An Outline of Functorial Semantics

F. E. J. Linton [1]

This paper is devoted to the elucidation of a very general structure-semantics adjointness theorem (Theorem 4.1), out of which follow all other structure-semantics adjointness theorems currently known to the author. Its reduction, in Section 10, to the classical theorem in the context of triples requires a representation theorem (see Section 9) asserting that the categories of algebras, in a category $\mathscr{A}$, tentatively described in Section 1 (and used in [Linton (1969)] in the special case $\mathscr{A} = \mathscr{S}$), "coincide" with the categories of algebras over suitably related triples, if such exist.

Sections 7 and 8 pave the way for this representation theorem. A detailed outline of the contents of Sections 3–11 is sketched in Section 2. Portions of this paper fulfill the promises made in [Linton (1966)] and at the close of Section 6 of [Linton (1966a)].

1. Introduction to algebras in general categories.

Functorial semantics generalizes to arbitrary categories the classical notion [Birkhoff (1935)] of an abstract algebra. This notion is usually [Cohn (1965), Słomiński (1959)] defined, in terms of a set Ω of "operations", a set-valued "arity" function[2] n defined on Ω, and a collection E of "laws[3] governing the operations of Ω", as a system $(\mathscr{A}, \mathfrak{A})$ consisting of a set A so equipped with an Ω-indexed family $\mathfrak{A} = \{\mathfrak{A}(\vartheta) \mid \vartheta \in \Omega\}$ of $n(\vartheta)$-ary operations

$$\mathfrak{A}(\vartheta)\colon A^{n(\vartheta)} \longrightarrow A \qquad\qquad (\vartheta \in \Omega)$$

that the body of laws is upheld. An algebra homomorphism from $(A, \mathfrak{A})$ to $(B, \mathfrak{B})$ is then, of course, a function $g\colon A \longrightarrow B$ commuting with all the operations, i.e., rendering commutative all the diagrams

$$\begin{array}{ccc} A^{n(\vartheta)} & \xrightarrow{\;g^{n(\vartheta)}\;} & B^{n(\vartheta)} \\ {\scriptstyle \mathfrak{A}(\vartheta)}\downarrow & & \downarrow{\scriptstyle \mathfrak{B}(\vartheta)} \\ A & \xrightarrow[\;g\;]{} & B \end{array} \qquad\qquad (\vartheta \in \Omega)$$

Among the algebras of greatest interest in functorial semantics are those arising by a very similar procedure from a functor

$$U\colon \mathscr{X} \longrightarrow \mathscr{A}$$

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[2]Its values are often constrained to be ordinals, or cardinals, or positive integers.

[3]E.g., associativity laws, unit laws, commutativity laws, Jacobi identities, idempotence laws, etc.

To spotlight the analogy, we first introduce some suggestive notation and terminology, writing $\mathscr{S}$ for a category of sets (in the sense either of universes [Sonner (1962)] or of Lawvere's axiomatic foundations [Lawvere (1964), Lawvere (1966)]) in which $\mathscr{A}$'s hom functor takes values.

Given the functor $U\colon \mathscr{X} \longrightarrow \mathscr{A}$, we define, for each $\mathscr{A}$-morphism $f\colon k \longrightarrow n$, a natural transformation

$$U^f\colon U^n \longrightarrow U^k$$

from the functor

$$U^n = \mathscr{A}(n, U(-))\colon \mathscr{X} \longrightarrow \mathscr{S}$$

to the functor

$$U^k = \mathscr{A}(k, U(-))\colon \mathscr{X} \longrightarrow \mathscr{S}$$

by posing

$$(U^f)_X = \mathscr{A}(f, UX)\colon U^nX \longrightarrow U^kX$$

($X \in |\mathscr{X}|$). Moreover, whenever, A, B, n, k are objects in $\mathscr{A}$ and $f\colon k \longrightarrow n$, $g\colon A \longrightarrow B$ are $\mathscr{A}$-morphisms, we set

$$\begin{aligned} A^n &= \mathscr{A}(n, A) \in |\mathscr{S}|, \\ A^f &= \mathscr{A}(f, A)\colon A^n \longrightarrow A^k, \text{ and} \\ g^n &= \mathscr{A}(n, g)\colon A^n \longrightarrow B^n. \end{aligned}$$

The class

$$\text{n. t.}(U^n, U^k) \quad (\text{resp. } \mathscr{S}(A^n, A^k))$$

is to be thought of as consisting of all *natural k-tuples of* (or all *k-tuple-valued*) *n-ary operations on U* (resp. *on A*).

A *U-algebra* is then defined to be a system $(A, \mathfrak{A})$ consisting of an object $A \in |\mathscr{A}|$ and a family

$$\mathfrak{A} = \{\mathfrak{A}_{n,k} \mid n \in |\mathscr{A}|, k \in |\mathscr{A}|\}$$

of functions

$$\mathfrak{A}_{n,k}\colon \text{ n. t.}(U^n, U^k) \longrightarrow \mathscr{S}(A^n, A^k)$$

satisfying the identities

$$\mathfrak{A}_{n,k}(U^f) = A^f \qquad (f \in \mathscr{A}(k, n)) \tag{1.1}$$

$$\mathfrak{A}_{n,m}(\vartheta' \circ \vartheta) = \mathfrak{A}_{n,m}(\vartheta') \circ \mathfrak{A}_{n,m}(\vartheta) \qquad (\vartheta\colon U^n \longrightarrow U^k, \vartheta'\colon U^k \longrightarrow U^m) \tag{1.2}$$

Writing (compare [Eilenberg & Wright (1967)][a])

$$\{\mathfrak{A}_{n,k}(\vartheta)\}(a) = \vartheta \star a \ (= \vartheta \star_{\mathfrak{A}} a, \text{ when one must remember } \mathfrak{A}) \tag{1.3}$$

whenever $\vartheta\colon U^n \longrightarrow U^k$ and $a \in A^n$, the identities (1.1) and (1.2) become

$$U^f \star a = a \circ f \qquad (f \in \mathscr{A}(k,n),\ a \in A^n) \tag{ALG 1}$$

$$(\vartheta' \circ \vartheta) \circ a = \vartheta' \star (\vartheta \star a) \qquad (\vartheta\colon U^n \longrightarrow U^k,\ \vartheta'\colon U^k \longrightarrow U^m,\ a \in A^n) \tag{ALG 2}$$

As U-algebra homomorphisms from $(A, \mathfrak{A})$ to $(B, \mathfrak{B})$ we admit all $\mathscr{A}$-morphisms $g\colon A \longrightarrow B$ making the diagrams

(1.4)

$$\begin{array}{ccc} \boldsymbol{A^n} & \xrightarrow{g^n} & \boldsymbol{B^n} \\ {\scriptstyle \mathfrak{A}_{n,k}(\vartheta)}\downarrow & & \downarrow{\scriptstyle \mathfrak{B}_{n,k}(\vartheta)} \\ \boldsymbol{A^k} & \xrightarrow[g^k]{} & \boldsymbol{B^k} \end{array}$$

commute, for each natural operation ϑ on U; in the notation of (1.3), this boils down to the requirement

$$g \circ (\vartheta \star a) = \vartheta \star (g \circ a) \qquad (\vartheta\colon U^n \longrightarrow U^k, a \in A^n) \tag{ALG 3}$$

We write U-**Alg** for the resulting category of U-algebras. The prime examples of U-algebras are the U-algebras $\Phi_U(X)$, available for each object $X \in |\mathscr{X}|$, given by the data

$$\Phi_U(X) = (UX, \mathfrak{A}_U(X)), \tag{1.5}$$

where, in the notation of (1.3), $\mathfrak{A}_U(X)$ is specified by

$$\vartheta \star a = \vartheta_X(a).$$

It is a trivial consequence of the defining property of a natural transformation that each $\mathscr{A}$-morphism $U(\xi)$ $(\xi\colon X \longrightarrow X')$ is a U-algebra homomorphism

$$U(\xi)\colon \Phi_U(X) \longrightarrow \Phi_U(X')$$

and that these passages provide a functor

$$\varphi_U\colon \mathscr{X} \longrightarrow U\text{-}\mathbf{Alg}\colon \begin{cases} X \mapsto \Phi_U(X), \\ \xi \mapsto U(\xi) \end{cases}$$

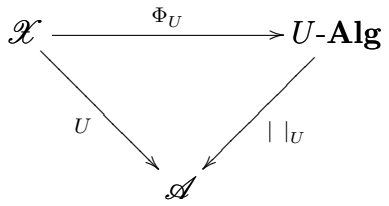
[a] Editor's footnote: Although this paper has a different title from the original reference, this is the only paper by Eilenberg and Wright found in MathSciNet.

called the *semantical comparison functor for* U.

We must not ignore the *underlying* $\mathscr{A}$*-object functor*

$$|\ |_U\colon U\text{-}\mathbf{Alg}\longrightarrow\mathscr{A}\colon\begin{cases}(A,\mathfrak{A})\mapsto A,\\ g\mapsto g.\end{cases}$$

For one thing, the triangle



commutes. For another, awareness of $|\ |_U$ is the first prerequisite for a much more concise description of U-**Alg** as a certain pullback. The only other prerequisite for this is the recognition that the system $\mathfrak{A}$ in a U-algebra $(A,\mathfrak{A})$ is nothing but the effects on morphisms of a certain set-valued functor, again denoted by $\mathfrak{A}$, defined on the following category $\mathfrak{T}_U$, the (*full*) *clone of operations on* U: the objects and maps of $\mathfrak{T}_U$ are given by

$$|\mathfrak{T}_U|=|\mathscr{A}|,$$
$$\mathfrak{T}_U(n,k)=\text{n. t.}(U^n,U^k);$$

the composition in $\mathfrak{T}_U$ is the usual composition of natural transformations. We point out the functor

$$\exp_U\colon\mathscr{A}^\star\longrightarrow\mathfrak{T}_U\colon\begin{cases}n\mapsto n,\\ f\mapsto U^f,\end{cases}$$

and remark that the functions $\mathfrak{A}_{n,k}$ are obviously the effects on morphisms of a functor (necessarily unique)

$$\mathfrak{A}\colon\mathfrak{T}_U\longrightarrow\mathscr{S}$$

whose effect on objects is simply

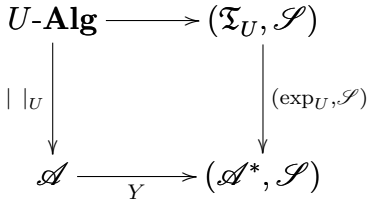
$$\mathfrak{A}(n)=A^n.$$

(Proof: (1.1) and (1.2).) Likewise, given a U-algebra homomorphism $g\colon(A,\mathfrak{A})\longrightarrow(B,\mathfrak{B})$, the commutativity of (1.4) makes the system $\{g^n\colon A^n\longrightarrow B^n\mid n\in|\mathscr{A}|\}$ a natural transformation from $\mathfrak{A}$ to $\mathfrak{B}$. In this way, the functor

$$U\text{-}\mathbf{Alg}\longrightarrow(\mathfrak{T}_U,\mathscr{S})$$

making the square

(1.6)



commute; here functor categories and induced functors between them are denoted by parentheses, and Y is the Yoneda embedding $A \longrightarrow \mathscr{A}(-, A)$. In Section 5 we shall see (it can be proved right away, with virtually no effort)

OBSERVATION 1.1. *Diagram* (1.6) *is a pullback diagram.*

With this introduction to algebras in general categories behind us, we turn to a description of what lies ahead.

2. General plan of the paper.

Motivated both by Observation 1.1 and the desire to recapture the structure-semantics adjointness of [Lawvere (1963)], we spend the next two sections, with a fixed functor

$$j\colon \mathscr{A}_0 \longrightarrow \mathscr{A},$$

studying the passage from

$$V\colon \mathscr{A}_0^{\star} \longrightarrow \mathscr{C}$$

to the $\mathscr{A}$-valued functor $\mathfrak{M}^{(j)}(V)$ defined on the pullback of the pullback diagram

$$\begin{array}{ccccc} \text{pullback} & \longrightarrow & & & (\mathscr{C}, \mathscr{S}) \\ {\scriptstyle \mathfrak{M}^{(j)}(V)}\downarrow & & & & \downarrow{\scriptstyle (V,\mathscr{S})} \\ \mathscr{A} & \xrightarrow[Y]{} & (\mathscr{A}^{*}, \mathscr{S}) & \xrightarrow[(j^{*},\mathscr{S})]{} & (\mathscr{A}_0^{*}, \mathscr{S}) \end{array}$$

the passage from

$$U\colon \mathscr{X} \longrightarrow \mathscr{A}$$

to the composition

$$\mathfrak{S}^{(j)}(U)\colon \mathscr{A}_0^{\star} \xrightarrow[j^{\star}]{} \mathscr{A}^{\star} \xrightarrow[Y]{} (\mathscr{A}, \mathscr{S}) \xrightarrow[(U,\mathscr{S})]{} (\mathscr{X}, \mathscr{S}),$$

the adjointness relation between $\mathfrak{M}^{(j)}$ and $\mathfrak{S}^{(j)}$, and the modification of this adjointness that results from consideration of the full image cotriple on the (comma) category $(\mathscr{A}_0^{\star}, \mathbf{Cat})$. In Section 5 we present some remarks on the constructions of Sections 3-4, including a proof of Observation 1.1 and an indication of the manner in which the structure-semantics adjointnesses of [Bénabou (1966), Lawvere (1963), Linton (1966a)] are recaptured by specializing the functor j.

The next three sections digress from the main line of thought, to present tangential results, without which, however, the main line of thought cannot easily continue. In

Section 6, the least important of these digressions, we present two completeness properties of the categories of algebras arising in Section 4 (slightly less satisfying results along the same line can be achieved also from those arising in Section 3—we forego them here). The material of Sections 7-8 is necessitated by the frequent possibility of associating a triple [Beck (1967)] $\mathbf{T} = (T, \eta, \mu)$ to an $\mathscr{A}$-valued functor $U\colon \mathscr{X} \longrightarrow \mathscr{A}$, in the manner of [Appelgate (1965), Kock (1966), Tierney (1969)]. This can be done, for example, when U has a left adjoint $F\colon \mathscr{A} \longrightarrow \mathscr{X}$, with front and back adjunctions $\eta\colon \mathrm{id}_{\mathscr{A}} \longrightarrow UF$, $\beta\colon FU \longrightarrow \mathrm{id}_{\mathscr{X}}$, by setting

$$T = UF, \eta = \eta, \mu = U\beta F.$$

It will be seen in Section 9 that if $\mathbf{T}$ is a triple suitably associated with $U\colon \mathscr{X} \longrightarrow \mathscr{A}$, the category of U-algebras and the category $\mathscr{A}^{\mathbf{T}}$ (constructed in [Eilenberg & Moore (1965a), Th. 2.2], for example) of $\mathbf{T}$-algebras are canonically isomorphic. To this end, Section 7 reviews the definition of triples, of the categories $\mathscr{A}^{\mathbf{T}}$, and of the construction [Kleisli (1965)] of the Kleisli category of a triple, while Section 8 is devoted to a full elucidation of the manner in which a triple $\mathscr{A}$ can be associated to an $\mathscr{A}$-valued functor.

The above mentioned isomorphism theorem in Section 9 is proved there in two ways: once by appeal to a general criterion, which depends on a result of Section 6 and on the availability of a left adjoint to $|\ |_U\colon U\text{-}\mathbf{Alg} \longrightarrow \mathscr{A}$, and once (sketchily) by a somewhat more involved argument that constructs the isomorphism explicitly, still using, of course, the left adjoint just mentioned.

In section 10, the result of Section 9 is used to recover the structure-semantics adjointness for the context of triples from that of Section 4. Finally, in Section 11, we give a proof of the isomorphism theorem of Section 9 that is entirely elementary—in particular, that is quite independent of the knowledge that $|\ |_U$ has a left adjoint, and from which that fact follows. The exposition of this last section is so arranged that it can be read immediately after Section 1, without bothering about Sections 3-10.

3. Preliminary structure-semantics adjointness relation.

The granddaddy of all the structure-semantics adjointness theorems is the humble canonical isomorphism

$$(\mathscr{X}, (\mathscr{C}, \mathscr{S})) \cong (\mathscr{C}, (\mathscr{X}, \mathscr{S}))$$

expressing the symmetry [Eilenberg & Kelly (1966)] of the closed category **Cat** of categories. Here we are using $(\mathscr{X}, \mathscr{Y})$ to denote the category of all functors from $\mathscr{X}$ to $\mathscr{Y}$, with natural transformations as morphisms.

Until further notice, fix a functor $j\colon \mathscr{A}_0 \longrightarrow \mathscr{A}$.

The first prototype of structure and semantics (rel. j)) will be functors passing from the category $(\mathbf{Cat}, \mathscr{A})$ of $\mathscr{A}$-valued functors $U\colon \mathscr{X} \longrightarrow \mathscr{A}$, with domain $\mathscr{X} \in |\mathbf{Cat}|$, to the category $(\mathscr{A}_0^\star, \mathbf{Cat})$ of all functors $V\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}$ with codomain $\mathscr{C} \in |\mathbf{Cat}|$, and back again, as outlined in Section 2. Of course, we think of $(\mathbf{Cat}, \mathscr{A})$ and $(\mathscr{A}_0^\star, \mathbf{Cat})$ as

comma categories [Lawvere (1963)], so that the $(\mathbf{Cat}, \mathscr{A})$-morphisms from $U'\colon \mathscr{X}' \longrightarrow \mathscr{A}$ to $U\colon \mathscr{X} \longrightarrow \mathscr{A}$ are those functors $x\colon \mathscr{X}' \longrightarrow \mathscr{X}$ satisfying $U' = U \circ x$, while $(\mathscr{A}_0^\star, \mathbf{Cat})$-morphisms from $V\colon \mathscr{A}_0' \longrightarrow \mathscr{C}$ to $V'\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}'$ are those functors $c\colon \mathscr{C} \longrightarrow \mathscr{C}'$ satisfying $V' = c \circ V$.

The proof of the basic lemma below is so completely elementary that it will be omitted. To find it, just follow your nose.

LEMMA 3.1. *For each pair of functors $U\colon \mathscr{X} \longrightarrow \mathscr{A}$, $V\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}$, the canonical isomorphism*

$$(\mathscr{X}, (\mathscr{C}, \mathscr{S})) \cong (\mathscr{C}, (\mathscr{X}, \mathscr{S}))$$

(where $\mathscr{S}$ is a category of sets reeciving $\mathscr{A}$'s hom functor) mediates an isomorphism

$$M(j; U, V) \cong S(j; U, V) \tag{3.1}$$

between the full subcategory $M(j; U, V) \subset (\mathscr{X}, (\mathscr{C}, \mathscr{S}))$ whose objects are those functors $F\colon \mathscr{X} \longrightarrow (\mathscr{C}, \mathscr{S})$ making the diagram

(3.2)
$$\begin{array}{ccccc} \mathscr{X} & & \xrightarrow{\quad F \quad} & & (\mathscr{C}, \mathscr{S}) \\ {\scriptstyle U}\downarrow & & & & \downarrow{\scriptstyle (V, \mathscr{S})} \\ \mathscr{A} & \xrightarrow[Y]{} & (\mathscr{A}^*, \mathscr{S}) & \xrightarrow[(j^*, \mathscr{S})]{} & (\mathscr{A}_0^*, \mathscr{S}) \end{array}$$

commute, and the full subcategory $S(j; U, V) \subset (\mathscr{C}, (\mathscr{X}, \mathscr{S}))$ whose objects are those functors $G\colon \mathscr{C} \longrightarrow (\mathscr{X}, \mathscr{S})$ making the diagram

(3.3)
$$\begin{array}{ccccc} \mathscr{C} & & \xrightarrow{\quad G \quad} & & (\mathscr{X}, \mathscr{S}) \\ {\scriptstyle V}\uparrow & & & & \uparrow{\scriptstyle (U, \mathscr{S})} \\ \mathscr{A}_0^* & \xrightarrow[j^*]{} & \mathscr{A}^* & \xrightarrow[Y]{} & (\mathscr{A}, \mathscr{S}) \end{array}$$

commute. Moreover, the isomorphisms (3.1) *are natural in the variables $U \in (\mathbf{Cat}, \mathscr{A})$ and $V \in (\mathscr{A}_0^\star, \mathbf{Cat})$.*

The crudest structure and semantics functors (rel. j), to be denoted $\mathfrak{S}^{(j)}$ and $\mathfrak{M}^{(j)}$, respectively, are defined as follows.

Given $U\colon \mathscr{X} \longrightarrow \mathscr{A}$ in $(\mathbf{Cat}, \mathscr{A})$, $\mathfrak{S}^{(j)}(U)$ is the composition

$$\mathfrak{S}^{(j)}(U) = (U, \mathscr{S}) \circ Y \circ j^\star\colon \mathscr{A}_0^\star \longrightarrow \mathscr{A}^\star \longrightarrow (\mathscr{A}, \mathscr{S}) \longrightarrow (\mathscr{X}, \mathscr{S}).$$

It is clear that $(x, \mathscr{S})\colon (\mathscr{X}, \mathscr{S}) \longrightarrow (\mathscr{X}', \mathscr{S})$ is an $(\mathscr{A}_0^\star, \mathbf{Cat})$-morphism $\mathfrak{S}^{(j)}(U) \longrightarrow \mathfrak{S}^{(j)}(U')$ whenever $x\colon \mathscr{X}' \longrightarrow \mathscr{X}$ is a $(\mathbf{Cat}, \mathscr{A})$-morphism from $U'\colon \mathscr{X}' \longrightarrow \mathscr{A}$ to $U\colon \mathscr{X} \longrightarrow \mathscr{A}$.

In the other direction, given $V\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}$ in $(\mathscr{A}_0^\star, \mathbf{Cat})$, define $\mathfrak{M}^{(j)}(V)$ to be the $\mathscr{A}$-valued functor from the pullback $\mathscr{P}_V^j$ in the pullback diagram

$$\begin{array}{ccccc} \mathscr{P}_V^j & & \longrightarrow & & (\mathscr{C}, \mathscr{S}) \\ {\scriptstyle \mathfrak{M}^{(j)}(V)}\downarrow & & & & \downarrow{\scriptstyle (V,\mathscr{S})} \\ \mathscr{A} & \xrightarrow[Y]{} & (\mathscr{A}^*, \mathscr{S}) & \xrightarrow[(j^*,\mathscr{S})]{} & (\mathscr{A}_0^*, \mathscr{S}) \end{array} \tag{3.4}$$

It is clear, whenever $c\colon \mathscr{C} \longrightarrow \mathscr{C}'$ is an $(\mathscr{A}_0^\star, \mathbf{Cat})$-morphism from $V\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}$ to $V'\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}'$, that

$$(c, \mathscr{S})\colon (\mathscr{C}', \mathscr{S}) \longrightarrow (\mathscr{C}, \mathscr{S})$$

induces a functor $\mathscr{P}_{V'}^j \longrightarrow \mathscr{P}_V^j$ between the pullbacks that is actually a $(\mathbf{Cat}, \mathscr{A})$-morphism $\mathfrak{M}^{(j)}(V') \longrightarrow \mathfrak{M}^{(j)}(V)$.

With these observations, it is virtually automatic that $\mathfrak{S}^{(j)}$ and $\mathfrak{M}^{(j)}$ are functors

$$\begin{aligned} \mathfrak{S}^{(j)}&\colon (\mathbf{Cat}, \mathscr{A}) \longrightarrow (\mathscr{A}_0^\star, \mathbf{Cat})^\star, \\ \mathfrak{M}^{(j)}&\colon (\mathscr{A}_0^\star, \mathbf{Cat})^\star \longrightarrow (\mathbf{Cat}, \mathscr{A}). \end{aligned}$$

THEOREM 3.1. (Preliminary structure-semantics adjointness.) *The functor* $\mathfrak{S}^{(j)}$ *is* (*right*) *adjoint to* $\mathfrak{M}^{(j)}$.

PROOF. By the definition of pullbacks, a functor from $\mathscr{S}$ to $\mathscr{P}_V^j$ "is"
a pair of functors from $\mathscr{X}$ making the diagram

$$\begin{array}{ccccc} \mathscr{X} & & \longrightarrow & & (\mathscr{C}, \mathscr{S}) \\ \downarrow & & & & \downarrow \\ \mathscr{A} & \xrightarrow[Y]{} & (\mathscr{A}^*, \mathscr{S}) & \xrightarrow[(j^*,\mathscr{S})]{} & (\mathscr{A}_0^*, \mathscr{S}) \end{array}$$

commute. Hence a morphism from $U\colon \mathscr{X} \longrightarrow \mathscr{A}$ to $\mathfrak{M}^{(j)}(V)$ "is" a functor $F\colon \mathscr{X} \longrightarrow (\mathscr{C}, \mathscr{S})$ making diagram (3.2) commute, i.e., "is" an object of $M(j; U, V)$, as defined in Lemma 3.1.

It is even easier to see that the $(\mathscr{A}_0^\star, \mathbf{Cat})^\star$-morphisms from $\mathfrak{S}^{(j)}(U)$ to V coincide with the objects of the category $S(j; U, V)$ of Lemma 3.1. Consequently, the desired natural equivalence

$$\begin{aligned} (\mathbf{Cat}, \mathscr{A})(U, \mathfrak{M}^{(j)}(V)) &\cong (\mathscr{A}_0^\star, \mathbf{Cat})^\star(\mathfrak{S}^{(j)}(U), V) \\ &(\cong (\mathscr{A}_0^\star, \mathbf{Cat})(V, \mathfrak{S}^{(j)}(U))) \end{aligned}$$

is delivered by the isomorphism (3.1) of Lemma 3.1. ∎

REMARK. In fact, $(\mathbf{Cat}, \mathscr{A})$ and $(\mathscr{A}_0^\star, \mathbf{Cat})$ are hypercategories [Eilenberg & Kelly (1966)], both $\mathfrak{S}^{(j)}$ and $\mathfrak{M}^{(j)}$ are hyperfunctors, and the adjointness relation is a hyperadjointness. The same remark will apply to the adjointness of Theorem 4.1; however, we know of no use for the stronger information.

4. Full images and the operational structure-semantics adjointness theorem.

It is time to introduce the full image cotriple in $(\mathscr{A}_0^\star, \mathbf{Cat})$. We recall that the *full image* of a functor $V\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}$ is the category $\mathscr{T}_V$ whose objects and maps are given by the formulas

$$|\mathscr{T}_V| = |\mathscr{A}_0|$$
$$\mathscr{T}_V(n,k) = \mathscr{C}(Vn, Vk),$$

and whose composition rule is that of $\mathscr{C}$. Then V admits a factorization

$$V = \underline{V} \circ \overline{V}\colon \mathscr{A}_0^\star \longrightarrow \mathscr{T}_V \longrightarrow \mathscr{C}$$

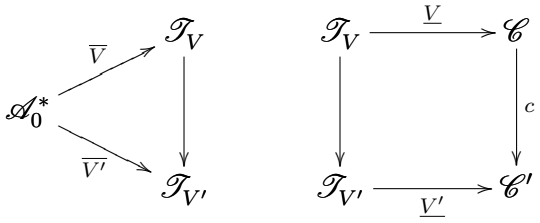
where $\overline{V}$ and $\underline{V}$ are functors

$$\overline{V}\colon \mathscr{A}_0^\star \longrightarrow \mathscr{T}_V\colon \begin{cases} n \mapsto n, \\ f \mapsto Vf, \end{cases}$$
$$\underline{V}\colon \mathscr{T}_V \longrightarrow \mathscr{C}\colon \begin{cases} n \mapsto Vn, \\ g \mapsto g. \end{cases}$$

Moreover, if $c\colon \mathscr{C} \longrightarrow \mathscr{C}'$ is an $(\mathscr{A}_0^\star, \mathbf{Cat})$-morphism from V to V' (i.e., if $c \circ V = V'$), then

$$\mathscr{T}_c\colon \mathscr{T}_V \longrightarrow \mathscr{T}_{V'}\colon \begin{cases} n \mapsto n \\ g \mapsto cg \end{cases}$$

(is the only functor that) makes the diagrams



commute. Thus $V \mapsto \overline{V}$, $c \mapsto \mathscr{T}_c$ is an endofunctor on $(\mathscr{A}_0^\star, \mathbf{Cat})$, and the maps $\underline{V}\colon \overline{V} \longrightarrow V$ are $(\mathscr{A}_0^\star, \mathbf{Cat})$-natural in V. Since clearly $\overline{\overline{V}} = \overline{V}$, we are in the presence of an idempotent cotriple on $(\mathscr{A}_0^\star, \mathbf{Cat})$.

We use this cotriple first to define a *clone over* $\mathscr{A}_0$ as a functor $V \in |(\mathscr{A}_0^\star, \mathbf{Cat})|$ for which $\overline{V} = V$—the full subcategory of $(\mathscr{A}_0^\star, \mathbf{Cat})$ consisting of clones will be denoted $\mathbf{Cl}(\mathscr{A}_0)$. Next, the formulas

$$\begin{aligned}\mathfrak{S}^j(U) &= \overline{\mathfrak{S}^{(j)}(U)} \colon \mathscr{A}_0^\star \longrightarrow \mathscr{T}_{\mathfrak{S}^{(j)}(U)},\\ \mathfrak{S}^j(x) &= \mathscr{T}_{\mathfrak{S}^{(j)}(x)},\\ \mathfrak{M}^j &= \mathfrak{M}^{(j)}|_{\mathbf{Cl}(\mathscr{A}_0)},\end{aligned}$$

serve to define functors

$$\begin{aligned}\mathfrak{S}^j &\colon (\mathbf{Cat}, \mathscr{A}) \longrightarrow (\mathbf{Cl}(\mathscr{A}_0))^\star,\\ \mathfrak{M}^j &\colon (\mathbf{Cl}(\mathscr{A}_0))^\star \longrightarrow (\mathbf{Cat}, \mathscr{A}),\end{aligned}$$

called operational structure and operation semantics (rel. j), respectively; they will be said to assign an $\mathscr{A}$-valued functor U (resp., a clone over $\mathscr{A}_0$) its structure clone (resp., its category of algebras in $\mathscr{A}$) (rel. j).

Given $V = \overline{V} \in |\mathbf{Cl}(\mathscr{A}_0)|$ and $U \in |(\mathbf{Cat}, \mathscr{A})|$, Theorem 3.1 and the idempotence of the full image cotriple immediately yield

$$\begin{aligned}(\mathbf{Cat}, \mathscr{A})(U, \mathfrak{M}^j(V)) &\cong (\mathscr{A}_0^\star, \mathbf{Cat})(V, \mathfrak{S}^{(j)}(U)) = (\mathscr{A}_0^\star, \mathbf{Cat})(\overline{V}, \mathfrak{S}^{(j)}(U))\\ &\cong \mathbf{Cl}(\mathscr{A}_0)(\overline{V}, \overline{\mathfrak{S}^{(j)}(U)}) = \mathbf{Cl}(\mathscr{A}_0)(V, \mathfrak{S}^j(U)),\end{aligned}$$

identifications whose obvious naturality in U and V completes the proof of

THEOREM 4.1. (Operational structure semantics adjointness.) *Operational structure (rel. j), $\mathfrak{S}^j$, is (right) adjoint to operational semantics (rel. j), $\mathfrak{M}^j$.*

5. Remarks on Section 4.

The first two remarks establish a generalization of Observation 1.1 to the (rel. j) case. They involve a fixed clone $V \colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}$ and a fixed functor $j \colon \mathscr{A}_0 \longrightarrow \mathscr{A}$.

1. A one-one correspondence is set up between V-algebras in $\mathscr{A}$ (rel. j), i.e., objects $(A, \mathfrak{A})$ of the pullback $\mathscr{P}_V^j$, and systems $(A, \star)$ consisting of

i) an object A of $\mathscr{A}$,

ii) pairing $(\vartheta, a) \mapsto \vartheta \star a \colon \mathscr{C}(n,k) \times A^{j(n)} \longrightarrow A^{j(k)}$

satisfying the identities

$$(\vartheta' \circ \vartheta) \star a = \vartheta' \star (\vartheta \star a) \qquad (\vartheta, \vartheta' \ \mathscr{C}\text{-morphisms}), \tag{5.1}$$

$$V(f) \star a = a \circ j(f) \qquad (f \text{ an } \mathscr{A}_0\text{-morphism}), \tag{5.2}$$

by the equations

$$\{\mathfrak{A}_{n,k}(\vartheta)\}(a) = \vartheta \star a, \tag{5.3}$$

$$\mathfrak{A}(n) = \mathscr{A}(jn, A) = A^{j(n)}. \tag{5.4}$$

PROOF. If $(A, \mathfrak{A}) \in |\mathscr{P}_V^j|$, formula (5.3) gives rise to a system ii) of pairings. That identities (5.1) and (5.2) are valid for the resulting $(A, \star)$ is a consequence of functoriality of $\mathfrak{A}$ and the relation

$$\mathfrak{A} \circ V = \mathscr{A}(j(-), A). \tag{5.5}$$

Conversely, if $(A, \star)$ is a system i), ii) satisfying (5.1) and (5.2), the attempt to define a functor $\mathfrak{A}$ satisfying (5.5) by means of (5.3) and (5.4) is successful precisely because of ii), (5.1) and (5.2), while (5.5) guarantees that $(A, \mathfrak{A})$ is in $\mathscr{P}_V^j$. The biunivocity of these correspondences is clear. ∎

2. With the functor $j\colon \mathscr{A}_0 \longrightarrow \mathscr{A}$ and the clone $V\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}$ still fixed, let $(A, \mathfrak{A})$ and $(B, \mathfrak{B})$ be objects of $\mathscr{P}_V^j$. Then, given $g \in \mathscr{A}(A, B)$, there is never more than one natural transformation $\varphi\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ with

$$(g, \varphi) \in \mathscr{P}_V^j((A, \mathfrak{A}), (B, \mathfrak{B})), \tag{5.6}$$

and there is one if and only if, in the notation of (5.3),

$$g \circ (\vartheta \star a) = \vartheta \star (ga) \tag{5.7}$$

for all $a \in A^{jn}$, all $\vartheta \in \mathscr{C}(n, k)$, and all $n, k \in |\mathscr{A}_0|$. Conversely, given the natural transformation $\varphi\colon \mathfrak{A} \longrightarrow \mathfrak{B}$, there is a $g \in \mathscr{A}(A, B)$ satisfying (5.6) if the composition

$$\mathscr{A} \xrightarrow[Y]{} (\mathscr{A}^\star, \mathscr{S}) \xrightarrow[(j^\star, \mathscr{S})]{} (\mathscr{A}_0^\star, \mathscr{S}) \tag{5.8}$$

is full, and there is at most one g if (5.8) is faithful. Hence if j is *dense* (this means that (5.8) is full and faithful cf. [Lawvere (1963)] or [Ulmer (1968a)]—[Isbell (1960)] uses the term *adequate*), the functor

$$\mathscr{P}_V^j \longrightarrow (\mathscr{C}, \mathscr{S})\colon \begin{cases} (A, \mathfrak{A}) \mapsto \mathfrak{A}, \\ (g, \varphi) \mapsto \varphi, \end{cases} \tag{5.9}$$

arising in the pullback diagram (3.4) is full and faithful (indeed, the density of j is a necessary and sufficient condition for (5.9) to be full and faithful for *every* clone V on $\mathscr{A}_0$).

PROOF. Given $g\colon A \longrightarrow B$, the requirement that (5.6) hold forces the components of φ to be

$$\varphi_n = \mathscr{A}(j(n), g)\colon A^{j(n)} \longrightarrow B^{j(n)},$$

and that takes care of uniqueness. That this system $\{\varphi_n\}_{n \in |\mathscr{A}_0|}$ is a natural transformation $\mathfrak{A} \longrightarrow \mathfrak{B}$ iff g satisfies the identities (5.7) is elementary definition juggling. The converse assertions are evident; the next assertion follows from them; and the statement in parentheses is seen to be true by taking $V = \mathrm{id}\colon \mathscr{A}_0^\star \longrightarrow \mathscr{A}_0^\star$ when j is not dense. ∎

The next remark points out some dense functors $j\colon \mathscr{A}_0 \longrightarrow \mathscr{A}$.

3. For any category $\mathscr{A}$, $\mathrm{id}_{\mathscr{A}}\colon \mathscr{A} \longrightarrow \mathscr{A}$ is dense (this is just part of the Yoneda Lemma). Moreover, if I is any set and $\mathscr{S}_{\aleph}$ is the full subcategory of the category $\mathscr{S}$ of sets and functions consisting of the cardinals (or sets of cardinality) $< \aleph$ ($\aleph \geq 2$), then the inclusion $\mathscr{S}_{\aleph} \longrightarrow \mathscr{S}$ and the induced inclusion $(\mathscr{S}_{\aleph})^I \longrightarrow \mathscr{S}^I$ are both dense.

The following remarks interpret the results of Section 4 in the settings indicated in Remark 3, using Remarks 1 and 2 when necessary.

4. When $j\colon \mathscr{A}_0 \longrightarrow \mathscr{A}$ is the inclusion in $\mathscr{S}$ of the full subcategory $\mathscr{S}_{\aleph_0}$ of finite cardinals, Theorem 4.1 is Lawvere's structure-semantics adjointness theorem [Lawvere (1963)].

5. If I is a set and $j\colon \mathscr{A}_0 \longrightarrow \mathscr{A}$ is the full inclusion $(\mathscr{S}_{\aleph})^I \longrightarrow \mathscr{S}^I$, Theorem 4.1 is Bénabou's structure-semantics adjointness theorem [Bénabou (1966)].

6. When $j = \mathrm{id}_{\mathscr{S}}$, Theorem 4.1 is the structure semantics adjointness theorem [Linton (1966a), Section 2]. 7. When $j = \mathrm{id}_{\mathscr{A}}$, then, for any $U\colon \mathscr{X} \longrightarrow \mathscr{A}$, $\mathfrak{S}^j(U) = \exp_U\colon \mathscr{A}^\star \longrightarrow \mathfrak{T}_U$, $\mathscr{P}^j_{\mathfrak{S}^j(U)} = U\text{-}\mathbf{Alg}$, $\mathfrak{M}^j\mathfrak{S}^j(U) = |\ |_U$, Observation 1.1 is the content of Remarks 1 and 2, and $\Phi_U\colon \mathscr{X} \longrightarrow U\text{-}\mathbf{Alg}$ is just the functor corresponding, under the adjointness of Theorem 4.1, to

$$\mathrm{id}_{\mathfrak{S}^j(U)} \in \mathbf{Cl}(A)(\mathfrak{S}^j(U), \mathfrak{S}^j(U)) \cong (\mathbf{Cat}, \mathscr{A})(U, \mathfrak{M}^j\mathfrak{S}^j(U)),$$

i.e., is the front adjunction for the operational structure-semantics adjointness.

8. When j is the inclusion $\mathscr{S}_{\aleph} \longrightarrow \mathscr{S}$, Theorem 4.1 is the adjointness implicit in the first paragraph of [Linton (1966a), Section 6].

6. Two constructions in algebras over a clone.

PROPOSITION 6.1. ($\mathfrak{M}^j(V)$ creates (inverse) limits.) *Let $V\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}$ be a clone over $\mathscr{A}_0$, and let $j\colon \mathscr{A}_0 \longrightarrow \mathscr{A}$ be a functor. Given a functor $X\colon \Delta \longrightarrow \mathscr{P}^j_V$, whose values at objects and morphisms of Δ are written $X_\delta = (A_\delta, \mathfrak{A}_\delta)$ and $X(i) = (g_i, \varphi_i)$, respectively, and given an object $A \in |\mathscr{A}|$ and $\mathscr{A}$-morphisms $p_\delta\colon A \longrightarrow A_\delta$ ($\delta \in |\Delta|$) making*

$$A = \varprojlim \mathfrak{M}^j(V) \circ X\colon \Delta \longrightarrow \mathscr{P}^j_V \longrightarrow \mathscr{A},$$

there are an object Q of $\mathscr{P}^j_V$ and maps $q_\delta\colon Q \longrightarrow X_\delta$ ($\delta \in |\Delta|$), uniquely determined by the requirements

$$\mathfrak{M}^j(V)(q_\delta) = p_\delta; \tag{6.1}$$

moreover, via the projections q_δ, $Q = \varprojlim X$.

PROOF. If $Q = (B, \mathfrak{B})$ and $q_\delta = (g_\delta, \varphi_\delta)$ satisfy (6.1), we must have $B = A$ and $g_\delta = p_\delta$. Remark 5.2 then identifies φ_δ. This it need only be seen that there is precisely one functor $\mathfrak{B}\colon \mathscr{C} \longrightarrow \mathscr{S}$ such that, in the notation of (5.3),

$$p_\delta \circ (\vartheta \star a) = \vartheta \star (p_\delta \circ a) \qquad (\vartheta \in \mathscr{C}(n,k), a \in A^{j(n)}). \tag{6.2}$$

But that's an obvious consequence of the limit property of A. That $(A, \mathfrak{B})$ is then an object of $\mathscr{P}_V^j$ is, again using the limit property of A and Remark 5.1, an automatic verification, and (6.2), using Remark 5.2, bespeaks the fact that p_δ "is" a $\mathscr{P}_V^j$-morphism from $(A, \mathfrak{B})$ to $(A_\delta, \mathfrak{A}_\delta)$. Finally, given a compatible system of $\mathscr{P}_V^j$-morphisms $(K, \mathfrak{K}) \longrightarrow (A_\delta, \mathfrak{A}_\delta)$, the $\mathscr{A}$-morphism component determine a unique $\mathscr{A}$-morphism $K \longrightarrow A$, which, using (6.2) and Remark 5.2, it is not hard to see "is" a $\mathscr{P}_V^j$-morphism $(K, \mathfrak{K}) \longrightarrow (A, \mathfrak{B})$. This completes the proof. ∎

PROPOSITION 6.2. $\mathfrak{M}^j(V)$ creates $\mathfrak{M}^j(V)$-split coequalizers.) *Let $V\colon \mathscr{A}_0^\star \longrightarrow \mathscr{C}$ be a clone over $\mathscr{A}_0$, let $j\colon \mathscr{A}_0 \longrightarrow \mathscr{A}$ be a functor, let $(A, \mathfrak{A})$ and $(B, \mathfrak{B})$ be two V-algebras, let $K \in |\mathscr{A}|$, and let*

$$(A, \mathfrak{A}) \underset{(g,\varphi)}{\overset{(f,\psi)}{\rightrightarrows}} (B, \mathfrak{B})$$

and

$$A \underset{d_1}{\longleftarrow} B \underset{d_0}{\overset{p}{\rightleftarrows}} K$$

be two $\mathscr{P}_V^j$-morphisms and three $\mathscr{A}$-morphisms satisfying

$$\begin{cases} pf = pg, \\ pd_0 = \mathrm{id}_K, \\ d_0 p = g d_1, \\ \mathrm{id}_B = f d_1. \end{cases} \tag{6.3}$$

Then there is a $\mathscr{P}_V^j$-morphism $(q, \rho)\colon (B, \mathfrak{B}) \longrightarrow (C, \mathfrak{K})$ uniquely determined by the requirement that $\mathfrak{M}^j(V)(q, \rho) = p$; moreover, (q, ρ) is then a coequalizer of the pair $((f, \psi), (g, \varphi))$.

