Acta Applicandae Mathematicae **49:** 235–248, 1997.

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Calculus over Commutative Algebras: A Concise User Guide

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(Received: 30 September 1996)

Abstract. A short introduction to the theory of linear differential operators over commutative algebras is exposed.

Mathematics Subject Classifications (1991). 13N10, 16S32, 58A20, 58G05.

Key words: linear differential operators, jets, forms, Spencer complexes, commutative algebras.

All papers of this collection are concerned with various aspects of Calculus over commutative algebras. To help the reader, we expose here the basic facts and definitions mutual for the majority of the papers that follow. The exposition is based on [1] and originates from [2].

The contents of this introductory text are as follows:

(1) Linear differential operators over commutative algebras.
(2) Diff-prolongations and compositions of differential operators.
(3) Derivations.
(4) The Spencer Diff-complex.
(5) Jets.
(6) Jet-prolongations and co-compositions.
(7) Differential forms and the de Rham complex.
(8) The Spencer Jet complex.
(9) Generalizations and specializations of the basic constructions.

1. Linear Differential Operators

Let K be a commutative ring with a unit and A be a commutative associative unitary K-algebra. Consider A-modules P and Q. Then two A-module structures exist in the K-module $\hom_K(P,Q)$:

$$\begin{aligned} &(af)p = a(fp), \qquad (a^+f)p = f(ap), \\ &a \in A,\ f \in \hom_K(P,Q),\ p \in P. \end{aligned} \tag{1}$$

We set $\delta_a(f) = af - a^+f$. Obviously, for any $a, b \in A$, we have

$$\delta_{ab}f = a\delta_b f + b\delta_a f \tag{2}$$

and if R is another A-module and $g \in \hom_K(Q, R)$, then

$$\delta_a(g \circ f) = \delta_a g \circ f + g \circ \delta_a f. \tag{3}$$

DEFINITION 1.1. A K-homomorphism $\Delta \in \hom_K(P, Q)$ is called a linear differential operator of order $\leqslant k$ acting from P to Q, if $(\delta_{a_0} \circ \delta_{a_1} \circ \ldots \circ \delta_{a_k})\Delta = 0$ for any $a_0, a_1, \ldots, a_k \in A$.

The set of all such operators inherits both A-module structures (1) and thus forms an A-bimodule. This bimodule is denoted by $\mathrm{Diff}_k^+(P, Q)$. When considering only one of these module structures, we write $\mathrm{Diff}_k(P, Q)$ or $\mathrm{Diff}_k^+(P, Q)$. From the definition, it follows that 0-order differential operators coincide with A-homomorphisms: $\mathrm{Diff}_0^+(P, Q) = \hom_A(PQ)$.

The correspondence $(P, Q) \Rightarrow \mathrm{Diff}_k^+(P, Q)$ is a bifunctor from the category $\mathcal{M}(A)$ of A-modules to the category of A-bimodules. We also use the notation $\mathrm{Diff}_k^+(P)$ for the bimodule $\mathrm{Diff}_k^+(A, P)$.

Consider the maps

$$i_+\colon \mathrm{Diff}_k(P, Q) \to \mathrm{Diff}_k^+(P, Q), \qquad i^+\colon \mathrm{Diff}_k^+(P, Q) \to \mathrm{Diff}_k(P, Q)$$

induced by the identity $\mathrm{id}\colon \hom_K(P, Q) \to \hom_K(P, Q)$.

PROPOSITION 1.2. *The maps i_+ and i^+ are differential operators of order $\leqslant k$.*

Let $\Delta\colon P \to Q$, $\nabla\colon Q \to R$ be differential operators. Then from (3) one obtains

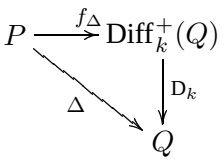
PROPOSITION 1.3. *For any two differential operators $\Delta \in \mathrm{Diff}_k^+(P, Q)$ and $\nabla \in \mathrm{Diff}_l^+(Q, R)$ their composition $\nabla \circ \Delta\colon P \to R$ is an operator of order $\leqslant k + l$.*

For any $k \leqslant l$ one has obvious embeddings $\mathrm{Diff}_k^+(P, Q) \subset \mathrm{Diff}_l^+(P, Q)$ and, consequently, the module $\mathrm{Diff}_*^+(P, Q) = \bigcup_{k \geqslant 0} \mathrm{Diff}_k^+(P, Q)$ is defined. Using Proposition 1.3, it can be seen that $\mathrm{Diff}_*^+(P, P)$ is a filtered associative A-algebra.

2. Diff-Prolongations and Compositions

Fix an A-module Q and consider the functor $\mathrm{Diff}_k^+(\cdot, Q)\colon \mathcal{M}(A) \Rightarrow \mathcal{M}(A)$. This functor is representable. Namely, one has the following proposition:

PROPOSITION 2.1. *For any differential operator $\Delta\colon P \to Q$ of order $\leqslant k$, there exists a uniquely defined homomorphism $f_\Delta\colon P \to \mathrm{Diff}_k^+(Q)$ such that the diagram*



where $\mathrm{D}(\nabla) = \nabla(1)$, $\nabla \in \mathrm{Diff}_k^+(Q)$, *is commutative. The correspondence* $\Delta \mapsto f_\Delta$ *establishes an isomorphism between the functors* $\mathrm{Diff}_k^+(\cdot, Q)$ *and* $\hom_A(\cdot, \mathrm{Diff}_k^+(Q))$. *We say the homomorphism* f_Δ *to be* Diff-*associated with the operator* Δ.

