SHEAVES IN PHYSICS - TWISTOR THEORY

Richard Jozsa

Mathematical Institute, Oxford, England
and Queen's College, Oxford, England.

Since the time of Descartes and Newton, physics has traditionally been developed in terms of differential equations on the real line, i.e. a local description of physical processes assuming a great deal about the small scale structure of fields and spacetime itself. From an operationalist point of view, however, it was apparent that this fine structure was physically unobservable: although the formalism of classical physics allows in general for infinitely precise measurements, this represents an extrapolation of the theory well outside its range of verifiability and there is little reason to believe that the predicted results have any relevance to physics. The development of quantum mechanics [20,21] causing radical changes in the notion of particle, the "smallest" physical object, also undermined the utility of the real line continuum structure as a physical model. A redeeming feature of the model, however, is its great mathematical utility but in the field of fundamental physics even this is doubtful in view of the difficulties being encountered in the problem of producing a quantised theory of spacetime and gravitation, which still remains one of the deepest unsolved problems of physics.

We are thus led to seek alternative mathematical models for spacetime. The twistor theory of Roger Penrose [5,6,7,8,9] provides such an alternative model and its mathematical formalism rests heavily on the theory of sheaves. It is interesting to contrast the twistor space view and the conventional spacetime view of physics. In twistor space a spacetime point is represented by a certain kind of subspace, i.e. a non-local object and correspondingly a twistor is represented in spacetime by a subspace. On this level the theories are mathematically equivalent but they diverge when we begin to set up physics. In particular the process of quantisation in twistor space [9] leads to a description in which the points of spacetime become "fuzzy" but certain relations associated with the causal structure are preserved. On the other hand conventional quantisation techniques in spacetime leave the points of the underlying space unchanged and quantum uncertainties are introduced into the fields defined on the space, in particular the metric and hence the causal structure. From the operationalist point of view, as was expounded in the works of P.W.Bridgman [13,19] the twistor approach is more satisfactory. Bridgman points out that a spacetime point can only be determined by the occurrence of a physical event which is required to have a certain amount of individuality or

identifiability generally giving rise to a fuzzy point since physical processes are never perfectly localised. Thus, for example, empty space (devoid of any physical processes) is amorphous and has no points of any kind. The existence and concept of points is thus closely related to an underlying system of physics which then determines the fuzziness structure of the points. A possible mathematical model of such a structure is given by a sheaf where the partial sections represent the fuzzy points, global sections giving precise points. For a section s of a sheaf A, the truth value of the statement $s \in A$ is measured by the open set on which s is defined [18] and the topological structure of the sheaf makes the resulting logic intuitionistic. The intuitionistic logic completely determines the fuzziness structure of the points and thus there is a close connection between the physics and the structure of the underlying logic of these models. There is some evidence that these models will be useful: classical physics fits into this framework with the underlying logic classical. Furthermore, twistor theory developed from a completely different point of view also seems to fit into this framework at least as far as the representation of spacetime is concerned. In this case the logic is intuitionistic with the structure of the Heyting algebra of the open sets of the 2-sphere S^2. Also there are general arguments [13,19] relating the constructive aspects of intuitionistic logic to the operational aspects of physics, indicating that intuitionistic models would provide better mathematical structures for the representation of experimental situations and measuring processes which form the crucial links between observation and theory.

The formalism of sheaf theory was applied in the 1950's with great success in the theory of complex manifolds and holomorphic bundles [22] and in the first instance it was through these results that it entered twistor theory. The curious and little understood relationship between complex structures and the equations of physics has turned up in many areas but nowhere is this connection better developed than in twistor theory [9,23]. In the following pages we give an annotated review of the formalism and basic results of twistor theory, particularly those aspects involving sheaves, beginning with an account of $SL(2,\mathbb{C})$-spinors and the zero rest-mass field equations.

The mathematical basis for special relativity is Minkowski spacetime M, i.e. $\mathbb{R}^4$ equipped with a metric of signature $(+---)$ (one timelike and three spacelike coordinates). The symmetry group which preserves this metric and also the direction of time and orientation of space is the restricted Lorentz group $O_+^\uparrow(1,3)$ [16]. Physics is now added into this framework by the statement that position measurements of a given event made by different inertial observers are related by a Lorentz transformation and that the equations of physics should be covariant with respect to these transformations, i.e. the equations are of the form $S = T$ where S and T are elements of the tensor algebra of the Lorentz group with

the same index structure. The description in terms of real structures is in a sense not the most fundamental one. There is an underlying complex representation based on the 2-1 local isomorphism [1,2,4]

$$SL(2,\mathbb{C}) \xrightarrow{2\text{-}1} O_+^{\uparrow}(1,3) \tag{1}$$

where $SL(2,\mathbb{C})$ is the group of complex 2x2 matrices with unit determinant. This is also the group of global automorphisms of the complex structure of the 2-sphere S^2 regarded as a complex manifold [2]. To see the isomorphism more explicitly, associate with each vector $x^a = (t,x,y,z)$ in Minkowski space the Hermitean matrix [1]

$$x^a \quad \leftrightsquigarrow \quad \begin{pmatrix} t+x & y+iz \\ y-iz & t-x \end{pmatrix} = x^{AB'} \tag{2}$$

We label the matrix with two indices of different kinds (primed and unprimed) each ranging over two values. The determinant of the matrix is the Minkowskian length of the vector and there is a 1-1 correspondence between 2x2 Hermitean matrices and vectors in Minkowski space. Thus a Lorentz transformation is one which preserves the determinant and Hermiteanness of the matrices. These are given by taking a matrix $L \in SL(2,\mathbb{C})$ and forming the matrix product $L^{\dagger}.x^{AB'}.L$ where $L^{\dagger}$ is the conjugate transpose of L. Under the action of $SL(2,\mathbb{C})$ a spacetime vector is represented as an object with two indices and thus is not the simplest representation of the group. The basic representation is given on the two complex dimensional spin space $\mathbb{C}^2$ and associated with this representation is the complex conjugate representation. We use primed and unprimed indices to distinguish spinor components transforming under the basic representation and its complex conjugate. The skew symmetric two index spinors ε_{AB} and $\varepsilon_{A'B'}$ (unique up to a constant multiplier) are invariant under $SL(2,\mathbb{C})$ transformations and play the role of a metric in the algebra of spinors. Given any spinor λ^A we can form its complex conjugate and the product $\lambda^A \bar{\lambda}^{B'}$ which then corresponds to a spacetime vector via (2), which has zero length, i.e. a null vector. This gives a spacetime interpretation of a spinor up to phase ($e^{i\upsilon}\lambda^A$ gives the same null vector). Thus projective spin space $\mathbb{C}P'$ ($= S^2$) corresponds to all the null directions at the origin in Minkowski space, i.e. to the celestial sphere of an observer. We will often be concerned with complexified Minkowski space $\mathbb{C}M$ whose points are of the form $x^a + iy^a$ with $x^a, y^a \in M$. In terms of spinors, these points are represented by matrices as in (2) with the Hermitean condition lifted. All of the usual spacetime equations can be expressed in spinor terms formally by replacing every spacetime index a by a pair of spinor indices AB'. The resulting equations can often be reduced to simpler forms using special properties of the two dimensionality of spin space.

As an example (which will be important later) consider the source-free Maxwell's

equations

$$\nabla_{[a} F_{bc]} = 0 \qquad \nabla^{a} F_{ab} = 0$$

where F_{ab} is a skew symmetric tensor describing the electromagnetic field on spacetime. Using the properties of spinors we can reduce the form of F_{ab} [4] :

$$F_{ab} \longleftrightarrow F_{AA'BB'} = \phi_{AB} \varepsilon_{A'B'} + \varepsilon_{AB} \tilde{\phi}_{A'B'} \tag{3}$$

where ϕ_{AB} and $\tilde{\phi}_{A'B'}$ are symmetric spinors and the field equations become

$$\nabla^{AA'} \phi_{AB} = 0 \qquad \nabla^{AA'} \tilde{\phi}_{A'B'} = 0$$

This is a special case of the zero restmass field equation [3]

$$\nabla^{AA'} \phi_{AB \ldots L} = 0 \qquad \nabla^{AA'} \tilde{\phi}_{A'B' \ldots L'} = 0$$

where $\phi_{AB \ldots L}$ and $\tilde{\phi}_{A'B' \ldots L'}$ are symmetric spinor fields with n indices describing zero restmass fields of spin $n/2$ (with opposite helicities) [3] . Three special cases of interest are ϕ_{A} describing the neutrino, ϕ_{AB} describing the photon and ϕ_{ABCD} describing linearized source free gravitation. The zero restmass equations are invariant under a larger group than the Lorentz group: we can include the translations

$$\mathrm{Tr}(b^a) : x^a \mapsto x^a + b^a \qquad (b^a \text{ constant})$$

and the conformal transformations

$$\text{Inversions} \quad \mathrm{Inv}(b^a) : x^a \mapsto x^a / \| x^a + b^a \|^2$$

$$\text{dilations} \quad D(k) : x^a \mapsto kx^a \qquad (k \text{ constant})$$

making up the fifteen-parameter conformal group denoted $C(1,3)$ [2,3] . To be precise, the inversion $\mathrm{Inv}(b^a)$ is not strictly defined on the null cone at $x^a = -b^a$ so we compactify Minkowski space by attaching a null cone at infinity and $C(1,3)$ then acts transitively on this compact space.

The formalism of twistor algebra is based on conformal invariance: the group $C(1,3)$ has a complex representation via the 4-1 local isomorphism

$$SU(2,2) \xrightarrow{4\text{-}1} C(1,3) \tag{4}$$

where $SU(2,2)$ is the group of $4x4$ complex matrices with unit determinant which preserve the Hermitean form

$$|Z_0|^2 + |Z_1|^2 - |Z_2|^2 - |Z_3|^2 \tag{5}$$

in $\mathbb{C}^4$. The basic representation space $\mathbb{C}^4$ for $SU(2,2)$ with the Hermitean form (5) is called twistor space $\mathbb{T}$ [5,7,9] . It is convenient to represent $\mathbb{T}$ as the sum of two spin spaces whereby twistor Z^{α} ($\alpha = 0,1,2,3$) is represented by a pair of spinors which we write as $(\omega^{A} , \pi_{A'})$ so that

$$Z^0 = \omega^0 \qquad Z^1 = \omega^1 \qquad Z^2 = \pi_{0'} \qquad Z^3 = \pi_{1'}$$

Defining the complex conjugate of Z^α by $\bar{Z}_\alpha = (\bar{\pi}_A, \bar{\omega}^{A'})$ the Hermitean form (5) can be expressed as

$$|Z^\alpha|^2 = Z^\alpha \bar{Z}_\alpha = \omega^A \bar{\pi}_A + \bar{\omega}^{A'} \pi_{A'}$$

This form divides $\mathcal{PT}$ into three regions $\mathcal{PT}^+$, $\mathcal{PT}^-$, $\mathbb{N}$ where $|Z^\alpha|^2$ is respectively positive, negative and zero. There is a dual correspondence between geometry in twistor space and geometry in Minkowski space. We say that a twistor $Z^\alpha = (\omega^A, \pi_{A'})$ passes through the point $x^{AA'}$ and that $x^{AA'}$ lies on the twistor Z^α if $\omega^A = ix^{AA'}\pi_{A'}$. The set of all $x^{AA'}$ passing through a fixed Z^α gives a spacetime representation of Z^α and forms a two-dimensional subspace of $\mathbb{C}M$. Dually the set of all twistors passing through $x^{AA'}$, representing $x^{AA'}$ in twistor space is a $\mathbb{C}^2$-subspace of $\mathcal{PT}$ (parametrised by the $\pi_{A'}$ spinor space at $x^{AA'}$). If we now pass to projective twistor space $\mathbb{P}\mathcal{PT}$ ($= \mathbb{C}P^3$, complex projective 3-space) these $\mathbb{C}^2$-subspaces become holomorphic $\mathbb{C}P^1$-submanifolds and we see that the points of Minkowski space are represented as S^2 submanifolds of projective twistor space. Removing a line from $\mathbb{P}\mathcal{PT}$ which represents a point at infinity the remaining space has the structure of a non-trivial bundle $\mathcal{B}$ with base space the projective $\pi_{A'}$ spin space and fibres the ω^A spin spaces. The global holomorphic sections of this bundle are the (finite) points of Minkowski space. Thus the sheaf $\mathcal{T}$ of germs of holomorphic sections of this bundle can be thought of as an intuitionistic representation of Minkowski space where the truth value structure of the intuitionistic logic is given by the Heyting algebra of open sets of S^2. The usual classical Minkowski space is recovered by taking the points (sections) p for which the statement $p \in \mathcal{T}$ has truth value True, i.e. the global sections of $\mathcal{T}$. This representation of flat spacetime has a remarkable generalisation to a basic class of complex curved solutions of Einstein's equations, called non-linear gravitons [8]. We take the bundle $\mathcal{B}$ and continuously deform its complex manifold structure in the sense of Kodaira and Spencer [14,17]. A basic theorem of Kodaira [15] guarantees that the S^2 submanifolds will be preserved in some small but finite deformation if the sheaf cohomology group $H^1(S^2, \mathcal{N})$ vanishes. Here $\mathcal{N}$ is the sheaf of germs of sections of the normal bundle of $S^2 \hookrightarrow \mathcal{B}$. This condition is satisfied here. Furthermore, the infinitesimal deformations are classified by $H^1(\mathcal{B}, \mathcal{V})$ where $\mathcal{V}$ is the sheaf of germs of sections of the holomorphic tangent bundle to $\mathcal{B}$. We now deform the complex structure of the bundle $\mathcal{B}$ while preserving (i) the fibre structure of ω^A over $\pi_{A'}$ (ii) the volume element (symplectic structure) $d\omega^0 \wedge d\omega^1$ on each fibre. Then the resulting bundle has a four-parameter family of global sections (S^2 submanifolds) which automatically:

1) has the structure of a complex manifold which is
2) endowed with a quadratic Riemannian metric which
3) satisfies Einstein's vacuum field equations and
4) is of the form of a non-linear graviton

and the most general such solution is obtained in this way. Actually, we have been a little imprecise: the above construction gives the graviton up to a conformal factor. To specify the metric uniquely we need to consider deformations of the non-projective twistor space $\mathbb{PT}$ preserving the structure stated above as well as the vector field $T = Z^{\alpha} \partial / \partial Z^{\alpha}$. Factoring out by the integral curves of T provides the way of passing from the deformed non projective twistor space to the deformed projective space, i.e. the bundle described above. An analogous construction has been developed by R. Ward [11] to represent electromagnetic fields in deformations of the complex structure of the canonical line bundle over $\mathbb{PT}^{+}$ and also solutions of the Yang-Mills equations in deformations of two-dimensional bundles over $\mathbb{PT}^{+}$. The remarkable feature of all these constructions is that the complicated nonlinear field equations are satisfied by the fields arising from arbitrary deformations of the complex structure. What is really happening is that the field equations are being represented by the integrability conditions (Newlander-Nirenberg theorem) for the existence of a complex structure.

Another basic application of sheaf theory in twistor theory concerns the solution of the zero restmass field equations [6,7,9]. Let $f(Z^{\alpha})$ be a twistor function, which is holomorphic in some part of twistor space and homogeneous of degree $-2S-2$ where S is a non-negative integer, i.e.

$$f(\lambda Z^{\alpha}) = \lambda^{-2S-2} f(Z^{\alpha})$$

For each fixed $x^{a} \in \mathbb{C}M$ consider all the twistors $(ix^{AA'} \pi_{A'}, \pi_{A'})$ passing through x^{a} and form the contour integral

$$\phi_{A' \ldots L'}(x^{a}) = \frac{1}{2\pi i} \oint \underbrace{\pi_{A'} \ldots \pi_{L'}}_{2S \text{ spinors}} f(ix^{AA'} \pi_{A'}, \pi_{A'}) \pi^{C'} d\pi_{C'} \qquad (6)$$

This gives a field on spacetime with $2S$ spinor indices. A beautiful result of Penrose states that the holomorphicity of $f(Z^{\alpha})$ guarantees that $\phi_{A' \ldots L'}(x^{a})$ automatically satisfies the spin S zero restmass field equations and all the analytic solutions are obtained in this way. A similar result holds for twistor functions of positive homogeneity using the contour integral formula

$$\phi_{A \ldots L}(x^{a}) = \frac{1}{2\pi i} \oint \left[\frac{\partial}{\partial \omega^{A}} \cdots \frac{\partial}{\partial \omega^{L}} f(\omega^{A}, \pi_{A'}) \right]_{\omega^{A} = ix^{AA'} \pi_{A'}} \pi^{C'} d\pi_{C'} \qquad (7)$$

Note that the integrands in (6) and (7) are both homogeneous of degree zero in

$\pi_{A'}$ and thus well defined on the projective spin space S^2. In order to obtain a non-zero integral (by Cauchy's theorem) the integrand must be singular in at least two disjoint regions and then the contour separates these regions. There is considerable freedom in the description of fields by twistor functions. We can (i) move the contour around freely in the region where $f(Z^\alpha)$ is holomorphic without changing the value of the integral, and also ii) we can add into f functions which are holomorphic all over one side of the contour. These freedoms are exactly the ones required to make $f(Z^\alpha)$ into a representative cocycle of a sheaf cohomology class [10]. Let K be the canonical line bundle over $\mathbb{P}\mathcal{PT}^+$. Then twistor functions homogeneous of degree n can be represented as sections of the tensor power K^n (where K^{-n} is defined as $(K^*)^n$). Denote by $\mathcal{O}(n)$ the sheaf of germs of holomorphic sections of K^n. Then the elements of $H^1(\mathbb{P}\mathcal{PT}^+, \mathcal{O}(-2S-2))$ are in 1-1 correspondence with analytic solutions of the spin S zero restmass field equations (with primed indices) and similarly for $H^1(\mathbb{P}\mathcal{PT}^+, \mathcal{O}(2S-2))$ and unprimed fields. The explicit transformation is via the contour integral formulas. Representing the cohomology in terms of coverings and the Čech construction, the freedoms (i) and (ii) in f stated above correspond respectively to (a) representing a cohomology class with respect to different coverings (the contour threads through the double intersection regions of the cover) and (b) the coboundary freedom in the cohomology class. Thus, for example, the two halves of the photon field as given in (3) are given respectively by elements of $H^1(\mathbb{P}\mathcal{PT}^+, \mathcal{O}(-4))$ and $H^1(\mathbb{P}\mathcal{PT}^+, \mathcal{O}(0))$. Note that again there are no restrictions on the cohomology classes whereas the corresponding fields automatically satisfy the correct field equations. In this case, the field equations are coded implicitly in the Cauchy-Riemann equations which guarantee holomorphicity. The above description has been heuristic at times but there also exists a rigorous argument based on exact sequences of sheaves which gives the same result [12].

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