PROOF. Clearly $C = K$ and $q = p$, so ρ will be forced; we must see there is a unique functor $\mathfrak{K}\colon \mathscr{C} \longrightarrow \mathscr{S}$ making $(K, \mathfrak{K})$ a V-algebra and p a $\mathscr{P}_V^j$-morphism. Now, since $p\colon B \longrightarrow K$ is a split epimorphism, each function $p^{j(n)}\colon B^{j(n)} \longrightarrow K^{j(n)}$ is onto. This fact ensures the uniqueness of any function $\mathfrak{K}(\vartheta)$ $(\vartheta \in \mathscr{C}(n, k))$ making the diagram

$$\begin{array}{ccc} B^{j(n)} & \xrightarrow{\;p^{j(n)}\;} & C^{j(n)} \\ {\scriptstyle \mathfrak{B}(\vartheta)}\downarrow & & \downarrow{\scriptstyle \mathfrak{K}(\vartheta)} \\ B^{j(k)} & \xrightarrow[\;p^{j(k)}\;]{} & C^{j(k)} \end{array}$$

commute. Their existence is ensured, using the section d_0, by the formula

$$\mathfrak{K}(\vartheta) = p^{j(k)} \circ \mathfrak{B}(\vartheta) \circ d_0^{j(n)} \qquad (\text{i.e., } \vartheta \star a = p \circ (\vartheta \star (d_0 \circ a))),$$

as the following calculations relying on (6.3) show:

$$\begin{aligned}\mathfrak{K}(\vartheta)\circ p^{j(n)} &= p^{j(k)}\circ\mathfrak{B}(\vartheta)\circ d_0^{j(n)}\circ p^{j(n)} = p^{j(k)}\circ\mathfrak{B}(\vartheta)\circ g^{j(n)}\circ d_1^{j(n)}\\ &= p^{j(k)}\circ g^{j(k)}\circ\mathfrak{A}(\vartheta)\circ d_1^{j(n)} = p^{j(k)}\circ f^{j(k)}\circ\mathfrak{A}(\vartheta)\circ d_1^{j(n)}\\ &= p^{j(k)}\circ\mathfrak{B}(\vartheta)\circ f^{j(n)}\circ d_1^{j(n)} = p^{j(k)}\circ\mathfrak{B}(\vartheta).\end{aligned}$$

That the resulting $(K,\mathfrak{K})$ is in $\mathscr{P}_V^j$ easy to see, using only the fact that each $p^{j(n)}$ is surjective. Finally, to see that $p\colon (B,\mathfrak{B})\longrightarrow(K,\mathfrak{K})$ is the coequalizer of (f,ψ) and (g,φ), note that the $\mathscr{A}$-morphism component of any $\mathscr{P}_V^j$-morphism from $(B,\mathfrak{B})$ having equal compositions with f and g factors uniquely through K (via its composition with d_0); but this factorization is a $\mathscr{P}_V^j$-morphism from $(K,\mathfrak{K})$ (in the sense of Remark 5.2) by virtue simply of the surjectivity of each $p^{j(n)}$. This completes the description of the proof. ∎

7. Constructions involving triples.

We recall [Eilenberg & Moore (1965a)] that a *triple* $\mathbf{T}$ on a category $\mathscr{A}$ is a system $\mathbf{T}=(T,\eta,\mu)$ consisting of a functor

$$T\colon \mathscr{A}\longrightarrow\mathscr{A}$$

and natural transformations

$$\eta\colon \mathrm{id}_{\mathscr{A}}\longrightarrow T,\quad \mu\colon TT\longrightarrow T$$

satisfying the relations

$$\mu\circ T\eta = \mathrm{id}_T, \tag{7.1}$$

$$\mu\circ\eta_T = \mathrm{id}_T, \tag{7.2}$$

$$\mu\circ\mu_T = \mu\circ T\mu. \tag{7.3}$$

It is often possible to associate a triple on $\mathscr{A}$ to an $\mathscr{A}$-valued functor $U\colon\mathscr{X}\longrightarrow\mathscr{A}$. For example, whenever U has a left adjoint $F\colon\mathscr{A}\longrightarrow\mathscr{X}$ with front and back adjunctions $\eta\colon\mathrm{id}_{\mathscr{A}}\longrightarrow UF$, $\beta\colon FU\longrightarrow\mathrm{id}_{\mathscr{X}}$, it is well known (cf. [Eilenberg & Moore (1965a), Prop. 2.1] or [Huber (1961), Th. 4.2*]) that

$$(UF,\eta,U\beta F) \tag{7.4}$$

is a triple on $\mathscr{A}$. More general situations in which a triple can be associated to U are discussed in Section 8. In any event, it will turn out (in Section 9) that, when $\mathbf{T}$ is a triple on $\mathscr{A}$ suitably associated to an $\mathscr{A}$-valued functor $U\colon\mathscr{X}\longrightarrow\mathscr{A}$, the category U-**Alg** of Section 1 is canonically isomorphic with the category $\mathscr{A}^{\mathbf{T}}$ (constructed in [Eilenberg & Moore (1965a), Th. 2.2], for example) of $\mathbf{T}$-algebras and $\mathbf{T}$-homomorphisms. For the

reader's convenience, the definition of $\mathscr{A}^{\mathbf{T}}$ will be reviewed. Since Kleisli's construction [Linton (1966)] of (what we shall call) the Kleisli category associated to a triple is needed in Section 8, enters into one proof of the isomorphism theorem of Section 9, and is relatively unfamiliar, we shall review it, here, too. Thereafter, we pave the way for Section 10 with some trivial observations.

Given the triple $\mathbf{T} = (T, \eta, \mu)$ on the category $\mathscr{A}$, a $\mathbf{T}$-algebra in $\mathscr{A}$ is a pair (A, α), where

$$\alpha\colon TA \longrightarrow A \tag{7.5}$$

is an $\mathscr{A}$-morphism satisfying the relations

$$\alpha \circ \eta_A = \mathrm{id}_A, \tag{7.6}$$
$$\alpha \circ \mu_A = \alpha \circ T\alpha. \tag{7.7}$$

For example, equations (7.2) and (7.3) bespeak the fact that

$$F^{\mathbf{T}}(A) = (TA, \mu_A)$$

is a $\mathbf{T}$-algebra, whatever $A \in |\mathscr{A}|$.

The category $\mathscr{A}^{\mathbf{T}}$ of $\mathbf{T}$-algebras has as objects all $\mathbf{T}$-algebras in $\mathscr{A}$ and as morphisms from (A, α) to (B, β) all $\mathscr{A}$-morphisms $g\colon A \longrightarrow B$ satisfying

$$g \circ \alpha = \beta \circ Tg; \tag{7.8}$$

the composition rule is that induced by composition of $\mathscr{A}$-morphisms. It follows that the passages

$$(A, \alpha) \mapsto A,$$
$$g \mapsto g$$

define a functor $U^{\mathbf{T}}\colon \mathscr{A}^{\mathbf{T}} \longrightarrow \mathscr{A}$, the underlying $\mathscr{A}$-object functor for $\mathbf{T}$-algebras.

On the other hand, it is easy to see that $Tf\colon TA \longrightarrow TB$ is an $\mathscr{A}^{\mathbf{T}}$-morphism from $F^{\mathbf{T}}(A)$ to $F^{\mathbf{T}}(B)$ ($f \in \mathscr{A}(A, B)$), and it readily follows that the passages

$$A \mapsto F^{\mathbf{T}}(A) = (TA, \mu_A),$$
$$f \mapsto Tf$$

define a functor $F^{\mathbf{T}}\colon \mathscr{A} \longrightarrow \mathscr{A}^{\mathbf{T}}$. Finally, it can be shown that $F^{\mathbf{T}}$ is left adjoint to $U^{\mathbf{T}}$ with front adjunction $\mathrm{id}_{\mathscr{A}} \longrightarrow U^{\mathbf{T}}F^{\mathbf{T}} = T$ given by η and back adjunction $\beta\colon F^{\mathbf{T}}U^{\mathbf{T}} \longrightarrow \mathrm{id}_{\mathscr{A}^{\mathbf{T}}}$ given by

$$\beta_{(A,\alpha)} = \alpha\colon (TA, \mu_A) = F^{\mathbf{T}}U^{\mathbf{T}}(A, \alpha) \longrightarrow (A, \alpha).$$

Consequently, the triple (7.4) arising from this adjunction is precisely the original triple $\mathbf{T} = (T, \eta, \mu)$ itself.

The Kleisli category associated to the triple $\mathbf{T} = (T, \eta, \mu)$ is the category $\mathscr{K}^{\mathbf{T}}$ whose objects and maps are given by

$$|\mathscr{K}^{\mathbf{T}}| = |\mathscr{A}|,$$
$$\mathscr{K}^{\mathbf{T}}(k, n) = \mathscr{A}(k, Tn);$$

the composition rule sends the pair

$$(s, t) \in \mathscr{K}^{\mathbf{T}}(m, k) \times \mathscr{K}^{\mathbf{T}}(k, n) = \mathscr{A}(m, Tk) \times \mathscr{A}(k, Tn)$$

to the element

$$t \circ s = \mu_n \circ Tt \circ s \in \mathscr{A}(m, Tn) = \mathscr{K}^{\mathbf{T}}(m, n),$$

the composition symbol on the right denoting composition in $\mathscr{A}$.

Functors $f^{\mathbf{T}} \colon \mathscr{A} \longrightarrow \mathscr{K}^{\mathbf{T}}$, $u^{\mathbf{T}} \colon \mathscr{K}^{\mathbf{T}} \longrightarrow \mathscr{A}$ are defined by

$$\begin{aligned} &f^{\mathbf{T}}(n) = n, && f^{\mathbf{T}}(f) = \eta_n \circ f && (n \in \mathscr{A}, f \in \mathscr{A}(k, n)), \\ &u^{\mathbf{T}}(n) = Tn, && u^{\mathbf{T}}(t) = \mu_n \circ Tt && (n \in \mathscr{A}, t \in \mathscr{K}^{\mathbf{T}}(k, n)), \end{aligned}$$

where, again, the composition symbols on the right denote the composition in $\mathscr{A}$. One observes that the equalities

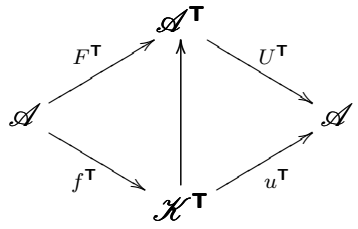
$$\mathscr{K}^{\mathbf{T}}(f^{\mathbf{T}}k, n) = \mathscr{K}^{\mathbf{T}}(k, n) = \mathscr{A}(k, Tn) = \mathscr{A}(k, u^{\mathbf{T}}n)$$

are $\mathscr{A}$-natural in k and $\mathscr{K}^{\mathbf{T}}$-natural in n, hence bespeak the adjointness of $u^{\mathbf{T}}$ to $f^{\mathbf{T}}$. Moreover, the triple (7.4) arising from this adjointness turns out, once again, to be just $\mathbf{T}$.

Linking $\mathscr{A}^{\mathbf{T}}$ with $\mathscr{K}^{\mathbf{T}}$ is the observation that the passages

$$\begin{aligned} &n \mapsto (Tn, \mu_n) && (n \in |\mathscr{A}|), \\ &t \mapsto \mu_n \circ Tt && (t \in \mathscr{K}^{\mathbf{T}}(k, n) = \mathscr{A}(k, Tn)) \end{aligned}$$

set up a full and faithful functor $\mathscr{K}^{\mathbf{T}} \longrightarrow \mathscr{A}^{\mathbf{T}}$ making the diagram



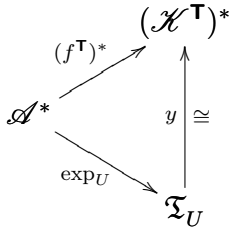
commute. This observation is based on the identifications

$$\mathscr{K}^{\mathbf{T}}(k, n) = \mathscr{A}(k, Tn) \cong \mathscr{A}^{\mathbf{T}}(F^{\mathbf{T}}k, F^{\mathbf{T}}n),$$

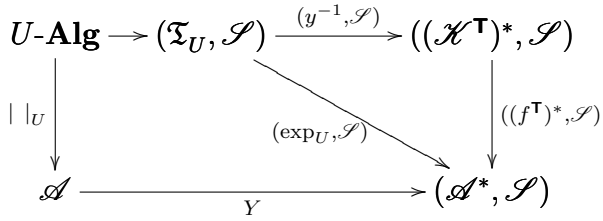
and results in an isomorphism (in $(\mathscr{A}, \mathbf{Cat})$) between $f^{\mathbf{T}}$ and the full image of $F^{\mathbf{T}}$.

8. Codensity triples.

Given a functor $U\colon \mathscr{X} \longrightarrow \mathscr{A}$, there may be a triple $\mathbf{T}$ on $\mathscr{A}$ whose Kleisli category $\mathscr{K}^{\mathbf{T}}$ is isomorphic to $(\mathfrak{T}_U)^*$ in such a way—say by an isomorphism $y\colon \mathfrak{T}_U \longrightarrow (\mathscr{K}^{\mathbf{T}})^*$—that the triangle

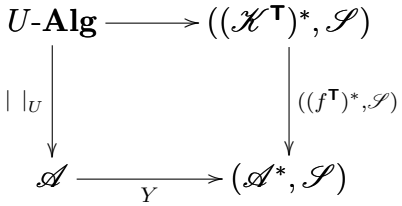


commutes. In that event, the diagram



(8.0)

commutes, and its vertices form a pullback diagram



(8.1)

Since this pullback representation of U-**Alg** is more convenient, for the purposes of Section 9, than that (established in Section 5) of Observation 1.1, the present section is devoted to the establishment of necessary and sufficient conditions for, and the interpretation of, the availability, given U, of such a triple and such an isomorphism.

In the ensuing discussion, we therefore fix an $\mathscr{A}$-valued functor $U\colon \mathscr{X} \longrightarrow \mathscr{A}$. We will need the comma categories $(n, U) = (\{\text{pt.}\}, U^n)$ constructed (see [Lawvere (1963)] for related generalities) as follows for each $n \in |\mathscr{A}|$. The objects of (n, U) are all pairs (f, X) with $X \in |\mathscr{X}|$ and $f \in U^n X = \mathscr{A}(n, UX)$. As morphisms from (f, X) to (f', X') are admitted all $\mathscr{X}$-morphisms $\xi\colon X \longrightarrow X'$ satisfying $U(\xi) \circ f = f'$. They are composed using the composition rule in $\mathscr{X}$, so that the passages

$$\begin{aligned} (f, X) &\mapsto X \\ \xi &\mapsto \xi \end{aligned}$$

constitute a functor from (n, U) to $\mathscr{X}$, to be denoted

$$C_n\colon (n, U) \longrightarrow \mathscr{X}$$

Now assume, *for this paragraph only*, that the functor U has a left adjoint. Then it is known (cf., e.g., [Bénabou (1965)] for details, including the converse) that U must preserve inverse limits and that the values of the left adjoint F serve as inverse limits for the functors C_n. (Indeed, the $(f, X)^{\text{th}}$ projection from Fn (to X) can be chosen to be the $\mathscr{X}$-morphism $Fn \longrightarrow X$ corresponding by the adjointness to $f\colon n \longrightarrow UX$.) It follows that UFn serves as inverse limit of the composite

$$(n, U) \xrightarrow[C_n]{} \mathscr{X} \xrightarrow[U]{} \mathscr{A} \tag{8.2}$$

Definition [Cf. [Appelgate (1965), Kock (1966), Tierney (1969)]]. *U admits a codensity triple if $\varprojlim UC_n$ exists for each $n \in |\mathscr{A}|$. A functor $T\colon |\mathscr{A}| \longrightarrow |\mathscr{A}|$ will be said to be a codensity triple for U if each Tn ($n \in |\mathscr{A}|$) is accompanied with a system of maps*

$$\{\langle f, X\rangle_n\colon Tn \longrightarrow UX \mid (f, X) \in |(n, U)|\} \tag{8.3}$$

by virtue of which $Tn = \varprojlim UC_n$.

The reader who is disturbed by the fact that a codensity triple for U seems not to be a triple may use the maps $\langle f, X\rangle_n$ (which we shall often abbreviate to $\langle f\rangle_n$ or even $\langle f\rangle$) to define $\mathscr{A}$-morphisms $Tg\colon Tk \longrightarrow Tn$ $(g \in \mathscr{A}(k, n))$, $\eta_n\colon n \longrightarrow Tn$ $(n \in |\mathscr{A}|)$, and $\mu_n\colon TTn \longrightarrow Tn$ $(n \in |\mathscr{A}|)$ by requiring their compositions with the projections $\langle f\rangle = \langle f\rangle_n$ to be

$$\langle f\rangle \circ Tg = \langle f \circ g\rangle, \tag{8.4}$$

$$\langle f\rangle \circ \eta_n = f, \quad \text{and} \tag{8.5}$$

$$\langle f\rangle \circ \mu_n = \langle\langle f\rangle\rangle \quad (= \langle\langle f\rangle_n\rangle_{Tn}), \tag{8.6}$$

respectively. He may then verify that T becomes a functor, that η and μ are natural transformations, and that (T, η, μ) is thus a triple (the same triple as (7.4) if U has a left adjoint F and T is obtained by the prescription in the discussion preceding (8.2)). Finally, he can prove that $\mathfrak{T}_U$ *is isomorphic with the dual $(\mathscr{K}^{\mathbf{T}})^*$ of the Kleisli category $\mathscr{K}^{\mathbf{T}}$ of $\mathbf{T}$ in the manner described at the head of this section.* Since we are after somewhat more information, including a converse to the emphasized statement above, we prefer what may seem a more roundabout approach.

A functor $U\colon \mathscr{X} \longrightarrow \mathscr{A}$ and a functor $T\colon |\mathscr{A}| \longrightarrow |\mathscr{A}|$ may be related in five apparently different ways, if certain additional information is specified; that T be a codensity triple for U is one of these ways. The five kinds of information we have in mind are:

I. maps $\langle f\rangle\colon Tn \longrightarrow UX$ (one for each $f\colon n \longrightarrow UX$ and $X \in |\mathscr{X}|$), making T a codensity triple for U;

II. functions $y\ (= y_{n,k})\colon \mathfrak{T}_U(n, k) \longrightarrow \mathscr{A}(k, Tn)$ making Tn represent the functor $\mathfrak{T}_U(n, \exp_U(-))$;

III. a left adjoint to $\exp_U : \mathscr{A}^* \longrightarrow \mathfrak{T}_U$, with specified front and back adjunctions, such that T is the object function of the composition $\mathscr{A}^* \longrightarrow \mathfrak{T}_U \longrightarrow \mathscr{A}^*$;

IV. a triple $\mathbf{T}$ whose functor component has object function T, and an isomorphism $y : \mathfrak{T}_U \longrightarrow (\mathscr{K}^{\mathbf{T}})^*$ satisfying $y \circ \exp_U = (f^{\mathbf{T}})^*$;

V. a triple $\mathbf{T}$ whose functor component has object function T and functions $y = y_{n,k} : \mathfrak{T}_U(n,k) \longrightarrow \mathscr{A}(k, Tn)$ fulfilling the four conditions:

o) each $y_{n,k}$ is a one-one correspondence,

i) $y_{n,m}(\vartheta' \circ \vartheta) = \mu_n \circ T(y_{n,k}(\vartheta)) \circ y_{k,m}(\vartheta')$,

ii) $U^{\eta_n} \circ y^{-1}_{n,Tn}(\mathrm{id}_{Tn}) = U^{\mathrm{id}_n}$,

iii) $y_{n,k}(U^f) = \eta_n \circ f$.

The theorem coming up asserts that if U and T are related in any one of these five ways, they are related in all of them. Section 11 exploits the computational accessibility of the fifth way; the other ways are more satisfactory from a conceptual point of view.

THEOREM 8.1. *There are canonical one-one correspondences, given $T : |\mathscr{A}| \longrightarrow |\mathscr{A}|$ and $U : \mathscr{X} \longrightarrow \mathscr{A}$, among the five specified classes of information relating U and T. In particular, each codensity triple for U "is", in one and only one way, a triple $\mathbf{T}$ the dual $(\mathscr{K}^{\mathbf{T}})^*$ of whose Kleisli category is isomorphic to $\mathfrak{T}_U$ in a manner compatible with the injections $(f^{\mathbf{T}})^*$ and $\exp_U$ of $\mathscr{A}^*$. Moreover, this triple structure on T is the one described above in the formulas (8.4), (8.5) and (8.6)*

PROOF. What is completely obvious is the one-one correspondence between information of type II and of type III: all that is being used is the fact (cf. [Mac Lane (1965), Prop. 8.3]) that left adjoints are defined pointwise. To go from information of type IV to that of type III, observe simply that $(y^*)^{-1} \circ f^{\mathbf{T}}$ serves as left adjoint to $\exp_U$ in the desired way; that this sets up a one-one correspondence between these kinds of information is due to the universal property (described in in [Maranda (1966), Th. 1]) of the Kleisli category. The major portion of the proof therefore consists in showing that information of types I and II (resp., of type IV and V) are in one-one correspondence with each other.

For types I and II, we have to recourse to the

LEMMA 8.1. *Let $U : \mathscr{X} \longrightarrow \mathscr{A}$ be a functor, let n and k be objects of $\mathscr{A}$ and let $\bar{k} : (n, U) \longrightarrow \mathscr{A}$ be the constant functor with value k. Then there are canonical one-one correspondences, $\mathscr{A}$-natural in k, among n.t.$(\bar{k}, UC_n)$, n.t.(U^n, U^k), and the class of all functors $\vartheta : (n, U) \longrightarrow (k, U)$ satisfying $C_k \circ \vartheta = C_n$. Indeed, the information needed to specify a member of any of these classes is the same.*

PROOF. An element ϑ of any of these classes involves a function assigning to each object $X \in |\mathscr{X}|$ and each map $f\colon n \longrightarrow UX$ a new map $\vartheta(f,X) = \vartheta_X(f)\colon k \longrightarrow UX$, subject to side conditions. In the first instance, the side conditions are

$$U\xi \circ \vartheta(f,X) = \vartheta(U\xi \circ f, X) \quad (f\colon n \longrightarrow UX, \xi\colon X \longrightarrow X')$$

In the second instance, the side conditions are the commutativity of all the squares

$$\begin{array}{ccc} U^nX & \xrightarrow{\vartheta_X} & U^kX \\ {\scriptstyle U^n\xi}\downarrow & & \downarrow{\scriptstyle U^k\xi} \\ U^nX' & \xrightarrow[\vartheta_{X'}]{} & U^kX' \end{array} \qquad (\xi\colon X \longrightarrow X').$$

In the third instance, the side conditions, in view of the requirement $C_k \circ \vartheta = C_n$ and the faithfulness of C_k and C_n, are the same as in the first instance. It now takes but a moment's reflection to see that the side conditions in the first two instances are also the same. The naturality in k will be left to the reader.

Continuing with the proof of Theorem 8.1, information of type I results in one-one correspondences $\mathscr{A}(k,Tn) \xrightarrow{\cong} \text{n.t.}(\bar{k}, UC_n)$, natural in k, obtained by composing with the $\langle f\rangle$'s. Information of type II results in one-one correspondences, natural in k, $\text{n.t.}(U^n, U^k) \cong \mathscr{A}(k,Tn)$. ∎

The free passage, natural in k, allowed by Lemma 8.1, between $\text{n.t.}(\bar{k}, UC_n)$ and $\text{n.t.}(U^n, U^k)$ thus takes ample care of the I $\Longleftrightarrow$ II relation.

[For later use, we remark that the resulting functions

$$y_{n,k}^{-1}\colon \mathscr{A}(k,Tn) \longrightarrow \text{n.t.}(U^n, U^k)$$

send $t\colon k \longrightarrow Tn$ to the natural transformation $y_{n,k}^{-1}(t)$ given by

$$\{y_{n,k}^{-1}(t)\}_X(a) = \langle \mathrm{id}_{UX}\rangle \circ Ta \circ t \quad (X \in |\mathscr{X}|, a \in U^nX)] \tag{8.7}$$

At this point, the reader can easily verify for himself that, in the passage (via II and III) from I to IV, the codensity triple

$$T, \{\{\langle f\rangle \mid f \in |(n,U)|\} \mid n \in |\mathscr{A}|\}$$

inherits the triple structure described in (8.4), (8.5) and (8.6): he need only use the fact that the triple $\mathbf{T}$ appearing in IV is the interpretation in $\mathscr{A}$ of the co-triple in $\mathscr{A}^*$ arising (by [Eilenberg & Moore (1965a), Prop. 2.1.*] or [Huber (1961), Th.4.2]) from the adjoint pair $\exp_U\colon \mathscr{A}^* \longrightarrow \mathfrak{T}_U$, $\mathfrak{T}_U \longrightarrow \mathscr{A}^*$ resulting in III from the reinterpretation of the codensity triple as information of type II.

The relation between information of types IV and V is taken care of by another lemma.

LEMMA 8.2. *Let $U\colon \mathscr{X} \longrightarrow \mathscr{A}$ be a functor, and let $\mathbf{T} = (T, \eta, \mu)$ be a triple on $\mathscr{A}$. A one-one correspondence between the class of all systems of functions*

$$y_{n,k}\colon n.t.(U^n, U^k) \longrightarrow \mathscr{A}(k, Tn) \quad (n, k \in |\mathscr{A}|)$$

fulfilling the four requirements in V above, and the class of all functors $y\colon \mathfrak{T}_U \longrightarrow (\mathscr{K}^{\mathbf{T}})^*$ satisfying the two conditions

iv) $y \circ \exp_U = (f^{\mathbf{T}})^*$

v) y is an isomorphism of categories

is induced by the passage from the functor y to the system $\{y_{n,k}\}$ in which $y_{n,k}$ is the effect of the functor y on the $\mathfrak{T}_U$-morphisms from n to k.

PROOF. Given an isomorphism $y\colon \mathfrak{T}_U \longrightarrow (\mathscr{K}^{\mathbf{T}})^*$ satisfying iv), and given n and k in $|\mathscr{A}|$, define $y_{n,k}\colon \text{n.t.}(U^n, U^k) \longrightarrow \mathscr{A}(k, Tn)$ to be the composition of the sequence

$$\text{n.t.}(U^n, U^k) = \mathfrak{T}_U(n, k) \xrightarrow[y]{\cong} (\mathscr{K}^{\mathbf{T}})^*(n, k) = \mathscr{K}^{\mathbf{T}}(k, n) = \mathscr{A}(k, Tn)$$

Condition V.o) follows from condition v). Condition V.i) follows immediately from the functoriality of y, once composition in $\mathfrak{T}_U$ and in $\mathscr{K}^{\mathbf{T}}$ are recalled. Condition V.iii) follows from iv). To establish V.ii), apply the isomorphism y to both sides. The right side is $y(U^{\mathrm{id}_n}) = y(\exp_U(\mathrm{id}_n)) = (f^{\mathbf{T}})^*(\mathrm{id}_n)$, which, viewed as an $\mathscr{A}$-morphism, is just η_n. In view of the validity of V.i), V.iii) and a triple identity, the left side is

$$\begin{aligned} y(U^{\eta_n} \circ y^{-1}(\mathrm{id}_{Tn})) &= \mu_n \circ Tyy^{-1}\mathrm{id}_{Tn} \circ y(U^{\eta_n}) = \\ &= \mu_n \circ T\mathrm{id}_{Tn} \circ \eta_{Tn} \circ \eta_n = \mu_n \circ \eta_{Tn} \circ \eta_n = \eta_n. \end{aligned}$$

Since this is what the right side of V.ii) is, after applying y, half the lemma is proved.

For the converse, take a system of functions as envisioned in the lemma, and attempt to define a functor $y\colon \mathfrak{T}_U \longrightarrow (\mathscr{K}^{\mathbf{T}})^*$ by setting $y(n) = n$ and, for $\vartheta\colon U^n \longrightarrow U^k$,

$$y(\vartheta) = y_{n,k}(\vartheta) \in \mathscr{A}(k, Tn) = \mathscr{K}^{\mathbf{T}}(k, n) = (\mathscr{K}^{\mathbf{T}})^*(n, k)$$

This attempt is successful because V.i) and the definition of composition in $\mathscr{K}^{\mathbf{T}}$ show that y preserves composition, while V.iii) (with $f = \mathrm{id}_n$) and the functoriality of $f^{\mathbf{T}}$ show that y preserves identity maps. Finally, V.iii) yields iv), using nothing but the definitions of $\exp_U$ and $f^{\mathbf{T}}$, and V.o) yields v), which completes the proof of the lemma, and hence of the theorem, too. ∎

One last comment. If T is a codensity triple for U (made into a triple $\mathbf{T} = (T, \eta, \mu)$ by the procedure of (8.4), (8.5) and (8.6), say), define $\Phi_{U,T}\colon \mathscr{X} \longrightarrow \mathscr{A}^{\mathbf{T}}$ by

$$\begin{aligned} \Phi_{U,T}(X) &= (UX, \langle \mathrm{id}_{UX} \rangle), \\ \Phi_{U,T}(\xi) &= U\xi, \end{aligned} \tag{8.8}$$

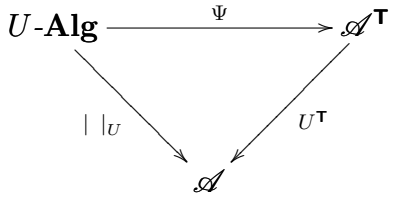
where $\langle \mathrm{id}_{UX} \rangle\colon TUX \longrightarrow UX$ is the $\mathrm{id}_{UX}{}^{\mathrm{th}}$ projection. There is no trouble in checking that $\Phi_{U,T}$ is well defined, is a functor, and satisfies $U^{\mathbf{T}} \circ \Phi_{U,T} = U$. This functor will turn out to be the front adjunction for the structure-semantics adjointness in the context of triples. If $\mathbf{T}$ arises as the adjunction triple $(UF, \eta, U\beta F)$ resulting from a left adjoint F for U, with front and back adjunctions η, β, then the effect of $\Phi_{U,T}$ on objects is given equivalently by

$$\Phi_{U,T}(X) = (UX, U\beta_X).$$

9. The isomorphism theorem

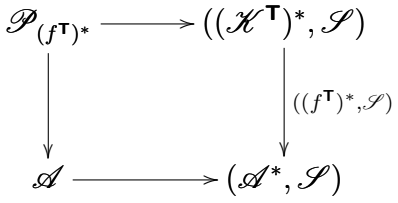
In this and the following section, we shall write $\mathfrak{M} = \mathfrak{M}^{\mathrm{id}_{\mathscr{A}}}$ and $\mathscr{P}_V = \mathscr{P}_V^{\mathrm{id}_{\mathscr{A}}}$.

THEOREM 9.1. *If $\mathbf{T}$ is a codensity triple for the $\mathscr{A}$-valued functor $U\colon \mathscr{X} \longrightarrow \mathscr{A}$, there is a canonical isomorphism $\Psi\colon U\text{-}\mathbf{Alg} \longrightarrow \mathscr{A}^{\mathbf{T}}$ making the triangle*



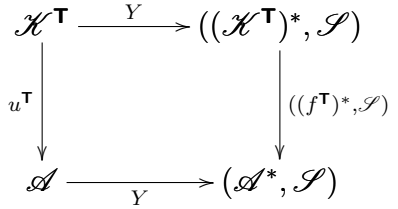
commute.

PROOF. Step 1. $|\ |_U\colon U\text{-}\mathbf{Alg} \longrightarrow \mathscr{A}$ is isomorphic to the $\mathscr{A}$-valued functor in the pullback diagram



because of the isomorphism $\mathfrak{T}_U \xrightarrow[y]{\cong} (\mathscr{K}^{\mathbf{T}})^*$ provided in Section 8.

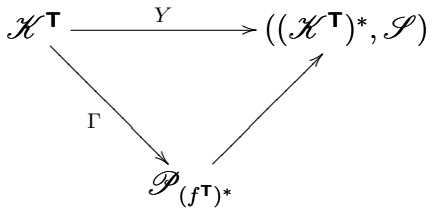
Step 2. $\mathscr{P}_{(f^{\mathbf{T}})^*} \longrightarrow \mathscr{A}$ has a left adjoint. Indeed, the commutativity of the diagram



provides a functor $\Gamma\colon \mathscr{K}^{\mathbf{T}} \longrightarrow \mathscr{P}_{(f^{\mathbf{T}})^*}$.

LEMMA 9.1. *$\Gamma\colon \mathscr{K}^{\mathbf{T}} \longrightarrow \mathscr{P}_{(f^{\mathbf{T}})^*}$ is full and faithful, $\Gamma \circ f^{\mathbf{T}}\colon \mathscr{A} \longrightarrow \mathscr{P}_{(f^{\mathbf{T}})^*}$ serves as left adjoint to $\mathfrak{M}((f^{\mathbf{T}})^*)$, and the resulting adjunction triple is $\mathbf{T}$.*

PROOF. Since the diagram



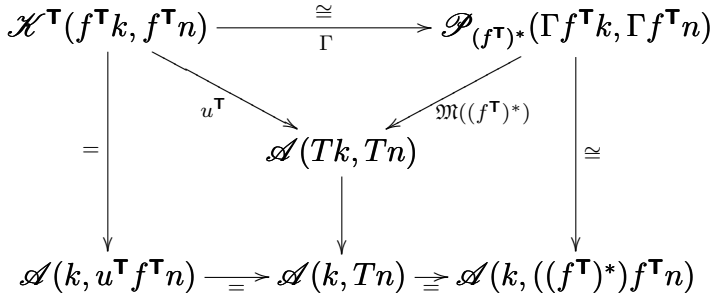
commutes, and both the Yoneda functor and

$$\mathscr{P}_{(f^{\mathbf{T}})^*} \longrightarrow ((\mathscr{K}^{\mathbf{T}})^*, \mathscr{S}) \tag{9.1}$$

are full and faithful (see Remarks 5.2 and 5.3), Γ is full and faithful, too. For the adjointness statement, the Yoneda Lemma and the fullness and faithfulness of (9.1) deliver

$$\begin{aligned}\mathscr{P}_{(f^{\mathbf{T}})^*}(\Gamma \circ f^{\mathbf{T}}k, (A, \mathfrak{A})) \cong \text{n.t.}(Y f^{\mathbf{T}}k, \mathfrak{A}) \cong \mathfrak{A}(f^{\mathbf{T}}(k)) = \\ = \mathscr{A}(k, A) = \mathscr{A}(k, \mathfrak{M}((f^{\mathbf{T}})^*)(A, \mathfrak{A})),\end{aligned}$$

whose naturality in $k \in |\mathscr{A}|$ and $(A, \mathfrak{A}) \in |\mathscr{P}_{(f^{\mathbf{T}})^*}|$ is left to the reader's verification. To compute the adjunction triple, note that $\mathfrak{M}((f^{\mathbf{T}})^*) \circ \Gamma \circ f^{\mathbf{T}} = u^{\mathbf{T}} \circ f^{\mathbf{T}} = T$, and that, when $(A, \mathfrak{A}) = \Gamma \circ f^{\mathbf{T}}k$, the front adjunction, which is whatever $\mathscr{A}$-morphism $k \longrightarrow Tk$ arises from the identity on $\Gamma f^{\mathbf{T}}k$, is the $\mathscr{A}$-morphism serving as the identity, in $\mathscr{K}^{\mathbf{T}}$, on k, namely η_k. It follows that, whatever $n, k \in |\mathscr{A}|$, the diagram



commutes, since $u^{\mathbf{T}} = \mathfrak{M}((f^{\mathbf{T}})^*) \circ \Gamma$ and the front adjunctions are the same. But then it follows that those back adjunctions that are obtained when $k = u^{\mathbf{T}}f^{\mathbf{T}}n$ (by reversing the vertical arrows and chasing the identity maps in $\mathscr{A}$ upwards) correspond to each other under Γ. This completes the proof of the lemma. ∎

Step 3. Apply the most precise tripleableness theorem (e.g., [Manes (1967), Th. 1.2.9])—it asserts that if the functor $U\colon \mathscr{X} \longrightarrow \mathscr{A}$ has a left adjoint and creates U-split coequalizers, the canonical functor $\Phi_{U,T}\colon \mathscr{X} \longrightarrow \mathscr{A}^{\mathbf{T}}$ (defined in (8.8), where $\mathbf{T}$ is the adjunction triple) is an isomorphism—using Lemma 9.1 and Proposition 6.2, to the functor $\mathfrak{M}((f^{\mathbf{T}})^*)\colon \mathscr{P}_{(f^{\mathbf{T}})^*} \longrightarrow \mathscr{A}$ to get an isomorphism $\mathfrak{M}((f^{\mathbf{T}})^*) \xrightarrow{\cong} U^{\mathbf{T}}$. Finally, combine this isomorphism with the isomorphism $|\ |_U \xrightarrow{\cong} \mathfrak{M}((f^{\mathbf{T}})^*)$ of Step 1, to obtain the desired isomorphism Ψ. ∎

Because this proof is (relatively) short and conceptual, it is somewhat uninformative. We collect the missing information in

THEOREM 9.2. *If $\mathbf{T}$ is a codensity triple for $U\colon \mathscr{X} \longrightarrow \mathscr{A}$, with associated isomorphism $y\colon \mathfrak{T}_U \longrightarrow (\mathscr{K}^{\mathbf{T}})^*$, the isomorphism $\Psi\colon U\text{-}\mathbf{Alg} \longrightarrow \mathscr{A}^{\mathbf{T}}$ provided by the proof of Theorem 9.1 has the following properties.*

1. $\Psi((A,\mathfrak{A})) = (A, \alpha(\mathfrak{A}))$, *where* $\alpha(\mathfrak{A}) = y^{-1}(\mathrm{id}_{TA}) *_{\mathfrak{A}} \mathrm{id}_A$.
2. $\Phi_{U,T} = \Psi \circ \Phi_U\colon \mathscr{X} \longrightarrow U\text{-}\mathbf{Alg} \longrightarrow \mathscr{A}^{\mathbf{T}}$.
3. $\Psi^{-1}((A,\alpha)) = (A, \mathfrak{A}(\alpha))$, *where* $\vartheta *_{\mathfrak{A}(\alpha)} a = \alpha \circ Ta \circ y(\vartheta)$;

furthermore, Ψ^{-1} is the functor portion $\Phi\colon \mathscr{A}^{\mathbf{T}} \longrightarrow U\text{-}\mathbf{Alg}$ of the $(\mathbf{Cat}, \mathscr{A})$-morphism $U^{\mathbf{T}} \longrightarrow \mathfrak{M}(\exp_U)$ corresponding by structure-semantics adjointness to the isomorphism $\exp_U \longrightarrow \exp_{U^{\mathbf{T}}}$ arising from the identifications

$$\mathfrak{T}_U(n,k) \cong \mathscr{A}(k, Tn) = (U^{\mathbf{T}})^k(F^{\mathbf{T}}(n)) \cong n.t.((U^{\mathbf{T}})^n, (U^{\mathbf{T}})^k) = \mathfrak{T}_{U^{\mathbf{T}}}(n,k)$$

PROOF. For the first assertion, we calculate the back adjunction for the adjointness of $\mathfrak{M}((f^{\mathbf{T}})^*)$ to $\Gamma \circ f^{\mathbf{T}}$, and then modify the result appropriately by y. Given the $\mathscr{P}_{(f^{\mathbf{T}})^*}$-object $(A,\mathfrak{A})$, we chase $\mathrm{id}_A \in \mathscr{A}(A,A)$ through the adjunction identification to the natural transformation

$$\mathscr{K}^{\mathbf{T}}(-, f^{\mathbf{T}}A) \longrightarrow \mathfrak{A} \tag{9.2}$$

sending $t \in \mathscr{K}^{\mathbf{T}}(k, f^{\mathbf{T}}A)$ to $\{\mathfrak{A}(t)\}(a) \in \mathfrak{A}(k) = A^k$. To find the $\mathscr{A}$-morphism component of the $\mathscr{P}_{(f^{\mathbf{T}})^*}$-morphism having (9.2) as its natural transformation component, we must apply the Yoneda Lemma to its value under $((f^{\mathbf{T}})^*, \mathscr{S})$. The resulting functors are

$$\mathscr{K}^{\mathbf{T}}(f^{\mathbf{T}}(-), f^{\mathbf{T}}A) = \mathscr{A}(-, TA)$$

and

$$\mathfrak{A} \circ f^{\mathbf{T}} = \mathscr{A}(-, A);$$

the natural transformation still has the same components; so the Yoneda Lemma produces $\mathfrak{A}(\mathrm{id}_{TA})(\mathrm{id}_A)$. Hence $\Phi_{U,T}((A,\mathfrak{A})) = (A, \mathfrak{A}(\mathrm{id}_{TA})(\mathrm{id}_A))$, and the effect of Ψ is therefore as asserted.

For point 2, it suffices to observe that $\alpha(\mathfrak{A}_U(X)) = \langle \mathrm{id}_{UX} \rangle$ (see (1.5) and (8.8)). But, in fact, using (8.7),

$$\begin{aligned}\alpha(\mathfrak{A}_U(X)) &= y^{-1}(\mathrm{id}_{TUX}) *_{\mathfrak{A}_U(X)} \mathrm{id}_{UX} = \{y^{-1}(\mathrm{id}_{TUX})\}_X(\mathrm{id}_{UX}) \\ &= \langle \mathrm{id}_{UX} \rangle \circ T(\mathrm{id}_{UX}) \circ \mathrm{id}_{TUX} = \langle \mathrm{id}_{UX} \rangle.\end{aligned}$$

To settle point 3, it is enough to show, where $\Phi\colon \mathscr{A}^{\mathbf{T}} \longrightarrow U\text{-}\mathbf{Alg}$ is obtained by the indicated adjointness, that $\Phi(A, \alpha) = (A, \mathfrak{A}(\alpha))$ and that $\Psi \circ \Phi = \mathrm{id}_{\mathscr{A}^{\mathbf{T}}}$. Now Φ is the composition

$$\mathscr{A}^{\mathbf{T}} \xrightarrow{\Phi_{U^{\mathbf{T}}}} U^{\mathbf{T}}\text{-}\mathbf{Alg} \xrightarrow{\cong} U\text{-}\mathbf{Alg}, \tag{9.3}$$

$\Phi_{U^{\mathbf{T}}}((A, \alpha)) = (A, \mathfrak{A}_{U^{\mathbf{T}}}(A, \alpha))$, and, since $\alpha\colon (TA, \mu_A) \longrightarrow (A, \alpha)$ is the back adjunction $F^{\mathbf{T}}U^{\mathbf{T}}(A, \alpha) \longrightarrow (A, \alpha)$, it follows from (1.5) that

$$\vartheta' *_{\mathfrak{A}_{U^{\mathbf{T}}}(A,\alpha)} a = \alpha \circ Ta \circ \vartheta'.$$

Applying the isomorphism, therefore, $\Phi(A, \alpha) = (A, \mathfrak{A}(\alpha))$. That $\Psi \circ \Phi = \mathrm{id}_{\mathscr{A}^{\mathbf{T}}}$ now follows immediately from the computation

$$\begin{aligned}\alpha(\mathfrak{A}(\alpha)) &= y^{-1}\mathrm{id}_{TA} *_{\mathfrak{A}(\alpha)} \mathrm{id}_A = \alpha \circ T\mathrm{id}_A \circ y(y^{-1}\mathrm{id}_{TA}) \qquad (9.4)\\ &= \alpha \circ \mathrm{id}_{TA} \circ \mathrm{id}_{TA} = \alpha \qquad \blacksquare\end{aligned}$$

Theorems 9.1 and 9.2 conspire jointly to prove Theorem 9.3 below, a more elementary, though less conceptual, proof of which appears in Section 11. To set up Theorem 9.3, we place ourselves (at first) in a more general setting, letting $U\colon \mathscr{X} \longrightarrow \mathscr{A}$ and $\mathbf{T} = (T, \eta, \mu)$ be an arbitrary $\mathscr{A}$-valued functor and a possibly unrelated triple on $\mathscr{A}$. Then, given functions

$$y_{n,k}\colon \mathrm{n.t.}(U^n, U^k) \longrightarrow \mathscr{A}(k, Tn) \quad (n, k \in |\mathscr{A}|)$$

and an $\mathscr{A}$-morphism $\alpha\colon TA \longrightarrow A$, define a system $\mathfrak{A}_y(\alpha) = \{(\mathfrak{A}_y(\alpha))_{n,k} | n, k \in |\mathscr{A}|\}$ of functions

$$(\mathfrak{A}_y(\alpha))_{n,k}\colon \mathrm{n.t.}(U^n, U^k) \longrightarrow \mathscr{S}(A^n, A^k)$$

by setting

$$\{(\mathfrak{A}_y(\alpha))_{n,k}(\vartheta)\}(a) (= \vartheta * a) = \alpha \circ Ta \circ y_{n,k}(\vartheta) \tag{9.5}$$

whenever $\vartheta\colon U^n \longrightarrow U^k$ and $a \in A^n$. Conversely, given functions

$$z_{n,k}\colon \mathscr{A}(k, Tn) \longrightarrow \mathrm{n.t.}(U^n, U^k) \quad (n, k \in |\mathscr{A}|)$$

and a U-algebra $(A, \mathfrak{A})$, define an $\mathscr{A}$-morphism

$$\alpha_z(\mathfrak{A})\colon TA \longrightarrow A$$

by posing

$$\alpha_z(\mathfrak{A}) = z_{A,TA}(\mathrm{id}_{TA}) *_{\mathfrak{A}} \mathrm{id}_A = \{\mathfrak{A}_{A,TA}(z_{A,TA}(\mathrm{id}_{TA}))\}(\mathrm{id}_A). \tag{9.6}$$

Before stating Theorem 9.3, which closes the section, we use the formalism above to suggest another proof, not using the tripleableness argument we have employed, of the core of Theorems 9.1 and 9.2. Where $\mathbf{T}$ is a codensity triple for U, functions y as above are provided in Section 8. Let $z_{n,k} = (y_{n,k})^{-1}$. The left adjoint $\Gamma \circ f^{\mathbf{T}}$ to $|\ |_U$ provided by steps 1 and 2 in the proof of Theorem 9.1 permits construction of $\Psi = \Phi_{|\ |_U,T} \colon U\text{-}\mathbf{Alg} \longrightarrow \mathscr{A}^{\mathbf{T}}$ and one can prove $\Psi(A, \mathfrak{A}) = (A, \alpha_z(\mathfrak{A}))$. Explicit analysis shows the back adjunction $\Gamma \circ f^{\mathbf{T}} \circ |\ |_U \longrightarrow \mathrm{id}_{U\text{-}\mathbf{Alg}}$ at $(A, \mathfrak{A})$ is just $\alpha_z(\mathfrak{A})$, mapping $(TA, \mathfrak{A}_y(\mu_A))$ to $(A, \mathfrak{A})$. With this, one proves

$$\mathfrak{A}_y(\alpha_z(\mathfrak{A})) = \mathfrak{A} \tag{9.7}$$

just as in Lemma 11.6. Since there is a functor $\Phi \colon \mathscr{A}^{\mathbf{T}} \longrightarrow U\text{-}\mathbf{Alg}$ (defined as in the proof of assertion 3 of Theorem 9.2) sending (A, α) to $(A, \mathfrak{A}_y(\alpha))$, it follows from (9.4), (9.7), and the fact that Φ and Ψ are compatible with the underlying $\mathscr{A}$-object functors that $\Phi = \Psi^{-1}$.