The homomorphism f_Δ is defined by $(f_\Delta(p))a = \Delta(ap)$.

Note that the above defined D_k: $\mathrm{Diff}_k^+(Q) \to Q$ is a differential operator of order $\leqslant k$. Hence, for any two integers $k, l \geqslant 0$, using Propositions 1.3 and 2.1, one obtains the commutative diagram

$$\begin{array}{ccc} \mathrm{Diff}_k^+\mathrm{Diff}_l^+(Q) & \xrightarrow{\mathrm{D}_k} & \mathrm{Diff}_l^+ Q \\ {\scriptstyle c_{k,l}}\downarrow & & \downarrow{\scriptstyle \mathrm{D}_l} \\ \mathrm{Diff}_{k+l}^+ Q & \xrightarrow{\mathrm{D}_{k+l}} & Q \end{array}$$

where $c_{k,l} = c_{k,l}(Q) \stackrel{\text{def}}{=} f_{\mathrm{D}_l \circ \mathrm{D}_k}$. The correspondence $Q \Rightarrow c_{k,l}(Q)$ is a natural transformation of functors $\mathrm{Diff}_k^+\mathrm{Diff}_l^+(\cdot)$ and $\mathrm{Diff}_{k+l}^+(\cdot)$; the homomorphism $c_{k,l}(Q)$ is called *the gluing*, or *the composition homomorphism*. Let $\nabla \in \mathrm{Diff}_k^+\mathrm{Diff}_l^+ Q$ and $a \in A$. Then from the definitions, it follows that

$$c_{k,l}(\nabla)(a) = f_{\mathrm{D}_l \circ \mathrm{D}_k}(\nabla)(a) = \mathrm{D}_l\mathrm{D}_k(a^+\nabla) = \nabla(a)(1). \tag{4}$$

In the sequel, we use the notation $\mathrm{Diff}_{i_1,\ldots,i_n}^+(\cdot)$ for the iterated functor $\mathrm{Diff}_{i_1}^+(\ldots(\mathrm{Diff}_{i_n}^+(\cdot))\ldots)$.

Consider now an operator Δ: $P \to Q$ of order $\leqslant k$. Then the operator $\Delta_{(l)} = \Delta \circ \mathrm{D}_l$: $\mathrm{Diff}_l^+(P) \to Q$ is of order $\leqslant k + l$ and is called *the* lth Diff-*prolongation* of Δ. Since for any $l \leqslant l'$, one has $\mathrm{D}_{l'} \mid_{\mathrm{Diff}_l^+(P)} = \mathrm{D}_l$, from where it follows that $\Delta_{(l')} \mid_{\mathrm{Diff}_l^+(P)} = \Delta_{(l)}$. Hence, one obtains *the infinite* Diff-*prolongation* $\Delta_{(*)}$: $\mathrm{Diff}_*^+(P) \to Q$ of the operator Δ. From the definition, it also follows that there exists the commutative diagram

$$\begin{array}{ccc} \mathrm{Diff}_l^+ P & \xrightarrow{f_{\Delta_{(l)}}} & \mathrm{Diff}_{k+l}^+ P \\ & {\scriptstyle \Delta_{(l)}}\searrow & \downarrow{\scriptstyle \mathrm{D}_{k+l}} \\ & & Q \end{array}$$

and $f_{\Delta_{(l')}} \mid_{\mathrm{Diff}_l^+ P} = f_{\Delta_{(l)}}$ for any $l' \geqslant l$. Thus, we obtain the filtered A-module homomorphism $f_{\Delta_{(l')}}$: $\mathrm{Diff}_*^+(P) \to \mathrm{Diff}_*^+(Q)$ of filtration k. It is called *the* Diff-*prolongation homomorphism*. In particular, the composition homomorphism

$c_{*,l}$: $\mathrm{Diff}^+_{*,l}Q \to \mathrm{Diff}^+_*Q$ coincides with the prolongation homomorphism for the operator D_l. Note also that the commutative diagram

$$\begin{array}{ccc} \mathrm{Diff}^+_{l,l'}P & \xrightarrow{c_{l',l}} & \mathrm{Diff}^+_{l+l'}P \\ {\scriptstyle \mathrm{D}_{l'}}\downarrow & \searrow^{(\Delta_{(l)})_{(l')}} & \downarrow{\scriptstyle \Delta_{(l+l')}} \\ \mathrm{Diff}^+_l P & \xrightarrow{\Delta_{(l)}} & Q \end{array}$$

establishes the relations between iterated prolongations and compositions.

3. Derivations

DEFINITION 3.1. Let P be an A-module. A K-linear map $X\colon A \to P$ satisfying the Leibnitz rule

$$X(ab) = aX(b) + bX(a), \quad a, b \in A,$$

is called a P-valued derivation of A, or P-derivation.

The set of all such derivation is denoted by $D(P)$. Obviously, any $X \in D(P)$ is a differential operator of order $\leqslant 1$ and $D(P)$ forms a submodule in $\mathrm{Diff}_1 P$.

EXAMPLE 3.2. Fix an operator $\Delta \in \mathrm{Diff}^+_k(P, Q)$. Then, due to (2) the correspondence $a \mapsto \delta_a(\Delta)$ is a $\mathrm{Diff}^+_{k-1}(P, Q)$-derivation.

