus 0_n)$ *and* $j_{2*}(p_2 \oplus 0_n)$ *are conjugated by* $U := u \oplus u^{-1} \in \mathbf{GL}_{2n}(\Lambda_\mu')$, *i.e.*

$$j_{1*}(p_1) \oplus 0_n \;=\; j_{1*}(p_1 \oplus 0_n) \;=\; U\; j_{2*}(p_2 \oplus 0_n)\; U^{-1} \;=\; (\,u\; j_{2*}(p_2)\; u^{-1}\,) \oplus 0_n. \tag{7.66}$$

(ii) Suppose that j_2 *is surjective; the invertible matrix* U *lifts to an invertible matrix* $\tilde{U} \in \mathbb{M}_{2n}(\Lambda_{2,\mu})$. *Let*

$$\tilde{p}_2 := \tilde{U}\; (p_2 \oplus 0_n)\; \tilde{U}^{-1}. \tag{7.67}$$

Then

$$j_{1,*}(p_1 \oplus 0_n) \;=\; (u\; j_{2,*}(p_2)\; u^{-1}) \oplus 0_n \;=\; j_{2,*}(\tilde{p}_2); \tag{7.68}$$

i.e. the matrices $p_1 \oplus 0_n$, $\tilde{p}_2$ *form a double matrix idempotent in* $\mathbb{I}demp_{2n}(\Lambda_\mu)$, *denoted* $p := (p_1, p_2, u) \in \mathbb{I}demp_{2n}(\Lambda_\mu)$ *and*

$$(i_{1*}, i_{2*})p \;=\; (p_1, \tilde{p}_2). \tag{7.69}$$

(iii) The idempotents $p_1 \in \mathbb{I}demp_n(\Lambda_{1,\mu})$, $p_2 \in \mathbb{I}demp_n(\Lambda_{2,\mu})$ *is stably equivalent to the pair of idempotents* $p_1 \oplus 0_n$, $p_2 \oplus 0_n$ *and* $p_2 \oplus 0_n \sim_l \tilde{p}_2$. *In other words*

$$(\,[p_1], [p_2]\,) \;=\; ([p_1], [\tilde{p}_2]) \in T_0(\Lambda_{1,\mu}) \oplus T_0(\Lambda_2, \mu). \tag{7.70}$$

Note that in this lemma, in comparison with the preceding Lemma 7.3 the size of the desired idempotent doubles; otherwise, the important modifications still occur onto matrices associated with $\Lambda_{2,\mu}$.

Proof of Lemma 7.4 Part (i) is clear.

The proof of (ii) uses $\mathscr{O}_{2n}(u)$, where $u \in \mathbb{GL}_n(\mathscr{A}_\mu)$. Recall that $\mathscr{O}_{2n}(u)$ may be written as a product of *elementary matrices* and a *scalar matrix* (7.35)

$$U := \mathscr{O}_{2n}(u) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbf{M}_{2n}(\Lambda_{2,\mu}). \tag{7.71}$$

Having in mind the localisation of the ring $\mathscr{A}$, it is important to note that the entries of the formula (7.71) depend on u and u^{-1} only.

The proof will be complete after we will have shown that the invertible matrix U has an invertible lifting $\tilde{U} \in \Lambda_{2,\mu}$. This follows from the properties of elementary matrices (valid for matrices and block matrices) to be discussed next.

Definition 7.24 (Elementary Matrices) The matrix $E_{ij}(a) \in GL_n(\mathscr{A}_\mu)$ having all entries equal to zero, except for the diagonal entries equal to 1 and just one (i, j)-entry $a \in \mathscr{A}_\mu$, $0 \leq i \neq j \leq n$ is called *elementary matrix with entry in* $\mathscr{A}_\mu$.

Definition 7.25 The space of *elementary matrices with entries in* $\mathscr{A}_\mu$ is by definition

$$E_n(\mathscr{A}_\mu) := \{E_{ij}(a) \mid 1 \leq i \neq j \leq n,\ a \in \mathscr{A}_\mu\}. \tag{7.72}$$

Let $\mathbf{E}_n(\mathscr{A})$ the sub-group generated by all elementary matrices and

$$\mathbf{E}(\mathscr{A}) := \operatorname*{inj\,lim}_{n \in \mathbf{N}} \mathbf{E}_n(\mathscr{A}). \tag{7.73}$$

Lemma 7.5 *The elementary matrices satisfy*

$$E_{i,j}(a)\ .\ E_{i,j}(b) = E_{i,j}(a+b) \tag{7.74}$$

$$(E_{i,j}(a))^{-1} = E_{i,j}(-a), \tag{7.75}$$

therefore $E_{i,j}(a) \in \mathbb{GL}(\mathscr{A}_\mu)$. *Any elementary matrix is a commutator*

$$E_{ij}(A) = [E_{ik}(A),\ E_{kj}(1)], \quad \textit{for any } i, j, k \textit{ distinct indices}. \tag{7.76}$$

We return to the proof of Lemma 7.4(ii). We assumed j_2 is surjective; therefore, each of the entries of factors of the RHS of (7.71) has a lifting in $\mathbf{M}_{2n}(\Lambda_{2\mu})$; each of the elementary matrix factor lifts as an invertible elementary matrix. The last factor

lifts as it is. Therefore, U has an invertible lifting $\tilde{U} \in \mathbb{GL}_{2n}(\Lambda_{2,\mu})$. Lemma 7.3 completes the proof of Lemma 7.4(ii).

(iii) follows from the definition of $T_0^{loc}(\Lambda)$. This completes the proof of Lemma 7.4.

Theorem 7.5 follows from Lemma 7.3 combined with Lemma 7.4.

Theorem 7.5 refers to the construction and description of idempotents over Λ_μ. We need to extend Theorem 7.5 to formal differences of *local* idempotents.

Lemma 7.6 (Compare [66, Lemma 1.1]) *Let $p_1, p_2\ q_1, q_2 \in \mathbb{I}dem p_n(\mathscr{A}_\mu)$ be idempotents. Let [ ] denote their $T_0(\mathscr{A}_\mu)$ class. Assume also*

$$[p_1] - [p_2] = [q_1] - [q_2] \in T_0(\mathscr{A}_\mu). \tag{7.77}$$

Then $p_1 + q_2$ and $p_2 + q_1$ are locally, stably *isomorphic, $p_1 + q_2 \sim_{ls} p_2 + q_1$.*

Proof The stabilisation and Grothendieck completion imply that there exists an idempotent $s \in \mathbb{I}dem p_m(\mathscr{A}_\mu)$ such that the idempotents

$$p_1 + q_2 + s, \quad p_2 + q_1 + s$$

are *locally, stably* isomorphic. We assume that the idempotent s is already sufficiently stabilised. This means there exists an invertible matrix $u \in \mathbb{GL}_{2n+m}(\mathscr{A}_\mu)$ such that

$$p_1 + q_2 + s, = u\,(p_2 + q_1 + s)\,u^{-1}.$$

We add to both sides of this equality the idempotent $1_m - s$ and we extend u to be the identity on the last summand. We get further

$$p_1 + q_2 + s + (1-s), = u\,(p_2 + q_1 + s + (1-s))\,u^{-1}.$$

From this we get further

$$p_1 + q_2 + 1_{2m}, = u\,(p_2 + q_1 + 1_{2m})\,u^{-1},$$

that is, the idempotents $p_1 + q_2, p_2 + q_1$ are *locally, stably* isomorphic

$$p_1 + q_2 \sim_{sl} p_2 + q_1.$$

Lemma 7.7 *Let $p, q \in \mathbb{I}dem p_n(\mathscr{A}_\mu)$ be idempotents. Suppose*

$$[p] - [1_n] = [q] - [1_n] \in T_0(\mathscr{A}_\mu). \tag{7.78}$$

Then

(i) p *and* q *are* locally, stably *isomorphic,* $p \sim_{ls} q$,
(ii) there exists an $N \in \mathbb{N}$ *and an* $u \in \mathbb{GL}_{n+N}(\mathscr{A}_\mu)$ *such that*

$$p \ + \ 1_N = u \ q \ u^{-1} \ + \ 1_N = u \ (q \ + \ 1_N) \ u^{-1}. \tag{7.79}$$

Proof

(i) Lemma 7.7(i) says that the idempotents $p \ + \ 1_n$, $q \ + \ 1_n$ are *locally, stably* isomorphic. This means that the idempotents p and q are *locally, stably* isomorphic. Part (ii) tells precisely this.

Theorem 7.6 *Let*

$$[p_1] \ = \ [p_{11}] \ - \ [p_{12}] \in T_0(\Lambda_{1,\mu}) \tag{7.80}$$

$$[p_2] \ = \ [p_{21}] \ - \ [p_{22}] \in T_0(\Lambda_{2,\mu}), \tag{7.81}$$

where π_{ij} *are idempotents.*
Suppose also that

$$j_{1*}[p_1] \ = \ j_{2*}[p_2] \in T_0(\Lambda'_\mu).$$

Then there exists the idempotents p_{01} , p_{02}

$$[p] \ = \ [p_{01}] \ - \ [p_{02}] \in T_0(\Lambda_\mu) \tag{7.82}$$

such that

$$i_{1*}[p] \ = \ [p_1] \ \textit{and} \ \ i_{2*}[p] \ = \ [p_2]. \tag{7.83}$$

Proof The two T-theory classes may be described differently

$$[p_1] \ = \ [p_{11}] \ - \ [p_{12}] \ = \ [p_{11} \ + \ (1 \ - \ p_{12})] \ - \ [p_{12} \ + \ (1 \ - \ p_{12})] \ = \ [p'_{12}] \ - \ [1_n] \in T_0(\Lambda_{1,\mu}) \tag{7.84}$$

$$[p_2] \ = \ [p_{21}] \ - \ [p_{22}] \ = \ [p_{21} \ + \ (1 \ - \ p_{22})] \ - \ [p_{22} \ + \ (1 \ - \ p_{22})] \ = \ [p'_{22}] \ - \ [1_n] \in T_0(\Lambda_{2,\mu}). \tag{7.85}$$

From this will follow

$$j_{1*}[p_1] \ = \ j_{1*}([p'_{12}] \ - \ [1_n]) \ = \ (j_{1*}[p'_{12}]) \ - \ [1_n] \tag{7.86}$$

$$j_{2*}[p_2] \ = \ j_{2*}([p'_{22}] \ - \ [1_n]) \ = \ (j_{2*}[p'_{22}]) \ - \ [1_n]. \tag{7.87}$$

The hypotheses say that

$$(j_{1*}[p'_{12}]) - [1_n] = (j_{2*})[p'_{22}]) - [1_n]. \tag{7.88}$$

Lemma 7.7 implies that the idempotents $j_{1*}[p'_{12}]$ $j_{2*}[p'_{22}]$ are *locally, stably* isomorphic. We are in the position to use Theorem 7.5. Let $u \in \mathbb{GL}(\Lambda_2(\mathscr{A}_\mu))$ and consider the conjugation

$$j_{1*}(p'_{12}) = u\ j_{2*}(p'_{22})\ u^{-1}. \tag{7.89}$$

Theorem 7.5 provides the idempotent

$$p = (j_{1*}(p'_{12}), (j_{2*}(p'_2), u). \tag{7.90}$$

The desired idempotents are

$$p_{10} = p = (j_{1*}(p'_{12}), (j_{2*}(p'_2), u) \in \mathbb{I}demp(\Lambda_\mu) \tag{7.91}$$

$$p_{20} = 1_N \in \mathbb{I}demp(\Lambda_\mu). \tag{7.92}$$

7.3.11.1 Constructing Invertibles Over Λ_μ

Theorem 7.7 (Definition) *Suppose $s_1 \in \mathbb{GL}_n(\Lambda_{1,\mu})$ and $s_2 \in \mathbb{GL}_n(\Lambda_{2,\mu})$ are invertible matrices with entries in $\Lambda_{1,\mu}$, resp. $\Lambda_{2,\mu}$, such that*

$$j_{1,*}(u_1) = u\ j_{2,*}(u_2)\ u^{-1}, \quad where\ u \in \mathbb{GL}_n(\Lambda'_\mu). \tag{7.93}$$

(i) Then there exists an invertible matrix $s \in \mathbb{GL}_{2n}(\Lambda_\mu)$ such that

$$i_1(s) = s_1 \oplus 1_n \ \ and \ \ i_2(s) = \tilde{s}_2, \tag{7.94}$$

where the invertible matrix $\tilde{s}_2 \in \mathbb{GL}_{2n}(\Lambda_{2,\mu})$ is conjugated to s_2 through an inner auto-morphism $\tilde{U} \in \mathbb{GL}(\Lambda_{2,\mu})$. This means

$$\tilde{s}_2 = \tilde{U}\ (s_2 \oplus 1_n)\ \tilde{U}^{-1} \tag{7.95}$$

$$j_{2,*}\, s_2 = (u\ j_{2,*}\, (s_2)\ u^{-1}) \oplus 1_n = j_{1,*}\, (s_1 \oplus 1_n). \tag{7.96}$$

(ii) The matrix s is denoted $s = (s_1, s_2, \tilde{U})$.

Proof The proof of Theorem 7.7 follows the same lines as the proof of Theorem 7.5. Theorem 7.5 is based on the following facts: (a) operations with double matrices respect Remark (b) lifting of the invertible element $U = \mathscr{O}_{2n}(u) := u \oplus u^{-1} \in \mathbb{GL}_{2n}(\Lambda_{2,\mu})$ $(u \in \mathbb{GL}_n(\Lambda'_{2,\mu}))$ by means of the factorisation of U by elementary

matrices, see (7.35), and (c) the fact that any inner auto-morphism keeps invariant any zero vector sub-space.

To prove Theorem 7.7 we use the same elements (a)–(c) with the following changes: idempotents are of course replaced by invertible elements and for (c) we use the fact that the inner auto-morphism transforms the mapping 1_n into itself. This ends the proof of the theorem.

The next theorem is the analogue of Theorem 7.6 in the $T_1(\mathscr{A}_\mu)$ case.

Theorem 7.8 *Let j_1 and j_2 be epimorphisms.*
Suppose $[s_1] \in T_1^{loc}(\Lambda_{1,\mu})$ and $[s_2] \in T_1^{loc}(\Lambda_{2,\mu})$ have the property that

$$j_{1*}[s_1] \;=\; j_{2*}[s_2] \in T_1^{loc}(\Lambda'_\mu). \tag{7.97}$$

Then there exists $[s] \in T_1^{loc}(\Lambda_\mu)$ such that

$$i_{1*}[\,s\,] \;=\; [\,s_1\,] \in T_1^{loc}(\Lambda_{1,\mu}) \tag{7.98}$$

and

$$i_{2*}[\,s\,] \;=\; [\,s_2\,] \in T_1^{loc}(\Lambda_{2,\mu}). \tag{7.99}$$

Proof The definition of $T_1^{loc}(\Lambda'_\mu)$ involves an ambiguity belonging to the sub-module $\mathbf{O}(\Lambda'_\mu)$. We can make the assumption that the elements s_1 and s_1 are sufficiently stabilised. Equality (7.97) tells there exist two elements $\xi_1, \xi_2 \in \mathbf{O}_{2n}(\Lambda'_\mu)$ with the property that the invertible matrices $j_{1*}(s_1) + \xi_1$, $j_{2*}(s_2) + \xi_2$ are *locally conjugated* by means of a matrix $u \in \mathbf{GL}_{2n}(\Lambda')$

$$j_{1*}(s_1) \;+\; \xi_1 \;=\; u\,(j_{2*}(s_2) \;+\; \xi_2)\,u^{-1}. \tag{7.100}$$

We split the problem of finding the element s in two parts

(1) find $\tilde{\xi}_1 \in \mathscr{O}(\Lambda_{1,\mu})$, resp. $\tilde{\xi}_2 \in \mathscr{O}(\Lambda_{2,\mu})$, as lifts of the elements ξ_1, resp. ξ_2,
(2) apply Theorem 7.8 with s_1, resp. s_2, replaced by $s_1 + \tilde{\xi}_1$, resp. $s_2 + \tilde{\xi}_2$ and the invertible element u.
(1) The first lift is obtained in two steps: (i) we find invertible lifts $\tilde{\xi}_1 \in \Lambda_{1,\mu}$, resp. $\tilde{\xi}_2 \in \Lambda_{2,\mu}$, of the elements ξ_1, resp. ξ_2 , and we verify that (ii) such lifts belong to $\mathbf{O}_{2n}(\Lambda_{i,\mu})$, $i = 1, 2$.

The lifts of the elements ξ_i belong to $\mathbf{O}_{2n}(\Lambda_{i,\mu})$. The lifted elements will not change the corresponding T-completion classes. The proof continues through the next section.

7.3.11.2 Lifting of $\mathbb{O}(\Lambda'_\mu)$

We show the procedure on the element ξ_1; for ξ_2 we use the same procedure.

The element ξ_1 has the form

$$\xi_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix} = \alpha_1 \oplus \alpha_1^{-1} \in \mathbf{O}_{2n}(\Lambda'_\mu). \tag{7.101}$$

We use the factorisation (7.35) to decompose ξ_1

$$\xi_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.102}$$

The homomorphism j_1 being an epimorphism, there exists $\tilde{\alpha}_1$, resp. $\tilde{\beta}_1 \in \Lambda_{1,\mu}$ such that $j_1(\tilde{\alpha}_1) = \alpha_1$, resp. $j_1(\tilde{\beta}_1) = \alpha_1^{-1}$. The lifted element is

$$\bar{\xi}_1 = \begin{pmatrix} 1 & \tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tilde{\beta}_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{7.103}$$

$$j_1(\bar{\xi}_1) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix} = \xi_1.$$

We use formula (7.35) and property (7.75) of elementary matrices to find the inverse of $\bar{\xi}_1$

$$\bar{\xi}_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{\beta}_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix}. \tag{7.104}$$

We have

$$j_1(\bar{\xi}_1^{-1}) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix}^{-1} \sim_{cl} \xi_1^{-1}$$

because j_1 is a unital ring homomorphism.

The lifted element is

$$\tilde{\xi}_1 = \tilde{\xi}_1 \oplus \tilde{\xi}_1^{-1} \in \mathbb{M}_{4n}(\Lambda_{1,\mu}).$$

7.3.11.3 Back to the Proof of Theorem 7.8

(2) The elements $s_1 + \tilde{\xi}_1 \in \mathbb{GL}_{4n}(\Lambda_{1,\mu})$, $s_2 + \tilde{\xi}_2 \in \mathbb{GL}_{4n}(\Lambda_{2,\mu})$ and the invertible element u satisfy the relation

$$j_{1,*}(s_1 + \tilde{\xi}_1) = u\,(s_2 + \tilde{\xi}_2)\,u^{-1}. \tag{7.105}$$

Theorem 7.8 follows from Theorem 7.7.

7.3.12 $K_1(\mathscr{A})$ vs. $T_1(\mathscr{A})$

The following identities are well known. They are used as building blocks of K-theory, see [31, 66, 89, 100, 108, 110, 126].

We compare properties the classical algebraic K_1-theory with properties of T_1.

Theorem 7.9

(i) The group $K_1(\mathscr{A})$ is

$$K_1(\mathscr{A}) := \operatorname*{inj\,lim}_{n\in\mathbb{N}} \mathbb{GL}_n(\mathscr{A})/[\mathbb{GL}_n(\mathscr{A}), \mathbb{GL}_n(\mathscr{A})]; \tag{7.106}$$

it is called the Whitehead group.

The group structure in $K_1(\mathscr{A})$ is given by matrix multiplication *and* direct sum addition

$$[A] + [B] := [\ A\ .\ B\ ]. \tag{7.107}$$

Therefore, $T_1(\mathscr{A})$ is the set of Jordan canonical forms of matrices over $\mathscr{A}$ modulo $\mathscr{C}(\mathscr{A})$. The sum *in the group $T_1(\mathscr{A})$ is the direct sum.*

(ii) Any commutator is stably isomorphic to a product of elements of the form $\mathscr{C}_{2n}(\mathscr{A})$. More specifically, for any $A, B \in \mathbb{GL}(\mathscr{A}_\mu)$ we have the formula

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}. \tag{7.108}$$

(iii) If $A \in \mathbb{GL}_n(\mathscr{A}_\mu)$, then (see (7.35))

$$\mathscr{C}_{2n}(A) := \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7.109}$$

(iv) Any elementary matrix is a commutator

$$E_{ij}(A) \;=\; [\; E_{ik}(A), E_{kj}(1)\;], \quad for\ any\ i, j, k\ distinct\ indices. \tag{7.110}$$

(iv) Let $A,\ B \in \mathbb{GL}_n(\mathscr{A}_\mu)$. *Then* $A+B$ *is stably equivalent to* AB *and* BA modulo multiplicatively *elements of the form* $\mathscr{O}_{2n}(\mathscr{A})$

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} BA & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.111}$$

(v) For any two elements $A,\ B \in \mathbb{GL}_n(\mathscr{A}_\mu)$ *one has the identity*

$$\begin{pmatrix} ABA^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}^{-1}. \tag{7.112}$$

Theorem 7.10 (See [31, 37, 43, 66, 89, 100])

(i) $[\mathbb{GL}(\mathscr{A}), \mathbb{GL}_n(\mathscr{A})] \;=\; \mathbb{E}_n(\mathscr{A})$.
(ii) $[ABA^{-1}B^{-1}] \;=\; 0 \in K_1(\mathscr{A})$.
(iii.1) $[A] \;+\; [B] \;=\; [\; A\; B\;] \;=\; [\; B\; A\;] \;=\; [B] \;+\; [A] \in K_1(\mathscr{A})$.
(iii.2) $[A] \;+\; [B] \;=\; [B] \;+\; [A] \in T_1^{loc}(\mathscr{A})$. *Therefore* $T_1^{loc}(\mathscr{A})$ *is an Abelian group.*
(iv)

$$[\mathscr{O}_{2n}(\mathscr{A})] \;=\; [1_n] \;=\; 0\ in\ K_1(\mathscr{A}),$$

$$T_1(\mathscr{A})\ and\ T_1^{loc}(\mathscr{A}).$$

(v) $-[A] \;=\; [A^{-1}]\ in\ K_1(\mathscr{A})\ , T_1(\mathscr{A})\ and\ T_1(\mathscr{A})^{loc}$.
(vi) $[ABA^{-1}] \;=\; [B] \in K_1(\mathscr{A})$.

Proof

(i) The relation (7.108) says that any commutator is a product of matrices of type $\mathscr{O}_n(\mathscr{A})$. Formula (7.35) says that any matrix $\mathscr{O}_n(\mathscr{A})$ is a product of *elementary matrices* and a scalar matrix (7.35). This proves the property (i).
(ii) This property follows from the definition of $K_1(\mathscr{A})$.
(iii) (iii.1) and (iii.2) follow from the definitions.
(iv) It is clear that $[1] = [1].[1]$. From this one obtains that the *zero element* of the group K_1 is represented by the class of the element 1.

In the definition of $T_1(\mathscr{A})$ and $T_1(\mathscr{A})$, the elements $\mathscr{O}(\mathscr{A})$ represent the zero element of these groups.

(v) This property follows from (iv).
(vi) This property follows from formula (ii) in $K_1(\mathscr{A})$

$$A\ B\ A^{-1} = (\ A\ B\ A^{-1}\ B^{-1}\ )\ B = 0 + B = B. \tag{7.113}$$

The next remark shows the specific parts of the construction of T_1^{loc} make our construction different from the classical one.

Remark 7.6

(i) In our next construction the factorisation of the elements of $\mathbb{GL}(\mathscr{A})$ through the commutator sub-group (or the sub-group generated by elementary matrices) is avoided because *the number of products needed to generate these sub-groups might be un-bounded*. Here we have in mind the algebra of integral operators, or pseudo-differential operators. Any product increases the size of the support; an un-bounded number of products does not allow a support control.

In the whole chapter we do not use more than *three products* of elements in the algebra. Products matrices of replaced by sums and direct sums. The size of matrices replaces the need to perform multiple products. In our definition of $T_1^{loc}(\mathscr{A})$—where products could not be avoided—the corresponding increase in the size of the supports is absorbed by the projective limit.

(ii) To observe, our construction of T_1 uses the elements $\mathscr{O}(\mathscr{A})$ *additively* and not *multiplicatively*. In the classical construction of $K_1(\mathscr{A})$, the elements of $\mathscr{O}(\mathscr{A})$ are used multiplicatively. To generate the commutator sub-group it is necessary to perform an *un-bounded number of products*. In our construction an un-bounded number of products is not allowed.

(iii) Our construction of $T_1^{loc}(\mathscr{A})$ uses the factorisation of $\mathbb{GL}(\mathscr{A})$ through the smaller sub-group of inner auto-morphisms. Observe, the class of an invertible element u modulo inner auto-morphisms is nothing but the abstract Jordan form $J(u)$. Our construction of T_1 contains more information than the class of the invertible element modulo the commutator sub-group.

Suppose $\mathscr{A}$ were the algebra of complex matrices and $u \in \mathbb{M}(\mathbb{C})$. Then $J(u)$ could be identified with the Jordan canonical form of the matrix u. The Jordan canonical form of the matrix $u \oplus u^{-1}$ is precisely $J(u) \cup J(u^{-1})$. If u is not the identity, it is clear that $u \oplus u^{-1}$ could never be conjugated to the identity element.

Suppose $\mathbb{C}$ is the algebra of complex numbers endowed with the trivial filtration. In this case, for the computation of the groups $T_i^{loc}(\mathbb{C})$, see [131] and Sect. 22.

For the definition of $T_1^{loc}(\mathscr{A})$, it natural to consider the quotient space of $\mathbb{GL}(\mathscr{A})$ modulo the equivalence relation $\sim_{sl}$. We use the factorisation (7.35) of the *additive sub-group* $\mathscr{O}(\mathscr{A})$. This factorisation shows that the Jordan canonical forms of the elements u and u^{-1} are opposite one to each other in T_1^{loc}. The *additive* group generated by elements $\mathscr{O}(\mathscr{A})$ is contained in the

commutator sub-group. This property insures the fact that there exists a natural epimorphism from $T_1^{loc}(\mathscr{A})$ to $K_1(\mathscr{A})$.

(iv) It does not appear that the factorisation through the sub-group $\mathscr{O}(\mathscr{A})$ would kill much information. To see this, suppose $u, v \in \mathbb{GL}_n(\mathscr{A})$. Then, $(u \oplus u^{-1}) \sim_l (v \oplus v^{-1})$ if and only if $(u \oplus 1_n) \sim_l (v \oplus 1_n)$.

(v) The projective limit (Alexander–Spanier type construction, made possible by the filtration $\mathscr{A}_\mu$ of the algebra), makes the algebraic T_i^{loc}-theory richer than the classical K_i-theory, $i = 0, 1$.