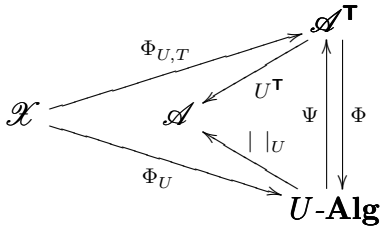
THEOREM 9.3. *If $\mathbf{T}$ is a codensity triple for $U \colon \mathscr{X} \longrightarrow \mathscr{A}$, if $y \colon \mathfrak{T}_U \longrightarrow (\mathscr{K}^{\mathbf{T}})^*$ is the resulting isomorphism, and if $z = y^{-1}$, then*

$$(A, \alpha) \mapsto (A, \mathfrak{A}_y(\alpha)), \quad (A, \mathfrak{A}) \mapsto (A, \alpha_z(\mathfrak{A}))$$

are the (bijective) object functions of a mutually inverse pair

$$\Phi \colon \mathscr{A}^{\mathbf{T}} \longrightarrow U\text{-}\mathbf{Alg}, \quad \Psi \colon U\text{-}\mathbf{Alg} \longrightarrow \mathscr{A}^{\mathbf{T}}$$

of isomorphisms making commutative the diagram



10. Structure and semantics in the presence of a triple.

In this section, we use the isomorphism of Section 9 to compare the structure-semantics adjointness of Section 4, when $j = \mathrm{id}_{\mathscr{A}}$, with that of Appelgate-Barr-Beck-Eilenberg-Huber-Kleisli-Maranda-Moore-Tierney in the context of triples. For notational convenience, we shall write $\mathfrak{T} = \mathfrak{T}^{\mathrm{id}_{\mathscr{A}}}$, and, as earlier, $\mathfrak{M} = \mathfrak{M}^{\mathrm{id}_{\mathscr{A}}}$, $\mathscr{P}_V = \mathscr{P}_V^{\mathrm{id}_{\mathscr{A}}}$. It will be used to take terminological and notational account of the canonical isomorphism

$$(\mathscr{A}^*, \mathbf{Cat}) = (\mathscr{A}, \mathbf{Cat})$$

obtained by reinterpreting $V \colon \mathscr{A}^* \longrightarrow \mathscr{C}$ as $V^* \colon \mathscr{A} \longrightarrow \mathscr{C}^*$, by speaking of V^* as a theory over $\mathscr{A}$ if V is a clone over $\mathscr{A}$; by referring to

$$\mathfrak{T}^*(U) = (\mathfrak{T}(U))^* = (\exp_U)^*$$

as the theory of $U\colon \mathscr{X} \longrightarrow \mathscr{A}$; and by speaking of

$$\mathfrak{M}^*(\varphi) = \mathfrak{M}(\varphi^*)\colon \mathscr{P}_{\varphi^*} \longrightarrow \mathscr{A}$$

as (the underlying $\mathscr{A}$-object functor on) the category of φ-algebras in $\mathscr{A}$, where φ is a theory over $\mathscr{A}$. It is clear that a clone has a left adjoint iff the corresponding theory has a right adjoint. By the same argument as was used in Section 8, one can prove the first part of

LEMMA 10.1. *To give a right adjoint u, with front and back adjunctions η, β for a theory φ on $\mathscr{A}$ is the same as to give an isomorphism $\varphi \cong f^{\mathbf{T}}$, where $\mathbf{T}$ is the triple $(u\varphi, \eta, u\beta\varphi)$ on $\mathscr{A}$. Moreover, given a triple $\mathbf{T}$, $\Theta = \mathscr{K}^{\mathbf{T}}$, $\varphi = f^{\mathbf{T}}$, $u = u^{\mathbf{T}}$ provide the only theory $\varphi\colon \mathscr{A} \longrightarrow \Theta$ with left adjoint u satisfying*
1) the adjunction equivalences

$$\mathscr{A}(k, un) \xrightarrow{\cong} \Theta(\varphi k, n)$$

are identity maps, and
2) the adjunction triple is $\mathbf{T}$.

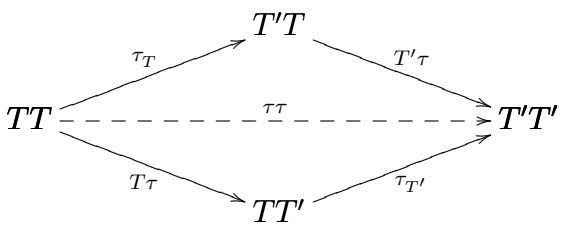
PROOF. We skip the proof of the first assertion, it being just like the proof of III$\Longleftrightarrow$IV in Theorem 8.1. For the second assertion, it is clear that the objects of Θ must be those of $\mathscr{A}$. Then Θ-morphisms $k \longrightarrow n$ must be $\mathscr{A}$-morphisms $k \longrightarrow Tn$, the identity in $\Theta(n, n)$ must be $\eta_n \in \mathscr{A}(n, Tn)$, and, finally, for the identity functions to be natural, as required by 1), it is forced that the composition rule of Θ is that of $\mathscr{K}^{\mathbf{T}}$. ∎

We introduce the categories $\mathrm{Ad}(\mathbf{Cat}, \mathscr{A})$ (resp. $\mathrm{Tr}(\mathbf{Cat}, \mathscr{A})$) of $\mathscr{A}$-valued functors having specified left adjoints (resp. specified codensity triples), and $\mathrm{AdTheo}(\mathscr{A})$ (resp. $\mathrm{AdCl}(\mathscr{A})$) of theories (resp. clones) over $\mathscr{A}$ having specified right (resp. left) adjoints. These are so constructed, per definitionem, as to make the obvious forgetful functors to the similarly named categories, with the prefix Ad or Tr omitted, full and faithful. We shall also need the category $\mathrm{Trip}(\mathscr{A})$ whose objects are triples on $\mathscr{A}$: a triple morphism from $\mathbf{T} = (T, \eta, \mu)$ to $\mathbf{T}' = (T', \eta', \mu')$ will be any natural transformation $\tau\colon T \longrightarrow T'$ for which

$$\tau \circ \eta = \eta', \quad \text{and} \tag{10.1}$$

$$\tau \circ \mu = \mu' \circ \tau\tau \tag{10.2}$$

(where $\tau\tau$ denotes either of the compositions



which are equal because τ is natural).

LEMMA 10.2. *The attempts to define functors*

$$\mathrm{AdTheo}(\mathscr{A}) \xrightarrow{t} \mathrm{Trip}(\mathscr{A})\colon \varphi, u, \eta, \beta \mapsto (u\varphi, \eta, u\beta\varphi),$$
$$\mathrm{Trip}(\mathscr{A}) \xrightarrow{k} \mathrm{AdTheo}(\mathscr{A})\colon \mathbf{T} \mapsto \textit{Kleisli cat. } w/f^{\mathbf{T}}, u^{\mathbf{T}},$$

are successful and represent $\mathrm{Trip}(\mathscr{A})$ *isomorphically as the full subcategory of* $\mathrm{AdTheo}(\mathscr{A})$*, equivalent to* $\mathrm{AdTheo}(\mathscr{A})$*, consisting of those adjointed theories for which condition 1) of Lemma 10.1 is valid.*

PROOF. Elementary. For related information, see [Barr (1965)] or [Maranda (1966)]. ∎

Theorem 8.1 shows that $\mathrm{Tr}(\mathbf{Cat}, \mathscr{A})$ is the pullback of the pullback diagram

$$\begin{array}{ccc} \mathrm{Tr}(\mathbf{Cat}, \mathscr{A}) & \xrightarrow{\mathfrak{S}_t} & (\mathrm{AdTheo}(\mathscr{A}))^* \\ \downarrow & & \downarrow \\ (\mathbf{Cat}, \mathscr{A}) & \xrightarrow[\mathfrak{S}^*]{} & (\mathrm{Theo}(\mathscr{A}))^* \end{array}$$

and arguments like those for Lemma 9.1 provide a lifting $\mathfrak{M}_t$

$$\begin{array}{ccc} (\mathrm{AdTheo}(\mathscr{A}))^* & \xrightarrow{\mathfrak{M}_t} & \mathrm{Ad}(\mathbf{Cat}, \mathscr{A}) \\ \downarrow & & \downarrow \\ (\mathrm{Theo}(\mathscr{A}))^* & \xrightarrow[\mathfrak{M}^*]{} & (\mathbf{Cat}, \mathscr{A}) \end{array}$$

of $\mathfrak{M}^*$.

THEOREM 10.1. *Let* $I\colon \mathrm{Ad}(\mathbf{Cat}, \mathscr{A}) \longrightarrow \mathrm{Tr}(\mathbf{Cat}, \mathscr{A})$ *be the obvious functor (sending* $(U; F, \eta, \beta)$ *to* $(U; (UF, \eta, U\beta F))$*), and let* $\mathfrak{M}'\colon (\mathrm{Trip}(\mathscr{A}))^* \longrightarrow \mathrm{Ad}(\mathbf{Cat}, \mathscr{A})$ *be the functor sending* $\mathbf{T}$ *to* $(U^{\mathbf{T}}; F^{\mathbf{T}}, \eta, \beta)$ *(where* $\beta_{(A,\alpha)} = \alpha$*). Then:*

1. $\mathfrak{S}_t \circ I$ *(resp.* $\mathfrak{S}_t$*) is adjoint to* $\mathfrak{M}_t$ *(resp.* $I \circ \mathfrak{M}_t$*),*
2. $\mathfrak{M}' \circ t$ *(resp.* $\mathfrak{M}_t \circ k$*) is equivalent to* $\mathfrak{M}_t$ *(resp.* $\mathfrak{M}'$*),*
3. $t \circ \mathfrak{S}_t \circ I$ *(resp.* $t \circ \mathfrak{S}_t$*) is adjoint to* $\mathfrak{M}'$ *(resp.* $I \circ \mathfrak{M}'$*).*

The proof, which is easy, uses Theorem 4.1, the above lemmas, and the isomorphisms produced in Section 9. $t \circ \mathfrak{S}_t \circ I$ and $\mathfrak{M}'$ are the most familiar structure and semantics functors in the context of triples and adjoint pairs; $t \circ \mathfrak{S}_t$ and $\mathfrak{M}'$ are those needed in the work of Appelgate and Tierney [Appelgate (1965)], [Tierney (1969)].

Motivating the presentation of [Eilenberg & Wright (1967)] is the realization that the Kleisli category arising from the adjunction triple of an adjoint pair U, F is isomorphic

with the full image of F. This makes "free algebras" more amenable, and encourages yet another (equivalent) structure functor in the setting of adjointed theories and adjoint $\mathscr{A}$-valued functors.

A pleasant exercise (for private execution) is to tabulate all the isomorphisms and equivalences that have arisen in this work and will arise from them by composition with an $\mathfrak{S}$ or an $\mathfrak{M}$.

11. Another proof of the isomorphism theorem

This section is devoted to a straightforward computational proof of Theorem 9.3. The proof itself follows a sequence of lemmas; these lemmas depend only on the "information of type V" arising from the assumption that $\mathbf{T}$ is a codensity triple for U (see Section 8). For convenience of reference, we recall the equations

$$y(\vartheta' \circ \vartheta) = \mu_n \circ T(y\vartheta) \circ y\vartheta', \tag{V.i}$$

$$U^{\eta_n} \circ y^{-1}(\mathrm{id}_{Tn}) = U^{\mathrm{id}_n}, \tag{V.ii}$$

$$y(U^f) = \eta_n \circ f, \tag{V.iii}$$

imposed on the one-one correspondences

$$y = y_{n,k} \colon \text{n.t.}(U^n, U^k) \longrightarrow \mathscr{A}(k, Tn) \quad (n, k \in |\mathscr{A}|),$$

the diagrams

$$\begin{array}{ccccc} & & TA & & \\ & \overset{\eta A}{\nearrow} & & \overset{\alpha}{\searrow} & \\ A & & \xrightarrow[\mathrm{id}_A]{} & & A \end{array} \tag{7.6}$$

$$\begin{array}{ccc} TTA & \xrightarrow{\mu_A} & TA \\ {\scriptstyle T\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ TA & \xrightarrow[\alpha]{} & A \end{array} \tag{7.7}$$

whose commutativity betokens the assertion that $\alpha \colon TA \longrightarrow A$ is a $\mathbf{T}$-algebra, and the diagram

$$\begin{array}{ccc} TA & \xrightarrow{Tg} & TB \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[g]{} & B \end{array} \tag{7.8}$$

on the basis of whose commutativity $g \colon A \longrightarrow B$ is an $\mathscr{A}^{\mathbf{T}}$-morphism from (A, α) to (B, β).

We begin to chip away at Theorem 9.3 by proving

LEMMA 11.1. *Suppose* (A, α) *is a* $\mathbf{T}$*-algebra. Then* $(A, \mathfrak{A}(\alpha))$ *is a* U*-algebra (here* $\mathfrak{A}(\alpha) = \mathfrak{A}_y(\alpha)$ *is defined by (9.5)).*

PROOF. The definition of $\mathfrak{A}(\alpha)$, (V.iii), naturality of η, and (7.6) allow us to verify ALG 1:

$$U^f * a = \alpha \circ Ta \circ y(U^f) = \alpha \circ Ta \circ \eta_n \circ f = \alpha \circ \eta_A \circ a \circ f = a \circ f.$$

Similarly, (9.5), (V.i), naturality of μ, (7.7), functoriality of T, and (twice more) (9.5) again, deliver ALG 2:

$$\begin{aligned}(\vartheta' \circ \vartheta) * a &= \alpha \circ Ta \circ y(\vartheta' \circ \vartheta) = \alpha \circ Ta \circ \mu_n \circ T(y\vartheta) \circ y\vartheta' \\ &= \alpha \circ \mu_a \circ TTa \circ T(y\vartheta) \circ y\vartheta' = \alpha \circ T\alpha \circ TTa \circ T(y\vartheta) \circ y\vartheta' \\ &= \alpha \circ T(\alpha \circ Ta \circ y\vartheta) \circ y\vartheta' = \alpha \circ T(\vartheta * a) \circ y\vartheta' \\ &= \vartheta' * (\vartheta * a)\end{aligned}$$

■

As a start in going the other way, we offer

LEMMA 11.2. *Suppose* $(A, \mathfrak{A})$ *is a* U*-algebra. Then*

$$\alpha(\mathfrak{A}) \circ \eta_A = \mathrm{id}_A.$$

PROOF. Using ALG 1, (9.6), (V.ii), and ALG 1 again, we see

$$\begin{aligned}\alpha(\mathfrak{A}) \circ \eta_A &= U^{\eta_A} * \alpha(\mathfrak{A}) = U^{\eta_A} * (y^{-1}(\mathrm{id}_{TA}) * \mathrm{id}_A) \\ &= (U^{\eta_A} \circ y^{-1}(\mathrm{id}_{TA})) * \mathrm{id}_A = U^{\mathrm{id}_A} * \mathrm{id}_A \\ &= \mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A\end{aligned}$$

■

To know that $\alpha(\mathfrak{A})$ is a $\mathbf{T}$-algebra, there remains the identity $\alpha(\mathfrak{A}) \circ \mu_A = \alpha(\mathfrak{A}) \circ T\alpha(\mathfrak{A})$. This identity, as well as the fact that each U-homomorphism $(A, \mathfrak{A}) \longrightarrow (B, \mathfrak{B})$ is also a $\mathbf{T}$-homomorphism $(A, \alpha(\mathfrak{A})) \longrightarrow (B, \alpha(\mathfrak{B}))$, will result from the fact (Lemma 11.8) that each such U-homomorphism g makes diagram (7.8) (with $\alpha = \alpha(\mathfrak{A})$, $\beta = \alpha(\mathfrak{B})$) commute, and the fact (Lemma 11.5) that $\alpha(\mathfrak{A})\colon TA \longrightarrow A$ is a U-homomorphism. The next two lemmas pave the way for a proof of Lemma 11.5.

LEMMA 11.3. *For any* U*-algebra* $(A, \mathfrak{A})$ *and any* $\mathscr{A}$*-morphism* $f\colon k \longrightarrow n$*, the diagram with solid arrows*

$$\begin{array}{ccccccc}
\mathscr{A}(n, TA) & \xrightarrow{y^{-1}} & \boldsymbol{n.t.}(\boldsymbol{U^A}, \boldsymbol{U^n}) & \xrightarrow{\mathfrak{A}_{A,n}} & \mathscr{S}(\boldsymbol{A^A}, \boldsymbol{A^n}) & \xrightarrow{\mathrm{ev}_{\mathrm{id}_A}} & \boldsymbol{A^n} \\
\downarrow{\scriptstyle \mathscr{A}(f,TA)} & & \vdots & & \vdots & & \downarrow{\scriptstyle A^f} \\
\mathscr{A}(k, TA) & \xrightarrow[y^{-1}]{} & \boldsymbol{n.t.}(\boldsymbol{U^A}, \boldsymbol{U^k}) & \xrightarrow[\mathfrak{A}_{A,k}]{} & \mathscr{S}(\boldsymbol{A^A}, \boldsymbol{A^k}) & \xrightarrow[\mathrm{ev}_{\mathrm{id}_A}]{} & \boldsymbol{A^k}
\end{array}$$

commutes. Moreover, starting with $n = TA$, the effect of the top row on $\mathrm{id}_{TA} \in \mathscr{A}(n, TA)$ is $\alpha(\mathfrak{A}) \in \mathscr{A}(TA, A) = A^n$. Hence the diagram

$$\begin{array}{ccc} n.t.(U^A, U^n) & \xrightarrow{\mathfrak{A}_{A,n}} & \mathscr{S}(A^A, A^n) \\ \Big\downarrow{\scriptstyle y} & & \Big\downarrow{\scriptstyle \mathrm{ev}_{\mathrm{id}_A}} \\ \mathscr{A}(n, TA) & \xrightarrow[\mathscr{A}(n,\alpha(A))]{} & \mathscr{A}(n, A) = A^n \end{array}$$

commutes, for each $n \in |\mathscr{A}|$.

Proof. For the dotted arrows use "composition with U^f" and "composition with A^f", respectively. That the right hand square then commutes follows from the naturality of $\mathrm{ev}_{\mathrm{id}_A}$. The central square commutes because

$$(\vartheta * a) \circ f = U^f * (\vartheta * a) = (U^f \circ \vartheta) * a,$$

using ALG 1 and ALG 2. The left hand square commutes because (V.i), (V.iii), naturality of η, and one of the triple identities deliver the chain of equalities

$$\begin{aligned} y(U^f \circ \vartheta) &= \mu_A \circ T(y\vartheta) \circ y(U^f) = \mu_A \circ T(y\vartheta) \circ \eta_n \circ f = \\ &= \mu_A \circ \eta_{TA} \circ y\vartheta \circ f = y\vartheta \circ f; \end{aligned}$$

setting $\vartheta = y^{-1}(t)$ and applying y^{-1} to both ends of this chain provides the identity expressing commutativity of the left hand square. The assertion regarding $\alpha(\mathfrak{A})$ is just the definition of $\alpha(\mathfrak{A})$. The Yoneda Lemma then applies: the given natural transformation $\mathscr{A}(-, TA) \longrightarrow \mathscr{A}(-, A)$ is of the form $\mathscr{A}(-, \alpha(\mathfrak{A}))$. This proves the last assertion. ∎

Lemma 11.4. *For any U-algebra $(A, \mathfrak{A})$ and any natural transformation $\vartheta\colon U^n \longrightarrow U^k$, the diagram*

$$\begin{array}{ccccccc} \mathscr{A}(n, TA) & \xrightarrow{y^{-1}} & n.t.(U^A, U^n) & \xrightarrow{\mathfrak{A}_{A,n}} & \mathscr{S}(A^A, A^n) & \xrightarrow{\mathrm{ev}_{\mathrm{id}_A}} & A^n \\ \Big\downarrow{\scriptstyle \mathfrak{A}(\mu_A)_{n,k}(\vartheta)} & & \Big\downarrow{\scriptstyle \text{compose with } \vartheta} & & & & \Big\downarrow{\scriptstyle \mathfrak{A}_{n,k}(\vartheta)} \\ \mathscr{A}(k, TA) & \xrightarrow[y^{-1}]{} & n.t.(U^A, U^k) & \xrightarrow[\mathfrak{A}_{A,k}]{} & \mathscr{S}(A^A, A^k) & \xrightarrow[\mathrm{ev}_{\mathrm{id}_A}]{} & A^k \end{array}$$

commutes.

Proof. The commutativity of the large right hand square is guaranteed by ALG 2. To deal with the small left hand square, note that (9.5) and (V.i) yield

$$\{\mathfrak{A}(\mu_A)_{n,k}(\vartheta)\}(a) = \mu_A \circ Ta \circ y\vartheta = y(\vartheta \circ y^{-1}(a)).$$

Apply y^{-1} to this equation to obtain the equation expressing the commutativity of the left hand square. ∎

We can now prove that $\alpha(\mathfrak{A})$ is a U-homomorphism.

LEMMA 11.5. *For every U-algebra $(A, \mathfrak{A})$, the $\mathscr{A}$-morphism $\alpha(\mathfrak{A})\colon TA \longrightarrow A$ is a U-algebra morphism from $(TA, \mathfrak{A}(\mu_A))$ to $(A, \mathfrak{A})$.*

PROOF. Given $a\colon n \longrightarrow TA$ and $\vartheta\colon U^n \longrightarrow U^k$, the clockwise composition in the diagram of Lemma 11.4 sends a, according to the last assertion of Lemma 11.3, to $\vartheta * (\alpha(\mathfrak{A}) \circ a)$. The counterclockwise composition, for the same reason, sends a to $\alpha(\mathfrak{A}) \circ (\vartheta * a)$. Hence $\alpha(\mathfrak{A}) \circ (\vartheta * a) = \vartheta * (\alpha(\mathfrak{A}) \circ a)$, and the lemma is proved. ∎

The only thing standing in the way of Lemma 11.8 is

LEMMA 11.6. *Whenever $(A, \mathfrak{A})$ is a U-algebra, $a\colon n \longrightarrow A$ is an $\mathscr{A}$-morphism, and $\vartheta\colon U^n \longrightarrow U^k$ is a natural transformation, then $\{\mathfrak{A}_{n,k}(\vartheta)\}(a) = \alpha(\mathfrak{A}) \circ Ta \circ y\vartheta$.*

PROOF. Lemmas 11.2 and 11.5 allow us to write

$$\vartheta * a = \vartheta * (\alpha(\mathfrak{A}) \circ \eta_A \circ a) = \alpha(\mathfrak{A}) \circ (\vartheta * (\eta_A \circ a)).$$

But, by (9.5), the definition of $\mathfrak{A}(\mu_A)$, we have

$$\vartheta * (\eta_A \circ a) = \mu_A \circ T(\eta_A \circ a) \circ y\vartheta.$$

Combining these equations, using the functoriality of T, and applying one of the triple identities, we obtain

$$\vartheta * a = \alpha(\mathfrak{A}) \circ \mu_A \circ T\eta_A \circ Ta \circ y\vartheta = \alpha(\mathfrak{A}) \circ Ta \circ y\vartheta,$$

which proves the lemma. ∎

Because it has to be proved sometime, we postpone the dénoument by means of

LEMMA 11.7. *Whenever (A, α) is a $\mathbf{T}$-algebra and $\mathfrak{A} = \mathfrak{A}(\alpha)$, then $\alpha = \alpha(\mathfrak{A})$.*

PROOF. Repeat the computation (9.4). ∎

LEMMA 11.8. *Let $(A, \mathfrak{A})$ and $(B, \mathfrak{B})$ be U-algebras. For each U-algebra morphism $f\colon A \longrightarrow B$ between them, the diagram*

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle \alpha(\mathfrak{A})}\downarrow & & \downarrow{\scriptstyle \alpha(\mathfrak{B})} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes.

PROOF. Using (9.6), the hypothesis, and Lemma 11.6, we see

$$\begin{aligned} f \circ \alpha(\mathfrak{A}) &= f \circ (y^{-1}(\mathrm{id}_{TA}) * \mathrm{id}_A) = y^{-1}(\mathrm{id}_{TA}) * f \\ &= \alpha(\mathfrak{B}) \circ Tf \circ yy^{-1}(\mathrm{id}_{TA}) = \alpha(\mathfrak{B}) \circ Tf \end{aligned}$$

∎

It is time to reap our corollaries.

COROLLARY 1. *If $(A, \mathfrak{A})$ is a U-algebra, $(A, \alpha(\mathfrak{A}))$ is a $\mathbf{T}$-algebra.*

PROOF. One of the necessary identities was proved as Lemma 11.2. The other is given by Lemma 11.8, applied (by virtue of Lemma 11.5) to $f = \alpha(\mathfrak{A})\colon (TA, \mathfrak{A}(\mu_A)) \longrightarrow (A, \mathfrak{A})$, and modified by taking into account Lemma 11.7:

$$\alpha(\mathfrak{A}) \circ T\alpha(\mathfrak{A}) = \alpha(\mathfrak{A}) \circ \alpha(\mathfrak{A}(\mu_A)) = \alpha(\mathfrak{A}) \circ \mu_A. \qquad \blacksquare$$

COROLLARY 2. *If $f\colon A \longrightarrow B$ is a U-algebra homomorphism from $(A, \mathfrak{A})$ to $(B, \mathfrak{B})$, it is also a $\mathbf{T}$-morphism from $(A, \alpha(\mathfrak{A}))$ to $(B, \alpha(\mathfrak{B}))$.*

PROOF. This follows immediately from the diagram of Lemma 11.8 and from Corollary 1. $\blacksquare$

COROLLARY 3. *If $f\colon A \longrightarrow B$ is a $\mathbf{T}$-algebra homomorphism from (A, α) to (B, β), it is also a U-algebra map from $(A, \mathfrak{A}(\alpha))$ to $(B, \mathfrak{A}(\beta))$.*

PROOF. Let $a\colon n \longrightarrow A$, $\vartheta\colon U^n \longrightarrow U^k$. By (9.5), the hypothesis, functoriality of T, and (9.5) again, we have

$$\begin{aligned} f \circ (\vartheta * a) &= f \circ \alpha \circ Ta \circ y\vartheta = \beta \circ Tf \circ Ta \circ y\vartheta \\ &= \beta \circ T(f \circ a) \circ y\vartheta = \vartheta * (f \circ a) \end{aligned} \qquad \blacksquare$$

PROOF OF THEOREM 9.3. Lemma 11.1, Corollary 1, and Lemmas 11.6 and 11.7 set up the desired isomorphism $|U\text{-}\mathbf{Alg}| \Longleftrightarrow |\mathscr{A}^{\mathbf{T}}|$. Corollaries 2 and 3, taken together with Lemmas 11.6 and 11.7, extend this to an isomorphism of categories $U\text{-}\mathbf{Alg} \Longleftrightarrow \mathscr{A}^{\mathbf{T}}$. It is clear from the constructions and from Corollaries 2 and 3 that the underlying $\mathscr{A}$-object functors are respected. The relation with the Φ's is settled by the proof of point 2) of Theorem 9.2. $\blacksquare$

Applied Functorial Semantics, II

F. E. J. Linton [1]

Introduction.

In this note, we derive from Jon Beck's precise triple-ableness theorem (stated as Theorem 1—for proof see [Beck (1967)]) a variant (appearing as Theorem 3) which resembles the characterization theorem for varietal categories (see [Linton (1966a), Prop. 3]—in the light of [Linton (1969)], varietal categories are just categories tripleable over $\mathscr{S}$). It turns out that this variant not only specializes to the theorem it resembles, but lies at the heart of a short proof of M. Bunge's theorem [Bunge (1966)] (known also to P. Gabriel [Gabriel (unpublished)]) characterizing functor categories $\mathscr{S}^{\mathfrak{C}} = (\mathfrak{C}, \mathscr{S})$ of all set-valued functors on a small category $\mathfrak{C}$.

1. The precise tripleableness theorem.

Our starting point is the assumption of familiarity with the precise tripleableness theorem [Beck (1967), Theorem 1] and its proof. This is summarized below as Theorem 1. The basic situation is a functor $U\colon \mathfrak{C} \longrightarrow \mathfrak{A}$ having a left adjoint $F\colon \mathfrak{A} \longrightarrow \mathfrak{C}$ with front and back adjunctions $\eta\colon \mathrm{id}_{\mathfrak{A}} \longrightarrow UF$, $\beta\colon FU \longrightarrow \mathrm{id}_{\mathfrak{C}}$. In this situation, one obtains a triple $\mathbf{T} = (UF, \eta, U\beta F)$ on $\mathfrak{A}$ and a functor $\Phi\colon \mathfrak{C} \longrightarrow \mathfrak{A}^{\mathbf{T}}$ (satisfying $U^{\mathbf{T}} \circ \Phi = U$), defined by

$$\Phi X = (UX, U\beta_X)$$

$$\Phi\xi = U\xi$$

The concern of all tripleableness theorems is whether Φ is an equivalence.

We will have repeated occasion to consider so-called U-split coequalizer systems. These consist of a pair

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \tag{1.1}$$

of $\mathfrak{C}$-morphisms and three $\mathfrak{A}$-morphisms

$$UX \underset{d_1}{\overset{}{\leftarrow}} UY \underset{d_0}{\overset{p}{\rightleftarrows}} Z \tag{1.2}$$

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for which the four identities

$$\left\{\begin{array}{lcl} pUf & = & pUg \\ pd_0 & = & \mathrm{id}_Z \\ d_0p & = & Ugd_1 \\ \mathrm{id}_{UY} & = & Ufd_1 \end{array}\right. \tag{1.3}$$

are valid. An $\mathrm{id}_{\mathfrak{A}}$-split coequalizer system will be called simply a split coequalizer system.

LEMMA 1. *If*

$$A \underset{g}{\overset{f}{\underset{\longleftarrow d_1 \longrightarrow}{\rightrightarrows}}} B \underset{d_0}{\overset{p}{\rightleftarrows}} C \tag{1.4}$$

is a split coequalizer system in $\mathfrak{A}$*, then* $p = \mathrm{coeq}(f, g)$*. Conversely, if* $B \xrightarrow{p} C$ *is a split epimorphism, with section* $d_0 \colon C \longrightarrow B$*, and* $A \underset{g}{\overset{f}{\rightrightarrows}} B$ *is its kernel pair, defining* $d_1 \colon B \longrightarrow A$ *by the requirements* $fd_1 = \mathrm{id}_B$*,* $gd_1 = d_0p$ *provides a split coequalizer diagram* (1.4).

PROOF. Let $x \colon B \longrightarrow ?$ be any map. Then if $xf = xg$, $x = xfd_1 = xgd_1 = xd_0p$. Conversely, if $x = xd_0p$, then

$$xf = xd_0pf = xd_0pg = xg.$$

Consequently, $xf = xg$ iff x factors through p by xd_0. That settles the first statement. The second is even more trivial. ∎

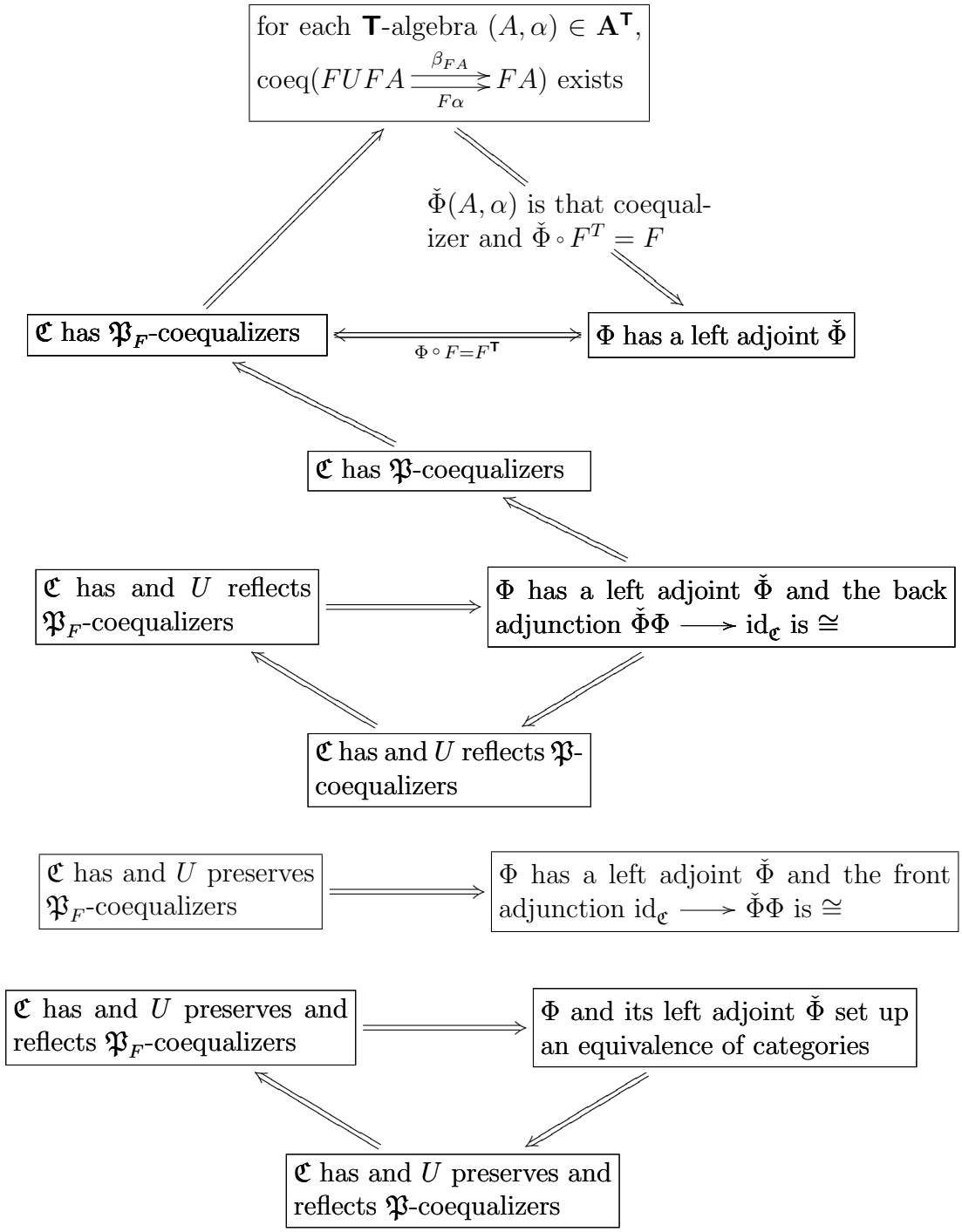
The class of all pairs of $\mathfrak{C}$-morphisms arising as (1.1) in a U-split coequalizer system (1.1), (1.2) will be denoted $\mathfrak{P}$. $\mathfrak{P}_F$ will denote those pairs in $\mathfrak{P}$ whose domain and codomain are values of F. Since we shall have to deal with yet other subclasses of $\mathfrak{P}$, we formulate the next three definitions in terms of an arbitrary class $\mathfrak{G}$ of pairs (1.1) of $\mathfrak{C}$-morphisms.

DEFINITION. $\mathfrak{C}$ **has** $\mathfrak{G}$**-coequalizers** *if each pair* $(f, g) \in \mathfrak{G}$ *has a coequalizer in* $\mathfrak{C}$*;* U **reflects** $\mathfrak{G}$**-coequalizers** *if, given a diagram*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{p} Z \tag{1.5}$$

in $\mathfrak{C}$*, with* $(f, g) \in \mathfrak{G}$ *and* $Up = \mathrm{coeq}(Uf, Ug)$*, it follows that* $p = \mathrm{coeq}(f, g)$*;* U **preserves** $\mathfrak{G}$**-coequalizers** *if, given a diagram* (1.5) *with* $(f, g) \in \mathfrak{G}$ *and* $p = \mathrm{coeq}(f, g)$*, it follows that* $Up = \mathrm{coeq}(Uf, Ug)$*.*

THEOREM 1. [Beck (1967), Theorem 1]. *If* $U, F, \mathbf{T}$ *and* $\Phi \colon \mathfrak{C} \longrightarrow \mathfrak{A}^{\mathbf{T}}$ *are as in the basic situation above, then* Φ *is an equivalence if and only if* $\mathfrak{C}$ *has and* U *preserves and reflects* $\mathfrak{P}$*-coequalizers. More precisely, we have the following implications, some accompanied by their reasons.*



REMARK. Φ will be an isomorphism if and only if it is an equivalence and U **creates isomorphisms**, in the sense: given X in $\mathfrak{C}$ and an isomorphism

$$f\colon A \longrightarrow UX$$

in $\mathfrak{A}$, there is one and only one $\mathfrak{C}$-morphism $g\colon X' \longrightarrow X$ satisfying the single requirement

$$Ug = f$$

(which, of course, entails $UX' = A$), and that $\mathfrak{C}$-morphism is an isomorphism. For details on this fact, which will enter tangentially in Section 5, consult [Manes (1967), Section 0.8 and (1.2.9)].

2. When $\mathfrak{A}$ has enough kernel pairs.

For the first variation on the theme of Theorem 1, we introduce the class $\mathfrak{P}_c$ of all pairs of $\mathfrak{C}$-morphisms

$$FE \underset{g}{\overset{f}{\rightrightarrows}} X \tag{2.1}$$

arising as follows:

i) there is a split epimorphism $p\colon UX \longrightarrow B$ in $\mathfrak{A}$;

ii) $E \underset{g_0}{\overset{f_0}{\rightrightarrows}} UX$ is its kernel pair;

iii) $f = \beta_X \circ Ff_0$, $g = \beta_X \circ Fg_0$;

iv) $pUf = pUg$.

It follows that $p = \mathrm{coeq}(Uf, Ug)$ and that

$$E \underset{\eta_E}{\longrightarrow} UFE \underset{Ug}{\overset{Uf}{\rightrightarrows}} UX$$

is p's kernel pair (for iv) $\Longrightarrow$ $\exists!\varepsilon\colon UFE \longrightarrow E$ with $f_0\varepsilon = Uf$, $g_0\varepsilon = Ug$, whence

$$f_0 \circ \varepsilon \circ \eta_E = Uf \circ \eta_E = f_0$$

$$g_0 \circ \varepsilon \circ \eta_E = Ug \circ \eta_E = g_0$$

whence $\varepsilon \circ \eta_E = \mathrm{id}_E$; hence $qUf = qUg$ iff $qf_0 = qg_0$, and $p = \mathrm{coeq}(f_0, g_0) = \mathrm{coeq}(Uf, Ug)$; the second assertion is obvious).

Conversely, if (2.1) is a pair of $\mathfrak{C}$-morphisms for which (Uf, Ug) has a coequalizer p, if p is a split epimorphism, and if

$$E \underset{\eta_E}{\longrightarrow} UFE \underset{Ug}{\overset{Uf}{\rightrightarrows}} UX \tag{2.2}$$

is a kernel pair for p, then where $f_0 = Uf \circ \eta_E$, $g_0 = UG \circ \eta_E$, (2.1) arises from p and f_0, g_0 through steps i) ... iv).

We use these remarks to prove

LEMMA 2. $\mathfrak{P}_c \subseteq \mathfrak{P}$. *Moreover, if* (2.1)$\in \mathfrak{P}_c$ *and* $\xi\colon X \longrightarrow X'$, *then* $\xi f = \xi g$ *iff* $U(\xi f) = U(\xi g)$ *(*$f_0 = Uf \circ \eta_E$, $g_0 = Ug \circ \eta_E$*).*

PROOF. If (2.1) depicts a pair in $\mathfrak{P}_c$, (Uf, Ug) has as coequalizer a split epimorphism $p\colon UX \longrightarrow B$ in $\mathfrak{A}$, with section $d_0\colon B \longrightarrow UX$, whose kernel pair is (2.2). By Lemma 1, there is a map $d_1\colon UX \longrightarrow E$ making

$$E \underset{Ug \circ \eta E}{\overset{Uf \circ \eta_E}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} d_1 \underset{\longrightarrow}{\longrightarrow}}} UX \underset{d_0}{\overset{p}{\rightleftarrows}} B$$

a split coequalizer diagram. Then so is

$$UFE \underset{Ug}{\overset{Uf}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \eta E \circ d_1 \underset{\longrightarrow}{\longrightarrow}}} UX \underset{d_0}{\overset{p}{\rightleftarrows}} B$$

whence $\mathfrak{P}_c \subseteq \mathfrak{P}$. For the second assertion, the adjointness results in the equivalence of $\xi f = \xi g$ with $U\xi f_0 = U\xi g_0$. But the relation $\mathrm{coeq}(Uf, Ug) = p = \mathrm{coeq}(f_0, g_0)$ shows that $U\xi f_0 = U\xi g_0$ iff $U\xi Uf = U\xi Ug$, which completes the proof. ∎

Write $\mathfrak{P}_{Fc} = \mathfrak{P}_c \cap \mathfrak{P}_F$.

LEMMA 3. *Assume* $\mathfrak{A}$ *has kernel pairs of split epimorphisms. Whenever* $FX \underset{g}{\overset{f}{\rightrightarrows}} FY$ *is a pair in* $\mathfrak{P}_F$, *there is a pair* $FE \underset{g'}{\overset{f'}{\rightrightarrows}} FY$ *in* $\mathfrak{P}_{Fc}$ *satisfying* $qf = qg$ *iff* $qf' = qg'$, *for every* $\mathfrak{C}$*-morphism* $q\colon FY \longrightarrow ?$.

