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No. 5

NIPPON KAGAKUSI GAKKAI

The History of Science Society of Japan

TOKYO 1966

On the Meaning of "Virtualité" in the History of the Set Theory.

Tamotsu MURATA*

§1. Introduction

The subject treated in this paper is the meaning of the French word "virtualité" within the context of the history of the set theory. Actually this is not a technical term, and we seldom find it in mathematical books or papers these days. However, in the beginning of the set theory, there were some publications in which it was used, and after a minute examination of them, "virtualité" turns out to be a significant word, which suggests to us the fact that there was unique criticism of the set theory ——— fundamental criticism, which, however, has been nearly forgotten ever since. (Even in Cavaillès' "philosophie mathématique" (Cavaillès [1]) one of the most reliable critical histories of the set theory, it is treated as if it were an ordinary word, never explained or examined.)

Let us take first of all from Lusin's "Lecons sur la théorie des ensembles analytiques" (1930) (Lusin [1]), a certain sentence in which a remarkable usage of "virtualité" is found, and in which we can see what Lusin conceived by this word. Actually Lusin did not refer this sentence to any other authors in spite of his habit of giving references in detail, but we can find (almost) the same sentence in one of Baire's critical remarks concerning the set theory (quoted in one of Borel's papers (1905), Borel [2]).

It is, of course, possible to consider that this is a coincidence, but Lusin's mathematical works were so much influenced by Baire, as well as by Borel and Lebesgue, that it may perhaps be more reasonable to ascribe the sentence in question originally to Baire. Now the interesting thing is that these two sentences, Lusin's and Baire's, nearly the same as they are, do not seem to agree with each other as to what is meant by "virtualité". Where is the difference between the two, and what is the reason for the difference? These are the first points to be

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discussed.

Baire's sentence in question, on the other hand, seems to have a connection with another sentence in one of Cantor's papers, the third part of his important article "Über unendlicher lineare Punktmannigfaltigkeitslehre" (Cantor [1]).

Baire's sentence is not only a critical remark directed against the Cantor theory, but also has some verbal correspondences with it, although there were some twenty years between their publications. (Cantor's paper was published in 1883, Baire's in 1905.) The year of 1883 was really the year when Cantor had awakened to the idea of the actual infinity (Eigentlich-Unendliches) and had advanced the last and decisive step towards the formation of the abstract set theory. And it was this idea that Baire criticized most fundamentally. Here are again the same questions as before: where is the real difference between the two, and what is the reason for it? But this time these questions are perhaps more significant than the foregoing ones, because Baire's criticism is so fundamental that it may give us a chance to search to the bottom of set thoretical thought once again, even if it might end in reassuring the basic idea of the present-day set theory.

In the beginning we should like to clarify the common meaning of "virtualité".

According to Lalande [1], the word "virtuel" denotes what exists in power (en puissance) but not in act (en acte). In the weak sense it means "simply possible" (as white marble which is "virtuellement" "God, a table or a scoop"). But this usage is rare. In the strong sense it means "already predetermined (to be something), and latently provided with all the essential conditions for the actualization (of it)"; this sense is exemplified thus:

> ——Toute l'arithmétique et toute la géométrie sont innées, et sont en nous d'une manière virtuelle .....
>
> (Leibniz: Nouveaux Essais, 1er partie, ch. 1, §5.)

This exemplification of "virtuel" is noteworthy for our present purpose, because these two branches of mathematics quoted here, arithmetic and geometry, imply respectively the totality of natural numbers and the continuum, two of the most fundamental objects of the set theory.

The synonyms and anonyms of "virtuel" or "virtualité" are also noteworthy. The word "puissance", which is the (current) French translation of "Mächtigkeit", is one of its synonyms —— a fact which will have a certain significance later. On the other hand, "formel" or "formalité", as well as "en acte" or "réel" stand for something in opposition to "virtuel". In the following observation, too, we think we can find "virtualité" as an anonym of some concepts in the mathematical for-

mal-realism of Cantor.

§2. "Virtualité" in Lusin and in Baire

In 1930 Nicholas Lusin, professor of mathematics in Moscow University at that time, wrote a book "Leçons sur la théorie des ensembles analytiques" (Lusin [1]), which has been one of the standard textbooks in this field from that time on.

Our sentence in question is found in the "Conclusion" of these "Leçons"; it is as follows:

> —— Et il y a bien des chances que ces virtualités soient irréductibles deux à deux.

This sentence is italicized in the original text; there are, however, some questionable points in the context concerning the meaning of "virtualités". In order to examine them, first we give an outline of Lusin's "Leçons" and summarize its "Conclusion".

In the first two chapters, Lusin continued Baire's investigation of discontinuous functions (Baire-functions) and Borel's investigation of measurable sets (Borel-sets). The next two chapters contain the theory of analytic sets, which is a direct extension of the theory of Baire-functions, as well as of Borel-sets. It is well known that this theory happened to start from a thought-provoking mistake in Lebesgue's (excellent) article "Sur les fonctions représentable analytiquement" (Lebesgue [1]). The mistake was discovered and corrected by Souslin, a brilliant student of Prof. Lusin, and afterwards this study formed itself into the theory of analytic sets, which was advanced by Lusin, Souslin and others. Then this theory was developed again into a wider theory —— the theory of projective sets, which includes the theory of analytic sets as its own special case, and this is the object of the last and fifth chapter.