Theorem 7.11 *Let $\Pi : T_1^{loc}(\mathscr{A}) \longrightarrow K_1(\mathscr{A})$ be the* canonical epimorphism. *Then*

$$Ker\ \Pi\ =\ [\ \mathbf{GL}(\mathscr{A}),\ \mathbf{GL}(\mathscr{A})\ ]\ /\ Inner(\mathscr{A}). \tag{7.114}$$

7.3.13 Connecting Homomorphism $\partial : T_1^{loc}(\Lambda') \longrightarrow T_0^{loc}(\Lambda) \otimes \mathbb{Z}[\frac{1}{2}]$

In this section we assume that the diagram (Sect. 7.3.4) satisfies Hypotheses 1–3. We define here the *connecting homomorphism* $\partial : T_1^{loc} \longrightarrow T_0^{loc} \otimes \mathbb{Z}[\frac{1}{2}]$.

The reader will notice that the construction of ∂ involves from the very beginning idempotents.

Definition 7.26 Let $[\ u\ ] \in T_1(\Lambda'_\mu)$. Recall that the elements of $T_1(\Lambda'_\mu)$ are equivalence classes of invertible matrices modulo $\sim_{\mathscr{O}_{\mathscr{A}_\mu}}$; we may assume that $u \in \mathbb{GL}_n(\Lambda'_\mu)$.

Define the *connecting homomophism* $\partial : T_1(\Lambda'_\mu) \longrightarrow T_0(\mathscr{A}_\mu) \otimes \mathbb{Z}[\frac{1}{2}]$

$$\partial\ [\ u\ ]\ =\ [\ p(1_n,\ 1_n,\ \tilde{U}(u))\ ]\ -\ [\ \Lambda^n(\mathscr{A}_\mu)\ ], \tag{7.115}$$

where $\tilde{U}_1(u)$ is obtained through the decomposition of $u \oplus u^{-1}$ as in (7.35), and lifted in $\Lambda_{1,\mu}$ as in (7.103).

Proposition 7.8 *Suppose the homomorphisms j_1, j_2 are epimorphisms. Then the connecting homomorphism is well defined.*

Proof The connecting homomorphism is additive. We have to show that $\partial\ (\ u \oplus u^{-1}\ )\ =\ 0$.

To do this we decompose $u \oplus u^{-1}$

$$\begin{aligned} u \oplus u^{-1}\ &=\ \frac{1}{2}\ (\ (\ u \oplus u^{-1}\ )\ \oplus\ (\ u \oplus u^{-1})\ )\ = \\ &\Lambda_{1,\mu} \oplus \Lambda_{1,\mu} \oplus \Lambda_{2,\mu} \oplus \Lambda_{2,\mu}. \end{aligned} \tag{7.116}$$

We lift the two components in $\Lambda_{1,\mu}$, resp. $\Lambda_{1,\mu}$. For this purpose we use the formula (7.35). The lifted elements are

$$\tilde{\xi} \;=\; \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{M}_{2n}(\mathscr{A}). \tag{7.117}$$

where $j_i(\alpha) \;=\; u$ and $j_i(\beta) \;=\; u^{-1}$. The elements α and β exist because the mappings j_1, j_2, are epimorphisms.

These elements produce invertible matrices as they are products of invertible matrices. In addition, they produce a double matrix of the Mayer–Vietoris diagram. We have

$$\tilde{\xi}\ (\mathscr{A}^{2n})\ \tilde{\xi}^{-1} \;=\; \mathscr{A}^{2n}. \tag{7.118}$$

From this we get

$$\partial\ (\ u \otimes u^{-1}\ ) \;=\; \mathscr{A}^{2n} \otimes \mathscr{A}^{2n} \;-\; \mathscr{A}^{4n} \;=\; 0. \tag{7.119}$$

7.3.14 Six Terms Exact Sequence

Theorem 7.12 *The following sequences are exact*

(i)

$$T_0^{loc}(\Lambda) \overset{(i_1,i_2)}{\longrightarrow} T_0^{loc}(\Lambda_1) \oplus T_0^{loc}(\Lambda_2) \overset{j_{1*}-j_{2*}}{\longrightarrow} T_0^{loc}(\Lambda') \tag{7.120}$$

(ii)

$$T_1^{loc}(\Lambda) \overset{(i_{1*},i_{2*})}{\longrightarrow} T_1^{loc}(\Lambda_1) \oplus T_1^{loc}(\Lambda_2) \overset{j_{1*}-j_{2*}}{\longrightarrow} T_1^{loc}(\Lambda') \tag{7.121}$$

(iii)

$$T_1^{loc}(\Lambda_1) \oplus T_1^{loc}(\Lambda_2) \overset{j_{1*}-j_{2*}}{\longrightarrow} T_1^{loc}(\Lambda') \overset{\partial}{\longrightarrow}$$
$$T_0^{loc}(\Lambda) \otimes \mathbb{Z}[\frac{1}{2}] \overset{(i_{1*},i_{2*})}{\longrightarrow} (T_0^{loc}(\Lambda_1) \otimes \mathbb{Z}[\frac{1}{2}]) \oplus (T_0^{loc}(\Lambda_2) \otimes \mathbb{Z}[\frac{1}{2}]). \tag{7.122}$$

Therefore, the following six terms sequence is exact

$$T_1^{loc}(\Lambda) \stackrel{(i_{1*},i_{2*})}{\longrightarrow} T_1^{loc}(\Lambda_1) \oplus T_1^{loc}(\Lambda_2) \stackrel{j_{1*}-j_{2*}}{\longrightarrow} T_1^{loc}(\Lambda') \stackrel{\partial}{\longrightarrow}$$
$$T_0^{loc}(\Lambda) \otimes \mathbb{Z}[\frac{1}{2}] \stackrel{(i_{1*},i_{2*})}{\longrightarrow} T_0^{loc}(\Lambda_1) \otimes \mathbb{Z}[\frac{1}{2}] \oplus T_0^{loc}(\Lambda_2) \otimes \mathbb{Z}[\frac{1}{2}] \stackrel{j_{1*}-j_{2*}}{\longrightarrow} T_0^{loc}(\Lambda') \otimes \mathbb{Z}[\frac{1}{2}] \tag{7.123}$$

Proof

(i) **(i.1)** $Im(i_{1*}, i_{2*}) \subset Ker(j_{1*} - j_{2*})$.
Easy to verify.

(i.2) $Ker(j_{1*} - j_{1*}) \subset Im(i_{1*}, i_{2*})$
We are going to verify it. If p is an idempotent we convene to write $\bar{p} := 1_n - p$.
Suppose $([p_1] - [p_2]\ ,\ [q_1] - [q_2]) \in T_0^{loc}(\Lambda_{1,\mu}) \oplus T_0^{loc}(\Lambda_{2,\mu})$ be such that

$$0 = j_{1,*}([p_1] - [p_2]) - j_{2,*}([q_1] - [q_2]), \tag{7.124}$$

where p_1, p_2, q_1, q_2 are idempotents. The pair of T-theory classes may be re-written

$$0 = j_{1,*}([p_1 + \bar{p}_2] - [p_2 + \bar{p}_2]) - j_{2,*}([q_1 + \bar{q}_2] - [q_2 + \bar{q}_2]) \tag{7.125}$$

or

$$0 = j_{1,*}([p_1 + \bar{p}_2] - [1_r]) - j_{2,*}([q_1 + \bar{q}_2] - [1_s]). \tag{7.126}$$

By adding to all sides a trivial idempotent of sufficiently large size, we may assume that $r = s$. This new relation may be re-written

$$0 = j_{1,*}([p_1 + \bar{p}_2] - j_{2,*}([q_1 + \bar{q}_2]). \tag{7.127}$$

This implies that there exists an idempotent $\xi \in \mathbf{I}demp_N(\Lambda'_\mu)$ such that the idempotents

$$(j_{1,*}([p_1 + \bar{p}_2] + \xi) - (j_{2,*}([q_1 + \bar{q}_2] + \xi) \tag{7.128}$$

are isomorphic. We add the idempotent $\bar{\xi} := 1_N - \xi$; we to get isomorphic idempotents

$$(j_{1,*}([p_1 + \bar{p}_2] + \xi + \bar{\xi}),\ (j_{2,*}([q_1 + \bar{q}_2] + \xi + \bar{\xi}) \in \mathbb{I}dem p_{\bar{N}}(\Lambda'_\mu) \tag{7.129}$$

We have

$$(j_{1,*}([p_1 + \bar{p}_2] + 1_N),\ \ (j_{2,*}([q_1 + \bar{q}_2] + 1_N). \tag{7.130}$$

This implies there exists $u \in \mathbb{GL}_{\bar{N}}(\Lambda')$ which conjugates these two idempotents.

From Theorem 7.6 we know that there exits the idempotent $p \in \mathbb{I}dem p_{2\bar{N}}(\Lambda_\mu)$ such that

$$i_{1*}p = (p_1 + \bar{p}_2 + 1_N) \oplus 1_{\bar{N}} \tag{7.131}$$

and

$$i_{2*}p = U\ ((q_1 + \bar{q}_2 + 1_N) \oplus 1_{\bar{N}})\ U^{-1} \tag{7.132}$$

where

$$U \in \mathbb{GL}_{2\bar{N}}(\Lambda'_\mu). \tag{7.133}$$

From here we deduce the image of the T-theory class of $p - 1_{\bar{N}} \in \Lambda_\mu$ through the pair of homomorphisms $(i_{1,*}, i_{2,*})$ is

$$\begin{gathered}(i_{1,*}, i_{2,*})[p - 1_{\bar{N}}] = \\ (\ (p_1 + \bar{p}_2 + 1_N) \oplus 1_{\bar{N}} - 1_{\bar{N}},\ U\ ((q_1 + \bar{q}_2 + 1_N) \oplus 1_{\bar{N}})\ U^{-1} - 1_{\bar{N}}\ ) = \\ (\ (p_1 + \bar{p}_2 + 1_N) \oplus 1_{\bar{N}} - 1_{\bar{N}},\ U\ ((q_1 + \bar{q}_2 + 1_N) \oplus 1_{\bar{N}} - 1_{\bar{N}})\ U^{-1} = \\ = ([p_1 - p_2], [q_1 - q_2],\end{gathered} \tag{7.134}$$

which completes the proof of the part (**i**).

(**ii**) (**ii.1**) We have to prove $Im(i_1, i_2)_* \subset Ker(j_{1,*} - j_{1,*})$.

Suppose $u = (u_1, u_2) \in \mathbb{GL}(\Lambda_\mu)$ belongs to $Im(i_1, i_2)_*$. Then according to the definition of Λ_μ, u may be chosen so that $j_1(u_1) = j_2(u_2)$ and hence $[u] \in Ker(j_{1,*} - j_{2,*})$.

(**ii.2**) Suppose now $Ker(j_{1,*} - j_{2,*}) \subset Im(i_1, i_2)_*$.

Let $u = (u_1, u_2) \in Ker(j_{1,*} - j_{1,*})$. Therefore, there exist two elements $\xi_1, \xi_2 \in \mathbb{O}(\Lambda'_\mu)$ so that $j_1(u_1) + \xi_1 = j_2(u_2) + \xi_2$. $j_1(u_1)$, $j_2(u_2)$ lift to $u_1 \in \Lambda_{1,\mu}$, resp. $u_2 \in \Lambda_{2,\mu}$. On the other hand, see Sect. 7.3.11.2, the elements ξ_1, resp. ξ_2, lift to elements $\tilde{\xi}_1 \in \mathbb{O}(\Lambda_{1,\mu})$, resp. $\tilde{\xi}_2 \in \mathbb{O}(\Lambda_{2,\mu})$. From here it follows that

$$u_1 + \tilde{\xi}_1 \in \Lambda_{1,\mu} \ \ and \ \ u_1 + \tilde{\xi}_2 \in \Lambda_{2,\mu}$$

are so that

$$j_1(u_1 + \tilde{\xi}_1) = j_2(u_2 + \tilde{\xi}_2). \tag{7.135}$$

Hence,

$$[(u_1 + \tilde{\xi}_1),\ (u_2 + \tilde{\xi}_2)] = [(u_1,\ u_2)] \in T_1(\Lambda_{1\mu}) \oplus T_1(\Lambda_{2\mu}); \tag{7.136}$$

the statement is proven.

(iii) **(iii.1)** It is required to prove that $Im(j_{1*} - j_{2*}) \subset Ker\partial$.

We have to prove $\partial \circ j_{1*} = 0$. Let $u \in \Lambda_{1,\mu}$. We need to compute $\partial \circ j_1(u)$. We know the result is independent of the lift of

$$(j_1(u)) \otimes (j_1(u))^{-1}.$$

We have to use the lift $u \otimes u^{-1}$; we get

$$\partial \circ j_1(u) = (u \otimes u^{-1})(1_n \oplus 1_n)(u \otimes u^{-1})^{-1} = 1_n \oplus 1_n.$$

(iii.2) Need to prove first that $Ker\partial \subset Im(j_{1*} - j_{2*})$. Suppose $u \in \Lambda'_\mu$ has the property

$$\partial u = [(1_n, 1_n, U(u))] - [\Lambda(\mathscr{A}^n_\mu)] = 0.$$

We have to prove that there exist $v_1 \in \Lambda_{1,\mu}$, $v_2 \in \Lambda_{2,\mu}$ such that

$$[u] = [j_{1,*}v_1 - j_{2,*}v_2].$$

The hypothesis says that

$$[(1_n, 1_n, U(u))] = [\Lambda^n_\mu].$$

We add an idempotent q and its complementary to both sides of this equation to get

$$(1_{m+n}, 1_{m+n}, U(u \oplus 1_m)) \sim_c \Lambda^{m+n}_\mu.$$

We conjugate the second term of this equation to get

$$\partial(u) = \Lambda^{m+n}_\mu.$$

This proves the statement.

(**iii.3**)

$$Im\ \partial \subset Ker\ (i_1, i_2)_*.$$

Let then $u \in T_1(\Lambda'_\mu)$ and $p = \partial u \in T_0(\Lambda_\mu)$. We need to prove that $(i_1, i_2)_*(p) = 0$.

We have

$$[i_1 \circ \partial u] = [i_1\ (1_n \oplus 0_n,\ u(1_n \oplus 0_n)u^{-1}) - \Lambda^n(\mathscr{A}_\mu)] =$$

$$[1_n \oplus 0_n] - [\Lambda^n(\mathscr{A}_\mu)] = 0.$$

On the other hand

$$\begin{gathered} = [i_2\ (1_n \oplus 0_n,\ u(1_n \oplus 0_n)u^{-1})] - [\Lambda^n(\mathscr{A}_\mu] = \\ [u(1_n \oplus 0_n)u^{-1})] - [\Lambda^n(\mathscr{A}_\mu)] = 0. \end{gathered} \tag{7.137}$$

(**iii.4**) Let $Ker(i_1, i_2)_* \subset Im(\partial \otimes \mathbb{Z}[\frac{1}{2}])$ and $p \in T_0(\Lambda_\mu)$ and $(i_1, i_2)_*(p) = 0$. We need to prove that there exists $u \in T_1(\Lambda'_\mu)$ such that $p = \partial u \in T_0(\Lambda_\mu)$.

We look for an invertible matrix $u \in \mathbb{M}_n(\Lambda'_\mu)$ such that

$$\begin{gathered} 0 = (i_1, i_2)_*([p]) = (i_1, i_2)_*[(1_n \oplus 0_n, u(1_n \oplus 0_n)u^{-1})] - [\Lambda^n_\mu] = \\ ([1_n \oplus 0_n], [u(1_n \oplus 0_n)u^{-1}]) - [\Lambda^n_\mu]. \end{gathered} \tag{7.138}$$

We stabilise the idempotents

$$1_n \oplus 0_n, \quad \Lambda^n_\mu. \tag{7.139}$$

The conjugation transforms trivial idempotents in trivial idempotents. The stabilised idempotents $1_{n+m} \oplus 0_{n+m}$, Λ^{n+m}_μ still satisfy the condition

$$p_{1,*}(1_{n+m} \oplus 0_{n+m}) = 0 \tag{7.140}$$

$$p_{2,*}(\Lambda_\mu)^{n+m} = 0. \tag{7.141}$$

We choose an isomorphism $\tilde{u}$ between the two trivial idempotents

$$1_{n+m} \oplus 0_{n+m}, \quad \Lambda^{n+m}_\mu. \tag{7.142}$$

We reduce these idempotents modulo the ideal J. We obtain idempotents in $\Lambda^{'}(\mathscr{A}_\mu)$; let u be the restriction of $\tilde{u}$ modulo J. u is the element we are looking for.

If we started with the element $p \otimes 1/2^m \in T_0(\Lambda_\mu) \otimes \mathbb{Z}[\frac{1}{2}]$ instead, the corresponding element would be $u \otimes 1/2^m$.

7.3.15 Relative T-Groups: $T_i(\mathscr{A}_\mu, J)$

Definition 7.27 Let $\mathscr{A}$ be a localised ring; let $\mathscr{J}$ be a localised bi-lateral ideal in $\mathscr{A}$.

In this section we make reference to the Mayer–Vietoris diagram (Sect. 7.3.4). We take

$$\Lambda_\mu \;=\; \mathscr{J}_\mu, \tag{7.143}$$

$$\begin{gathered}\Lambda_{1,\mu} \;=\; \Lambda_{2,\mu} \;=\; \mathscr{A}_\mu(\Lambda_\mu) \;= \\ \{(\lambda_1, \lambda_2) \mid \lambda_1 \in \Lambda_{1,\mu},\ \lambda_2 \in \Lambda_{2,\mu},\ \lambda_1 \,-\, \lambda_2 \in \mathscr{J}_\mu\}\end{gathered} \tag{7.144}$$

and

$$\Lambda'_\mu \;=\; \mathscr{A}_\mu / \mathscr{J}_\mu. \tag{7.145}$$

The above structure will be called $(\mathscr{A}, \mathscr{J})$ *localised ideal.*
Define, for $i = 1, 2$

$$j_i : \Lambda_\mu \longrightarrow \mathscr{A}/\mathscr{J}, \quad j_i\ (\lambda_1, \lambda_2) := \lambda_i\ mod.\ \mathscr{J}_\mu. \tag{7.146}$$

Definition 7.28

$$T_0^{loc}(\mathscr{A}, \mathscr{J}) := \; Ker\ (\ T_0^{loc}(\Lambda_\mu) \xrightarrow{j_{2,*}} T_0^{loc}(\mathscr{A}_\mu/\mathscr{J}_\mu)\ ) \tag{7.147}$$

$$T_1^{loc}(\mathscr{A}, \mathscr{J}) := \; Ker\ (\ T_1^{loc}(\Lambda_\mu) \xrightarrow{j_{2,*}} T_1^{loc}(\mathscr{A}_\mu/\mathscr{J}_\mu)\ ). \tag{7.148}$$

Theorem 7.13

(i) One has the exact sequence

$$0 \longrightarrow \mathscr{J}_\mu \xrightarrow{\iota} \mathscr{A}_\mu \xrightarrow{\pi} \mathscr{A}_\mu/\mathscr{J}_\mu \longrightarrow 0, \tag{7.149}$$

where ι is the inclusion and π is the canonical projection.

(ii) The exact sequence (7.122) becomes

$$T_1^{loc}(\mathscr{A},\ \mathscr{J}) \xrightarrow{i_*} T_1^{loc}(\mathscr{A}) \xrightarrow{\pi_*} T_1^{loc}(\mathscr{A}/\mathscr{J}) \xrightarrow{\partial}$$
$$T_0^{loc}(\mathscr{A},\ \mathscr{J}) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{i_*} T_0^{loc}(\mathscr{A}) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{\pi_*} T_0^{loc}(\mathscr{A}/\mathscr{J}) \otimes \mathbb{Z}[\frac{1}{2}]. \tag{7.150}$$

Proof The reader is invited to make the proof.

7.3.16 Connecting Homomorphism: Second Form

In this section we define Mayer–Vietoris diagrams associated with elliptic operators between two different Hilbert spaces. We exhibit the corresponding connecting homomorphism.

Let $D : H_0 \longrightarrow H_1$ be an integral or pseudo-differential elliptic operator.

Consider the following the ring

$$\Lambda_1 \ = \ \Lambda_2 := \ Hom_{vect}(H_0 \oplus H_1,\ H_0 \oplus H_1) \tag{7.151}$$

and the ideal

$$J := \ Compact\ operators\ on\ H_0 \oplus H_1. \tag{7.152}$$

Definition 7.29 Let next be the following structure.

Let $\Lambda_1 \ = \ \Lambda_2$ be a ring. Let $J \subset \Lambda_i$, $i = 1, 2$, be a bi-lateral ideal.

Introduce the rings

$$\Lambda := \ \{(\lambda_1, \lambda_2) \mid \lambda_1 \in \Lambda_1,\ \lambda_2 \in \Lambda_2,\ \lambda_1 - \lambda_2 \in J\} \tag{7.153}$$

$$\Lambda^{'} := \ \Lambda_1/J \ = \ \Lambda_2/J. \tag{7.154}$$

The ring structure in Λ is

$$(\lambda_1,\ \lambda_2) \ + \ (\lambda_1',\ \lambda_2') := \ (\lambda_1 \ + \ \lambda_1',\ \lambda_2 \ + \ \lambda_2') \tag{7.155}$$

$$(\lambda_1,\ \lambda_2)\ .\ (\lambda_1',\ \lambda_2') := (\lambda_1.\lambda_1',\ \lambda_2.\lambda_2'). \tag{7.156}$$

Define $i_i,\ i = 1, 2,$

$$i_i : \Lambda \longrightarrow \Lambda_1 \oplus \Lambda_2, \quad i_i\ (\lambda_1,\ \lambda_2) := \lambda_i. \tag{7.157}$$

and

$$j_i : \Lambda_1 \oplus \Lambda_2 \longrightarrow \Lambda^{'}, \quad j_i\ (\lambda_1, \lambda_2) := \lambda_i / J.$$

Proposition 7.9 $\{\Lambda, \Lambda_1, \Lambda_2, \Lambda^{'}, i_1, i_2, j_1, j_2\}$ *is a Mayer–Vietoris diagram, see Sect. 7.3.4.*

If Λ_i, J *are* localised *rings, then this is a localised Mayer–Vietoris diagram; to this localised Mayer–Vietoris diagram there corresponds a six term exact sequence, see Theorem 7.12.*

We are going to exhibit the corresponding *connecting homomorphism* $\partial : T_1^{loc}(\Lambda^{'}) \longrightarrow T_0^{loc}(\Lambda)$.

Definition 7.30 Let $[u] \in T_1^{loc}(\Lambda^{'})$, where $u \in \mathbb{GL}_n(\Lambda^{'}_{\mu})$ for some n and μ.

As j_1 is surjective, let $A,\ resp.\ B, \in \mathbb{M}_n(\Lambda_{1,\mu})$ be liftings of u, resp. u^{-1}, in $\mathbf{M}_n(\Lambda_{1,\mu})$. With A and B one associates

$$S_0 \ = \ 1 \ - \ BA \in \mathbf{M}_n(\Lambda_{1,\mu-1}), \quad S_1 \ = \ 1 \ - \ AB \in \mathbf{M}_n(\Lambda_{1,\mu-1}). \tag{7.158}$$

The matrices S_0, S_1 satisfy

$$j_1(S_0) \ = \ j_2(S_1) \ = \ 0. \tag{7.159}$$

There corresponds the invertible matrix [96]

$$L = \begin{pmatrix} S_0 & -(1+S_0)B \\ A & S_1 \end{pmatrix} \in \mathbb{GL}_{2n}(\Lambda_{1,\ \mu-2}). \tag{7.160}$$

The inverse of the matrix L is

$$L^{-1} \ = \ \begin{pmatrix} S_0 & (1+S_0)B \\ -A & S_1 \end{pmatrix} \in \mathbb{GL}_{2n}(\Lambda_{1,\ \mu-2}). \tag{7.161}$$

We denote by e_1, e_2 be the idempotents

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \Lambda_{1,\ \mu-2}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda_{1,\ \mu-2}. \tag{7.162}$$

The matrix L is used to produce the idempotent

$$P := \ L\ e_1\ L^{-1} \ = \ \begin{pmatrix} S_0^2 & S_0(1+S_0)B \\ S_1 A & 1 - S_1^2 \end{pmatrix} \in Idemp_{2n}(\Lambda_{1,\ \mu-2}). \tag{7.163}$$

Then

$$j_1(P) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathbb{P}_U$ be the idempotent

$$\mathbf{P}_U := \begin{pmatrix} (S_0^2,\ 0) & (S_0(1+S_0)B,\ 0) \\ (S_1 A,\ 0) & (1-S_1^2,\ 1) \end{pmatrix} \in Idemp_{2n}(\Lambda_{1,\mu-2}) \oplus \mathbb{I}demp_{2n}(\Lambda_{2,\mu-2}). \tag{7.164}$$

Then

$$\pi\ (\mathbf{P}_U) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{I}demp(\Lambda_{2,\mu}).$$

The idempotent $\mathbb{P}_U$ satisfies the equation $(j_{1,*} \;-\; j_{2,*})(\mathbb{P}_U) \;=\; 0$. Therefore, $\mathbb{P}_U$ belongs to $\mathbb{M}_{2n}(\Lambda_{\mu-2})$.

Definition 7.31 *Connecting homomorphism—second form* (for *K-theory* see [96]).

Let $[U] \in T_1^{loc}(\Lambda')$. We define the *second form of the connecting homomorphism*

$$\partial_{II} : T_1^{loc}(\Lambda') \longrightarrow T_0^{loc}(\Lambda) \tag{7.165}$$

$$\partial_{II}[U] := [\mathbf{P}_U] \;-\; [(e_2, e_2)] \in T_0^{loc}(\Lambda). \tag{7.166}$$

It is easy to check that ∂ is compatible with the equivalence relation $\sim_{\mathbb{O}}$. In fact, any element $\xi \;=\; u \oplus u^{-1} \in \Lambda'_{\mu}$ has an invertible lifting in $\Lambda_{1,\,\mu}$, see Sect. 7.3.11.2. For this reason, both A and B may be chosen to be inverse one to each other. Therefore, $S_0 \;=\; 0$ and $S_1 \;=\; 0$ and hence

$$P \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \partial[U] \;= 0. \tag{7.167}$$

7.3.16.1 Connecting Homomorphism and Stabilisation

In this sub-section we see how ∂ depends on stabilisations. In other words, we need to know how Theorem 7.12(iii), and how the connecting homomorphism, Definition 7.31, behaves with respect to stabilisations. We consider a more general situation than that considered in the previous section (Sect. 7.31).

For this purpose, let $U \in \mathbb{GL}_{m+n}(\Lambda')$ and

$$e_{(0,n)} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 1_n \end{pmatrix} \in \mathbf{M}_{m+n}(\Lambda') \tag{7.168}$$

be such that the diagram

$$\begin{array}{ccc} \Lambda'^{m+n} & \xrightarrow{U} & \Lambda'^{m+n} \\ \downarrow e_{(0,n)} & & \downarrow e_{(0,n)} \\ \Lambda'^{m+n} & \xrightarrow{U} & \Lambda'^{m+n} \end{array} \tag{7.169}$$

is commutative. This condition will be needed to show that $\mathbb{P}_U \in \mathbb{I}demp_{m+n}(\Lambda)$.