PROOF. Since $(f, g) \in \mathfrak{P}_F$, there are $\mathfrak{A}$-morphisms

$$UFX \underset{d_1}{\longleftarrow} UFY \underset{d_0}{\overset{p}{\rightleftarrows}} B$$

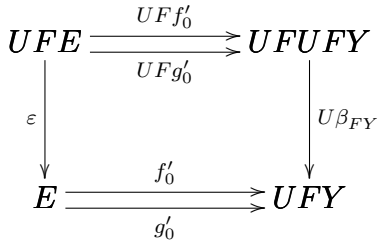
which, with (Uf, Ug), make a split coequalizer diagram in $\mathfrak{A}$. Since $Uf = U^{\mathbf{T}}\Phi f$, $Ug = U^{\mathbf{T}}\Phi g$, we see that $\Phi f, \Phi g\colon \Phi FX \rightrightarrows \Phi FY$ and d_1, p, d_0 make a $U^{\mathbf{T}}$-split coequalizer system in $\mathfrak{A}^{\mathbf{T}}$. Hence there is a $\mathbf{T}$-algebra structure $TB \longrightarrow B$ on B making $p\colon UFY \longrightarrow B$ a $\mathbf{T}$-homomorphism; letting (E, ε) be its kernel pair (possible because the kernel pair exists in $\mathfrak{A}$ by hypothesis and lifts to $\mathfrak{A}^{\mathbf{T}}$ by a property (*cf.* [Eilenberg & Moore (1965a), Prop. 5.1], [Linton (1969), Section 6], or [Manes (1967), (1.2.1)]) of $U^{\mathbf{T}}$), we obtain an $\mathfrak{A}$-object E, a pair of maps

$$E \underset{g'_0}{\overset{f'_0}{\rightrightarrows}} UFY$$

serving as a kernel pair of p, and a map $\varepsilon\colon UFE \longrightarrow E$ satisfying (at least)

$$\varepsilon \circ \eta_E = \mathrm{id}_E \tag{2.3}$$

and making the squares



serially commute. Let $f' = \beta_{FY} \circ Ff'_0$, $g' = \beta_{FY} \circ Fg'_0$. Then $Uf' \circ \eta_E = f'_0$, $Ug' \circ \eta_E = g'_0$ (by adjointness) and, since $p = \mathrm{coeq}(f'_0, g'_0)$, (2.3) shows $p = \mathrm{coeq}(Uf', Ug')$, and

$$E \longrightarrow UFE \rightrightarrows UFY$$

is its kernel pair. Thus $(f', g') \in \mathfrak{P}_{Fc}$; finally, $\xi f = \xi g$ iff $U\xi f_0 = U\xi g_0$ iff $U\xi$ factors through p iff $U\xi f'_0 = U\xi g'_0$ iff $\xi f' = \xi g'$ (by adjointness, constructions, and Lemma 2.) ∎

Corollary. *Assume $\mathfrak{A}$ has kernel pairs of split epimorphisms. Then $\mathfrak{C}$ has $\mathfrak{P}_F$-coequalizers iff it has $\mathfrak{P}_{Fc}$-coequalizers, U preserves $\mathfrak{P}_F$-coequalizers iff it preserves $\mathfrak{P}_{Fc}$-coequalizers, and U reflects $\mathfrak{P}_F$-coequalizers iff it reflects $\mathfrak{P}_{Fc}$-coequalizers.*

Proof. The inclusion $\mathfrak{P}_{Fc} \subseteq \mathfrak{P}_F$ guarantees three of the implications. For the other three, we rely on Lemma 3: given a pair $(f, g) \in \mathfrak{P}_F$, let (f', g') be a pair in $\mathfrak{P}_{Fc}$ having the property $\xi f = \xi g \iff \xi f' = \xi g'$. Then any coequalizer for (f', g') must be a coequalizer for (f, g), and conversely. Hence, if $\mathfrak{C}$ has $\mathfrak{P}_{Fc}$-coequalizers, (f', g'), and consequently (f, g), has a coequalizer. Similarly, if p is a coequalizer for (f, g) and U preserves $\mathfrak{P}_{Fc}$-coequalizers, Up is a coequalizer for (Uf', Ug'), hence a coequalizer for the kernel pair of the coequalizer of (Uf, Ug), hence a coequalizer of (Uf, Ug). Finally, if p is a map with Up a coequalizer of (Uf, Ug), Up is also a coequalizer for the kernel pair of p, hence p is a coequalizer of (f', g'), hence of (f, g). ∎

From this corollary and Theorem 1 follows

Theorem 2. *Let $\mathfrak{A}$ be a category having kernel pairs of split epimorphisms, and let $U, F, T, \Phi\colon \mathfrak{C} \longrightarrow \mathfrak{A}^{\mathsf{T}}$ be as in the basic situation. Then the following statements are equivalent:*

1) Φ is an equivalence

2) $\mathfrak{C}$ has and U preserves and reflects $\mathfrak{P}_c$-coequalizers

3) $\mathfrak{C}$ has and U preserves and reflects $\mathfrak{P}_{Fc}$-coequalizers

Indeed, the statements

4) Φ has a left adjoint

5) $\mathfrak{C}$ has $\mathfrak{P}_F$-coequalizers

6) $\mathfrak{C}$ *has* $\mathfrak{P}_{Fc}$*-coequalizers*

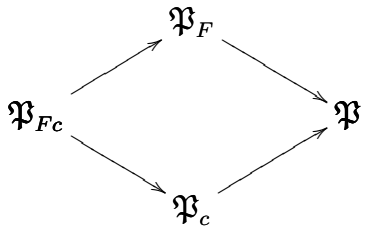
are mutually equivalent, as are

7) Φ *has a left adjoint and* $\check{\Phi}\Phi \longrightarrow \mathrm{id}_{\mathfrak{C}}$ *is an equivalence*

8) $\mathfrak{C}$ *has and* U *reflects* $\mathfrak{P}_c$*-coequalizers*

9) $\mathfrak{C}$ *has and* U *reflects* $\mathfrak{P}_{Fc}$*-coequalizers*

Proof. Apply the Corollary to Lemma 3, and the inclusions



to Theorem 1, to prove $3n-2 \Longrightarrow 3n-1 \Longrightarrow 3n \Longrightarrow 3n-2$ $(n = 1, 2, 3)$. ∎

3. When $\mathfrak{A}$ is very like {sets}.

The second variation on Theorem 1 will eventually require more stringent restrictions on $\mathfrak{A}$. As in Section 2, we do the hypothesis juggling first, imposing the restrictions on $\mathfrak{A}$ as required. We stay in the basic situation of an $\mathfrak{A}$-valued functor $U\colon \mathfrak{C} \longrightarrow \mathfrak{A}$ having a left adjoint $F\colon \mathfrak{A} \longrightarrow \mathfrak{C}$. $\mathbf{T}$ is the resulting triple, and $\Phi\colon \mathfrak{C} \longrightarrow \mathfrak{A}^{\mathbf{T}}$ the semantical comparison functor for U, as before.

Lemma 4. *Assume* $\mathfrak{A}$ *has kernel pairs of split epimorphisms and that* U *reflects* $\mathfrak{P}_c$*-coequalizers. Let* $p\colon X \longrightarrow Y$ *be a* $\mathfrak{C}$*-morphism with* Up *a split epimorphism. Then* p *is a coequalizer.*

Proof. Let $E \underset{g_0}{\overset{f_0}{\rightrightarrows}} UX$ be a kernel pair of Up. Then there is a map $\varepsilon\colon UFE \longrightarrow E$ making

$$(E, \varepsilon) \underset{g_0}{\overset{f_0}{\rightrightarrows}} \Phi X$$

a kernel pair of Φp. As in the proof of Lemma 3

$$FE \underset{g}{\overset{f}{\rightrightarrows}} X$$

(where $f = \beta_X \circ Ff_0$, $g = \beta_X \circ Fg_0$), is in $\mathfrak{P}_c$, and so, since $Up = \mathrm{coeq}(Uf, Ug)$, $p = \mathrm{coeq}(f, g)$. ∎

LEMMA 5. *Assume nothing about $\mathfrak{A}$, but only that Up epi $\Longrightarrow$ p is a coequalizer. Then U is faithful, reflects monomorphisms, and reflects isomorphisms, and $\beta_X: FUX \longrightarrow X$ is a coequalizer.*

PROOF. For a functor U with left adjoint, the implication Up epi $\Longrightarrow$ p epi guarantees (see [Eilenberg & Moore (1965), Prop. II.1.5]) U to be faithful. A faithful functor obviously reflects monomorphisms. Finally, if $U(p)$ is an isomorphism, p is a coequalizer and a monomorphism, hence an isomorphism. ∎

LEMMA 6. *Assume $\mathfrak{C}$ has kernel pairs, $\mathfrak{A}$ has coequalizers, and every epimorphism in $\mathfrak{A}$ splits. Suppose U is faithful and preserves $\mathfrak{P}_c$-coequalizers. Then, if the $\mathfrak{C}$-morphism p is a coequalizer, Up is (split) epi.*

PROOF. Let $p: X \longrightarrow Y$ be a coequalizer, let $E \underset{g_0}{\overset{f_0}{\rightrightarrows}} X$ be its kernel pair. Then p is a coequalizer of (f_0, g_0). Now (Uf_0, Ug_0) is a kernel pair of Up (since U has a left adjoint) and hence fits in a split coequalizer diagram

$$UE \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} UX \underset{\longleftarrow}{\longrightarrow} B$$

Let $f, g: FUE \longrightarrow X$ correspond by adjointness to Uf_0, Ug_0. Then $(f, g) \in \mathfrak{P}_c$ (roughly because $UE \longrightarrow UFUE \longrightarrow UE = \mathrm{id}_{UE}$) and, for any map $q: E \longrightarrow ?$, $qf_0 = qg_0$ iff $UqUf_0 = UqUg_0$ iff $qf = qg$ (using faithfulness of U and the adjointness naturality). So p is a coequalizer of $(f, g) \in \mathfrak{P}_c$, and since U preserves $\mathfrak{P}_c$-coequalizers, Up is a coequalizer, too (of Uf, Ug), hence is (split) epi.
(Remark: need only suppose $\mathfrak{A}$ has coeq of kernel pairs, not of everything.) ∎

LEMMA 7. *Assume $\mathfrak{C}$ has kernel pairs and $\mathfrak{P}_c$-coequalizers, $\mathfrak{A}$ has coequalizers, and every epimorphism in $\mathfrak{A}$ splits. Suppose U is faithful, reflects isomorphisms, and preserves $\mathfrak{P}_c$-coequalizers. Then a pair of $\mathfrak{C}$-morphisms*

$$E \underset{g}{\overset{f}{\rightrightarrows}} X$$

is a kernel pair if (and, in view of U's left adjoint, only if)

$$UE \underset{Ug}{\overset{Uf}{\rightrightarrows}} UX$$

is a kernel pair.

PROOF. Assume (Uf, Ug) is a kernel pair. Let $p: UX \longrightarrow Z$ be its coequalizer: then, since p is split epi and (Uf, Ug) is a kernel pair for p, we obtain a split coequalizer diagram

$$UE \rightrightarrows UX \overset{p}{\longrightarrow} Z$$

Now $FUE \longrightarrow E \underset{g}{\overset{f}{\rightrightarrows}} X$ is therefore a $\mathfrak{P}_c$-pair (since $UE \longrightarrow UFUE \longrightarrow UE = \mathrm{id}_{UE}$), has a coequalizer $q: X \longrightarrow Y$ in $\mathfrak{C}$, which, because of the faithfulness of U, is a coequalizer

for (f,g) too. Let $E' \underset{g'}{\overset{f'}{\rightrightarrows}} X$ be a kernel pair for q. We shall prove $E' \underset{g'}{\overset{f'}{\rightrightarrows}} X$ is isomorphic to $E \underset{g}{\overset{f}{\rightrightarrows}} X$ by using the hypothesis that U reflects isomorphisms. We have, in any case, a $\mathfrak{C}$-morphism $E \xrightarrow{e} E'$ with $f'e = f$, $g'e = g$, and the knowledge that $(Uf', Ug') =$ ker pair (Uq), $(Uf, Ug) =$ ker pair (p). Since U preserves $\mathfrak{P}_c$-coequalizers, however, $Uq = \mathrm{coeq}(Uf, Ug)$. Thus p is isomorphic with Uq, whence $Ue: UE \longrightarrow UE'$ is an isomorphism, whence e is an isomorphism, and (f,g) is a kernel pair. ∎

We can now prove one half of

THEOREM 3. *Let $\mathfrak{A}$ be a category in which every epimorphism splits, and in which kernel pairs and difference cokernels are available. Let $U: \mathfrak{C} \longrightarrow \mathfrak{A}$, F, $\mathbf{T}$, $\Phi: \mathfrak{C} \longrightarrow \mathfrak{A}^{\mathbf{T}}$ be as in the basic situation. Then Φ is an equivalence of categories if and only if*

1) *$\mathfrak{C}$ has kernel pairs and $\mathfrak{P}_c$-coequalizers*

2) *Up epi $\Longleftrightarrow$ p is a coequalizer*

3) *(f,g) is a kernel pair if (and only if) (Uf, Ug) is a kernel pair.*

PROOF. If Φ is an equivalence, Theorem 2 guarantees the $\mathfrak{P}_c$-coequalizers, and general principles guarantee the kernel pairs. Theorem 2 and Lemma 4 guarantee the implication Up epi $\Longrightarrow$ p a coequalizer. Lemma 5 applied to this implication, Theorem 2, and Lemma 6 then provide the converse implication. Statement 3 follows from Lemma 7. The converse argument is outlined in statement 1 and the parenthetical remarks in statements 2 and 3 of the following theorem, whose proof, outlined below, is entirely contained in the three lemmas in Section 4. ∎

THEOREM 4. *With the situation as in Theorem 3, suppose throughout that $\mathfrak{C}$ has $\mathfrak{P}_c$-coequalizers and kernel pairs. Then*

1) *Φ has a left adjoint.*

2) *If condition 2 of Theorem 3 holds, then U reflects $\mathfrak{P}_c$-coequalizers (whence the back adjunction $\check{\Phi}\Phi \longrightarrow \mathrm{id}_{\mathfrak{C}}$ is an equivalence) and any $\mathbf{T}$-algebra (A,α) admitting a jointly monomorphic family of maps to values of Φ is itself (isomorphic to) a value of Φ, namely $\Phi\check{\Phi}(A,\alpha)$.*

3) *If conditions 2 and 3 of Theorem 3 hold, then U preserves (and reflects) $\mathfrak{P}_c$-coequalizers (whence the front adjunction $\mathrm{id}_{\mathfrak{A}^{\mathbf{T}}} \longrightarrow \Phi\check{\Phi}$ is an equivalence too, and Φ and $\check{\Phi}$ set up an equivalence of categories).*

OUTLINE OF PROOF. Theorem 2 proves 1). Lemma 8, Theorem 2, and Lemma 9 prove 2). 2), Lemma 9, Lemma 10, and Theorem 2 prove 3). Theorem 3 obviously follows. Lemmas 8, 9, 10 are proved in Section 4. ∎

4. Proof of Theorem 4.

LEMMA 8. *With the situation as in Theorem 3, conditions 1 and 2 of Theorem 3 imply that U reflects $\mathfrak{P}_c$-coequalizers.*

PROOF. Let $FE \underset{g}{\overset{f}{\rightrightarrows}} X$ be in $\mathfrak{P}_c$ and suppose $\xi\colon X \longrightarrow X'$ is a $\mathfrak{C}$-morphism for which

$$U\xi = \mathrm{coeq}(Uf, Ug) \tag{4.1}$$

From condition 2 of Theorem 3, it follows that ξ is itself a coequalizer of something. Next, the equality $U\xi Uf = U\xi Ug$ (consequence of (4.1)), taken with the faithfulness of U (consequence of Lemma 5), shows

$$\xi f = \xi g. \tag{4.2}$$

Condition 1 of Theorem 3 permits us to take a coequalizer $p\colon X \longrightarrow Z$ of the pair (f, g). Equation (4.2) then entails a unique $\mathfrak{C}$-morphism $z\colon Z \longrightarrow X'$ satisfying

$$z \circ p = \xi. \tag{4.3}$$

Since $pf = pg$, (4.1) affords a unique $\mathfrak{A}$-morphism $z'\colon UX' \longrightarrow UZ$ satisfying

$$z' \circ U\xi = Up. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain the equations

$$Up = z' \circ U\xi = z' \circ Uz \circ Up \tag{4.5}$$

$$U\xi = Uz \circ Up = Uz \circ z' \circ U\xi \tag{4.6}$$

But Up is epi, since p is a coequalizer (using condition 2 of Theorem 3) and $U\xi$ is epi, being itself a coequalizer, so from (4.5) and (4.6) it follows that

$$z' \circ Uz = \mathrm{id}, \qquad Uz \circ z' = \mathrm{id}$$

whence Uz is an isomorphism. Another appeal to Lemma 5 demonstrates that z is an isomorphism, from $p = \mathrm{coeq}(f, g)$ to ξ, whence ξ is a coequalizer of (f, g), as needed to be shown. ∎

LEMMA 9. *With the situation as in Theorem 3, conditions 1 and 2 of Theorem 3 imply any object $X \in \mathfrak{A}^{\mathsf{T}}$ admitting a jointly monomorphic family of maps to values of Φ is itself (isomorphic to) a value of Φ, namely $\Phi\check{\Phi}X$.*

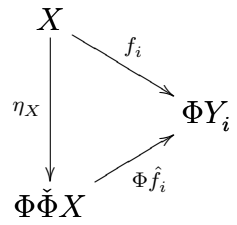
PROOF. Condition 1 of Theorem 3 and Theorem 2 guarantee a left adjoint $\check{\Phi}$ for Φ. Now, given a family of $\mathfrak{A}^{\mathbf{T}}$-morphisms

$$f_i\colon X \longrightarrow \Phi Y_i \qquad (Y_i \in |\mathfrak{C}|, X = (A,\alpha), i \in I)$$

for which the implication $f_i \circ a = f_i \circ b,\ \forall i \Longrightarrow a = b$ holds for all $\mathfrak{A}^{\mathbf{T}}$-morphisms a, b with codomain X, form the maps

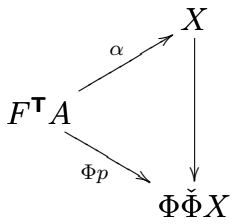
$$\hat{f}_i\colon \check{\Phi}X \longrightarrow Y_i$$

resulting by adjointness, and, applying Φ to them, consider the diagrams



where η_X is the front adjunction for the adjointness of Φ to $\check{\Phi}$. If $\eta \circ a = \eta \circ b$, then $f_i \circ a = \Phi f_i \circ \eta \circ a = \Phi f_i \circ \eta \circ b = f_i \circ b$, whence η is a monomorphism. It is a matter of indifference whether this statement is understood in $\mathfrak{A}$ or in $\mathfrak{A}^{\mathbf{T}}$, for, being faithful and having a left adjoint, $U^{\mathbf{T}}$ preserves and reflects monomorphisms. To show η is an isomorphism, as required, it thus suffices to prove $U^{\mathbf{T}}\eta$ is (split) epi, since $U^{\mathbf{T}}$ certainly reflects isomorphisms.

To do this, we must recall the construction of $\check{\Phi}X$. $\check{\Phi}X$ is the coequalizer, via some projection $p\colon FA \longrightarrow \check{\Phi}X$, of the $\mathfrak{P}_c$-pair $FE \rightrightarrows FA$ arising by adjointness from the kernel pair of $\alpha\colon UFA \longrightarrow A$. Now the coequalizer of $\Phi FE \rightrightarrows \Phi FA$ (which is $F^{\mathbf{T}}E \rightrightarrows F^{\mathbf{T}}A$) is just $\alpha\colon F^{\mathbf{T}}A \longrightarrow X = (A,\alpha)$ itself, hence there is a unique map $X = (A,\alpha) \longrightarrow \Phi\check{\Phi}X$ making the diagram



commute: that map is η_X. Since p is a coequalizer, $Up = U^{\mathbf{T}}\Phi p$ is epi; hence $U^{\mathbf{T}}\eta$ is (split) epi. This completes the proof. ∎

LEMMA 10. *With the situation as in Theorem 3, U preserves $\mathfrak{P}_c$-coequalizers if*

i) $\mathfrak{C}$ has $\mathfrak{P}_c$-coequalizers (all that's really needed is a left adjoint $\check{\Phi}$ for Φ)

ii) Up is epi if p is a coequalizer

iii) (f,g) is a kernel pair if (Uf, Ug) is a kernel pair, and

iv) the conclusion of Lemma 9 holds.

PROOF. Given a pair $FE \underset{g}{\overset{f}{\rightrightarrows}} X$ in $\mathfrak{P}_c$, and a map $p\colon X \longrightarrow Z$, coequalizer of f, g, we must show

$$Up = \mathrm{coeq}(Uf, Ug).$$

Since $E \xrightarrow[\eta_E]{} UFE \underset{Ug}{\overset{Uf}{\rightrightarrows}} X$ is the kernel pair of (Uf, Ug)'s coequalizer in $\mathfrak{A}$, there is a unique $\mathfrak{A}$-morphism $\varepsilon\colon UFE \longrightarrow E$ making the diagrams

$$\begin{array}{ccc} UFE & \xrightarrow{Uf} & UX \\ {\scriptstyle\varepsilon}\downarrow & & \uparrow{\scriptstyle Uf} \\ E & \xrightarrow[\eta_E]{} & UFE \end{array} \qquad \begin{array}{ccc} UFE & \xrightarrow{Ug} & UX \\ {\scriptstyle\varepsilon}\downarrow & & \uparrow{\scriptstyle Ug} \\ E & \xrightarrow[\eta_E]{} & UFE \end{array}$$

commute. It is left as an easy exercise to prove that (E, ε) is then a $\mathbf{T}$-algebra and that

$$(E, \varepsilon) \underset{Ug \circ \eta_E}{\overset{Uf \circ \eta_E}{\rightrightarrows}} \Phi X \tag{4.7}$$

is a jointly monomorphic pair of $\mathbf{T}$-homomorphisms. By iv), $(E, \varepsilon) \cong \Phi\check{\Phi}(E, \varepsilon)$; and there are maps

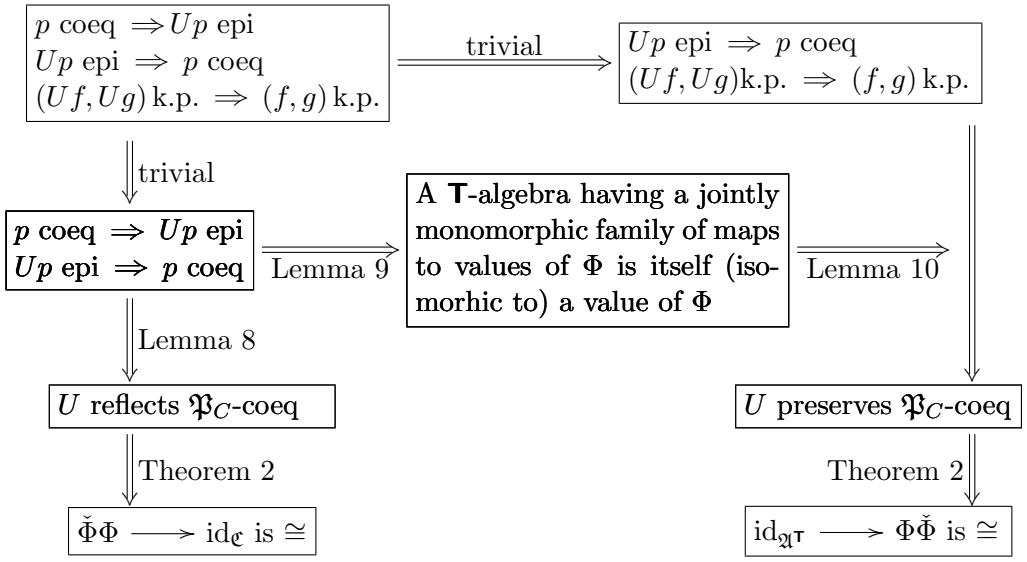
$$\check{\Phi}(E, \varepsilon) \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} X \tag{4.8}$$

corresponding, by the adjointness of Φ to $\check{\Phi}$, to (4.7). Using the adjointness relations and the definition of Φ, a $\mathfrak{C}$-morphism $q\colon X \longrightarrow Z$ satisfies $qf = qg$ iff $Uq(Uf \circ \eta_E) = Uq(Ug \circ \eta_E)$ iff $\Phi q \circ (Uf \circ \eta_E) = \Phi q \circ (Ug \circ \eta_E)$ iff $q \circ \hat{f} = q \circ \hat{g}$. Consequently, $p = \mathrm{coeq}(f, g) = \mathrm{coeq}(\hat{f}, \hat{g})$. Next, since

$$(U\hat{f}, U\hat{g}) = (U^{\mathbf{T}}\Phi\hat{f}, U^{\mathbf{T}}\Phi\hat{g}) \cong (Uf \circ \eta_E, Ug \circ \eta_E) \tag{4.9}$$

and the latter is a kernel pair (since $f, g \in \mathfrak{P}_c$), $(U\hat{f}, U\hat{g})$ is a kernel pair, too, whence, by iii), (4.8) is a kernel pair. Since p is its coequalizer, (4.8) is a kernel pair for p. It follows, since U has a left adjoint, that $(U\hat{f}, U\hat{g})$ is a kernel pair for Up. Then (4.9) shows (4.7) is a kernel pair for Up, too. On the other hand, Up is (split) epi, by ii), since p is a coequalizer. Consequently, Up is the coequalizer of its kernel pair, namely of (4.7). Finally, since (4.7) has the same coequalizer as (Uf, Ug), $Up = \mathrm{coeq}(Uf, Ug)$, as had to be shown. ∎

Schematically, the proof of Theorem 4 and the rest of Theorem 3 follows the following pattern: if $\mathfrak{A}$ has kernel pairs and coequalizers and every $\mathfrak{A}$-epimorphism splits, and if $\mathfrak{C}$ has kernel pairs and $\mathfrak{P}_c$-coequalizers, then Φ has a left adjoint $\check{\Phi}$ (by Theorem 2) and:



5. Applications.

For the first application of Theorem 3, we take $\mathfrak{A} = \mathscr{S} = \{\text{sets}\}$. Then, modulo the easily supplied information that any category tripleable over sets has all small limits and colimits, this instance of Theorem 3 is just the characterization theorem of [Linton (1966a)] for varietal categories, since, by [Linton (1969)], varietal and tripleable over $\mathscr{S}$ mean the same thing. For the second application of Theorem 3, which will also be the last to be presented here, we prove the theorem of Bunge–Gabriel.

THEOREM 5. [Bunge (1966), Gabriel (unpublished)]. *A category $\mathfrak{B}$ is equivalent to the functor category $\mathscr{S}^{\mathfrak{C}} = (\mathfrak{C}, \mathscr{S})$ of all set valued functors on a small category $\mathfrak{C}$ if and only if*

1. *$\mathfrak{B}$ has kernel pairs and coequalizers*
2. *there is a set X and a function $\psi\colon X \longrightarrow |\mathfrak{B}|$:*
   a. *$\mathfrak{B}$ contains all small coproducts of images of ψ*
   b. *$p\colon B \longrightarrow B'$ is a coequalizer if and only if $\mathfrak{B}(\psi x, p)\colon \mathfrak{B}(\psi x, B) \longrightarrow \mathfrak{B}(\psi x, B')$ is onto, $\forall x \in X$*
   c. *$E \underset{g}{\overset{f}{\rightrightarrows}} B$ is a kernel pair if and only if $\mathfrak{B}(\psi x, E) \underset{\mathfrak{B}(\psi x, g)}{\overset{\mathfrak{B}(\psi x, f)}{\rightrightarrows}} \mathfrak{B}(\psi x, B)$ is a kernel pair, $\forall x \in X$*

d. $x \in X \Longrightarrow \mathfrak{B}(\psi x, -)$ *preserves coproducts.*

Indeed, if $\mathfrak{B} \approx \mathscr{S}^{\mathfrak{C}}$, $\psi\colon X \longrightarrow |\mathfrak{B}|$ *may be taken as the object function of* $\mathfrak{C}^* \longrightarrow \mathscr{S}^{\mathfrak{C}} \xrightarrow{\approx} \mathfrak{B}$, *while if* $\psi\colon X \longrightarrow |\mathfrak{B}|$ *is given,* $\mathfrak{C}$ *may be taken as the full image of* $X \longrightarrow |\mathfrak{B}| \longrightarrow \mathfrak{B}^*$.

PROOF. We dispense with the easy part of the proof first. To begin with, suppose $\mathfrak{B} = \mathscr{S}^{\mathfrak{C}}$. Then surely condition 1. is valid. To check condition 2. where $X = |\mathfrak{C}|$ and $\psi(x) = \mathfrak{C}(x, -)$, note that

$$\mathscr{S}^{\mathfrak{C}}(\psi(x), -) = \mathrm{ev}_x\colon \mathscr{S}^{\mathfrak{C}} \longrightarrow \mathscr{S}$$

and so conditions 2b, 2c, 2d are automatic. So far as condition 2a is concerned, $\mathscr{S}^{\mathfrak{C}}$ has all small coproducts. These, however, are all properties preserved under equivalence of categories, and that finishes the "only if" part of the proof.

For the converse, view the set X as a discrete category and let

$$X \xrightarrow{\overline{\varphi}} \mathfrak{C} \xrightarrow{\underline{\varphi}} \mathfrak{B}^*$$

be the full image factorization of the composition

$$\varphi\colon X \xrightarrow{\psi} |\mathfrak{B}| \xrightarrow{\text{incl.}} \mathfrak{B}^*$$

of the function ψ given by condition 2. with the inclusion of (the discrete category) $|\mathfrak{B}|$ as the class of objects of $\mathfrak{B}^*$. Here is an outline of the argument that

$$\mathscr{S}^{\underline{\varphi}} \circ Y\colon \mathfrak{B} \longrightarrow \mathscr{S}^{(\mathfrak{B}^*)} \longrightarrow \mathscr{S}^{\mathfrak{C}}$$

is an equivalence.

Step 1. The $\mathscr{S}^X$-valued functors

$$U = \mathscr{S}^{\varphi} \circ Y\colon \mathfrak{B} \longrightarrow \mathscr{S}^{(\mathfrak{B}^*)} \longrightarrow \mathscr{S}^X$$

$$U_1 = \mathscr{S}^{\varphi}\colon \mathscr{S}^{(\mathfrak{B}^*)} \longrightarrow \mathscr{S}^X$$

$$U_2 = \mathscr{S}^{\overline{\varphi}}\colon \mathscr{S}^{\mathfrak{C}} \longrightarrow \mathscr{S}^X$$

all have left adjoints.

Step 2. If $\mathbf{T}, \mathbf{T}_1, \mathbf{T}_2$ are the triples on $\mathscr{S}^X$ and

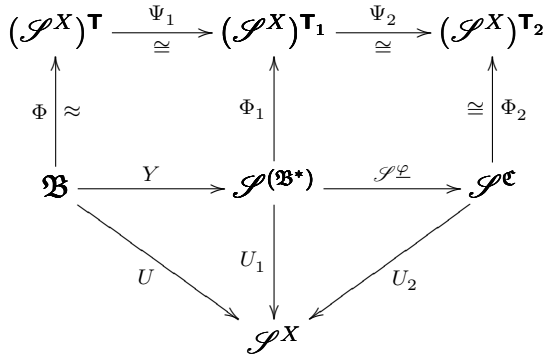
$$\Phi\colon \mathfrak{B} \longrightarrow (\mathscr{S}^X)^{\mathbf{T}}$$

$$\Phi_1\colon \mathfrak{B} \longrightarrow (\mathscr{S}^X)^{\mathbf{T}_1}$$

$$\Phi_2\colon \mathfrak{B} \longrightarrow (\mathscr{S}^X)^{\mathbf{T}_2}$$

are the semantical comparison functors arising from U, U_1 and U_2, respectively, then Φ and Φ_2 are equivalences (in fact, Φ_2 is an isomorphism!)

Step 3. The commutativity of the lower triangles in the diagram



gives triple maps $\mathbf{T}_1 \xrightarrow{\tau_1} \mathbf{T}$, $\mathbf{T}_2 \xrightarrow{\tau_2} \mathbf{T}_1$, whose semantical interpretations Ψ_1, Ψ_2 on the categories of algebras make the upper squares commute; both τ_1 and τ_2 will be shown to be isomorphisms.

It follows that Ψ_1 and Ψ_2 are isomorphisms and that

$$\mathscr{S}^{\varphi} \circ Y = \Phi^{-1} \circ \Psi_2 \circ \Psi_1 \circ \Phi \colon \mathfrak{B} \longrightarrow \mathscr{S}^{\mathfrak{C}}$$

is an equivalence (indeed, an isomorphism if (and only if) $\Phi \colon \mathfrak{B} \longrightarrow (\mathscr{S}^X)^{\mathbf{T}}$ is an isomorphism, a condition which can be expressed as an additional requirement on ψ:

2.e Given $b \in \mathfrak{B}$, sets A_x $(x \in X)$ and one-one correspondences $f_x \colon A_x \xrightarrow{\cong} \mathfrak{B}(x, b)$, there is a $\mathfrak{B}$-morphism $f \colon b' \longrightarrow b$ uniquely determined by the single requirement

$$\mathfrak{B}(x, f) = f_x$$

moreover, this f is an isomorphism.

The details regarding this refinement will be omitted, bring easy, and of little interest. See the remark following Theorem 1.)

Step 1. Using condition 2a, we produce a left adjoint F to $U = \mathscr{S}^{\varphi} \circ Y \colon \mathfrak{B} \longrightarrow \mathscr{S}^X$

$$F(G) = \bigoplus_{x \in X} Gx \cdot \psi x \qquad (G \in \mathscr{S}^X)$$

The identifications

$$\begin{aligned} \mathfrak{B}(F(G), b) &\cong \mathfrak{B}\left(\bigoplus_{x \in X} Gx \cdot \psi x, b\right) \\ &\cong \bigtimes_{x \in X} \mathfrak{B}(Gx \cdot \psi x, b) \\ &\cong \bigtimes_{x \in X} (\mathfrak{B}(\psi x, b))^{Gx} \\ &= \bigtimes_{x \in X} \mathscr{S}(Gx, U(b)(x)) \\ &= \mathscr{S}^X(G, U(b)) \end{aligned}$$

show that this works. In the same way, the fact that $\mathscr{S}^{(\mathfrak{B}^*)}$ and $\mathscr{S}^{\mathfrak{C}}$ have all small coproducts allows us to define

$$F_1(G) = \bigoplus_{x \in X} (Gx \cdot Y(\psi x))$$

$$F_2(G) = \bigoplus_{x \in X} Gx \cdot (\mathscr{S}^{\varphi} Y(\psi x))$$

and to prove, by much the same calculations, that F_1 and F_2 serve as left adjoints to U_1 and U_2.

Step 2. Conditions 1), 2), and 3) of Theorem 3 are provided precisely by conditions 1, 2b, and 2c of Theorem 5. Since $\mathfrak{A} = \mathscr{S}^X$ obviously has the properties envisioned of it in Theorem 3, the functor

$$\Phi\colon \mathfrak{B} \longrightarrow (\mathscr{S}^X)^{\mathsf{T}}$$

is an equivalence. In the same way, it is obvious that $U_2\colon \mathscr{S}^{\mathfrak{C}} \longrightarrow \mathscr{S}^X$ fulfills the hypotheses of Theorems 1 and 3 (whichever the reader prefers to think of), and so $\Phi_2\colon \mathscr{S}^{\mathfrak{C}} \longrightarrow (\mathscr{S}^X)^{\mathsf{T}_2}$ is an equivalence (in fact, an isomorphism, since U_2 creates isomorphisms).

Step 3. To see that $\tau_2\colon \mathsf{T}_2 \longrightarrow \mathsf{T}_1$ is an isomorphism, refer back to step 1, and observe that $F_2 = \mathscr{S}^{\varphi} F_1$. Indeed,

$$\mathscr{S}^{\varphi} F_1(G) = \mathscr{S}^{\varphi} \bigoplus_{x \in X} Gx \cdot Y(\psi x) = \bigoplus_{x \in X} Gx \cdot \mathscr{S}^{\varphi} Y(\psi x) = F_2(G)$$

since $\mathscr{S}^{\varphi}$ preserves coproducts. Thus

$$U_2 F_2 = U_2 \mathscr{S}^{\varphi} F_1 = U_1 F_1$$

and this identity is the triple map τ_2.

To see that $\tau_1\colon \mathsf{T}_1 \longrightarrow \mathsf{T}$ is an isomorphism, we need to invoke condition 2d, which bespeaks the fact that $U = \mathscr{S}^{\varphi} \circ Y$ preserves coproducts. Now

$$\begin{aligned} U_1 F_1(G) &= U_1 \bigoplus_{x \in X} Gx \cdot Y(\psi x) \\ &= \mathscr{S}^{\varphi} \left(\bigoplus_{x \in X} Gx \cdot Y(\psi x) \right) \\ &= \bigoplus_{x \in X} Gx \cdot \mathscr{S}^{\varphi} Y(\psi x) \end{aligned}$$

and

$$\begin{aligned} UF(G) &= U\left(\bigoplus_{x\in X} Gx\cdot\psi x\right) \\ &= \bigoplus_{x\in X} Gx\cdot U\psi x \\ &= \bigoplus_{x\in X} Gx\cdot\mathscr{S}^{\varphi}Y(\psi x) \end{aligned}$$

and it is clear that $\tau_1\colon U_1F_1\longrightarrow UF$ is this isomorphism.

This completes the proof. ∎

Coequalizers in Categories of Algebras

F. E. J. Linton [1]

Introduction

It is well known [Linton (1969), Section 6] that (inverse) limits in a category of algebras over $\mathscr{A}$—in particular, in the category $\mathscr{A}^{\mathbf{T}}$ of algebras over a triple $\mathbf{T} = (T, \eta, \mu)$ on $\mathscr{A}$—can be calculated in $\mathscr{A}$. Despite the fact that such a statement is, in general, false for colimits (direct limits), a number of colimit constructions can be carried out in $\mathscr{A}^{\mathbf{T}}$ provided they can be carried out in $\mathscr{A}$ and $\mathscr{A}^{\mathbf{T}}$ has enough coequalizers.

The coequalizers $\mathscr{A}^{\mathbf{T}}$ should have, at a minimum, are, as we shall see in Section 1, those of *reflexive pairs*: a pair

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

of maps f, g in a category $\mathscr{X}$ is *reflexive* if there is an $\mathscr{X}$-morphism

$$\Delta\colon Y \longrightarrow X$$

satisfying the identities

$$f \circ \Delta = \mathrm{id}_Y = g \circ \Delta.$$

(This terminology arises form the fact that, when $\mathscr{X} = \mathscr{S} = \{\text{sets}\}$, (f, g) is reflexive if and only if the image of the induced function

$$X \xrightarrow[f,\,g]{} Y \times Y$$

contains the diagonal of $Y \times Y$.)

In Section 2 we give two criteria for $\mathscr{A}^{\mathbf{T}}$ to have coequalizers of reflexive pairs, neither of them necessary, of course. In Section 1, it will turn out, so long as $\mathscr{A}^{\mathbf{T}}$ has such coequalizers, that each functor

$$\mathscr{A}^{\tau}\colon \mathscr{A}^{\mathbf{T}} \longrightarrow \mathscr{A}^{\mathbf{S}},$$

induced by a map of triples $\tau\colon \mathbf{S} \longrightarrow \mathbf{T}$, has a left adjoint, that $\mathscr{A}^{\mathbf{T}}$ has coproducts if $\mathscr{A}$ does, indeed, has all small colimits if $\mathscr{A}$ has coproducts, and that $\mathscr{A}^{\mathbf{T}}$ has tensor products if $\mathscr{A}$ does. These are, of course, known facts when $\mathscr{A} = \mathscr{S} = \{\text{sets}\}$; however, at the time of this writing, it is unknown, for example, whether the category of contramodules over an associative coalgebra, presented (in [Eilenberg & Moore (1965a)]) as $\mathscr{A}^{\mathbf{T}}$ with $\mathscr{A} = \{\text{ab. groups}\}$, has coequalizers of reflexive pairs.

[1]Research supported by an N.A.S.-N.R.C. Postdoctoral Research Grant while the author was at E. T. H. Zürich, on leave from Wesleyan University.

1. Constructions using coequalizers of reflexive pairs.

We begin with a lemma that will have repeated use. It concerns the following definition, which clarifies what would otherwise be a recurrent conceptual obscurity in the proofs of this section.

Let $U\colon \mathscr{X} \longrightarrow \mathscr{A}$ be a functor, let $X \in |\mathscr{X}|$, and let $(f, g) = \{(f_i, g_i) | i \in I\}$ be a family of $\mathscr{A}$-morphisms

$$A_i \underset{g_i}{\overset{f_i}{\rightrightarrows}} UX \quad (i \in I). \tag{1.1}$$

An $\mathscr{X}$-morphism $p\colon X \longrightarrow P$ is a *coequalizer* (*rel. U*) of the family of pairs (1.1) if

1) $\forall i \in I, Up \circ f_i = Up \circ g_i$, and

2) if $q\colon X \longrightarrow Y$ satisfies $Uq \circ f_i = Uq \circ g_i$ $(\forall i \in I)$, then $\exists! x\colon P \longrightarrow Y$ with $q = x \circ p$.

If $U = \mathrm{id}_{\mathscr{X}}\colon \mathscr{X} \longrightarrow \mathscr{X}$, a coequalizer (rel. U) of the family (1.1) will be called simply a *coequalizer* of (1.1).

Lemma 1. *If U has a left adjoint $F\colon \mathscr{A} \longrightarrow \mathscr{X}$ and $\bar{f}_i, \bar{g}_i\colon FA_i \longrightarrow X$ are the $\mathscr{X}$-morphisms corresponding to f_i, g_i by adjointness, then $p\colon X \longrightarrow P$ is a coequalizer (rel. U) of (f, g) if and only if it is a coequalizer of $(\bar{f}, \bar{g})$. If U is faithful and $\bar{f}_i, \bar{g}_i\colon X_i \longrightarrow X$ are $\mathscr{X}$-morphisms with $U\bar{f}_i = f_i$, $U\bar{g}_i = g_i$, then $p\colon X \longrightarrow P$ is a coequalizer (rel. U) of (f, g) if and only if it is a coequalizer of $(\bar{f}, \bar{g})$.*

Proof. In the first case, the naturality of the adjunction isomorphisms yields

$$q \cdot \bar{f}_i = q \cdot \bar{g}_i \Leftrightarrow Uq \cdot f_i = Uq \cdot g_i$$

for every $\mathscr{X}$-morphism q defined on X. In the second case, that relation follows from the faithfulness of U. Clearly, that relation is all the proof required. ∎

Proposition 1. *Let $\mathbf{S} = (S, \eta', \mu')$ and $\mathbf{T} = (T, \eta, \mu)$ be triples on $\mathscr{A}$, suppose the natural transformation $\tau\colon S \longrightarrow T$ is a map of triples from $\mathbf{S}$ to $\mathbf{T}$, and let (A, α) be an $\mathbf{S}$-algebra, (B, β) a $\mathbf{T}$-algebra, $p\colon TA \longrightarrow B$ a $\mathbf{T}$-homomorphism from (TA, μ_A) to (B, β), and $\iota = p \circ \eta_A\colon A \longrightarrow B$. Then the following statements are equivalent.*

1) p is a coequalizer of the pair

$$(TSA, \mu_{SA}) \xrightarrow{T(\tau_A)} (TTA, \mu_{TA}) \xrightarrow{\mu_A} (TA, \mu_A), \qquad (TA, \mu_A) \xrightarrow{T(\alpha)} (B, \beta)$$

2) p is a coequalizer (rel. $U^{\mathbf{T}}$) of the pair

$$SA \xrightarrow{\tau_A} TA, \qquad SA \xrightarrow{\alpha} A \xrightarrow{\eta_A} TA$$

3) ι is an $\mathbf{S}$-homomorphism $(A,\alpha) \longrightarrow (B, \beta \cdot \tau_B)$ making the composition

$$\begin{array}{c} \mathscr{A}^{\mathbf{T}}((B,\beta),X) \\ \downarrow \\ \mathscr{A}^{\mathbf{S}}(\mathscr{A}^{\tau}(B,\beta),\mathscr{A}^{\tau}X) \\ \downarrow = \\ \mathscr{A}^{\mathbf{S}}((B,\beta\cdot\tau_B),\mathscr{A}^{\tau}X) \\ \downarrow \iota \\ \mathscr{A}^{\mathbf{S}}((A,\alpha),\mathscr{A}^{\tau}X) \end{array}$$

a one-one correspondence, $\forall X \in |\mathscr{A}^{\mathbf{T}}|$.