Facing these circumstances, we may suspect that Lusin successfully opened a new field in mathematics. But according to his own words, it was not truly the case, at least so it seems to a certain extent. In the "Conclusion" we read the following opinion of Lusin (summarized by the present author):

> —— The role of Borel-sets in the set theory compares well with those of rational numbers in analysis. As the introduction of irrational numbers causes great trouble in analysis, so does the introduction of some new kinds of sets in the set theory. On the other hand, as some simple irrational numbers are obliged to be introduced naturally, say, by simple geometrical construction, into analysis, so are, or so it seems to be, some (complicated but) relatively simple sets introduced into the set theory ...

From the context these relatively simple sets are (some) analytic set, and

at first glance we may think he is about to say that the family of analytic sets is a veritable mathematical existence. But he does not actually say so; he is hesitating over several difficult problems, and vacillating; thus:

> ——— It is true we can "name (nommer)" an analytic set as an individual object; however, for this purpose we are obliged to use the totality of irrational numbers, or that of transfinite numbers (of the 2nd class) or the concept of negation, each of which is difficult to be accepted from our (i.e. Lusin's) standpoint.

Following is a paragraph which includes the sentence in question:

> ——— L'auteur de ce Livre adopte le point de vue empiriste et incline à considérer les exemples construits par lui comme formés de mots et ne définissant pas des êtres veritablement achevés, mais seulement des *virtualités*. En particulier, il considère les ensembles projectifs comme des êtres dont la définition ne peut pas être cemplètement achevée: ce sont des virtualités purement négatives qui échappent à tout mode de définition positive. *Et il y a bien des chances que ces virtualités soient irréductibles deux à deux.* Pour citer un exemple, l'auteur considère comme insoluble la question de savoir si tous les ensembles projectifs sont mesurables ou non, puisque, à son avis, les procédés mêmes de définition des ensembles projectifs et de la mesure au sens de M. H. Lebesgue sont des virtualités incomparables et, par suite, privées de relations logiques mutuelles. (Lusin [1] pp. 322 - 323)

Lusin's perplexity in these respects continues ever after. But as far as our present question is concerned, there is no more to quote from the text. His perplexity itself will be discussed later in our §4.

From the above quotation we can see how, or in what meaning, Lusin used "virtualité". In fact in that context "virtualité" stands for "a being which is formed by words alone", in contrast to "a being of which a definition can be completely achieved". This meaning seemingly agrees with the explanation by Lalande's "Vocabulaire" (Lalande [1]), but it does not correspond to the Leibniz's phrase cited in our §1. As is shown in the last half of the above quotation, Lusin speaks neither arithmetic nor geometry; he only offers two other examples of "virtualité": a process of defining projective sets, on one hand, and a process of defining L-measurability, on the other hand. It is true these two processes are "virtuels" in the sense of being "possible but not yet realized". However, as these are really technical matters, it should have been explained beforehand how this "virtualité" was connected with these two processes. Actually, "virtualité" appears here rather abruptly in the text, and the same is true to a certain degree

for "irréductibles" in the text.

Beside these, the phrase "deux à deux" in the said sentence seems to have a remote relationship with the foregoing part of that quoted paragraph; since there are no "two by two","virtualités" there.

It is under these circumstances that we find an article of Borel in which almost the same sentence as the above is cited from one of Baire's letters, as we have already mentioned in §1. Here are the two:

——— Et il y a bien des chances que ces virtualités soient irréductibles deux à deux. (Lusin)

——— et il y a des chances pour que ces deux virtualités soient irréductibles. (Baire)

Just before this quoted line, there are several lines (which will be cited later), which state that "ces deux virtualités" in Baire's opinion are the continuum and the set of all natural numbers, in compatibility not only with the daily usage of "virtualité" but also with the above mentioned Leibniz's quotation. The word "irréductibles" has also a clear meaning in this context. Frankly speaking, the sentence in Lusin's "Leçons", if it was taken from this letter of Baire, does not seem to be inserted so cleverly as to give a good effect upon the original text. It is true there may be a possibility that these two sentences come from another common source. But, as far as the style of the sentences is concerned, the present author has not found any clue to it——— he rather thinks that it would be too intentional to take into account such a possibility, if it be logically true.

In short it would be safe to conclude that Lusin's "virtualité" comes from Baire's, and although that has a similar meaning to that which we mentioned in §1, there is a delicate difference between the denotations of these two "virtualités". Actually it is worthwhile to examine what the real cause of this difference is, but before entering into this problem several preliminary observations are necessary.

§3. Cantor versus Borel and Baire

In 1883 Georg Cantor published a most epoch-making article (Cantor [1] - Nr. 5) where the idea of the actual infinity (Eigentlich-Unendliches) appeared for the first time in the history of mathematics. A theorem which was to be called afterwards the *well-ordered theorem* was, among many others, proposed here to be proved, and after some twenty years it was really proved by Ernst Zermelo (Zermelo [1]). Soon after this Zermelo's work, Émil Borel objected to Zermelo's proof (Borel [2]), especially to the axiom used there, which was later to be called the *axiom of choice*, and a long and serious debate took place among many

mathematicians (Zermelo [2]).

Now in this paper of Borel, specifically in a footnote in it, there appeared our sentence in question. Borel concluded this paper by saying that Zermelo's argument could not stay within the domain of mathematics (!), and on this very point Borel gave the above-mentioned footnote which, in full, is as follows:

> —— On me permettra de citer quelques lignes d'une lettre de M. Baire (de Montpellier), qui me paraissent résumer avec beaucoup de netteté une opinion que je crois très juste et qui est sans doute très répandue: "Personnellement, je doute qu'une commune mesure puisse jamais se trouver entre le continue, ou ce qui, dans l'espèce, revient au même, l'ensemble de toutes les suites d'entiers positifs, et les ensembles bien ordonnés; il y a là, pour moi, deux choses, dont chacune n'est définie que virtuellement, et il y a des chances pour que ces deux virtualités soient irréductibles". (Math. Ann. Bd. 60, S. 195)

At a cursory glance Baire's opinion may appear to be out-of-date. In fact he does not accept even the concept of "Mächtigkeit"! But, in thinking over its significance, we come to the conclusion that the weight of Baire's opinion has become heavier and heavier, until at last it would seem to be one of the most fundamental criticisms of the Cantorian set theory. Moreover, in the present author's opinion also, "virtualité" should be considered as a counteractive concept to the (ordinary) concept of set. Here is a sentence in Nr. 3 of Cantor's article [1] (1883) which seems to be refuted by Baire.