The case discussed here corresponds in Sect. 7.31 to $m = 0$.

We proceed as in Sect. 7.31. Let $A,\ B \in \mathbb{GL}_{m+n}(\Lambda_{1,\mu})$ be liftings of U, resp. U^{-1}, in $\mathbb{M}_{m+n,\mu}(\Lambda_1)$. We assume j_1 is surjective.

As usually, with A and B one associates the matrices $S_0 = 1 - BA \in \mathbf{GL}_{m+n}(\Lambda_{1,\mu})$ and $S_1 = 1 - AB \in \mathbb{GL}_{m+n}(\Lambda_{2,\mu})$. The matrices S_0, S_1 satisfy the condition

$$j_{1,*}(S_0) = j_{1,*}(S_1) = 0.$$

We associate the invertible matrix

$$L = \begin{pmatrix} S_0 & -(1+S_0)B \\ A & S_1 \end{pmatrix} \in \mathbf{GL}_{2(m+n)}(\Lambda_{1,\mu-2}). \tag{7.170}$$

The inverse of the matrix L is

$$L^{-1} = \begin{pmatrix} S_0 & (1+S_0)B \\ -A & S_1 \end{pmatrix} \in \mathbf{GL}_{2(m+n)}(\Lambda_{1,\mu-2}). \tag{7.171}$$

We consider the idempotent matrices

$$e_1 = \begin{pmatrix} e_{(0,n)} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{GL}_{2(m+n)}(\Lambda_{1,\mu}) \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & e_{(0,n)} \end{pmatrix} \in \mathbb{GL}_{2(m+n)}(\Lambda_{1,\mu}). \tag{7.172}$$

The invertible matrix we produce is used to produce the idempotent

$$P_U := L\ e_1\ L^{-1} = \begin{pmatrix} S_0\ e_{(0,n)}\ S_0 & S_0\ e_{(0,n)}\ (1+S_0)B \\ A\ e_{(0,n)}\ S_0 & A\ e_{(0,n)}\ (1+S_0)B \end{pmatrix} \in Idemp_{2(m+n)}(\Lambda_{1,\mu-2}). \tag{7.173}$$

Definition 7.32 Let $[u]$ be an invertible element and $R(U)$ be the corresponding operator. Then

$$R(U) := P_U - e_2. \tag{7.174}$$

The idempotent P_U is used to construct the double-matrix idempotent

$$\mathbb{P}_U := \begin{pmatrix} (S_0\ e_{(0,n)}\ S_0,\ 0) & (S_0\ e_{(0,n)}\ (1+S_0)B,\ 0) \\ (A\ e_{(0,n)}\ S_0,\ 0) & (A\ e_{(0,n)}\ (1-S_0)B,\ e_{(0,n)}) \end{pmatrix} \in \mathbf{I}dem p_{2(m+n)}(\Lambda_{1,\mu-2} \oplus \Lambda_{2,\mu-2}). \tag{7.175}$$

The matrix $\mathbb{P}_U$ is an idempotent in $\mathbb{M}_{2(m+n)}(\Lambda_{\mu-2})$. Indeed, we may verify directly that $(j_{1,*} - j_{2,*})\mathbb{P}_U = 0$. For this end, j_{1*} is a ring homomorphism and $j_{1*}(S_0) = j_{1*}(S_1) = 0$. Finally we get

$$j_1(A\ e_{(0,n)}\ (1-S_0)B) = U\ e_{(0,n)}\ U^{-1} = e_{(0,n)} = j_2(e_{(0,n)}). \tag{7.176}$$

Therefore, it follows that $\mathbf{P}_U \in \mathbf{M}_{2n}(\Lambda_{\mu-2})$. The fact that the matrix $\mathbb{P}_U$ is an idempotent in $\mathbf{M}_{2(m+n)}(\Lambda_{\mu-2})$ follows from the fact that $\mathbb{P}$ and e_2 are idempotents along with the discussion above.

Definition 7.33 *Connecting homomorphism—third form*

We suppose the assumptions of Sect. 7.31 above are in place. For any $[U] \in T_1^{loc}(\Lambda')$ one defines the *connecting homomorphism*

$$\partial : T_1^{loc}(\Lambda') \longrightarrow T_0^{loc}(\Lambda) \tag{7.177}$$

by

$$\partial[U] := [\mathbf{P}_U] - [(e_2, e_2)] = \begin{pmatrix} (0,\ 0) & (0,\ 0) \\ (0,\ 0) & (Ae_{(0,n)}B,\ e_{(0,n)}) \end{pmatrix} - [(e_2, e_2)] \in T_0^{loc}(\Lambda). \tag{7.178}$$

We have to follow how $\mathbb{P}_U$ depends on the choice of the lifts A, B, U and U^{-1}. Let $\tilde{\mathbb{P}}_U$ be the new operator.

Let $\tilde{A}$ and $\tilde{B}$ be two such different lifts. We assume first that the operator A is changed; with the operator B we use the same procedure.

Let $\tilde{A} = A + T$ with $j_1(T) = 0$. Let $\tilde{S}_0$, $\tilde{S}_1$ be the corresponding operators; let $\tilde{L} \in \mathbb{GL}_{2(m+n)}(\Lambda)$ and let $\mathbb{P}'_U$ be the corresponding new operators.

The new operator is $\mathbb{P}_U := \tilde{L}\ e_1\ \tilde{L}^{-1}$.

The $\mathbb{P}_U$ is conjugated with the old operator $\mathbb{P}_U := L\ e_1\ L^{-1}$, $\mathbb{P}'_U := \tilde{L}\ e_1\ \tilde{L}^{-1}$. The operator $\tilde{L}$ is given by

$$\tilde{L} = \begin{pmatrix} 1 - BT & -BTB \\ T & 1 + TB \end{pmatrix} L^{-1}. \tag{7.179}$$

We obtain

$$\tilde{L}L^{-1} = \begin{pmatrix} 1-BT & -BTB \\ T & 1+TB \end{pmatrix}. \tag{7.180}$$

We have

$$\tilde{L} = \begin{pmatrix} 1-BT & -BTB \\ T & 1+TB \end{pmatrix} L. \tag{7.181}$$

We know already that $\tilde{L}$ and L are invertible matrices. Therefore the right hand side of (7.181) is an invertible matrix.

The operator $\tilde{P}$ is given by the matrix

$$\begin{aligned} \tilde{P} = \tilde{L}\, e_1\, \tilde{L}^{-1} &= \begin{pmatrix} 1-BT & -BTB \\ T & 1+TB \end{pmatrix} L.e_1.L^{-1} \begin{pmatrix} 1-BT & -BTB \\ T & 1+TB \end{pmatrix}^{-1} = \\ &= \begin{pmatrix} 1-BT & -BTB \\ T & 1+TB \end{pmatrix} P \begin{pmatrix} 1-BT & -BTB \\ T & 1+TB \end{pmatrix}^{-1}. \end{aligned} \tag{7.182}$$

Therefore,

$$\tilde{\mathbf{P}}_U = (\begin{pmatrix} 1-BT & -BTB \\ T & 1+TB \end{pmatrix}, 1)\, \mathbf{P}_U\, (\begin{pmatrix} 1-BT & -BTB \\ T & 1+TB \end{pmatrix}, 1)^{-1}. \tag{7.183}$$

In conclusion, $[\ \tilde{\mathbf{P}}_U\ ] = [\ \mathbf{P}_U\ ] \in T_0^{loc}(\Lambda)$.

This completes the discussion about the choice of the liftings of A and B.

Chapter 8
Topological Index and Analytical Index: Reformulation of Index Theory



Abstract In this part we present the main algebraic structures on *topological manifolds* and we reformulate the index theory.

8.1 Level I: Index Theory at the T_*^{loc}-Theory Level

Definition 8.1 Let $\mathscr{J} \subset \mathscr{A}$ be a localised bi-lateral ideal of the unital ring $\mathscr{A}$. There corresponds the short ring exact sequence

$$0 \longrightarrow \mathscr{J}_\mu \xrightarrow{\iota} \mathscr{A}_\mu \xrightarrow{\pi} \mathscr{A}_\mu/\mathscr{J}_\mu \longrightarrow 0. \tag{8.1}$$

Definition 8.2 The ring $\mathscr{A}_\mu/\mathscr{J}_\mu$ is the analogue of the Calkin ring. If $U \in \mathscr{A}_\mu$ then $\pi(U) \in \mathscr{A}/\mathscr{J}$ is called the *symbol* of U.

Consider the 6-term exact sequence in T^{loc}-theory (8.2)

$$\begin{aligned} &T_1^{loc}(\mathscr{A}, \mathscr{J}) \xrightarrow{i_*} T_1^{loc}(\mathscr{A}) \xrightarrow{\pi_*} T_1^{loc}(\mathscr{A}/\mathscr{J}) \xrightarrow{\partial} \\ &T_0^{loc}(\mathscr{A}, \mathscr{J}) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{i_*} T_0^{loc}(\mathscr{A}) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{\pi_*} T_0^{loc}(\mathscr{A}/\mathscr{J}) \otimes \mathbb{Z}[\frac{1}{2}]. \end{aligned} \tag{8.2}$$

1. *Topological index* of u is

$$Top^T\, Index\ (u) := \delta[u] \in T_0^{loc}(\mathscr{A}) \otimes \mathbb{Z}[\frac{1}{2}], \tag{8.3}$$

   where $[u] \in T_1^{loc}(\mathscr{A}/\mathscr{J})$.
2. *Analytical Index* of u is

$$An^T\, Index\ (u) := \mathbf{R}\,(\delta[u]) = \delta_{II}[u] \in T_0^{loc}(\mathscr{A}) \otimes \mathbb{Z}[\frac{1}{2}] \tag{8.4}$$

where δ_{II} is the second definition of boundary map 7.31.

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Case 1

Problem 8.1 Define significant classes of extensions (8.1) for which the difference

$$\Delta(\mathscr{A}, \mathscr{J}) := Top^T\, Index\ (u) - An^T\, Index(u) \tag{8.5}$$

can be computed.

Case 2

Conjecture 8.1

(1) Let M^{2l} be a closed compact quasi-conformal manifold. Let $\Omega^*(M)$ be the algebra of differential forms on M^{2l}. Let H be the Hilbert space of L_2-forms of degree l. Let (8.1) be the exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathscr{L}^{(1,\infty)} \xrightarrow{\iota} \Psi Diff \xrightarrow{\pi} \Psi Diff/\mathscr{L}^{1,\infty} \longrightarrow 0.$$

For any $u \in \mathbb{GL}_N(\Psi Diff/\mathscr{L}^{1,\infty})$ one has

$$(Top^T\, Index)\ (u) \otimes \mathbb{Q} = (An^T\, Index)\ (u) \otimes \mathbb{Q}. \tag{8.6}$$

(2) Let A be an elliptic pseudo-differential operator on M. Let $u = \pi(A) \in \mathbb{GL}_N(\mathscr{A}/\mathscr{J})$ be the image of A in the quotient space. Then

$$An^T\, Index\ (u) \otimes \mathbb{Q} = Top^T\, Index\ (u) \otimes \mathbb{Q} = \frac{1}{(2\pi i)^q}\frac{q!}{(2q)!}(-1)^{dimM}\ Ch(u) \cap [T^*M]. \tag{8.7}$$

$Ch(u)$ in the formula (8.7) is the periodic cyclic homology of u, discussed next in Sect. 8.2

(3) Let A be a pseudo-differential elliptic operator on the quasi-conformal manifold M. Let u be its *classical* symbol. Produce the residue operator $\mathbb{R}(u)$ (6.1), (K_1 is replaced by T_1^{loc}). Let $f \in C^q_{AS}(M)$ be an Alexander–Spanier co-cycle; let $[f]$ be its cohomology class. Let $\tau(T^*(M))$ be the *Todd class* of M.

Then

$$An^T\, Index\ (u) \otimes_{\mathbb{Q}} [f] = Top^T\, Index\ (u) \otimes_{\mathbb{Q}} [f] = \frac{1}{(2\pi i)^q}\frac{q!}{(2q)!}(-1)^{dimM}\ Ch(u) \cup \tau(T^*(M)) \cap [T^*M]. \tag{8.8}$$

8.2 Level II: Index Theory in *Local* Periodic Cyclic Homology

We consider the exact sequence

Definition 8.3 Let $\mathscr{A}_\mu$ be a localised ring. Consider the exact sequence (8.1). Let $u \in T_1^{loc}(\mathscr{A}_\mu/\mathscr{J}_\mu)$.

1. *Topological index* of u is

$$Top^{Ch} Index\ (u) := Ch\delta[u] \in C^{loc,per,\lambda}(\mathscr{A}), \tag{8.9}$$

2. *Analytical Index* of u is

$$An^{Ch} Index\ (u) := Ch_{ev}\ (\delta\ [u]) = Ch_{ev}(\delta_{II}\ [u]) \in H_0^{loc,per,\lambda}(\mathscr{A}). \tag{8.10}$$

see Definition 7.31.
The topological index could be defined in a different way. One could consider the connecting homomorphism δ_λ in the local cyclic periodic homology instead of the connecting homomorphism in the T_*^{loc} exact sequence. This leads to the *topological index*

$$Top^{Ch_\lambda} Index\ (u) := Ch\delta_\lambda[u] \in H_{ev}^{loc,per,\lambda}(\mathscr{A}), \tag{8.11}$$

Case 1
Problem 8.2 Define significant classes of extensions (8.1) for which the difference

$$\Delta^{Ch}(\mathscr{A}, \mathscr{J})\ : An^{Ch} Index\ (u) - Top^{Ch} Index\ (u) \tag{8.12}$$

can be computed.

Case 2
Conjecture 8.2 Let M be a quasi-conformal closed manifold. Let $u \in \mathbb{GL}_N(\mathscr{A}_\mu/\mathscr{J}_\mu)$, where $(\mathscr{A}_\mu)$ is the algebra of pseudo-differential operators on M and $\mathscr{J}) = \mathscr{L}^{(\frac{1}{n},\infty)}$, localised by the support of operators about the diagonal.
Then

$$Top^{Ch} Index(u) = An^{Ch} Index(u). \tag{8.13}$$

8.3 Level III: Index Theory Restricted at the Diagonal

This situation applies only when the local periodic cyclic homology has a limit to the diagonal. It depends on the regularity of the structure. The *classical index theorems* belong to this class.

8.4 Topological Expectations

This book goes in two directions. On one side it proposes new ways of research in index theory and on the other hand it generalises the notion of *topological spaces*. Sections 9.1–9.3 treated the *index theory* in various situations. The purpose of this section is to propose the developing of topology. This section should be read together with the last section of this book where the Chern classes of bundles over generalised topological spaces are defined.

8.5 $T_i^{loc}(\mathbb{C}), i = 0, 1$

In this section we compute the *local algebraic T-theory*, $i = 0, 1$, of the algebra of complex numbers $\mathbb{C}$ endowed with the trivial filtration ($\mathbb{C}_\mu = \mathbb{C},\ \mu \in \mathbb{N}$); *local algebras* and *local* algebraic T_i^{loc}-theory were introduced in the previous section. Theorem 8.1 is the main result of this section. This case corresponds in the simplest case, the Alexander–Spanier *local T-theory* of the point. The results of this section constitute a part of a comprehensive program aimed at re-stating the index theorem, see [125, 127, 129, 130].

8.5.1 Introduction

For the definition of the local algebraic T_i^{loc}-groups see Chap. 7.

8.5.2 Notation: Preliminaries

Definition 8.4 Let $J_{n,\lambda} \in \mathbb{GL}_n(\mathbb{C})$, $2 \leqslant n$, denote the Jordan cell matrix of size n having the entry λ on the diagonal

$$J_{n,\lambda} = \begin{pmatrix} \lambda & 1 & 0 & 0 & \ldots\ldots & 0 & 0 \\ 0 & \lambda & 1 & 0 & \ldots\ldots & 0 & 0 \\ \ldots\ldots\ldots\ldots & & \ldots & & \ldots\ldots & & \\ 0 & 0 & 0 & 0 & \ldots\ldots & \lambda & 1 \\ 0 & 0 & 0 & 0 & \ldots\ldots & 0 & \lambda \end{pmatrix}; \tag{8.14}$$

it is understood that $J_{1,\lambda} = \lambda.1_1$.

For of any matrix A, let $J(A)$ denote the Jordan canonical form of A.

Definition 8.5 Let $\bar{\mathbb{C}} := \mathbb{C} \setminus \{-1, 0, 1\}$.

Introduce on $\bar{\mathbb{C}}$ the following equivalence relation $\sim_r$: $\lambda \sim_r \lambda'$ if and only if $\lambda = \lambda'$ or $\lambda^{-1} = \lambda'$.

The values $\lambda = -1, +1$ are the fixed points of the $\mathbb{Z}_2$-group action $\mathscr{I} : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$ $\mathscr{I}(\lambda) := \lambda^{-1}$.

Define

$$\hat{\mathbb{C}} := \bar{\mathbb{C}}/\sim_r . \tag{8.15}$$

8.5.3 The Main Result

Theorem 8.1 *Let $\mathbb{C}$ be endowed with the trivial filtration ($\mathbb{C}_\mu = \mathbb{C}$, $\mu \in \mathbb{N}$).*

Then

(i)

$$T_0^{loc}(\mathbb{C}) = \mathbb{Z} \tag{8.16}$$

(ii)

$$T_1^{loc}(\mathbb{C}) = \tag{8.17}$$

$$\oplus_{1 \leqslant n \in \mathbb{N}} \ \mathbb{Z}_2[J_{n,-1}] \ \oplus_{2 \leqslant n \in \mathbb{N}} \ \mathbb{Z}_2[J_{n,1}] \oplus_{1 \leqslant n \in \mathbb{N},\ \hat{\lambda} \in \hat{\mathbb{C}}} \ \mathbb{Z}[J_{n,\hat{\lambda}}].$$

The group structure in $T_1^{loc}(\mathbb{C})$ is provided by the relations

$$m_1[J_{n,\lambda}] + m_2[J_{n,\lambda^{-1}}] = (m_1 - m_2)[J_{n,\lambda}], \quad for\ any\ m_1, m_2 \in \mathbb{N}. \tag{8.18}$$

$$-m[J_{n,\lambda}] = m[J_{n,\lambda^{-1}}], \quad for\ any\ m \in \mathbb{Z}. \tag{8.19}$$

The proof of this theorem gives a taste of the algebraic structure behind the construction of T_0^{loc} and T_1^{loc}.

Proof

(i) It is easy to compute $T_0^{loc}(\mathbb{C})$. Any two idempotent matrices in $\mathbb{M}_n(\mathbb{C})$ are conjugated if and only if they have the same rank; this could be seen, for example, by using their Jordan canonical forms. Therefore

$$T_0^{loc}(\mathbb{C}) = T_0(\mathbb{C}) = \mathbb{Z}. \tag{8.20}$$

(ii) We pass now to compute $T_1^{loc}(\mathbb{C})$. We will need a few lemmas.

In the computation of T_1^{loc}, the Jordan canonical form of the matrix $J_{n,\lambda}^{-1}$ plays a central role. For this reason we present two proofs of the result (10.21). The first proof is based on explicit formulas; the second proof is more conceptual.

We begin by determining the formula for $J_{n,\lambda}^{-1}$.

Definition 8.6 Consider the matrix

$$M_n := J_{n,\lambda} - \lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{8.21}$$

Then, for any $1 \leqslant k \leqslant n-1$

$$M_n^{\,k} = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 \dots & 0 \\ 0 & \dots & 0 & 0 & 1 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{8.22}$$

i.e. in this matrix the entries of the $(k+1)$th diagonal, above the main diagonal, are equal to 1 while all other entries are equal to 0.

In consecutive powers of M_n the diagonal of 1's moves towards the upper right end of the matrix. From this it follows that any distinct powers k, $1 \leqslant k \leqslant n-1$, of M_n are linearly independent over $\mathbb{C}$. For larger powers of this matrix one has

$$M_n^{\,k} = 0, \quad n \leqslant k. \tag{8.23}$$

These well-known formulas allow us to find $J(J_{n,\lambda}^{-1})$. Indeed,

$$I = I - (-\frac{1}{\lambda}M_n)^n = (I - (\frac{-1}{\lambda}M_n)) \sum_{0 \leqslant k \leqslant n-1} (-\frac{1}{\lambda}M_n)^{\,k} = \tag{8.24}$$

$$= (\lambda J_{n,\lambda}) \sum_{0 \leqslant k \leqslant n-1} (-\frac{1}{\lambda}M_n)^k,$$

which shows

$$(\lambda\ J_{n,\lambda})^{-1}\ =\ \sum_{0\leqslant k\leqslant n-1}(-\frac{1}{\lambda}M_n)^k, \tag{8.25}$$

or

$$J_{n,\lambda}^{-1}\ =\ \sum_{0\leqslant k\leqslant n-1}\frac{(-1)^k}{\lambda^{k+1}}M_n{}^k. \tag{8.26}$$

Therefore, for any $1 \leqslant l \leqslant n-1$

$$(J_{n,\lambda}^{-1}\ -\ \frac{1}{\lambda}I)^l\ =\ (\sum_{1\leqslant k\leqslant n-1}\frac{(-1)^k}{\lambda^{k+1}}M_n{}^k)^l\ =\ \frac{(-1)^l}{\lambda^{2l}}M_n^l\ +\ higher\ powers\ of\ M_n. \tag{8.27}$$

These show that $(2 \leq n)$

$$1 \leqslant Rank\ (J_{n,\lambda}^{-1} - \frac{1}{\lambda}I)^l \quad (2 \leqslant l \leqslant n-1) \tag{8.28}$$

and

$$Rank\ (J_{n,\lambda}^{-1} - \frac{1}{\lambda}I)^n\ = 0. \tag{8.29}$$

All eigen-values of $J_{n,\lambda}$ are equal to λ. The minimum power k for which

$$1 \leqslant Rank(J_{n,\lambda} - \lambda.1_n)^k \tag{8.30}$$

$$(J_{n,\lambda} - \lambda.1_n)^{k+1}\ =\ 0 \tag{8.31}$$

is $n-1$. For this reason, by definition, the number n will be called the *Jordan rank* of the matrix $J_{n,\lambda}$, denoted $\mathscr{JR}(J_{n,\lambda})$.

Formulas (8.28), (8.29) show, on the other hand, that

$$\mathscr{JR}(J_{n,\lambda}^{-1})\ =\ n\ =\ \mathscr{JR}(J_{n,\lambda}) \tag{8.32}$$

Given that there is just one Jordan cell with given Jordan rank and spectrum, this relation says that

$$J\ (J_{n,\lambda}^{-1})\ =\ J_{n,\lambda^{-1}}. \tag{8.33}$$

The next lemma states the same formula (10.21), but provides a different proof of it.

Lemma 8.1 *Let $J_{n,\lambda} \in \mathbb{GL}(\mathbb{C})$, $1 \leq n$. Then*

$$J\ (J_{n,\lambda}^{-1}) = J_{n,\lambda^{-1}}. \tag{8.34}$$

Proof This is a second proof of (8.33); it does not use the explicit formula for $J_{n,\lambda}^{-1}$ discussed above. As $J_{n,\lambda} \in \mathbb{GL}(\mathbb{M}(\mathbb{C}))$, $\lambda \neq 0$.

The matrix $J_{n,\lambda}$ is invertible. We get, for any $1 \leqslant k \leq n-1$

$$1 \leqslant Rank(J_{n,\lambda} - \lambda 1_n)^k \ = \ Rank[\ J_{n,\lambda}^k.(1_n - \lambda.J_{n,\lambda}^{-1})^k\ ] = \tag{8.35}$$

$$Rank(1_n - \lambda.J_{n,\lambda}^{-1})^k \ = \ Rank(\frac{1}{\lambda}1_n \ - \ J_{n,\lambda}^{-1})^k$$

$$0 = Rank(J_{n,\lambda} - \lambda 1_n)^n \ = \ Rank[\ J_{n,\lambda}^n.(1_n - \lambda.J_{n,\lambda}^{-1})^n\ ] = \tag{8.36}$$

$$Rank(1_n - \lambda.J_{n,\lambda}^{-1})^n \ = \ Rank(\frac{1}{\lambda}1_n - J_{n,\lambda}^{-1})^n$$

The spectrum of $J_{n,\lambda}^{-1}$ consists of the sole value $1/\lambda$. Formulas (8.35) and (8.36) say that

$$\mathscr{J}\mathscr{R}(J_{n,\lambda}^{-1}) \ = \ n \ = \ \mathscr{J}\mathscr{R}(J_{n,\lambda}). \tag{8.37}$$

Given that there is just one Jordan cell with given Jordan rank and spectrum, this relation proves formula (10.22)

$$J\ (J_{n,\lambda}^{-1}) \ = \ J_{n,\lambda^{-1}}.$$

Lemma 8.2 *Let $U \ = \ (A \oplus A^{-1}) \in \mathbb{O}_{2n}(\mathbb{C})$, any $1 \leqslant n$.*

(i) The Jordan canonical form of U is

$$U \sim_c J(U) = \oplus_{\sum n_i = n}(\ J_{n_i,\lambda_i} \oplus J_{n_i,\ \lambda_i^{-1}}\ ) = \sum_{\sum n_i = n}(\ J_{n_i,\lambda_i} + J_{n_i,\ \lambda_i^{-1}}\ ) \tag{8.38}$$

with

$$(\ J_{n_i,\lambda_i} + J_{n_i\ ,\ \lambda_i{}^{-1}}\ ) \in \mathbb{O}_{2n}(\mathbb{C}). \tag{8.39}$$

(ii)

$$[J_{n_i\ ,\ \lambda_i{}^{-1}}] = -[J_{n_i,\lambda_i}] \in T_1^{loc}(\mathbb{C}). \tag{8.40}$$

Proof A consequence of Lemma 8.1 is that the equivalence relation $\sim_{\mathbb{O}}$ could produce relations between the Jordan canonical forms with eigen-values λ and λ^{-1}. Inner auto-morphisms (and therefore the equivalence relation $\sim_I$) do not mix up Jordan canonical forms with different values of the class $\hat{\lambda}$. Therefore, the only relations between generators of $T_1^{loc}(\mathbb{C})$ could occur within the Jordan canonical forms with *eigen-values within the same class* $\hat{\lambda}$, or *separately, within* the Jordan canonical forms with eigen-values -1 and $+1$. These cases will be discussed separately.

(i) Recall that the filtration of $\mathbb{C}$ is trivial. This means any inner auto-morphism in $\mathbb{GL}(\mathbb{M}(\mathbb{C}))$ is allowed and there is no limitation on the number of products.

The matrix U is the direct sum of the matrices A and A^{-1}. Although we would be allowed to use any inner auto-morphism of $\mathbb{C}^n \oplus \mathbb{C}^n$ to bring the matrix U to its Jordan canonical form, a direct sum of such inner auto-morphisms acting separately onto the two terms will produce the result.

We use an inner auto-morphisms to bring the matrix A to its Jordan canonical form

$$A \sim_c \oplus_{\sum n_i = n} J_{n_i,\lambda_i}. \tag{8.41}$$

We use this decomposition to replace both A and A^{-1} of the expression of U; then we use Lemma 8.2 to get

$$U = A \oplus A^{-1} \sim_I \oplus_{\sum n_i = n} (J_{n_i,\lambda_i} \oplus J^{-1}_{n_i,\lambda_i}) \sim_c \oplus_{\sum n_i = n} (J_{n_i,\lambda_i} \oplus J_{n_i,\lambda_i^{-1}}) \tag{8.42}$$

in which, any term $J_{n_i,\lambda_i} \oplus J_{n_i,\lambda_i^{-1}} \sim_c J_{n_i,\lambda_i} + J^{-1}_{n_i,\lambda_i} \in \mathbb{O}_{2n}(\mathbb{C})$.

The uniqueness part of the Jordan canonical form theorem assures that the decomposition (10.30) is precisely (except for permutation of the pairs in the direct sum) the Jordan canonical form of U

$$J(U) = \oplus_{\sum n_i = n} (J_{n_i,\lambda_i} \oplus J_{n_i,\lambda_i^{-1}}). \tag{8.43}$$

Therefore, any element $A \oplus A^{-1} \in \mathbb{O}_{2n}(\mathbb{C})$ splits-up in a direct sum of elements in $\mathbb{O}_{2n}(\mathbb{C}^n)$; each of these terms represents the zero element in $T_1^{loc}(\mathbb{C})$.

(ii) As $(J_{n_i,\lambda_i} \oplus J^{-1}_{n_i,\lambda_i}) \in \mathbb{O}_{2n}(\mathbb{C})$, one has

$$0 = [J_{n_i,\lambda_i} \oplus J^{-1}_{n_i,\lambda_i}] = [J_{n_i,\lambda_i}] + [J_{n_i,\lambda_i^{-1}}] \in T_1^{loc}(\mathbb{C}). \tag{8.44}$$

Therefore

$$[J_{n_i,\lambda_i^{-1}}] = -[J_{n_i,\lambda_i}] \in T_1^{loc}(\mathbb{C}). \tag{8.45}$$

Lemma 8.3

(i) The Jordan cells are generators of $T_1^{loc}(\mathbb{C})$ *subject to the following relations*

$$[J_{1,1}] = 0 \tag{8.46}$$

$$2\,[J_{n,-1}] = 0 \quad for\ any \quad 1 \leqslant n \in \mathbb{N} \tag{8.47}$$

$$2\,[J_{n,1}] = 0 \quad for\ any \quad 2 \leqslant n \in \mathbb{N} \tag{8.48}$$

$$[J_{n\,,\,\lambda}] + [J_{n\,,\,\lambda^{-1}}] = 0 \quad for\ any \quad 1 \leqslant n \in \mathbb{N} \ \ and \ \ \lambda \neq -1, 1. \tag{8.49}$$

(ii) In $T_1^{loc}(\mathbb{C})$ *there no other relations between the classes of Jordan cells besides (8.46)–(8.49).*

Proof

(i) The first relation (8.46) is a consequence of the stabilisation equivalence relation

$$A \sim_s A + 1_1 \ \ implies \ \ [A] = [A + 1_1] = [A] + [1_1] \in T_1^{loc}(\mathbb{C}). \tag{8.50}$$

Formulas (8.47) and (8.48) are consequences of formula (8.45) for the only two values of λ which satisfy the condition $\lambda = \lambda^{-1}$.

Formula (8.49) is formula (8.45) in the remaining cases.

(ii) In what follows we show that between the classes of the Jordan canonical forms, in $T_1^{loc}(\mathbb{C})$, there are no other relations besides the relations (8.46)–(8.49).

Any generator A of $T_1^{loc}(\mathbb{C})$, i.e. invertible matrix, decomposes in a direct sum of invertible Jordan cells; this decomposition is unique modulo a permutation of them.

Any element of $u \in T_1^{loc}$ is an equivalence class belonging to $\mathbb{GL}_n(\mathbb{C})/\sim_{sc}$. We start with two generic generators in $T_1^{loc}(\mathbb{C})$ which represent the same element in this group and we look for all relations which will deem necessary at the level of $T_1^{loc}(\mathbb{C})$. Suppose then

$$(u_1 + \xi_1) \sim_c (u_2 + \xi_2) \ \ with \ \ \xi_1, \xi_2 \in \mathbb{O}_{2n}(\mathbb{C}). \tag{8.51}$$

(in this formula we have assumed that both sides of the equation were sufficiently stabilised). We get the *necessary relations* between the involved Jordan canonical forms

$$[u_1] = [u_2] \in T_1^{loc}(\mathbb{C}) \ \ and \ \ [\xi_1] = [\xi_2] = 0. \tag{8.52}$$

We will show that relations (8.46)–(8.49) of this lemma imply formula (8.52).

Indeed, Eq. (8.51) says that at the level of Jordan canonical forms of both sides

$$J(u_1 + \xi_1) = J(u_2 + \xi_2). \tag{8.53}$$

The discussion made in the proof of Lemma 8.3(i) shows that each of the four elements of Eq. (8.51) may be individually brought to their canonical Jordan form. The uniqueness part of the Jordan canonical form theorem tells that

$$J(u_1) \cup J(\xi_1) = J(u_2) \cup J(\xi_2). \tag{8.54}$$

Both sides of this formula have to be seen as consisting of the union of the Jordan cells, each cell counted with its multiplicity; i.e. the two sides of Eqs. (8.53), (8.54) consist precisely of the same Jordan cells counted with their multiplicities.

We recall, Definition 8.5, that $\hat{\mathbb{C}} := \mathbb{C} \setminus \{-1, 0, 1\}/\sim_r$ where $\sim_r$ is the equivalence relation $\lambda \sim_r \lambda'$ if and only if $\lambda = \lambda'$ or $\lambda^{-1} = \lambda'$. Let $\hat{\lambda}$ be the equivalence class of λ in $\hat{\mathbb{C}}$.

We decompose the set of Jordan cells appearing in the formula (8.54) in disjoint groups $J_{n_i,\lambda}$, according to the size n_i and eigen-value $\lambda \in \mathbb{C}$

$$J_{n_i,\lambda}(u_1) := \{\text{ Jordan cells of } u_1 \text{ of size } n_i \text{ and eigen-value } \lambda\}. \tag{8.55}$$

$$J_{n_i,\lambda}(\xi_1) := \{\text{ Jordan cells of } \xi_1 \text{ of size } n_i \text{ and eigen-value } \lambda\}. \tag{8.56}$$

$$J_{n_i,\lambda}(u_2) := \{\text{ Jordan cells of } u_2 \text{ of size } n_i \text{ and eigen-value } \lambda\}. \tag{8.57}$$

$$J_{n_i,\lambda}(\xi_2) := \{\text{ Jordan cells of } \xi_2 \text{ of size } n_i \text{ and eigen-value } \lambda\}. \tag{8.58}$$

We begin by considering two classes $J_{n_i,\hat{\lambda}}$, $\hat{\lambda} \in \hat{\mathbb{C}}$, with *distinct* eigen-values $\lambda_0 \neq \lambda_0^{-1}$.

Let $a_{n_i,\lambda_0}(u_\alpha)$, resp. $a_{n_i,\lambda_0^{-1}}(u_\alpha)$, represent the number of times the Jordan cell J_{n_i,λ_0}, resp. $J_{n_i,\lambda_0^{-1}}$, appears in $J_{n_i,\hat{\lambda}}(u_\alpha)$, $\alpha = 1, 2$.

Analogously, let $b_{n_i,\lambda_0}(\xi_\alpha)$, resp. $b_{n_i,\lambda_0^{-1}}(\xi_\alpha)$, be the number of times the Jordan cell J_{n_i,λ_0}, resp. $J_{n_i,\lambda_0^{-1}}$, appears in $J_{n_i,\hat{\lambda}}(\xi_\alpha)$, $\alpha = 1, 2$.

The formula (8.54), for the chosen size and eigen-values, gives

$$(a_{n_i,\lambda_0}(u_1) + b_{n_i,\lambda_0}(\xi_1))J_{n_i,\lambda_0} = (a_{n_i,\lambda_0}(u_2) + b_{n_i,\lambda_0}(\xi_2))J_{n_i,\lambda_0}. \tag{8.59}$$

$$(a_{n_i,\lambda_0^{-1}}(u_1) + b_{n_i,\lambda_0^{-1}}(\xi_1))J_{n_i,\lambda_0^{-1}} = (a_{n_i,\lambda_0^{-1}}(u_2) + b_{n_i,\lambda_0^{-1}}(\xi_2))J_{n_i,\lambda_0^{-1}}. \tag{8.60}$$

We extract the coefficients from the formulas (8.59) and (8.60); we obtain the relations

$$(a_{n_i,\lambda_0}(u_1) + b_{n_i,\lambda_0}(\xi_1)) = (a_{n_i,\lambda_0}(u_2) + b_{n_i,\lambda_0}(\xi_2)). \tag{8.61}$$

$$(a_{n_i,\lambda_0^{-1}}(u_1) + b_{n_i,\lambda_0^{-1}}(\xi_1)) = (a_{n_i,\lambda_0^{-1}}(u_2) + b_{n_i,\lambda_0^{-1}}(\xi_2)). \tag{8.62}$$

On the other hand, formulas (8.38), (8.39), (8.40) give us the second group of relations between the coefficients

$$b_{n_i,\lambda_0}(\xi_1) = b_{n_i,\lambda_0^{-1}}(\xi_1) \tag{8.63}$$

$$b_{n_i,\lambda_0^{-1}}(\xi_2) = b_{n_i,\lambda_0^{-1}}(\xi_2).$$

Equations (8.59)–(8.61), (8.60)–(8.62) and (8.63), describe the Jordan composition of the conjugate elements $\tilde{u}_1 := (u_1 + \xi_1)$, $\tilde{u}_2 := (u_2 + \xi_2)$.

We pass to the T_1^{loc} classes of these invertible elements appearing in the formulas (8.59)–(8.60). We get

$$(a_{n_i,\lambda_0}(u_1) + b_{n_i,\lambda_0}(\xi_1))[J_{n_i,\lambda}] = (a_{n_i,\lambda_0}(u_2) + b_{n_i,\lambda_0}(\xi_2))[J_{n_i,\lambda}] \tag{8.64}$$

and

$$(a_{n_i,\lambda_0^{-1}}(u_1) +; b_{n_i,\lambda_0^{-1}}(\xi_1))[J_{n_i,\lambda_0^{-1}}] = (a_{n_i,\lambda_0^{-1}}(u_2) + b_{n_i,\lambda_0^{-1}}(\xi_1))[J_{n_i,\lambda_0^{-1}}]. \tag{8.65}$$

We sum up formulas (8.64), (8.65); we use formulas (8.62), (8.63) to get

$$a_{n_i,\lambda_0}(u_1)[J_{n_i\lambda_0}] + a_{n_i,\lambda_0^{-1}}(u_1)[J_{n_i\lambda_0^{-1}}] + b_{n_i,\lambda_0}(\xi_1))([J_{n_i\lambda_0}] + [J_{n_i\lambda_0^{-1}}]) = \tag{8.66}$$

$$= a_{n_i,\lambda_0}(u_2)[J_{n_i\lambda_0}] + a_{n_i,\lambda_0^{-1}}(u_2)[J_{n_i\lambda_0^{-1}}] + b_{n_i,\lambda_0}(\xi_2))([J_{n_i\lambda_0}] + [J_{n_i\lambda_0^{-1}}]).$$

We take into account formula (8.49) again to get rid of the terms coming from ξ_1 and ξ_2; formula (8.66) becomes

$$a_{n_i,\lambda_0}(u_1)[J_{n_i,\lambda_0}] + a_{n_i,\lambda_0^{-1}}(u_1)[J_{n_i,\lambda_0^{-1}}] = \tag{8.67}$$

$$= a_{n_i,\lambda_0}(u_2)[J_{n_i,\lambda}] + a_{n_i,\lambda_0^{-1}}(u_2)[J_{n_i,\lambda_0^{-1}}].$$

This relation is precisely formula (8.52).

The formulas (8.55) to (8.67) have referred to the case of Jordan canonical forms with $\lambda^{-1} \neq \lambda$. Next we consider the remaining cases.

We assume again the situation described by Eqs. (8.51), (8.53). We need to show that (8.52) holds in the special case in which the class $\hat{\lambda} \in \hat{\mathbb{C}}$ contains just one element instead of two.

We use again formulas (8.55)–(8.58) and the corresponding coefficients $a_{n_i,\lambda}(u_\alpha)$, $b_{n_i,\lambda}(u_\alpha)$, $\alpha = 1, 2$.

Keeping in mind that $\lambda^{-1} = \lambda$, one has $J(J_{n_i,\lambda^{-1}}) = J(J_{n_i,\lambda})$ and hence

$$\xi_\alpha = J_{n_i,\lambda} \oplus J_{n_i,\lambda^{-1}} = 2J_{n_i,\lambda} \tag{8.68}$$

and

$$[\xi_\alpha] = [J_{n_i,\lambda} \oplus J_{n_i,\lambda}] = 2[J_{n_i,\lambda}] = 0 \in T_1^{loc}(\mathbb{C}) \tag{8.69}$$

With regard to the previous equations (8.59)–(8.62), we see that in this special case we do not obtain pairs of such equations, but just one

$$(a_{n_i,\lambda}(u_1) + 2b_{n_i,\lambda}(\xi_1))J_{n_i,\lambda} = (a_{n_i,\lambda}(u_2) + 2b_{n_i,\lambda}(\xi_2))J_{n_i,\lambda}. \tag{8.70}$$

We extract the coefficients from the formula (8.71) below; the composition of the Jordan form of the invertible matrices appearing in the formulas (8.51), (8.53) leads to the relations

$$a_{n_i,\lambda_0}(u_1) + 2b_{n_i,\lambda_0}(\xi_1) = a_{n_i,\lambda_0}(u_2) + 2b_{n_i,\lambda_0}(\xi_2). \tag{8.71}$$

Formula (8.71) gives

$$a_{n_i,\lambda_0}(u_1) \cong a_{n_i,\lambda_0}(u_2) \quad mod.2. \tag{8.72}$$

We come back to the formula (8.71) (which expresses the Jordan composition of the four invertible elements of formula (8.51)) to look for necessary relations inside $T_1^{loc}(\mathbb{C})$

$$a_{n_i,\lambda}(u_1)[J_{n_i,\lambda}] = a_{n_i,\lambda}(u_2)[J_{n_i,\lambda}]. \tag{8.73}$$

Relation (8.72) says that the formula (8.73) is indeed verified if the coefficients $a_{n_i,\lambda}(u_1)$, $a_{n_i,\lambda}(u_2)$ are taken modulo 2. This is precisely what formulas (8.47), (8.48) of the lemma say.

This completes the proof of Lemma 8.3.

We come back to the proof of Theorem 8.1. Lemma 8.3 provides most of the information stated by Theorem 8.1. We only need to explain how the last term of the formula (8.6) appears. We already established formula (8.68)

$$a_{n_i,\lambda_0}(u_1)[J_{n_i,\lambda_0}] + a_{n_i,\lambda_0^{-1}}(u_1)[J_{n_i,\lambda_0^{-1}}] = a_{n_i,\lambda_0}(u_2)[J_{n_i,\lambda}] + a_{n_i,\lambda_0^{-1}}(u_2)[J_{n_i,\lambda_0^{-1}}]. \tag{8.74}$$

We use relation (8.45) to eliminate $[J_{n_i,\lambda^{-1}}]$ from the formula (8.74); we get

$$(a_{n_i,\lambda_0}(u_1) - a_{n_i,\lambda_0^{-1}}(u_1))[J_{n_i,\lambda_0}] = (a_{n_i,\lambda_0}(u_2) - a_{n_i,\lambda_0^{-1}}(u_2))[J_{n_i,\lambda_0}] \in \mathbb{Z}. \tag{8.75}$$

which explains how integer coefficients appear in the formula (8.6).

This ends the proof of Theorem 8.1.

Chapter 9
Local Hochschild Homology of the Algebra of Hilbert–Schmidt Operators on Simplicial Spaces



Abstract In this section we compute the *local* Hochschild homology of the algebra of Hilbert–Schmidt operators on homogeneous simplicial complexes.

9.1 The Main Result

Theorem 9.1 *The* local *continuous Hochschild homology of the algebra of real, resp. complex, Hilbert–Schmidt operators on the countable, locally finite, homogeneous n-dimensional simplicial space X is isomorphic to the real, resp. complex, Alexander–Spanier homology of X.*

Proof The remaining part of this chapter is devoted to the proof of the theorem.

9.2 Preliminaries and Notation

9.2.1 The Space: Hilbert–Schmidt Kernels and Operators

Let X be a connected, locally finite, countable simplicial set of dimension n. Let Δ_α, $\alpha \in \Lambda$, denote all n-dimensional simplices of X. We assume that any simplex of X is contained in an n-dimensional simplex of X; such a simplicial complex will be called *homogeneous*. In particular, X might be an n-dimensional pseudo-manifold or manifold. We assume that each simplex Δ_α is endowed with a Lebesgue measure μ_{Δ_α}.

Let $\{e^n_\alpha\}_{n\in N}$ be an ortho-normal basis of L_2 real/complex valued functions on Δ_α.

Then the complex conjugates $\{\bar{e}^n_\alpha\}_{n\in N}$ form too an ortho-normal basis of $L_2(\Delta_\alpha)$.

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A Hilbert–Schmidt kernel on X is an L_2-function on $X \times X$. It is given by an L_2-convergent series

$$K = \sum_{\alpha\beta,ij} K_{ij}^{\alpha\beta} \, (e_\alpha^i \times \bar{e}_\beta^j) \tag{9.1}$$

with real/complex coefficients $K_{ij}^{\alpha\beta}$. Given the Hilbert–Schmidt kernel K, the decomposition (9.1) is unique.

A Hilbert–Schmidt kernel of type $(e_\alpha^i \times \bar{e}_\beta^j)$ will be called *elementary*.

Any Hilbert–Schmidt kernel K defines a bounded Hilbert–Schmidt operator $Op(K) : L_2(X) \to L_2(X)$

$$(Op(K)\phi)(x) := \int_X K(x, y) \, \phi(y) \, d\mu(y). \tag{9.2}$$

The composition of two elementary Hilbert–Schmidt operators is given by

$$OpK_1 \circ OpK_2 = OpK, \tag{9.3}$$

where

$$K(x, z) = \int_X K_1(x, y).K_2(y, z)d\mu(y). \tag{9.4}$$

This kernel K is by definition the composition of the kernels K_1, K_2, written $K = K_1 \circ K_2$. In other words,

$$Op(K_1 \circ K_2) = Op(K_1) \circ Op(K_2). \tag{9.5}$$

In particular,

$$(e_\alpha^i \times \bar{e}_\beta^j) \circ (e_\gamma^k \times \bar{e}_\eta^l) = \delta^{jk}\delta_{\beta\gamma} \, (e_\alpha^i \times \bar{e}_\eta^l), \tag{9.6}$$

where δ^{jk} and $\delta_{\beta\gamma}$ are the Kronecker symbols.

The Hilbert–Schmidt operators form an associative algebra denoted $HS(X)$.

The algebra of Hilbert–Schmidt operators is the Schatten class L^2 of compact operators on the separable Hilbert space $H = L_2(X)$.

9.3 Hochschild and *Local* Hochschild Homology of Hilbert–Schmidt Operators

We recall the basic definitions regarding the Hochschild homology of associative algebras A. In this paper we compute the *local* Hochschild homology of the algebra $A = HS(X)$. *Local* Hochschild homology was defined by Teleman [125]. It is the analogue of the Alexander–Spanier construction implanted into the Hochschild complex. This is done by considering only Hochschild chains which have small support about the main diagonal of the powers of the space X.

In Sects. 9.4 and 9.5 of this chapter we introduce certain algebraic constructions within the Hochschild complex of this algebra, ignoring the supports of the chains. It is important to stress here that all algebraic manipulations we introduce are *local*, and therefore, they are well defined in the *local* Hochschild complex. These will enable us, at the end of this chapter in Sect. 9.8, to complete the computation of the *local* Hochschild homology of the algebra of Hilbert–Schmidt operators and to connect it naturally with the Alexander–Spanier homology.

The vector space of Hochschild p-chains of the algebra $HS(X)$ with values in itself is by definition

$$C_p(HS(X)) \;=\; \otimes_C^{p+1} HS(X). \tag{9.7}$$

The Hochschild boundary $b_{(p)} : C_p(HS(X)) \longrightarrow C_{p-1}(HS(X))$ is

$$b_{(p)} = \sum_{k=0}^{k=p-1} b_{(p)k} \;+\; b_{(p)p}, \tag{9.8}$$

where

$$b_{(p)k} = (-1)^k \partial_k^H, \quad \textit{and} \quad b_{(p)p} = (-1)^p \partial_p^H \tag{9.9}$$

with

$$\partial^H_{(p)k}(K_0 \otimes_C K_1 \otimes_C \ldots \otimes_C K_p) \;=\; K_0 \otimes_C \ldots \otimes_C K_{k-1} \otimes_C (K_k \circ K_{k+1}) \otimes_C \ldots \otimes_C K_p \tag{9.10}$$

and

$$\partial^H_{(p)p}(K_0 \otimes_C K_1 \otimes_C \cdots \otimes_C K_p) \;=\; (K_p \circ K_0) \otimes_C K_1 \otimes_C \cdots \otimes_C K_{p-1}. \tag{9.11}$$

The operator $b^H_{(p)k}$ is called *Hochschild face operator of order* k. When no confusion occurs, the index (p), indicating the degree of chains, could be omitted.

Definition 9.1 Let $K = K_0 \otimes_C K_1 \otimes_C \cdots \otimes_C K_p$ where each of the factors K_i is elementary. Then K is called *elementary chain*.

When the ground algebra A is a locally convex topological algebra, it is customary to replace the algebraic tensor products by projective tensor products, see Connes [99]. The homology of the corresponding completed complex is called *continuous Hochschild homology* of the algebra A. In this paper we consider the completion $\bar{C}_p(HS)$ defined below.

Definition 9.2 Let $\bar{C}_p(HS)$ denote the space of L_2-functions on $(X \times X)^{p+1}$.

All considerations made in the sequel refer to the computation of the homology of the completed complex $\{\bar{C}_*(HS), b\}_*$.

The elementary chains form an ortho-normal basis of the Hilbert space $\bar{C}_p(HS)$. To simplify the notation we agree to denote $\bar{C}_p(HS)$ by $C_p(HS)$.

9.4 Algebraic Constructions

Definition 9.3 Let K be an elementary p-chain and let k any index $0 \leq k \leq p$. We say that K has a *k-gap* provided $\partial_k^H K = 0$.

Remark 9.1 The elementary chain K has a k-gap (for $k = p$ we intend $p + 1$ to be 0) provided the consecutive elementary factors

$$(e_{\alpha_k}^{i_k} \times \bar{e}_{\beta_k}^{j_k}) \otimes_C (e_{\alpha_{k+1}}^{i_{k+1}} \times \bar{e}_{\beta_{k+1}}^{j_{k+1}}) \tag{9.12}$$

either have different supports $\beta_k \neq \alpha_{k+1}$ or they involve different elements of the Hilbert space basis, $e^{j_k} \neq e^{i_{k+1}}$, or both.

Definition 9.4

(i) Let $C_p^0(HS(X)) \subset \bar{C}_p(HS(X))$ be the vector sub-space generated by those elementary chains K which contain at least one gap.

By definition,

$$C_0^0(HS(X)) = \sum_{(i_0,\alpha_0)\neq(j_0,\beta_0)} K_{i_0 j_0}^{\alpha_0 \beta_0}\, e_{\alpha_0}^{i_0} \times \bar{e}_{\beta_0}^{j_0}. \tag{9.13}$$

(ii) $\{C_p^0(HS(X)),\ b\}_{0\leq p}$ is a sub-complex of the Hochschild complex.

(iii) Let $C_p^{\Delta}(HS(X)) \subset \bar{C}_p(HS_0(X))$ be the vector sub-space generated by those elementary chains K which possess no gap.

Any chain belonging to $C_p^{\Delta}(HS(X))$ is the sum of an L_2-convergent series of elementary chains

$$K = \sum_{\alpha_k, i_k} K_{i_0,\ldots,i_p}^{\alpha_{i_0},\ldots,i_p}\ (e_{\alpha_0}^{i_0} \times \bar{e}_{\alpha_1}^{i_1}) \otimes_C (e_{\alpha_1}^{i_1} \times \bar{e}_{\alpha_2}^{i_2}) \otimes_C \cdots \otimes_C (e_{\alpha_p}^{i_p} \times \bar{e}_{\alpha_0}^{i_0}), \tag{9.14}$$

with

$$\sum_{\alpha_k, i_k} |K_{i_0,\ldots i_p}^{\alpha_{i_0},\ dots, i_p}|^2 < \infty. \tag{9.15}$$

By definition,

$$C_0^{\Delta}(HS(X)) \ = \ \sum_{(i_0,\alpha_0)} K_{i_0 i_0}^{\alpha_0 \alpha_0}\ e_{\alpha_0}^{i_0} \times \bar{e}_{\alpha_0}^{i_0}. \tag{9.16}$$

(iv) $C_p^{\Delta}(HS(X)),\ b\}_{0\leq p}$ is a complex. This complex is called *diagonal complex.*

Concerning (ii), we observe that if a Hochschild boundary face $\partial_{(p)k}^{H}$ acts on a gap, the result is the zero chain; if the Hochschild boundary does not involve the gap, the gap survives, which shows that $C_p^0(HS(X)),\ b\}_{0\leq p}$ is indeed a homology complex.

Concerning (iv), formula (9.6) shows that all Hochschild boundary faces keep the structure of the chains (9.14) unaltered, which shows that $C_p^{\Delta}(HS(X)),\ b\}_{0\leq p}$ is a homology complex too.

Proposition 9.1 *The Hochschild complex* $\{\bar{C}_*(HS(X)),\ b\}_*$ *decomposes into a direct sum of Hochschild sub-complexes*

$$\{\bar{C}_*(HS(X)),\ b\}_* \ = \ \{C_*^0(HS(X)),\ b\}_* \oplus \{C_*^{\Delta}(HS(X)),\ b\}_* \tag{9.17}$$

Proof Obvious.

9.4.1 Homotopy Operator s: The Splitting

Proposition 9.2

(i) The complex $\{C_*^0(HS(X)),\ b\}_*$ *is acyclic.*
(ii) The inclusion of the diagonal sub-complex $\{C_*^{\Delta}(HS(X)),\ b\}_*$ *in the complex* $\{C_*(HS(X)),\ b\}_*$ *induces isomorphism in homology.*
(iii) Therefore, the homology of the diagonal sub-complex $\{C_*^{\Delta}(HS(X)),\ b\}_*$ *is the continuous Hochschild homology of the algebra of Hilbert–Schmidt operators.*

In view of Proposition 9.8, only part (i) needs to be proven.

The proof of (i) is based on a homotopy operator $s := \{s_{(p)}\}$,

$$s_{(p)} : C_p^0(HS(X)) \longrightarrow C_{p+1}^0(HS(X)) \tag{9.18}$$

defined below.

The construction of the homotopy operator $s_{(p)}$ keeps track of the *first* gap present in each elementary monomial. For, suppose K is an elementary p-chain which possesses an r-gap, with r minimal, $0 \leq r \leq p$

$$K = (e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \cdots \otimes_C (e^{i_r}_{\alpha_r} \times \bar{e}^{j_r}_{\beta_r}) \otimes_C (e^{i_{r+1}}_{\alpha_{r+1}} \times \bar{e}^{j_{r+1}}_{\beta_{r+1}}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{j_p}_{\beta_p}). \tag{9.19}$$

The operator $s_{(p)}(K)$ is defined by inserting the factor $(e^{j_r}_{\beta_r} \times \bar{e}^{j_r}_{\beta_r})$

$$s_{(p)}(K) :=$$

$$= (-1)^r (e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) \otimes_C \ldots \otimes_C (e^{i_r}_{\alpha_r} \times \bar{e}^{j_r}_{\beta_r}) \otimes_C (e^{j_r}_{\beta_r} \times \bar{e}^{j_r}_{\beta_r}) \otimes_C (e^{i_{r+1}}_{\alpha_{r+1}} \times \bar{e}^{j_{r+1}}_{\beta_{r+1}}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{j_p}_{\beta_p}). \tag{9.20}$$

If K is an elementary chain of degree 0, then the homotopy $s_{(0)}$ is defined by

$$s_{(0)}(e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) := (e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) \otimes_C (e^{j_0}_{\beta_0} \times \bar{e}^{j_0}_{\beta_0}). \tag{9.21}$$

We postpone to Sect. 9.5.3. the checking that the operators $s_{(p)}$ are well defined on the spaces $C^0_p(HS(X))$.

Lemma 9.1 *The operators s satisfy*

$$(\, b\, s_{(p)} \, + \, s_{(p-1)}\, b\,)(K) \, = \, K \tag{9.22}$$

on $C^0_*(HS)$.

Proof One has

$$b\, s_{(p)}(K) = \sum_{k=0}^{k=r-1} b_{(p+1)k}(s_{(p)}K) \, + \, b_{(p+1)r}(s_{(p)}K) \, + \, b_{(p+1)r+1}(s_{(p)}K) \, + \sum_{k=r+2}^{k=p+1} b_{(p+1)k}(s_{(p)}K) \, =$$

$$= \sum_{k=0}^{k=r-1} b_{(p+1)k}(s_{(p)}K) \, + \, K \, + \, 0 \, + \sum_{k=r+2}^{k=p+1} b_{(p+1)k}(s_{(p)}K).$$

Therefore

$$b\, s_{(p)}(K) = \sum_{k=0}^{k=r-1} b_{(p+1)k}(s_{(p)}K) \, + \, K \, + \sum_{k=r+2}^{k=p+1} b_{(p+1)k}(s_{(p)}K). \tag{9.23}$$

On the other hand, we split the boundary bK in three parts and we apply the operator $s_{(p-1)}$

$$s_{(p-1)}\, b(K) = \sum_{k=0}^{k=r-1} s_{(p-1)} b_{(p)k} K \; + \; s_{(p-1)} b_{(p)r} K \; + \sum_{k=r+1}^{k=p} s_{(p-1)} b_{(p)k} K \; =$$

$$= \sum_{k=0}^{k=r-1} s_{(p-1)} b_{(p)k} K \; + \sum_{k=r+1}^{k=p} s_{(p-1)} b_{(p)k} K. \tag{9.24}$$

A direct check shows that both the first and the last terms from Eqs. (9.23) and (9.24) cancel out, respectively. Here, in particular, we intend to check the formula (9.22) when the first gap of K_p is a p-gap

$$K = (e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots .. \otimes_C (e^{i_{p-1}}_{\alpha_{p-1}} \times \bar{e}^{i_p}_{\alpha_p}) \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{j_p}_{\beta_p}) \tag{9.25}$$

with $(\beta_p, j_p) \neq (\alpha_0, i_0)$. Then

$$\begin{aligned}
&bs_{(p)}(K) \; = \\
&. = (-1)^p \{ \sum_{k=0}^{k=p-1} b_{(p+1)k} (e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots .. \otimes_C (e^{i_{p-1}}_{\alpha_{p-1}} \times \bar{e}^{i_p}_{\alpha_p}) \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{j_p}_{\beta_p}) \} \otimes_C (e^{j_p}_{\beta_p} \times \bar{e}^{j_p}_{\beta_p}) + \\
&+ (-1)^p (-1)^p \partial_{(p+1)p} \{ (e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots .. \otimes_C (e^{i_{p-1}}_{\alpha_{p-1}} \times \bar{e}^{i_p}_{\alpha_p}) \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{j_p}_{\beta_p}) \otimes_C (e^{j_p}_{\beta_p} \times \bar{e}^{j_p}_{\beta_p}) \} + \\
&+ (-1)^{p+1} (-1)^p \partial_{(p+1)p+1} \{ (e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots .. \otimes_C (e^{i_{p-1}}_{\alpha_{p-1}} \times \bar{e}^{i_p}_{\alpha_p}) \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{j_p}_{\beta_p}) \otimes_C (e^{j_p}_{\beta_p} \times \bar{e}^{j_p}_{\beta_p}) \} \; = \\
&= (-1)^p \sum_{k=0}^{k=p-1} b_{(p+1)k} (e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots .. \otimes_C (e^{i_{p-1}}_{\alpha_{p-1}} \times \bar{e}^{i_p}_{\alpha_p}) \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{j_p}_{\beta_p}) \} \otimes_C (e^{j_p}_{\beta_p} \times \bar{e}^{j_p}_{\beta_p}) + K.
\end{aligned} \tag{9.26}$$

On the other hand,

$$s_{(p-1)} b(K) \; = \; s_{(p-1)} \{ \sum_{k=0}^{k=p-1} b_{(p)k}(K) \; + \; b_{(p)p}(K) \} \; = \; (-1)^{p-1} \sum_{k=0}^{k=p-1} b_{(p)k}(K) \otimes_C (e^{j_p}_{\beta_p} \times \bar{e}^{j_p}_{\beta_p}). \tag{9.27}$$

Summing up formulas (9.26) and (9.27) shows that the identity (9.22) holds also for p-elementary chains whose first gap is a p-gap.

Finally, we verify the homotopy formula (9.22) for chains of degree zero. For such chains, one has

$$\begin{aligned}
(bs_{(0)} + sb)(e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) = bs_{(0)}(e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) \; &= \; b(\; (e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) \otimes_C (e^{j_0}_{\beta_0} \times \bar{e}^{j_0}_{\beta_0}) \;) \; = \\
&= \; (e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) \; - \; (e^{j_0}_{\beta_0} \times \bar{e}^{j_0}_{\beta_0}) \;) \circ (e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) =
\end{aligned} \tag{9.28}$$

$$= (e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) \; - \; \delta^{j_0 i_0}\delta_{\beta_0\alpha_0}(e^{j_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) \; = \; (e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0})$$

because $\delta^{j_0 i_0}\delta_{\beta_0\alpha_0} = 0$, which proves the desired formula.

These computations complete the proof of Lemma 9.1. This completes the proof of Proposition 9.8(i).

Part (ii) of Proposition 9.8 follows from part (i) along with Proposition 9.7. This completes the proof of Proposition 9.8.

9.4.2 Homotopy Operator S

For each n-simplex Δ_α we choose and fix an element e^{I_α} of the corresponding ortho-normal Hilbert basis. To simplify the notation we denote e^{I_α} by I.

Definition 9.5 Let $S_{(p)} : C^{\Delta}_p(HS(X)) \longrightarrow C^{\Delta}_{p+1}(HS(X))$ be defined on elementary chains by the formula

$$S_{(p)}[(e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{i_0}_{\alpha_0})] :=$$
$$:= (e^{i_0}_{\alpha_0} \times \bar{e}^{I}_{\alpha_0}) \otimes_C (e^{I}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{i_0}_{\alpha_0}), \qquad (9.29)$$

i.e. S is defined by inserting the factor $\bar{e}^{I}_{\alpha_0} \otimes_C e^{I}_{\alpha_0}$ into the expression of K.

9.4.2.1 The Homology of the Sub-Complex $\{C^{\Delta}_p(HS(X)), b'\}$

Proposition 9.3 *The operator S satisfies*

$$b' \, S + S \, b' = Id. \qquad (9.30)$$

Proof We compute first $b'S_{(p)}(K)$. One has

$$b'S_{(p)}(K) = b'S_{(p)}[(e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{i_0}_{\alpha_0})] :=$$
$$:= b'[(e^{i_0}_{\alpha_0} \times \bar{e}^{I}_{\alpha_0}) \otimes_C (e^{I}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \cdots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{i_0}_{\alpha_0})] =$$
$$= (e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{i_0}_{\alpha_0})$$
$$- (e^{i_0}_{\alpha_0} \times \bar{e}^{I}_{\alpha_0}) \otimes_C b'[(e^{I}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{i_0}_{\alpha_0})] =$$
$$= K - (e^{i_0}_{\alpha_0} \times \bar{e}^{I}_{\alpha_0}) \otimes_C b'[(e^{I}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{i_0}_{\alpha_0})]. \qquad (9.31)$$

On the other hand

$$
\begin{aligned}
S_{(p-1)}b'(K) \;=\; & \\
&= S_{(p-1)}\,[(e^{i_0}_{\alpha_0}\times\bar e^{i_2}_{\alpha_2})\otimes_C(e^{i_2}_{\alpha_2}\times\bar e^{i_3}_{\alpha_3})\otimes_C\cdots\otimes_C(e^{i_p}_{\alpha_p}\times\bar e^{i_0}_{\alpha_0}) \\
&\quad-(e^{i_0}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})\otimes_C(e^{i_1}_{\alpha_1}\times\bar e^{i_3}_{\alpha_3})\otimes_C\cdots\otimes_C(e^{i_p}_{\alpha_p}\times\bar e^{i_0}_{\alpha_0})+\ldots. \\
+\;(-1)^k&(e^{i_0}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})\otimes_C(e^{i_1}_{\alpha_1}\times\bar e^{i_2}_{\alpha_2})\otimes_C\cdots\otimes_C(e^{i_k}_{\alpha_k}\times\bar e^{i_{k+2}}_{\alpha_{k+2}})\otimes_C\cdots\otimes_C(e^{i_p}_{\alpha_p}\times\bar e^{i_0}_{\alpha_0})\;+\;\ldots \\
&\quad+(-1)^{p-1}(e^{i_0}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})\otimes_C(e^{i_1}_{\alpha_1}\times\bar e^{i_2}_{\alpha_2})\otimes_C\ldots\otimes_C(e^{i_{p-1}}_{\alpha_{p-1}}\times\bar e^{i_0}_{\alpha_0})]= \\
&= (e^{i_0}_{\alpha_0}\times\bar e^{I}_{\alpha_0})\otimes_C(e^{I}_{\alpha_0}\times\bar e^{i_2}_{\alpha_2})\otimes_C(e^{i_2}_{\alpha_2}\times\bar e^{i_3}_{\alpha_3})\otimes_C\ldots\otimes_C(e^{i_p}_{\alpha_p}\times\bar e^{i_0}_{\alpha_0}) \\
&\quad-(e^{i_0}_{\alpha_0}\times\bar e^{I}_{\alpha_0})\otimes_C(e^{I}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})\otimes_C(e^{i_1}_{\alpha_1}\times\bar e^{i_3}_{\alpha_3})\otimes_C\cdots\otimes_C(e^{i_p}_{\alpha_p}\times\bar e^{i_0}_{\alpha_0})+\ldots. \\
+(-1)^k&(e^{i_0}_{\alpha_0}\times\bar e^{I}_{\alpha_0})\otimes_C(e^{I}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})\otimes_C(e^{i_1}_{\alpha_1}\times\bar e^{i_2}_{\alpha_2})\otimes_C\cdots\otimes_C(e^{i_k}_{\alpha_k}\times\bar e^{i_{k+2}}_{\alpha_{k+2}})\otimes_C\cdots\otimes_C(e^{i_p}_{\alpha_p}\times\bar e^{i_0}_{\alpha_0})\;+\;\ldots \\
&\quad+\;(-1)^{p-1}(e^{i_0}_{\alpha_0}\times\bar e^{I}_{\alpha_0})\otimes_C(e^{I}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})\otimes_C(e^{i_1}_{\alpha_1}\times\bar e^{i_2}_{\alpha_2})\otimes_C\ldots\otimes_C(e^{i_{p-1}}_{\alpha_{p-1}}\times\bar e^{i_0}_{\alpha_0})]= \\
&= (e^{i_0}_{\alpha_0}\times\bar e^{I}_{\alpha_0})\otimes_C b'[(e^{I}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})\otimes_C(e^{i_1}_{\alpha_1}\times\bar e^{i_2}_{\alpha_2})\otimes_C\ldots\otimes_C(e^{i_p}_{\alpha_p}\times\bar e^{i_0}_{\alpha_0})]. \qquad (9.32)
\end{aligned}
$$

Summing up formulas (9.31) and (9.32) proves the proposition.

Corollary 9.1 *The complex* $\{C^{\Delta}_{p}(HS(X)), b'\}$ *is acyclic.*

9.4.2.2 The Homology of the Sub-Complex $\{C^{\Delta}_{p}(HS(X)), b\}$

Proposition 9.4 *The operator S establishes on the complex* $K\{C^{\Delta}_{p}(HS(X)), b\}$ *a homotopy between the identity and the operator* θ

$$
b\,S_{(p)}\;+\;S_{(p-1)}\,b\;=\;Id.-\theta, \qquad (9.33)
$$

where

$$
\begin{aligned}
&\theta[(e^{i_0}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})\otimes_C(e^{i_1}_{\alpha_1}\times\bar e^{i_2}_{\alpha_2})\otimes_C\cdots\otimes_C(e^{i_p}_{\alpha_p}\times\bar e^{i_0}_{\alpha_0})]= \\
&=\;(-1)^p[\;(e^{i_p}_{\alpha_p}\times\bar e^{I}_{\alpha_0})\otimes_C(e^{I}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})-(e^{i_p}_{\alpha_p}\times\bar e^{I}_{\alpha_p})\otimes_C(e^{I}_{\alpha_p}\times\bar e^{i_1}_{\alpha_1})\;]\otimes_C[(e^{i_1}_{\alpha_1}\times\bar e^{i_2}_{\alpha_2})\otimes_C\cdots\otimes_C(e^{i_{p-1}}_{\alpha_{p-1}}\times\bar e^{i_p}_{\alpha_p})]. \qquad (9.34)
\end{aligned}
$$

Proof For the proof of this proposition we use Proposition 9.11. One has

$$
\begin{aligned}
(\;bS_{(p)}+S_{(p-1)}b\;)(K)= & \\
&=\;[\;b'+(-1)^{p+1}\partial_{(p+1)p+1}\;]S_{(p)}\;+\;S_{(p-1)}[\;b'+(-1)^p\partial_{(p)p}\;)](K)= \\
&=\;K\;+\;(-1)^{p+1}\partial_{(p+1)p+1}S_{(p)}\;(K)\;+\;(-1)^pS_{(p-1)}\partial_{(p)p}\;(K)= \\
&=\;K\;+\;(-1)^{p+1}\partial_{(p+1)p+1}[(e^{i_0}_{\alpha_0}\times\bar e^{I}_{\alpha_0})\otimes_C(e^{I}_{\alpha_0}\times\bar e^{i_1}_{\alpha_1})\otimes_C(e^{i_1}_{\alpha_1}\times\bar e^{i_2}_{\alpha_2})\otimes_C\cdots\otimes_C(e^{i_p}_{\alpha_p}\times\bar e^{i_0}_{\alpha_0})]\;+
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^p S_{(p-1)}[(e_{\alpha_p}^{i_p} \times \bar{e}_{\alpha_1}^{i_1}) \otimes_C (e_{\alpha_1}^{i_1} \times \bar{e}_{\alpha_2}^{i_2}) \otimes_C \cdots \otimes_C (e_{\alpha_{p-1}}^{i_{p-1}} \times \bar{e}_{\alpha_p}^{i_p})] = \\
&= K + (-1)^{p+1}(e_{\alpha_p}^{i_p} \times \bar{e}_{\alpha_0}^{I}) \otimes_C (e_{\alpha_0}^{I} \times \bar{e}_{\alpha_1}^{i_1}) \otimes_C (e_{\alpha_1}^{i_1} \times \bar{e}_{\alpha_2}^{i_2}) \otimes_C \cdots \otimes_C (e_{\alpha_{p-1}}^{i_{p-1}} \times \bar{e}_{\alpha_p}^{i_p}) \otimes_C (e_{\alpha_p}^{i_p} \times \bar{e}_{\alpha_0}^{i_0}) + \\
&+ (-1)^p[(e_{\alpha_p}^{i_p} \times \bar{e}_{\alpha_p}^{I}) \otimes_C (e_{\alpha_p}^{I} \times \bar{e}_{\alpha_1}^{i_1}) \otimes_C (e_{\alpha_1}^{i_1} \times \bar{e}_{\alpha_2}^{i_2})] \otimes_C \cdots \otimes_C (e_{\alpha_{p-1}}^{i_{p-1}} \times \bar{e}_{\alpha_p}^{i_p})] = \\
&= K - (-1)^p[\,(e_{\alpha_p}^{i_p} \times \bar{e}_{\alpha_0}^{I}) \otimes_C (e_{\alpha_0}^{I} \times \bar{e}_{\alpha_1}^{i_1}) - (e_{\alpha_p}^{i_p} \times \bar{e}_{\alpha_p}^{I}) \otimes_C (e_{\alpha_p}^{I} \times \bar{e}_{\alpha_1}^{i_1})\,] \otimes_C [(e_{\alpha_1}^{i_1} \times \bar{e}_{\alpha_2}^{i_2}) \otimes_C \cdots \otimes_C (e_{\alpha_{p-1}}^{i_{p-1}} \times \bar{e}_{\alpha_p}^{i_p})],
\end{aligned}
\tag{9.35}
$$

which completes the proof of Proposition 9.13.

We intend to describe the Hochschild boundary acting on $C_p^{\Delta}(HS(X))$ in a more topological fashion; this will help us to understand better the hidden geometry and suggest the next constructions. For doing this we introduce the following:

Notation 9.2 *To the elementary chain* $K \in C_p^{\Delta}(HS(X))$

$$
K = (e_{\alpha_0}^{i_0} \times \bar{e}_{\alpha_1}^{i_1}) \otimes_C (e_{\alpha_1}^{i_1} \times \bar{e}_{\alpha_2}^{i_2}) \otimes_C \cdots \otimes_C (e_{\alpha_p}^{i_p} \times \bar{e}_{\alpha_0}^{i_0}) \tag{9.36}
$$

we bi-univocally associate the symbol

$$
[K] = \begin{bmatrix} i_1 & \dots & i_{p-1} & i_p & i_0 \\ \alpha_1 & \dots & \alpha_{p-1} & \alpha_p & \alpha_0 \end{bmatrix} \tag{9.37}
$$

Proposition 9.5 *Using the Notation 9.2, the Hochschild boundary faces are*

$$
[b_{(p)k}K] = (-1)^{k-1} \begin{bmatrix} i_1 & \dots & \hat{i_k} & \dots & i_{p-1} & i_p & i_0 \\ \alpha_1 & \dots & \hat{\alpha_k} & \dots & \alpha_{p-1} & \alpha_p & \alpha_0 \end{bmatrix}, \text{for } 0 \le k \le p-1 \tag{9.38}
$$

and

$$
[b_{(p)p}K] = (-1)^p \begin{bmatrix} i_1 & \dots & i_k & \dots & i_{p-1} & i_p & \hat{i_0} \\ \alpha_1 & \dots & \alpha_k & \dots & \alpha_{p-1} & \alpha_p & \hat{\alpha_0}. \end{bmatrix} \tag{9.39}
$$

Definition 9.6 Let $[X]^{\Delta}$ be the simplicial set whose vertices are the column-pairs $[i_k \alpha_k]$ and whose boundary faces $\partial_{(p)k}$ are the simplicial boundary faces.

Proposition 9.5 implies the next result.

Theorem 9.3 *The linear bijective mapping* $[\ ] : K \to [K]$ *interchanges*

(i) the Hochschild boundary faces with the simplicial boundary faces and hence
(ii) $[\ ]$ *establishes an isomorphism from the complex* $C_p^{\Delta}(HS(X))$ *to the simplicial chain complex of the simplicial space* $[X]^{\Delta}$

$$
[bK] = \partial[K], \qquad \text{for any } K \in C_p^{\Delta}(HS(X)). \tag{9.40}
$$

Theorem 9.4

(i) The Hochschild homology of the algebra of Hilbert–Schmidt operators on X is isomorphic to the simplicial homology of $[X]^{\Delta}$*, (see Definition 9.16).*
(ii) The simplicial complex $[X]^{\Delta}$ *is an infinite dimensional simplex.*

Proof

(i) is a consequence of Theorem 9.3(ii).
(ii) In the simplicial complex $[X]^{\Delta}$ any column-vertex $[i_k, \alpha_k]$ may be joint with any column-vertex $[i_{k+1}, \alpha_{k+1}]$ to create the simplex (9.37). In other words, the simplicial complex $[X]^{\Delta}$ is an infinite dimensional simplex and therefore its homology is trivial.

Theorem 9.5 *Let X be a countable, locally finite simplicial complex of dimension n. Then, the Hochschild homology of the algebra of real or complex Hilbert–Schmidt operators on X is trivial.*

It is known that the Hochschild homology of Banach algebras is not interesting, see Connes [99], or that in many cases it is trivial, or that "cyclic homology is degenerate on Banach or C^*-algebras", see [115, p. 42 and Proposition 3.5, Corollary 3.6., p. 39]". The basic intent of this section is to contribute to the better understanding of the interplay between the analysis and the topology at the level of Hochschild homology. More specifically, we state that the relationship between the content of Theorems 9.5 and 9.1, or the relationship between the Hochschild homology with no control on the chain-supports and *local* Hochschild homology [125], is the same relationship which occurs in the definition of the Alexander–Spanier (co)-homology before and after the consideration of the control on the supports, see Sect. 9.8.1.

Proof Given that the homology of the simplex is trivial, Theorem 9.5 is an immediate consequence of Theorem 9.4.

Notation 9.2 allows us to describe the operator θ in a more geometrical fashion, allowing simplices to be multiplied by integers

$$\theta \begin{bmatrix} i_1 & \dots & i_{p-1} & i_p & i_0 \\ \alpha_1 & \dots & \alpha_{p-1} & \alpha_p & \alpha_0 \end{bmatrix} =$$

$$= (-1)^{p+1}[\begin{bmatrix} I & i_1 & i_2 & \dots & i_{p-1} & i_p \\ \alpha_p & \alpha_1 & \alpha_2 & \dots & \alpha_{p-1} & \alpha_p \end{bmatrix} - \begin{bmatrix} I & i_1 & i_2 & \dots & i_{p-1} & i_p \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_{p-1} & \alpha_p \end{bmatrix}], \tag{9.41}$$

which may be written further

$$\theta \begin{bmatrix} i_1 & \dots & i_{p-1} & i_p & i_0 \\ \alpha_1 & \dots & \alpha_{p-1} & \alpha_p & \alpha_0 \end{bmatrix} =$$

$$= (-1)^{p+1} \left(\begin{bmatrix} I \\ \alpha_p \end{bmatrix} - \begin{bmatrix} I \\ \alpha_0 \end{bmatrix} \right) \begin{bmatrix} i_1 & i_2 & \dots & i_{p-1} & i_p \\ \alpha_1 & \alpha_2 & \dots & \alpha_{p-1} & \alpha_p \end{bmatrix}. \tag{9.42}$$

Formula (9.42) helps us to iterate the operator θ easily.

Remark 9.2

1. The operator θ removes the last vertex $[i_0, \alpha_0]$, replaces it with the difference $[I, \alpha_p] - [I, \alpha_0]$ and places it in the first position
2. The operator θ does not affect the other vertices, except for a cyclical shift of them to the right.

Proposition 9.6 *The operator θ satisfies*

(i)

$$\theta^p \begin{bmatrix} i_1 & \dots & i_{p-1} & i_p & i_0 \\ \alpha_1 & \dots & \alpha_{p-1} & \alpha_p & \alpha_0 \end{bmatrix} =$$

$$= \left(\begin{bmatrix} I \\ \alpha_1 \end{bmatrix} - \begin{bmatrix} I \\ \alpha_2 \end{bmatrix} \right) \left(\begin{bmatrix} I \\ \alpha_2 \end{bmatrix} - \begin{bmatrix} I \\ \alpha_3 \end{bmatrix} \right) \dots \left(\begin{bmatrix} I \\ \alpha_{p-1} \end{bmatrix} - \begin{bmatrix} I \\ \alpha_p \end{bmatrix} \right) \left(\begin{bmatrix} I \\ \alpha_p \end{bmatrix} - \begin{bmatrix} I \\ \alpha_0 \end{bmatrix} \right) \begin{bmatrix} i_1 \\ \alpha_1 \end{bmatrix} \tag{9.43}$$

(ii)

$$(-1)^{p+1}\, \theta^{p+1} \begin{bmatrix} i_1 & \dots & i_{p-1} & i_p & i_0 \\ \alpha_1 & \dots & \alpha_{p-1} & \alpha_p & \alpha_0 \end{bmatrix} =$$

$$= \left(\begin{bmatrix} I \\ \alpha_p \end{bmatrix} - \begin{bmatrix} I \\ \alpha_1 \end{bmatrix} \right) \left(\begin{bmatrix} I \\ \alpha_2 \end{bmatrix} - \begin{bmatrix} I \\ \alpha_1 \end{bmatrix} \right) \dots \left(\begin{bmatrix} I \\ \alpha_{p-1} \end{bmatrix} - \begin{bmatrix} I \\ \alpha_{p-2} \end{bmatrix} \right) \left(\begin{bmatrix} I \\ \alpha_p \end{bmatrix} - \begin{bmatrix} I \\ \alpha_{p-1} \end{bmatrix} \right) \begin{bmatrix} I \\ \alpha_p \end{bmatrix}$$

$$- \left(\begin{bmatrix} I \\ \alpha_0 \end{bmatrix} - \begin{bmatrix} I \\ \alpha_1 \end{bmatrix} \right) \left(\begin{bmatrix} I \\ \alpha_2 \end{bmatrix} - \begin{bmatrix} I \\ \alpha_1 \end{bmatrix} \right) \dots \left(\begin{bmatrix} I \\ \alpha_{p-1} \end{bmatrix} - \begin{bmatrix} I \\ \alpha_{p-2} \end{bmatrix} \right) \left(\begin{bmatrix} I \\ \alpha_p \end{bmatrix} - \begin{bmatrix} I \\ \alpha_{p-1} \end{bmatrix} \right) \cdot \begin{bmatrix} I \\ \alpha_0 \end{bmatrix}$$

$$= \left(\begin{bmatrix} I \\ \alpha_p \end{bmatrix} - \begin{bmatrix} I \\ \alpha_1 \end{bmatrix} \right) \left(\begin{bmatrix} I \\ \alpha_2 \end{bmatrix} - \begin{bmatrix} I \\ \alpha_1 \end{bmatrix} \right) \dots \left(\begin{bmatrix} I \\ \alpha_{p-1} \end{bmatrix} - \begin{bmatrix} I \\ \alpha_{p-2} \end{bmatrix} \right) \left(\begin{bmatrix} I \\ \alpha_p \end{bmatrix} - \begin{bmatrix} I \\ \alpha_{p-1} \end{bmatrix} \right) \left(\begin{bmatrix} I \\ \alpha_p \end{bmatrix} - \begin{bmatrix} I \\ \alpha_0 \end{bmatrix} \right). \tag{9.44}$$

9.4.3 Homological Consequences

Definition 9.7

(i) Let $C_p^I(HS) \subset C_p^{\Delta}(HS)$ denote the subset consisting of all p-chains whose elementary components contain exclusively the Hilbert space basis index I.

$\{C_p^I(HS),\ b\}_{0\le p}$ is complex. This complex will be called *reduced diagonal complex.*

(ii) Let $[X]^I$ be the sub-simplex of the simplex $[X]^\Delta$ whose vertices are the column pairs $[I, \alpha]$.

Lemma 9.2 *The spaces $C_p^I(HS)$ have the properties*

(i) for any $0 \le p+1 \le \tilde{p}$,

$$\theta_{(p)}^{\tilde{p}}(C_p^\Delta(HS)) \subset C_p^I(HS) \tag{9.45}$$

(ii) $b,\ S,\ \theta^k$ transform the spaces $C_*^I(HS)$ into themselves, for any $0 \le k$,

(iii) $\{\ C_p^I(HS), b\ \}_{0\le p}$ is a sub-complex of $\{\ C_p^\Delta(HS), b\ \}_{0\le p}$

Proof Relations (9.42), (9.43) along with (9.40) imply (i).

(ii) Equations (9.37)–(9.39) imply the property relative to b. Definition 9.10, Eq. (9.29) implies the property concerning S and Eqs. (9.41)–(9.43) prove the property concerning θ^k.

Definition 9.8 For any $0 \le \tilde{p}$, we define

$$\tilde{S}_{(p)} : C_p^\Delta(HS) \longrightarrow C_{p+1}^\Delta(HS) \tag{9.46}$$

be given by

$$\tilde{S}_{(p)} := (1 + \theta_{(p)} + \theta_{(p)}^2 + \cdots + \theta_{(p)}^{p+1}) S_{(p)}. \tag{9.47}$$

We agree to represent the homomorphisms $\tilde{S}_{(p)}$ with the same notation when they are restricted to the sub-spaces $C_p^I(HS)$.

Remark 9.3 In the formula (9.47) the extra power $\theta_{(p)}^{p+1}$ is inserted. This extra power will be used in the proof of Proposition 9.7(ii) below, formula (9.51).

Proposition 9.7

(i) The operators $\theta_{(p)}$ commutes with b

$$\theta_{(p-1)}\, b \ = \ b\, \theta_{(p)} \tag{9.48}$$

(ii) The operators $\tilde{S}_{(p)}$ satisfy

$$\tilde{S}_{(p-1)} b_{(p)} + b_{(p+1)} \tilde{S}_{(p)} = 1 - \theta_{(p)}^{p+2} \tag{9.49}$$

both on the spaces $C_p^\Delta(HS)$ and $C_p^I(HS)$, for any $0 \le p$.

Proof

(i) The commutation relation is obtained by multiplying the relation (9.33) to the left and to the right by b

$$bSb = b(bS + Sb) = b - b\theta, \qquad bSb = (bS + Sb)b = b - \theta b. \tag{9.50}$$

(ii) The identity (9.49) is obtained by multiplying Eq. (9.33) to the left by $(1 + \theta_{(p)} + \theta_{(p)}^2 + \ldots + \theta_{(p)}^{p+1})$, along with the commutation identity (9.48).

Proposition 9.8

(i) Any p-cycle $K \in C_p^{\Delta}(HS)$ is co-homologous, inside the complex $\{ C_p^{\Delta}(HS), b \}_{0 \leq p}$ to the cycle $\theta_{(p)}^{\tilde{p}} K \in C_p^I(HS)$, for any $p + 1 \leq \tilde{p}$

(ii) If $K \in C_p^I(HS)$ is a boundary inside the complex $\{ C_p^{\Delta}(HS), b \}_{0 \leq p}$ ($K_{(p)} = bL_{(p+1)}$, where $L_{(p+1)} \in C_{p+1}^{\Delta}(HS)$), then K is a boundary inside the complex $\{ C_p^I(HS), b \}_{0 \leq p}$ ($K_{(p)} = b\tilde{L}_{(p+1)}$, where $\tilde{L}_{(p+1)} = \tilde{S}_{(p)} K_{(p)} + \theta_{(p+1)}^{p+2} L_{(p+1)} \in C_p^I(HS)$).

Proof

(i) For any cycle $K \in C_p^{\Delta}(HS)$, the relation (9.49) gives

$$(\tilde{S}_{(p-1)} b_{(p)} + b_{(p+1)} \tilde{S}_{(p)})(K) = (1 - \theta_{(p)}^{p+2})(K). \tag{9.51}$$

and then

$$K = \theta_{(p)}^{p+2} K + b_{(p+1)} (\tilde{S}_{(p)} K). \tag{9.52}$$

Equation (9.45) insures that $\theta_{(p)}^{p+2} K \in C_p^I(HS)$ while the commutation relation (9.48) implies that $\theta_{(p)}^{p+2} K$ is a cycle

$$b \left(\theta_{(p)}^{p+2} K \right) = \theta_{(p)}^{p+2} (bK) = 0. \tag{9.53}$$

Equation (9.52) complemented with this information proves part (i).

Notice that here we do not make any statement about whether or not $\tilde{S}_{(p)} K$ does belong to the sub-complex $C_{p+1}^I(HS)$. Part (ii) of Proposition 9.8 clarifies this point.

(ii) By hypothesis, $K \in C_p^I(HS)$ is a boundary inside the complex $\{ C_p^{\Delta}(HS), b \}_{0 \leq p}$; then K is a cycle in the complex $\{ C_p^I(HS), b \}_{0 \leq p}$.

We plug $K = bL$, into the identity (9.49) to get

$$(\tilde{S}_{(p-1)} b_{(p)} + b_{(p+1)} \tilde{S}_{(p)})(bL) = (1 - \theta_{(p)}^{p+2})(bL)$$

or

$$b_{(p+1)}\tilde{S}_{(p)}(bL) \;=\; K - \theta_p^{p+2}(bL),$$

which gives

$$K_{(p)} = b_{(p+1)}\tilde{S}_{(p)}(bL_{(p+1)}) \;+\; \theta_{(p)}^{p+2}(bL_{(p+1)}). \tag{9.54}$$

The commutation relation (9.48) gives us further

$$\begin{aligned} K_{(p)} \;&=\; b_{(p+1)}\tilde{S}_{(p)}(bL_{(p+1)}) \;+\; b_{(p+1)}\theta_{(p+1)}^{p+2}(L_{(p+1)}) \;= \\ &= b_{(p+1)}(\;\tilde{S}_{(p)}(bL_{(p+1)}) \;+\; \theta_{(p+1)}^{p+2}(L_{(p+1)})\;) \end{aligned}$$

which can be re-written

$$K_{(p)} = b_{(p+1)}\;(\;\tilde{S}_{(p)}K_{(p)} \;+\; \theta_{p+1}^{p+2}L_{(p+1)}\;). \tag{9.55}$$

By hypotheses $K_{(p)} \in C_p^I(HS)$. Then Lemma 9.2(ii) gives that

$$\tilde{S}_{(p)}(K_{(p)}) \in C_{p+1}^I(HS). \tag{9.56}$$

On the other hand, Lemma 9.2(i), formula (9.45) insures that

$$\theta_{(p+1)}^{p+2}(L_{(p+1)}) \in C_{p+1}^I(HS). \tag{9.57}$$

Equation (9.55) may be re-written

$$K = b\;\tilde{L}_{(p+1)},$$

where, in virtue of the relations (9.56), (9.57) one has

$$\tilde{L}_{(p+1)} \;=\; \tilde{S}_{(p)}K_{(p)} \;+\; \theta_{p+1}^{p+2}L_{(p+1)} \;\in\; C_{p+1}^I(HS). \tag{9.58}$$

This completes the proof of (ii).

Theorem 9.6 *The inclusion of complexes* $\iota : \{\; C_p^I(HS), b\;\}_{0\le p} \longrightarrow \{\; C_p^\Delta(HS),$ $b\;\}_{0\le p}$ *induces isomorphism in homology.*

Therefore, the homology of the reduced diagonal complex is the homology of the diagonal complex.

Proof Part (i) of Proposition 9.8 tells that any homology class in the complex $\{\; C_p^\Delta(HS), b\;\}_{0\le p}$ has a cycle representative in the sub-complex $\{\; C_p^I(HS), b\;\}_{0\le p}$. Therefore, the inclusion ι induces epimorphisms in homology.

The second part (ii) of the same proposition tells us that if the homology class of the cycle K of the sub-complex $\{ C_p^I(HS), b \}_{0\leq p}$ is a boundary in the complex $\{ C_p^{\Delta}(HS), b \}_{0\leq p}$, then it is a boundary in the sub-complex $\{ C_p^I(HS), b \}_{0\leq p}$ too. In other words, the inclusion ι induces monomorphisms in homology. This completes the proof of Theorem 9.6.

Proposition 9.8(ii) and Theorem 9.6 imply the first part of the following:

Theorem 9.7

(i) For any locally finite, countable, homogeneous, simplicial complex X, the continuous Hochschild homology of the algebra of Hilbert–Schmidt operators on X is isomorphic to the homology of the reduced diagonal sub-complex $\{ C_p^I(HS), b \}_{0\leq p}$.

(ii) The mapping $[\ ]$ induces an isomorphism from the sub-complex $\{ C_p^I(HS), b \}_{0\leq p}$ to the space of simplicial chains of the complex $[X]^I$,

(iii) the homology of the sub-complex $\{ C_p^I(HS), b \}_{0\leq p}$ is trivial.

Proof Looking back at Theorem 9.3, it is clear that the mapping $[\ ]$ establishes also an isomorphism from the complex $\{C_*^I(HS), b\}$ to the chain complex, with complex coefficients, of the space $[X]^I$ (see Definition 9.7(ii)). As the simplicial complex $[X]^I$ is a simplex, the result follows.

Remark 9.4

(i) Theorem 9.7 implies the triviality of the continuous Hochschild homology of the algebra of Hilbert–Schmidt operators. This result is not new, see [99]. In Sect. 9.8. we shall discuss how this result changes when we will be considering continuous Hochschild chains with small supports about the main diagonal, i.e. when we will be going to compute the *local* Hochschild homology of the algebra of Hilbert–Schmidt operators.

To complete the proof of Theorem 9.1, we will use the sub-complex $\{ C_p^{I,loc}(HS), b \}_{0\leq p}$ of $\{ C_p^I(HS), b \}_{0\leq p}$ consisting of those chains which have small supports. To be able to do this, we shall observe that the quasi-isomorphisms treated in Chaps. 4 and 5

$$\{ C_*(HS), b \} \longleftarrow \{ C_*^{\Delta}(HS), b \} \longleftarrow \{ C_*^I(HS), b \}$$

pass to the *local* sub-complex.

(ii) It is interesting to investigate more closely the homotopy operator $\tilde{S}$.

9.5 Analytic Considerations

In this section we regard the chains of the complex $C_*(HS(X))$ and we discuss both the continuity of the Hochschild boundary b and of the homotopy operators $s_{(p)}$, $S_{(p)}$ on this complex.

9.5.1 Continuous Hochschild Chains Over the Algebra of Hilbert–Schmidt Operators

Let $\{e^n_\alpha\}_{n\in N}$ be an ortho-normal basis of L_2 complex valued functions on Δ_α. Then $\{e^i_\alpha \times \bar{e}^j_\beta\}_{i,j\in N}$ is an ortho-normal basis of L_2 complex valued functions on $\Delta_\alpha \times \Delta_\beta$.

Any Hilbert–Schmidt kernel on $X \times X$ is given by an L_2-convergent series

$$K \;=\; \sum_{\alpha\beta,ij} K^{\alpha\beta}_{ij}\ (e^i_\alpha \times \bar{e}^j_\beta), \qquad \sum_{i,j} |K^{\alpha\beta}_{i,j}|^2 < \infty \tag{9.59}$$

with complex coefficients $K^{\alpha\beta}_{ij}$.

Given the Hilbert–Schmidt operator K, the decomposition (9.1) is unique. A Hilbert–Schmidt operator of type $(e^i_\alpha \times \bar{e}^j_\beta)$ was called *elementary*.

The composition of two elementary Hilbert–Schmidt operators is given by

$$(e^i_\alpha \times \bar{e}^j_\beta) \circ (e^k_\gamma \times \bar{e}^l_\eta) \;=\; \delta^{jk}\delta_{\beta\gamma}\ (e^i_\alpha \times \bar{e}^l_\eta), \tag{9.60}$$

where δ^{jk} and $\delta_{\beta\gamma}$ are the Kronecker symbols.

Given two Hilbert–Schmidt kernels

$$K \;=\; \sum_{\alpha\beta,ij} K^{\alpha\beta}_{ij}\ (e^i_\alpha \times \bar{e}^j_\beta), \qquad L \;=\; \sum_{\alpha\beta,ij} L^{\alpha\beta}_{ij}\ (e^i_\alpha \times \bar{e}^j_\beta), \tag{9.61}$$

their composition is given by the formula

$$K \circ L; = \sum_{\alpha\beta,ij} K^{\alpha\beta}_{ij}\ L^{\alpha\beta}_{kl}\ \delta^{j,k}\ (e^i_\alpha \times \bar{e}^l_\beta) \;=\; \sum_{\alpha,\beta,\gamma}^{i,k,j} K^{\alpha\beta}_{ik}\ L^{\beta\gamma}_{kl}\ (e^i_\alpha \times \bar{e}^l_\beta), \tag{9.62}$$

i.e.

$$(K \circ L)^{\alpha,\gamma}_{i,l} \;=\; \sum_{k}^{\beta} K^{\alpha\beta}_{ik}\ L^{\beta\gamma}_{kl} \tag{9.63}$$

Definition 9.9 The *space of continuous p*-chains of the algebra of Hilbert–Schmidt operators on X is

$$C_p(HS)(X) := \tag{9.64}$$

$$= \{\sum_{i,j}^{\alpha,\beta} K_{i_0\ldots i_p,j_0,\ldots,p}^{\alpha_0,\ldots,\alpha_p,\beta_0,\ldots,\beta_p} (e_{\alpha_0}^{i_0} \times \bar{e}_{\beta_0}^{j_0}) \otimes_C (e_{\alpha_1}^{i_1} \times \bar{e}_{\beta_1}^{j_1}) \otimes_C \cdots \otimes_C (e_{\alpha_p}^{i_p} \times \bar{e}_{\beta_p}^{j_p}) \mid$$

$$\text{such that } \sum_{i,j}^{\alpha,\beta} |K_{i_0,..,i_p,j_0,\ldots,j_p}^{\alpha_0,\ldots,\alpha_p,j_0,\ldots,j_p}|^2 < \infty\}.$$

The coefficients $K_{i_0,\ldots,i_p,j_0,\ldots,j_p}^{\alpha_0,\ldots,\alpha_p,\beta_0,\ldots,\beta_p}$ represent the components of the L_2-decomposition of the chain K relative to the Hilbert basis

$$\{(e_{\alpha_0}^{i_0} \times \bar{e}_{\beta_0}^{j_0}) \otimes_C (e_{\alpha_1}^{i_1} \times \bar{e}_{\beta_1}^{j_1}) \otimes_C \ldots . \otimes_C (e_{\alpha_p}^{i_p} \times \bar{e}_{\beta_p}^{j_p})\}_{\alpha,\beta,i,j}. \tag{9.65}$$

By definition, the norm of K is given by

$$||K||^2 := \sum_{i,j}^{\alpha,\beta} |K_{i_0,..,i_p,j_0,\ldots,j_p}^{\alpha_0,\ldots,\alpha_p,j_0,\ldots,j_p}|^2 < \infty. \tag{9.66}$$

We discuss here the continuity property both of the Hochschild boundary and of the homotopy operators s, S defined on the spaces of continuous chains $C_p(HS(X))$.

9.5.2 Continuity of the Hochschild Boundary

We begin by addressing the continuity property of the Hochschild boundary.

Proposition 9.9

(i) The series

$$(K \circ L)_{i,l} := \sum_k K_{ik}^{\alpha\beta} L_{kl}^{\beta\gamma} \tag{9.67}$$

is absolutely summable.

(ii) The coefficients $(K \circ L)_{i,l}$ *satisfy*

$$\sum_{i,j} |(K \circ L)_{i,l}|^2 \leq (\sum_{i,j} |K_{i,l}|^2).(\sum_{i,j} |L_{i,l}|^2). \tag{9.68}$$

Proof

(i) The Cauchy-Schwartz inequality gives

$$\sum_k |K_{ik}\ L_{kl}| \ \leq\ ||(K_{ik})_k||.||(L_{kl})_k|| \ =\ (\sum_k |K_{ik}|^2)^{1/2}.(\sum_k |L_{kl}|^2)^{1/2} \tag{9.69}$$

From (i) we get

$$\sum_{i,l}(\sum_k |K_{ik}\ L_{kl}|)^2 \ \leq\ \sum_{i,j}||(K_{ik})_k||^2.||(L_{kl})_k||^2 \ =\ \sum_{i,l}(\sum_k |K_{ik}|^2).(\sum_k |L_{kl}|^2) \leq \tag{9.70}$$

$$\leq\ \sum_{i,l}(\sum_k |K_{ik}|^2).(\sum_k |L_{kl}|^2) \ \leq\ (\sum_{i,j} |K_{ij}|^2).(\sum_{k,l} |L_{kl}|^2) = ||K||^2_{HS}.||L||^2_{HS}$$

Recall that the Hochschild boundary $b_{(p)} : C_p(HS(X)) \longrightarrow C_{p-1}(HS(X))$ is

$$b_{(p)} = \sum_{k=0}^{k=p-1} b_{(p)k} + b_{(p)p},$$

where

$$b_{(p)k} = (-1)^k \partial_k^H, \quad and \quad b_{(p)p} = (-1)^p \partial_p^H$$

with

$$\partial_k^H(K_0 \otimes_C K_1 \otimes_C \ldots . \otimes_C K_p) \ =\ K_0 \otimes_C \ldots \otimes_C K_{k-1} \otimes_C (K_k \circ K_{k+1}) \otimes_C \ldots \otimes_C K_p$$

and

$$\partial_p^H(K_0 \otimes_C K_1 \otimes_C \ldots . \otimes_C K_p) \ =\ (K_p \circ K_0) \otimes_C K_1 \otimes_C \ldots . \otimes_C K_{p-1}.$$

Proposition 9.10 *The Hochschild boundary face operators are well defined operators from the vector space $C_p(HS(X))$ of continuous Hochschild chains of the algebra of Hilbert–Schmidt operators into the space $C_{p-1}(HS(X))$ of continuous Hochschild chains of the algebra of Hilbert–Schmidt operators.*

9.5.3 *Continuity of the Homotopy Operators s*

Here we show that the homotopy operators s are well defined on the spaces of continuous Hochschild chains generated by elementary chains which contain at least one gap.

Suppose K is an elementary chain containing at least one gap. Suppose the first gap is of order r. Recall the operator $s_{(p)}(K)$ is defined by inserting the factor $(e^{j_r}_{\beta_r} \times \bar{e}^{j_r}_{\beta_r})$ inside this gap.

$$s_{(p)}(K) :=$$

$$= (-1)^r (e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) \otimes_C \ldots \otimes_C (e^{i_r}_{\alpha_r} \times \bar{e}^{j_r}_{\beta_r}) \otimes_C (e^{j_r}_{\beta_r} \times \bar{e}^{j_r}_{\beta_r}) \otimes_C (e^{i_{r+1}}_{\alpha_{r+1}} \times \bar{e}^{j_{r+1}}_{\beta_{r+1}}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{j_p}_{\beta_p}). \tag{9.71}$$

Proposition 9.11 *The homotopy operators s satisfy*

$$||sK|| = ||K||. \tag{9.72}$$

and hence are well defined on the Hochschild sub-complex of continuous chains over the algebra of Hilbert–Schmidt operators.

Proof

(i) Let K the above chain and let $K^{\alpha_0,\ldots,\alpha_p,\beta_0,\ldots,\beta_p}_{i_0,\ldots,i_p,j_0,\ldots,j_p}$ be its components. Then the components of sK are

$$(sK)^{\alpha_0,\ldots,\alpha_r,\beta_r,\alpha_{r+1},\ldots,\alpha_p,\beta_0,\ldots,\beta_r,\beta_r,\beta_{r+1},\ldots,\beta_p}_{i_0,\ldots,i_r,j_r,i_{r+1},\ldots,\ i_p,\ j_0,\ldots,\ j_r,j_r,\ j_{r+1},\ldots,j_p} = (-1)^r K^{\alpha_0,\ldots,\alpha_r,\alpha_{r+1},\ldots,\alpha_p,\beta_0,\ldots,\beta_r,\beta_{r+1},\ldots,\beta_p}_{i_0,\ldots,i_r,\ i_{r+1},\ldots,\ i_p,\ j_0,\ldots,\ j_r,j_{r+1},\ldots,\ j_p}. \tag{9.73}$$

We have to check that the components of sK satisfy the condition (9.67). This is clearly true. Indeed, the formula (9.73) tells us that the passage from the components of K to the components of sK involves two operations: a shift (given by the addition of two columns $(j_r, j_r, \beta_r, \beta_r)$ and the multiplication by $(-1)^r$. All other components, which do not contain the additional columns, are equal to zero. The additional columns are uniquely characterised by the choice of the *first* gap. The shift prevents non-trivial linear combinations between the components of K to produce the components of sK. These arguments prove

$$||sK|| = ||K||.$$

which completes the proof of the proposition.

9.5.4 Continuity of the Homotopy Operators S

Proposition 9.12 *The operator* $S_{(p)} : C_p^{\Delta}(HS(X)) \longrightarrow C_{p+1}^{\Delta}(HS(X))$, *defined on elementary chains by the formula*

$$S_{(p)}[(e^{i_0}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{i_0}_{\alpha_0})] :=$$

$$:= (e^{i_0}_{\alpha_0} \times \bar{e}^{I}_{\alpha_0}) \otimes_C (e^{I}_{\alpha_0} \times \bar{e}^{i_1}_{\alpha_1}) \otimes_C (e^{i_1}_{\alpha_1} \times \bar{e}^{i_2}_{\alpha_2}) \otimes_C \ldots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{i_0}_{\alpha_0}), \tag{9.74}$$

satisfies

$$||SK|| = ||K|| \tag{9.75}$$

and hence, it is a continuous operator.

Proof The proof goes along the same lines as in the proof of Proposition 9.11.

We have

$$(SK)^{\alpha_0,\alpha_0,\alpha_1,\ldots,\alpha_p,\ ,\alpha_0,\ \alpha_1,\alpha_2,\ldots,\alpha_p,\alpha_0}_{i_0,\ I,\ i_1,\ldots,\ i_p,\quad I\quad i_1,\ i_2,\ldots,\ i_p,\ i_0} = K^{\alpha_0,\alpha_1,\ldots,\alpha_p,\ \alpha_1,\ldots,\beta_p,\alpha_0}_{i_0,\ i_1,\ldots,\ i_p,\ \ i_1,\ldots,\ i_p,\ i_0}. \tag{9.76}$$

The formula (9.76) tells that the passage from the components of K to the components of SK involves a shift (given by the addition of two columns $[I, \alpha_0]$ $[I, \alpha_0]$ placed on the second and third positions. All other components, which are not obtained by this procedure, are equal to zero. The shift prevents non-trivial linear combinations between the components of K to produce the components of SK. These arguments complete the proof of the proposition.

9.6 Topological Considerations

9.6.1 Alexander–Spanier Cohomology

We introduced the Alexander–Spanier cohomology in Sect. 1.3 very schematically. Here we present the *local* Alexander–Spanier cohomology (9.6.1) and homolog (9.6.2).

To simplify the presentation of the construction of the Alexander–Spanier cohomology $H_*^{AS}(X, G)$, we assume the spaces X are countable, locally finite simplicial complexes and G is an arbitrary commutative group.

Let $C^p_{AS}(X, G) := \{f \mid f : X^{p+1} \longrightarrow G\}$, where f is an arbitrary function. Let us call any such function f a *non-localised* Alexander–Spanier p co-chain on X with coefficients in G.

The Alexander–Spanier co-boundary of f, df, is the $p+1$ co-chain given by the formula

$$df(x_0, x_1, \ldots, x_p, x_{p+1}); = \sum_{k=0}^{k=p+1} (-1)^k f(x_0, x_1, \ldots, \hat{x}_k, \ldots, x_{p+1}). \qquad (9.77)$$

The complex $\{C^*_{AS}(X, G), d\}_*$ is not interesting because it is *acyclic*. Formally, the acyclicity of this complex is provided by the homotopy operator $\tilde{s}$

$$(\tilde{s} f)(x_0, x_1, \ldots, x_{p-1}) := f(P, x_0, x_1, \ldots, x_{p-1}), \qquad (9.78)$$

where P is an arbitrarily chosen point in X.

This fact may be immediately understood if we agree to think of the point $(x_0, x_1, \ldots, x_p)$ as representing the simplex $[x_0, x_1, \ldots, x_p]$ having as vertices the *arbitrary* points $x_0, x_1, \ldots, x_p$ of X. The set of all such simplexes form a simplicial complex $\tilde{X}$ in which every point of X is a vertex and any such vertices are allowed to be connected to form a simplex of this simplicial complex. Clearly, the simplicial complex $\tilde{X}$ is an infinite (if X is an infinite set) dimensional simplex whose vertices are all points of X.

The simplicial complex $\tilde{X}$ is, homotopically, a point. The point P in the construction of the homotopy operator $\tilde{s}$ becomes the vertex of the cone over the simplicial complex $\tilde{X}$.

On the other hand, obviously, any non-localised Alexander–Spanier p co-chain f is a simplicial co-chain of the simplicial complex $\tilde{X}$ with coefficients in G. The Alexander–Spanier co-boundary df is nothing but the usual co-boundary of this simplicial co-chain.

The basic idea of the Alexander–Spanier cohomology is to consider a *sub*-complex $\tilde{X}^{loc}$ of the simplicial complex $\tilde{X}$. A simplex $[x_0, x_1, \ldots, x_p]$ will be allowed to belong to this sub-complex provided the points $x_0, x_1, \ldots, x_p$ are sufficiently close one to each other, i.e. iff they belong to a tubular neighbourhood U_{p+1} of the main diagonal $\nabla^{p+1}_X := \{(x, x, \ldots, x) \mid for\ any\ x \in X\} \subset X^{p+1}$, for any p. Such a sub-complex will be denoted by $\tilde{X}_U$. Let $\tilde{U}$ be the collection of such neighbourhoods.

We assume that tubular neighbourhoods U_{p+1} are compatible, i.e., that by removing any component point x_k of a point in U_{p+1} one gets a point in U_p. This could be realised, for example, by choosing a distance function on X and allow the points $x_0, x_1, \ldots, x_p$ to belong to U_{p+1} provided their mutual distances do not exceed a sufficiently small fixed number $0 < \epsilon$.

The sub-complex $\tilde{X}_U$ has X as a deformation retract. Therefore the sub-complexes $\tilde{X}_U$ are homotopically equivalent to the original simplicial complex X and hence they have isomorphic simplicial cohomologies.

Let us denote the simplicial co-chain complex associated to the sub-complex $\tilde{X}_U$ by $\{C^{U,*}_{AS}(X, G),\ d\}_*$. Therefore, its homology is isomorphic to the simplicial homology of the complex X.

Definition 9.10 Any co-chain $f \in C_{AS}^{U,p}(X, G)$ will be called *U-local* Alexander–Spanier co-chain.

The tubular neighbourhoods U form an inductive system by declaring $U \preccurlyeq V$ iff $V \subseteqq U$.

Let $C_{AS}^{*}(X, G)$ denote the projective limit of the complexes $\{C_{AS}^{U,*}(X, G),\ d\}$ with respect to this filtration.

Definition 9.11 Any co-chain belonging to the complex $C_{AS}^{p}(X, G)$ is called an Alexander–Spanier co-chain of degree p.

As all cohomology complexes $C_{AS}^{p}(U, G)$ have isomorphic homologies, compatible with the deformation retractions discussed above, one gets

Theorem 9.8 (Alexander–Spanier) *For any countable, locally finite simplicial complex X, the homology of the Alexander–Spanier complex $C_{AS}^{*}(X, G)$ is canonically isomorphic to the simplicial cohomology $H^{*}(X, G)$.*

If the space X does not have an additional analytical structure (differential structure, e.g.), any Alexander–Spanier cohomology *class* is the homology class $[f]$ of a function $f \in C_{AS}^{U,p}(X, G)$. Its support may be chosen in an arbitrarily small neighbourhood U of the diagonal.

However, if X does possess a differentiable structure, then already at the level of co-chains f, it is possible to associate with f a de Rham differential form $DR(f)$, see e.g. [96]

$$DR(f) := \frac{\partial^p f(x_0, x_1, \ldots, x_p)}{\partial x_1^{i_1}, \partial x_2^{i_2}, , \ldots, \partial x_p^{i_p}} |\nabla_p \, dx_1^{i_1} \wedge dx_2^{i_2} \wedge, , \ldots, \wedge dx_p^{i_p}. \tag{9.79}$$

Theorem 9.9 *The mappings DR form a co-chain homomorphism which induces an isomorphism from the Alexander–Spanier cohomology to the de Rham cohomology of X.*

Remark 9.5 If the space X possesses a sufficiently regular differentiable structure (at least C^2), then any Alexander–Spanier cohomology class $[f]$ may be represented by a closed differential form $DR(f)$ by *going to the diagonal*, where this differential form lives, and therefore, entering the classical differential geometry.

If the space X does not possess a sufficiently regular analytical structure (as e.g. a C^2-differentiable structure), then Alexander–Spanier cohomology classes of degree p may not be represented by classical differential forms. They can, however, be represented by *arbitrary* functions (even not continuous), with support in an *arbitrarily small* tubular neighbourhood of the diagonal ∇_{p+1}. These are examples of elements in the *non-commutative geometry* of X.

If the space X possesses intermediate analytical structures, e.g. a Lipschitz structure, one may consider the Whitney complex consisting of measurable, bounded differential forms, with measurable, bounded exterior derivatives, see e.g. [83]. If the manifold X has a quasi-conformal structure, then the manifold possesses differential

forms whose components satisfy weaker conditions, see e.g. [92], [101]. Both, in the Lipschitz and quasi-conformal case, the corresponding differential forms live in the classical differential geometry, i.e. they live on the diagonals ∇, although their components are defined almost everywhere and are discontinuous. In these two situations, the DR-homomorphisms are defined *provided* the corresponding Alexander–Spanier co-chains f belong to the *projective* tensor products of the algebra of Lipschitz, resp. the Royden algebra, see Connes et al. [101] for more details.

9.6.2 Alexander–Spanier Homology

The Alexander–Spanier homology is dual to the Alexander–Spanier cohomology.

Formally, an Alexander–Spanier chain γ of degree p with coefficients in the commutative group G would be an *infinite* formal sum

$$\gamma = \sum_{[x_0,x_1,\dots,x_p]} \gamma_{[x_0,x_1,\dots,x_p]}[x_0, x_1, \dots, x_p], \tag{9.80}$$

where $\gamma_{[x_0,x1,\dots,x_p]} \in G$.

The Alexander–Spanier boundary of γ would be

$$\partial\gamma = \sum_{k=0}^{k=p}(-1)^k \sum_{[x_0,x_1,\dots,x_p]} \gamma_{[x_0,x1,\dots,x_p]}[x_0, \dots, \hat{x_k}, \dots, x_p]. \tag{9.81}$$

Although infinite chains (9.80) could be considered in a homology theory, if X is an infinite set, the boundary $\partial\gamma$ does not make sense because the coefficient of the simplex $[x_0, \dots, \hat{x_k}, \dots, x_p]$ would be given by an infinite sum with respect of $x_k \in X$.

The meaning of the formula (9.81) could be recovered provided the infinite sum would be replaced by an integral.

Definition 9.12 A real/complex Alexander–Spanier chain γ of degree p on the space X is a real/complex valued Lebesgue integrable function γ on X^{p+1} with support in a tubular neighbourhood of the diagonal $\nabla_X \in X^{p+1}$.

Let $C_p^{AS}(X)$ denote the set of all Alexander–Spanier p-chains on X.

The Alexander–Spanier kth boundary face $\partial_{(p)k}^{AS}\gamma$ of γ is

$$(\partial_{(p)k}^{AS}\gamma)(x_0, \dots, x_{p-1}) := \int_X \gamma(x_0, \dots, x_{k-1}, t, x_k, \dots, x_p)d\mu(t). \tag{9.82}$$

The Alexander–Spanier boundary of the chain γ is

$$\partial^{AS}_{(p)}\gamma \ := \ \sum_{k=0}^{k=p} (-1)^k \partial^{AS}_{(p)k}\gamma. \tag{9.83}$$

The directed system of neighbourhoods of the diagonal U discussed in the previous sub-section lead to a projective system of complexes $\{C^{AS}_*, \ \partial^{AS}_*\}_*$.

Definition 9.13 A real/complex Alexander–Spanier chain on the space X is by definition any element of the complex $\{C^{AS}_*, \ \partial^{AS}_*\}_*$.

Theorem 9.10 *(Alexander–Spanier) The homology of the real/complex Alexander–Spanier complex of X, $\{C^{AS}_*, \ \partial^{AS}_*\}_*$, is isomorphic to the singular real/complex homology of X.*

Proposition 9.13 *Let X be a countable, locally finite simplicial complex of dimension n and let $\{\Delta_\alpha\}_\alpha$ denote its n-dimensional simplices. Suppose μ is a Lebesgue measure on X such that each simplex Δ_α has measure* 1

$$\int_{\Delta_\alpha} 1 \, d\mu = 1. \tag{9.84}$$

Let G be an arbitrary Abelian group and let $C^{AS}_{(p)}(X, G)$ be the set of all G-valued Alexander–Spanier p-chains on X which are constant on each poly-top $\Delta_{\alpha_0} \times \Delta_{\alpha_1} \times \ldots \times \Delta_{\alpha_p}$.

Then $\{C^{AS}_{(*)}(X, G), \partial^{AS}\}_*$ is a complex and its homology is the singular (simplicial) homology $H_*(X, G)$.

Proof The Alexander–Spanier homology is a homology functor. The five-lemma applied onto the inclusion of the CW-homology complex of X into the Alexander–Spanier complex completes the argument.

Remark 9.6 The condition (9.84) is not important for the overall homology result and may be easily removed accordingly.

9.6.3 Isomorphism Between $\{ C^{I,loc}_*, \ b \}_*$ and $\{C^{AS}_{(*)}(X, G), \partial^{AS}\}_*$, $G = R$, or C

Definition 9.14 The space of *local* chains in $C^I_{(p)(HS)}$ is by definition $C^{I,loc}_{(p)(HS)}$ consisting of all chains $K \in C^I_{(p)(HS)}$ whose supports lay in a tubular neighbourhood of the diagonal $\bigtriangledown_X \in X^{p+1}$.

Recall that any $K \in C^{I,loc}_{(p)}(HS(X))$ has the expression

$$K = \sum_{\alpha_0,\alpha_1,\ldots,\alpha_p} K^{\alpha_0,\alpha_1,\ldots,\alpha_p}(e^I_{\alpha_0} \times \bar{e}^I_{\alpha_1}) \otimes_C (e^I_{\alpha_1} \times \bar{e}^I_{\alpha_2}) \otimes_C \cdots \otimes_C (e^I_{\alpha_p} \times \bar{e}^I_{\alpha_0}), \tag{9.85}$$

where $K^{\alpha_0,\alpha_1,\ldots,\alpha_p}$ are real or complex numbers.

The corresponding chain $[K]$ is given by formula (9.37), in which all indices $i_k = I$

$$[K] = \sum_{\alpha_0,\alpha_1,\ldots,\alpha_p} K^{\alpha_0,\alpha_1,\ldots,\alpha_p}[\alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_0], \tag{9.86}$$

and the repeated index I was omitted.

Recall that the formula (9.40) states

$$[bK] = \partial[K]. \tag{9.87}$$

Now we are going to interpret the simplicial chain $[K]$ as an Alexander–Spanier p-chain $\{K\} \in C^{AS}_p(X, G)$ ($G = R, C$). To do this, we agree to think of this function as taking the constant value $K^{\alpha_0,\alpha_1,\ldots,\alpha_p}$ on the poly-top $\Delta_{\alpha_1} \times \Delta_{\alpha_2} \times \ldots \times \Delta_{\alpha_p} \times \Delta_{\alpha_0}$.

Theorem 9.11

(i) Suppose each simplex Δ_α has measure 1, see formula (9.84). Then

$$\partial^{AS}_{(p)k}\{K\} = \partial_{(p)k}[K], \quad for\ any\ 0 \le k \le p. \tag{9.88}$$

Therefore,

$$\partial^{AS}\{K\} = \partial[K]. \tag{9.89}$$

(ii) If the supports of the chains of $\{ C^{I,loc}_*,\ b \}_*$ lay in a tubular neighbourhood of the diagonals ∇_p, for any p, then the homology of the complex $\{ C^{I,loc}_*,\ b \}_*$ is isomorphic to the Alexander–Spanier homology $H^{AS}_*(X)$.

Proof

(i) It is sufficient to prove formula (9.83). One has

$$\partial^{AS}_{(p)k}\{K\}(x_0, x_1, \ldots, x_{p-1}) := \int_X \{K\}(x_0, x_1, \ldots, x_{k-1},\ t,\ x_k, .., x_{p-1})\, d\mu(t) =$$

$$= \sum_{\alpha_0,\alpha_1,\ldots,\alpha_p} K^{\alpha_0,\alpha_1\ldots\alpha_p} \int_X \chi_{\alpha_0}(x_0) \ldots \chi_{\alpha_{k-1}}(x_{k-1})\ \chi_{\alpha_k}(t)\ \chi_{\alpha_{k+1}}(x_k) \ldots \chi_{\alpha_p}(x_{p-1})\, d\mu(t) =$$

$$= \sum_{\alpha_0,\alpha_1,\ldots,\hat{\alpha}_k,\ldots,\alpha_p} \sum_{\alpha_k} K^{\alpha_0,\alpha_1\ldots\alpha_p} \chi_{\alpha_0}(x_0)\ldots\chi_{\alpha_{k-1}}(x_{k-1})\ \chi_{\alpha_{k+1}}(x_k)\ldots\chi_{\alpha_p}(x_{p-1}) = \tag{9.90}$$

$$= \partial_{(p)k}\{K\}(x_0, x_1, \ldots, x_{p-1}).$$

where χ_Δ are characteristic functions.

Part (i) implies (ii).

9.7 Control of the Supports of Hochschild Chains

In this section we are going to show that if the triangulation of the space X is sufficiently fine, then the quasi-isomorphisms treated in Sects. 9.3 and 9.4

$$\{\ C_*(HS), b\ \} \longleftarrow \{\ C_*^{\Delta}(HS), b\ \} \longleftarrow \{\ C_*^{I}(HS), b\ \} \tag{9.91}$$

pass to the *local* sub-complexes. The computation of the homology of the sub-complex $\{\ C_*^{I,loc}(HS), b\ \}$ will complete the proof of Theorem 9.1.

We remark that the Hochschild boundary b as well as the homotopy operators s and S discussed in Sects. 9.4.2. and 9.4.2.2. send continuous Hochschild chains which have *small support* about the diagonal into chains of the same type. For the purpose of controlling the support of Hochschild chains we introduce the *diameter* of chains (see Definition 9.15 below).

Assume for simplicity that X is connected. We say that the *simplicial distance* between the top dimensional simplexes Δ_α and Δ_β is d provided d is the minimum number of 1-simplices needed to connect a vertex of the simplex Δ_α with a vertex of the simplex Δ_β.

The simplicial distance leads to an increasing filtration of the space of elementary chains.

Definition 9.15 Given the elementary chain

$$K \ = \ (e^{i_0}_{\alpha_0} \times \bar{e}^{j_0}_{\beta_0}) \otimes_C \cdots \otimes_C (e^{i_p}_{\alpha_p} \times \bar{e}^{j_p}_{\beta_p}).$$

we define its *diameter*

$$Diam(K) := \text{minimal simplicial distance between the simplices } \Delta_{\alpha_0}, \ldots, \Delta_{\alpha_p}, \Delta_{\beta_0}, \ldots, \Delta_{\beta_p} \tag{9.92}$$

Definition 9.16 For any natural number N, let $C_p(HS(X))_N \subset C_p(HS(X))$ be

$$C_p(HS(X))_N := \{K \mid K = \sum_{\alpha_0,\beta_0,\ldots,\alpha_p,\beta_p}^{i_0,j_0,\ldots,i_p,j_p} K_{i_0,j_0,\ldots,i_p,j_p}^{\alpha_0,\beta_0,\ldots,\alpha_p,\beta_p} (e_{\alpha_0}^{i_0} \times \bar{e}_{\beta_0}^{j_0}) \otimes_C \ldots \otimes_C (e_{\alpha_p}^{i_p} \times \bar{e}_{\beta_p}^{j_p}), \tag{9.93}$$

$$Diam(e_{\alpha_0}^{i_0} \times \bar{e}_{\beta_0}^{j_0}) \otimes_C \ldots \otimes_C (e_{\alpha_p}^{i_p} \times \bar{e}_{\beta_p}^{j_p}) \leq N.\}$$

A chain belonging to $C_p(HS(X))_N$ will be called N-*local*.

In particular, the simplicial distance leads to an increasing filtration of the space of Hilbert–Schmidt operators $HS(X)_N$

$$HS(X)_N := \sum_{\alpha\beta,ij} K_{ij}^{\alpha\beta} (e_\alpha^i \times \bar{e}_\beta^j), \qquad Diam(e_\alpha^i \times \bar{e}_\beta^j) \leq N. \tag{9.94}$$

It is clear that

$$HS(X)_m \circ HS(X)_n \subset HS(X)_{m+n+1}. \tag{9.95}$$

Definition 9.17 Let N be any natural number such that the supports of the chains of the complex $C_*(HS)_N$ lay in a tubular neighbourhood of the diagonals ∇_p, for any p. Define

$$C_*^\Delta(HS)_N := C_*^\Delta(HS) \cap C_*(HS)_N \tag{9.96}$$

$$C_*^I(HS)_N := C_*^I(HS) \cap C_*(HS)_N. \tag{9.97}$$

Proposition 9.14 *The Hochschild boundary operator b, as well as the homotopy operators s and S, do not increase the diameter of chains.*

Therefore, the homotopy operators s, S are well defined in the corresponding complexes $C*(HS)_N$, $C_*^\Delta(HS)_N$, $C_*^I(HS)_N$.

Theorem 9.12 *Let N be any natural number with the property that the supports of the chains of the complex $C_*(HS)_N$ lay in a tubular neighbourhood of the diagonals ∇_p, for any p. Then*

$$\{ C_*(HS)_N, b \} \longleftarrow \{ C_*^\Delta(HS)_N, b \} \longleftarrow \{ C_*^I(HS)_N, b \} \longrightarrow \{ C_*^{AS}(X)_N, \partial \} \tag{9.98}$$

are quasi-isomorphisms.

Proof The notion of gap passes to the *local* spaces $C_*(HS)_N$. Therefore, the splitting stated by Proposition 9.1 continues to hold in these spaces.

The operator s continues to satisfy the identity (9.22), so that Proposition 9.8 holds inside the complex $\{ C_*(HS)_N, b \}$ too. These prove that the first arrow of (9.98) is a quasi-isomorphism.

The S continues to satisfy the identity (9.33) of Proposition 9.13 inside the complex $\{ C_*^{\Delta}(HS)_N, b \}$. The algebraic modifications discussed in the proof of Proposition 9.8 may be carried out inside the sub-complex $\{ C_*^{I}(HS)_N, b \}$. Therefore, the second arrow of the relation (9.98) is a quasi-isomorphism.

The discussion made in Sect. 9.6.3. passes to sub-complexes $\{ C_*^{I}(HS)_N, b \}$, $\{ C_*^{AS}(X)_N, \partial \}$. Theorem 9.11 remains valid in these contexts. These prove that the last arrow of the relation (9.98) is a quasi-isomorphism. This completes the proof of the theorem.

Theorem 9.12 implies the following result:

Theorem 9.13 *Let $HS(X)$ denote the algebra of real/complex Hilbert–Schmidt operators on the countable, locally finite, homogeneous, simplicial complex X of dimension N.*

Let N be any natural number with the property that the supports of the chains of the complex $C_*(HS)_N$ lay in a tubular neighbourhood of the diagonals ∇_p, for any p.

Then the homology of the Hochschild sub-complex

$$\{ C_*(HS)_N, b \} \tag{9.99}$$

is isomorphic to the real/complex Alexander–Spanier homology of X

$$H_*^{AS}(X). \tag{9.100}$$

9.8 *Local* Hochschild Homology of the Algebra of Hilbert–Schmidt Operators

9.8.1 Preliminaries

We discuss here the *local* Hochschild homology of the algebra of Hilbert–Schmidt operators on the simplicial complex X. Recall that *local* Hochschild and cyclic homology were introduced in [125] as a tool to set up the index theorem in a more natural environment. To begin with, for this reason, we are interested in describing this notion primarily on the algebra of *quasi-local, bounded operators on a Hilbert space of L_2-sections in vector bundles*. This class of operators contains pseudo-differential operators and integral operators.

To start this argument we assume that the n-dimensional simplicial complex X is embedded in an Euclidean space. Let r be the induced metric on X.

As explained in Sect. 9.2, any Hilbert–Schmidt operator on X is defined by its kernel $K : X \times X \longrightarrow C$. The space of Hilbert–Schmidt operators on X is filtrated by the support if its elements. We have seen in Sect. 9.5.1. that the description of the elements of the Hochschild complex over this algebra involves real/complex valued L_2-functions over $(X \times X)^{p+1}$. In Sect. 9.6. we introduced a filtration of the Hochschild chains based on the simplicial structure of X. This filtration has its own interest.

In this section we are going to introduce a new filtration based on the size of the support measured in terms of *distance to the diagonal*.

Definition 9.18 Let $P = (x_0, y_0)|x_1, y_1|, \ldots, |x_p, y_p) \in (X \times X)^{p+1}$. Define the *distance from the point P to the main diagonal* $\nabla_X^{2(p+1)}$ by

$$r((x_0, y_0), (x_1, y_1), \ldots, (x_p, y_p),\ \nabla_X^{p+1}) := Max_{k=0}^{k=p}\{r(x_k, y_k), r(y_k, x_{k+1})\}.\quad (x_{p+1} := x_0) \tag{9.101}$$

Let ϵ be any positive number. Let

$$U_\epsilon^{2(p+1)} := \{P \mid P \in (X \times X)^{p+1}, r(P) < \epsilon\}. \tag{9.102}$$

Denote by $HS(X)_\epsilon$ the set of all Hilbert–Schmidt operators whose kernel have support in $U_\epsilon^{2(p+1)}$. The main idea of *local* Hochschild homology is to consider the homology of the sub-complex of the Hochschild complex consisting of chains which have small support about the diagonal. For chains of degree zero we intend elements of $HS(X)_\epsilon$ with ϵ small. Smoothing operators with arbitrarily small support appear in the Connes, Moscovici local index theorem [96] and they hold the topological information relating to the index formula.

Looking for chains of degree p which have small support about the diagonal $\nabla_X^{2(p+1)}$, we start by considering elements of $\otimes_C^{p+1} HS(X)_\epsilon$. For any element in this set, for any point in its support, one has $r(x_k, y_k) < \epsilon$, for any $0 \leq k \leq p$. For the elements in this set there is, so far, no condition on the distances $r(y_k, x_{k+1})$. The support (Supp) of $K_0 \otimes K_1 \otimes, \ldots, \otimes K_p$ is $Supp(K_0) \times Supp(K_1) \times, \ldots, \times Supp(K_p)$. To insure that the support of this element is small about the diagonal $\nabla_X^{2(p+1)} := \{(x, x, \ldots, x) \in (X \times X)^{p+1}\}$, we impose also the conditions $r(y_k, x_{k+1}) < \epsilon$, for any $0 \leq k \leq p$. Now we are in the position to define precisely the ϵ-*local* Hochschild chains in the completed Hochschild complex.

Definition 9.19 An ϵ-*local* Hochschild chain of degree p is by definition an element of

$$C_p(HS(X))_\epsilon := \{K \mid K \in C_p(HS(X)),\ Supp(K) \subset U_\epsilon^{2(p+1)}\}. \tag{9.103}$$

Multiplying Hilbert–Schmidt operators increases their support. For any natural numbers k_1, k_2

$$HS(X)_{k_1\epsilon} \circ HS(X)_{k_2\epsilon} \subset HS(X)_{(k_1+k_2)\epsilon}. \tag{9.104}$$

Given that the multiplication of Hilbert–Schmidt operators increases the support (9.104), the vector spaces $C_p(HS(X))_\epsilon$ satisfy

$$b\, C_p(HS(X))_\epsilon \subset C_p(HS(X))_{2\epsilon}. \tag{9.105}$$

To simplify the notation, we write $C_p(HS(X))_\epsilon = C_{p,\epsilon}$.

Definition 9.20 For any $0 < \epsilon$ we define

$$H_p^\epsilon(HS(X)) := \tag{9.106}$$

$$= \frac{Ker\ b : C_{p,\epsilon} \to C_{p,2\epsilon}}{\{Im\ b : C_{p+1,k\epsilon} \to C_{p,2k\epsilon}\} \cap \{Ker\ b : C_{p,\epsilon} \to C_{p,2\epsilon}\}}, \quad 2 \leq k,$$

where k is fixed number.

For any $\epsilon' < \epsilon$ one has $C_{p,\epsilon'} \subset C_{p,\epsilon}$ and therefore there is an induced mapping in homology $\Delta(\epsilon, \epsilon') : H_p^{\epsilon'}(HS(X)) \to H_p^\epsilon(HS(X))$.

$$\begin{aligned} \Delta(\epsilon, \epsilon') \frac{Ker\ b : C_{p,\epsilon'} \to C_{p,2\epsilon'}}{\{Im\ b : C_{p+1,k\epsilon'} \to C_{p,2k\epsilon'}\} \cap \{Ker\ b : C_{p,\epsilon'} \to C_{p,2\epsilon'}\}} \longrightarrow \\ \longrightarrow \frac{Ker\ b : C_{p,\epsilon} \to C_{p,2\epsilon}}{\{Im\ b : C_{p+1,k\epsilon} \to C_{p,2k\epsilon}\} \cap \{Ker\ b : C_{p,\epsilon} \to C_{p,2\epsilon}\}} \end{aligned} \tag{9.107}$$

Definition 9.21 The *local* Hochschild homology of the algebra of Hilbert–Schmidt operators is given by the formula

$$H_p^{loc}(HS(X)) := \underset{\epsilon \searrow 0}{ProjLim}\ H_p^\epsilon(HS(X)). \tag{9.108}$$

9.8.2 *Distance Control of Supports vs. Simplicial Control: The Result*

Let $K \in C_{p,\epsilon}(HS(X))$, $0 \leq p \leq n = dim X$.

We intend to look on the size of the support of K after all algebraic modifications used in Sects. 9.3 and 9.4 are performed. To this purpose notice that the Hochschild boundary b doubles the "diameter" of chains, see (9.105). On the other hand, the homotopy operators s, (see Sect. 9.4.2) and S, (see Sect. 9.4.2.2), do not modify the size of the support.

We chose an ϵ. We suppose that the diameter of each of the simplices Δ_α is less than ϵ. If this condition is not satisfied, we consider a higher order barycentric subdivision. The condition (9.84) of Proposition 9.13 changes by replacing the measure 1 of each simplex by a constant, depending only on the number of barycentric

sub-divisions. To avoid further complications of the homological picture, we may assume that for each maximal simplex Δ_α the chosen function I_α is a normalised constant function, see Definition 9.7, Sect. 9.4.3. With these precautions taken, the elementary chains belong to $C_{p,\epsilon}(HS(X))$.

For any $K \in C_{p,\epsilon}(HS(X))$ the decomposition (9.17) of Proposition 9.1 holds.

Given that the homotopy formulas (9.22) and (9.33) involve only once the Hochschild boundary, Lemma 9.1 formula (9.22)and Proposition 9.10 hold inside $C_{p,2\epsilon}(HS(X))$. The operator θ_p is available in the space $C_{p,2\epsilon}(HS(X))$.

Furthermore, Proposition 9.8 involves the operator θ_p raised to the maximum power $p+1$.

To summarise, we may state that all considerations made in Sect. 9.4 hold in the vector spaces $C_{p,2^{p+2}\epsilon}(HS(X))$. The connection between the *local* Hochschild homology and the Alexander–Spanier homology discussed in Sect. 9.6 holds provided the supports of the chains sit inside a tubular neighbourhood of the diagonal. Now we are in the position to state

Theorem 9.14 *Let X be any countable, locally finite, homogeneous simplicial complex of dimension n. Let r be a distance function.*

Let ϵ be a positive number. Suppose each maximal dimension simplex of X has diameter less than ϵ.

Let p be a natural number. Suppose the simplicial decomposition of X is sufficiently fine so that the set

$$U^{2(p+1)}_{2^{p+2}\epsilon} \quad \textit{is a tubular neighbourhood of the diagonal.} \tag{9.109}$$

Then

$$H^{loc}_p(HS(X)) \quad \textit{is naturally isomorphic to } H^{AS}_p(X). \tag{9.110}$$

The condition (9.109) tells us that in order to prove Theorem 9.1 we need to consider a sufficiently fine decomposition of the simplicial complex X before the constructions made in Sects. 9.4 and 9.5 start. This modification does not change the Hilbert–Schmidt algebra; it only changes the representation of its elements. This completes the proof of Theorem 9.1.

Part V
Non-commutative Topology

Chapter 10
Non-commutative Topology



Abstract We intend to produce a theory which *generalises topological spaces*; we call it *non-commutative topology*. In non-commutative *differential* geometry the *basic homology theory* is the *periodic cyclic homology*, based on the bi-complex (b, B). In non-commutative topology this structure will be replaced by the bi-complex $(\tilde{b}, d)$; the boundary $\tilde{b}$ is called *topological Hochschild boundary*. These ideas combine Connes' work (Noncommutative geometry, Academic Press, 1994) with ideas of the articles by Teleman and Teleman (St Cerc Math Tom 18(5):753–762, Bucarest, 1966; Rev Roumaine Math Pures Applic 12:725–731, 1967; Central European journal of mathematics. C*-algebras and elliptic theory II. Trends in mathematics, 251–265, Birkhauser, 2008; $Local^3$ index theorem. arXiv: 1109.6095v1, math.KT, 28 Sep. 2011; textitLocal Hochschild homology of Hilbert-Schmidt operators on simplicial spaces. arXiv hal-00707040, Version 1, 11 June 2012; Local algebraic K-theory, 26 Aug 2013 - arXiv.org > math > arXiv:1307.7014, 2013; The local index theorem, HAL-00825083, arXiv 1305.5329, 22 May 2013; $K_i^{loc}(\ mathbbC)$, $i\ =\ 0, 1$ 10 set 2013 - arXiv:1309.2421v1 [math.KT] 10 Sep 2013).

Remark 10.1 The Alexander–Spanier cohomology uses solely the *topology* of the space; it does not require any kind of analytical regularity. We use the Alexander–Spanier construction and the definition of local periodic cyclic cohomology as the departure point for non-commutative topology. Recall the Alexander–Spanier cohomology to arbitrary *localised rings*, see Part V. As an application we compute the *local topological periodic cyclic homology* of the algebra of *smooth* functions, Theorem 10.3, of the Banach algebra of *continuous* functions, Theorem 10.4, and of the algebra of arbitrary functions, Theorem 10.7, on a smooth manifold. These results are significant because it is known that the Hochschild and (periodic) cyclic homology of *Banach algebras* are either trivial or not interesting, see Connes [82, 96, 99]. *Entire cyclic cohomology*, due to Connes [99] ([95, 96]) gives a different solution to the problem of defining the cyclic homology of Banach algebras. The chains of the entire cyclic cohomology are elements of the infinite product (b, B) which satisfy a certain bi-degree asymptotic growth condition. Connes constructed

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a Chern character of θ-summable Fredholm modules with values in the entire cyclic cohomology.

10.1 The Idempotent Π

1. The operator σ was introduced in Sect. 2.4.9. It is well defined on the whole complex $C_*(\mathscr{A})$

$$d\,b\,+\,b\,d\,=\,1\,-\,\sigma, \tag{10.1}$$

   where d is the non-localised Alexander–Spanier co-boundary and b is the Hochschild boundary.
2. σ commutes both with d and b.
3. σ is a chain homomorphism both in the Alexander–Spanier and in the Hochschild complex. Hence, the range of the operator σ and its powers are sub-complexes both in the Alexander–Spanier and Hochschild complexes.
4. The range of the homomorphism Π consists of *non-degenerate chains*.
5. σ is homotopic to the identity. Therefore, the inclusions of these sub-complexes into the Alexander–Spanier, resp. Hochschild, complexes induce isomorphisms between their homologies.

10.2 Topological Hochschild Homology

On the range of the idempotent Π, i.e. $\{\tilde{C}_*(\mathscr{A})\}$, one has the identity

$$\sigma^k\,=\,1\,+\,b\,d\,\sigma^k. \tag{10.2}$$

From this formula we get

Proposition 10.1 *On the complex* $\{\tilde{C}_*(\mathscr{A})\}$ *one has the identity*

$$0\,=\,\sum_{k=1}^{n+1}(-1)^{k-1}C_{n+1}^k\,(bd)^k\,+\,\sum_{k=1}^{n}(-1)^{k-1}C_n^k\,(db)^k. \tag{10.3}$$

Proof The relation (10.3) is obtained from the formula

$$1\,=\,(1\,-\,b\,d)\,\sigma^n. \tag{10.4}$$

by making the substitution $\sigma\,=\,1\,-\,(d\,b\,+\,b\,d)$; for more details see [123].

This formula suggests us to introduce on the spaces $C_*(\mathscr{A})$ the operators $\tilde{b}$, $\tilde{d}$.

Definition 10.1 The operators $\tilde{b}_n$ and $\tilde{d}_n$ acting on $C_n(\mathscr{A})$ are defined by the formulae

$$\tilde{b}_n := b \sum_{k=1}^{n} (-1)^{k-1} C_n^k (d\ b)^{k-1} \tag{10.5}$$

$$\tilde{d}^n := d \sum_{k=1}^{n} (-1)^{k-1} C_{n+1}^k (b\ d)^{k-1}. \tag{10.6}$$

We do not use the operator $\tilde{d}$ in this book.

Proposition 10.2 *The operators $\tilde{b}$ and d anti-commute in $\tilde{C}(\mathscr{A})$; the same relation holds for the operators b and $\tilde{d}^n$*

$$0 = \tilde{b}_{n+1}\, d_n + d_{n-1}\, \tilde{b}_n \tag{10.7}$$

$$0 = \tilde{d}_{n-1}\, b_n + b_{n+1}\, \tilde{d}_n. \tag{10.8}$$

This formula is going to be proved below.

Definition 10.2 The homology of the complex $\{C_*(\mathscr{A},\ \tilde{b})\}_*$ is called *topological Hochschild homology*.

Compare to [123, 125, 127, 129–131].

The expression of the operator $\tilde{b}_n$ contains the factor b both to the left and to the right. From this it follows that the topological Hochschild homology contains the Hochschild homology.

Definition 10.3

(1) $\tilde{b}\,\tilde{b} = 0$. Therefore $\tilde{b}$ is a boundary operator.
(2) The complex $\{C_*(\mathscr{A}),\ \tilde{b}\}_*\}$ is called *topological Hochschild complex*. It makes sense even for non-localised algebras.
    The homology of the topological Hochschild complex is called *topological Hochschild homology*.
(3) We assume $\mathscr{A}$ to be a localised ring; the Alexander–Spanier and the Hochschild complex are localised. On the space of normalised chains $(\tilde{b},\ d)$ introduces a bi-complex structure. The boundary $\tilde{b}$ defined on the space of normalised chains is called *local topological Hochschild homology*.

The homology of the bi-complex $(\tilde{b}, d)$ is called *local topological periodic cyclic homology*; it is $\mathbb{Z}_2$-graded.

The definition of the local topological periodic cyclic homology is analogues to the definition of periodic cyclic homology, comp. Connes [99], see also Loday [97, Sect. 5.1.7., pag.159].

More specifically, given the localised ring $\mathscr{A}$, one has

$$(\tilde{b}, d)^{\lambda, per}(\mathscr{A}) \;=\; \{C_{p,q}(\mathscr{A})\}_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}, \tag{10.9}$$

where $C_{p,q}(\mathscr{A}) := \otimes^{p-q}\mathscr{A}$, for $q \leq p$. The boundary maps are

$$\tilde{b} : C_{p,q} \longrightarrow C_{p,q-1} \tag{10.10}$$

and

$$d : C_{p,q} \longrightarrow C_{p+1,q};$$

Proposition 10.3 *For any associative ring $\mathscr{A}$ the homology of the topological Hochschild complex contains the Hochschild homology.*

Proof As said before, the operator $\tilde{b}$ contains the factor b both to the left as well as to the right. For this reason any Hochschild cycle is a $\tilde{b}_n$ cycle; for the same reason any $\tilde{b}$ boundary is a b boundary.

Proposition 10.4 *The boundary $\tilde{b}$ verifies the identity (10.10).*

Proof Replacing σ in the expression of Π one gets

$$1 \;=\; (1 \;-\; bd)\sigma^n \;=\; (1 \;-\; bd)\,[1 - (bd + db)]^n \;=$$

$$(1 \;-\; b\,d)\,[1 + \sum_{k=1}^{n}(-1)^k C_n^k (b\,d + d\,b)^k] = (1 \;-\; b\,d)\{1 + \sum_{k=1}^{n}(-1)^k C_n^k [(bd)^k + (db)^k]\} =$$

$$1 \;+\; \sum_{k=1}^{n}(-1)^k C_n^k [(bd)^k \;+\; (db)^k] - bd - \sum_{k=1}^{n}(-1)^k C_n^k (bd)^{k+1}.$$

Therefore we get further

$$0 \;=\; bd + \sum_{k=1}^{n}(-1)^{k-1} C_n^k [(bd)^k \;+\; (db)^k] + \sum_{k=1}^{n}(-1)^k C_n^k (bd)^{k+1} =$$

$$bd + \sum_{k=1}^{n}(-1)^{k-1} C_n^k (bd)^k + \sum_{k=1}^{n}(-1)^k C_n^k (bd)^{k+1} + \sum_{k=1}^{n}(-1)^{k-1} C_n^k (db)^k \;=$$

$$(1+n)bd + \sum_{k=2}^{n}(-1)^{k-1} C_n^k (bd)^k + \sum_{k=1}^{n}(-1)^k C_n^k (bd)^{k+1} + \sum_{k=1}^{n}(-1)^{k-1} C_n^k (db)^k =$$

$$(1+n)bd + \sum_{k=2}^{n}(-1)^{k-1} C_n^k (bd)^k + \sum_{k=1}^{n-1}(-1)^k C_n^k (bd)^{k+1} + (-1)^n C_n^n (bd)^{n+1} + \sum_{k=1}^{n}(-1)^{k-1} C_n^k (db)^k =$$

$$(1+n)bd + \sum_{k=2}^{n}(-1)^{k-1} C_n^k (bd)^k + \sum_{k=2}^{n}(-1)^{k-1} C_n^{k-1} (bd)^k + (-1)^n C_n^n (bd)^{n+1} + \sum_{k=1}^{n}(-1)^{k-1} C_n^k (db)^k =$$

$$(1+n)bd + \sum_{k=2}^{n}(-1)^{k-1} (C_n^k + C_n^{k-1}) (bd)^k + (-1)^n C_n^n (bd)^{n+1} + \sum_{k=1}^{n}(-1)^{k-1} C_n^k (db)^k =$$

and as $C_n^k + C_n^{k-1} = C_{n+1}^k$

$$= (1 + n)bd + \sum_{k=2}^{n}(-1)^{k-1}C_{n+1}^k(bd)^k + (-1)^n C_{n+1}^{n+1}(bd)^{n+1} + \sum_{k=1}^{n}(-1)^{k-1}C_n^k(db)^k,$$

or

$$0 = \sum_{k=1}^{n+1}(-1)^{k-1}C_{n+1}^k(bd)^k + \sum_{k=1}^{n}(-1)^{k-1}C_n^k(db)^k. \tag{10.11}$$

Finally, we get the formula

$$\tilde{b}_{n+1}\, d_n + d_{n-1}\, \tilde{b}_n = 0. \tag{10.12}$$

Proposition 10.5 *For any ring $\mathscr{A}$ the homology of the topological Hochschild complex contains the Hochschild homology.*

Proof As said before, the operator $\tilde{b}$ contains the factor b both to the left as well as to the right. For this reason any Hochschild cycle is a $\tilde{b}_n$ cycle; for the same reason any $\tilde{b}$ boundary is a b boundary.

10.2.1 Local Topological Periodic Cyclic Homology of the Algebra of Smooth Functions

In this sub-section we consider the Fréchét topological algebra of *smooth* functions over a smooth manifold M. We have seen Sects. 3.1.2, 3.1.3 that the Hochschild boundary is well defined on germs at the diagonals and that the Hochschild homology depends only on the quotient complex, see Teleman [107].

The first two terms of the spectral sequence associated to the first filtration (with respect to the d-degree of chains) of the $(\tilde{b}, d)$ bi-complex are

$$E_{p,q}^1 = H_p(C_{*,q}, d) \cong H_{dR}^{q-p}(M)$$

and

$$E_{p,q}^2 = H_q(E_{p,*}^1, \tilde{b}) \cong H_{dR}^{q-p}(M).$$

The term E^1 is given by the Alexander–Spanier cohomology of the complex of smooth chains. Any Alexander–Spanier cycle may be represented by a skew-symmetric chain. The $\tilde{b}$ boundary of any skew-symmetric chain vanishes. Therefore $E_{p,q}^2 = H_q(E_{p,*}^1, \tilde{b}) = 0$.

We have proved the following result.

Theorem 10.1 *The local topological periodic cyclic homology of the algebra of smooth functions on a smooth manifold is isomorphic to the de Rham cohomology of the manifold.*

Analogously one has the theorem.

Theorem 10.2 *The local topological periodic cyclic homology of the algebra of* continuous functions on a topological manifold *is isomorphic to the de Rham cohomology of the manifold.*

10.3 Local Topological Periodic Cyclic Homology of the Algebra of *Arbitrary* Functions on a Smooth Manifold

We will proceed as in the preceding section (Sect. 10.2.1). We arrive at the point of computing $E^2_{p,q}$. We may again represent the Alexander–Spanier cohomology classes by *skew-symmetric* co-chains. This proves the next result.

Theorem 10.3 *The local topological periodic cyclic homology of the algebra of* continuous *functions on a smooth manifold is isomorphic to the de Rham cohomology of the manifold.*

Theorem 10.4 *The local topological periodic cyclic homology of the algebra of arbitrary functions on a smooth manifold is isomorphic to the de Rham cohomology of the manifold.*

10.4 Characteristic Classes of Idempotents in Non-commutative Topology

Definition 10.4 Let $\mathscr{A}$ be a localised ring. Let $e \in \mathbb{M}_N(\mathscr{A})$ be an idempotent.

Then the *Chern character* of e is the even local periodic cyclic homology class given by

$$Ch\,(\,e\,) := e + \frac{(-1)^1}{1\,!}\, e\,(de)^2 + \frac{(-1)^2}{2\,!} e\,(de)^4 + \ldots + \frac{(-1)^n}{n\,!} e\,(de)^{2n} + \ldots \tag{10.13}$$

Theorem 10.5 *Let $\mathscr{A}$ be a localised ring. Let $e \in \mathbb{M}_N(\mathscr{A})$ be an idempotent. Then the RHS of Eq. (10.13) is a cycle in the bi-complex $(\tilde{b},\ d)$.*

Proof We compute $\tilde{b}\ e(de)^{2n}$. We have

$$\tilde{b}_n(e\ (de)^{2n}) = b\sum_{k=1}^{n}(-1)^{k-1}C_{2n}^{k}(db)^{k-1} =$$

$$bC_{2n}^{1} + b\sum_{k=2}^{n}(-1)^{k-1}C_{2n}^{k}(db)^{k-1} =$$

$$\tilde{b}_{2n}(e(de)^{2n}) = 2n\ e\ (de)^{2n-1}\ +\ b\sum_{k=2}^{2n}(-1)^{k-1}C_{2n}^{k}(db)^{k-2}(db)(e(de)^{2n}) =$$

$$2n\ e\ (de)^{2n-1}\ +\ b\sum_{k=2}^{2n}(-1)^{k-1}C_{2n}^{k}(db)^{k-2}d(e\ de)^{2n-1}) =$$

$$2ne(de)^{2n-1}\ +\ b\sum_{k=2}^{2n}(-1)^{k-1}C_{2n}^{k}(db)^{k-2}(de)^{2n}) =$$

$$2ne\ (de)^{2n-1}\ +\ b\sum_{k=2}^{2n}(-1)^{k-1}C_{2n}^{k}2^{k-2}(de)^{2n}) =$$

$$2ne\ (de)^{2n-1}\ +\ \sum_{k=2}^{2n}(-1)^{k-1}C_{2n}^{k}2^{k-2}(2e-1)(de)^{2n-1}) =$$

$$(2n\ +\ \sum_{k=2}^{2n}(-1)^{k-1}C_{2n}^{k}2^{k-1})e\ (de)^{2n-1}\ +\ \sum_{k=2}^{2n}(-1)^{k}C_{2n}^{k}2^{k-2}(de)^{2n-1}. \tag{10.14}$$

Lemma 10.1

(1)

$$2n\ +\ \sum_{k=2}^{2n}(-1)^{k-1}C_{2n}^{k}2^{k-1}\ =\ 0, \tag{10.15}$$

(2)

$$\sum_{k=2}^{2n}(-1)^{k}C_{2n}^{k}2^{k-2}\ =\ n. \tag{10.16}$$

Proof

(1)

$$2n \ + \ \sum_{k=2}^{2n}(-1)^{k-1}C_{2n}^{k}2^{k-1} \ = \ 2n - \frac{1}{2}\sum_{k=2}^{2n}(-1)^{k}C_{2n}^{k}2^{k} \ =$$

$$2n - \frac{1}{2}[(1-2)^{2n} - (1 \ - \ 2.2n)] \ = \ 2b - \frac{1}{2}[1 \ - \ 1 \ + \ 4n] \ = \ 0. \tag{10.17}$$

(2)

$$\sum_{k=2}^{2n}(-1)^{k}C_{2n}^{k}2^{k-2} \ = \ \frac{1}{4}\sum_{k=2}^{2n}(-1)^{k}C_{2n}^{k}2^{k} \ = \ \frac{1}{4}[\ (1-2)^{2n} - (1-2n.2)\ ] \ = \ n. \tag{10.18}$$

From Lemma 10.1 we get

$$\tilde{b}_{2n}\ (\ e(de)^{2n}\ ) \ = \ n\ (de)^{2n-1}. \tag{10.19}$$

On the other hand

$$d\ (\ e\ (de)^{2n-2}\ ) \ = \ (de)^{2n-1}. \tag{10.20}$$

These prove the theorem.

Remark 10.2 It make sense to speak about *discontinuous bundles*; they are invariant under conjugations.

Remark 10.3 In a similar manner one obtains the definition of characteristic classes $Ch_{odd} : T_1^{loc,\lambda}(\mathscr{A}) \longrightarrow H_{odd}^{loc,\lambda}(\mathscr{A})$.

Definition 10.5 Let $\Delta \subset \tilde{C}_*(\mathscr{A})$ be the $\mathbb{K}$ sub-module generated by all chains

$$e(de)^n, \quad (de)^n, \quad b\ (e(de)^n), \quad b\ (de)^n, \quad B(\ e(de)^n\ ), \quad B\ (de)^n, \quad d(\ e(de)^n), \quad d\ (de)^n \tag{10.21}$$

where $n \in \mathbb{N}$ and $e \in \mathbb{M}_*(\mathscr{A})$ is an arbitrary idempotent.

Δ is called *characteristic sub-complex*.

Proposition 10.6 *Δ is a sub-complex both in the (b, B) and $(\tilde{b}, d)$ complexes.*

Although not all relations below are used in this book, we provide them for the benefit of the reader.

$$B(a_0 da_1 \dots da_n) \;=\; \sum_{j=0}^{j=n} (-1)^{nj} da_j \dots da_n da_0 \dots da_{j-1}. \tag{10.22}$$

On the same space the formulas hold

$$B^2 \;=\; 0, \quad b\,B \;+\; B\,b \;=\; 0. \tag{10.23}$$

The following relation is needed next, see (2.334):

$$\Pi_{p+1}\,(B\,x) \;=\; d\,\Pi_p\,(x). \tag{10.24}$$

Theorem 10.6 *Let e be an indempotent of the factorised algebra* $\mathscr{A}$.

Then the non-commutative topology Hochschild Chern character $Ch(e)$ *is co-homologous to the* non-commutative Chern character.

Remark 10.4 In Theorem 10.6 we start with an idempotent e. We state the equality of the Chern character of this idempotent both in the cyclic homology of the algebra $\mathscr{A}$ in the non-commutative geometry and in the non-commutative topology. In the next section we start with a direct connection A instead.

Proof The relations

$$\tilde{b}(e(de)^{2n-1}) \;=\; 0, \tag{10.25}$$

$$d(e(de)^{2n-1}) \;=\; e(de)^{2n} \tag{10.26}$$

show that the homology classes

$$e(de)^{2n} \;=\; (e-\frac{1}{2})(de)^{2n} - (\tilde{b} \;+\; d)\frac{1}{2}(e(de)^{2n-1}). \tag{10.27}$$

are $\{\tilde{b}, d\}$-homologous. Therefore, the homology classes $e(de)^{2n}$, $(e \;-\; \frac{1}{2})(de)^{2n}$ are co-homologous in the local and non-local complex $\sigma(\tilde{C}_*(\mathscr{A}))$.

Proposition 10.7 *The class* $(e \;-\; \frac{1}{2})(de)^{2n}$ *is the Connes–Chern character of the idempotent e in the* entire cyclic homology. *For the definition of* entire cyclic homology *see Getzler and Szenes [93] and [111, Pg. 447].*

Theorem 10.7 *The Chern character defines a homomorphism*

$$Ch \;: T_0(\mathscr{A}) \longrightarrow H_{ev}^{per,loc}(\mathscr{A}). \tag{10.28}$$

Proof The Chern character is compatible with all equivalences which define $T_0(\mathscr{A})$: stabilisation and conjugation.

10.5 Rational Pontrjagin Classes of Topological Manifolds

In this part we give a construction of the rational Pontrjagin classes of topological manifolds.

10.5.1 Existence of Direct Connections on Topological Manifolds

Let M be a topological manifold. We intend to construct a direct connection A on M. Recall that a direct connection on M consists of a set of *homeomorphisms* $A(x, y)$ where

$$A(x, y) = \text{ homeomorphisms from a neighbourhood of } y \text{ to a neighbourhood of } x \tag{10.29}$$

$$A(x, x) = \textit{identity} \text{ from a neighbourhood of } x \text{ to itself.} \tag{10.30}$$

The connection A will be constructed by induction using a handlebody decomposition of M, see [56, Essay III, §2, Pg. 104, Pg. 319] and Ranicki [106]. The second condition of a direct connection assures that the inductive construction may be performed without meeting homotopic obstructions in $\pi_i(TOP)$.

10.5.2 The Rational Pontrjagin Chain

We consider the chain

$$\tilde{\Psi}_{2n}(x_0 \dots x_{2n}) \;=\; \wedge\,(\, A(x_{i_0}, x_{i_{2n}}).dA(x_{i_{2n}}, x_{i_{2n-1}}) \dots .dA(x_{i_1}, x_{i_0})\,), \tag{10.31}$$

where $\wedge$ is the skew-symmetrisation with respect to the variables x and

$$dA(y, x) \;:=\; 1 \otimes A(y, x) - A(y, x) \otimes 1. \tag{10.32}$$

We intend to prove that $(\tilde{b} + d)(\Psi_{2n}) \;=\; 0$.

The effect of the skew-symmetrisation is $b\ \Psi_{2n} \;=\; 0$; therefore (10.5)

$$\tilde{b}(\Psi_{2n}) \;= 0. \tag{10.33}$$

On the other hand

$$d\ (\ \Psi_{2n}\ )\ =\ dA(x_{i_0}, x_{i_{2n}})\ .\ dA(x_{i_{2n}}, x_{i_{2n-1}})\ldots .dA(x_{i_1}, x_{i_0}). \tag{10.34}$$

Let $\mathscr{S}_{2n}$ be the operator defined on the permutation group Π_{2n+1} which *inverts* the order of the points x_i

$$\mathscr{S}_{2n}\ :(\ x_0, x_1, \ldots, x_{2n-1}, x_{2n}\ ) \longrightarrow (\ x_0, x_{2n}, x_{2n-1}\ldots, x_2, x_1\ ). \tag{10.35}$$

The effect of $\mathscr{S}_{2n}$ on $d\ \Psi_{2n}$ is

$$\begin{aligned} &dA(x_0, x_1)\ .\ dA(x_1, x_2), \ldots, dA(x_{2n-1}, x_{2n})\ .\ dA(x_{2n}, x_0) \longrightarrow \\ &dA(x_0, x_{2n})\ .\ dA(x_{2n-1}, x_{2n-2})\ \ldots .\ dA(x_2, x_1)\ .\ dA(x_1, x_0). \end{aligned} \tag{10.36}$$

The two sides of the previous relation (10.36) contain an odd number of factors. All factors of the two expressions are respectively equal in absolute value, but with opposite signs. Remember that Ψ_{2n} involves skew-symmetrisation; these show that in the expression of Ψ_{2n} the terms corresponding to the left and right sides of the (10.36) have opposite signs. The operation $\mathscr{S}_{2n}$ does not produce two times the same sequence of points because the number of factors is odd.

10.5.3 The Rational Pontrjagin Class

Any Alexander–Spanier class may be represented by a skew-symmetrisation chain, see, for example, [82]. For this reason the skew-symmetrisation involved in the expression of Ψ_{2n} does not change the homology class. The relation (10.27) shows that the class of Ψ_{2n}, belonging to the homology of the $\tilde{b} + d$, coincides with the Hochschild homology of

$$A(x_0, x_1)\ .\ dA(x_1, x_2)\ \ldots .\ dA(x_{2n}, x_0) \tag{10.37}$$

in the local periodic cyclic homology of the algebra of *continuous functions*, see [121, §12.3]. The class of (10.37) is one of the Pontrjagin classes of M when M is a smooth manifold.

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