PROOF. The equivalence of statements 1) and 2) follows from Lemma 1, since $\mu_A \cdot T(\tau_A)$ is the $\mathscr{A}^{\mathbf{T}}$-morphism corresponding to τ_A by adjointness and $\eta_A \cdot \alpha = T(\alpha) \cdot \eta_{SA}$ is the $\mathscr{A}$-morphism corresponding to $T(\alpha)$ by adjointness.

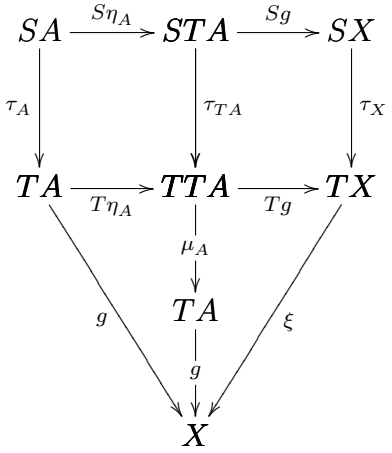
Next, if $g\colon TA \longrightarrow X$ is a $\mathbf{T}$-homomorphism from (TA, μ_A) to a $\mathbf{T}$-algebra (X,ξ), having equal compositions with τ_A and $\eta_A \cdot \alpha$, we show that $g \cdot \eta_A\colon A \longrightarrow X$ is an $\mathbf{S}$-morphism from (A,α) to $\mathscr{A}^{\tau}(X,\xi) = (X, \xi \cdot \tau_X)$, *i.e.*, that

$$g \cdot \eta_A \cdot \alpha = \xi \cdot \tau_X \cdot S(g \cdot \eta_A).$$

Clearly this requires only the proof of

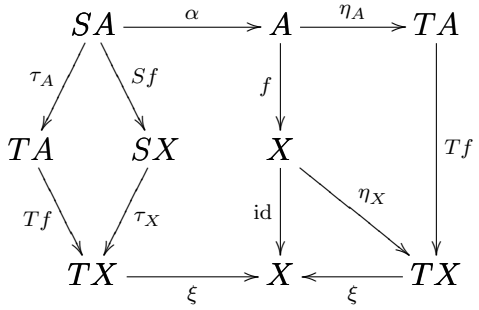
$$g \cdot \tau_A = \xi \cdot \tau_X \cdot Sg \cdot S\eta_A,$$

for which, consider the diagram



The upper squares commute because τ is natural, the left hand triangle, because $\mu_A \cdot T\eta_A = \mathrm{id}_{TA}$, the right hand triangle, because g is a $\mathbf{T}$-homomorphism.

Finally, given an $\mathbf{S}$-homomorphism $f\colon A \longrightarrow X$ from (A,α) to $\mathscr{A}^{\tau}(X,\xi) = (X, \xi\cdot\tau_X)$, it turns out that $\xi\cdot Tf$ is a $\mathbf{T}$-homomorphism $(TA,\mu_A)\longrightarrow(X,\xi)$ having equal compositions with τ_A and $\eta\cdot\alpha$. For, the diagram



commutes, since τ is natural, f is an $\mathbf{S}$-homomorphism, η is natural, and $\xi\cdot\eta_X = \mathrm{id}_X$.

These arguments form the core of a proof of Proposition 1. ∎

Corollary 1. *If $\mathscr{A}^{\mathbf{T}}$ has coequalizers of reflexive pairs, then each functor $\mathscr{A}^{\tau}\colon \mathscr{A}^{\mathbf{T}}\longrightarrow\mathscr{A}^{\mathbf{S}}$, induced by a triple map $\tau\colon \mathbf{S}\longrightarrow\mathbf{T}$, has a left adjoint $\hat{\tau}$.*

Proof. For each $(A,\alpha)\in|\mathscr{A}^{\mathbf{S}}|$, the pair

$$F^{\mathbf{T}}SA \xrightarrow{F^{\mathbf{T}}(\tau_A)} F^{\mathbf{T}}TA \xrightarrow{\mu_A} F^{\mathbf{T}}A, \qquad F^{\mathbf{T}}SA \xrightarrow[F^{\mathbf{T}}(\alpha)]{} F^{\mathbf{T}}A$$

whose coequalizer, if any, is (by Proposition 1) the value $\hat{\tau}(A,\alpha)$ of $\hat{\tau}$ at (A,α), is reflexive by virtue of

$$\Delta = F^{\mathbf{T}}(\eta'_A).$$ ∎

Proposition 2. *Let (A_i,α_i) $(i\in I)$ be a family of $\mathbf{T}$-algebras, and assume the coproduct $\oplus_{i\in I}A_i$ exists in $\mathscr{A}$, say with injections $j_i\colon A_i\longrightarrow\oplus A_i$. Let $p\colon T(\oplus A_i)\longrightarrow P$ be a $\mathbf{T}$-homomorphism. Then the following statements are equivalent.*

1) p is a coequalizer (rel. $U^{\mathbf{T}}$) of the family of pairs

$$TA_i \xrightarrow{T(j_i)} T(\oplus A_i), \qquad TA_i \xrightarrow{\alpha_i} A_i \xrightarrow{j_i} \oplus A_i \xrightarrow{\eta_{\oplus A_i}} T(\oplus A_i) \qquad (i\in I)$$

2) each map $h_i = p\cdot\eta_{\oplus A_i}\cdot j_i\colon A_i\longrightarrow P$ is a $\mathbf{T}$-homomorphism and the family $(h_i)_{i\in I}$ serves to make P the coproduct in $\mathscr{A}^{\mathbf{T}}$ of $(A_i)_{i\in I}$.

Moreover, if $\oplus TA_i$ is available in $\mathscr{A}$, statements 1) and 2) are equivalent to each of the following statements about p:

3) p is a coequalizer (rel. $U^{\mathbf{T}}$) of the pair

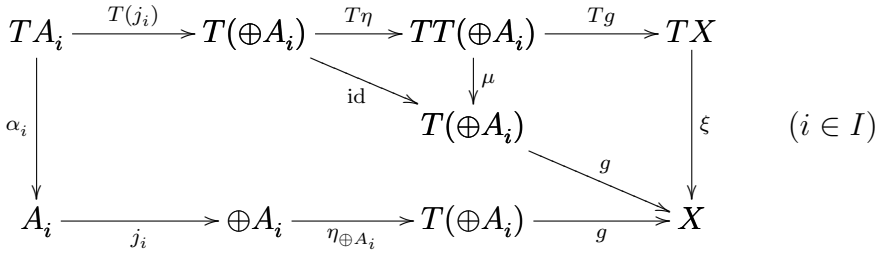
$$\oplus TA_i \underset{\oplus\alpha_i}{\overset{(\cdots T(j_i)\cdots)}{\rightrightarrows}} T(\oplus A_i) \quad \eta_{\oplus A_i}$$

4) p is a coequalizer of the pair

$$(T(\oplus TA_i),\mu) \underset{T(\oplus\alpha_i)}{\overset{T(\cdots T(j_i)\cdots)}{\rightrightarrows}} (TT(\oplus A_i),\mu) \xrightarrow{\ \mu\ } (T(\oplus A_i),\mu)$$

Proof. The equivalence of statements 1) and 3) is obvious. The equivalence of 3) with 4) is due to Lemma 1, since the top (bottom) maps correspond to each other by adjointness.

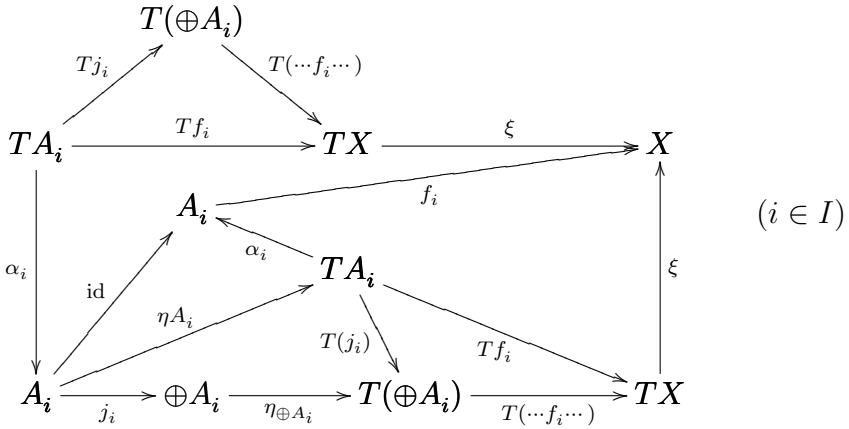
Next, let $g\colon T(\oplus A_i) \longrightarrow X$ be a $\mathbf{T}$-homomorphism from $F^{\mathbf{T}}(\oplus A_i)$ to (X,ξ), having equal compositions with both components of all the pairs in 1). Then $g\cdot\eta_{\oplus A_i}\cdot j_i\colon A_i \longrightarrow X$ is a $\mathbf{T}$-homomorphism $(A_i,\alpha_i)\longrightarrow(X,\xi)$, for all i, as is shown by the commutativity of the diagrams



Finally, if $f_i\colon A_i \longrightarrow X$ is a family of $\mathbf{T}$-homomorphisms $(A_i,\alpha_i)\longrightarrow(X,\xi)$, then the map

$$g = \xi\cdot T(\cdots f_i\cdots)\colon T(\oplus A_i) \to TX \xrightarrow{\ \xi\ } X$$

is a $\mathbf{T}$-homomorphism (the only one) having $g\cdot\eta_{\oplus A_i}\cdot j_i = f_i$, and, as the diagram below shows, has equal compositions with both members of all of the pairs in 1).



This essentially concludes the proof of Proposition 2. ∎

COROLLARY 2. *If $\mathscr{A}^{\mathbf{T}}$ has coequalizers of reflexive pairs, and if $\mathscr{A}$ has all small coproducts, then $\mathscr{A}^{\mathbf{T}}$ has all small colimits (direct limits).*

PROOF. In the first place, $\mathscr{A}^{\mathbf{T}}$ has small coproducts, because, given $(A_i, \alpha_i) \in |\mathscr{A}^{\mathbf{T}}|$ $(i \in I)$, the pair

$$F^{\mathbf{T}}(\oplus TA_i) \underset{F^{\mathbf{T}}(\oplus \alpha_i)}{\overset{F^{\mathbf{T}}(\cdots T(j_i)\cdots)}{\rightrightarrows}} F^{\mathbf{T}}T(\oplus A_i) \xrightarrow{\ \mu\ } F^{\mathbf{T}}(\oplus A_i)$$

whose coequalizer, according to Proposition 2, serves as coproduct in $\mathscr{A}^{\mathbf{T}}$ of the family $\{(A_i, \alpha_i) | i \in I\}$, is reflexive by virtue of

$$\Delta = F^{\mathbf{T}}(\oplus \eta_{A_i}).$$

But then, having coproducts and coequalizers of reflexive pairs, $\mathscr{A}^{\mathbf{T}}$ has all small colimits. Indeed, the pair

$$\bigoplus_{\delta \in |\mathscr{D}^{\mathbf{2}}|} D_{\operatorname{dom}\delta} \underset{(\cdots j_{\operatorname{dom}\delta} \cdots)}{\overset{(\cdots j_{\operatorname{cod}\delta} \cdot D_\delta \cdots)_{\delta \in |\mathscr{D}^{\mathbf{2}}|}}{\rightrightarrows}} \bigoplus_{i \in |\mathscr{D}|} D_i,$$

whose coequalizer is well known to serve as colimit of the functor $D\colon \mathscr{D} \longrightarrow ?$, is reflexive by virtue of

$$\Delta = (\cdots j_{\mathrm{id}_i} \cdots)_{i \in |\mathscr{D}|}.$$

REMARK. If $\mathscr{A}$ is a monoidal category [Eilenberg & Kelly (1966)] and $\mathbf{T} = (T, \eta, \mu)$ is a suitable triple (meaning at least that $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ is a monoidal functor [Eilenberg & Kelly (1966)], so that there are maps $\widetilde{T}\colon TA \otimes TB \longrightarrow T(A \otimes B)$ subject to conditions, and η is a monoidal natural transformation, as should probably be μ), then, given $\mathbf{T}$-algebras (A, α), (B, β), a coequalizer (rel. $U^{\mathbf{T}}$) of the pair

$$T(TA \otimes TB) \underset{T(\alpha \otimes \beta)}{\overset{T(\widetilde{T})}{\rightrightarrows}} TT(A \otimes B) \xrightarrow{\ \mu\ } T(A \otimes B)$$

which is reflexive by virtue of

$$\Delta = T(\eta_A \otimes \eta_B)\ ,$$

serves equally well as a coequalizer (rel. $U^{\mathbf{T}}$) of the pair

$$TA \otimes TB \underset{\alpha \otimes \beta}{\overset{\widetilde{T}}{\rightrightarrows}} A \otimes B \xrightarrow{\ \eta\ } T(A \otimes B)$$

and, if $\mathscr{A}$ is closed monoidal [Eilenberg & Kelly (1966)], can be interpreted (in terms of "bilinear maps") as a tensor product, in $\mathscr{A}^{\mathbf{T}}$, of (A, α) and (B, β). Such phenomena hope to be treated in detail elsewhere.

2. Criteria for the existence of such coequalizers.

In view of Section 1, it behooves us to find workable sufficient conditions, on $\mathscr{A}$, on $\mathbf{T}$, or on both, that $\mathscr{A}^{\mathbf{T}}$ have coequalizers of reflexive pairs. The first such condition, though rather special, depends on knowing when coequalizers in $\mathscr{A}^{\mathbf{T}}$ can be calculated in $\mathscr{A}$.

PROPOSITION 3. *Let $\mathbf{T} = (T, \eta, \mu)$ be a triple in $\mathscr{A}$, and let*

$$(A, \alpha) \underset{g}{\overset{f}{\rightrightarrows}} (B, \beta)$$

be a pair of $\mathscr{A}^{\mathbf{T}}$-morphisms. Assume

1) *There is an $\mathscr{A}$-morphism $p\colon B \longrightarrow C$ which is a coequalizer (in $\mathscr{A}$) of $A \underset{g}{\overset{f}{\rightrightarrows}} B$;*

2) *Tp is a coequalizer of (Tf, Tg);*

3) *TTp is epic.*

Then: there is a map $\gamma\colon TC \longrightarrow C$, uniquely determined by the single requirement that

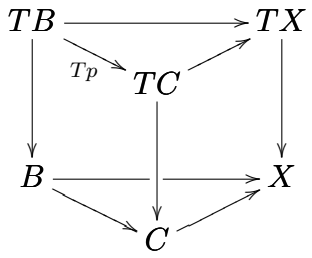
$$\begin{array}{ccc} TB & \xrightarrow{Tp} & TC \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ B & \xrightarrow[p]{} & C \end{array}$$

commute; (C, γ) is a $\mathbf{T}$-algebra; and $p\colon (B, \beta) \longrightarrow (C, \gamma)$ is a coequalizer in $\mathscr{A}^{\mathbf{T}}$ of (f, g).

PROOF. The equations $p \circ \beta \circ Tf = p \circ f \circ \alpha = p \circ g \circ \alpha = p \circ \beta \circ Tg$, occurring because of assumption 1, force, because of assumption 2, a unique $\gamma\colon TC \longrightarrow C$ with $\gamma Tp = p\beta$. Then $\gamma \circ \eta_C \circ p = \gamma \circ Tp \circ \eta_B = p \circ \beta \circ \eta_B = p$, but p is epic (by 1) and so $\gamma \circ \eta_C = \mathrm{id}_C$. Similarly, using assumption 3, the equation $\gamma \circ T\gamma = \gamma \circ \mu_C$ follows from the calculation

$$\begin{aligned} \gamma \circ T\gamma \circ TTp &= \gamma \circ Tp \circ T\beta = p \circ \beta \circ T\beta \\ &= p \circ \beta \circ \mu_B = \gamma \circ Tp \circ \mu_B \\ &= \gamma \circ \mu_C \circ TTp\ . \end{aligned}$$

Finally, if $q\colon B \longrightarrow X$ is an $\mathscr{A}^{\mathbf{T}}$ morphism from (B, β) to (X, ξ) factoring through C, the factorization must be an $\mathscr{A}^{\mathbf{T}}$-morphism, because the diagram



commutes everywhere else, and Tp is epic, by assumption 2. ∎

COROLLARY 3. *If $\mathscr{A}$ has and T preserves coequalizers of reflexive pairs, then $\mathscr{A}^{\mathsf{T}}$ has coequalizers of reflexive pairs, and Corollaries 1 and 2 apply.*

EXAMPLES.

1. If T is an adjoint triple, then T preserves all coequalizers (and all colimits, in fact). Samples of such triples:

   a) $\mathsf{T} = (\text{-} \otimes \Lambda, \mathrm{id} \otimes u, \mathrm{id} \otimes m)$ where the ground category $\mathscr{A}$ is $\{k\text{-modules}\}$ (k comm.) and Λ is an associative k-algebra with unit $u\colon k \longrightarrow \Lambda$ and multiplication $m\colon \Lambda \otimes \Lambda \longrightarrow \Lambda$ ($\otimes = \otimes_k$). $\mathscr{A}^{\mathsf{T}} = \Lambda\text{-modules}$.

   b) $\mathsf{T} = (\text{-} \times \mathbf{2}, \mathrm{id} \times (\mathbf{1} \xrightarrow{0} \mathbf{2}), \mathrm{id} \times (\mathbf{2} \times \mathbf{2} \xrightarrow{\max} \mathbf{2}))$, where the ground category $\mathscr{A}$ is $\mathbf{Cat}$ and $\mathbf{2}$ is the p.o. set $0 \longrightarrow 1$. $\mathbf{Cat}^{\mathsf{T}} = \{\text{categories with idempotent triples}\}$. Where S is constructed like T, replacing $\mathbf{2}$ by the category Δ given by

   $$\begin{aligned} |\Delta| &= \{0, 1, 2, \ldots, n, \ldots\} \\ \Delta(n, k) &= \text{order preserving maps } \{0 \ldots n-1\} \longrightarrow \{0 \ldots k-1\} \end{aligned}$$

   with the obvious composition, $0\colon \mathbf{1} \longrightarrow \Delta$ the inclusion of the object 0, $m\colon \Delta \times \Delta \longrightarrow \Delta$ the functor given by

   $$n,\ n' \longmapsto n + n'$$

   $$n \xrightarrow{f} k,\ n' \xrightarrow{f'} k' \longmapsto \begin{array}{ccc} 0 & \xrightarrow{\ f\ } & 0 \\ \vdots & & \vdots \\ n-1 & & k-1 \end{array}$$

   $$\begin{array}{ccc} n & \xrightarrow{\ k+f'(\text{-}-n)\ } & k \\ \vdots & & \vdots \\ n+n'-1 & & n+k'-1 \end{array},$$

   $\mathbf{Cat}^{\mathsf{S}}$ is $\{\text{categories equipped with a triple}\}$. Define $\tau\colon \mathsf{S} \longrightarrow \mathsf{T}$ by crossing with the only functor $\Delta \longrightarrow \mathbf{2}$ sending $n \neq 0$ to 1 and 0 to 0. Then $\mathbf{Cat}^{\tau}\colon \mathbf{Cat}^{\mathsf{T}} \longrightarrow \mathbf{Cat}^{\mathsf{S}}$ is the functor interpreting an idempotent triple as a triple on the same category. These constructions and observations are all due to Lawvere. Since $\mathbf{Cat}$ has coequalizers and T is an adjoint triple, $\mathbf{Cat}^{\tau}$ has a left adjoint, by Corollary 1; roughly speaking, it assigns to a triple in a category, a best idempotent triple on an as closely related other category as possible.

2. Let $\mathscr{A}$ be an additive category, let $m\colon G \times G \longrightarrow G$ be an $\mathscr{A}$-morphism satisfying $m(m \times G) = m(G \times m)$, $m(\mathrm{id}, 0) = \mathrm{id} = m(0, \mathrm{id})$. Define a triple $\mathsf{T} = (\text{-} \times G, (\mathrm{id}, 0), (\mathrm{id} \times m))$ on $\mathscr{A}$. Then T preserves all coequalizers because $A \times G = A \oplus G$.

3. Any functor preserves split coequalizer systems [Linton (1969a)]. In particular every triple does, and so Proposition 3 guarantees that coequalizers of $U^{\mathbf{T}}$-split pairs of $\mathscr{A}^{\mathbf{T}}$-morphisms can be computed in $\mathscr{A}$, as was stated in greater generality in [Linton (1969), Section 6].

The other criterion involves images. We treat images axiomatically, in a manner suggestive of (and perhaps equivalent to) bicategories. Recall that **1**, **2** and **3** are the categories depicted as the partially ordered sets

$$\begin{aligned} \mathbf{1} &= \{0\}, \\ \mathbf{2} &= \{0 \longrightarrow 1\}, \\ \mathbf{3} &= \left\{ 0 \underset{c=b\circ a}{\overset{a}{\nearrow} 1 \overset{b}{\searrow} 2} \right\}. \end{aligned}$$

We will need the functors $\mathbf{2} \xrightarrow{c} \mathbf{3}$ and $\mathbf{1} \xrightarrow{1} \mathbf{3}$ (whose values serve as their names). These induce functors $\mathscr{A}^c\colon \mathscr{A}^{\mathbf{3}} \longrightarrow \mathscr{A}^{\mathbf{2}}$ and $\mathscr{A}^1\colon \mathscr{A}^{\mathbf{3}} \longrightarrow \mathscr{A}^{\mathbf{1}} \cong \mathscr{A}$, for any category $\mathscr{A}$.

By an *image factorization functor* for the category $\mathscr{A}$, we mean a functor

$$\mathscr{I}\colon \mathscr{A}^{\mathbf{2}} \longrightarrow \mathscr{A}^{\mathbf{3}},$$

having the property

1) $\mathscr{A}^{\mathbf{2}} \xrightarrow{\mathscr{I}} \mathscr{A}^{\mathbf{3}} \xrightarrow{\mathscr{A}^c} \mathscr{A}^{\mathbf{2}} =$ identity on $\mathscr{A}^{\mathbf{2}}$, and three more properties which we state using the notations

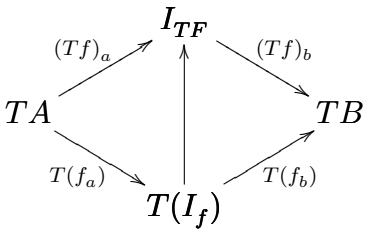
$$\begin{aligned} \mathscr{A}^1(\mathscr{I}(f)) &= I_f \\ \mathscr{I} f &= \circ \xrightarrow{f_a} I_f \xrightarrow{f_b} \circ\colon \end{aligned}$$

2) $f \in |\mathscr{A}^{\mathbf{2}}| \implies f_a$ is an epimorphism,

3) $f \in |\mathscr{A}^{\mathbf{2}}| \implies f_b$ is an monomorphism,

4) $f \in |\mathscr{A}^{\mathbf{2}}| \implies (f_b)_a$ and $(f_a)_b$ are isomorphisms.

A functor T *preserves $\mathscr{I}$-images* if there is a natural equivalence, whose composition with $\mathscr{A}^c$ is the identity, between $T^{\mathbf{3}} \circ \mathscr{I}$ and $\mathscr{I} \circ T^{\mathbf{2}}$. This entails, for each $f \in \mathscr{A}(A, B)$, an isomorphism $\iota_f\colon T(I_f) \longrightarrow I_{Tf}$ making the triangle



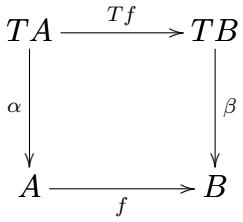
commute.

A triple $\mathbf{T} = (T, \eta, \mu)$ on $\mathscr{A}$ preserves $\mathscr{I}$-images if the functor T does.

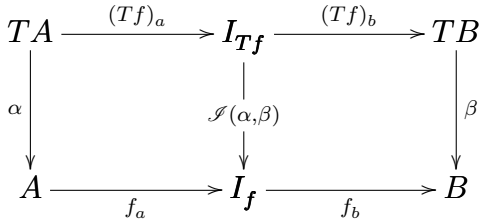
LEMMA 2. *If $\mathscr{I}\colon \mathscr{A}^{\mathbf{2}} \longrightarrow \mathscr{A}^{\mathbf{3}}$ is an image factorization functor for $\mathscr{A}$ and $\mathbf{T}$ is a triple that preserves $\mathscr{I}$-images, then there is one and only one image factorization functor $\mathscr{I}^{\mathbf{T}}$ for $\mathscr{A}^{\mathbf{T}}$ with the property*

$$(U^{\mathbf{T}})^{\mathbf{3}} \circ \mathscr{I}^{\mathbf{T}} = \mathscr{I}.$$

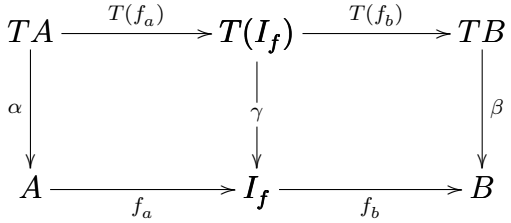
PROOF. Given $f\colon A \longrightarrow B$, an $\mathscr{A}^{\mathbf{T}}$-morphism from (A, α) to (B, β), the commutativity of the square



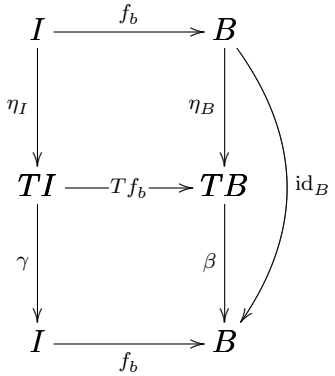
yields a commutative diagram



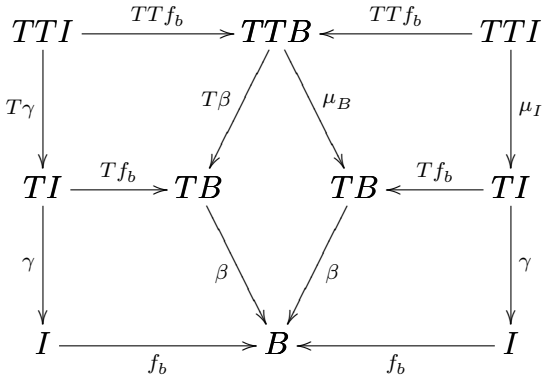
Combining this with the commutative diagram arising from the definition of "T preserves $\mathscr{I}$-images", we obtain a map $\gamma = \mathscr{I}(\alpha, \beta) \circ \iota_f\colon T(I_f) \longrightarrow I_f$ making the diagram



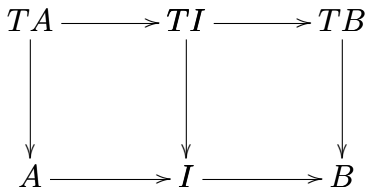
commute. There is only one such map γ because $T(f_a)$ is epic and f_b is monic. We show that (I_f, γ) is a $\mathbf{T}$-algebra. Write simply $I = I_f$. $\gamma \circ \eta_I = \mathrm{id}_I$ follows from the commutativity of



and the fact that f_b is monic. $\gamma \circ T\gamma = \gamma \circ \mu_I$ follows from the commutativity of



and the fact that f_b is monic. Since $(U^{\mathbf{T}})^{\mathbf{3}} \circ \mathscr{I}^{\mathbf{T}} = \mathscr{I}$, the axioms $\mathscr{I}^{\mathbf{T}}$ must satisfy, to be an image factorization functor, are easily verified. The uniqueness is taken care of, essentially, by the obvious uniqueness of $\gamma\colon TI \longrightarrow I$, subject to the commutativity relations expressed in the diagram



■

PROPOSITION 4. *Let $\mathscr{I}$ be an image factorization functor for $\mathscr{A}$, and let $\mathbf{T}$ be a triple on $\mathscr{A}$ preserving $\mathscr{I}$-images. Assume $\mathscr{A}$ has small products and is co-well-powered (or even just that the isomorphism classes of each class*

$$\mathscr{I}\text{-epi}(A) = \{f | f\colon A \longrightarrow B, f_b\colon I_f \xrightarrow{\cong} B, B \in |\mathscr{A}|\}$$

constitute a set (isomorphisms that are the identity on $\mathscr{A}$, of course)). Then $\mathscr{A}^{\mathbf{T}}$ has all coequalizers.

PROOF. Given a pair $(E, \varepsilon) \underset{g}{\overset{f}{\rightrightarrows}} (A, \alpha)$ of $\mathscr{A}^{\mathbf{T}}$-morphisms, let

$$\mathscr{E}_{f,g} = \{h | h \in |\mathscr{A}^{\mathbf{2}}|, h\colon (A, \alpha) \longrightarrow (X, \xi), hf = hg, h = h_a\}.$$

Observe that an isomorphism class of $\mathscr{E}_{f,g}$ in the sense of $\mathscr{A}^{\mathbf{T}}$ or in the sense of $\mathscr{A}$ is the same thing, because $\mathbf{T}$ preserves $\mathscr{I}$-images, and the maps h, Th are epic. Pick representatives of the isomorphism classes of $\mathscr{E}_{f,g}$, say

$$h_i\colon (A, \alpha) \longrightarrow (X_i, \xi_i) \qquad (i \in I)$$

and form the induced map (an $\mathscr{A}^{\mathbf{T}}$-morphism by [Linton (1969), Section 6])

$$k = \langle \cdots h_i \cdots \rangle\colon (A, \alpha) \longrightarrow (\Pi_i X_i, \Pi_i \xi_i).$$

Then k_a is a coequalizer of (f,g) in $\mathscr{A}^{\mathbf{T}}$. Indeed, given $h\colon (A,\alpha)\longrightarrow(Z,\zeta)$ with $hf=hg$, we have $h_a\in\mathscr{E}_{f,g}$ and so $\exists i_0\in I$ with $h_a\simeq h_{i_0}$. Then the composition

$$(I^{\mathbf{T}})_k \xrightarrow{k_b} (\Pi_i X_i, \Pi_i\xi_i) \xrightarrow{pr_{i_0}} (X_{i_0},\xi_{i_0}) \xrightarrow{\cong} (I^{\mathbf{T}})_h \xrightarrow{h_b} (Z,\zeta)$$

makes the triangle

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{k_a} & (I^{\mathbf{T}})_k \\ & \searrow^{h} & \downarrow \\ & & (Z,\zeta) \end{array}$$

commute, and since k_a is epic, it is the only such map. That $k_a\in\mathscr{E}_{f,g}$ is obvious, and completes the proof of the existence of coequalizers. ∎

Remark. Much the same arguments prove, under the same hypotheses, that $\mathscr{A}^{\mathbf{T}}$ has coequalizers of families of pairs of maps.

Example. $\mathscr{A}=\mathscr{S}$, $\mathscr{I}=$ usual epic-monic factorization. Then any triple preserves $\mathscr{I}$-images (proof below), and consequently $\mathscr{S}^{\mathbf{T}}$ has coequalizers, and, by virtue of Corollary 2, all colimits. That $\mathscr{A}^{\mathbf{T}}$ has all limits if $\mathscr{A}$ has is well known, and this then takes care of the completeness properties of varietal categories.

To see that every triple in $\mathscr{S}$ preserves images, it suffices to see that every triple in $\mathscr{S}$ preserves monomorphisms since the usual epic-monic factorization is determined to within isomorphism by the requirement that it be an epic-monic factorization, and every functor preserves epimorphisms, since they split. So let $\mathbf{T}=(T,\eta,\mu)$ be a triple in $\mathscr{S}$. The only monomorphisms f that T has a chance of not preserving are those that are not split, *i.e.*, those with empty domain. Now if $T(\emptyset)$ is $\emptyset$, Tf is surely monic. But if $T\emptyset$ has at least one element, Tf, which may be thought of as a $\mathbf{T}$-morphism from $F^{\mathbf{T}}(\emptyset)$ to $F^{\mathbf{T}}(n)$, admits a retraction, namely the extension to a $\mathbf{T}$-homomorphismof any function

$$n\longrightarrow U^{\mathbf{T}}F^{\mathbf{T}}\emptyset = T\emptyset.$$

That the composition on $F^{\mathbf{T}}(\emptyset)$ is the identity is due to the fact that $F^{\mathbf{T}}(\emptyset)$ is a left zero (is initial, is a copoint) in $\mathscr{S}^{\mathbf{T}}$.

A Triple Theoretic Construction of Compact Algebras

Ernest Manes [1]

Let $\mathbf{T}$ be a triple in the category of sets. Using the Yoneda Lemma, it is possible to reinterpret $\mathbf{T}$-algebras in the classical way as sets with (not necessarily finitary) operations; (the "equations" are built into $\mathbf{T}$ and need not be mentioned). The objects in the *category of compact* $\mathbf{T}$*-algebras* are defined to be sets provided with $\mathbf{T}$-algebra structure and compact T2 topology in such a way that $\mathbf{T}$-operations are continuous, whereas the morphisms are defined to be continuous $\mathbf{T}$-homomorphisms. The end result of this paper is the proof that "compact $\mathbf{T}$-algebras" is itself the category of algebras over a triple in the category of sets; that is, a compact $\mathbf{T}$-algebra—a compact T2 space in particular—is an example of a set with algebraic structure. When $\mathbf{T}$-algebras = G-sets, compact $\mathbf{T}$-algebras = compact topological dynamics with discrete phase group G. The general case of compact topological dynamics, when G is a (not necessarily compact) topological group, is also algebraic, indeed is a Birkhoff subcategory (= variety) of the discrete case. (For more on the interplay between compact topological dynamics and universal algebra see [Manes (1967), Sections 2.4, 2.5]). This motivates our general study of Birkhoff subcategories in Section 3. Otherwise, the paper pursues a suitably geodesic course to our main result 7.1, so long as "suitable" means "intended to convince the reader with little background in triple theoretic methods".

1. Preliminaries.

We assume the reader is conversant with elementary category theory at the level of, say, the first five chapters of [Mitchell (1965)]. Most of the main prerequisites are listed in this section.

1.1 Miscellaneous preliminaries. If f, g are morphisms in a category we compose first on the left so that fg (which we also write $f \cdot g$) $= \xrightarrow{f} \xrightarrow{g}$. We use "$=_{\text{df}}$" for "is defined to be" and "$=_{\text{dn}}$" for "is denoted to be". We write "$\rightarrowtail$" (resp. "$\twoheadrightarrow$") to assert that the morphism f is mono (resp., epi); "mono" and "epi" are defined below in 1.2. A function f is bijective $=_{\text{df}}$ f is 1-to-1 and onto. If f is a function and if x is an element of the domain of f, we write "xf" or "$\langle x, f\rangle$" for the element of Y that f assigns to x. "End of proof" $=_{\text{dn}}\blacksquare$.

Let $\mathscr{K}$ be a category. $|\mathscr{K}|$ or obj$\mathscr{K}$ $=_{\text{dn}}$ the class of $\mathscr{K}$-objects. For $X \in \text{obj}\mathscr{K}$, 1_X or $X \xrightarrow{1} X =_{\text{dn}}$ the identity morphism of X. $\mathscr{S} =_{\text{df}}$ the category of sets and functions. $\mathscr{K}$ is legitimate $=_{\text{df}}$ for all $X, Y \in \text{obj}\mathscr{K}$ the class $(X, Y)\mathscr{K}$ of $\mathscr{K}$-morphisms from X to Y is a set. A class $\mathscr{F}$ of $\mathscr{K}$-morphisms has a representative set $=_{\text{df}}$ there exists a set

[1] Virtually all of this paper appears in the author's thesis [Manes (1967)]. Many of the ideas were developed in conversations with Jon Beck and F. E. J. Linton to whom the author is grateful.

$\mathscr{R}$ of $\mathscr{K}$-morphisms such that for every $X \xrightarrow{f} X \in \mathscr{F}$ there exists $A \xrightarrow{r} B \in \mathscr{R}$ and $\mathscr{K}$-isomorphisms $X \xrightarrow{a} A$, $Y \xrightarrow{\beta} B$ such that $f\beta = ar$.

1.2 Monos and epis.

Definition 1.2.1. *Let $A \xrightarrow{f} B \in \mathscr{K}$. f is a* split epi *if there exists $B \xrightarrow{\widetilde{f}} A \in \mathscr{K}$ with $\widetilde{f}f = 1_B$. f is a* coequalizer *if there exist $g, h \in \mathscr{K}$ with $f = \mathrm{coeq}(g, h)$. Define*

$$\mathrm{reg}(f) =_{\mathrm{df}} \{A \xrightarrow{g} Y \in \mathscr{K} : \textit{for every } (a,b)\colon X \longrightarrow A, af = bf \textit{ implies } ag = bg\}$$

Then f is a regular epi *if for every $g \in \mathrm{reg}(f)$ there exists a unique $\widetilde{g} \in \mathscr{K}$ with $f\widetilde{g} = g$. f is* epi *if for every $(a,b)\colon B \longrightarrow X$ in $\mathscr{K}$, $fa = fb$ implies $a = b$.*[a] *Dually, we have* split mono*,* equalizer*,* regular mono*,* mono*.*

Proposition 1.2.2. *Let $A \xrightarrow{f} B \in \mathscr{K}$. Then f split epi implies f coequalizer implies f regular epi implies f epi.*

Proof. If $\widetilde{f}f = 1_B$, $f = \mathrm{coeq}(1_A, f\widetilde{f})$. If $f = \mathrm{coeq}(a, b)$ then for every $g \in \mathrm{reg}(f)$ we have $ag = bg$ so that the coequalizer property induces unique $\widetilde{g}$ with $f\widetilde{g} = g$. Finally, suppose f is regular epi and that $fa = fb$. Defining $g =_{\mathrm{df}} fa, g \in \mathrm{reg}(f)$ so there exists unique $\widetilde{g}$ with $f\widetilde{g} = g$, and $a = \widetilde{g} = b$. ∎

Proposition 1.2.3. *In $\mathscr{S}$ the following notions are equivalent: split epi, coequalizer, regular epi, epi, onto function.*

Proof. To see that epis are onto, consider functions to a two-element set. The axiom of choice implies that onto functions are split epi (and conversely, by the way). ∎

Proposition 1.2.4. *Let $A \xrightarrow{f} B \xrightarrow{g} C \in \mathscr{K}$. Then f (split) epi and g (split) epi implies fg (split) epi. fg (split) epi implies g (split) epi.*

Proposition 1.2.5. *Let $A \xrightarrow{f} B \in \mathscr{K}$. Then f iso iff f regular epi and mono.*

Proof. [iso] implies [split epi and mono] implies [regular epi and mono]. Conversely, if f is regular epi and mono, $1_A \in \mathrm{reg}(f)$ and so induces $\widetilde{f}$ with $f\widetilde{f} = 1_A$. As $f\widetilde{f}f = f$ and f is epi, $\widetilde{f}f = 1_B$. ∎

Definition 1.2.6. *Let $A \xrightarrow{f} B \in \mathscr{K}$. A* regular coimage factorization *of f is a factorization $f = A \xrightarrow{p} Q \xrightarrow{i} B$ with p regular epi and i mono. $\mathscr{K}$* has regular coimage factorizations *if every $\mathscr{K}$-morphism admits a regular coimage factorization.*

Proposition 1.2.7. *Regular coimage factorizations are unique within isomorphism.*

Proof. Suppose p, p' are regular epis and i, i' are monos with $pi = p'i'$. p' is in $\mathrm{reg}(p)$ as i' is mono, so h is uniquely induced with $ph = p'$. $hi' = i$ because p is epi. h^{-1} is induced similarly. ∎

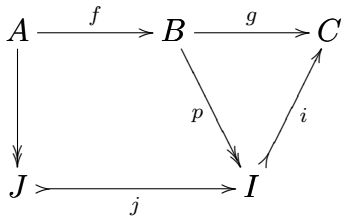
[a] Editor's footnote: Note that $\mathrm{reg}(f)$ is a proper class in general. This definition—which could be reworded to avoid the proper class—defines "regular epi" without requiring that it be a coequalizer of any single map.

PROPOSITION 1.2.8. *Assume that $\mathscr{K}$ has regular coimage factorizations. Let*

$$A \xrightarrow{f} B \xrightarrow{g} C \in \mathscr{K}$$

Then f, g regular epi implies fg regular epi. fg regular epi implies g regular epi. The hypothesis on $\mathscr{K}$ is necessary in both cases.

PROOF. Suppose fg is regular epi. Consider the diagram



where pi is the regular coimage factorization of g and then J is the regular coimage factorization of fp. By 1.2.7, ji is an isomorphism. But then i is mono and split epic, hence an isomorphism, by 1.2.5; and then g is regular epi because p is.

Now suppose that f, g are regular epi and let $fg = pi$ be a regular coimage factorization of fg. As i is mono, p is in $\mathrm{reg}(f)$ inducing $\widetilde{p}$ such that $f\widetilde{p} = p$. As just proved, i is regular epi (noting that $\widetilde{p}i = g$ because f is epi). As i is also mono, i is iso and hence fg is regular epi because p is.

The third assertion is left to the reader with the hint to look at some simple finite categories. ∎

1.3 LIMITS. If D is a $\mathscr{K}$-valued functor, the inverse limit of D (determined only within isomorphism if it exists at all) is denoted "$\varprojlim D$", or more precisely "$\varprojlim D \longrightarrow D$"; similarly, we use "$D \longrightarrow \varinjlim D$" for direct limits. The i^{th} projection of a product $=_{\text{dn}} \Pi X_i \xrightarrow{\text{pr}_i} X_i$. The coequalizer of $(f, g)\colon X \longrightarrow Y =_{\text{dn}} \mathrm{coeq}(f, g)$. If $(A_a \xrightarrowtail{i_a} X\colon a \in I)$ is a family of monomorphisms, their inverse limit $=_{\text{dn}} \bigcap A_a \xrightarrowtail{i} X$ (i is, in fact, a monomorphism). The class of monomorphisms into X is partially ordered by $A \xrightarrowtail{i} X \leq B \xrightarrowtail{j} X =_{\text{df}}$ there exists $A \xrightarrow{k} B$ such that $kj = i$ (in which case k is unique and is a monomorphism); $\cap A_a = \inf A_a$ with respect to this ordering.

1.4 GODEMONT'S CINQ RÈGLES; SEE [GODEMENT (1958)]. Suppose that W, X, Y, Z are functors and that a is a natural transformation from X to Y. Natural transformations $WX \xrightarrow{Wa} WY$ and $XZ \xrightarrow{aZ} YZ$ are induced by defining $K(Wa) =_{\text{df}} (KW)a$ and $K(aZ) =_{\text{df}} (Ka)Z$ for every object K. The five rules concerning these operations are as follows.

$$(WX)a = W(Xa)\colon WXY \longrightarrow WXZ;$$

$$a(YZ) = (aY)Z\colon WYZ \longrightarrow XYZ;$$

$WaZ =_{\text{df}} (Wa)Z = W(aZ)\colon WXZ \longrightarrow WYZ;$

$V(a \cdot b)Z = VaZ \cdot VbZ\colon VWZ \longrightarrow VYZ;$

$ab =_{\text{df}} aY \cdot Xb = Wb \cdot aZ\colon WY \longrightarrow XZ.$

1.5 The Yoneda Lemma. Let $\mathscr{K} \xrightarrow{H} \mathscr{S}$ be a set-valued functor, and let X be a $\mathscr{K}$-object such that $(X,\text{-})\mathscr{K}$ is set-valued. Then the passages

$$\begin{array}{rcl@{\qquad}rcl} ((X,\text{-})\mathscr{K}, H)\text{n.t.} & \longrightarrow & XH, & XH & \longrightarrow & ((X,\text{-})\mathscr{K}, H)\text{n.t.} \\ a & \mapsto & \langle 1_A, Xa\rangle & x \mapsto (X,\text{-})\mathscr{K} & \xrightarrow{a} & H \\ & & & (X,Y)\mathscr{K} & \xrightarrow{Ya} & YH \\ & & & X \xrightarrow{f} Y & \mapsto & \langle x, fH\rangle \end{array}$$

(where "n.t." means natural transformations) are mutually inverse. In particular, $((X,\text{-})\mathscr{K}, H)$n.t. is a set. For a proof see [Mitchell (1965), pp. 97–99].

