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LAUTMAN ON PROBLEMS AS THE CONDITIONS OF EXISTENCE OF SOLUTIONS

Albert Lautman (1908–44) was a philosopher of mathematics who postulated a conception of mathematics that is both formalist and structuralist in the Hilbertian sense. The reference to the axiomatic structuralism of Hilbert is foundational for Lautman, and it is because of this that his views on mathematical reality and on the philosophy of mathematics parted with the dominant tendencies of mathematical epistemology of the time. Lautman outlined a "critical" program in mathematics that was intended to displace the previous foundational discussions that were occupied with the criticism of classical analysis. Against the logicist claim that the development of mathematics is dominated a priori by logic, Lautman proposes a "metaphysics of logic," and calls for the development of a "philosophy of mathematical genesis." Lautman considered the role of philosophy, and of the philosopher, in relation to mathematics to be quite specific. He writes that "in the development of mathematics, a reality is asserted that mathematical philosophy has as a function to recognize and describe" (*Mathematics* 87). He goes on to characterise this reality as an "ideal reality" that "governs" the development of mathematics. He maintains that

> what mathematics leaves for the philosopher to hope for, is a truth which would appear in the harmony of its edifices, and in this field as in all others, the search for the primitive concepts must yield place to a synthetic study of the whole. (Ibid.)

One of the challenges that Lautman set himself, but never carried through, was the task of deploying the mathematical philosophy that he developed in other domains. The structural feature of his mathematical philosophy that aimed to facilitate this deployment is an account of the relation between mathematical problems as they arise in the historical development of mathematics and the solutions that are provided to these problems by mathematicians, in the form of mathematical proofs and theories, all of which are governed by what Lautman characterises as a dialectics of mathematics. The aim of this paper is to give an account of both this Lautmanian dialectic and of how it can be understood to govern the development of solutions to mathematical problems, and of the concept of the mathematical real that Lautman uses to frame this dialectic. I argue that it is by clarifying Lautman's relation

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to the work of Plato and Heidegger that his account of the mathematical real and the dialectic operating in relation to it can best be understood.

lautman's axiomatic structuralism

What is quite clear in Lautman's work is that he was not concerned with specific foundational questions in mathematics, neither with those relating to its origins, to its relationship to logic or to the problem of foundations. What he is interested in, rather, is shifting the ground of this very problematic by presenting an account of the nature of mathematical problematics in general.

Lautman, along with Cavaillès, is one of the introducers of the German axiomatic into the French context that at the time was dominated by the "intuitionisms" of Poincaré, Borel, Baire and Lebesgue (Petitot 83). The two main ideas that are foregrounded in his primary theses in the philosophy of mathematics (*Essai sur les notions de structure et d'existence en mathématiques*; *Essai sur l'unité des sciences mathématiques dans leur développement actuel*) and which dominate the development of his subsequent work are "the concept of *mathematical structure* and the idea of the essential *unity* underlying the apparent multiplicity of diverse mathematical disciplines" (Dieudonné 16). It should be noted that, "in 1935, the concept of structure" in mathematics "had not yet been made completely explicit" (ibid.). Lautman's project is therefore novel. Lautman was inspired by the work of Hilbert on the axiomatic concept of mathematics to deploy the potential of an axiomatic-structuralism in mathematics. The essential point that motivated this move was Lautman's conviction "that a mathematical theory is predominantly occupied with the relations between the objects that it considers, more so than with the nature of those objects" (ibid.).

Lautman considers the understanding that there is "an independence of mathematical entities compared to the theories in which they are defined" (*Mathematics* 186) to be steeped in the analysis and geometry of the nineteenth century. Lautman, on the contrary, championed the modern algebra, and maintains that "if classical mathematics was constructivist [...] modern algebra is on the contrary axiomatic" (Loi 13). The introduction of the axiomatic method[1] into mathematics means that there is an "essential dependence between the properties of a mathematical entity and the axiomatic of the domain to which it belongs" (*Mathematics* 186). The isolation of "elementary mathematical facts" that would function as building blocks is ruled out. Lautman can therefore claim that "the problem of mathematical reality arises neither at the level of facts, nor at that of entities, but [rather] at that of theories" (187). This is not, of course, to put mathematical facts per se into question. Lautman considered mathematics to be constituted like physics: "the facts to be explained were throughout history the paradoxes that the progress of reflexion rendered intelligible by a constant renewal of the meaning of the essential notions" (88). Rather than being isolatable elementary objects, mathematical facts, such as the "irrational numbers, the infinitely small, continuous functions without derivatives, the transcendence of e and of π, and the transfinite," "were admitted by an incomprehensible necessity of fact before there was a deductive theory of them" (ibid.). He argues that mathematical and physical facts "are thus organized under the unity of the notion that generalizes them" (184).

Lautman's "axiomatic structuralism" was the new mathematics that inspired the Bourbaki project which was influential in mathematics for a number of the decades that followed.[2] Notably in the figure of Jean Dieudonné, who wrote the preface to Lautman's collected works (16). The structuralist point of view has been so influential on the development of mathematics since 1940 that it has become rather commonplace.[3] However, this was not yet the case when Lautman was writing (ibid.).

The first move that Lautman makes to develop his structural conception of mathematics is against the logical positivism of the Vienna Circle logicists. Lautman considered their effort "to construct all mathematical notions from a

small number of notions and from primitive logical propositions" to be in vain, because it loses sight of what he refers to as "the qualitative and integral character of the constituted theories" (*Mathematics* 87). He argues that "it is impossible to consider a mathematical 'whole' as resulting from the juxtaposition of elements defined independently of any overall consideration relative to the structure of the whole in which these elements are integrated" (108). For Lautman, this impoverishment of logical positivism is the consequence of its conception of mathematics in propositional terms, as "nothing more than a language that is indifferent to the content that it expresses" (87).

Lautman also protests against the use made of Hilbert by the Vienna Circle logicists. Despite their claims to endorse the Hilbert program,[4] Lautman is critical of the logicist interpretation of the term "formalism," which he considers to be unrepresentative of Hilbert's thought (17). While the logicists are deriving theorems in a formal system, such that the theorems are genetic or constitutive of the system, for Lautman, Hilbert is rather looking for theorems about formal systems, such as consistency or non-contradiction, completeness, decidability, etc.[5] Rather than confounding mathematical philosophy with the study of the different logical formalisms, Lautman considered it necessary to try to characterise mathematical reality "from the point of view of its own structure" (Loi 9). Lautman considered this to be a more accurate characterisation of Hilbert's metamathematical program, which, he argued, "internalised the epistemological problem of foundations by transforming it into a purely mathematical problem" (Petitot 98).[6]

Against the logicist interpretation of Hilbert's work Lautman argues that

> Hilbert has replaced the method of genetic definitions with that of axiomatic definitions, and far from claiming to reconstruct the whole of mathematics from logic, introduced on the contrary, by passing from logic to arithmetic and from arithmetic to analysis, new variables and new axioms which extend each time the domain of consequences. (*Mathematics* 89)

The (Hilbertian) axiomatic structural conception of mathematics that Lautman mobilises in his work is a non-constructivist axiomatic, and he argues that "Mathematics thus presents itself as successive syntheses in which each step is irreducible to the previous step" (ibid.). He continues by making the important point, again drawn from Hilbert, that

> a theory thus formalized is itself incapable of providing the proof of its internal coherence. It must be overlaid with a metamathematics that takes the formalized mathematics as an object and studies it from the dual point of view of consistency and completeness. (89–90)

This dual point of view distinguishes Lautman's concept of mathematics from the formalism of the logicists, which considered the study of mathematical reality to consist solely in the demonstration of the consistency of the axioms which define it. The consequence of this "duality of plans" that Hilbert establishes between "formalized mathematics and the metamathematic study of this formalism" is that while the formalism is governed by the concepts of "consistency and completeness," these concepts are not themselves defined by this formalism (90). Hilbert expresses this governing role of metamathematical concepts over formalised mathematics when he writes that

> The axioms and provable theorems (i.e. the formulas that arise in this alternating game [namely formal deduction and the adjunction of new axioms]) are images of the thoughts that make up the usual procedure of traditional mathematics; but they are not themselves the truths in the absolute sense. Rather, the absolute truths are the insights (*Einsichten*) that my proof theory furnishes into the provability and the consistency of these formal systems. (Hilbert 180 cited in Ewald 1138)

So, according to Lautman, the value of a mathematical theory is determined by "the metamathematical properties that its structure incarnates" (*Mathematics* 90).

While Lautman took a position against the version of logicism and formalism proposed by the Vienna Circle, he also distances himself

from the empirico-psychologising perspective of French mathematicians such as Brunschvicg. Brunschvicg developed "the idea that the objectivity of mathematics was the work of intelligence, in its effort to overcome the resistance that is opposed to it by the material on which it works" (*Mathematics* 88). Brunschvicg goes so far as to maintain that "any effort of *a priori* deduction tends [...] to reverse the natural order of the mind in mathematical discovery" (89).[7] While Lautman follows Brunschvicg in distrusting all attempts "to deduce the unity of mathematics starting from a small number of initial principles," including "the reduction of mathematics to logic" (88), he does not endorse Brunschvicg's concept of mathematical philosophy "as a pure psychology of creative invention" (89). For Lautman, the task of characterising the mathematical real must be undertaken, rather, by mediating between these two extreme positions. By extracting the minimal elements of each, the "logical rigour" of the former and "the movement of the intelligence" of the latter (ibid.), Lautman proposes a third alternative characterisation of the mathematical real that is both axiomatic-structural and dynamic; where the fixity or temporal independence of the logical concepts and the dynamism of the temporal development of mathematical theories are combined.

the metaphysics of logic: a philosophy of mathematical genesis

In order to do this, Lautman distinguishes two periods in mathematical logic. The first he characterises as "the naive period," which goes from "the early work of Russell until 1929," which is the "date of the metamathematical work of Herbrand and Gödel." The latter marks the beginning of what Lautman calls "the critical period." He characterises the first period as "that where formalism and intuitionism are opposed in discussions that extend those raised by Cantor's theory of sets" (*Mathematics* 141). These involved the criticism of classical analysis and the foundational disputes which were largely characterised by the dispute over the legitimacy of the actual infinite. While the formalists, as partisans of the actual infinite, claim the right to identify a mathematical object "as a result of its implicit definition by a system of non-contradictory axioms" (ibid.), the intuitionists, on the contrary, maintain that "asserting the possibility of an unrealizable operation," for example, "with regard to an object whose construction would require an infinite number of steps, or a theorem that is impossible to verify" because it relies on impredicative definitions,[8] "is to assert something which is either meaningless, or false, or at least unproven" (141–42).

Lautman's interpretation of the unity of mathematics distinguishes him from the constructivist perspective of his French intuitionist contemporaries (including Brouwer) because Lautman considered the actual infinite to be legitimate in its algebraic-axiomatic presentation. And, contrary to the intuitionists and constructivists, Lautman grants to mathematical logic all the consideration which it deserves, that is, Lautman accepts the logical principle of the excluded middle.[9] However, he maintains that "logic is not *a priori* in relation to mathematics but that for logic to exist a mathematics is necessary" (109). He considered the simple idea that the logicists of the "naive period" had made of "an absolute and univocal anteriority of logic in relation to mathematics" to be "out-of-date" (Loi 13).

For Lautman, the philosophy of mathematics is not reducible to a secondary epistemological commentary on problematic logical foundations, nor to historical or *a fortiori* psychosociological research, nor to reflections on marginal movements such as intuitionism.[10] It is, however, precisely in the research of the critical period relating to the consistency of arithmetic that Lautman considers a new theory of the mathematical real to have been affirmed. One that is "as different from the logicism of the formalists as from the constructivism of the intuitionist" (*Mathematics* 143). Lautman claims that between the naive and critical periods there is an "evolution internal to logic," and he sets himself the task of

disengaging from this new mathematical real "a philosophy of mathematical genesis, whose scope goes far beyond the domain of logic" (ibid.).

While Hilbert's metamathematics proposes to examine mathematical theories from the point of view of the logical concepts of non-contradiction and completeness, Lautman notes that "this is only an ideal toward which the research is oriented, and it is known at what point this ideal currently appears difficult to attain" (90). This is an implicit reference to Gödel's second incompleteness theorem which demonstrates that any consistent formal system cannot demonstrate its completeness by way of its own axioms. Lautman concludes from this that "Metamathematics can thus envisage the idea of certain perfect structures, possibly realizable by effective mathematical theories, and this independently of the fact of knowing whether theories making use of the properties in question exist" (ibid.). What we have with this conception of the mathematical real is "the statement of a logical problem is possessed without any mathematical means to resolve it" (ibid.). What this means for Lautman is that the critical period marks the appearance of innovation in mathematics, not only at the level of results but also at that of the problematic (143). Lautman proposes to characterise the problematic "distinction between the position of a logical problem and its mathematical solution" (91) by means of an "exposé" of what he calls "the metaphysics of logic" (141). This takes the form of "an introduction to a general theory of the connections which unite the structural considerations" of the critical axiomatic-structural conception of mathematics with the "assertions of existence" of a particular dynamic conception (ibid.). The particular dynamic conception of mathematics that Lautman deploys is further characterised when he qualifies his conception of the essential nature of mathematical truth as follows: "Any logical attempt that would profess to dominate *a priori* the development of mathematics therefore disregards the essential nature of mathematical truth, because it is connected to the creative activity of the mind, and participates in its temporal character" (187). Lautman is careful here to point out that mathematical truth is only partially related to the creative activity of the mind of the mathematician. In order to distinguish his account of dynamism from Brunschvicg's, Lautman considers it "necessary to grasp, beyond the temporal circumstances of a discovery, the ideal reality which is solely capable of giving its sense and value to the mathematical experience" (Cavaillès and Lautman 39). The lynchpin of this distinction is that Lautman conceives "this ideal reality as independent of the activity of the mind." For Lautman, the activity of the mind of the mathematician "only intervenes […] once it is a matter of creating effective mathematics," that is, effective mathematical theories (ibid.). This ideal reality is constituted by what he refers to as "abstract Ideas." Lautman proposes to call the relation between the independent activity of the mind of the mathematician in the development of mathematical theories and the ideas of this ideal reality "dialectical," and he refers to these ideas as "dialectical Ideas" (*Mathematics* 199). Lautman's principal thesis is that mathematics, and the mathematical problems that it grapples with, participates in a dialectic that governs it, its problems and the possible solutions to those problems, in an abstract way. He argues that the Ideas "which seem to govern [*domines*] the movement of certain mathematical theories" (91–92), and which are conceivable as independent of the mathematics, "are nevertheless not amenable to direct study" (92). When Lautman maintains that they are conceivable as independent of the mathematics, he means of the mathematical theories themselves, but not of the mathematical real, of which they are a component. Lautman is working with a broader concept of mathematics, that of the mathematical real, which is greater than the sum of mathematical theories. He goes on to claim that it is these dialectical Ideas that "confer on mathematics its eminent philosophical value" (ibid.). This is why Lautman considers mathematics, and especially "modern mathematics," and here Lautman is referring to the post-critical developments in

algebra, group theory and topology, to tell, "in addition to the constructions in which the mathematician is interested, another more hidden story made for the philosopher" (91). The gist of the story is that there is a "dialectical action [that] is always at play in the background and it is towards its clarification" that Lautman directs his research (ibid.). Lautman characterises this dialectical action, which is always in response to a mathematical problem, as follows:

> Partial results, comparisons stopped midway, attempts that still resemble groupings, are organized under the unity of the same theme, and in their movement allow a connection to be seen which takes shape between certain abstract Ideas, that we propose to call dialectical. (Ibid.)

Lautman argues that the nature of the mathematical real, and indeed the nature of physical reality, "its structure and the conditions of its genesis are only knowable by ascending to the Ideas" (193).

lautman's platonism

At the time, opinion amongst mathematicians and philosophers was largely unfavourable to Lautman. Mathematicians were at odds with what was for them his incomprehensible "philosophical speculation" and its "subtleties" (Petitot 99), while the philosophers reproached him for what they considered to be a certain inaccuracy in his use of the term "dialectical" (*Mathematics* 28): was it Socratic, Kantian or Hegelian?[11] The confusion over the nature of the dialectic in Lautman remains pretty much intact, with quite recent commentators such as Jean Petitot – a French mathematician and philosopher of mathematics who, contrary to Lautman's peers, considers Lautman to be one of the most inspiring philosophers of the twentieth century (Petitot 80) – suggesting that the dialectic proposed by Lautman is a Hegelian one (Petitot 113).[12] I would like to suggest, to the contrary, that the dialectic Lautman is proposing is rather Platonic. His account of Ideas does seem to commit him to a version of Platonism. It is, however, a Platonism that is quite distinct from what is usually called "platonism" in mathematics, which consists rather in the practice of summarily indicating with the name platonism any mathematical philosophy for which the existence of a mathematical object is held as assured. Lautman considers this to be only one superficial understanding of platonism (*Mathematics* 190). Nor does he "understand by Ideas the models whose mathematical entities would merely be copies" (199). Lautman is here opposed to the platonism traditionally founded on a certain realm of Ideas, which interprets mathematical theories as copies, reproductions, translations, or simple transpositions of eternal ideal models or forms. Instead he wants to "remove the idea of an irreducible distance between the '*eidos*' and its representation to affirm the productive power of ideas which are incarnated in the theories."[13] What Lautman wants to do is restore to Ideas what he considers to be "the true Platonic meaning of the term." The role that Lautman assigns to mathematics is the result of a "Platonic interpretation." "Platonism" in mathematics is usually understood to be a simple abbreviation for the realism of Plato's Ideas applied to the existence of mathematical things, or objects. This meaning corresponds to the Platonic dualism popularised in the myth of the Cave: to the Visible world of bodies is superimposed the Intelligible world of Ideas. But in the Analogy of the Line which precedes it, Plato makes a double division: the Line is initially divided into the Intelligible and the Visible or Ontological, then the superior segment of the understandable is subdivided in turn into the distinction between visible forms, which include mathematical objects, and forms themselves, which are the objects of the Dialectic or "Ideas."

Lautman's philosophy of mathematics embraces pure and applied mathematics in a unitary theory. Applied mathematics, which is occupied with the physical real, is deployed in the "Visible" or Ontological – one of the initial divisions of the line – whereas the treatment of pure mathematics requires the "Intelligible." The intelligible is then subdivided into the inferior segment, occupied by the visible forms of mathematics, and the superior

segment by the "Ideas" of the Dialectic. A practising geometer or mathematician, working with visible lines and figures, is thinking about the square itself and the diagonal itself, which are not visible. The visible forms with which he works are useful as images in coming to see truths about intelligible objects. One and the same thing (a visible triangle) can therefore serve both as an original in the visible or ontological division (when compared to its shadows and reflections – the inferior subdivision of the ontological) and as an image in the intelligible (when compared to the triangle itself, which is among the contents of knowledge/understanding). This is reflected in the two attitudes one can take to the numbers, lines, angles, and figures of the mathematician:

(1) One is to take them as known, as starting-points or first principles, and to reason from them without ever examining them. This is what Plato considers to be the customary attitude, which involves reasoning from hypotheses to end-points or conclusions. But then what one is entitled to conclude can be only as clear as one's hypotheses will allow; and since they are never themselves examined, this may leave room for unclarity or doubt.
(2) The other attitude is to regard hypotheses not as starting-points, but as what they really are, things set down at the beginning of an inquiry to enable one to work one's way toward something else. This is what the dialectician does. He begins with hypotheses just as the geometer does, but by subjecting these hypotheses to dialectical examination he can work his way to a clearer grasp, of even the geometer's conclusions.

The practising mathematician takes for granted the entities with which he works and gives no account of them, but treats them as starting-points; his state or condition is thought/reasoning (Plato 510C2–D3). He may deal with forms, for example the square itself or the diagonal itself, but he simply takes them for granted. It is with such an attitude in mind that, after describing the mathematical curriculum in Book VII of the *Republic*, Plato says that the mathematical sciences are evidently dreaming about reality. There is no chance of their having a conscious glimpse of reality as long as they refuse to disturb the things they take for granted and remain incapable of explaining them. Plato is concerned to stress that one must not rely on unexamined hypotheses, but must rather subject all one's hypotheses to dialectical scrutiny. The dialectician, as distinguished from the practising scientist or mathematician, sees things holistically, and leaves no assumption unexamined. There is thus a difference in method and attitude between the dialectician and the mathematician; however, Plato insists that knowledge can be had of those things the practising mathematician grasps by thought/reasoning. Plato goes so far as to suggest that by diligent use of dialectic one can work one's way to what is unhypothesised, the starting-point for everything. Having got that far, one can draw conclusions, even mathematical and geometrical ones, without using anything perceptible at all, but simply "by means of forms alone, in and of themselves, and [ending] with forms" (Plato 511B8–C2, 511C8–D2).

It is important to note that Lautman's Platonism and the dialectic that he employs does not go this far. The most important point for Lautman is that Platonic Ideas are by no means reducible to "universals." The Platonic Idea is, moreover, an Archetype or Ideal, which makes them the touchstone for the selective function which is that of the Dialectic. Lautman wants to restore the Platonic understanding of these abstract dialectical Ideas, not as universal forms but as "the structural schemas according to which effective theories are organized" (*Mathematics* 199).[14] Lautman characterises these structural schemas as establishing specific connections between contrary concepts such as: local–global; intrinsic–extrinsic; essence–existence; continuous–discontinuous; and finite–infinite. Lautman provides many examples of these contrary concepts, including the introduction of analysis into arithmetic; of topology into the theory of functions; and the effect of the penetration of the

structural and finitist methods of the algebra into the field of analysis and the debates about the continuum.[15]

The nature of mathematical reality for Lautman is therefore such that "mathematical theories [...] give substance to a dialectical ideal" (240). This dialectic is constituted "by pairs of opposites" and the Ideas or structural schemas of this dialectic are presented in each case "as the problem of establishing connections between opposing notions" (ibid.), or concepts. Lautman makes a firm distinction between concepts and dialectical Ideas: the Ideas "consider possible relations between dialectical notions" (204), or conceptual pairs,[16] and "these connections can only be made within the domains in which the dialectic is incarnated" (240). What Lautman is proposing is a philosophical logic that considerably broadens the field and range of the metamathematics that he adopts from Hilbert. While metamathematics examines mathematical theories from the point of view of the concepts of non-contradiction and completeness, or consistency, Lautman argues that there are "other logical notions," or concepts, "equally likely to be potentially linked to one another within a mathematical theory" (91). These other logical concepts are the conceptual pairs of the structural schemas,[17] and Lautman argues that, "contrary to the preceding cases (of non-contradiction and completeness)," each of which is bivalent, "the mathematical solutions to the problems" which these conceptual pairs pose can comprise "an infinity of degrees" (ibid.).

So, for Lautman, Ideas constitute, along with mathematical facts, objects and theories, a fourth point of view of the mathematical real. "Far from being opposed these four conceptions fit naturally together: the facts consist in the discovery of new entities, these entities are organized in theories, and the movement of these theories incarnates the schema of connections of certain Ideas" (183). For this reason, the mathematical real depends not only on the base of mathematical facts but also on dialectical Ideas that govern the mathematical theories in which they are actualised. Lautman thus reconsiders metamathematics in metaphysical terms, and postulates the metaphysical regulation of mathematics. However, he is not suggesting the application of metaphysics to mathematics. Mathematical philosophy such as Lautman conceives it does not consist "in finding a logical problem of classical metaphysics within a mathematical theory" (189). Rather, it is from the mathematical constitution of problems that it is necessary to turn to the metaphysical, that is to the dialectic, in order to give an account of the Ideas which govern the mathematical theories. Lautman maintains that the philosophical meaning of mathematical thought appears in the incorporation of a metaphysics (or dialectic), of which mathematical theories are the necessary consequence. "We would like to have shown," he argues, "that this rapprochement of metaphysics and mathematics is not contingent but necessary" (197). Lautman does not consider this to be "a diminution for mathematics"; on the contrary, "it confers on it an exemplary role" (224).[18] Lautman's work can therefore be characterised as metaphysical, which, in the history of modern epistemology, characterises it as "simultaneously original and solitary."[19]

problematic ideas and the concept of genesis

A key point for Lautman is that dialectical Ideas only exist in so far as incarnated in mathematical theories. Lautman insists on this point. He argues that "the reality inherent to mathematical theories comes to them from their participation in an ideal reality that is dominating with respect to mathematics, but that is only knowable through it" (*Mathematics* 30). This is what distinguishes Lautman's conception from "a naive subjective idealism" (Petitot 86). The dialectical Ideas are therefore characterised by Lautman as constituting a problematic.[20] He argues that "while the mathematical relations describe connections existing in fact between distinct mathematical entities, the Ideas of dialectical relations are not assertive of any connection whatsoever that in fact exists between notions," or concepts (*Mathematics* 204). They constitute, rather, a problematic, that is, they are "posed problems,

relating to the connections that are [only] likely to support certain dialectical notions" or concepts (205). As such, they are characterised by Lautman as "transcendent (in the usual sense) with respect to mathematics" (ibid.). The effective mathematical theories are constructed in an effort to bring a response to the problem posed by these connections, and Lautman interprets "the overall structure of these theories in terms of immanence for the logical schemas of the solution sought after" (205–06). That is, the conceptual pairs of the logical schemata "*are not anterior to their realization within a theory*" (188). They lack what Lautman calls "the extra-mathematical intuition of the exigency of a logical problem" (188–89). The fundamental consequence is that the constitution of new logical schemas and problematic Ideas "*depend on the progress of mathematics itself*" (189). Mathematical philosophy such as Lautman conceives it consists in "grasping the structure of" a mathematical "theory globally in order to identify the logical problem that" is mathematical and is "both defined and resolved by the very existence of this theory" (ibid.). "An intimate link thus exists," for Lautman, "between the transcendence of the Ideas and the immanence of the logical structure of the solution to a dialectical problem within mathematics" (206). It is in direct relation to this link that Lautman characterises the concept of "genesis" (ibid.) that he considers to be operative in the relation between the dialectic and mathematics. However, "the order implied by the concept of genesis is not about the order of the logical reconstruction of mathematics" as undertaken by the logicists. For the latter, the genetic definitions or initial axioms of a theory give rise to "all the propositions of the theory" (203), whereas for Lautman, although the dialectic is anterior to mathematics, it "is not part of mathematics," that is, the mathematical theories, and its concepts "are without relationship to the primitive notions," or concepts, "of a theory" (204). Nor is the genesis conceived in the Platonic sense as "the material creation of the concrete from the Idea," but rather as what Lautman describes as the "advent of notions relative to the concrete within an analysis of the Idea" (200). Lautman defines the "anteriority of the dialectic" as that of "the 'question' with respect to the response": "it is of the nature of the response to be an answer to a question already posed […] even if the idea of the question comes to mind only after having seen the response" (204).

The dialectic therefore functions by extracting logical problems from mathematical theories. The apprehension of the conceptual pair, that is, the logical schema of the problematic Idea, comes only after having extracted the logical problem from the mathematical theory. This is the basis for Lautman's understanding of the genesis of concepts from the concrete that is operating in the dialectic. And, it is the logical problem itself, rather than the problematic Idea, that directly drives the development of mathematics. The problematic Idea governs the extraction process that deploys the logical problem in the further development of new mathematical theories. So for Lautman, "The philosopher has neither to extract the laws, nor to envisage a future evolution, his role only consists in becoming aware of the logical drama which is played out within the theories" (189). This effort of understanding on the part of the philosopher that is "adequate to dialectical Ideas" is itself "creative of the system of more concrete notions," or concepts, where the connections between the concepts are defined (200). The only "*a priori* element" that is able to be conceived "is given in the experience of the exigency of the problems," which is anterior not only to "the discovery of their solutions" (189) but also to the extraction of the logical problem from the mathematical theory under scrutiny.

So there are dialectical Ideas, which are referred to as "problematic" because they are presented solely as the problem of establishing the connections between conceptual pairs. This relation between conceptual pairs is recognised as being characteristic of a mathematical theory only after the logical problem, which is specific to the mathematical theory, has been extracted from it. So the starting point is with a mathematical theory, and the extraction of a logical problem from it leads to the

retrospective recognition that the development of the mathematical theory was governed by the problematic relation between the relevant conceptual pairs, which is representative of a particular dialectical Idea.

heidegger and the naive period in the history of mathematical logic

Lautman suggests that the anteriority of the dialectic in relation to mathematics "is a matter of an 'ontological' anteriority, to use the words of Heidegger" (*Mathematics* 204). Lautman describes this as a "transcendental conception of the relation of governing […] between the dialectic and mathematics" (200), by which he means mathematical theories, and he draws conceptual support for this by analogy with Heidegger, who, Lautman argues, "independently of any reference to mathematical philosophy […] presented analogous views to explain how the production of notions," or concepts, "relative to concrete existence arise from an effort to understand more abstract concepts" (ibid.).[21]

Lautman considers such an analysis to be "masked" in *Being and Time* by the importance that Heidegger places on "the existential considerations relating to being-in-the-world" (201). However, he claims that

> in the second part of *The Essence of Reasons*, Heidegger relies precisely on the distinction of the ontological point of view and ontical point of view to explain the link that exists between human reality and existence-in-the-world. (Ibid.)

Heidegger describes it as "a genesis of the ontical concept of the World from the idea of human reality" (202).[22] Lautman maintains that the "primacy of the anthropological preoccupations" in Heidegger's philosophy "should not prevent his conception of the genesis of notions," or concepts, "relating to the Entity," or to what already exists, "from having a very general bearing," despite being almost exclusively deployed by Heidegger "within the analysis of Ideas relating to Being" (ibid.). Lautman notes that Heidegger himself applies this analysis "to physical concepts" such as "space, locus, time, movement, mass, force and velocity" (ibid.).[23] However, those "questions that do not come out of the anthropology […] remain," for Heidegger, "very brief" (203). Despite this, Lautman considers it to be "possible, in the light of these conceptions of Heidegger, to see the utility of mathematical philosophy for metaphysics in general" (ibid.), a move that Heidegger himself is highly sceptical of. What is encouraging in this respect with Heidegger is that in *Being and Time* he does not collapse mathematics completely into "the mathematical," or the "theoretical attitude" that Dasein adopts as a way of being that is ostensibly inauthentic (*Being and Time* 408–15. See Woodard 9). Neither does he do so in the much later work *The Question Concerning Technology*, where he characterises this attitude as "Enframing," or "the mode of ordering as standing-reserve" (20. See Woodard 14). Heidegger's objections against modern mathematics, together with his sporadic endorsements of the intuitionist rejoinders against the principle of the excluded middle, can be presented as targeting the formalism of the logicists (see Woodard 17). However, there is nothing to indicate that these Heideggerian comments deal with anything but what Lautman has characterised as the "naive period" in the history of mathematical logic.

Lautman does, however, acknowledge that "the restrictions and delimitations" in Heidegger's text "should not be conceived as an impoverishment, but on the contrary as an enrichment of knowledge, due to the increase in precision and the certainty provided" (*Mathematics* 205). What Lautman draws from Heidegger, then, is the understanding that

> the constitution of the being of the entity, on the ontological plane, is inseparable from the determination, on the ontic plane, of the factual existence of a domain in which the objects of a scientific knowledge receive life and matter. (201)

So when he is referring to the governing role of the problematic Ideas of the dialectic over the

development of mathematical theories, Lautman claims that

> It then happens to be once again exactly as in Heidegger's analysis, that the Ideas that constitute this problematic are characterized by an essential insufficiency, and it is yet once again in this effort to complete the understanding of the Idea, that more concrete notions are seen to appear relative to the entity, i.e., true mathematical theories. (204)

And that, "As in the philosophy of Heidegger, in the philosophy of mathematics, as we conceive it, the rational activity of foundation can be seen transformed into the genesis of notions relating to the real" (218).

Here, Lautman presents the structure of Heidegger's philosophy as providing an analogy for the structure of the mathematical real, without commitment to the question of fundamental ontology. Indeed, Lautman's aim is to displace the question of fundamental ontology with a structural genetic approach to the mathematical real that serves as a model for developing an understanding of ontology in general. Lautman's claim of the utility of mathematical philosophy for metaphysics, indeed of the necessity of bringing metaphysics and mathematics together, runs counter to the aesthetic move that Heidegger eventually makes against the risks posed by mathematics towards the poiesis of the fine arts; specifically poetry. Lautman's approach can therefore be understood to be offering an alternative point of view that champions the utility of the mathematical real and thereby challenges Heidegger's turn away from mathematics.

the virtual in lautman

The method that Lautman uses in his mathematical philosophy is "descriptive analysis." The particular mathematical theories that he deploys throughout his work constitute for him "a given" in which he endeavours "to identify the ideal reality with which this matter is involved" (*Mathematics* 92). That is, Lautman starts with mathematical theories that are already in circulation. For example, he incorporates all the new work in algebraic topology of the German mathematicians Alexandroff, Hopf, and Weyl, and connects it to the work of Elie Cartan in complex analysis and to that of André Weil in what was then the emerging field of algebraic geometry (Barot 22). He is also one of the first to anticipate the philosophical interest in algebraic topology, a branch of mathematics that was then under development. In relation to these mathematical theories Lautman argues that while

> it is necessary that mathematics exists, as examples in which the ideal structure of the dialectic can be realized, it is not necessary that the examples which correspond to a particular dialectical structure are of a particular kind. What most often happens on the contrary is that the organizing power of a same structure is asserted in different theories; they then present the affinities of specific mathematical structures that reflect this common dialectical structure in which they participate. (*Mathematics* 207)

One of the examples that is developed by Lautman is the operation of the local–global conceptual pair in the theory of the approximate representation of functions (46, 60, 95–109). The "global conception" of the analytic function that one finds with Cauchy and Riemann (*Mathematics* 95) is posed as a conceptual pair in relation to Weierstrass's approximation theorem, which is a local method of determining an analytic function in the neighbourhood of a complex point by a power series expansion that, by a series of local operations, converges around this point (105–06).

The same conceptual pair is illustrated in geometry (95–99), by the connections between "topological properties and the differential properties of a surface," that is, between the curvature of the former and the determination of second derivatives of the latter, both in the "metric formulation" of geometry in the work of Hopf (*Mathematics* 95–98), and "in its topological formulation" in Weyl and Cartan's theory of closed groups (*Mathematics* 98–99). Distinct mathematical theories can therefore be structured by the same conceptual pair.[24]

Lautman sees in the local–global conceptual pair the source of a dialectical movement in mathematics that produces new theories. He argues that "one can follow […] the mechanism of this operation closely in which the analysis of Ideas is extended in effective creation, in which the virtual is transformed into the real" (203). In the case of the Cauchy and Riemann–Weierstrass example, one of the new mathematical theories that was effectively created is Poincaré's qualitative theory of differential equations.

According to Lautman, the problematic nature of the connections between conceptual pairs "can arise outside of any mathematics, but the effectuation of these connections is immediately mathematical theory" (28). As a consequence, he maintains that "Mathematics thus plays with respect to the other domains of incarnation, physical reality, social reality, human reality, the role of model where the way that things come into existence is observed" (209). Lautman's final word on mathematical logic is that it "does not enjoy in this respect any special privilege. It is only one theory among others and the problems that it raises or that it solves are found almost identically elsewhere" (28). Lautman maintains that

> For the mathematician, it is in the choice of original definitions and judicious axioms that true invention resides. It is by the introduction of new concepts, much more than by transformations of symbols or blind handling of algorithms that mathematics has progressed and will progress. (Loi 12)

critical considerations

Lautman explicated the philosophical logic of the actualisation of Ideas within the framework of mathematics. It is important to note that Lautman's references to a dialectic of Ideas should therefore not be understood as being references to a general dialectic that exists independently of the mathematics. Lautman is quite explicit in claiming that the dialectic of Ideas is the fourth point of view of the mathematical real (*Mathematics* 183). The mathematical real is not just the sum of all mathematical theories. The former should therefore not be collapsed into the latter. To do so would lead to the mistaken thesis that mathematics provides evidence of an external and more general dialectic that is equally accessible by means of some kind of analysis performed in regard to or from within other discourses. What seems to be clear in Lautman's work is that he considers himself to be working within the constraints of the discourse of mathematics, and the structure of the dialectic that he presents is determined as operating within the expanded concept of mathematics that he makes claim to: the mathematical real. The dialectic of Ideas is independent of the mathematical theories, or the mathematics per se, but not of the expanded understanding of the mathematical real.

Lautman does claim that the structure of the dialectic is not the sole privy of the mathematical real, and that it can therefore also "be found" in other discourses. However, he does not claim that this is the case because the dialectic is able to be generalised, or in so far as it is transcendent with respect to the mathematical real. While Lautman makes strong claims to the unity of mathematics, which was controversial at the time and remains so today, he does not make any claim whatsoever as to the unity of all discourses. What Lautman argues, rather, is that this is the case because the way that the structure of the dialectic operates in the mathematical real functions as a model for recognising how it can be understood to operate in other discourses.

Lautman maintains that we are able to recognise the logic of relations structured by the dialectic in other discourses solely by virtue of the mathematical theories in which these relations are incarnated; as Lautman argues, "the effectuation of these connections is immediately mathematical theory" (*Mathematics* 28). That is to say that it is the way in which the mathematical logic is deployed in other discourses that allows such a discourse to be understood to operate according to the dialectic. By dialectic Lautman means here the dialectic of the

mathematical real. So mathematics is not privileged over other discourses according to Lautman because, on the one hand, he doesn't consider there to be a definite system of mathematical laws at the base of nature, and, on the other, he does consider it to be intimately involved in our understanding of the very dialectical structure of those discourses. What this amounts to is that mathematical theories are not the sole privy of mathematics, or the mathematical real; they also provide the ground for understanding how the dialectic operates in other discourses. So when Lautman argues that "mathematical logic does not enjoy in this respect any special privilege. It is only one theory among others and the problems that it raises or that it solves are found almost identically elsewhere" (ibid.), by privileged we should also understand exclusive to the mathematical real.

What is important about mathematics for Lautman is its *a priority*, which allows the dialectic of Ideas to be recognised as a component of the mathematical real in a way that is not directly accessible in other discourses. This sets up the mathematical real, and the structure of the dialectic as it operates in the mathematical real in relation to mathematical problems, as a model for the structure of problems in other discourses, and for how we can understand problems in these other discourses to operate as the conditions of the existence of solutions in those discourses.

Lautman's work does more than merely provide a descriptive account of an overarching dialectic of Ideas that subsumes the operations of all discourses. Rather, his work makes claim to a particular kind of dialectic of Ideas characteristic of mathematical problems that can be understood to operate in relation to other discourses by virtue of the way that it operates in the mathematical real. The detail of the structure of the dialectic can only be offered in terms of mathematical problems in the mathematical real; however, this structure can be used to model the structure and mode of operation of problems in other discourses. So the structure of the dialectic of Ideas can indeed be found in other discourses, as Lautman states; however, it is only by virtue of the extent to which the structure of problems in other discourses can be determined to operate according to the model of problems in the mathematical real that this can be achieved.



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notes

1 The axiomatic method is a way of developing mathematical theories by postulating certain primitive assumptions, or axioms, as the basis of the theory, while the remaining propositions of the theory are obtained as logical consequences of these axioms.

2 The Bourbaki project explicitly espoused a set-theoretic version of mathematical structuralism.

3 According to mathematical structuralism, mathematical objects are defined by their positions in mathematical structures, and the subject matter that mathematics concerns itself with is structural relationships in abstraction from the intrinsic nature of the related objects. See Hellman 556.

4 The main aim of Hilbert's program, which was first clearly formulated in 1922, was to establish the logical acceptability of the principles and modes of inference of modern mathematics by formalising each mathematical theory into a finite, complete set of axioms, and to provide a proof that these axioms were consistent. The point of Hilbert's approach was to make mathematical theories fully precise, so that it is possible to obtain precise results about properties of the theory. In 1931 Gödel showed that the program as it stood was not possible. Revised efforts have since emerged as continuations of the program that concentrate on relative results in relation to specific mathematical theories, rather than all mathematics. See Ferreirós chapter 2.6.3.2.

5 See Largeault 215, 264.

6 The term "metamathematics" is introduced by Hilbert in "Uber das Unendliche."

7 See Brunschvicg.

8 A mathematical definition is impredicative if it depends on a certain set, **N**, being defined and introduced by appeal to a totality of sets which includes **N** itself. That is, the definition is self-referencing.

9 The law of the excluded middle states that every proposition is either true or false. In propositional logic, the law is written "$P \vee \neg P$" ("P or not-P").

10 See Petitot 81.

11 See ibid. 113.

12 See also Barot 6, 16 n. 1.

13 See Chevalley 61.

14 See also *Mathematics* 189–90, 40–42; Barot 7 n. 2.

15 See Chevalley 60.

16 Which are also referred to and operate as "dualities." See Alunni 78.

17 Which he therefore also refers to as "logical schemas." See *Mathematics* 83.

18 From Lautman's correspondence with Fréchet dated 1 Feb. 1939.

19 See Chevalley 50.

20 See *Mathematics* 211.

21 A cautionary word along the lines of Cavaillès' warning of "possible misunderstandings" of Lautman's references to Heidegger (from correspondence dated 7 Nov. 1938, cited in Granger 299): if Lautman were to be considered Heideggerian, or to be embarking on a project of fundamental ontology, because of the few places in his work where he makes brief allusion to specific conceptual distinctions in Heidegger's work that serve as analogies for his own undertakings in relation to mathematics, a more detailed reading of Lautman, which I hope to have provided here, should lead to revising such an understanding. It is in this vein that I briefly clarify Lautman's relation to Heidegger.

22 This is Lautman's gloss of Heidegger.

23 See Heidegger, *The Essence of Reasons* 160–61.

24 See Barot 10; Chevalley 63–64.

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