> "Die *Mannigfaltigkeitslehre* in der ihr hier zuteil gewordenen Auffassung, umspannt, wenn wir das Mathematische allein ins Auge fassen und die übrigen Begriffssphären vorläufig unberücksichtigt lassen, die Gebiete der Arithmetik, der Funktionenlehre und der Geometrie; sie fasst sie auf Grund des Mächtigkeitsbegriffs zu einer höheren Einheit zusammen. *Unstetiges* und *Stetiges* findet sich solcherweise von demselben Gesichtspunkte aus betrachtet und mit gemeinschaftlichem Masse gemessen. (Gesam. Abh. S. 152)

It can be easily seen that these two paragraphs, Baire's and Cantor's, have some verbal connections between them, although their style is not so akin to each other as is found between Lusin's and Baire's. While Cantor speaks of the One-ness which unifies arithmetic, geometry and analysis by means of "Mächtigkeit (power)", Baire speaks of the Two-ness, so to speak, which consists of the continuum and of the totality of all natural numbers. While Cantor speaks of the common measure, i.e. "Mächtigkeit", which measures equally both the "continuous" and the "discontinuous", Baire doubts the existence of such a common

measure.

We may be able to distinguish these two ways of thinking by referring to the *axiom of extentionality;* that is, the axiom by which any set is to be determined, if and only if each of its elements is determined. But here it would be better to distinguish them by saying somewhat vaguely that Cantor's is atomistic and Baire's is non-atomistic, because our consideration is not limited to axiomatically arranged theories alone, but sometimes encompasses the historical and the philosophical.

From this broader viewpoint, it should be noted that it must have been a serious question at that time as to whether it is possible or not to consider the continuum not as something beyond a collection of points, but as a simple collection of points. And it must also have been a serious question as to whether it is possible or not to compare the continuum with the set of all natural numbers with respect to the degree of their infinity, so to speak, by means of the concept of power (*Mächtigkeit*). There were some similar problems which had perhaps troubled the Eleatic School and the like more than 2000 years ago in Greece, or some other problems which perhaps had an effect to develop some primitive-atomistic treatments of measure (before the beginning of the 17th century) into more exact theory in analysis, and in the end opened a way to the present-day measure theory. And there were also similar problems which Cantor had deliberated for several years in creating the set theory before publication of his important paper cited above. Thus it was not strange in those days that Baire submitted such a criticism of the Cantorian set theory, even if it looks like something out-of-date from the present-day viewpoint.

As to the "non-atomistic" way of thinking in our sense, we cannot omit É. Borel's opinions, especially his early ones, say before 1908. This 1908 is the year when we think we can see a delicate change in Borel's non-atomistic opinion, which change we shall have a chance to take into account later on in this paper. For the time being we restrict ourselves to the observation of Borel's early articles, that is, those written before 1908, and we can find many places in them where the "non-atomistic" undertone is clearly discernible.

The most important source material gleaning Borel's opinion, with its transition, is his "Leçons sur la théorie des fonctions" (Borel [1], 1 éd; 1898, 4 éd, 1952). From the fourth edition we can see his early non-atomistic opinion, as well as a delicate transition of his opinion thereafter for over half a century. Moreover we can find a usage of Baire's "virtualité" other than the above cited one.

Borel's "Lecons" has two aspects: in one aspect it played a considerable role in advancing the young set theory into a wider circle of mathematicians; in

another aspect it presented his critical opinion against the set theory. Several remarkable points can be pointed out as to his (early) criticism, and Baire's opinion seems to be included in Borel's.

In the first place the concept of set is differently understood by Cantor and Borel-Baire. As for the Cantorian set theory, a concept of "wohldefinit (well-defined)" is one of its cornerstones (Cantor [1] - Nr. 3). According to this concept, a set is completely determined if and only if each element of it is determined logically. This is a sort of atomistic way of thinking, which states that the totality —— the One —— is nothing but a collection of its elements —— the Many consisting of atoms! Borel quoted this part of Cantor's paper (in French translation published in Acta Mathematica T. II) in a footnote of his "Lecons" not to support it, but to argue against it. For Borel, the totality of natural numbers and the continuum are two different kinds of sets, even if they are equally called "sets (ensembles)".

Then could we not sat that Borel accepted this translation "ensemble" but Baire translated "Mannigfaltigkeit" into "virtualité"? In the latter case a connection between "virtualité" and "Mächtigkeit (puissance)", too, may have been taken into account. Incidentally, according to Cantor's own words, "Mächtigkeit" had no relationship with Leibniz's quotation or the like. It was used as a technical term by J. Steiner in his textbook of projective geometry (Cantor [1]-Nr. 3).