1.6 Adjoint functors. Let $\mathscr{L} \xrightarrow{i} \mathscr{K}$ be a (not necessarily full) subcategory of $\mathscr{K}$, and let X be a $\mathscr{K}$-object. A reflection of X in $\mathscr{L} =_{\text{df}}$ a $\mathscr{K}$-morphism $X \xrightarrow{X\eta} X_{\mathscr{L}}$ such that $X_{\mathscr{L}} \in \text{obj}\mathscr{L}$ and such that whenever $X \xrightarrow{f} L \in \mathscr{K}$ with $L \in \text{obj}\mathscr{L}$ there exists a unique $X_{\mathscr{L}} \xrightarrow{\widetilde{f}} L \in \mathscr{L}$ such that $X\eta \cdot \widetilde{f} = f$. If every $\mathscr{K}$-object has a reflection in $\mathscr{L}$ then $\mathscr{L}$ is a *reflective sub category of* $\mathscr{K}$ and there is a *reflector functor* $\mathscr{K} \xrightarrow{R} \mathscr{L}$ defined so as to make $1 \xrightarrow{\eta} Ri$ natural. R is determined within natural equivalence. $\mathscr{L}$ is full iff R may be chosen with $iR = 1_{\mathscr{L}}$; (however, the definition of reflectors requires a suitable axiom of choice).

A *left adjointness* consists of functors $\mathscr{K} \xrightarrow{F} \mathscr{A}$, $\mathscr{A} \xrightarrow{U} \mathscr{K}$ and natural transformations $UF \xrightarrow{\varepsilon} 1$, $1 \xrightarrow{\eta} FU$ (called adjunctions) subject to the *adjointness axioms* $F \xrightarrow{\eta F} FUF \xrightarrow{F\varepsilon} F = 1_F$, $U \xrightarrow{U\eta} UFU \xrightarrow{\varepsilon U} U = 1_U$. We denote this by $F \dashv U$, read "F is left adjoint to U" and let η, ε be understood. U has a left adjoint $=_{\text{df}}$ there exists $F \dashv U$. If $\mathscr{A}$ and $\mathscr{K}$ are legitimate, then a left adjointness may be expressed in terms of a natural equivalence $((\text{-})F,\text{-})\mathscr{A} \xrightarrow{\alpha} (\text{-},(\text{-})U)\mathscr{K}$ where $\langle f,(X,A)\alpha\rangle = X\eta \cdot fU$, $\langle f,(X,A)\alpha^{-1}\rangle = gF \cdot A\varepsilon$ and conversely $X\eta = \langle 1_{XF},(X,XF)\alpha\rangle$, $A\varepsilon = \langle 1_{AU},(AU,A)\alpha^{-1}\rangle$.

If $f \dashv U$ and $\widetilde{F} \dashv U$ then F and $\widetilde{F}$ are naturally equivalent. A subcategory is reflective iff its inclusion functor has a left adjoint. Notice that a subcategory inclusion i is a full reflective subcategory iff there exists $R \dashv i$ with $iR \xrightarrow{\varepsilon} 1$ a natural equivalence.

Finally, we state the *adjoint functor theorem* first proved by Freyd. Let $\mathscr{A} \xrightarrow{U} \mathscr{K}$ be a functor. *U satisfies the solusion set condition* $=_{\text{df}}$ for every $K \in \text{obj}\mathscr{K}$ there exists a set, R_K, of $\mathscr{A}$-objects such that whenever $A \in \mathscr{A}$ and $K \xrightarrow{f} AU \in \mathscr{K}$, there exist $R \in R_K$, $K \xrightarrow{a} RU \in \mathscr{K}$, $R \xrightarrow{b} A \in \mathscr{A}$ with $f = a \cdot bU$; (such a set is called a *solution*

set for $\mathscr{K}$). Let $\mathscr{A}, \mathscr{K}$ be legitimate and assume further that $\mathscr{A}$ has $\varprojlim$. The adjoint functor theorem says: there exists $F \dashv U$ iff U preserves $\varprojlim$ and satisfies the solution set condition.

1.7 Regular categories. The category $\mathscr{K}$ is *regular* if it satisfies the following four axioms.

REG 1. $\mathscr{K}$ has regular coimage factorizations.

REG 2. $\mathscr{K}$ has $\varprojlim$.

REG 3. $\mathscr{K}$ is legitimate.

REG 4. For every X in obj$\mathscr{K}$ the class of regular epimorphisms with domain X has a representative set.

Contractible pairs (Jon Beck).

Definition 1.8.1. *Let* $(f,g)\colon X \longrightarrow Y$ *be* $\mathscr{K}$*-morphisms.* (f,g) *is contractible* $=_{\text{df}}$ *there exists* $Y \xrightarrow{d} X$ *such that* $df = 1_Y$ *and* $fdg = gdg$.

Proposition 1.8.2. *A coequalizer of a contractible pair is a split epi.*

Proof. If (f,g) is contractible and $q = \text{coeq}(f,g)$ then as $fdg = gdg$ there exists $Q \xrightarrow{h} Y$ with $qh = dg$. As q is epi and $qhq = dgq = dfq = q$, $hq = 1_Q$. ∎

For more on the theory of contractible pairs see [Manes (1967), Section 0.7]

Creation of constructions. Let $\mathscr{A} \xrightarrow{U} \mathscr{K}$ be a functor. U creates $\varprojlim =_{\text{df}}$ for each functor $\Delta \xrightarrow{H} \mathscr{A}$ and for each model $X \xrightarrow{\alpha} HU$ for $\varprojlim HU$ there exists a unique natural transformation $A \xrightarrow{\widetilde{\alpha}} H$ with codomain H such that $\widetilde{\alpha}U = \alpha$; and moreover $\widetilde{\alpha} = \varprojlim H$.

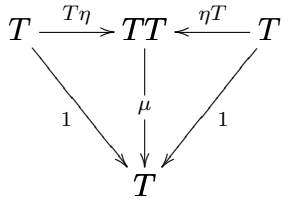
U creates regular coimage factorizations $=_{\text{df}}$ for each $\mathscr{A}$-morphism $A \xrightarrow{\widetilde{f}} B$ and for each regular coimage factorization $AU \xrightarrow{p} I \xrightarrow{i} BU$ of $\widetilde{f}U$ there exists unique $\widetilde{p}, \widetilde{i} \in \mathscr{A}$ with $\widetilde{p}U = p$ and $\widetilde{i}U = i$; and moreover, $\widetilde{p}$, $\widetilde{i}$ is a regular coimage factorization of $\widetilde{f}$.

U creates coequalizers of U-contractible pairs $=_{\text{df}}$ for each pair of $\mathscr{A}$-morphisms $(\widetilde{f}, \widetilde{g})\colon A \longrightarrow B$ such that $(\widetilde{f}U, \widetilde{g}U)$ is contractible and for each model $BU \xrightarrow{q} Q$ of $\text{coeq}(\widetilde{f}U, \widetilde{g}U)$ in $\mathscr{K}$, there exists unique $B \xrightarrow{\widetilde{q}} Q$ with domain B such that $\widetilde{q}U = q$; and moreover, $\widetilde{q} = \text{coeq}(\widetilde{f}, \widetilde{g})$.

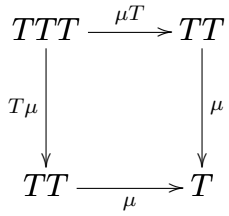
2. Algebras over a triple.

In this section we study just enough about the category of algebras over a triple to suit our later needs. See [Manes (1967), Chapter 1] for more results in a similar vein.

DEFINITION 2.1. *Let $\mathscr{K}$ be a category.* $\mathsf{T} = (T, \eta, \mu)$ *is a* triple in $\mathscr{K}$ *with* unit η *and* multiplication μ *if* $\mathscr{K} \xrightarrow{T} \mathscr{K}$ *is a functor and if* $1 \xrightarrow{\eta} T$, $TT \xrightarrow{\mu} T$ *are natural transformations subject to the three axioms:*

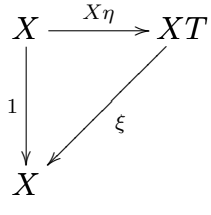


T-*unitary axioms*

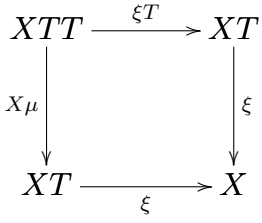


T-*associativity axiom*

Let $\mathsf{T} = (T, \eta, \mu)$ *be a triple in* $\mathscr{K}$. *A* T-*algebra* $=_{\text{df}}$ *a pair* (X, ξ) *with* $X \in |\mathscr{K}|$ *and* $XT \xrightarrow{\xi} X$ *a* $\mathscr{K}$-*morphism subject to the two axioms*

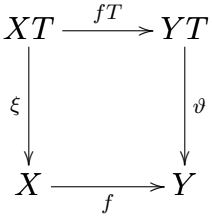


ξ-*unitary axiom*



ξ-*associativity axiom*

X *is the underlying* $\mathscr{K}$-*object of* (X, ξ) *and* ξ *is the* structure map of (X, ξ). *If* (X, ξ) *and* (Y, ϑ) *are* T-*algebras, a* T-homomorphism $f\colon (X, \xi) \longrightarrow (Y, \vartheta)$ from (X, ξ) to (Y, ϑ) *is a* $\mathscr{K}$-*morphism* $f\colon X \longrightarrow Y$ *subject to the*



T-*homomorphism axiom*

$\mathscr{K}^{\mathsf{T}} =_{\text{dn}}$ *the resulting category of* T-*algebras.* $U^{\mathsf{T}} =_{\text{dn}}$ *the faithful underlying* $\mathscr{K}$-*object functor.*

A functor $U\colon \mathscr{A} \longrightarrow \mathscr{K}$ is *tripleable* if there exists a triple T in $\mathscr{K}$ and an isomorphism of categories $\mathscr{A} \xrightarrow{\Phi} \mathscr{K}^{\mathsf{T}}$ such that $\Phi U^{\mathsf{T}} = U$. [b]

[b] Editor's footnote: This definition is non-standard. What is usually required is categorical equivalence, not isomorphism. Moreover, from the use the author makes of tripleability, it seems likely that the standard definition is what he really wants.

2.2 Heuristics in $\mathscr{K}^{\mathbf{T}}$. Categories of algebras in the classical sense are tripleable (see [Linton (1969), Section 9]; also [Manes (1967), 1.1.7]). We observe now that there are always free $\mathbf{T}$-algebras for $\mathbf{T} = (T, \eta, \mu)$ a triple in $\mathscr{K}$. If X is a $\mathscr{K}$-object, then $(XT, X\mu)$ is a $\mathbf{T}$-algebra (as is immediate from the triple axioms). Observe that if (X, ξ) is a $\mathbf{T}$-algebra, then $\xi\colon (XT, X\mu) \longrightarrow (X, \xi)$ is a $\mathbf{T}$-homomorphism by the ξ-associativity axiom. Since μ is natural,for each $\mathscr{K}$-morphism $f\colon X \longrightarrow Y$, $fT\colon (XT, X\mu) \longrightarrow (YT, Y\mu)$ is a $\mathbf{T}$-homomorphism. We wish to think of $(XT, X\mu)$ as the "free $\mathbf{T}$-algebra on X generators" with $X\eta\colon X \longrightarrow XT$ as "inclusion of the generators". Indeed, if (Y, ϑ) is a $\mathbf{T}$-algebra and if $f\colon X \longrightarrow Y$ is a $\mathscr{K}$-morphism, then it is easy to check that there exists a unique $\mathbf{T}$-homomorphism $\widetilde{f}\colon (XT, X\mu) \longrightarrow (Y, \vartheta)$ such that $X\eta.\widetilde{f} = f$, namely $\widetilde{f} =_{\text{df}} fT.\vartheta$. Note that a $\mathbf{T}$-algebra is characterized by the unique extension of the identity map on generators to $(XT, X\mu)$.

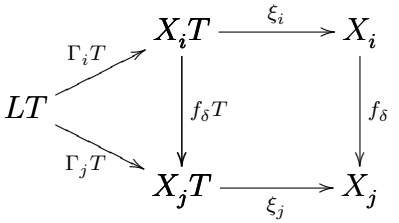
2.3 Example: the triple associated with a monoid. Let G be a monoid. "Cartesian product with the underlying set of G" is a functor

$$\mathscr{S} \xrightarrow{\text{-}\times G} \mathscr{S}$$

Define $X\eta =_{\text{df}} (1, e)\colon X \longrightarrow X \times G$ and $X\mu =_{\text{df}} 1 \times m\colon X \times G \times G \longrightarrow X \times C$, where e is the monoid unit and m is the monoid multiplication. Then $\mathbf{G} = (\text{-} \times G, \eta, \mu)$ is a triple in $\mathscr{S}$. $\mathbf{G}$-algebras are right G-sets. $\mathbf{G}$ is called the *triple associated with* G.

Proposition 2.4. *$U^{\mathbf{T}}$ creates* $\varprojlim$.

Proof. Suppose $D\colon \Delta \longrightarrow \mathscr{K}^{\mathbf{T}}$ is a functor and $\Gamma_i\colon L \longrightarrow X_i$ is a model for $\varprojlim DU^{\mathbf{T}}$. For every $\delta\colon i \longrightarrow j \in \Delta$, we have



which induces a unique $\mathscr{K}$-morphism ξ such that $\Gamma_iT.\xi_i = \xi.\Gamma_i$ for all i. It is routine to check that $\Gamma_i\colon (L, \xi) \longrightarrow (X_i, \xi_i)$ is the created $\varprojlim$ of D. ∎

2.5 Subalgebras. Let (X, ξ) be a $\mathbf{T}$-algebra and let $i\colon A \rightarrowtail X$ be a $\mathscr{K}$-monomorphism. Say that i (or by abuse of language A) is a subalgebra of (X, ξ) if there exists a $\mathscr{K}$-morphism $\xi_0\colon AT \longrightarrow A$ such that $\xi_0.i = iT.\xi$. It is easy to check that, indeed, (A, ξ_0) is a $\mathbf{T}$-algebra. To denote that (A, ξ_0) is a subalgebra of (X, ξ), we write "$(A, \xi_0) \leq (X, \xi)$".

Proposition 2.6. *Let T preserve regular coimage factorizations. Then $U^{\mathbf{T}}$ creates regular coimage factorizations.*

PROOF. Let $f\colon (X,\xi)\longrightarrow(Y,\vartheta)$ be a $\mathbf{T}$-homomorphism and suppose f has regular coimage factorization $f = X \xrightarrow{p} I \rightarrowtail X$ in $\mathscr{K}$. By hypothesis, $fT = XT \xrightarrow{pT} IT \rightarrowtail YT$ is a regular coimage factorization in $\mathscr{K}$. Since $\xi.f = fT.\vartheta$ and i is mono, $\xi.p \in \mathrm{reg}(pT)$ which induces unique ϑ_0 with $pT.\vartheta_0 = \vartheta.p$. $\vartheta_0.i = iT.\vartheta$ as pT is epi. We have $(X,\xi)\xrightarrow{p}(I,\vartheta_0)\xrightarrow{i}(Y,\vartheta)$ and that ϑ_0 is unique with this property. To complete the proof, we have only to show that $p\colon (X,\xi)\longrightarrow(I,\vartheta_0)$ is regular epi in $\mathscr{K}^{\mathbf{T}}$. Let $a\colon (X,\xi)\longrightarrow(A,\kappa)\in \mathrm{reg}_{\mathbf{T}}(p)$. Suppose $\zeta,\ \chi\colon B\longrightarrow X$ are $\mathscr{K}$-morphisms with $\zeta.p = \chi.p$. Let $\widetilde{\zeta},\ \widetilde{\chi}$ be the induced homomorphic extensions. Since $\widetilde{\zeta}.p,\ \widetilde{\chi}.p$ are homomorphisms agreeing on generators, $\widetilde{\zeta}.p = \widetilde{\chi}.p$. This proves that $a\in \mathrm{reg}_{\mathscr{K}}(p)$. As $p\colon X\longrightarrow I$ is a regular epi in $\mathscr{K}$, there exists a unique $\mathscr{K}$-morphism $\widetilde{a}$ with $p.\widetilde{a} = a$. Since a is a $\mathbf{T}$-homomorphism and pT is epi, $\widetilde{a}$ is forced to be a $\mathbf{T}$-homomorphism. ∎

DEFINITION 2.7. $\mathbf{T}$ *is a* regular triple *in* $\mathscr{K}$ *if* $\mathscr{K}$ *is a regular category and if* T *preserves regular coimage factorizations.*

PROPOSITION 2.8. *If* $\mathbf{T}$ *is a regular triple, then* $\mathscr{K}^{\mathbf{T}}$ *is a regular category.*

PROOF. REG 1, REG 2, REG 3 follow respectively from 2.6, 2.4 and the fact that $U^{\mathbf{T}}$ is faithful. Now let $p\colon (X,\xi)\longrightarrow(Y,\vartheta)$ be a regular epi in $\mathscr{K}^{\mathbf{T}}$. Combining the way the regular coimage factorization of p was created at the level $\mathscr{K}$ in 2.6 with 1.2.7, we see that $p\colon X\longrightarrow Y$ is regular epi in $\mathscr{K}$. But then REG 4 is clear. ∎

PROPOSITION 2.9 THE PRECISE TRIPLEABILITY THEOREM (JON BECK). *Let* $U\colon \mathscr{A}\longrightarrow\mathscr{K}$ *be a functor. Then* U *is tripleable iff* U *has a left adjoint and* U *creates coequalizers of* U*-contractible pairs.*

PROOF. See [Manes (1967), 1.2.9].[c] ∎

DEFINITION 2.10. *Let* (X,ξ) *be a* $\mathbf{T}$*-algebra and let* $i\colon A\rightarrowtail X$. *The* subalgebra of (X,ξ) *generated by* $A =_{\mathrm{dn}} \langle A\rangle =_{\mathrm{df}}$ *the intersection*

$$\bigcap\{(D,\alpha)\le(X,\xi)\mid A\subseteq D\}\rightarrowtail(X,\xi)$$

When $\langle A\rangle$ *exists it is in fact the smallest subobject of* (X,ξ) *containing* A. *If* $f\colon X\longrightarrow Y$ *is a* $\mathscr{K}$*-morphism and if* $f = X\xrightarrow{p} I\xrightarrow{j} Y$ *is a regular coimage factorization of* f, $\mathrm{im}\, f =_{\mathrm{dn}} j\colon I\rightarrowtail Y$. *If* $A\xrightarrow{i} X\xrightarrow{f} Y$ *we also denote* $\mathrm{im}\, i.f$ *by "*$Af\rightarrowtail Y$*". If* $X\xrightarrow{f}Y\xleftarrow{k}B$, *we denote the pullback of* k *along* f *by "*$Bf^{-1}\rightarrowtail X$. *(That* $Bf^{-1}\rightarrowtail X$ *is a monomorphism is easily verified.)*

PROPOSITION 2.11. *Let* $f\colon (X\xi)\longrightarrow(Y,\vartheta)$ *be a* $\mathbf{T}$*-homomorphism and let* $A\rightarrowtail X$, $B\rightarrowtail Y$. *The following statements are valid*

a. $\langle A\rangle = \mathrm{im}\, iT.\xi$, *provided both exist and* T *preserves coimage factorization.*

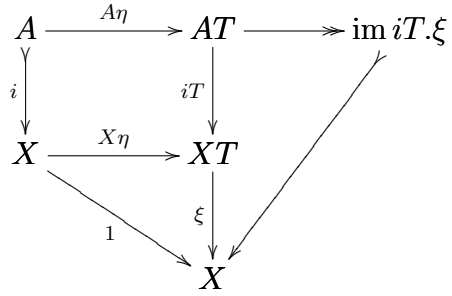
[c] Editor's footnote: See also p. 8ff. of the TAC reprint of [Beck (1967)].

b. $\langle A\rangle f = \langle Af\rangle$ *provided both exist and* T *preserves coimage factorization.*

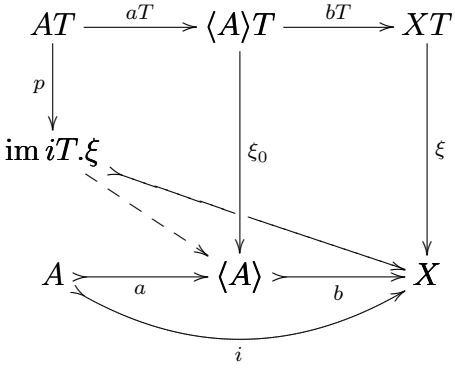
c. Bf^{-1}*, if it exists, is a subalgebra of* (X, ξ).

PROOF.

a. The diagram



proves that $A \subseteq \operatorname{im} iT.\xi$. But by 2.6, $\operatorname{im} iT \leq (X, \xi)$. Therefore $\langle A\rangle \subseteq \operatorname{im} iT.\xi$. Conversely, consider the diagram



$aT.\xi_0 \in \operatorname{reg}(p)$ because b is mono. Therefore, $\operatorname{im} iT.\xi \subseteq \langle A\rangle$.

b. Suppose $A \overset{p}{\twoheadrightarrow} Af \overset{c}{\rightarrowtail} Y$ is a regular coimage factorization. By hypothesis, pT is a regular epimorphism. We have $\langle Af\rangle = \operatorname{im} cT.\vartheta = \operatorname{im} pT.cT.\vartheta$ (by 1.2.7) $= \operatorname{im} iT.\xi.f = \langle A\rangle f$.

c. $Bf^{-1} \longrightarrow (X, \xi)$ is a monomorphism in $\mathscr{K}^{\mathsf{T}}$. But by 2.4, it is clear that the underlying $\mathscr{K}$-morphism of $Bf^{-1} \longrightarrow (X, \xi)$ is a monomorphism, and hence $Bf^{-1} \leq (X, \xi)$.

Alternate proof: since U preserves kernel pairs it preserves monos. ∎

3. Birkhoff subcategories

DEFINITION 3.1. *Let* $\mathscr{K}$ *be a category and let* $\mathscr{B}$ *be a full subcategory of* $\mathscr{K}$*.* $\mathscr{B}$ *is* closed under products *if every model for a product in* $\mathscr{K}$ *of a set of* $\mathscr{B}$*-objects lies in* $\mathscr{B}$*.* $\mathscr{B}$ *is*

closed under subobjects *if every monomorphism in $\mathscr{K}$ with range in $\mathscr{B}$ lies in $\mathscr{B}$. Let $\mathscr{C}$ be any subcategory of $\mathscr{K}$. Define $\widehat{\mathscr{C}} =_{\text{df}}$ the intersection of all full subcategories of $\mathscr{K}$ containing $\mathscr{C}$ and closed under products and subobjects.*

Evidently "$\widehat{\ }$" is a closure operator on the (large) lattice of subcategories of $\mathscr{K}$ and $\mathscr{C} = \widehat{\mathscr{C}}$ iff $\mathscr{C}$ is closed under products and subobjects.

PROPOSITION 3.2. *Let $\mathscr{K}$ be a regular category and let $i\colon \mathscr{B} \longrightarrow \mathscr{K}$ be a full subcategory. The following statements are equivalent.*

a. $\mathscr{B} = \widehat{\mathscr{B}}$

b. $\mathscr{B}$ is a reflective subcategory of $\mathscr{K}$ in such a way that for every $\mathscr{K}$-object X the reflection $X\eta\colon X \longrightarrow X_{\mathscr{B}}$ of X in $\mathscr{B}$ is a regular epimorphism; also $|\mathscr{B}|$ is a union of $\mathscr{K}$-isomorphism classes.

PROOF.

a $\Longrightarrow$ b. Viewing an isomorphism as a unary product, $|\mathscr{B}|$ is a union of $\mathscr{K}$-isomorphism classes. $\mathscr{B}$ has $\varprojlim$s and i preserves them. i satisfies the solution set condition by REG 1 and REG 4. By the adjoint functor theorem it follows that $\mathscr{B}$ is a reflective subcategory. Let X be a $\mathscr{K}$-object with reflection $X\eta\colon X \longrightarrow X_{\mathscr{B}}$. Factor $X\eta = p.k$ through its regular coimage. As k is mono, x is induced with $X\eta.x = p$. By the uniqueness of reflection-induced maps, $x.k = 1$. Therefore x is epi and split mono, hence iso, and $X\eta$ is regular epi because p is.

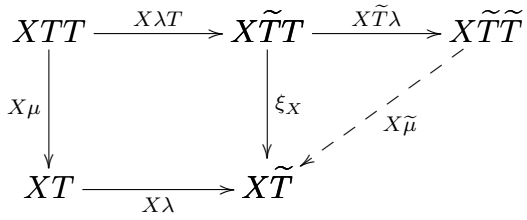
b $\Longrightarrow$ a. Let X be a product in $\mathscr{K}$ of a set of $\mathscr{B}$-objects. Each projection factors through $X\eta$ inducing a map $a\colon X_{\mathscr{B}} \longrightarrow X$ such that $X\eta.a = 1_X$. Hence $X\eta$ is a split mono. Since we assume $X\eta$ is epi, $X\eta$ is an isomorphism. Now suppose X is a $\mathscr{K}$-object admitting a monomorphism j to some object in $\mathscr{B}$. Then j factors through $X\eta$, and hence $X\eta$ is mono. But then $X\eta$ is mono and regular epi and hence iso. ∎

For the balance of this section, let $\mathbf{T} = (T, \eta, \mu)$ be a regular triple in $\mathscr{K}$.

PROPOSITION 3.3. *Let $\lambda\colon T \longrightarrow \widetilde{T}$ be a pointwise regular epi natural transformation, and suppose further that for every $\mathscr{K}$-object X there exists a $\mathscr{K}$-morphism $X\widetilde{\mu}$ such that $X\lambda\lambda.X\widetilde{\mu} = X\mu.X\lambda$. Then $\widetilde{\mathbf{T}} =_{\text{df}} (\widetilde{T}, \widetilde{\eta}, \widetilde{\mu})$ (where $\widetilde{\eta} = \eta.\lambda$) is a triple in $\mathscr{K}$ and $\lambda\lambda.\widetilde{\mu} = \mu.\lambda$.*

PROOF. The fact that $X\lambda$ is epi yields the unitary axioms. It is also true that $X\lambda\lambda$ and $X\lambda\lambda\lambda$ are epi, i.e., $X\lambda\lambda\lambda = X\lambda TT.X\widetilde{T}\lambda T.X\widetilde{T}\widetilde{T}\lambda$ so use 1.2.8 and the fact that T preserves regular epis. $X\lambda\lambda$ epi implies $\widetilde{\mu}$ is natural, and $X\lambda\lambda\lambda$ epi implies the associativity axiom. The reader may provide the requisite diagrams. ∎

3.4. The regular triple induced by a $\widehat{\ }$-closed subcategory.. Let $\mathscr{B} \subseteq \mathscr{K}$ be a subcategory such that $\mathscr{B} = \widehat{\mathscr{B}}$. By 2.8 $\mathscr{K}^{\mathbf{T}}$ is a regular category, so that by 3.2 $\mathscr{B}$ is a full coreflective subcategory with regular epi reflections. In particular, for each $\mathscr{K}$-object X let $X\lambda\colon (XT, X\mu) \longrightarrow (X\widetilde{T}, \xi_X)$ be a regular epi reflection of $(XT, X\mu)$ in $\mathscr{B}$. By the reflection property, each $\mathscr{K}$-morphism $f\colon X \longrightarrow Y$ induces unique $f\widetilde{T}$ such that $X\lambda.f\widetilde{T} = fT.Y\lambda$ which establishes a functor $\widetilde{T}\colon \mathscr{K} \longrightarrow \mathscr{K}$ and a pointwise regular epi natural transformation $\lambda\colon T \longrightarrow \widetilde{T}$. For ever $\mathscr{K}$-object X, the fact that ξ_X is a $\mathbf{T}$-homomorphism and the reflection property induce unique $X\widetilde{\mu}$:



By 3.3, $\widetilde{\mathbf{T}} = (\widetilde{T}, \eta\lambda, \widetilde{\mu})$ is a triple in $\mathscr{K}$ and $\lambda\lambda.\widetilde{\mu} = \mu.\lambda$. $\widetilde{\mathbf{T}}$ is called the *regular triple induced by* $\mathscr{B}$.

Definition 3.5. *A full subcategory $\mathscr{B}$ of $\mathscr{K}^{\mathbf{T}}$ is closed under $U^{\mathbf{T}}$-split epis $=_{\mathrm{df}}$ whenever $q\colon (X,\xi) \longrightarrow (Q,\alpha) \in \mathscr{K}^{\mathbf{T}}$ with $q\colon X \longrightarrow Q$ split epi in $\mathscr{K}$ and $(X,\xi) \in |\mathscr{B}|$ then $(Q,\alpha) \in |\mathscr{B}|$. For each subcategory $\mathscr{C}$ of $\mathscr{K}^{\mathbf{T}}$ define $\widehat{\mathscr{C}} =_{\mathrm{df}}$ the intersection of all full subcategories of $\mathscr{K}^{\mathbf{T}}$ closed under products, subalgebras ($=_{\mathrm{df}}$ subobjects of $\mathscr{K}^{\mathbf{T}}$) and $U^{\mathbf{T}}$-split epis. A $\widehat{\ }$-closed subcategory of $\mathscr{K}^{\mathbf{T}}$ is called a* Birkhoff subcategory of $\mathscr{K}^{\mathbf{T}}$ *(because the following theorem is a triple-theoretic version of [Birkhoff (1935)]).*

Proposition 3.6. *Let $\mathscr{B}$ be a Birkhoff subcategory of $\mathscr{K}^{\mathbf{T}}$ and $U =_{\mathrm{df}}$ the restriction of $U^{\mathbf{T}}$ to $\mathscr{B}$. Then U is tripleable.*

Proof. Let $\widetilde{\mathbf{T}} = (\widetilde{T}, \widetilde{\eta}, \widetilde{\mu})$, λ be as in 3.4. We will construct an isomorphism $\Phi\colon \mathscr{K}^{\widetilde{\mathbf{T}}} \longrightarrow \mathscr{B}$ such that $\Phi U = U^{\widetilde{\mathbf{T}}}$. If $(X,\widetilde{\xi})$ is a $\widetilde{\mathbf{T}}$-algebra, $(X,\widetilde{\xi})\Phi =_{\mathrm{df}} (X, XT \xrightarrow{X\lambda} X\widetilde{T} \xrightarrow{\widetilde{\xi}} X)$. $\widetilde{\eta} = \eta\lambda$ implies $X\eta.X\lambda.\widetilde{\xi} = 1_X$ and $\lambda\lambda.\widetilde{\mu} = \mu.\lambda$ implies $X\mu.X\lambda.\widetilde{\xi} = X\lambda T.\widetilde{\xi}T.X\lambda.\widetilde{\xi}$. If $f\colon (X,\widetilde{\xi}) \longrightarrow (Y,\widetilde{\vartheta})$ is a $\widetilde{\mathbf{T}}$-homomorphism, the naturality of λ guarantees that

$$f\colon (X, X\lambda.\widetilde{\xi}) \longrightarrow (Y, Y\lambda, \widetilde{\vartheta})$$

is a $\mathbf{T}$-homomorphism. Hence $\Phi\colon \mathscr{K}^{\widetilde{\mathbf{T}}} \longrightarrow \mathscr{K}^{\mathbf{T}}$ is a well-defined functor with $\Phi U^{\mathbf{T}} = U^{\widetilde{\mathbf{T}}}$. Because λ is pointwise epic, Φ is 1-to-1 on objects. If $(X,\widetilde{\xi})$, $(Y,\widetilde{\vartheta})$ are $\widetilde{\mathbf{T}}$ objects and if $f\colon (X, X\lambda.\widetilde{\xi}) \longrightarrow (Y, Y\lambda.\widetilde{\vartheta})$ is a $\mathbf{T}$-homomorphism, $f\colon (X,\widetilde{\xi}) \longrightarrow (Y,\widetilde{\vartheta})$ is a $\widetilde{\mathbf{T}}$-homomorphism because $X\lambda$ is epi. Hence Φ is an isomorphism onto the full subcategory $\operatorname{im}\Phi$. We must show $\operatorname{im}\Phi = \mathscr{B}$ on objects. If $(X, X\lambda.\widetilde{\Phi}) \in \operatorname{im}\xi$ then $(XT, X\widetilde{T}\lambda.X\widetilde{\mu}) \in |\mathscr{B}|$ by definition of $\widetilde{\mu}$ and $X\widetilde{\eta}\colon X \longrightarrow XT$ is a $\mathscr{K}$-splitting of the $\mathbf{T}$-homomorphism $\widetilde{\xi}\colon (X\widetilde{T}, X\widetilde{T}\lambda.X\widetilde{\mu}) \longrightarrow (X, X\lambda.\widetilde{\xi})$, so that $(X, X\lambda.\widetilde{\xi}) \in |B|$. Conversely, let $(X,\xi) \in |\mathscr{B}|$. By the reflection property there exists a unique $\mathbf{T}$-homomorphism $\widetilde{\xi}\colon X\widetilde{T} \longrightarrow X$ with

$X\lambda.\widetilde{\xi} = \xi$. Using the fact that $X\lambda\lambda$ is epi, it is easy to check that $(X, \widetilde{\xi})$ is a $\widetilde{\mathbf{T}}$-algebra. The proof is complete. ∎

4. The category $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$

For this section, let $\mathbf{T}$, $\widetilde{\mathbf{T}}$ be regular triples in $\mathscr{K}$.

DEFINITION 4.1. *Define a new category $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$ with objects $(X, \xi, \widetilde{\xi})$ such that $(X, \xi) \in |\mathscr{K}^{\mathbf{T}}|$ and $(X, \widetilde{\xi}) \in |\mathscr{K}^{\widetilde{\mathbf{T}}}|$ and morphisms $f\colon (X, \xi, \widetilde{\xi}) \longrightarrow (Y, \vartheta, \widetilde{\vartheta})$ such that $f\colon (X, \xi) \longrightarrow (Y, \vartheta) \in \mathscr{K}^t$ and $f\colon (X, \widetilde{\xi}) \longrightarrow (Y, \widetilde{\vartheta}) \in \mathscr{K}^{\widetilde{\mathbf{T}}}$ with identities and compositions defined at the level $\mathscr{K}$. $U^{(\mathbf{T},\widetilde{\mathbf{T}})} =_{\mathrm{df}}$ the obvious underlying $\mathscr{K}$-object functor.*

PROPOSITION 4.2. *$\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$ is a regular category and $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ creates $\varprojlim$ and regular coimage factorizations.*

PROOF. That $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ creates $\varprojlim$ follows easily from 2.4; in particular we have REG 2. REG 3 is clear as $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ is faithful.

Now suppose that $f\colon (X, \xi, \widetilde{\xi}) \longrightarrow (Y, \vartheta, \widetilde{\vartheta}) \in \mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$ is such that f is regular epi in $\mathscr{K}^{\mathbf{T}}$ and $\mathscr{K}^{\widetilde{\mathbf{T}}}$. Then f is regular epi in $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$. To prove it, it is enough to to let g be in $\mathrm{reg}_{(\mathbf{T},\widetilde{\mathbf{T}})}(f)$ and show that g is in $\mathrm{reg}_{\mathbf{T}}(f)$. Let a, $b\colon (A, \alpha) \longrightarrow (X, \xi)$ in $\mathscr{K}^{\mathbf{T}}$ such that $af = bf$. Let $(t, u)\colon P \longrightarrow X$ be a pullback of f with itself in $\mathscr{K}$. Since $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ creates $\varprojlim$, t, u lift to $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$-morphisms $(P, \gamma, \widetilde{\gamma}) \longrightarrow (X, \xi, \widetilde{\xi})$. Since $U^{\mathbf{T}}$ creates $\varprojlim$, $(t, u)\colon (P, \gamma) \longrightarrow (X, \xi)$ is the pullback of $f\colon (X, \xi) \longrightarrow (Y, \vartheta)$ with itself, which induces $h\colon (A, \alpha) \longrightarrow (P, \gamma)$ such that $ht = a$, $hu = b$. Since g is in $\mathrm{reg}_{(\mathbf{T},\widetilde{\mathbf{T}})}(f)$ and $tf = uf$ we have $tg = ug$. Therefore $ag = htg = hug = bg$.

That $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ creates regular coimage factorizations now follows easily from 2.6; in particular, REG 1 is established. Let $f\colon (X, \xi, \widetilde{\xi}) \longrightarrow (Y, \vartheta, \widetilde{\vartheta})$ be regular epi in $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$. REG 4 will be clear from 2.8 if we show f is regular epi in $\mathscr{K}^{\mathbf{T}}$. This is immediate from 1.2.7 and the way the regular coimage factorization of f in $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$ was constructed. ∎

DEFINITION 4.3. *Let $\mathscr{B}$ be a full subcategory of $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$. $\mathscr{B}$ is a* Birkhoff subcategory *if $\mathscr{B}$ is closed under products, subobjects and $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$-split epis (the meaning of "closed under $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$-split epis" is clear).*

PROPOSITION 4.4. *Let $\mathscr{B}$ be a Birkhoff subcategory of $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$ and let $U =_{\mathrm{df}}$ the restriction of $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ to $\mathscr{B}$. Then U is tripleable iff U satisfies the solution set condition.*

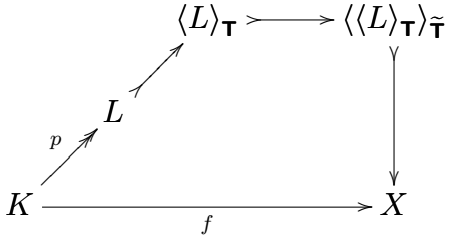
PROOF. We use 2.9. It is trivial to check that $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ creates coequalizers of $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ contractible pairs. Now suppose $(X, \xi, \widetilde{\xi}) \underset{g}{\overset{f}{\rightrightarrows}} (Y, \vartheta, \widetilde{\vartheta})$ with $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ contractible and $q\colon Y \longrightarrow Q = \mathrm{coeq}(f, g)$ in $\mathscr{K}$. Since q is a split epimorphism in $\mathscr{K}$, by 1.8.2, the created coequalizer $q\colon (Y, \vartheta, \widetilde{\vartheta}) \longrightarrow (Q, \alpha, \widetilde{\alpha})$ is, in fact, in $\mathscr{B}$. Therefore U creates coequalizers of

U-contractible pairs. $\mathscr{B}$ is closed under the $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$-$\varprojlim$ so that $\mathscr{B}$ has and U preserves $\varprojlim$. By the adjoint functor theorem, the proof is complete. ∎

DEFINITION 4.5. *Let $(X,\xi,\widetilde{\xi}) \in |\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}|$. $(X,\xi,\widetilde{\xi})$ is a $\mathbf{T}$-$\widetilde{\mathbf{T}}$ quasicomposite algebra $=_{\mathrm{df}}$ for every K-monomorphism $i\colon A \rightarrowtail X$, the $K^{(\mathbf{T},\widetilde{\mathbf{T}})}$- subalgebra generated by A is $\langle\langle A\rangle_{\mathbf{T}}\rangle_{\widetilde{\mathbf{T}}}$. (What we mean by "subalgebra generated by" is clear.) Equivalently, if A is a $\widetilde{\mathbf{T}}$-subalgebra, so is $\langle A\rangle_{\widetilde{\mathbf{T}}}$.*[d]

PROPOSITION 4.6. *Let $\mathscr{B}$ be a Birkhoff subcategory of $\mathscr{K}^{(\mathbf{T},\widetilde{\mathbf{T}})}$ and let $U =_{\mathrm{df}} U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ restricted to $\mathscr{B}$. If every $\mathscr{B}$-object is a $\mathbf{T}$-$\widetilde{\mathbf{T}}$ quasicomposite algebra then U is tripleable.*

PROOF. By 4.4 we have only to show that U satisfies the solution set condition. Let K be a $\mathscr{K}$-object. Let $\mathscr{S}_1$ be a representative set of regular epis with domain K. Let $\mathscr{S}_2$ be a representative set of split epis with domain of the form LT for some L which is in the range of some element of $\mathscr{S}_1$. Let $\mathscr{S}_3$ be a representative set of split epis with domain of the form $L\widetilde{T}$ for some L which is in the range of some element of $\mathscr{S}_2$. Now suppose $(X,\xi,\widetilde{\xi})$ is an object of $\mathscr{B}$ and that $f\colon K \longrightarrow X$ is a $\mathscr{K}$-morphism. There exists p in $\mathscr{S}_1$ with $f = K \overset{p}{\twoheadrightarrow} L \overset{i}{\rightarrowtail} X$. There exists a model for $\langle L\rangle_{\mathbf{T}}$ such that the canonical split epi $\vartheta\colon LT \twoheadrightarrow \langle L\rangle_{\mathbf{T}}$ is in $\mathscr{S}_2$ (as we can always transport a structure map through a $\mathscr{K}$-isomorphism). Similarly there exists a split epi $\langle L\rangle_t\widetilde{\mathbf{T}} \twoheadrightarrow \langle\langle L\rangle_{\mathbf{T}}\rangle_{\widetilde{\mathbf{T}}}$ in $\mathscr{S}_3$. Hence the diagram



proves that f factors through a set of objects $\{\langle\langle L\rangle_{\mathbf{T}}\rangle_{\widetilde{\mathbf{T}}}\}$. The crucial point is our hypothesis that each $\langle\langle L\rangle_{\mathbf{T}}\rangle_{\widetilde{\mathbf{T}}}$ is in $|\mathscr{B}|$. ∎

5. Compact spaces

DEFINITION 5.1. *Let X be a set, $\mathscr{F} \subset 2^X$. $\mathscr{F}^c =_{\mathrm{df}} [A \subset X$: there exists $F \in \mathscr{F}$ with $F \subset A]$. $\mathscr{F}$ is a* filter on X *if $\mathscr{F} \neq \emptyset$, $\emptyset \notin \mathscr{F}$, $A,B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$ and $\mathscr{F} = \mathscr{F}^c$. An* ultrafilter on X *is an inclusion maximal filter on X. $X\beta =_{\mathrm{df}} [\mathscr{U}\colon \mathscr{U}$ is an ultrafilter on $X]$. If $A \subset X$, $\mathscr{F} \wedge A =_{\mathrm{df}} [F \cap A\colon F \in \mathscr{F}]$. If $\mathscr{F}$ is a filter on X, it is easy to verify that $A \notin \mathscr{F}$ iff $\mathscr{F} \wedge A$ is a filter on A' iff $(\mathscr{F} \wedge A')^c$ is a filter on X (where $A' =_{\mathrm{dn}}$ complement of A in X.)*

[d]Editor's footnote: The notation is not defined but it seems from the context and $\langle A\rangle_{\widetilde{\mathbf{T}}}$ is meant to denote the $\widetilde{\mathbf{T}}$-subalgebra generated by A and similarly for $\langle\langle A\rangle_{\mathbf{T}}\rangle_{\widetilde{\mathbf{T}}}$. This concept seems to be awfully close to distributive laws, [Beck (1969)].