Obviously, an operator $\Delta \in \mathrm{Diff}_1 P$ lies in $D(P)$ if and only if $\Delta(1) = 0$. Hence, we have the exact sequence

$$0 \to D(P) \xrightarrow{S^1_1} \mathrm{Diff}^+_1 P \xrightarrow{\mathrm{D}_1} P \to 0, \tag{5}$$

where S^1_1 is the composition of the natural embedding $D(P) \hookrightarrow \mathrm{Diff}_1 P$ with the operator $i_+\colon \mathrm{Diff}_1 P \to \mathrm{Diff}^+_1 P$. This the first-order Spencer Diff-complex. In the next subsection we construct its nth-order generalizations. To do this, we shall need the notion of polyderivations.

Let P be an A-module and $N \subset P$ be a subset. Define

$$D(N \subset P) \stackrel{\text{def}}{=} \{X \in D(P) \mid X(A) \subset N\}.$$

Now we inductively define modules $D_i(P),\ i \geqslant 0$, together with monomorphic differential operators $D_i(P) \hookrightarrow (\mathrm{Diff}^+_1)^i P$, where

$$(\mathrm{Diff}^+_1)^i \stackrel{\text{def}}{=} \mathrm{Diff}^+_{\underbrace{1, \ldots, 1}_{i \text{ times}}}.$$

Namely, we set $D_0(P) = P, D_1(P) = D(P)$ and assuming that the embedding $D_{i-1}(P) \hookrightarrow (\mathrm{Diff}_1^+)^{i-1}P$ has been defined, set

$$D_i(P) \stackrel{\text{def}}{=} D(D_{i-1}(P) \subset (\mathrm{Diff}_1^+)^i P).$$

Corresponding embedding is defined as

$$D_i(P) = D(D_{i-1}(P) \subset (\mathrm{Diff}_1^+)^i P) \subset D((\mathrm{Diff}_1^+)^{i-1}P \subset (\mathrm{Diff}_1^+)^i P).$$

From the definition one obtains

PROPOSITION 3.3. *For any $i \geqslant 0$ the set $D_i(P)$ is an A-module while the correspondence $P \Rightarrow D_i(P)$ is a functor in $\mathcal{M}(A)$.*

Note that for any $X \in D_i(P)$ and $a \in A$ one has $X(a) \in D_{i-1}(P)$ and by induction $X(a_1, \ldots, a_j) \in D_{i-j}(P)$ for any $a_1, \ldots, a_j \in A$ and $i \geqslant j$. For $j > i$, we formally set $X(a_1, \ldots, a_j) = 0$.

PROPOSITION 3.4. *Let $i \geqslant 2$ and $X \in \hom_K(A, D_{i-1}(P))$. Then $X \in D_i(P)$ if and only if $X(ab) = aX(b) + bX(a)$ and $X(a, b) + X(b, a) = 0$ for any $a, b \in A$.*

This result motivates the following definition:

DEFINITION 3.5. Elements of $D_i(P)$ are called P-valued i-derivations of the algebra A.

4. The Spencer Diff-Complex

The construction of the Spencer Diff-complex in the general situation is based on the following lemma:

LEMMA 4.1. *Let P^+ be an A-bimodule and $P_1, P_2 \subset P$ be submodules with respect to the first A-module structure. Consider a homomorphism $\varphi\colon\ P^+ \to P^+$ such that $\varphi\,|_{P_1}\colon\ P_1 \to P_2$ is a differential operator of order $\leqslant 1$. Then the map $D\varphi\colon\ D(P_1 \subset P^+) \to D(P_2 \subset P^+)$ defined by $(D\varphi)\Delta =_{\text{def}} \varphi \circ \Delta$ is an operator of order $\leqslant 1$ as well. Moreover,* $\ker \varphi = 0$ *implies* $\ker D\varphi = 0$.

Now we inductively define the sequence of injective operators $\delta^i\colon\ D_i(P) \to D_{i-1}\mathrm{Diff}_1^+ P$ of order $\leqslant 1$, starting with $\delta^1 =_{\text{def}} S_1^1$ (see (5)). Assume that $\delta^1, \ldots, \delta^{i-1},\ i \geqslant 2$, have been defined. By definition, one has

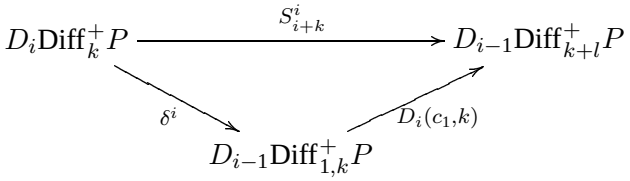
$$D_i(P) = D(D_{i-1}(P) \subset (\mathrm{Diff}_1^+)^{i-1}P)$$

and

$$D_{i-1}\mathrm{Diff}_1^+ P = D(D_{i-2}\mathrm{Diff}_1^+ P \subset (\mathrm{Diff}_1^+)^{i-1}P).$$

Hence, using Lemma 4.1, we may set $\delta^i =_{\text{def}} D\delta^{i-1}$.

Let operators S^i_{i+k}: $D_i\text{Diff}^+_k P \to D_{i-1}\text{Diff}^+_{k+1}P$ be defined by commutativity of the diagram



PROPOSITION 4.2. *For any $i, k \geqslant 0$, one has $S^i_{i+k} \circ S^{i+1}_{i+k} = 0$.*

DEFINITION 4.3. The sequence

$$\begin{aligned} 0 \longrightarrow D_{i+k}(P) &\xrightarrow{S^{i+k}_{i+k}} D_{i+k-1}\text{Diff}^+_1 P \longrightarrow \ldots \longrightarrow \\ D_i\text{Diff}^+_k P &\xrightarrow{S^i_{i+k}} D_{i-1}\text{Diff}^+_{k+1}P \longrightarrow \ldots \longrightarrow \\ \text{Diff}^+_{i+k}P &\xrightarrow{S^0_{i+k}=D_{i+k}} P \longrightarrow 0 \end{aligned} \tag{6}$$

is called the Spencer Diff-complex of order $k+i$ for the module P.

Denote this complex by $\mathcal{S}_{i+k}P$. Consider now the natural embeddings

$$D_i\text{Diff}^+_k P \hookrightarrow D_i\text{Diff}^+_{k+1}P$$

induced by the embeddings $\text{Diff}^+_k P \hookrightarrow \text{Diff}^+_{k+1}P$. Then the diagram

$$\begin{array}{ccc} D_i\text{Diff}^+_k P & \xrightarrow{S^i_{i+k}} & D_{i-1}\text{Diff}^+_{k+1}P \\ \downarrow & & \downarrow \\ D_i\text{Diff}^+_{k+1}P & \xrightarrow{S^i_{i+k+1}} & D_{i-1}\text{Diff}^+_{k+2}P \end{array}$$

is commutative. Consequently, morphisms $\mathcal{S}_{i+k}P \to \mathcal{S}_{i+k+1}$ of complexes exist and one can define the complex $\mathcal{S}_*P$

$$\begin{aligned} \ldots \longrightarrow D_i\text{Diff}^+_* P &\xrightarrow{S^i_*} D_{i-1}\text{Diff}^+_* P \longrightarrow \ldots \longrightarrow \\ D_1\text{Diff}^+_* P &\xrightarrow{S^0_*} \text{Diff}^+_* P \longrightarrow 0 \end{aligned}$$

as the inductive limit of $\mathcal{S}_{i+k}$ with respect to these morphisms. Both $\mathcal{S}_{i+k}(\cdot)$ and $\mathcal{S}_*(\cdot)$ are functors from $\mathcal{M}(A)$ to the category of differential complexes over A.

Moreover, let P, Q be A-modules and $\Delta\colon\ P \to Q$ be an sth order differential operator. Then the map $f_{\Delta_{(*)}}\colon\ \mathrm{Diff}^+_* P \to \mathrm{Diff}^+_* P$ (see above) induces a corresponding cochain map of the corresponding Spencer complexes:

$$\begin{array}{ccc} D_i\mathrm{Diff}^+_k P & \xrightarrow{S^i_{i+k}} & D_{i-1}\mathrm{Diff}^+_{k+1} P \\ \downarrow{\scriptstyle D_i(f_{\Delta_{(k)}})} & & \downarrow{\scriptstyle D_{i-1}(f_{\Delta_{(k+1)}})} \\ D_i\mathrm{Diff}^+_{k+s} Q & \xrightarrow{S^i_{i+k+s}} & D_{i-1}\mathrm{Diff}^+_{k+s+1} Q \end{array}$$

Denote by $C_{i,k}(\Delta)$ the cokernel of the map $D_i(f_{\Delta_{(k)}})$. Then one obtains the following complex

$$\ldots \longrightarrow C_{i,k}(\Delta) \xrightarrow{S^i_{i+k}} C_{i-1,k+1}(\Delta) \longrightarrow \ldots \longrightarrow C_{0,i+k} \longrightarrow \mathrm{coker}\ \Delta \longrightarrow 0$$

which is called the $(i+k)$th order Spencer Diff-complex of the operator Δ.

5. Jets

Let P be an A-module. Consider the tensor product $A \otimes_K P$ endowed with an A-module structure by

$$a'(a \otimes p) \stackrel{\text{def}}{=} (a'a) \otimes p,\ \ a, a' \in A,\ \ p \in P.$$

For any $a' \in A$ set

$$\delta^{a'}(a \otimes p) \stackrel{\text{def}}{=} a \otimes a'p - a'a \otimes p$$

and denote by μ_k the submodule in $A \otimes_K P$ spanned by

$$(\delta^{a_0} \circ \cdots \circ \delta^{a_k})(a \otimes p), \qquad a, a_0, \ldots, a_k \in A,\ \ p \in P.$$

Set

$$\mathcal{J}^k(P) \stackrel{\text{def}}{=} (A \otimes_K P)/\mu_k,\ \ j_k(p) \stackrel{\text{def}}{=} (1 \otimes p) \mathrm{mod}\, \mu_k.$$

The map $j_k\colon\ P \to \mathcal{J}^k(P)$ is a differential operator of order $\leqslant k$.

PROPOSITION 5.1. *Fix a module P. Then the functor* $\mathrm{Diff}_k(P, \cdot)$ *is representable. Namely, for any differential operator $\Delta\colon\ P \to Q$ there exists a uniquely defined homomorphism $f^\Delta\colon\ P \to \mathcal{J}^k(P)$ such that the diagram*

$$\begin{array}{ccc} P & \xrightarrow{j^k} & \mathcal{J}^k(P) \\ & {\scriptstyle\Delta}\searrow & \downarrow{\scriptstyle f^\Delta} \\ & & Q \end{array}$$

is commutative. The correspondence $\Delta \mapsto f^\Delta$ establishes an isomorphism between the functors $\mathrm{Diff}_k(P, \cdot)$ *and* $\mathrm{hom}_A(\mathcal{J}^k(P), \cdot)$. *We say the homomorphism f_Δ to be Jet-associated with the operator Δ.*

DEFINITION 5.2. $\mathcal{J}^k(P)$ is called the module of k-jets of the module P.

Obviously, $\mathcal{J}^k(P)$ as an A-module is generated by the elements of the form $j_k(p),\ p \in P$.

Introduce a second A-module structure into $\mathcal{J}^k(P)$ by setting

$$a^+(a'j_k(p)) \stackrel{\text{def}}{=} a'j_k(ap)$$

and denote corresponding bimodule by $\mathcal{J}^k_{(+)}(P)$. Let Q be another A module. For any $\varphi \in \hom_A(P,Q)$ set

$$\mathcal{J}^k(\varphi)(aj_k(p)) \stackrel{\text{def}}{=} aj_k(\varphi(p)).$$

Then $\mathcal{J}^k_{(+)}(\cdot)$ is a functor from $\mathcal{M}(A)$ to the category of A-bimodules.

Denote by $\mathcal{J}^k_+$ the bimodule $\mathcal{J}^k_+(A)$. For any $\theta = aj_k(b) \in \mathcal{J}^k_+$ and $\rho = a'j_k(p) \in \mathcal{J}^k(P)$ set $\theta \cdot \rho =_{\text{def}} aa'j_k(bp)$. Then $\mathcal{J}^k_+$ acquires the structure of a commutative unitary A-algebra while $\mathcal{J}^k_+(P)$ becomes a $\mathcal{J}^k_+$-module. Define the maps $\tau_k\colon\ \mathcal{J}^k_+ \to \mathcal{J}^k_+$ and $i_k\colon\ A \to \mathcal{J}^k_+$ by setting

$$\tau_k(aj_k(b)) \stackrel{\text{def}}{=} bj_k(a), \qquad i_k(a) \stackrel{\text{def}}{=} aj_k(1), \quad a,b \in A.$$

Then τ_k is a homomorphism of the algebra $\mathcal{J}^k_+$,

$$\tau_k\colon\ \mathcal{J}^k \to \mathcal{J}^k_+, \qquad \tau_k\colon\ \mathcal{J}^k_+ \to \mathcal{J}^k$$

are A-module homomorphisms and the following identities hold:

$$\tau_k^2 = \text{id}, \qquad \tau_k \circ j_k = i_k, \qquad \tau_k \circ i_k = j_k.$$

Consider an A-bimodule $P^{(+)}$ and a module Q. Then two module structures exist in the tensor product $P^+ \otimes Q$:

$$a^+(p \otimes q) = a^+p \otimes q, \qquad a(p \otimes q) = ap \otimes q, \quad a \in A,\ p \in P,\ q \in Q.$$

The module endowed with the second structure will be denoted by $P^+ \otimes_{\text{in}} Q$.

PROPOSITION 5.3. *Let P be an A-module. Then the following isomorphisms take place*

$$\mathcal{J}^k(P) = \mathcal{J}^k_+ \otimes_{\text{in}} P, \qquad \mathcal{J}^k_+(P) = \mathcal{J}^k_+ \otimes P.$$

Note also that the modules $\mathcal{J}^k_+$ and $\mathcal{J}^k$ are representative objects for the functors $\text{Diff}_k(\cdot)$ and $\text{Diff}^+_k(\cdot)$, respectively.

6. Jet-Prolongations and Co-Compositions

Take two integers $k, l \geqslant 0$. Then from Propositions 1.3 and 5.1, one obtains the following commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{j_l} & \mathcal{J}^l(P) \\ {\scriptstyle j_{k+l}}\downarrow & & \downarrow{\scriptstyle j_k} \\ \mathcal{J}^{k+l}(P) & \xrightarrow{c^{k,l}} & \mathcal{J}^k(\mathcal{J}^l(P)) \end{array}$$

where $c^{k,l} = c^{k,l}(P) =_{\text{def}} f^{j_k \circ j_l}$. The correspondence $P \Rightarrow c^{k,l}(P)$ is a natural transformation of functors $\mathcal{J}^{k+l}(\cdot)$ and $\mathcal{J}^k(\mathcal{J}^l(\cdot))$; the homomorphism $c^{k,l}(P)$ is called *the co-composition homomorphism*.

Let now $\Delta\colon\ P \to Q$ be an operator of order k. Then the operator $\Delta^{(l)} =_{\text{def}} j_l \circ \Delta\colon\ P \to \mathcal{J}^l(Q)$ is called *the* lth Jet-*prolongation* of the operator Δ. For such a prolongation the diagram

$$\begin{array}{ccc} P & \xrightarrow{\Delta} & Q \\ {\scriptstyle j_{k+l}}\downarrow & \searrow^{\Delta^{(l)}} & \downarrow{\scriptstyle j_l} \\ \mathcal{J}^{k+l}(P) & \xrightarrow{f^{\Delta^{(l)}}} & \mathcal{J}^l(Q) \end{array}$$

is commutative.

Let $k \leqslant l$. Then, since any kth order differential operator may be considered as an operator of order l, we have the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{j_l} & \mathcal{J}^l(P) \\ & \searrow^{j_k} & \downarrow{\scriptstyle \nu_{l,k}} \\ & & \mathcal{J}^k(P) \end{array} \tag{7}$$

where $\nu_{l,k} =_{\text{def}} f^{j_k}$. The maps $\nu_{l,k}$ are epimorphisms dual to the embeddings $\text{Diff}_k(P,Q) \hookrightarrow \text{Diff}_l(P,Q)$. For any $k \leqslant l \leqslant s$, one has $\nu_{s,k} = \nu_{l,k} \circ \nu_{s,l}$ and the projective limit of the system $\{\mathcal{J}^k(P), \nu_{l,k}\}_{l \geqslant k}$ is denoted by $\mathcal{J}^*(P)$ and is called *the module of infinite jets* of the module P. For any $k \geqslant 0$ a projection $\nu_{*,k}\colon\ \mathcal{J}^*(P) \to \mathcal{J}^k(P)$ is defined such that $\nu_{*,k} = \nu_{l,k} \circ \nu_{*,l},\ l \geqslant k$.