> —— Den Ausdruck *"Mächtigkeit"* habe ich J. Steiner entlehnt, der ihn in einem ganz speziellen, immerhin jedoch verwandten Sinne gebraucht, um auszusprechen, dass zwei Gebilde durch *projektivische* Zuordnung so auf einander bezogen sind, dass jedem Element des einen ein und nur ein Element des andern entspricht; bei dem hier gemeinten absoluten Mächtigkeitsbegriff wird zwar an der gegenseitig-eindeutigen Beziehbarkeit festgehalten, dagegen für das Gesetz der Zuordnung keinerlei Beschränkung, namentlich keine Beschränkung in bezug auf Stetigkeit und Unstetigkeit gemacht, so dass zweien Mengen dann, aber auch nur dann *gleiche* Mächtigkeit zugestanden wird, wenn sie nach irgendeinem Gesetze einander gegenseitig-eindeutig zugeordnet werden können;..... (Gesam. Abh. S. 151)

In the second place, the concept of continuum is grasped in completely different ways. For Cantor, the continuum is nothing but a collection of (atomic) points, while for Borel and Baire it is something beyond a simple collection of points, even if the totality of natural numbers is considered as a collection of numbers. The following is cited from Note IV- § VI of Borel's "Leçons", original-

ly written in 1908, nearly the last paper which shows us his non-atomistic thinking.

> Il est nécessaire de dire un mot de la notion du *continu*, qui est le seul exemple d'ensemble non dénombrable qui soit bien connu, c'est-à-dire duquel les mathématiciens aient une notion claire et commune (ou croient avoir, ce qui est pratiquement la même chose). Je regarde cette notion comme acquise par l'intuition géométrique; on sait que la notion arithmétique complète du continue exige qu'on admette la légitimité d'une infinité dénombrable de choix successifs et arbitraires. Cette légitimité me paraît fort discutable, mais on doit cependant la distinguer essentiellement de la légitimité non dénombrable de choix (successifs ou simultanés). (Borel [1], p. 160)

It might be added in passing that, as to the totality of natural numbers, Baire's opinion seems to have been more radical than Borel's. An example which shows us this fact is at the same time an example of another usage of Baire's "virtualité". This time also it is found in Baire's letter, which is the second of the famous "Cinq lettres sur la théorie des ensembles". The part in question is as follows:

> L'expression *ensemble donné* est employée à chaque instant: a-t-elle un sens? Pas toujours, selon moi. Dès qu'on parle d'infini (même dénombrable, et c'est ici que je suis tenté d'être plus radical que Borel), l'assimilation, *consciente ou inconsciente*, avec un sac de billes qu'on donne de la main à la main, doit complètement disparaître, et nous sommes, à mon avis, dans le *virtuel*, c'est-à-dire que nous faisons des coventions qui nous permettent ultérieurement, un objet étant défini *par une nouvelle convention*, d'affirmer certaines propriétés de cet objet. Mais croire qu'on est allé plus loin ne me paraît pas légitime. En particulier, de ce qu'un ensemble est donné (nous serons d'accord pour dire, par exemple, que nous nous donnons l'ensemble des suites d'entires positifs), *il est faux pour moi de considérer les parties de cet ens emble comme données*. (Borel [1] pp. 151-152)

Actually, Lusin's interpretation of "virtualité" seems to be much influenced by this opinion of Baire, and it may well be said that Lusin took our sentence out of Baire's (first) letter cited in §2 (Borel [2]), used it in a sense implied in this second quotation: "we are in the 'virtuel'; in other words, we establish some conventions which permit us ultimately, an object to be defined by a new convention, to affirm certain properties of that object. But it seems to me improper to think anyone goes too far."... For Lusin, either the L-measurability or

the projectivity of a set was perhaps a "convention"; and these properties were "virtualités" in the sense mentioned in §1, although a connection with the usage according to Leibniz was destroyed. Thus the present author surmises.

Now we return to our main problem. As to the opinion of Borel or Baire, the trouble is that neither Borel nor Baire left us clear information about it. However, through a minute examination, we often come across manifestations of their opinions, say in Borel's "Leçons". Here is an example: in Chapter III of his "Leçons", Borel cites a theorem concerning a measure-theoretic character of the continuum, which is logically true but paradoxical to our intuition; after its proof he adds thus:

> Nous avons insisté sur cette démonstration parce qu'elle nous a paru de nature à éclairer un peu la conception que chacun peut essayer de se faire du *continu*. Après avoir réfléchi à ce fait, qu'*on peut enlever d'une droite tous les points compris dans chacun des intervalles*
>
> $$\frac{p}{q} - \frac{1}{q^3}, \qquad \frac{p}{q} + \frac{1}{q^3},$$
>
> *et qu'il reste encore des points, en infinité non dénombrable*, on sera moins disposé à croire qu'on sait ce que c'est que le continu et à raisonner sur lui comme sur une notion intuitive et parfaitement claire. (Borel [1], pp. 44-45)

Really here is a difference which is not confined within such and such mathematical existences, but is combined with the very concept of *mathematical existence*. For Cantor, any mathematical existence can be defined formally so long as it implies no contradiction, and no intuitive factors are required in defining it; the continuum is not exceptional; in his article [1] - Nr. 5, especially in §10 of it, we can read his firm belief in considering the continuum as a point-set. Indeed he writes as if he had got the "Idee (idea)" of the continuum for the first time since the Greek period!

> Der Begriff des "Kontinuums" hat in der Entwicklung der Wissenschaften überall nicht nur eine bedeutende Rolle gespielt, sondern auch stets die grössten Meinungsverschiedenheiten und sogar heftige Streitigkeiten hervorgerufen. Dies liegt vielleicht daran, dass die ihm zugrunde liegende Idee in ihrer Erscheinung bei den Dissentierenden einen verschiedenen Inhalt aus dem Grunde angenommen hat, weil ihnen die genaue und vollständige Definition des Begriffs nicht überliefert worden war; vielleicht aber auch, und dies ist mir das wahrscheinlichste, ist die Idee des Kontinuums schon von denjenigen Griechen, welche sie zuerst gefasst haben mögen, nicht mit der

Klarheit und Vollständigkeit gedacht worden, welche erforderlich gewesen wäre, um die Möglichkeit verschiedener Auffassungen seitens der Nachfolger auszuschliessen. (Gesam. Abh. S. 190)

For Cantor, neither the time-concept nor the space-concept plays any role; conversely they themselves are defined by the mathematical continuum, he says. A concept of *motion* or the like is of little account. This is, in the present author's opinion, one of the most remarkable points in his theory of the continuum.

Zunächst habe ich zu erklären, dass meiner Meinung nach die Heranziehung des *Zeitbegriffs* oder der *Zeitanschauung* bei Erörtrung des viel ursprünglicheren und allgemeineren Begriffs des Kontinuums *nicht* in der Ordnung ist; die Zeit ist meines Erachtens eine Vorstellung, die zu ihrer deutlichen Erklärung den von ihr unabhängigen Kontinuitätsbegriff *Zeit* Voraussetzung hat und sogar mit Zuhilfenahme desselben weder objektiv als eine Substanz, noch subjektiv als eine notwendige apriorische Anschauungsform aufgefasst werden kann, sondern nichts anderes als ein *Hilfs- und Beziehungsbegriff* ist, durch welchen die Relation zwischen verschiedenen in der Natur vorkommenden und von uns wahrgenommenen Bewegungen festgestellt wird.

Ebenso ist es meine Überzeugung, dass man mit der sogenannten *Anschauungsform* des *Raumes* gar nichts anfangen kann, um Aufschluss über das *Kontinuum* zu gewinnen, da auch der *Raum* und die in ihm gedachten Gebilde nur mit Hilfe eines begrifflich bereits *fertigen* Kontinuums denjenigen Gehalt erlangen, mit welchen sie Gegenstand nicht bloss ästhetischer Betrachtungen oder philosophischen Scharfsinnes oder ungenauer Vergleiche, sondern nüchtern-exakter mathematischer Untersuchungen werden können. (Gesam. Abh. SS. 191-192)

So any opinions as to the continuum, other than the atomistic one, are rejected or neglected. Thomas Aquinas, for example, who advocated an opinion that (according to Cantor's statement - Cantor [1] - Nr. 5 -) the continuum consists neither of the infinity of parts nor of a finite number of parts, but of no parts, is criticized most severely and rejected. Here may perhaps be a piece of historical background concerning something philosophical or theological, which we can not discuss here.

..... andere wieder statuierten, um diesem Streit fern zu bleiben, mit *Thomas von Aquino*, dass das Kontinuum weder aus unendlich vielen, noch aus einer endlichen Anzahl von Teilen, sondern aus *gar keinen* Teilen bestehe; diese letztere Meinung scheint mit weniger eine Sacherklärung

als das stillschweigende Bekenntnis zu enthalten, dass man der Sache nicht auf den Grund gekommen ist und es vorzieht, ihr vornehm aus dem Wege zu gehen. Hier sehen wir den *mittelalterlichscholastischen Ursprung* einer Ansicht, die wir noch heutigentages vertreten finden, wonach das Kontinuum ein unzerlegbarer Begriff oder auch, wie andere sich ausdrücken, eine reine *apriorische* Anschauung sei, die kaum einer Bestimmung durch Begriffe zugänglich wäre; jeder arithmetische Determinationsversuch dieses *Mysteriums* wird als ein unerlaubter Eingriff angesehen und mit gehörigem Nachdruck zurückgewiesen; schüchterne Naturen empfangen dabei den Eindruck, als ob es sich bei dem "Kontinuum" nicht um einen *mathematisch-logischen Begriff*, sondern viel eher um ein *religiöses Dogma* handle. (Gesam. Abh., SS. 190-191)

Anyway, reflecting on these aspects, Cantor is probably a genuine Platonist in mathematics, or in mathematical philosophy, who advocated in a most unique form a monistic and idealistic ontology in mathematics—— it is monistic, because there is only one concept of "set" there; "Mengenlehre" was originally called "Mannigfaltigkeitslehre", which probably meant the One-Many theory.

Contrasting with this, such a thoroughly monistic ontology as Cantor's can not be seen in Borel; a philosophy of Borel in Cantor's sense, if any, would not be Platonism, but Empiricism or Pragmatism, if not to say Aristotelianism. Here is rather a difference between "German philosophy" and "French philosophy". For Borel, the continuum is grasped, for example, as an intuitive totality, and regarded as an object of the measure theory. As to the measure theory it should be noted that, for him, it is one thing and the set theory is another. There is other evidence of this fact in Borel's "Leçons".

> Il est clair que, si deux ensembles A et B ont chacun même puissance qu'un troisième ensemble C, A a même puissance que B. Au lieu de dire que deux ensembles ont même puissance, on dit aussi que leurs puissances sont égales, de sorte que nous aurions pu exprimer le fait précédent en disant que deux puissances égales à une même troisième sont égales entre elles; mais nous préférons éviter ce langage, qui a l'inconvénient de suggérer une analogie entre les puissances et les grandeurs mesurables, analogie qui n'est pas suffisamment justifiée; (Borel [1], p. 5).

In short, he does not accept "A and B are equal", because it suggests an analogy between the powers and the measurable quantity because "it is not sufficiently justified."

Generally speaking, there is a difficult question as to the relationship between the measure and the set (collection); that is, how we can get a positive

measure by collecting some (infinity of) points, each of which has zero-measure. It is true that Cantor (seemingly) accepted this puzzling problem as an unavoidable fact and did not directly enter into it. But, could we not say his point-set theory as well as his continuum-problem show us his strong inclination towards this problem? Meanwhile, as for Borel and others, we have subsidiary evidence of their concerns about this problem. In the "Preface" of Lusin's "Lecons", Lebesgue informs us that students of mathematics at that time used to discuss such philosophical matters as idealism and materialism, on one hand, a Cretan liar and Zenon's paradox, on the other hand. (Lusin [1], p.x) These matters perhaps characterize the undertone of those days. From this point of view, Cantor's set theory was, to some extent, one of the amplifications of this undertone, however dominant it may have become later. It is not strange that there might have been theories completely at variance with this undertone, and actually there was, or there was about to be, one by Borel. (The following (iv) and "Théorie de la croissance" (Borel [3]) are its examples.) Let us surmise some aspects of it:

(i) *The theory of enumerable sets.* This developed afterwards into the theory of effectively enumerable sets, which was an origin of the constructive set theory in recent times (Kleere [1], Heything [1],). This is, of course, an interesting subject, but in the present paper we have no chance to go into it in detail.

(ii) *The power of the continuum.* This concept is used only as a basis for a negative proof. Thus, if A were true, then the continuum would be enumerable; so A should be denied. (This can be seen in the letter of Lebesque to Borel, one of the "cinq letters".

> ..... lorsque'on interprète un raisonnement bien connu de Cantor en disant: *il existe une infinité non dénombrable de nombres,* on ne donne cependant pas le moyen de nommer une telle infinité. On montre seulement, vous l'avez dit avant moi, que, chaque fois qu'on aura une infinité dénombrable de nombres, on pourra définir un nombre ne faisant pas partie de cette infinité. (Le mot *définir* à tout le temps le sens de: *nommer une propriété caractéristique du défini.*) Une existence de cette nature peut être utilisée dans un raisonnement et de la manière suivante: une propriété est vraie, si, la nier, conduit à admettre qu'on peut ranger tous les nombres en suite dénombrable. Je crois qu'elle ne peut intervenir que de cette manière. (Borel [1], p. 153)

(iii) *There is no "Continuum Problem"*, even if the power of the continuum could be accepted. The continuum is the limitation against enumeration, so to speak, and is nothing but a being which is never exhaustible by any enumeration.

(iv) *There is no third class of transfinite ordinals.* In Cantor, since the

power of the continuum is "larger" than that of the set of natural numbers, and since his idea of the actual infinity permits yielding new transfinite ordinals infinitely one after another, so there must surely be (an infinity of) ordinals "larger" than any second class ordinal. But in Borel, not only does the continuum not play such a role in his theory, but also the word "larger power" is forbidden, so he has no ground to yield some ordinals "larger" than any second class ordinals. He really said that he had once attempted to write a paper on a theory of his ordinals, but after all he had to give it up on account of being "too long and too incomplete." (Borel [1], Note III).

Among the above four points, the last three have, as far as the present author knows, never been taken up as something of weight by any writer.

Anyway thus we think we could make a little clearer the difference between Cantor's mathematics and Borel-Baire's, and, to a certain extent, the reasons for the difference. In the next section we shall return to the remaining half of our first problem posed in §1: what is the reason for the difference between Lusin and Baire?

§4. The retreat of the non-atomistic way of thinking

Before proceeding further, let us summarize what we have found out about "virtualité" in the foregoing sections.

There is no doubt that the mathematical meaning of "virtualité" was originally associated with the foundations of arithmetic and geometry, and that, as far as Baire's usage is concerned, this was still the case. When we speak of Baire's usage of "virtualité", we have only two examples of it: one is in Baire's letter cited in Borel's "Quelques remarques ..." (Borel [2]), and the other is in another of his letters, one of the "cinq lettres", both of which we have cited previously.

It is also probable that Lusin took the sentence in question out of Baire's first letter, but he did not follow its meaning in the original context. Rather he seems to have construed it in his own manner, by emphasizing an aspect of its meaning that Baire had suggested in his second letter (that is the one in "cinq lettres"), an aspect associated with "a verbal definition as a mathematical convention", say, a definition to be a projective set ——— though this may be a somewhat forced interpretation.

In addition to these, we can surmise, although without any positive evidence, that "virtualité" may be originally a (tentative) translation of "Mannigfaltigkeit" or of "Mächtigkeit". From the historical points of view, the former seems better, whereas from the literal meaning the latter better. But in reality,

it could be said that this was a concept of totality fundamentally different from both "Mannigfaltigkeit" and "Mächtigkeit", although it was surely provoked by them.

This then is a summary of what we have said so far concerning "virtualité".

Now we return to the difference between Baire and Lusin. Actually, if our study of "virtualité" is concerned with the verbal meaning alone, it would not be a question of much importance. But now that we are conscious of the significance of "virtualité" in Baire's mathematical thought, the question should not remain on a verbal level, but should have contact with this mathematical thought. So it is desirable in this sense to examine whether it is true or not, that Lusin's "virtualité" can be considered as a non-atomistic totality, as was seen in Baire's. However, through a close examination, we think we can conclude that the radical but somewhat vague non-atomistic way of thinking in Baire (or in early Borel) has been largely watered down in Lusin.

In the beginning of Chap. 1 of Lusin's "Leçons", he says that, to a certain extent, the main problem in his book is a new form of Eleatic problem (Lusin [1], p. 2); but thereafter he gives us little hint of neither the atomistic nor the non-atomistic way of thinking in the sense used here. At the outset he consciously permits the continuum, although provisionally, to be a set of points, and upon it developed his theory. Veritable existences are dug out, so to speak, from this provisional totality. This treatment of the continuum is very similar to that of the "virtualité" in Baire. Indeed, from his attitude we can consider that, for Lusin also, the continuum is something given *a priori* which latently plays a fundamental role in mathematics.

But in this respect Lusin's opinion vacillates delicately. The following quotation, which is taken from another source (Lusin [3]), shows it to us clearly:

> D'abord, il y a bien des virtualités qui semblent être, dans l'état actuel de la Science, des totalités légitimes (c'est-à-dire de véritables ensembles). Par exemple, il paraît difficile de nier, pendent qu'on fait un travail dans la Théorie des Fonctions, la possibilité de raisonner sur *tous* les *points* compris dans un intervalle donné, ou sur *toutes* les *fonctions continues*, cette possibilité paraît légitimée par la définition connue *tout finie* d'un point réel ou de la continuité d'une fonction due à Cauchy. On obtient ainsi les propriétés générales de la *totalité*, mais on ne raisonne jamais en réalité sur un individu *déterminé* de cette totalité, à moins qu'il ne s'agisse d'un individu très particulier qui a pu être distingué de tous les autres par une définition finie *). D'ailleures, si l'on donne, *par une loi quelconque*, un individu de la totalité, nous savons le déterminer *post factum* par un pro-

cédé régulier, en employant, par exemple, les divisions décimales ou les approximations polynomiales bien chosies.

(*) *Voir* É. Borel: *La Philosophie mathématique* et *l'infini* (Revue du Mois, aout 1912).

(Lusin [3], Fund. Math. vol. 10, pp 31-32)

As to the non-atomistic way of thinking, it would be safe to say that in Lusin it refreated largely from a stage on which Borel (in his early days) and Baire had played. As we have mentioned in § 2, in the theory of analytic sets, those mathematical existences of which the set theoretical structures are well defined are selected out (at least tentatively) as "veritable" mathematical existences. Interesting is the fact that in Lusin, the *axiom of extentionality* can be applied to those "veritable" existences; in other words, the atomistic character of the Cantorian mathematics has already entered secretly into Lusin's mathematics as one of the most fundamental characters of its "veritable" existences. We should like to note the fact that, as is shown in the above quotation, Lusin doubted the validity of the continuum on account of a doubt about the concept of "un individu de la totalité" ——— an "atom" of the continuum, so to speak. Apart from this philosophical opinion of Lusin, his actual way of developing the theory would be readily accepted by any Cantorian.

Facing these aspects of the matter, we suspect that Cantor's way of thinking, even if it is restricted within the domain of "veritable" existences, has already dominated unwittingly Lusin's way of thinking, although he was one of the most eminent and faithful successors of Baire and Borel, and probably considered himself as such. And this suspicion will be more strongly supported by the fact that Lusin later submitted his own "continuum-hypothesis", which was incompatible with the (ordinary) Cantorian one. (Lusin's continuum-hypothesis is as follows: any set of which the cardinality is aleph 1, is always a CA set. (Lusin [2], Sierpinski [2])) The question is not in the content of this or that hypothesis, but in the very fact that he submitted such a hypothesis. This fact really tells us that the continuum was already regarded by Lusin as a set of points.

Here is one more point we can not neglect. As we have mentioned in §3, Borel started a debate with Zermelo in 1905, and many mathematicians participated in it. In this period we have much evidence for Borel's non-atomistic way of thinking, for example, in the "cinq lettres". But after all, this long and serious debate may have left a certain influence upon Borel to change something in his way of thinking. On reading through his articles written soon after the debate, we can perceive the transition of his thought. There may also have been some influence of the atmosphere of the age upon Borel. Indeed the atmosphere

was transforming itself day by day in those days, and the Cantor theory, strange as it had been in the beginning, was gradually forming a new spirit of mathematics. In §3 we cited a paragraph in Note IV - §V from Borel's "Leçons" to show his non-atomistic opinion as to the continuum, but after the next part of the Note, we can no more read such an opinion: Borel's article which is cited in our above quotation from Lusin's book was also written in 1912. Towards the beginning of this period, in Note IV - § V (orginally published in Oct. 1908), we read that he submitted a new concept of "practical continuum", which consists of those points which are defined effectively.

> Il n'est pas possible d'indiquer le moyen de fixer sur la droite un point unique et bien déterminé qui n'appartienne pas à cet ensemble; la proposition d'après laquelle *il y a* de tels points est vraie ou fausse suivant qu'on admet ou non la possibilité d'une infinité dénombrable de choix successifs; mais c'est là une question *métaphysique*, en ce sens que la réponse positive ou négative n'aura jamais aucune influence sur le développement de la Science: tous les points dont on pourra jamais avoir besoin dans les raisonnements auront été définis au moyen d'un nombre fini de mots; ils constituent le *continu pratique* qu'utilisent les mathématiciens. (Borel [1], p. 164)

The "practical continuum" is nothing but a set of those points, of which the construction is well defined. In this stage, the non-atomistic way of thinking of Borel has already largely retreated. From this time on, this idea of "practical continuum" was developed in his articles, such as "Les probabilités dénombrables et leurs applications arithmétiques" (Borel [1], Note V; partly published in 1909), and "La théorie de la mesure et la théorie de l'intégration" (Borel [1], Note VI; supposed to be written around 1912); then Lusin's "continu arithmétique" succeeded it. Although this concept of Lusin appeared, without any reference, for the first time towards the end of the "Conclusion" of his "Leçons", some influence exerted by Borel's idea can be clearly seen, say, from the fact that there are many long and important footnotes taken from Borel's "Leçons".

Incidentally, it is not our arbitrary opinion to take the atmosphere of the age into account in these respects. For example, we have H. Lebesgue, who once participated in the debate (1905) concerning Zermelo's proof (of *the well-ordered theorem*) as a member of Borel's side, and confronted Jacque Hadamard, an advocate of Zermelo in the debate. It is very interesting that several years later Lebesgue himself offered his weakened opinion, thus:

> M. Hadamard et moi avions échangé sur ce sujet, à l'occasion du Congrès de philosophie de 1914, une correspondance passionnée dont la guerre a

> empêché la publication. On ne doit guère le regretter, car toutes ces discussions sont oiseuses; l'intérêt de l'axiome de Zermelo serait prouvé le jour où l'on en aurait tiré des résultats positifs utiles. On se familiariserait avec cet axiome en l'employant, et peu à peu on verrait se dissiper les difficultés d'ordre philosophique qui nous arrêtent actuellement. (Sierpinski [1], p 107; from Lebesgue's "Notice sur les travaux", Toulouse, 1922).

Taking these circumstances into account, we think we can conclude that it was the influence of Borel's new thought (after 1908), or more widely, the transition of the mathematical atmosphere towards the Cantor theory which surrounded Borel's thought at any rate, that had caused the difference between Baire and Lusin with respect to the meaning of "virtualité".

Here again is a question: why did Borel retreat from his early non-atomistic thought? Was the atmosphere of the age the only reason? Actually, we have no positive evidence concerning this matter; but we think we can surmise the reason for his retreat --- perhaps the first thing is not the atmosphere, but the difficulty in the problem concerning the treatment of the "Continuous" in mathematics.

It goes without saying that the root of this difficulty is so deep that there were no really mathematical thinkers who were not worried about it from the most ancient times to the present day. We have no intention now to confront this hard problem directly, or even enter into the complete history of it, but it would be worthwhile to make a survey of it, as far as our present problem is concerned.

First of all, it is noted that in Greek mathematics there seems to have been the same kinds of problems. According to one of Árpád Szabó's papers (Szabó [1]), the distinction among "definition", "postulates", and "axioms" in the (so-called) Euclid's "Elements" comes from various ancient micro-systems which had been fated, after having been tested and arranged, to form the frame of Euclidean Geometry. Szabó also said that the arithmetical part of the "Elements" (vols, 7, 8, 9) started with the uniform "principles" called "definitions", whereas the (continuous) geometrical part of it started with multiform "principles", such as "definitions", "postulates", "axioms" and others. Probably there may have been many attempts to treat the "Continuous" as exactly and as simply as possible in those days; and it had been as difficult a problem in ancient times to deal with the "Continuous" ingeniously as it was in the early days of the set theory. Did the perplexed mind of Lusin in founding his theory of analytic sets (summarized in our § 2) not show us his efforts to grope for a clue to the problem? And, was it not the case for Borel and Baire, too? Or, on the other hand, could we say that Cantor did surely speak the last word to this oldest and most puzzling problem?

Apart from these serious matters, here is another important historical point

---- the rise of notationalism in the Renaissance. It is very important, because it was fated to be the beginning of the decay of the (continuous) geometrical theory in the long run. Notational algebra in its early days was simply "analysis", that is to say, it was merely *a means* to find something, say, to find a root of an equation, and was not the *method* which offered a way to prove something as well as a way to find something. In those days, in order to prove such and such a thing, the most reliable theory was Euclidean geometry; it contained, indeed, the most exact geometrical theory of quantity (the theory of *Proportion*) that human beings had ever had, and this theory was fully comparable with the theory of real numbers in present-day mathematics. (Nakamura [1] [2]).

The transition from the geometrical theory to the notational algebraic theory, which was one of the greatest events in the history of mathematics, was, as is well known, first advocated by René Descartes; and after some two centuries, another decisive step along this line was taken by Richard Dedekind, G. Cantor and others. This was the birth of the modern (notational) theory of real numbers, and with this historical background, Borel started his mathematical works soon thereafter.

Frankly speaking, Borel's mathematical papers are not easy to read: his terminology is not always clearly defined; sometimes it changes, too. Moreover, at least seemingly, he did not even settle his own standpoints in discussing the foundations of mathematics (Murata [1]). But after looking over his books and papers, together with Baire's, Lebesgue's and others' on one hand, with Cantor's and others' on the other hand, gradually we come to realize that he was a person who, among others, had sincerely worried about the problem concerning the Continuous and attempted a construction of a firm concept of the continuum for mathematics' sake, even if the direction of his works was in opposition to the current of the history of mathematics at that time.

The theory of real numbers in present-day mathematics is essentially atomistic in the sense used here. For many years this theory has been examined to the core and has proved to be firm, except in the study of the Foundations of Mathematics. On account of its fluent notationalism, its use is far more convenient than that of the old geometrical theory of quantity, although this was as exact a theory as that. Under these circumstances, there might be an opinion that it is unnecessary to reexamine the concept of the continuum from the non-atomistic points of view which is too old in the present day. But it would be equally true that, since the atomistic way of thinking becomes too matter-of-course today, rather it is worthwhile to deliberate and discuss it from the non-atomistic

way of thinking ---- this was also one of the most fundamental thoughts through the history of mathematics, or rather, the history of thought.

Actually, there may be no other way better than the atomistic way of thinking in present-day mathematics. But the present author, although he knows this fact fairly well, still would like to say that he doubts that there can be a last word in a problem like this. ---- But this is no longer our present subject.

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