LEMMA 5.2. *The following statements are valid.*

a. *For every filter, $\mathscr{F}$, on X, $\mathscr{F} \in X\beta$ iff for every subset A of X either $A \in \mathscr{F}$ or $A' \in \mathscr{F}$.*

b. *For every filter, $\mathscr{F}$, on X, $\mathscr{F} = \bigcap[\mathscr{U} \in X\beta \colon \mathscr{F} \subset \mathscr{U}]$.*

PROOF. a. If $A \notin \mathscr{F}$, $(\mathscr{F} \wedge A')^c$ is a filter finer than, hence equal to, $\mathscr{F}$. Therefore $A' \in \mathscr{F}$. Conversely, let $\mathscr{G}$ be a filter containing $\mathscr{F}$. If $G \in \mathscr{G}$, $G' \notin \mathscr{F}$ so that $G \in \mathscr{F}$.

b. Let $A \subseteq X$, $A \notin \mathscr{F}$. $(\mathscr{F} \wedge A')^c$ is a filter on X. By Zorn's lemma (a nested union of filters is a filter) every filter is contained in an ultrafilter. Hence there exists $\mathscr{U} \in X\beta$ with $(\mathscr{F} \wedge A')^c \subseteq \mathscr{U}$. We have $\mathscr{F} \subseteq \mathscr{U}$, $A \notin \mathscr{U}$ proving $A \notin \bigcap[\mathscr{V} \in X\beta \colon \mathscr{F} \subseteq \mathscr{V}]$. ∎

DEFINITION 5.3. *Let $(X, \mathfrak{S})$ be a topological space, let $\mathscr{F} \subseteq 2^X$ and let $x \in X$. Recall that $\mathscr{F}$ converges to $x =_{\mathrm{df}} \mathscr{F}^c \supset \mathscr{N}_x$, (where $\mathscr{N}_x =_{\mathrm{df}}$ the neighborhood filter of x), $=_{\mathrm{dn}} \mathscr{F} \to x$. More generally, if $A \subseteq X, \mathscr{F} \to A =_{\mathrm{dn}}$ there exists $x \in A$ with $\mathscr{F} \to x$. If $f \colon X \longrightarrow Y$ is a function, $\mathscr{F}f =_{\mathrm{df}}$ the filter $[Ff \colon F \in \mathscr{F}]^c \subseteq 2^Y$.*

LEMMA 5.4. *The following statements are valid.*

a. *(Due to [Ellis & Gottschalk (1960), Lemma 7]). Let $(X, \mathfrak{S})$, $(X', \mathfrak{S}')$ be topological spaces, let $f \colon X \longrightarrow X'$ be a function and let $x \in X$. Then f is continuous at x iff for every $\mathscr{U} \in X\beta$, $\mathscr{U} \to x$ implies $\mathscr{U}f \to xf$.*

b. *Let $(X, \mathfrak{S})$ be a topological space, and let $A \subseteq X$. Then A is open iff for every $\mathscr{U} \in X\beta$, $\mathscr{U} \to A$ implies $A \in \mathscr{U}$.*

c. *Let $(X, \mathfrak{S})$ be a topological space, and let $f \colon X \longrightarrow X'$ be an onto function. Let $\mathfrak{S}'$ be the quotient topology induced by f. Then if $(X, \mathfrak{S})$ is compact T2 and if $f \colon (X, \mathfrak{S}) \longrightarrow (X', \mathfrak{S}')$ is closed (i.e. maps closed sets to closed sets) then $(X', \mathfrak{S}')$ is compact T2.*

PROOF. a. Let $\mathscr{U} \in X\beta$, $\mathscr{U} \to x$. Let $V \in \mathscr{N}_{xf}$. There exists $W \in \mathscr{N}_x$ with $Wf \subseteq V$. As $W \in \mathscr{U}$, $V \in \mathscr{U}f$. Now the converse. For every $\mathscr{U} \to x$ we have $\mathscr{U}f \supset \mathscr{N}_{xf}$. $\mathscr{U} \supset \mathscr{U}ff^{-1} \supset \mathscr{N}_{xf}f^{-1}$. By 5.2b, $\mathscr{N}_x = \bigcap[\mathscr{U} \colon \mathscr{U} \to x] \supset \mathscr{N}_{xf}f^{-1}$.

b. A is open iff $A \in \bigcap_{x \in A} \mathscr{N}_x = \bigcap_{x \in A} \bigcap_{\mathscr{U} \to x} \mathscr{U} = \bigcap_{\mathscr{U} \to A} \mathscr{U}$.

c. This is standard. See [Kelley (1955), Chapter 5, Theorem 20, p. 148]. ∎

PROPOSITION 5.5. *Let $\mathscr{C}$ be the category of compact T2 spaces with underlying set functor $U \colon \mathscr{C} \longrightarrow \mathscr{S}$. Then U is tripleable.*

PROOF. A fairly short proof could be given using 2.9. Instead, we offer an independent definition of "compact T2 space" by making the triple explicit. If X is a set with $x \in X$, $A \subseteq X$, define $\dot{x} =_{\mathrm{df}} \{B \subseteq X \colon x \in B\}$ and $\dot{A} =_{\mathrm{df}} \{\mathscr{U} \in X\beta \colon A \in \mathscr{U}\}$. The following five statements are trivial to verify:[e] $\dot{x} \in X\beta$, $\dot{\overline{(x)}} = (\dot{x})$, $\dot{A} \cap \dot{B} = \dot{\overline{A \cap B}}$, $\dot{\overline{A'}} = (\dot{A})'$, $\dot{\emptyset} = \emptyset$.

[e] Editor's footnote: In the original, the scope of the dots is not clear and the statement as a whole is hard to parse. We use the overline here in its old sense as a kind of horizontal parenthesis.

Define $\boldsymbol{\beta} = (\beta, \eta, \mu)$ by

$$\begin{array}{ccccc} \mathscr{S} & \xrightarrow{\beta} & \mathscr{S} & & \\ X & \xrightarrow{f} & Y & X\beta \xrightarrow{f\beta} Y\beta, & X \xrightarrow{X\eta} X\beta \\ & & & \mathscr{U} \longmapsto \mathscr{U} f & x \longmapsto \dot{x} \end{array}$$

$$\begin{array}{c} X\beta\beta \xrightarrow{X\mu} X\beta \\ \mathscr{H} \longmapsto [A \subset X \colon \dot{A} \in \mathscr{H}] \end{array}$$

The proof that $\boldsymbol{\beta}$ is a triple on $\mathscr{S}$ will be left to the reader; the details are routine providing one remembers that two ultrafilters are equal if one is contained in the other. (The details are written out in [Manes (1967), 2.3.3]). We will construct an isomorphism of categories $\Phi \colon \mathscr{C} \longrightarrow \mathscr{S}^{\boldsymbol{\beta}}$ such that $\Phi U^{\boldsymbol{\beta}} = U$. Let $(X, \mathfrak{S})$ be a compact T2 space. $\xi_{\mathfrak{S}} \colon X\beta \longrightarrow X =_{\mathrm{df}}$ the function sending an ultrafilter to the unique point to which it converges. $X\eta.\xi = 1$ because $\dot{x} \to x$ in all topologies. Now let $\mathscr{H} \in X\beta\beta$. $x =_{\mathrm{df}} \langle \mathscr{H}, \xi_{\mathfrak{S}}\beta.\xi_{\mathfrak{S}} \rangle = [A \subseteq X \colon \dot{A} \in \mathscr{H}]^{c}\xi_{\mathfrak{S}}$. To verify the $\xi_{\mathfrak{S}}$-associativity axiom we must show that $\langle \mathscr{H}, X\mu \rangle = [A \subseteq X \colon \dot{A} \in \mathscr{H}] \to x$. So let $B^{\mathrm{open}} \in \mathscr{N}_x$. There exists $\mathscr{L} \in \mathscr{H}$ such that $[\mathscr{U}\xi_{\mathfrak{S}} \colon \mathscr{U} \in \mathscr{L}] \subseteq B$. Therefore $\mathscr{U} \in \mathscr{L}$ implies $\mathscr{U}\xi_{\mathfrak{S}} \in B$ implies there exists $b \in B$ such that $\mathscr{U} \to b$. As $B \in \mathscr{N}_b$, $B \in \mathscr{U}$, so $\mathscr{U} \in \dot{B}$. Therefore $\dot{B} \supset \mathscr{L} \in \mathscr{H}$ and $\dot{B} \in \mathscr{H}$, as we wished to show. This defines Φ on objets. Now let $(X, \mathfrak{S}), (X', \mathfrak{S}')$ be compact T2 spaces and let $f \colon X \longrightarrow X'$ be a function. f is a $\boldsymbol{\beta}$-homomorphism iff $f\beta.\xi_{\mathfrak{S}'} = \xi_{\mathfrak{S}}.f$ iff for every $\mathscr{U} \in X\beta$ and for every $x \in X, \mathscr{U} \to x$ implies $\mathscr{U} f \to xf$ iff f is continuous. Summing up, Φ is a well-defined full and faithful functor such that $\Phi U^{\boldsymbol{\beta}} = U$ and such that Φ is 1-to-1 on objects (using 5.4b for the last statement). To complete the proof we show that Φ is onto on objects.

Let X be a set, and define a topology, $\mathfrak{S}_X$, on $X\beta$ by taking $[\dot{A} \colon A \in X]$ as a base, which we may do since the $\dot{A}$'s are closed under finite intersections: explicitly, every open set is a union of $\dot{A}$'s and conversely. Let $\mathscr{H} \in X\beta\beta$. $\mathscr{H} \to \mathscr{H}X\mu$, because if $\mathscr{H}X\mu = [A \subseteq X \colon \dot{A} \in \mathscr{H}] \in \dot{B}$ then $B \in [A \subseteq X \colon \dot{A} \in \mathscr{H}]$, that is $\dot{B} \in \mathscr{H}$. Moreover if $\mathscr{U} \in X\beta$ is such that $\mathscr{H} \to \mathscr{U}$ it follows that $\mathscr{U} = \mathscr{H}X\mu$, for if $A \in \mathscr{U}$, then $\mathscr{U} \in \dot{A} \in \mathscr{H}$ and hence $A \in [B \subseteq X \colon \dot{B} \in \mathscr{H}] = \mathscr{H}X\mu$. This proves that $(X\beta, \mathfrak{S}_X) \in \mathrm{obj}\ \mathscr{C}$ and $(X\beta, \mathfrak{S}_X)\Phi = (X\beta, X\mu)$.

Let $i \colon \mathscr{L} \rightarrowtail X\beta$, and consider the diagram

$$\begin{array}{ccc} \mathscr{L}\beta & \xrightarrow{i\beta} & X\beta\beta \\ \vdots & & \downarrow{\scriptstyle X\mu} \\ \mathscr{L} & \xrightarrow[i]{} & X\beta \end{array}$$

One sees immediately that $\mathscr{L}$ is a subalgebra of $(X\beta, X\mu)$ iff every ultrafilter on $\mathscr{L}$ converges in $\mathscr{L}$ iff (applying a well-known theorem of topology) $\mathscr{L}$ is closed.

Now let (X,ξ) be any $\boldsymbol{\beta}$-algebra. $\xi\colon (X\beta, X\mu) \longrightarrow (X,\xi)$ is a $\boldsymbol{\beta}$-homomorphism onto. Let $\mathfrak{S}$ be the quotient topology induced by ξ on X. Let $\mathscr{L} \subseteq X\beta$. $\mathscr{L}$ is closed iff $\mathscr{L} \leq (X\beta, X\mu)$ implies (by 2.11b) $\mathscr{L}\xi \leq (X,\xi)$ implies (by 2.11c) $(\mathscr{L}\xi)\xi^{-1} \leq (X\beta, \mathbf{X}\mu)$ iff $(\mathscr{L}\xi)\xi^{-1}$ is closed in $(X\beta, \mathfrak{S}_X)$ iff $\mathscr{L}$ is closed in $(X,\mathfrak{S})$. Therefore ξ is a closed mapping. By 5.4c, $(X,\mathfrak{S}) \in \text{obj}\ \mathscr{C}$. Finally, for $\mathscr{U} \in X\beta$ we show $\mathscr{U} \to_{\mathfrak{S}} \mathscr{U}\xi$. Let $\mathscr{U}\xi \in A \in \mathfrak{S}$. There exists $B \subseteq X$ with $\mathscr{U} \in \dot{B} \subseteq A\xi^{-1}$. For all $b \in B$, $b = \dot{b}\xi \in A\xi^{-1}\xi = A$. Therefore $A \supset B \in \mathscr{U}$ and $A \in \mathscr{U}$. ∎

Remarks. 5.6

a. Let (X,ξ) be a $\boldsymbol{\beta}$-algebra and let $A \subseteq X$. Then A is a subalgebra iff A is closed.

b. Free $\boldsymbol{\beta}$-algebras are totally disconnected.

c. We can easily prove the Tychonoff theorem in the weak form "the cartesian product of compact T2 spaces is compact"

Proof. To prove (a), use the argument given for free algebras in the proof of 5.5. (b) is easy using the properties of ".": notice that the class of clopen subsets of $(X\beta, \mathbf{X}\mu)$ is precisely $[\dot{A}\colon A \subseteq X]$. For the third statement, construct the product in $\mathscr{S}^{\boldsymbol{\beta}}$:

$$\begin{array}{ccc} (\prod X_i)\beta & \xrightarrow{\text{pr}_i} & X_i\beta \\ \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \xi_i} \\ \prod X_i & \xrightarrow[\text{pr}_i]{} & X_i \end{array}$$

Now observe that the diagram says that an ultrafilter on the product converges iff it converges pointwise, a characterization of the cartesian product topology of any family of topological spaces. ∎

6. Operations

For this section fix a triple $\mathbf{T} = (T, \eta, \mu)$ in $\mathscr{S}$.

Proposition 6.1. $\mathbf{T}$ *is a regular triple.*

Proof. That $\mathscr{S}$ is a regular category is well known: ordinary image factorizations provide the regular coimage factorizations. T preserves all epimorphisms and all monomorphisms with non-empty domain since these are split. To complete the proof we must show that for each set X, $(i\colon \emptyset \rightarrowtail X)T$ is mono. This is clear if $\emptyset T = \emptyset$. Otherwise there exists a function $f\colon X \longrightarrow \emptyset T$. By freeness, $\mathbf{T}$-homomorphisms from $(\emptyset T, \emptyset\mu)$ to any $\mathbf{T}$-algebra (A,α) are in bijective correspondence with functions from $\emptyset$ to A. Hence $\emptyset T \xrightarrow{iT} XT \xrightarrow{fT} \emptyset TT \xrightarrow{\emptyset\mu} \emptyset T$ is the identity map and iT is (split) mono. ∎

DEFINITION 6.2. *Let n be a set. "Raising to the n^{th} power" is a functor:*

$$1^n\colon \mathscr{S} \longrightarrow \mathscr{S}$$

$$f\colon X \longrightarrow Y \;\mapsto\; f^n\colon X^n \longrightarrow Y^n$$

Define[f]

$$\mathscr{O}_{\mathbf{T}}(n) =_{\text{df}} \{g\colon g \text{ is a natural transformation from } 1^n \text{ to } T\}$$

For (X,ξ) a $\mathbf{T}$-algebra and $g \in \mathscr{O}_{\mathbf{T}}(n)$, $\xi^g =_{\text{df}}$ the function $X^n \xrightarrow{XG} XT \xrightarrow{\xi} X$. ξ^g is called an n-ary operation *of (X,ξ) and the set of all such $=_{\text{dn}} \mathscr{O}_n(X,\xi)$.*

6.3 and 6.4 below are indications that $\mathbf{T}$-algebras are characterized by their operations. See [Manes (1967), Section 2.2] for further details.

PROPOSITION 6.3. *Let $(X,\xi),(Y,\vartheta)$ be $\mathbf{T}$-algebras, and let $f\colon X \longrightarrow Y$ be a function. The following statements are pairwise equivalent.*

a. f is a $\mathbf{T}$-morphism.

b. For every set n and for every $g \in \mathscr{O}_{\mathbf{T}}(n)$ the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{f^n} & Y^n \\ {\scriptstyle \xi^g}\downarrow & (*)_g & \downarrow{\scriptstyle \vartheta^g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes.

c. $(*)_g$ commutes for every $g \in \mathscr{O}_{\mathbf{T}}(X)$.

PROOF. a. implies b.

$$\begin{array}{ccccc} X^n & \xrightarrow{Xg} & XT & \xrightarrow{\xi} & X \\ {\scriptstyle f^n}\downarrow & & {\scriptstyle fT}\downarrow & & \downarrow{\scriptstyle f} \\ Y^n & \xrightarrow[Yg]{} & YT & \xrightarrow[\vartheta]{} & Y \end{array}$$

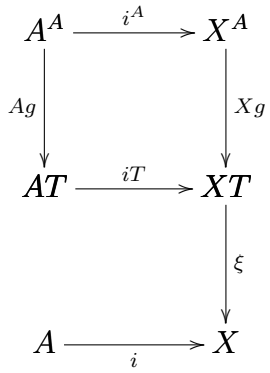
b. implies c. This is obvious.

c. implies a. Consider the diagram of "a implies b" with $n = X$. Let $x \in XT$. By the Yoneda Lemma there exists $g \in \mathscr{O}_{\mathbf{T}}(X)$ with $\langle 1_X, Xg\rangle = x$. We have $\langle x, \xi.f\rangle = \langle 1_X, \xi^g.f\rangle = \langle 1_X, Xg.fT.\vartheta\rangle = \langle x, fT.\vartheta\rangle$. ∎

[f] Editor's footnote: We have changed the original—rather ghastly—notation in which $\mathbf{T}$ denoted both the triple and what we have called $\mathscr{O}_{\mathbf{T}}$.

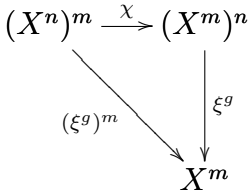
PROPOSITION 6.4. *Let (X,ξ) be a $\mathbf{T}$-algebra and let $i\colon A \rightarrowtail X$ be a subset. Then $A = \langle A\rangle$ iff for every $g \in \mathscr{O}_{\mathbf{T}}(A)$, $A^A \xrightarrow{i^A} X^A \xrightarrow{\xi^g} X$ factors through A.*

PROOF. If $(A,\xi_0) \leq (X,\xi)$, ξ_0^g is the desired factorization. Conversely, consider the diagram:



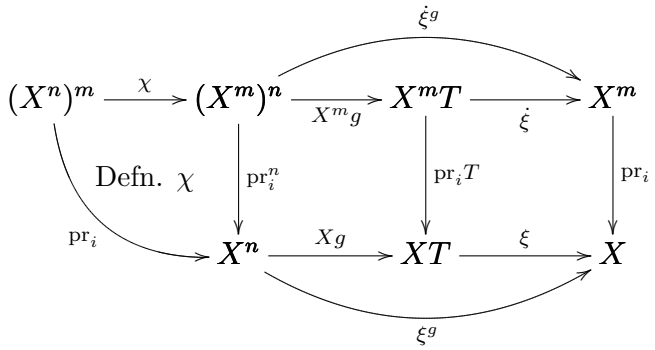
Let $x \in AT$. By the Yoneda Lemma there exists $g \in \mathscr{O}_{\mathbf{T}}(A)$ with $\langle 1_X, Ag\rangle = x$. Hence as $\operatorname{im} i^A.\xi^g \subseteq A$ by hypothesis, $\langle x, iT.\xi\rangle = \langle 1_A, i^A.\xi^g\rangle \in A$. Therefore $iT.\xi$ factors through i and $A = \langle A\rangle$. ∎

LEMMA 6.6. *Let n, m be sets, $g \in \mathscr{O}_{\mathbf{T}}(n)$, $(X,\xi) \in |\mathscr{S}^{\mathbf{T}}|$, $(X^m,\dot\xi) =_{\mathrm{df}} (X,\xi)^m$ and let $\chi\colon (X^n)^m \longrightarrow (X^m)^n$ be the canonical bijection. Then*



commutes.

PROOF.



∎

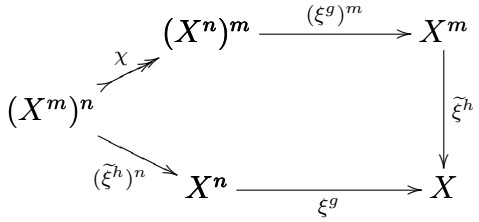
For the balance of this section fix another triple $\widetilde{\mathbf{T}} = (\widetilde{T}, \widetilde{\eta}, \widetilde{\mu})$ in $\mathscr{S}$.

PROPOSITION 6.6. *Let $(X, \xi, \widetilde{\xi}) \in |\mathscr{S}^{(\mathbf{T},\widetilde{\mathbf{T}})}|$. The following statements are equivalent.*

a. For all sets n, for all $g \in \mathscr{O}_{\mathbf{T}}(n)$, ξ^g is a $\widetilde{\mathbf{T}}$-homomorphism.

b. for all sets m, for all $h \in \widetilde{\mathbf{T}}(m)$, $\widetilde{\xi}^h$ is a $\mathbf{T}$-homomorphism.

PROOF. Use 6.3, 6.5 and the symmetry of the diagram

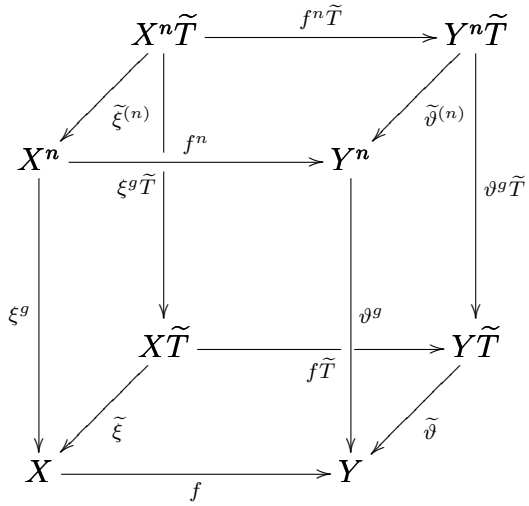


■

DEFINITION 6.7. *$(X, \xi, \widetilde{\xi}) \in |\mathscr{S}^{(\mathbf{T},\widetilde{\mathbf{T}})}|$ is a $\mathbf{T}$-$\widetilde{\mathbf{T}}$-*bialgebra *if it satisfies either of the equivalent conditions of 6.6. The full subcategory of $\mathbf{T}$-$\widetilde{\mathbf{T}}$-bialgebras $=_{\mathrm{dn}} \mathscr{S}^{[\mathbf{T},\widetilde{\mathbf{T}}]}$ and the restriction of $U^{(\mathbf{T},\widetilde{\mathbf{T}})}$ to $\mathscr{S}^{[\mathbf{T},\widetilde{\mathbf{T}}]} =_{\mathrm{dn}} U^{[\mathbf{T},\widetilde{\mathbf{T}}]}$. If $U^{[\mathbf{T},\widetilde{\mathbf{T}}]}$ is tripleable, the resulting triple, $=_{\mathrm{dn}} \mathbf{T} \otimes \widetilde{\mathbf{T}}$, $=_{\mathrm{df}}$ the tensor product of $\mathbf{T}$ and $\widetilde{\mathbf{T}}$. It is an open question whether or not $\mathbf{T} \otimes \widetilde{\mathbf{T}}$ always exists. A constructive proof can be given if both $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ have a rank (in the sense of [Manes (1967), 2.2.6]) by generalizing Freyd's proof in [Freyd (1966)]. By 4.4 and 6.8 below the problem reduces to showing that $U^{[\mathbf{T},\widetilde{\mathbf{T}}]}$ satisfies the solution set condition.*

PROPOSITION 6.8. *$\mathscr{S}^{[\mathbf{T},\widetilde{\mathbf{T}}]}$ is a Birkhoff subcategory of $\mathscr{S}^{(\mathbf{T},\widetilde{\mathbf{T}})}$.*

PROOF. The diagram of 6.5 shows "closed under products". Now consider the diagram:



If $f\colon (X, \xi, \widetilde{\xi}) \longrightarrow (Y, \vartheta, \widetilde{\vartheta}) \in \mathscr{S}^{(\mathbf{T},\widetilde{\mathbf{T}})}$, all commutes except possibly the left and right faces. Hence if f is mono then right implies left; if f is epi then so is $f^n\widetilde{T}$ so left implies right. ■

DEFINITION 6.9. *Let (X,ξ) be a $\mathbf{T}$-algebra, let A be a subset of X and let n be a set. Consider the factorization $A \overset{i}{\rightarrowtail} \langle A\rangle \overset{j}{\rightarrowtail} X$. Because $\langle A\rangle^n \overset{j^n}{\rightarrowtail} X^n \leq X^n$ we have a factorization*

$$A^n \xrightarrow{k} \langle A^n\rangle \xrightarrow{m} \langle A\rangle^n \xrightarrow{j^n} X^n, \qquad i^n : A^n \to \langle A\rangle^n$$

Say that subalgebras commute with powers in $\mathscr{S}^{\mathbf{T}}$ *if m is always an isomorphism for all (X,ξ), A, n.*

PROPOSITION 6.10. *Suppose that subalgebras commute with powers in $\mathscr{S}^{\widetilde{\mathbf{T}}}$. Then every $\mathbf{T}$-$\widetilde{\mathbf{T}}$-bialgebra is a $\mathbf{T}$-$\widetilde{\mathbf{T}}$ quasicomposite algebra, and hence $\mathbf{T}\otimes\widetilde{\mathbf{T}}$ exists.*

PROOF. Let $(X,\xi,\widetilde{\xi})$ be a $\mathbf{T}$-$\widetilde{\mathbf{T}}$-bialgebra and let $(A,\xi_0) \leq (X,\xi)$. For each $g \in \mathscr{O}_{\mathbf{T}}(\langle A\rangle_{\widetilde{\mathbf{T}}})$ consider the diagram

$$\begin{array}{ccccc} A^{\langle A\rangle_{\widetilde{\mathbf{T}}}} & \rightarrowtail & \langle A^{\langle A\rangle_{\widetilde{\mathbf{T}}}}\rangle = \langle A\rangle_{\widetilde{\mathbf{T}}}^{\langle A\rangle_{\widetilde{\mathbf{T}}}} & \rightarrowtail & X^{\langle A\rangle_{\widetilde{\mathbf{T}}}} \\ \downarrow{\scriptstyle \xi_0^g} & & & & \downarrow{\scriptstyle \xi^g} \\ A & \rightarrowtail & \langle A\rangle_{\widetilde{\mathbf{T}}} & \rightarrowtail & X \end{array}$$

Since ξ^g is a$\widetilde{\mathbf{T}}$-homomorphism it follows from 2.11b that ξ^g maps $\langle A^{\langle A\rangle_{\widetilde{\mathbf{T}}}}\rangle$ into $\langle A\rangle_{\widetilde{\mathbf{T}}}$. Since $\langle A^{\langle A\rangle_{\widetilde{\mathbf{T}}}}\rangle = \langle A\rangle_{\widetilde{\mathbf{T}}}^{\langle A\rangle_{\widetilde{\mathbf{T}}}}$, ξ^g maps $\langle A\rangle_{\widetilde{\mathbf{T}}}^{\langle A\rangle_{\widetilde{\mathbf{T}}}}$ into $\langle A\rangle_{\widetilde{\mathbf{T}}}$. It follows from 6.4 that $\langle A\rangle_{\widetilde{\mathbf{T}}}$ is a $\mathbf{T}$-algebra. The last statement is immediate from 6.8 and 4.6. ∎

7. Compact algebras

PROPOSITION 7.1. *For every triple, $\mathbf{T}$, in $\mathscr{S}$, every $\mathbf{T}$-$\boldsymbol{\beta}$ bialgebra is a $\mathbf{T}$-$\boldsymbol{\beta}$ quasicomposite algebra. In particular $\mathbf{T}\otimes\boldsymbol{\beta}$ always exists.*

PROOF. A well-known theorem of topology is: "the product of the closures is the closure of the product." Using the Tychonoff theorem and 5.6a we have that subalgebras commute with powers in $\mathscr{S}^{\boldsymbol{\beta}}$. Now use 6.10. ∎

A $\mathbf{T}\otimes\boldsymbol{\beta}$-algebra is, by definition, a $\mathbf{T}$-algebra whose underlying set is provided with a compact T2 topology in such a way that $\mathbf{T}$-operators are continuous. Hence $\mathbf{T}\otimes\boldsymbol{\beta}$-algebras deserve to be —and are— called *compact $\mathbf{T}$-algebras.*

7.2 EXAMPLE: DISCRETE ACTIONS WITH COMPACT PHASE SPACE. Let G be a discrete monoid with associated triple $\mathbf{G}$. If $g \in \mathbf{G}(n)$ it is easy to check, using the Yoneda Lemma, that for each G-set (X,α), α^g factors as a projection map followed by the "transition" map induced from X to X by the action of some element of G. Hence $\mathscr{S}^{\mathbf{G}\otimes\boldsymbol{\beta}}$ is the category of compact T2 transformation semigroups with phase semigroup G, that is since G is discrete $\alpha\colon X\times G \longrightarrow G$ is continuous iff each transition $\alpha^g\colon X \longrightarrow X$ is continuous.

PROPOSITION 7.3. *Compact topological dynamics is tripleable. More precisely, let G be a monoid with associated triple $\mathbf{G}$. Let $\mathfrak{S}$ be any topology whatever on the underlying set of G. $\mathscr{B} =_{\mathrm{df}}$ the full subcategory of $\mathscr{S}^{\mathbf{G}\otimes\boldsymbol{\beta}}$ generated by objects (X,ξ,α) such that $\alpha\colon (X,\xi)\times(G,\mathfrak{S})\longrightarrow(X,\xi)$ is continuous. Then $\mathscr{B}$ is a Birkhoff subcategory of $\mathscr{S}^{\mathbf{G}\otimes\boldsymbol{\beta}}$, and in particular $\mathscr{B}$ is tripleable. (Compact topological dynamics is recovered by insisting that $\mathfrak{S}$ be compatible with G.)*

PROOF. Consider a product of $\mathscr{B}$-objects, $(X,\alpha,\xi)=\prod(X_i,\alpha_i,\xi_i)$. At the level of sets we have

$$\begin{array}{ccc} X\times G & \xrightarrow{\ \alpha\ } & X \\ {\scriptstyle \mathrm{pr}_i\times 1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_i} \\ X_i\times G & \xrightarrow[\ \alpha_i\ ]{} & X_i \end{array}$$

$(X,\xi)=\prod(X_i,\xi_i)$ in $\mathscr{S}^{\boldsymbol{\beta}}$ by 4.2, and hence in the category of all topological spaces by the Tychonoff theorem. Therefore α is continuous because each $\alpha.\mathrm{pr}_i$ is.

Next, let $i\colon (A,\alpha_0,\xi_0)\rightarrowtail(X,\alpha,\xi)$ be a $\mathbf{G}\otimes\boldsymbol{\beta}$-subalgebra with $(X,\alpha,\xi)\in|\mathscr{B}|$. We have $\alpha_0.i=(i\times 1).\alpha$. Now all monomorphisms in $\mathscr{S}^{\boldsymbol{\beta}}$ become relative subspaces when viewed in the category of all spaces because every algebraic monomorphism is an isomorphism into. Therefore α_0 is continuous because $\alpha_0.i$ is.

To show that $\mathscr{B}$ is closed under quotients it suffices to prove the following topological lemma: Consider the situation

$$\begin{array}{ccc} X\times H & \xrightarrow{\ a\ } & X \\ {\scriptstyle f\times 1}\downarrow & & \downarrow{\scriptstyle f} \\ Y\times H & \xrightarrow[\ b\ ]{} & Y \end{array}$$

where X,H,Y are topological spaces with X compact and Y T2 and where a is continuous and f is continuous onto. Then b is continuous. To prove it we use 5.4a. Let $\mathscr{U}$ be an ultrafilter on $Y\times H$ such that $\mathscr{U}\to(y,h)\in Y\times H$. Because f is onto, $U(f\times 1)^{-1}\neq\emptyset$ for all $U\in\mathscr{U}$. Hence there exists an ultrafilter $\mathscr{V}$ on $X\times H$ with $\mathscr{V}\supset\mathscr{U}(f\times 1)^{-1}$. $U\supset U(f\times 1)^{-1}(f\times 1)$ proves $\mathscr{U}=\mathscr{V}(f\times 1)$. Since X is compact there exists $x\in X$ such that $\mathscr{V}\mathrm{pr}_X\to x$. $\mathscr{V}\mathrm{pr}_H=\mathscr{V}(f\times 1)\mathrm{pr}_H=\mathscr{U}\mathrm{pr}_H\to h$. If $N\in\mathscr{N}_x$, $M\in\mathscr{N}_h$ there exist $V,W\in\mathscr{V}$ with $N\supset V\mathrm{pr}_X$, $M\supset W\mathrm{pr}_H$ and then $N\times M\supset(V\cap W)\mathrm{pr}_X\times(V\cap W)\mathrm{pr}_H\supset V\cap W\in\mathscr{V}$ proves that $\mathscr{V}\to(x,h)$. Since $\mathscr{U}\mathrm{pr}_Y=\mathscr{V}(f\times 1)\mathrm{pr}_Y=\mathscr{V}\mathrm{pr}_X f\to(x,h)\mathrm{pr}_X f=xf$ and Y is T2, $xf=y$. Therefore $\mathscr{U}b=\mathscr{V}(f\times 1)b=\mathscr{V}af\to(x,h)af=(x,h)(f\times 1)b=(y,h)b$ as desired. ∎

Proposition 7.3 says that a compact T2 topological transformation group may equally well be viewed as a set with algebraic structure. Certain results of [Ellis (1960a)], [Ellis (1960)] can be conveniently proved by this approach, and certain questions originating

in topological dynamics may be asked in $\mathscr{S}^{\mathbf{T}}$. See [Manes (1967), Sections 2.4,2.5] for further details.

Distributive Laws

Jon Beck

The usual distributive law of multiplication over addition, $(x_0+x_1)(y_0+y_1) \longrightarrow x_0y_0 + x_0y_1 + x_1y_0 + x_1y_1$, combines mathematical structures of abelian groups and monoids to produce the more interesting and complex structure of rings. From the point of view of "triples", a distributive law provides a way of interchanging two types of operations and making the functorial composition of two triples into a more complex triple.

The main formal properties and different ways of looking at distributive laws are given in Section 1. Section 2 is about algebras over composite triples. These are found to be objects with two structures, and the distributive law or interchange of operations appears in its usual form as an equation which the two types of operations must obey. Section 3 is about some frequently-occurring diagrams of adjoint functors which are connected with distributive laws. Section 4 is devoted to Examples. There is an Appendix on compositions of adjoint functors.

I should mention that many properties of distributive laws, some of them beyond the scope of this paper, have also been developed by Barr, Linton and Manes. In particular, one can refer to Barr's paper *Composite cotriples* in this volume. Since Barr's paper is available, I omitted almost all references to cotriples.

I would like to acknowledge the support of an NAS-NRC (AFOSR) Postdoctoral Fellowship at the E.T.H., Zürich, while this paper was being prepared, as well as the hospitality of the Mathematics Institute of the University of Rome, where some of the commutative diagrams were found.

One general fact about triples will be used. If $\varphi\colon \mathbf{S} \longrightarrow \mathbf{T}$ is a map of triples in $\mathbf{A}$, the functor $\mathbf{A}^{\varphi}\colon \mathbf{A}^{\mathbf{S}} \longleftarrow \mathbf{A}^{\mathbf{T}}$ usually has a left adjoint, for which there is a coequalizer formula:

$$ASF^{\boldsymbol{T}} \underset{A\overline{\varphi}}{\overset{\sigma F^T}{\rightrightarrows}} AF^{\boldsymbol{T}} \longrightarrow (A,\sigma) \otimes_{\mathbf{S}} F^T.$$

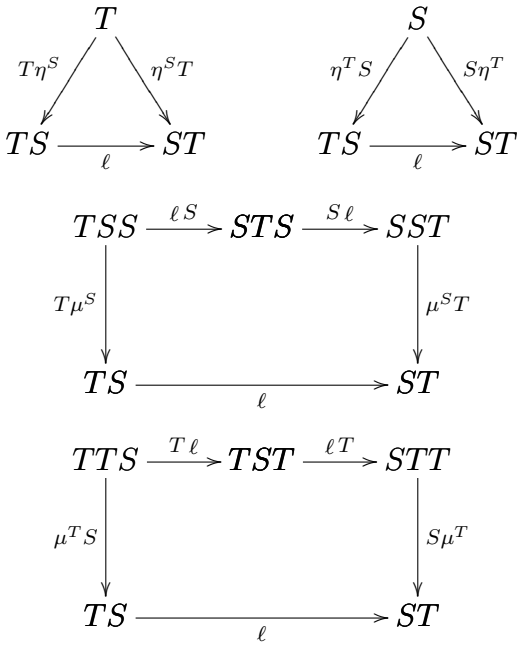
Here (A,σ) is an $\mathbf{S}$-algebra and the coequalizer is calculated in $\mathbf{A}^{\mathbf{T}}$. The natural operation $\overline{\varphi}$ of $\mathbf{S}$ on F^T is the composition

$$(AST, AS\mu^T) \xrightarrow{\varphi T} (ATT, AT\mu^T) \xrightarrow{\mu^T} (AT, A\mu^T)$$

The notation $(\ ) \otimes_{\mathbf{S}} F^T$ for the left adjoint is justifiable. Later on the symbol $\mathbf{A}^{\varphi}$ is replaced by a Hom notation. The adjoint pair $(\ ) \otimes_{\mathbf{S}} F^T, \mathbf{A}^{\varphi}$ is always tripleable.

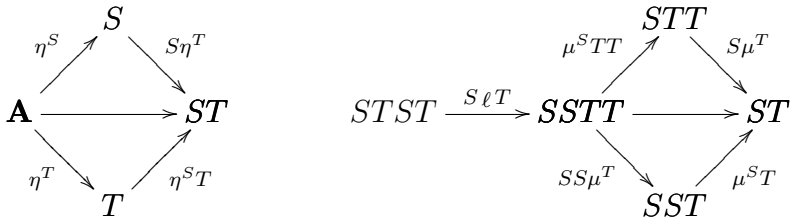
1. Distributive laws, composite and lifted triples

A *distributive law of* S *over* T is a natural transformation $\ell\colon TS \longrightarrow ST$ such that



commute.

The *composite triple* defined by ℓ is $\mathsf{ST} = (ST, \eta^S\eta^T, S\,\ell\,T \cdot \mu^S\mu^T)$. That is, the composite functor $ST\colon \mathbf{A} \longrightarrow \mathbf{A}$, with unit and multiplication



is a triple in $\mathbf{A}$. The units of S and T give triple maps

$$S\eta^T\colon \mathsf{S} \longrightarrow \mathsf{ST}$$

$$\eta^S T\colon \mathsf{T} \longrightarrow \mathsf{ST}$$

The proofs of these facts are just long naturality calculations. Note that the composite triple should be written $(\mathsf{ST})_\ell$ to show its dependence on ℓ, but that is not usually observed.

In addition to the composite triple, ℓ defines a *lifting* of the triple T into the category of S-algebras. This is the triple $\widetilde{\mathsf{T}}$ in $\mathbf{A}^{\mathsf{S}}$ defined by

$$\widetilde{T} = \begin{cases} \widetilde{T}\colon (A,\sigma)\widetilde{T} = (AT, A\,\ell\cdot\sigma T), \\ \widetilde{\eta}\colon (A,\sigma)\widetilde{\eta} = A\eta\colon (A,\sigma) \longrightarrow (A,\sigma)\widetilde{T}, \\ \widetilde{\mu}\colon (A,\sigma)\widetilde{\mu} = A\mu\colon (A,\sigma)\widetilde{T}\widetilde{T} \longrightarrow (A,\sigma)\widetilde{T}. \end{cases}$$

It follows from compatibility of ℓ with S that $A\,\ell\cdot\sigma T$ is an S-algebra structure, and from compatibility of ℓ with T that $\widetilde{\eta}$, $\widetilde{\mu}$ are maps of S-algebras.

That $\widetilde{\mathsf{T}}$ is a *lifting* of T is expressed by the commutativity relations

$$\widetilde{T}U^S = U^S T, \quad \widetilde{\eta}U^S = U^S\eta, \quad \widetilde{\mu}U^S = U^S\mu$$

$$\begin{array}{ccc} \mathbf{A}^{\mathsf{S}} & \xrightarrow{\widetilde{T}} & \mathbf{A}^{\mathsf{S}} \\ {\scriptstyle U^S}\downarrow & & \downarrow{\scriptstyle U^S} \\ \mathbf{A} & \xrightarrow[T]{} & \mathbf{A} \end{array}$$

Proposition. *Not only do distributive laws give rise to composite triples and liftings, but in fact these three concepts are equivalent:*

(1) distributive laws $\ell\colon TS \longrightarrow ST$,

(2) multiplications $m\colon STST \longrightarrow ST$ *with the properties:* $(\mathsf{ST})_m = (ST, \eta^S\eta^T, m)$ *is a triple in* $\mathbf{A}$, *the natural transformations*

$$\mathsf{S} \xrightarrow{S\eta^T} \mathsf{ST} \xleftarrow{\eta^S T} \mathsf{T}$$

are triple maps, and the middle unitary law

$$\begin{array}{ccc} & ST & \\ {\scriptstyle S\eta^T\eta^S T}\swarrow & & \searrow{\scriptstyle =} \\ STST & \xrightarrow[m]{} & ST \end{array}$$

holds,

(3) liftings $\widetilde{\mathsf{T}}$ *of the triple* T *into* $\mathbf{A}^{\mathsf{S}}$.

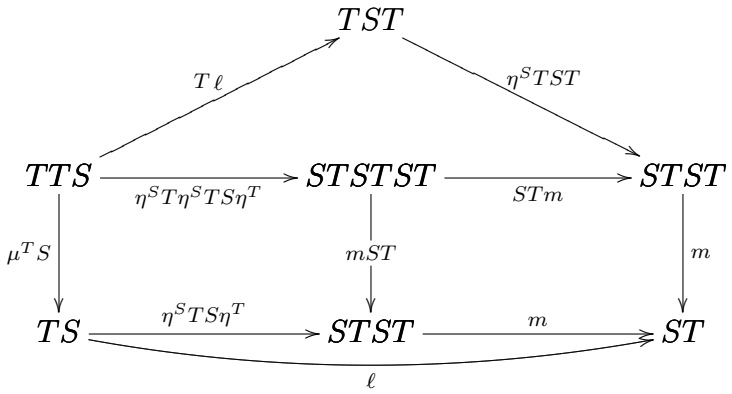
Proof. Maps (1) $\longrightarrow$ (2) and (1) $\longrightarrow$ (3) have been constructed above. It remains to construct their inverses and prove that they are equivalences.

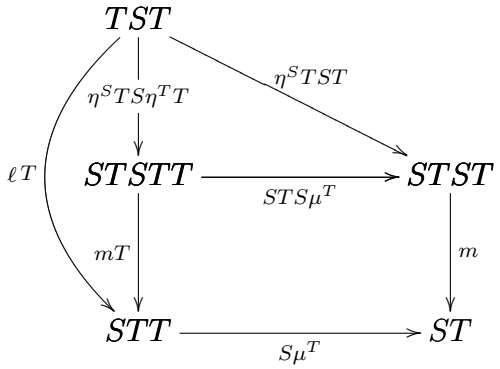
(2) $\longrightarrow$ (1). Given m, define ℓ as the composition

$$TS \xrightarrow{\eta^S TS\eta^T} STST \xrightarrow{m} ST$$

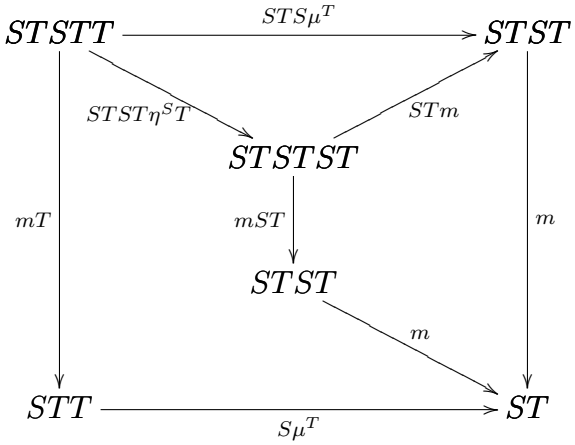
Compatibility of ℓ with the units of S and T is trivial. As to compatibility with the

multiplication in $\mathbf{T}$,





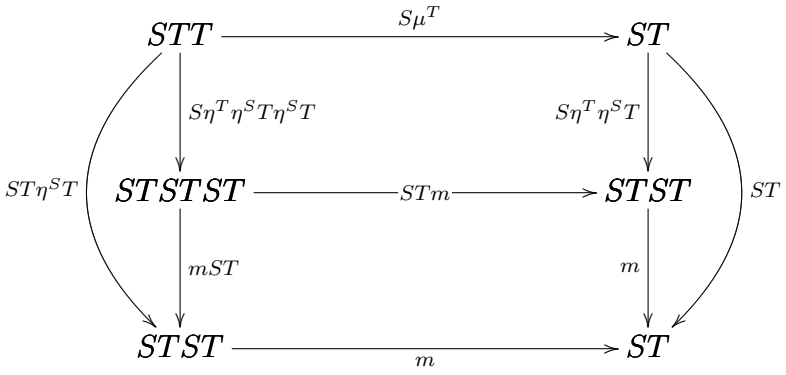
commute, the second because $\mathbf{T} \longrightarrow (\mathbf{ST})_m$ is a triple map[a]. This reduces the problem to showing that an associative law holds between μ^T and m:



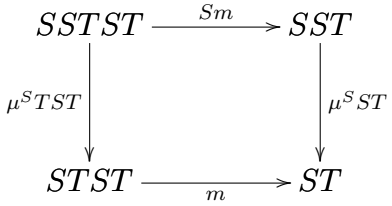
This commutes since $S\mu^T = ST\eta^S T \cdot m$, as follows from the fact that $\mathbf{T} \longrightarrow (\mathbf{ST})_m$ is a

[a]Editor's footnote: The material in the next two diagrams is needed to prove the assertion that the second diagram commutes. We are indebted to Gavin Seal, assisted by Francisco Marmalejo and Christof Schubert for supplying this information.