All natural constructions presented here for the modules $\mathcal{J}^k(P)$ are carried over to $\mathcal{J}^*(P)$. Namely, due to diagram (7) there exists an operator $j_*\colon\ P \to$

$\mathcal{J}^*(P)$ such that $j_k = \nu_{*,k} \circ j_*$. Further, let $k' \geqslant k$, $l' \geqslant l$ be nonnegative integers. Then the diagrams

$$\begin{array}{ccc} \mathcal{J}^{k'+l}(P) & \xrightarrow{c^{k',l}} & \mathcal{J}^{k'}(\mathcal{J}^l(P)) \\ \downarrow{\scriptstyle \nu_{k'+l,k+l}} & & \downarrow{\scriptstyle \nu_{k',k}} \\ \mathcal{J}^{k+l}(P) & \xrightarrow{c^{k,l}} & \mathcal{J}^k(\mathcal{J}^l(P)) \end{array} \qquad \begin{array}{ccc} \mathcal{J}^{k+l'}(P) & \xrightarrow{c^{k,l'}} & \mathcal{J}^k(\mathcal{J}^{l'}(P)) \\ \downarrow{\scriptstyle \nu_{k+l',k+l}} & & \downarrow{\scriptstyle \mathcal{J}^k(\nu_{l',l})} \\ \mathcal{J}^{k+l}(P) & \xrightarrow{c^{k,l}} & \mathcal{J}^k(\mathcal{J}^l(P)) \end{array}$$

are commutative. Consequently, one obtains the map

$$c^{*,*}\colon\ \mathcal{J}^*(P) \to \mathcal{J}^*(\mathcal{J}^*(P)).$$

Finally, for any operator $\Delta\colon\ P \to Q$ of order $\leqslant k$ from the definitions we have commutative diagrams

$$\begin{array}{ccc} P & \xrightarrow{\Delta^{(l')}} & \mathcal{J}^l(Q) \\ & {\scriptstyle \Delta^{(l)}} \searrow & \downarrow{\scriptstyle \nu_{l',l}} \\ & & \mathcal{J}^l(Q) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{J}^{k+l'}(P) & \xrightarrow{f^{\Delta^{(l')}}} & \mathcal{J}^{l'}(Q) \\ \downarrow{\scriptstyle \nu_{k+l',k+l}} & & \downarrow{\scriptstyle \nu_{l',l}} \\ \mathcal{J}^{k+l}(P) & \xrightarrow{f^{\Delta^{(l)}}} & \mathcal{J}^l(Q) \end{array}$$

Hence, *the infinite prolongation* $\Delta^{(*)}\colon\ P \to \mathcal{J}^*(Q)$ is defined, as well as the corresponding homomorphism $f^{\Delta^{(*)}}\colon\ \mathcal{J}^*(P) \to \mathcal{J}^*(P)$. Note also that for any $l, l' \geqslant 0$, the diagram

$$\begin{array}{ccc} P & \xrightarrow{\Delta^{(l')}} & \mathcal{J}^{l'}(Q) \\ {\scriptstyle \Delta^{(l'+l)}} \downarrow & \searrow^{(\Delta^{(l')})^{(l')}} & \downarrow{\scriptstyle j_l} \\ \mathcal{J}^{l+l'}(Q) & \xrightarrow{c^{l,l'}} & \mathcal{J}^l(\mathcal{J}^{l'}(Q)) \end{array}$$

is commutative and thus $c^{*,*} \circ \Delta^{(*)} = (\Delta^{(*)})^{(*)}$.

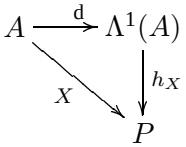
7. Forms and the de Rham Complex

Set $\Lambda^1(A) =_{\text{def}} \mathcal{J}^1/\operatorname{im} i_1$ and define $\mathrm{d}\colon\ A \to \Lambda^1(A)$ as the composition of $j_1\colon\ A \to \mathcal{J}^1$ with the natural projection $\mathcal{J}^1 \to \Lambda^1(A)$. Set also

$$\Lambda^i(A) \stackrel{\text{def}}{=} \underbrace{\Lambda^1(A) \wedge \ldots \wedge \Lambda^1(A)}_{i \text{ times}}.$$

DEFINITION 7.1. The elements of $\Lambda^i(A)$ are called differential i-forms over the algebra A.

PROPOSITION 7.2. *The modules $\Lambda^i(A)$ are representative objects for the functors $D_i(\cdot)$. In the case $i = 1$ an isomorphism between the modules $D_1(P)$ and $\hom_A(\Lambda^1(A), P)$ is established in the following way: for any derivation $X \in D_1(P)$ there exists a uniquely defined homomorphism h_X: $\Lambda^1(A) \to P$ such that the diagram*



is commutative.

In the sequel we use the notation $\Lambda^i =_{\text{def}} \Lambda^i(A)$.

Note that $\Lambda^1(A)$ as an A-module is generated by elements $\mathrm{d}(a)$ with the relations $\mathrm{d}(ab) = a\mathrm{d}(b) + b\mathrm{d}(a)$. Note also that the wedge product $\wedge$: $\Lambda^i \otimes_A \Lambda^j \to \Lambda^{i+j}$ is the representing morphism for the natural transformation of functors $D_{i+j}(\cdot) \Rightarrow D_j(D_i(\cdot))$.

Now we define by induction on i the 1st order operators d: $\Lambda^i \to \Lambda^{i+1}$. For $i = 0$ we take for d the operator d: $A \to \Lambda^1$ introduced above. If $i = 1$, we set $\mathrm{d}(a\mathrm{d}(b)) =_{\text{def}} \mathrm{d}a \wedge \mathrm{d}b$ and for $i > 1$ and any decomposable element $\omega \wedge \theta \in \Lambda^i$ set

$$\mathrm{d}(\omega \wedge \theta) \stackrel{\text{def}}{=} \mathrm{d}\omega \wedge \theta + (-1)^{\omega}\omega \wedge \mathrm{d}\theta.$$

PROPOSITION 7.3. $\mathrm{d} \circ \mathrm{d} = 0$.

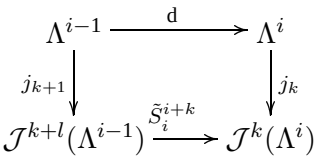
DEFINITION 7.4. The sequence

$$0 \longrightarrow A \xrightarrow{\mathrm{d}} \Lambda^1 \longrightarrow \ldots \longrightarrow \Lambda^i \xrightarrow{\mathrm{d}} \Lambda^{i+1} \longrightarrow \ldots$$

is called the de Rham complex of the algebra A.

8. The Spencer Jet-Complex

Consider the composition $\Lambda^{i+1} \xrightarrow{\mathrm{d}} \Lambda^i \xrightarrow{j_k} \mathcal{J}^k(\Lambda^i)$. Since it is a $(k+1)$st order operator, from Proposition 5.1 one obtains the following commutative diagram



where $\tilde{S}_i^{i+k} =_{\text{def}} f^{j_k \circ \mathrm{d}}$. Thus we get the sequence of homomorphisms

$$\begin{aligned} 0 \longrightarrow\ & \mathcal{J}^{k+i} \longrightarrow \ldots \longrightarrow \mathcal{J}^{k+i}(\Lambda^{i-1}) \longrightarrow^{\tilde{S}_i^{i+k}} \mathcal{J}^k(\Lambda^i) \longrightarrow \ldots \\ & \longrightarrow \Lambda^{i+k} \longrightarrow 0 \end{aligned} \tag{8}$$

which is a complex and is called *the Spencer* Jet-*sequence*. Denote it by SJ^{i+k}.

Let $P^{(+)}$ be an A-bimodule and P be an A-module. Consider the module $\hom_A(P, Q)$ and introduce the second module structure by setting

$$(a^+\varphi)(p) \stackrel{\text{def}}{=} \varphi(a^+p), \quad a \in A,\ p \in P,\ \varphi \in \hom_A(P, Q).$$

Denote such a module by $\hom_A^+(P, Q)$.

PROPOSITION 8.1. *The functors* $S_{i+k}(\cdot)$ *and* $\hom_A^+(SJ^{i+k}, \cdot)$ *are isomorphic, i.e. the Spencer* Jet-*sequence is the representative object for the functor* $S_{i+k}(\cdot)$.

Let now P, Q be A-modules. Define the map $\tau_k^{P,Q}\colon\ \mathcal{J}^k(P) \otimes_A Q$ by setting $\tau_k^{P,Q}(j_k(p) \otimes q) =_{\text{def}} j_k(q) \otimes p$. Obviously,

$$\tau_k^{P,Q} \in \mathrm{Diff}_k^+(\mathcal{J}^k(P) \otimes_A Q, \mathcal{J}^k(Q) \otimes_A P), \qquad \tau_k^{P,Q} \circ \tau_k^{Q,P} = \mathrm{id}.$$

Introduce the operators $S_i^{i+k}\colon\ \mathcal{J}^{k+1}(P) \otimes_A \Lambda^{i-1} \longrightarrow \mathcal{J}^k(P) \otimes_A \Lambda^i$ defined by commutativity of the diagram

$$\begin{array}{ccc} \mathcal{J}^{k+1}(\Lambda^{i-1}) \otimes_A P & \xrightarrow{\tilde{S}_i^{i+k} \otimes \mathrm{id}} & \mathcal{J}^k(\Lambda^i) \otimes_A P \\ \Big\downarrow{\scriptstyle \tau_{k+1}^{\Lambda^{i-1}, P}} & & \Big\downarrow{\scriptstyle \tau_k^{\Lambda^i, P}} \\ \mathcal{J}^{k+1}(P) \otimes_A \Lambda^{i-1} & \xrightarrow{S_i^{i+k}} & \mathcal{J}^k(P) \otimes_A \Lambda^i \end{array} \tag{9}$$

PROPOSITION 8.2. *The maps* $S_i^{i+k}\colon\ \mathcal{J}^{k+1}(P) \otimes_A \Lambda^{i-1} \longrightarrow \mathcal{J}^k(P) \otimes_A \Lambda^i$ *defined by* (9) *are operators of order* $\leqslant 1$ *and* $S_{i+1}^{i+k} \circ S_i^{i+k} = 0$.

DEFINITION 8.3. The sequence

$$\begin{aligned} 0 \longrightarrow\ & P \xrightarrow{j_{k+i}} \mathcal{J}^{k+1}(P) \longrightarrow \cdots \longrightarrow S_i^{i+k}\mathcal{J}^{k+1}(P) \otimes_A \Lambda^{i-1} \xrightarrow{S_i^{i+k}} \\ & \mathcal{J}^k(P) \otimes_A \Lambda^i \longrightarrow \cdots \longrightarrow P \otimes_A \Lambda^{i+k} \longrightarrow 0 \end{aligned} \tag{10}$$

is called the Spencer Jet-complex of the module P.

The action of the Spencer Jet-operators is expressed by the following formula

$$S_i^{i+k}(j_{k+1}(p) \otimes \omega) = j_k(p) \otimes \mathrm{d}\omega, \quad p \in P,\ \omega \in \Lambda^{i-1}. \tag{11}$$

Consider an lth order operator $\Delta\colon\ P \to Q$. Then the diagram

$$\begin{array}{ccc} \mathcal{J}^{k+l+1}(P) \otimes_A \Lambda^{i-1} & \xrightarrow{S_i^{k+l+i}} & \mathcal{J}^{k+l}(P) \otimes_A \Lambda^i \\ \downarrow{\scriptstyle f^{\Delta^{(k+1)}} \otimes \mathrm{id}} & & \downarrow{\scriptstyle f^{\Delta^{(k)}} \otimes \mathrm{id}} \\ \mathcal{J}^{k+1}(Q) \otimes_A \Lambda^{i-1} & \xrightarrow{S_i^{k+1}} & \mathcal{J}^k(Q) \otimes_A \Lambda^i \end{array}$$

is commutative. Hence, one can define *the Spencer* Jet-*complex* of the operator Δ as the corresponding kernel complex and introduce the Spencer Jet-cohomology groups of Δ.

9. Generalizations and Specializations

There are several directions to generalize or specialize the above described theory. Probably, the most important one, giving rize to various interesting specializations, is associated with the following concept.

DEFINITION 9.1. An Abelian subcategory $\mathcal{D}$ of $\mathcal{M}(A)$ is said to be differentially closed, if it is closed under tensor product over A and

(i) it is closed under the action of the functors $\mathrm{Diff}_k^+(\cdot,\cdot)$ and $D_i(\cdot)$,
(ii) the functors $\mathrm{Diff}_k^+(P,\cdot)$, $\mathrm{Diff}_k^+(\cdot,Q)$ and $D_i(\cdot)$ are representable in $\mathcal{D}$, where P, Q are objects of $\mathcal{D}$.

When it is necessary to indicate the category explicitely, we use the notations $\mathrm{Diff}_k^+(P,Q;\mathcal{D}), D_i(P;\mathcal{D})$, etc. In the case $\mathcal{D} = \mathcal{M}(A)$, we write simply $\mathrm{Diff}_k^+(P,Q;A), D_i(P;A)$, etc.

As an example consider the following situation. Let M be a smooth (i.e. C^∞-class) finite-dimensional manifold and set $A =_{\mathrm{def}} C^\infty(M)$. Let $\pi\colon\ E \to M$, $\xi\colon\ F \to M$ be two smooth locally trivial finite-dimensional vector bundles over M and $P = \Gamma(\pi)$, $Q = \Gamma(\xi)$ be corresponding A-modules of smooth sections.

PROPOSITION 9.2. *The module* $\mathrm{Diff}_k^+(P,Q)$ *coincides with the module of* k*th order differential operators acting from the bundle* π *to* ξ. *The module* $D(A)$ *coincides with the module of vector fields on the manifold* M.

However if one constructs representative objects for the functors $\mathrm{Diff}_k(P,\cdot)$ and $D_i(\cdot)$ in the category $\mathcal{M}(A)$, the modules $\mathcal{J}^k(P)$ and Λ^i will not coincide with 'geometrical' jets and differential forms.

DEFINITION 9.3. A module P over $C^\infty(M)$ is called geometrical, if $\bigcap_{x\in M} \mu_x P = 0$, where μ_x is the ideal in $C^\infty(M)$ consisting of functions vanishing at the point $x \in M$.

Denote by $\mathcal{G}(M)$ the full subcategory of $\mathcal{M}(A)$ whose objects are geometrical $C^\infty(M)$-modules. Let P be an A-module and set $\mathcal{G}(P) =_{\text{def}} P/\bigcap_{x\in M} \mu_x P$. Evidently, $\mathcal{G}(P)$ is a geometrical module while the correspondence $P \Rightarrow \mathcal{G}(P)$ is a functor form $\mathcal{M}(C^\infty(M))$ to $\mathcal{G}(M)$, if for any homomorphism $\varphi \in \hom_A(P, Q)$ one sets $\mathcal{G}(\varphi) = \varphi$.

PROPOSITION 9.4. *Let M be a smooth finite-dimensional manifold and $A = C^\infty(M)$. Then*

(i) *The category $\mathcal{G}(A)$ of geometrical A-modules is differentially closed.*
(ii) *The representative objects for the functors* $\operatorname{Diff}_k(P, \cdot)$ *and $D_i(\cdot)$ in $\mathcal{G}(A)$ coincide with $\mathcal{G}(\mathcal{J}^k(P))$ and $\mathcal{G}(\Lambda^i)$ respectively.*
(iii) *The module $\mathcal{G}(\Lambda^i)$ coincide with the module of differential i-forms on M.*
(iv) *If $P = \Gamma(\pi)$ for a smooth locally trivial finite-dimensional vector bundle $\pi\colon E \to M$, then the module $\mathcal{G}(\mathcal{J}^k(P))$ coincide with $\Gamma(\pi_k)$, where $\pi_k\colon J^k(\pi) \to M$ is the bundle of k-jets for the bundle π.*

Another example of a differentially closed category is the category of filtered geometrical modules over a filtered algebra. This category is essential to construct differential calculus over manifolds of infinite jets and infinitely prolonged differential equations (see details in [1]).

Note in conclusion that the logical structure of the above described theory is generalized onto a noncommutative case as it is shown in the paper by A. M. Verbovetsky [3] published in this issue of *Acta Applicandae Mathematicae*.

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