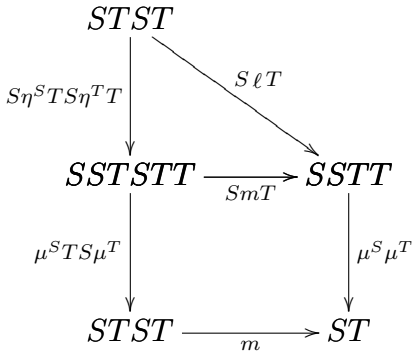
triple map, and also the middle unitary law:



The proof that ℓ is compatible with multiplication in $\mathbf{S}$ is similar; it uses the associative law



The composition $(1) \longrightarrow (2) \longrightarrow (1)$ is clearly the identity. $(2) \longrightarrow (1) \longrightarrow (2)$ is the identity because of

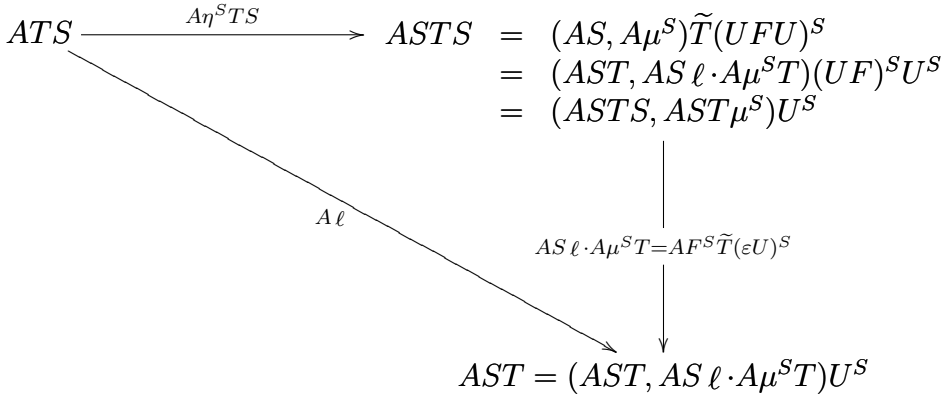


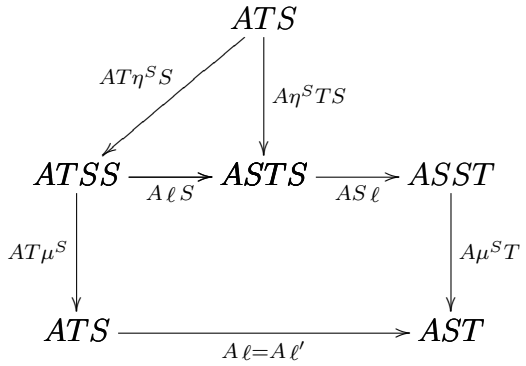
$(3) \longrightarrow (1)$. If $\widetilde{\mathbf{T}}$ is a lifting of $\mathbf{T}$ define ℓ as the composition

$$TS \xrightarrow{\eta^STS} STS = F^SU^STS = F^S\widetilde{T}U^S(FU)^S \xrightarrow{F^S\widetilde{T}(\varepsilon U)^S} F^S\widetilde{T}U^S = (FU)^ST = ST,$$

where the abbreviation $(FU)^S$ stands for $F^SU^S, \ldots$.

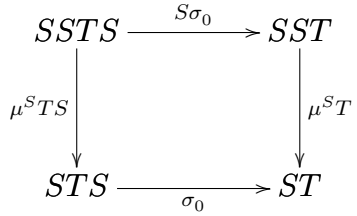
$(1) \longrightarrow (3) \longrightarrow (1)$ is the identity. If we write $\ell \longrightarrow \widetilde{\mathbf{T}} \longrightarrow \ell'$, then $\ell' = \ell$:





$(3) \longrightarrow (1) \longrightarrow (3)$ is the identity. Let us write $\widetilde{\mathbf{T}} \longrightarrow \ell \longrightarrow \widetilde{\widetilde{\mathbf{T}}}$ and prove $\widetilde{\mathbf{T}} = \widetilde{\widetilde{\mathbf{T}}}$.

Any lifting $\widetilde{\mathbf{T}}$ of $\mathbf{T}$ can be written $(A,\sigma)\widetilde{T} = (AT, (A,\sigma)\widetilde{\sigma})$, where $\widetilde{\sigma}\colon U^S TS \longrightarrow U^S T$ is a natural $\mathbf{S}$-structure on $U^S T$. Restricting the lifting to free $\mathbf{S}$-algebras, $AF^S\widetilde{T} = (AST, A\sigma_0)$, where $\sigma_0 = F^S\widetilde{\sigma}\colon STS \longrightarrow ST$ is a natural $\mathbf{S}$-structure on ST, which in addition satisfies an internal associativity relation involving μ^S:



This follows from the fact that $A\mu^S\colon ASF^S \longrightarrow AF^S$ is an $\mathbf{S}$-algebra map.

Similarly, write $(A,\sigma)\widetilde{\widetilde{T}} = (AT, (A,\sigma)\widetilde{\widetilde{\sigma}}), AF^S\widetilde{\widetilde{T}} = (AST, A\sigma_1)$.

We must show that $\widetilde{\sigma} = \widetilde{\widetilde{\sigma}}$. This is done first for free $\mathbf{S}$-algebras, i.e. $\sigma_0 = \sigma_1$, and then the result is deduced for all $\mathbf{S}$-algebras by means of the canonical epimorphism

$\sigma\colon AF^S \longrightarrow (A,\sigma)$. If $\widetilde{\mathbf{T}} \longrightarrow \ell$ then ℓ is the composition

$$\begin{array}{ccccc} ATS & \xrightarrow{\eta^S TS} & ASTS & = & AF^S\widetilde{T}U^SF^SU^S \\ & & & = & (AST, A\sigma_0)U^SF^SU^S \\ & & & = & (ASTS, AST\mu^S)U^S \\ & & & & \downarrow{\scriptstyle \sigma_0} \\ & \searrow^{\ell} & & & \\ & & AST = (AST, A\sigma_0)U^S & & \end{array}$$

Now, $(AS, A\mu^S)\widetilde{\widetilde{T}} = (AST, A\sigma_1)$. But since $\ell \longrightarrow \widetilde{\widetilde{\mathbf{T}}}$,

$$\begin{aligned} (AS, A\mu^S)\widetilde{\widetilde{T}} &= (AST, AS\,\ell\cdot A\mu^S T) \\ &= (AST, AS\eta^S TS\cdot AS\sigma_0\cdot A\mu^S T) \\ &= (AST, A\sigma_0). \end{aligned}$$

Thus $\sigma_1 = \sigma_0$. Applying $\widetilde{\widetilde{T}}$ and $\widetilde{T}$ to the canonical epimorphism of the free algebra,

$$\begin{array}{ccc} AF^S\widetilde{\widetilde{T}} = (AST, A\sigma_1) & \xrightarrow{\sigma T} & (AT, (A,\sigma)\widetilde{\widetilde{\sigma}}) = (A,\sigma)\widetilde{\widetilde{T}} \\ \| & & \| \\ AF^S\widetilde{T} = (AST, A\sigma_0) & \xrightarrow[\sigma T]{} & (AT, (A,\sigma)\widetilde{\sigma}) = (A,\sigma)\widetilde{T} \end{array}$$

But $A\eta^S T\cdot\sigma T = AT$. A general fact in any tripleable category is that if $f\colon (A,\sigma)\longrightarrow(A',\sigma')$, $f\colon (A,\sigma)\longrightarrow(A',\sigma'')$, and f is a split epimorphism in $\mathbf{A}$, then $\sigma' = \sigma''$. Thus $\widetilde{\sigma} = \widetilde{\widetilde{\sigma}}$.

Of course, $\widetilde{\widetilde{\eta}} = \widetilde{\eta}$, $\widetilde{\widetilde{\mu}} = \widetilde{\mu}$, since these are just the unique liftings of η, μ into $\mathbf{A}^{\mathbf{S}}$ and do not depend on ℓ. Thus $\widetilde{\mathbf{T}} = \widetilde{\widetilde{\mathbf{T}}}$. ∎

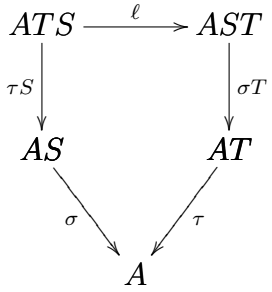
2. Algebras over the composite triple

Let $\ell\colon TS\longrightarrow ST$ be a distributive law, and $\mathbf{ST}, \widetilde{\mathbf{T}}$ the corresponding composite and lifted triples.

PROPOSITION. *Let (A,ξ) be an $\mathbf{ST}$-algebra. Since $\mathbf{S}, \mathbf{T} \longrightarrow \mathbf{ST}$ are triple maps, the compositions*

$$\begin{array}{ccccc} AS & \xrightarrow{AS\eta^T} & AST & \xleftarrow{A\eta^S T} & AT \\ & \searrow^{\sigma} & \downarrow{\scriptstyle \xi} & \swarrow_{\tau} & \\ & & A & & \end{array}$$

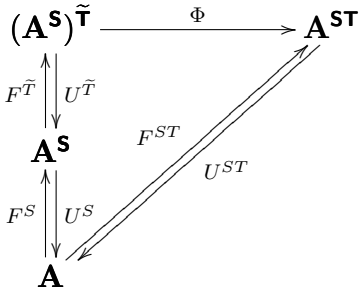
are $\mathbf{S}$- *and* $\mathbf{T}$-*structures on* A*, and it turns out that* σ *is "*ℓ*-distributive" over* τ*:*



A $\widetilde{\mathbf{T}}$*-algebra in* $\mathbf{A}^{\mathbf{S}}$ *consists of an* $\mathbf{S}$*-algebra* (A,σ) *with a* $\widetilde{\mathbf{T}}$*-structure* $\tau\colon (A,\sigma)\widetilde{T}\longrightarrow(A,\sigma)$*. Thus* τ *must be both a* $\mathbf{T}$*-structure* $AT\longrightarrow A$ *and an* $\mathbf{S}$*-algebra map. The latter condition is equivalent to* ℓ*-distributivity of* σ *over* τ*. The above therefore defines a functor*

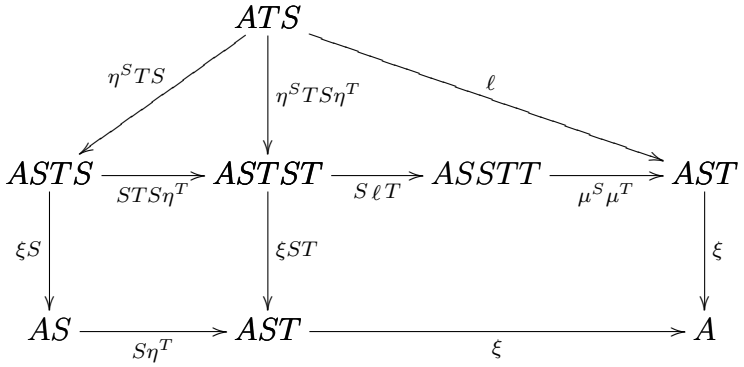
$$(\mathbf{A}^{\mathbf{S}})^{\widetilde{\mathbf{T}}} \xleftarrow{\ \Phi^{-1}\ } \mathbf{A}^{\mathbf{ST}}$$

Finally, the triple induced in $\mathbf{A}$ *by the composite adjoint pair below is exactly the* ℓ*-composite* $\mathbf{ST}$*. The "semantical comparison functor"* Φ *is an isomorphism of categories, with the above* Φ^{-1} *as inverse.*

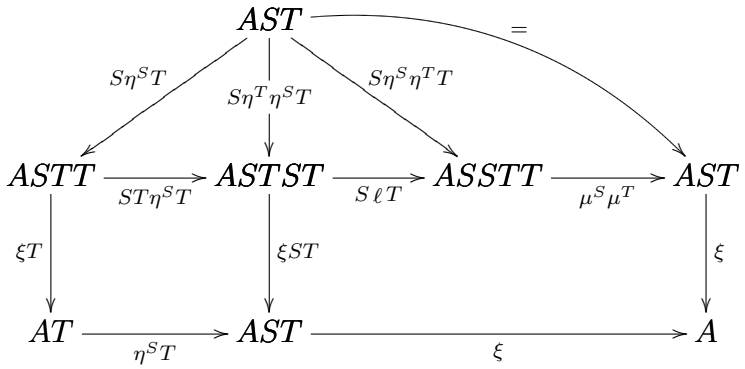


The formula for Φ *is* $(A,\sigma,\tau)\Phi=(A,\sigma T\cdot\tau)$ *in this context.*

Proof. First, distributivity holds between σ,τ.



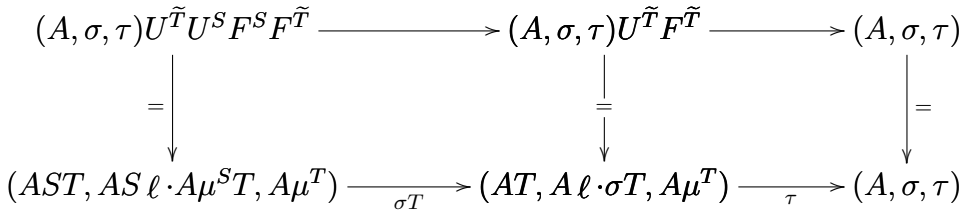
commutes, so we only need to show that $\xi = \sigma T \cdot \tau$. (This is also the essential part in proving that $\Phi\Phi^{-1} = \Phi^{-1}\Phi = \mathrm{id}$.)



Now compute the composite adjointness. The formula for $F^{\widetilde{T}}$ is

$$(A, \sigma) \longrightarrow (AT, A\,\ell \cdot \sigma T, A\mu^T)$$

Thus $F^S F^{\widetilde{T}} U^{\widetilde{T}} U^S = ST$. Clearly the composite unit is $\eta^S\eta^T$. As for the counit, that is the contraction



The multiplication in a triple induced by an adjoint pair is always the value of the counit on free objects, here objects of the form $AF^S F^{\widetilde{T}} = (AST, AS\,\ell\cdot A\mu^S T, AS\mu^T)$. Thus the multiplication in the induced triple is

$$AST\,ST \xrightarrow{AS\,\ell T\cdot A\mu^S TT} ASTT \xrightarrow{AS\mu^T} AST$$

which is exactly that defined by the given distributive law ℓ. The composite adjointness $\mathbf{A} \longrightarrow (\mathbf{A^S})^{\widetilde{\mathbf{T}}} \longrightarrow \mathbf{A}$ therefore induces the ℓ-composite triple $\mathbf{ST}$.

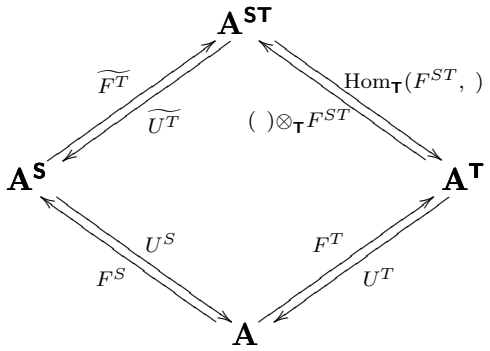
By the universal formula for Φ and the above counit formula,

$$\begin{aligned}(A,\sigma,\tau)\Phi &= (A, (A,\sigma,\tau)((U^{\widetilde{T}}\varepsilon^S F^{\widetilde{T}})\widetilde{\varepsilon^T})U^{\widetilde{T}}U^S) \\ &= (A, \sigma T\cdot\tau).\end{aligned}$$

■

3. Distributive laws and adjoint functors

A distributive law enables four pairs of adjoint functors to exist, all of which are tripleable.



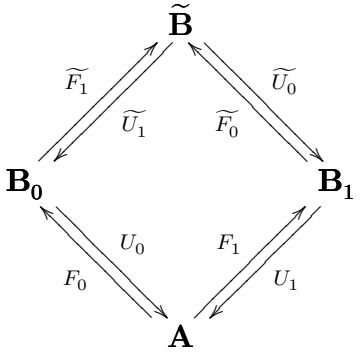
Here $\widetilde{F^T} = \widetilde{F^T}\Phi$, $\widetilde{U^T} = \Phi^{-1}U^{\tilde{T}}$ are the liftings of F^T, U^T into $\mathbf{A}^{\mathbf{S}}$ given by the Proposition of Section 2. $(\ ) \otimes_{\mathbf{T}} F^{ST}$ and its adjoint are induced by the triple map $\eta^S T\colon \mathbf{T} \longrightarrow \mathbf{ST}$ as described in the Introduction. $\widetilde{F^T}$ could be written $(\ ) \otimes_{\mathbf{S}} F^{ST}$, of course.

Since the composite underlying $\mathbf{A}$-object functors $\mathbf{A}^{\mathbf{ST}} \longrightarrow \mathbf{A}$ are equal, the natural map e described in the Appendix is induced. It is a functorial equality

$$U^S F^T \xrightarrow{\ =\ } \widetilde{F^T} \cdot \mathrm{Hom}_{\mathbf{T}}(F^{ST},\ ).$$

The above functorial equality, or isomorphism in general, will be referred to as "distributivity". I now want to demonstrate a converse, to the effect that if an adjoint square is commutative and distributive, then distributive laws hold between the triples and cotriples that are present.

Proposition. *Let*



be an adjoint square and assume it commutes by virtue of adjoint natural isomorphisms $u\colon \widetilde{U_1}U_0^{\sim} \longrightarrow \widetilde{U_0}U_1$, $f\colon F_1\widetilde{F_0} \xrightarrow{\sim} F_0\widetilde{F_1}$. *Let* $\mathbf{T}_0$, $\mathbf{T}_1$, $\widetilde{\mathbf{G}_1}$, $\widetilde{\mathbf{G}_0}$ *be the triples in* $\mathbf{A}$ *and cotriples in* $\mathbf{B}$ *which are induced. Let* $e\colon U_0F_1 \longrightarrow \widetilde{F_1}\widetilde{U_0}$, $e'\colon U_1F_0 \longrightarrow \widetilde{F_0}\widetilde{U_1}$ *be defined as in the Appendix. The adjoint square is* distributive *(in an asymmetrical sense,* 0 *over* 1*) if* e *is an isomorphism. Assume this. Then with* φ, ψ *the isomorphisms defined in the proof*

(and induced by e), we have that

$$\begin{array}{ccccc}
T_1T_0 & & \xrightarrow{\quad\ell\quad} & & T_0T_1 \\
\Big\downarrow{\scriptstyle =} & & & & \Big\uparrow{\scriptstyle \varphi^{-1}} \\
F_1U_1F_0U_0 & \xrightarrow[F_1e'U_0]{} & F_1\widetilde{F_0}\widetilde{U_1}U_0 & \xrightarrow[f\widetilde{U_1}U_0]{} & F_0\widetilde{F_1}\widetilde{U_1}U_0
\end{array}$$

$$\begin{array}{ccccc}
\widetilde{G_1}\widetilde{G_0} & & \xrightarrow{\quad\lambda\quad} & & \widetilde{G_0}\widetilde{G_1} \\
\Big\uparrow{\scriptstyle \psi^{-1}} & & & & \Big\downarrow{\scriptstyle =} \\
\widetilde{U_1}U_0F_0\widetilde{F_1} & \xrightarrow[uF_0\widetilde{F_1}]{} & \widetilde{U_0}U_1F_0\widetilde{F_1} & \xrightarrow[\widetilde{U_0}e'F_1]{} & \widetilde{U_0}\widetilde{F_0}\widetilde{U_1}\widetilde{F_1}
\end{array}$$

are distributive laws of $\mathbf{T}_0$ *over* $\mathbf{T}_1$, $\widetilde{\mathbf{G}_1}$ *over* $\widetilde{\mathbf{G}_0}$.

If the adjoint square is produced by a distributive law $TS \longrightarrow ST$ *as described at the start of Section 3, so that* $\mathbf{S}$ *corresponds to* $\mathbf{T}_0$ *and* $\mathbf{T}$ *to* $\mathbf{T}_1$*, then the distributive law given by the above formula is the original one.*

Proof. Let

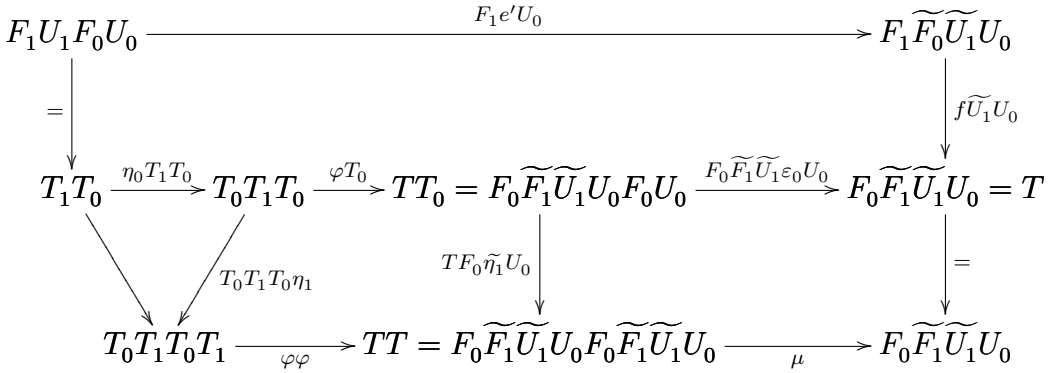
$$\mathbf{T} = \begin{cases} T = F_0\widetilde{F_1}\widetilde{U_1}U_0 \colon \mathbf{A} \longrightarrow \mathbf{A} \\ \eta = \eta_0(F_0\widetilde{\eta_1}U_0) \colon \mathbf{A} \longrightarrow T \\ \mu = F_0\widetilde{F_1}((\widetilde{U_1}\varepsilon_0\widetilde{F_1})\widetilde{\varepsilon_1})\widetilde{U_1}U_0 \colon TT \longrightarrow T \end{cases}$$

be the total triple induced by the left hand composite adjointness. e, u induce a natural isomorphism

$$\begin{array}{ccccc}
T_0T_1 & & \xrightarrow{\quad\varphi\quad} & & T \\
\Big\downarrow{\scriptstyle =} & & & & \Big\downarrow{\scriptstyle =} \\
F_0U_0F_1U_1 & \xrightarrow[F_0eU_1]{} & F_0\widetilde{F_1}\widetilde{U_0}U_1 & \xrightarrow[F_0\widetilde{F_1}u^{-1}]{} & F_0\widetilde{F_1}\widetilde{U_1}U_0
\end{array}$$

By transfer of structure, any functor isomorphic to a triple also has a triple structure. Thus we have an isomorphism of triples $\varphi \colon (\mathbf{T}_0\mathbf{T}_1)_m = (T_0T_1, \eta_0\eta_1, m) \longrightarrow \mathbf{T}$. Actually, the diagrams in the Appendix show that φ transfers units as indicated, and m is the quantity that is defined via the isomorphism. A short calculation also shows that m is middle-unitary. By (1)$\longleftrightarrow$(2), Proposition of Section 1, m is induced by the distributive

law $(\eta_0 T_1 T_0 \eta_1)m\colon T_1T_0 \longrightarrow T_0T_1$. Now, consider the diagram

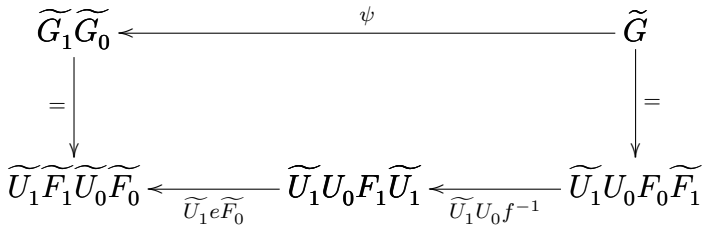


The upper figure commutes, by expanding φ, and naturality. Its top line is $\ell\varphi$. Thus $\ell = (\eta_0T_1T_0\eta_1)(\varphi\varphi)\mu\varphi^{-1} = (\eta_0T_1T_0\mu_1)m$ is a distributive law.

The proof that λ is a distributive law is dual. One defines the total cotriple

$$\mathbf{G} = \begin{cases} \widetilde{G} = \widetilde{U_1}U_0F_0\widetilde{F_1}\colon \widetilde{B} \longrightarrow \widetilde{\mathbf{B}} \\ \widetilde{\varepsilon} = (\widetilde{U_1}\varepsilon_0\widetilde{F_1})\widetilde{\varepsilon_1}\colon \widetilde{G} \longrightarrow \widetilde{\mathbf{B}} \\ \widetilde{\delta} = \widetilde{U_1}U_0(\eta_0(F_0\widetilde{\eta_1}U_0))F_0\widetilde{F_1}\colon \widetilde{G} \longrightarrow \widetilde{G}\widetilde{G} \end{cases}$$

and uses the isomorphism



to induce a similar isomorphism of cotriples $\psi\colon \mathbf{G} \longrightarrow (\mathbf{G}_1\mathbf{G}_0)_d$. Finally, if the original adjoint square is produced by a distributive law $\ell\colon TS \longrightarrow ST$, the Proposition of Section 2 shows that the total triple is the ℓ-composite $\mathbf{ST}$, and φ is an identity map. ∎

One can easily obtain distributive laws of mixed type, for example $\widetilde{\mathbf{T}}_1\mathbf{G}_0 \longrightarrow \mathbf{G}_0\widetilde{\mathbf{T}}_1$.

Remark on structure-semantics of distributive laws.. Triples in $\mathbf{A}$ give rise to adjoint pairs over $\mathbf{A}$, $\mathbf{A} \longrightarrow \mathbf{A}^{\mathbf{T}} \longrightarrow \mathbf{A}$, and adjoint pairs $\mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{A}$ give rise to triples in $\mathbf{A}$. This yields the structures-semantics adjoint pair for triples:

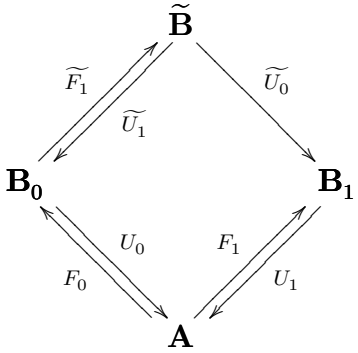
$$\text{Ad } \mathbf{A} \underset{\sigma}{\overset{\breve{\sigma}}{\rightleftarrows}} (\text{Trip } \mathbf{A})^*.$$

This adjoint pair is a reflection ($\sigma\breve{\sigma} = \text{id}$) and the comparison functor $\Phi\colon \text{id} \longrightarrow \breve{\sigma}\sigma$ is the unit.

Something similar can be done for distributive adjoint situations over $\mathbf{A}$ and distributive laws.

Define a *distributive law in* $\mathbf{A}$ to be a triple $(\mathsf{S}, \mathsf{T}, \ell)$ where S, T are triples and $\ell\colon TS \longrightarrow ST$ is a distributive law. A *map* $(\varphi, \psi)\colon (\mathsf{S}, \mathsf{T}\,\ell) \longrightarrow (\mathsf{S}', \mathsf{T}', \ell')$ is a pair of triple maps $\mathsf{S} \longrightarrow \mathsf{S}', \mathsf{T} \longrightarrow \mathsf{T}'$ which is compatible with ℓ, ℓ'. Let $\mathrm{Dist}(\mathbf{A})$ be this category.

A *distributive adjoint situation over* $\mathbf{A}$ means a diagram



where (F_0, U_0), $(\widetilde{F_1}, \widetilde{U_1})$, (F_1, U_1) are adjoint pairs, $\widetilde{U_1}\widetilde{U_0} = U_0U_1$, and the natural map $U_0F_1 \longrightarrow \widetilde{F_1}\widetilde{U_0}$ is an isomorphism. A *map* of such adjoint situations consists of functors $\mathbf{B}_0 \longrightarrow \mathbf{B}'_0$, $\mathbf{B}_1 \longrightarrow \mathbf{B}'_1$, $\widetilde{\mathbf{B}} \longrightarrow \widetilde{\mathbf{B}'}$ commuting with the underlying object functors $\widetilde{U_1}$, U_0, $\widetilde{U_0}$, U_1, ...

Distributive laws give rise to distributive adjoint situations over $\mathbf{A}$, and vice versa (note that $\ell = (\eta_0 T_1 T_0 \eta_1)m$ and m does not involve $\widetilde{F_0}$). Thus we have an adjoint pair

$$\textbf{Distributive Adj } \mathbf{A} \underset{\sigma}{\overset{\breve{\sigma}}{\rightleftarrows}} (\textbf{Dist } \mathbf{A})^*.$$

The structure functor $\breve{\sigma}$ is left adjoint to the semantics functor σ, $\sigma\breve{\sigma} = \mathrm{id}$, and the unit is a combination of Φ's. This is the correct formulation of the above Proposition.

4. Examples

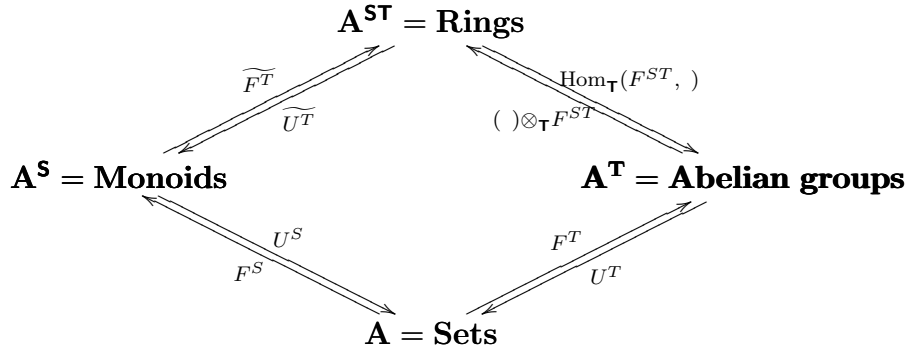
(1) Multiplication and addition. Let $\mathbf{A}$ be the category of sets, let S be the free monoid triple in $\mathbf{A}$, and T the free abelian group triple. The $\mathbf{A}^{\mathsf{S}}$ is the category of monoids and $\mathbf{A}^{\mathsf{T}}$ is the category of abelian groups. For every set X the usual interchange of addition and multiplication

$$\prod_{i=0}^{m} \sum_{j_i=0}^{n_i} x_{ij_i} \longrightarrow \sum_{j_0=0}^{n_0} \cdots \sum_{j_m}^{n_m} \prod_{i=0}^{m} x_{ij_i}$$

can be interpreted as a natural transformation $XTS \xrightarrow{\ell} XST$ and is a distributive law of multiplication over addition, that is, of S over T, in the formal sense.

The composite $\mathbf{ST}$ is the free ring triple. XST is the polynomial ring $\mathbf{Z}[X]$ with the elements of X as noncommuting indeterminates.

The canonical diagram of adjoint functors is:



$\widetilde{F^T}$ is the free abelian group functor lifted into the category of monoids, and is known as the "monoid ring" functor. $\mathrm{Hom}_{\mathbf{T}}(F^{ST},\ )$ is the forgetful functor. If A is an abelian group, the value of the left adjoint, $A\otimes_{\mathbf{T}} F^{ST}$, is the $\mathbf{Z}$-tensor ring generated by A, namely $\mathbf{Z}+A+A\otimes A+\ldots$. The natural map $U^SF^T \longrightarrow \widetilde{F^T}\cdot \mathrm{Hom}_{\mathbf{T}}(F^ST,\ )$ is the identity, that is distributivity holds. Both compositions give the free abelian groups generated by the elements of monoids.

The scheme is: the distributive law $\ell\colon TS\longrightarrow ST$ produces the adjoint square, which, being distributive (Section 3), induces a distributive law $\lambda\colon \mathbf{G_{Ab}G_{Mon}} \longrightarrow \mathbf{G_{Mon}G_{Ab}}$, where $G_{\mathbf{Mon}} = \widetilde{U^T}\widetilde{F^T}$, $G_{\mathbf{Ab}} = \mathrm{Hom}_{\mathbf{T}}(F^{ST},\ )\otimes_{\mathbf{T}} F^{ST}$. This λ is that employed by Barr in his *Composite cotriples*, this volume (Theorem 4.6).

A distributive law $ST\longrightarrow TS$ would have the air of a universal solution to the problem of factoring polynomials into linear factors. This suggests that the composite TS has little chance of being a triple.

(2) Constants. Any set C can be interpreted as a triple in the category of sets, $\mathbf{A}$, via the coproduct injection and folding map $X\longrightarrow C+X, C+C+X\longrightarrow C+X$. $\mathbf{A}^{C+(\ )}$ is the category of sets with C as constants. For example, if $C=1$, $\mathbf{A}^{1+(\ )}$ is the category of pointed sets.

Let $\mathbf{T}$ be any triple in $\mathbf{A}$. A natural map $\ell\colon C+XT\longrightarrow (C+X)T$ is defined in an obvious way, using $C\eta$. ℓ is a distributive law of $C+(\ )$ over $\mathbf{T}$. The composite triple $C+\mathbf{T}$ has as algebras $\mathbf{T}$-algebras furnished with the set C as constants.

(3) Group actions. Let π be a monoid or group. π can be interpreted as a triple in $\mathbf{A}$, the category of sets, via cartesian product:

$$X\xrightarrow{(x,1)} X\times\pi, \qquad X\times\pi\times\pi\xrightarrow{(X,\sigma_1\sigma_2)} X\times\pi.$$

$\mathbf{A}^{\pi}$ is the category of π-sets. If T is any functor $\mathbf{A}\longrightarrow\mathbf{A}$, there is a natural map

$$XT\times\pi\xrightarrow{\ \ell\ } (X\times\pi)T$$

Viewing $XT \times \pi$ as a π-fold coproduct of XT with itself, ℓ has the value

$$XT \xrightarrow{(X,\sigma)T} (X \times \pi)T$$

on the σ-th cofactor, if (X, σ) is the map $X \longrightarrow (X, \sigma)$. If $\mathbf{T} = (T, \eta, \mu)$ is a triple in $\mathbf{A}$, ℓ is a distributive law of π over $\mathbf{T}$. The algebras over the composite triple $\pi\mathbf{T}$ are $\mathbf{T}$-algebras equipped with π-operations. The elements of π act as $\mathbf{T}$-homomorphisms.

Example (3) can be combined with (2) to show that any triple $\mathbf{S}$ generated by constants and unary operations has a canonical distributive law over any triple $\mathbf{T}$ in $\mathbf{A}$. The $\mathbf{ST}$-algebras are $\mathbf{T}$-linear automata.

(4) No new equations in the composite triple. It is known that if $\mathbf{T}$ is a consistent triple in sets, then the unit $X\eta^T: X \longrightarrow XT$ is a monomorphism for every X. And every triple in sets, as a functor, preservers monomorphisms. Thus if $\mathbf{S}$, $\mathbf{T}$ are consistent triples, and $\ell: TS \longrightarrow ST$ is a distributive law, then the triple maps $\mathbf{S}, \mathbf{T} \longrightarrow \mathbf{ST}$ are monomorphisms of functors.

This means that the operations of $\mathbf{S}$ and of $\mathbf{T}$ are mapped injectively into operations of $\mathbf{ST}$, and no new equations hold among them in the composite.

The triples excluded as "inconsistent" are the terminal triple and one other:

(a) $XT = 1$ for all X,

(b) $0T = 0$, $XT = 1$ for all $X \neq 0$.

(5) Distributive laws on rings as triples in the category of abelian groups. Let S and T be rings. S and T can be interpreted as triples $\mathbf{S}$ and $\mathbf{T}$ in the category of abelian groups, $\mathbf{A}$, via tensor product:

$$A \xrightarrow{a \otimes 1} A \otimes S, \qquad A \otimes S \otimes S \xrightarrow{a \otimes s_0 s_1} A \otimes S.$$

$\mathbf{A}^{\mathbf{S}}$ and $\mathbf{A}^{\mathbf{T}}$ are the categories of S- and T-modules. The usual interchange map of the tensor product, $\ell: T \otimes S \longrightarrow S \otimes T$, gives a distributive law of $\mathbf{S}$ over $\mathbf{T}$. This is just what is needed to make the composite $S \otimes T$ into a ring:

$$S \otimes T \otimes S \otimes T \xrightarrow{S \otimes \ell \otimes T} S \otimes S \otimes T \otimes T \xrightarrow{\text{mult.} \otimes \text{mult.}} S \otimes T.$$

This ring multiplication is the multiplication in the composite triple $\mathbf{ST}$.

The interchange map is adjoint to a distributive law between the adjoint cotriples $\mathrm{Hom}(S, \ ), \mathrm{Hom}(T, \ )$. This is a general fact about adjoint triples.

The identities for a distributive law are especially easy to check in this example, as are certain conjectures about distributive laws.

Let $\mathbf{A}$ be the category of graded abelian groups, and let S and T be graded rings. Then two obvious transpositions of the graded tensor product, $T \otimes S \longrightarrow S \otimes T$, exist:

$$t \otimes s \longrightarrow s \otimes t,$$
$$t \otimes s \longrightarrow (-1)^{\dim s \cdot \dim t} s \otimes t.$$

Both are distributive laws of the triple $\mathbf{S} = (\ ) \otimes S$ over $\mathbf{T} = (\ ) \otimes T$. They give different graded ring structures on $S \otimes T$ and different composite triples $\mathbf{ST}$.

Finally, note the following ring multiplication in $S \otimes T$:

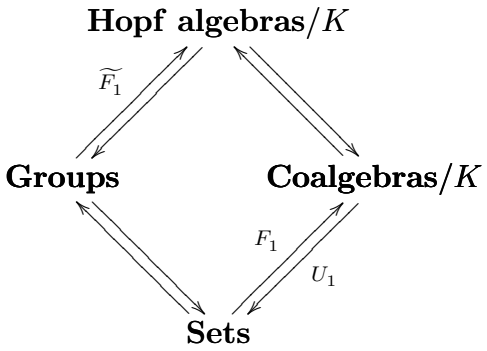
$$(s_0 \otimes t_0)(s_1 \otimes t_1) = (-1)^{\dim s_0 \cdot \dim t_1} s_0 s_1 \otimes t_0 t_1$$

The maps

$$S \xrightarrow{s \otimes 1} S \otimes T \xleftarrow{1 \otimes t} T$$

are still ring homomorphisms, but the "middle unitary" law described in Section 1 does not hold.

A number of problems are open, in such areas as homology, the relation of composites to tensor products of triples, and possible extension of the distributive law formalism to non-tripleable situations, for example, the following, suggested by Knus-Stammbach:



($XF_1 = K(X)$, $CU_1 =$ all $c \in C$ such that $\Delta(c) = c \otimes c$, and $\pi\widetilde{F_1} =$ the group algebra $K(\pi)$.)

5. Appendix

If there are adjoint pairs of functors

$$\mathbf{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{B} \underset{\widetilde{U}}{\overset{\widetilde{F}}{\rightleftarrows}} \widetilde{\mathbf{B}}$$

with

$$\eta\colon \mathbf{A} \longrightarrow FU, \qquad \widetilde{\eta}\colon \mathbf{B} \longrightarrow \widetilde{F}\widetilde{U}$$
$$\varepsilon\colon UF \longrightarrow \mathbf{B}, \qquad \widetilde{\varepsilon}\colon \widetilde{U}\widetilde{F} \longrightarrow \widetilde{\mathbf{B}}$$

then there is a *composite* adjoint pair

$$\mathbf{A} \underset{\widetilde{U}U}{\overset{F\widetilde{F}}{\rightleftarrows}} \widetilde{\mathbf{B}}$$

whose unit and counit are

$$\mathbf{A} \xrightarrow{\eta(F\widetilde{\eta}U)} F\widetilde{F}\widetilde{U}U, \qquad \widetilde{U}UF\widetilde{F} \xrightarrow{(\widetilde{U}\varepsilon\widetilde{F})\widetilde{\varepsilon}} \widetilde{\mathbf{B}}.$$

Given adjoint functors

$$\mathbf{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{B}, \qquad \mathbf{A} \underset{U'}{\overset{F'}{\rightleftarrows}} \mathbf{B}$$

then the diagrams

$$\begin{array}{ccc} F' & \xrightarrow{f} & F \\ {\scriptstyle \eta F'}\downarrow & & \uparrow{\scriptstyle F\varepsilon'} \\ FUF' & \xrightarrow[FuF']{} & FU'F' \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{u} & U' \\ {\scriptstyle U\eta'}\downarrow & & \uparrow{\scriptstyle \varepsilon U'} \\ UF'U' & \xrightarrow[UfU']{} & UFU' \end{array}$$

establish a 1-1 correspondence between morphisms $u\colon U \longrightarrow U'$, $f\colon F' \longrightarrow F$. Corresponding morphisms are called *adjoint*.

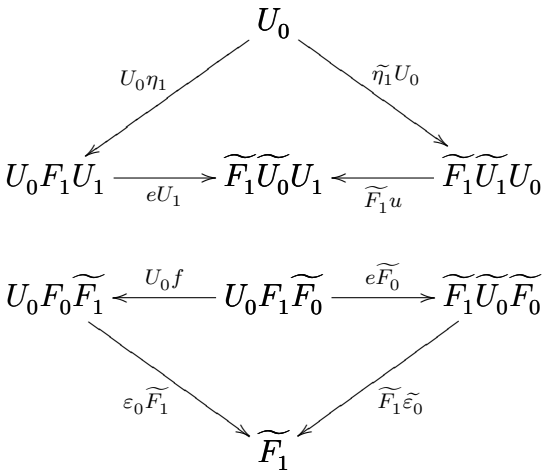
Let

$$\begin{array}{ccccc} & & \widetilde{\mathbf{B}} & & \\ & \overset{\widetilde{F_1}}{\swarrow} & & \overset{\widetilde{U_0}}{\searrow} & \\ & \underset{\widetilde{U_1}}{\searrow} & & \underset{\widetilde{F_0}}{\swarrow} & \\ \mathbf{B_0} & & & & \mathbf{B_1} \\ & \underset{F_0}{\searrow} & & \overset{U_1}{\swarrow} & \\ & \overset{U_0}{\swarrow} & & \underset{F_1}{\searrow} & \\ & & \mathbf{A} & & \end{array}$$

be adjoint pairs of functors, and let $u\colon \widetilde{U_1}U_0 \longrightarrow \widetilde{U_0}U_1$, $f\colon F_1\widetilde{F_0} \longrightarrow F_0\widetilde{F_1}$ be adjoint natural transformations. Then u, f induce a natural transformation e which plays a large role in Section 3:

$$\begin{array}{ccc} U_0F_1\widetilde{F_0}\widetilde{U_0} & \xrightarrow{U_0f\widetilde{U_0}} & U_0F_0\widetilde{F_1}\widetilde{U_0} \\ {\scriptstyle U_0F_1\widetilde{\eta_0}}\uparrow & & \downarrow{\scriptstyle \varepsilon_0\widetilde{F_1}\widetilde{U_0}} \\ U_0F_1 & \xrightarrow{e} & \widetilde{F_1}\widetilde{U_0} \\ {\scriptstyle \widetilde{\eta_1}U_0F_1}\downarrow & & \uparrow{\scriptstyle \widetilde{F_1}\widetilde{U_0}\varepsilon_1} \\ \widetilde{F_1}\widetilde{U_1}U_0F_1 & \xrightarrow[\widetilde{F_1}uF_1]{} & \widetilde{F_1}\widetilde{U_0}U_1F_1 \end{array}$$

The following diagrams commute:



If in addition adjoint maps $u^{-1}\colon \widetilde{U_0}U_1 \longrightarrow \widetilde{U_1}U_0$, $f^{-1}\colon F_0\widetilde{F_1} \longrightarrow F_1\widetilde{F_0}$ are available, they induce a natural map e' with similar unit and counit properties:

