Appl Categor Struct (2009) 17:1–28
DOI 10.1007/s10485-008-9149-0

$\mathcal{Q}$-$*$-Categories

Hans Heymans

Received: 14 September 2006 / Accepted: 25 May 2008 / Published online: 26 June 2008
© Springer Science + Business Media B.V. 2008

Abstract We consider the theory of categories enriched in an involutive quantaloid $\mathcal{Q}$: the $\mathcal{Q}$-$*$-categories. After giving an introduction to involutive quantaloids and nuclei, we use matrices with entries in $\mathcal{Q}$ to define $\mathcal{Q}$-$*$-categories. Then we examine the relations between two kinds of morphisms between them, the functors and the $*$-maps, to provide a basis to study completeness properties. These results are used to provide a definition of pseudo-presheaves, presheaves and sheaves on involutive quantaloids in order to get a generalization of presheaves and sheaves on sites. Finally a characterization of these sheaves in terms of covers and compatible families is presented.

Keywords Involutive quantaloids · Quantales · Sheaves · Nuclei · Relations · Matrices · Topos theory

Mathematics Subject Classifications (2000) Primary 06F07 · 18D20 · 18F20 · Secondary 06D22 · 18B05 · 18F10

1 Introduction

In mathematics, sets are the basic tools to work with. They have a very simple structure: elements x and y of a set A have two possible relations, $x = y$ (they are 'completely the same') or $x \neq y$ (they are 'completely different'). This can be

H. Heymans (✉)
Department of Mathematics and Computer Science, University of Antwerp,
Middelheimlaan 1, 2020 Antwerp, Belgium
e-mail: hans.heymans@ua.ac.be

expressed by a comparison-function $\delta : A^2 \to \{0, 1\}$ that measures the similarity between the elements, corresponding to the identity relation, being the smallest equivalence relation on A. δ has the evident properties:

$$\begin{aligned} \delta(x, x) &= 1 && \text{(reflexive)} \\ \delta(x, z) \wedge \delta(z, y) &\leq \delta(x, y) && \text{(transitive)} \\ \delta(x, y) &= \delta(y, x) && \text{(symmetric)} \\ \delta(x, y) &= 1 \Leftrightarrow x = y && \text{(separating)} \end{aligned}$$

However, in practice we encounter sets of elements with more complex relations. A metric space (A, d) provides a more expressive comparison-function, $\mathrm{d} : A^2 \to \mathbb{R}_+$. d can be regarded as a square *matrix* with values in $\mathbb{R}_+$, having certain properties. If we reverse the order on $\mathbb{R}_+$ and if we interpret $\wedge$ as addition then the equations above (with δ replaced by d and 1 by 0, representing the 'strongest similarity') are properties of d. It is very natural to generalize the concept of a metric by adding the possibility of points x and y being infinitely far away from each other: $\mathrm{d}(x, y) = \infty$ and $\mathrm{d} : A^2 \to \overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \infty$. In this way the lattice of comparison-values becomes a complete lattice with a monoidal operation $+$ that distributes over arbitrary joins. $\overline{\mathbb{R}}_+$ is in fact a commutative *quantale* [12]. The first case is included by the correspondence:

$$0 \;\; 1 \;\; \leq \;\; \wedge \qquad \mapsto \qquad \infty \;\; 0 \;\; \geq \;\; +$$

This description is in conformity with enriched category theory [8]: A is a category enriched in $\overline{\mathbb{R}}_+$ [9]. Reflexivity and transitivity correspond respectively to unit and composition. The extra condition of symmetry establishes the set-like character.

There are other generalizations of sets that are worth examining, such as the sheaves over a site $(\mathcal{C}, \mathfrak{F})$ (with $\mathfrak{F}$ a Grothendieck topology on the small category $\mathcal{C}$), which form the topos $\mathsf{Sh}(\mathcal{C}, \mathfrak{F})$. Standard books on this topic are [7] and [10]. The concept of a (pre)sheaf is perfectly translatable into the language of comparison-functions. This is done by using binary relations in $\mathsf{Sh}(\mathcal{C}, \mathfrak{F})$ as comparison-values between the elements (the local sections) of a sheaf [16], which can be seen as an extension of the approach in [4]. As a result, the structures in which we enrich are broadened to *involutive quantaloids* [13]. An involution is needed to be able to express symmetry in a non-commutative setting.

This is a strong motivation for the study of $\mathcal{Q}$-$*$-categories, categories enriched in an involutive quantaloid $\mathcal{Q}$, which can be regarded as generalizations of sets that comprise metric spaces and sheaves on sites. The central notion of these structures is *comparison*, which has the characteristics of an equivalence relation: reflexivity, transitivity and symmetry.

Enrichment in bicategories was introduced by Betti et al. [3]. An exhibition of the special case involving quantaloids without involution is provided by Stubbe [14]. We should also mention the connection with sheaves on right-sided idempotent quantales, as introduced by Borceux [1] and Mulvey [11]. The last part of this paper reconciles the work of Walters [16] with the material in [6], in which Gylys deals with the involutive setting (some significant differences with our paper will be addressed).

2 Involutive Quantaloids

2.1 Basics

Definition 2.1 A **quantaloid** $\mathcal{Q}$ is a (locally small) category with some extra structure:

1. For each two objects $A, B \in \mathcal{Q}_0$, the set of morphisms $\mathcal{Q}(A, B)$ is a complete lattice.
2. The composition of morphisms satisfies the following axiom (whenever the operations are defined):

$$a \circ \bigvee_i b_i = \bigvee_i (a \circ b_i) \qquad \left(\bigvee_i a_i\right) \circ b = \bigvee_i (a_i \circ b) \tag{1}$$

Equivalently, a quantaloid can be considered as a category enriched in SL, the category of complete lattices and functions that preserve arbitrary joins (in that context we speak of *sup-lattices* instead of complete lattices).

A quantaloid with only one object is called a unital quantale. We usually call the morphisms the *elements* of $\mathcal{Q}$. Each lattice $\mathcal{Q}(A, B)$ has a bottom and a top element: $\mathsf{0}_{B,A}$ and $\mathsf{T}_{B,A}$. We write T_A and $\mathsf{0}_A$ instead of $\mathsf{T}_{A,A}$ and $\mathsf{0}_{A,A}$. Note that for $A, B, C \in \mathcal{Q}_0$, $a \in \mathcal{Q}(A, B)$ and $b \in \mathcal{Q}(B, C)$ we have:

1. $b \circ \mathsf{0}_{B,A} = \mathsf{0}_{C,A} = \mathsf{0}_{C,B} \circ a$
2. $\mathsf{T}_{B,A} \circ \mathsf{T}_A = \mathsf{T}_{B,A} = \mathsf{T}_B \circ \mathsf{T}_{B,A}$ (because $\mathsf{id}_A \leq \mathsf{T}_A$)

$\mathsf{T}_{C,B} \circ \mathsf{T}_{B,A} = \mathsf{T}_{C,A}$ does not need to be the case (as shown in the first example).

Example 2.2

1. Suppose that $\mathcal{C}$ is a (locally small) category. We now define the quantaloid $\mathcal{PC}$. The objects of $\mathcal{PC}$ are the same as those of $\mathcal{C}$. If $C, D \in \mathcal{C}_0$ then we define $\mathcal{PC}(C, D)$ as the powerset of $\mathcal{C}(C, D)$, endowed with the inclusion order. The composition of morphisms is taken pointwise: if $A \subseteq \mathcal{C}(C, D)$ and $B \subseteq \mathcal{C}(D, E)$ then $B \circ A := \{g \circ f \mid f \in \mathcal{C}(C, D),\ g \in \mathcal{C}(D, E)\}$. It is clear that this defines a quantaloid, because the pointwise composition is associative and distributes over arbitrary unions. We also have:

$$\mathsf{id}_C^{\mathcal{PC}} = \{\mathsf{id}_C\} \qquad \mathsf{T}_{D,C} = \mathcal{C}(C, D)$$

   Now it is easy to construct a category $\mathcal{C}$ where $\mathcal{C}(D, E) \circ \mathcal{C}(C, D) \neq \mathcal{C}(C, E)$.
2. SL, the 2-category of sup-lattices and join-preserving morphisms is clearly a quantaloid.
3. Rel, the 2-category of sets and relations between sets is also a quantaloid. To be consistent with the notion of *distributor*, as developed in Section 4, we use the convention $\mathsf{Rel}(X, Y) := \mathcal{P}(Y \times X)$, such that the composition of relations becomes

$$R \circ S = \{(a, b) \mid \exists c : (a, c) \in R \text{ and } (c, b) \in S\}.$$

   The identity relation on a set X is denoted by Δ_X.

Suppose $a \in \mathcal{Q}(A, B)$. Then, because of Eq. 1, right and left multiplication with a gives morphisms in SL (i.e., join-preserving functions):

$$a \circ - : \mathcal{Q}\left(A', A\right) \to \mathcal{Q}\left(A', B\right) \qquad - \circ a : \mathcal{Q}\left(B, B'\right) \to \mathcal{Q}\left(A, B'\right)$$

Every join-preserving morphism $\phi : L \to M$ between sup-lattices has a right-adjoint $\phi^{\dashv}$, with:

$$\phi^{\dashv}(m) = \bigvee \{d \in L \mid \phi(d) \leq m\}$$

so we have two right-adjoints:

$$(a \circ -)^{\dashv} : \mathcal{Q}\left(A', B\right) \to \mathcal{Q}\left(A', A\right) : b \mapsto \bigvee \{x \mid a \circ x \leq b\}$$

$$(- \circ a)^{\dashv} : \mathcal{Q}\left(A, B'\right) \to \mathcal{Q}\left(B, B'\right) : b' \mapsto \bigvee \left\{y \mid y \circ a \leq b'\right\}$$

This gives two binary operations $\to$ and $\leftarrow$, **right** and **left residuation**:

$$a \to b := \bigvee \{x \mid a \circ x \leq b\} \qquad b' \leftarrow a := \bigvee \left\{y \mid y \circ a \leq b'\right\}$$

Note that $a \to b$ and $b' \leftarrow a$ are defined respectively when $\mathsf{cod}\, a = \mathsf{cod}\, b$ and $\mathsf{dom}\, a = \mathsf{dom}\, b'$.

Theorem 2.3 *The following properties are true for the elements of a quantaloid* (*whenever the operations are defined*)*:*

1. $a \circ b \leq c \;\Leftrightarrow\; b \leq a \to c \;\Leftrightarrow\; a \leq c \leftarrow b$
2. $(a \to c) \leftarrow b = a \to (c \leftarrow b) \qquad \mathsf{id}_{\mathsf{cod}\, a} \to a = a \leftarrow \mathsf{id}_{\mathsf{dom}\, a} = a$
3. $a \to (b \to c) = (b \circ a) \to c \qquad (c \leftarrow b) \leftarrow a = c \leftarrow (a \circ b)$
4. $\left(\bigvee_i a_i\right) \to b = \bigwedge_i (a_i \to b) \qquad b \leftarrow \left(\bigvee_i a_i\right) = \bigwedge_i (b \leftarrow a_i)$
5. $a \to \left(\bigwedge_i b_i\right) = \bigwedge_i (a \to b_i) \qquad \left(\bigwedge_i b_i\right) \leftarrow a = \bigwedge_i (b_i \leftarrow a)$

2.2 Functors and $*$-maps

Definition 2.4 An SL**-functor** F between quantaloids $\mathcal{Q}$ and $\mathcal{Q}'$ is a functor $F : \mathcal{Q} \to \mathcal{Q}'$ (between $\mathcal{Q}$ and $\mathcal{Q}'$ regarded as categories) that satisfies the condition (for all $A, B \in \mathcal{Q}_0$):

$$F_{A,B} : \mathcal{Q}(A, B) \to \mathcal{Q}'(FA, FB) : F\left(\bigvee_i a_i\right) = \bigvee_i F(a_i)$$

So $F_{A,B}$ needs to be a morphism in SL. If $\mathcal{Q}$ is a quantaloid then $\mathcal{Q}^{\mathsf{op}}$, the **opposite quantaloid**, has the same structure as $\mathcal{Q}$ but the order of the composition of elements is reversed ($\mathcal{Q}^{\mathsf{op}}(A, B) = \mathcal{Q}(B, A)$ and $a \circ_{\mathsf{op}} b = b \circ a$). $\mathcal{Q}^{\mathsf{co}}$ is the 2-category obtained by reversing the local order on the elements: $a \leq_{\mathsf{co}} a' \;\Leftrightarrow\; a \geq a'$. If the order and the composition are reversed one gets the 2-category $\mathcal{Q}^{\mathsf{coop}}$.

Definition 2.5 An **involution** on a quantaloid $\mathcal{Q}$ is an SL-functor of quantaloids $(-)^* : \mathcal{Q}^{\mathsf{op}} \to \mathcal{Q}$ that is the identity on objects and that is its own inverse, i.e.:

$$A^* = A \qquad (a \circ b)^* = b^* \circ a^* \qquad a^{**} = a$$

$$(-)^* : \mathcal{Q}(A, B) \to \mathcal{Q}(B, A) : \left(\bigvee_i a_i\right)^* = \bigvee_i a_i^*$$

An **involutive quantaloid** or a **$*$-quantaloid** $\mathcal{Q}$ is a quantaloid equipped with an involution.

Example 2.6

1. The most typical example of an involutive quantaloid is Rel, the 2-category of sets and relations between sets. The involution is given by taking the converse of relations: $R^* := R^{-1}$. More generally, one can take relations between objects in a topos $\mathcal{E}$, being a universe of 'generalized sets'. The involutive quantaloid $\mathsf{Rel}(\mathcal{E})$ obtained in this way is an example of a power allegory [5].
2. A locale Ω, being a commutative idempotent unital quantale, with involution the identity, is an involutive quantaloid.
3. Each involutive quantale Q gives rise to an involutive quantaloid $\mathsf{Pr}(Q)$ with objects the **projections** ($p \in Q$ with $p \circ p = p$ and $p^* = p$) and morphisms $\mathsf{Pr}(Q)(p, q) = \{a \in Q \mid a \circ p = a = q \circ a\}$ (order, composition, units and involution are evident). Virtually the same construction can be carried out for a quantaloid $\mathcal{Q}$. Note that in the trivial case of $Q = \Omega$, a locale, $\mathsf{Pr}(\Omega)(p, q) = \{a \in \Omega \mid a \leq p \wedge q\}$ and $\mathsf{Pr}(\Omega)$ is the Cauchy completion of $(\Omega, \wedge)$ as a *category* (monoid).

For an involutive quantaloid $\mathcal{Q}$ we consider the *subcategory* $\underline{\mathcal{Q}}$, with the same objects and with morphisms the '$*$-maps':

Definition 2.7 A **$*$-map** between objects A and B is a morphism $f \in \mathcal{Q}(A, B)$ satisfying:

$$\mathsf{id}_A \leq f^* \circ f \text{ and } f \circ f^* \leq \mathsf{id}_B.$$

Note that each $*$-map f is a left adjoint morphism (or a *map*), with $f^{\dashv} = f^*$. If one presupposes $\mathcal{Q}$ to satisfy *Freyd's law of modularity* $(a \circ b) \wedge c \leq (a \wedge c \circ b^*) \circ b$ [5, 13], then it is easy to prove the converse.

Example 2.8

1. The $*$-maps in Rel (see Example 2.2.3) can be identified with functions between sets: $\mathsf{Set} \cong \underline{\mathsf{Rel}}$ ($f \mapsto \langle f \rangle = \{(fx, x) \mid x \in \mathsf{dom}\, f\}$).
2. For a small category $\mathcal{C}$, we can construct the involutive quantaloid $\mathsf{R}(\mathcal{C})$, defined to be the restriction of $\mathsf{Rel}([\mathcal{C}^{\mathsf{op}}, \mathsf{Set}])$ to the objects of $\mathcal{C}$ (regarded as presheaves via the Yoneda embedding). A **span** from $D \in \mathcal{C}_0$ to $C \in \mathcal{C}_0$ is a pair (f, g) with $\mathsf{cod}\, f = C$, $\mathsf{cod} g = D$ and $\mathsf{dom}\, f = \mathsf{dom} g$. A **crible** R from D to C is a down-

closed set of spans from D to C (if $(f, g) \in R$ and $\mathsf{cod}h = \mathsf{dom}f$ then $(f \circ h, g \circ h) \in R$). The composition of cribles is done like in Example 2.2.3 and the involution on cribles is the operation that takes the converse, resulting in an explicit description of $\mathsf{R}(\mathcal{C})$. For consistency we conceived the spans and cribles as going in the reverse direction compared to [16], in which they are introduced (we do not write $\mathsf{Rel}(\mathcal{C})$, as in [2], to prevent confusion with relations *in* $\mathcal{C}$ [5]). The category of $*$-maps $\underline{\mathsf{R}}(\mathcal{C})$ is isomorphic to $\mathcal{C}$ ($f \mapsto \underline{f} := \{(f \circ h, h) \mid \mathsf{cod}h = \mathsf{dom}f\}$). The identities of $\mathsf{R}(\mathcal{C})$ are denoted by $\Delta_C := \underline{\mathsf{id}}_C$.

Proposition 2.9 *Let $\mathcal{Q}$ be a $*$-quantaloid. Then the following properties hold for elements of $\mathcal{Q}$, whenever the operations are defined:*

1. $(a \to b)^* = b^* \leftarrow a^* \qquad (a \leftarrow b)^* = b^* \to a^*$
2. *if a is a $*$-map then:*
   (a) $a = a \circ a^* \circ a \qquad a^* = a^* \circ a \circ a^*$
   (b) $a \circ c \leq b \;\Leftrightarrow\; c \leq a^* \circ b \qquad a \to b = a^* \circ b$
   (c) $c \circ a^* \leq b \;\Leftrightarrow\; c \leq b \circ a \qquad b \leftarrow a^* = b \circ a$
   (d) $a^* \circ \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a^* \circ b_i) \qquad \big(\bigwedge_{i\in I} b_i\big) \circ a = \bigwedge_{i\in I}(b_i \circ a)$
3. *if a and b are $*$-maps then:* $\quad a \leq b \;\Rightarrow\; a = b$

2.3 Nuclei

Definition 2.10 A **lax functor** $F : \mathcal{Q} \to \mathcal{Q}'$ between quantaloids sends objects/morphisms of $\mathcal{Q}$ to objects/morphisms of $\mathcal{Q}'$ such that (for all $A, B, C \in \mathcal{Q}_0$):

1. $F_{A,B} : \mathcal{Q}(A, B) \to \mathcal{Q}'(FA, FB)$ is a monotone $*$-map
2. $\mathsf{id}_{FA} \leq F_{A,A}(\mathsf{id}_A)$
3. $F_{B,C}(b) \circ F_{A,B}(a) \leq F_{A,C}(b \circ a)$

F is a **lax SL-functor** if in addition all $F_{A,B}$'s preserve arbitrary joins. If the inequalities in (2) and (3) are reversed, then we respectively speak of a **colax functor** and a **colax SL-functor**.

Definition 2.11 A **prenucleus** j on a quantaloid $\mathcal{Q}$ is a lax functor $j : \mathcal{Q} \to \mathcal{Q}$ that satisfies the following conditions:

1. j is the identity on objects,
2. $a \leq ja \quad (\forall a \in \mathcal{Q}_1)$.

A prenucleus j is a **nucleus** if also

3. $jja \leq ja \quad (\forall a \in \mathcal{Q}_1)$.

A nucleus j is **left-exact** if

4. $j(a \wedge b) = ja \wedge jb \quad (\forall a, b \in \mathcal{Q}_1)$.

A nucleus j on an involutive quantaloid $\mathcal{Q}$ is an **involutive nucleus** or a **$*$-nucleus** if it preserves the involution, i.e. if $j(a^*) = (ja)^*$.

Example 2.12 If $\mathcal{E}$ is a Grothendieck topos then it can be considered as the topos of sheaves on a small site $(\mathcal{C}, \mathfrak{F})$ (with $\mathcal{C}$ a small category and $\mathfrak{F}$ a Grothendieck topology on $\mathcal{C}$). In this case it is useful to consider the (small) involutive quantaloid $\mathsf{R}(\mathcal{C})$ Eq. 2.8. The left-exact nuclei on $\mathsf{R}(\mathcal{C})$ are automatically involutive and correspond bijectively to Grothendieck topologies on $\mathcal{C}$ [2]. We slightly adapt the presentation of this correspondence from [2] to Grothendieck topologies in their general form, such that we do not have to put restrictions on $\mathcal{C}$, like the existence of pullbacks. Let us present a nucleus j on $\mathsf{R}(\mathcal{C})$ in terms of the corresponding $\mathfrak{F}$ and vice versa:

$$jR := \left\{ C \xleftarrow{(f,g)} D \mid \exists S \in \mathfrak{F}(\mathsf{dom}\, f) : \forall s \in S : (f \circ s, g \circ s) \in R \right\}$$

$$\mathfrak{F}(C) := \{ S \in \Omega(C) \mid \Delta_C \subseteq jR_S \}$$

with $R \in \mathsf{R}(\mathcal{C})(D, C)$, $R_S := \{(s, s) \mid s \in S\}$ and Ω the subobject classifier.

Proposition 2.13

1. *Every nucleus j on a quantaloid $\mathcal{Q}$ induces a quantaloid $\mathcal{Q}_j$ and an* SL*-functor $\mathcal{Q} \xrightarrow{\mathbf{j}} \mathcal{Q}_j$ by the following data:*
   (a) $(\mathcal{Q}_j)_0 := \mathcal{Q}_0$
   (b) $\mathcal{Q}_j(A, B) := \{a \in \mathcal{Q}(A, B) \mid ja = a\}$ (*equipped with the order induced by the order of* $\mathcal{Q}(A, B)$)
   (c) $b \circ_j a := j_{A,C}(b \circ a)$, *for* $a \in \mathcal{Q}_j(A, B)$ *and* $b \in \mathcal{Q}_j(B, C)$
   (d) $\mathbf{j} : A \mapsto A$ *and* $a \mapsto ja$

   We have:

$$\bigvee_i^j a_i = j\left(\bigvee_i a_i\right) \quad \text{and} \quad \bigwedge_i^j a_i = \bigwedge_i a_i$$

2. *There is a one-to-one correspondence* (*up to isomorphism*) *between nuclei on $\mathcal{Q}$ and full* SL*-functors*

$$F : \mathcal{Q} \to \mathcal{Q}' : A \mapsto A$$

   from $\mathcal{Q}$ to a quantaloid $\mathcal{Q}'$ having 'the same' objects. With full we mean that the $*$-maps $F_{A,B} : \mathcal{Q}(A, B) \to \mathcal{Q}'(A, B)$ are surjective.

We say that two elements a and b of a quantaloid $\mathcal{Q}$ are **equivalent** with respect to a nucleus j on $\mathcal{Q}$ if their closures are equal, we write:

$$a \sim_j b \qquad \Leftrightarrow \qquad ja = jb$$

Note that the closure of a $*$-map in $\mathcal{Q}$ by a $*$-nucleus j is a $*$-map in $\mathcal{Q}_j$.

Example 2.14

1. If $(\mathcal{C}, \mathfrak{F})$ is a site, with corresponding nucleus j, then $\mathsf{R}_j(\mathcal{C})$ is the bicategory B of closed cribles that plays the central role in [16]. $\mathsf{R}_j(\mathcal{C})$ can be considered as the restriction of $\mathsf{Rel}(\mathsf{Sh}(\mathcal{C}, \mathfrak{F}))$ to the objects of $\mathcal{C}$.

2. If Ω is a locale and j the nucleus on $\mathsf{R}(\Omega)$ that corresponds to the canonical Grothendieck topology on Ω as a poset/category (for $S \in \Omega(p)$: $S \in \mathfrak{F}(p) \Leftrightarrow \bigvee S = p$), then $\mathsf{R}_j(\Omega) \cong \mathsf{Pr}(\Omega)$ (Example 2.6.3), because the closed cribles are exactly the *principal* ones [16].

Lemma 2.15 *Let $\mathcal{Q}$ be a $*$-quantaloid and j a nucleus on $\mathcal{Q}$. If f, a and b are elements of $\mathcal{Q}$, with f a $*$-map and $\mathsf{cod}\, f = \mathsf{dom}\, a = \mathsf{cod}\, b$, then:*

$$j(a \circ f) = ja \circ f \qquad j(f^* \circ b) = f^* \circ jb$$

In particular, if a (respectively b) is j-closed then $a \circ f$ ($f^* \circ b$) is j-closed.

Proof We just proof $j(a \circ f) \leq ja \circ f$.

$$\begin{aligned} & a \leq ja \qquad \text{and} \qquad f \circ f^* \leq \mathsf{id}_{\mathsf{cod}\, f} \\ & \Rightarrow a \circ f \circ f^* \leq ja \\ & \Leftrightarrow j(a \circ f) \circ f^* \leq ja \\ & \Leftrightarrow j(a \circ f) \leq ja \circ f \qquad (\mathsf{id}_{\mathsf{dom}\, f} \leq f^* \circ f) \end{aligned}$$ □

3 Constructions on $\mathcal{Q}$

3.1 $\mathsf{Matr}(\mathcal{Q})$

Let $\mathcal{Q}$ be a quantaloid. We now construct what we will call the quantaloid $\mathsf{Matr}(\mathcal{Q})$ of $\mathcal{Q}$-matrices. A $\mathcal{Q}_0$**-typed set** is a pair (X, t), where X is a set and $t : X \to \mathcal{Q}_0$ a function that assigns to every element of X an object of $\mathcal{Q}$. These sets will serve as the dimensions of the matrices we are about to define. If (X, t) and (Y, t') are two $\mathcal{Q}_0$-typed sets then a $\mathcal{Q}$**-matrix** $\mathbb{M}$ from (X, t) to (Y, t') is a function:

$$\mathbb{M} : Y \times X \to \mathcal{Q}_1 : (y, x) \mapsto \mathbb{M}(y, x) \in \mathcal{Q}\left(tx, t'y\right)$$

We usually write $\mathbb{M} : X \to Y$ (leaving t and t' unmentioned). It is evident that the set of all $\mathcal{Q}$-matrices from (X, t) to (Y, t') carries a natural order. Suppose $\mathbb{M}' : X \to Y$, then:

$$\mathbb{M} \leq \mathbb{M}' \qquad \Leftrightarrow \qquad \forall\, (y, x) \in Y \times X : \mathbb{M}(y, x) \leq \mathbb{M}'(y, x)$$

Moreover, this set is a sup-lattice because the joins can be taken pointwise. We can now define the composition of $\mathcal{Q}$-matrices: if $\mathbb{M} : X \to Y$ and $\mathbb{N} : Y \to Z$ then $\mathbb{N} \circ \mathbb{M} : X \to Z$ is the matrix given by:

$$(\mathbb{N} \circ \mathbb{M})(z, x) := \bigvee_{y \in Y} \mathbb{N}(z, y) \circ \mathbb{M}(y, x)$$

It is easily checked that this composition is associative and that it distributes over arbitrary joins. The composition has of course a unit:

$$X \xrightarrow{\mathbb{I}_X} X \qquad \mathbb{I}_X(x, y) = \begin{cases} \mathsf{id}_{tx} & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

In this way we get the quantaloid $\mathsf{Matr}(\mathcal{Q})$:

1. **objects:** $\mathcal{Q}_0$-typed sets
2. **morphisms:** $\mathcal{Q}$-matrices between $\mathcal{Q}_0$-typed sets.

Remark 3.1

1. An involution on $\mathcal{Q}$ induces an involution on $\mathsf{Matr}(\mathcal{Q})$: if $X \overset{\mathbb{M}}{\to} Y$ is a $\mathcal{Q}$-matrix then we obtain a $\mathcal{Q}$-matrix $Y \overset{\mathbb{M}^*}{\to} X$ as follows:
$$\mathbb{M}^*(y, x) := \mathbb{M}(x, y)^*$$
2. A nucleus j on $\mathcal{Q}$ induces a nucleus j on $\mathsf{Matr}(\mathcal{Q})$, by:
$$(j\mathbb{M})(x, y) := j(\mathbb{M}(x, y))$$
This nucleus is involutive if the original nucleus is involutive (in case of $\mathcal{Q}$ being an involutive quantaloid).

Example 3.2

1. Let **2** be the 2-element poset $(\{0, 1\}, 0 \leq 1)$ (which is a commutative one-object quantaloid, with $\circ = \wedge$). Then,
$$\mathsf{Matr}(\mathbf{2}) \cong \mathsf{Rel}.$$
2. Let us consider the involutive quantaloid $\mathsf{R}(\mathcal{C})$ from Example 2.8. Every presheaf $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ gives rise to a matrix $\mathbb{A}_F : X \to X$ in $\mathsf{Matr}(\mathsf{R}(\mathcal{C}))$. X consists of the elements of F and the type of an element $x \in FC$ is defined as C. For $x \in FC$ and $y \in FD$ we define $\mathbb{A}_F(x, y) := \left\{C \overset{(f,g)}{\longleftarrow} D \mid F(f)(x) = F(g)(y)\right\}$, which is a crible from D to C that measures the 'similarity' of x and y. If $\mathfrak{F} \leftrightarrow j$ is a Grothendieck topology on $\mathcal{C}$, then we can make the $\mathsf{R}_j(\mathcal{C})$-matrix $\mathbb{B}_F := j\mathbb{A}_F$, which measures the *local* similarity:
$$\mathbb{B}_F(x, y) = \left\{C \overset{(f,g)}{\longleftarrow} D \mid \exists S \in \mathfrak{F}(\mathsf{dom}\, f) : \forall s \in S : F(fs)(x) = F(gs)(y)\right\}$$
For a locale Ω, with the canonical Grothendieck topology, every presheaf F induces an $\mathsf{R}_j(\Omega)$-matrix or equivalently, a $\mathsf{Pr}(\Omega)$-matrix (see Example 2.14.2), with:
$$\mathbb{B}_F(x, y) = \bigvee\{p \leq tx \wedge ty \mid x_{|p} = y_{|p}\}.$$
The features of these matrices will be studied further in the following sections.

3.2 $\mathsf{Bim}_*(\mathcal{Q})$

Let $\mathcal{Q}$ be an involutive quantaloid. An endomorphism $m \in \mathcal{Q}_1$ is a **monad** if it is **transitive** and **reflexive**, i.e., if $m \circ m \leq m$ and $\mathsf{id}_{\mathsf{dom}m} \leq m$ (hence, $m \circ m = m$). A monad m is called **symmetric** or an **equivalence relation** if also $m^* = m$. A **bimodule** b from monad m to monad n is a morphism $b \in \mathcal{Q}(\mathsf{dom}m, \mathsf{dom}n)$ such that $b \circ m \leq b$ and $n \circ b \leq b$ (hence, $b \circ m = b = n \circ b$). The monads and the bimodules between them constitute the quantaloid $\mathsf{Bim}(\mathcal{Q})$ (order, composition and units are evident).

The equivalence relations and the bimodules between them give the involutive quantaloid $\mathsf{Bim}_*(\mathcal{Q})$, the involution being inherited from the one on $\mathcal{Q}$ (note that for a well-defined involution, the objects have to be *symmetric* monads).

More details about quantaloids can be found in [13].

4 $\mathcal{Q}$-$*$-categories

Let be $\mathcal{Q}$ a $*$-quantaloid.

Definition 4.1 A **$\mathcal{Q}$-category** is a monad in $\mathsf{Matr}(\mathcal{Q})$. A **$\mathcal{Q}$-$*$-category** is a *symmetric* $\mathcal{Q}$-category, i.e., an object of $\mathsf{Bim}_*(\mathsf{Matr}(\mathcal{Q}))$.

So a $\mathcal{Q}$-$*$-category is an equivalence relation in the category of $\mathcal{Q}$-matrices. It consists of a $\mathcal{Q}_0$-typed set $(\mathbb{A}_0, t)$ and a matrix $\mathbb{A} : \mathbb{A}_0 \to \mathbb{A}_0$:

$$t : \mathbb{A}_0 \to \mathcal{Q}_0 \qquad \mathbb{A} : \mathbb{A}_0 \times \mathbb{A}_0 \to \mathcal{Q}_1 \qquad \mathbb{A}(y, x) \in \mathcal{Q}(tx, ty)\,,$$

satisfying the following conditions:

$$\mathbb{I}_{\mathbb{A}_0} \leq \mathbb{A} \qquad \mathbb{A} \circ \mathbb{A} \leq \mathbb{A} \qquad \mathbb{A}^* = \mathbb{A}$$

In terms of the objects of $\mathbb{A}$ - we just call them **elements** - this means $(\forall\, x, y, z \in \mathbb{A}_0)$:

1. $\mathsf{id}_{tx} \leq \mathbb{A}(x, x)$
2. $\mathbb{A}(x, z) \circ \mathbb{A}(z, y) \leq \mathbb{A}(x, y)$
3. $\mathbb{A}(x, y)^* = \mathbb{A}(y, x)$

The elements of $\mathbb{A}$ of type $C \in \mathcal{Q}_0$ form the set $\mathbb{A}_C := \{x \in \mathbb{A}_0 \mid tx = C\}$. If $x \in \mathbb{A}_0$ then $\mathsf{E}x := \mathbb{A}(x, x)$ is called the **extent** of x. The morphisms of $\mathsf{Bim}_*(\mathsf{Matr}(\mathcal{Q}))$, being bimodules, are called **distributors** between $\mathcal{Q}$-$*$-categories and the category is denoted by $\mathsf{Dist}_*(\mathcal{Q}) := \mathsf{Bim}_*(\mathsf{Matr}(\mathcal{Q}))$. Hence, a distributor $\phi : \mathbb{A} \multimap \mathbb{B}$ between $\mathcal{Q}$-$*$-categories is given by a $\mathcal{Q}$-matrix

$$\phi : \mathbb{B}_0 \times \mathbb{A}_0 \to \mathcal{Q}_1 \qquad \phi(y, x) \in \mathcal{Q}(tx, ty)\,,$$

satisfying the properties $\phi \circ \mathbb{A} \leq \phi$ and $\mathbb{B} \circ \phi \leq \phi$, i.e. $(\forall\, x \in \mathbb{A}_0, y \in \mathbb{B}_0)$:

$$\bigvee_{z \in \mathbb{A}_0} \phi(y, z) \circ \mathbb{A}(z, x) \leq \phi(y, x) \qquad \bigvee_{z \in \mathbb{B}_0} \mathbb{B}(y, z) \circ \phi(z, x) \leq \phi(y, x)$$

This is equivalent to $\phi \circ \mathbb{A} = \phi$ and $\mathbb{B} \circ \phi = \phi$, because $\mathbb{I}_{\mathbb{A}_0} \leq \mathbb{A}$ and $\mathbb{I}_{\mathbb{B}_0} \leq \mathbb{B}$. The distributor ϕ is a **$*$-map** if in addition we have $\mathbb{A} \leq \phi^* \circ \phi$ and $\phi \circ \phi^* \leq \mathbb{B}$, meaning:

$$\mathbb{A}(y, x) \leq \bigvee_{z \in \mathbb{B}_0} \phi(z, y)^* \circ \phi(z, x) \qquad \bigvee_{w \in \mathbb{A}_0} \phi(v, w) \circ \phi(u, w)^* \leq \mathbb{B}(v, u)$$

$\underline{\mathsf{Dist}}_*(\mathcal{Q})$ is the category of $*$-maps between $\mathcal{Q}$-$*$-categories. We denote the unit $*$-map $\mathbb{A} : \mathbb{A} \multimap \mathbb{A}$ by $\iota\delta_{\mathbb{A}}$ and the composition of distributors by '$\otimes$', to stress the context. However, for the conception of the category of $\mathcal{Q}$-$*$-categories we define:

Definition 4.2 A **functor** $f : \mathbb{A} \to \mathbb{B}$ between $\mathcal{Q}$-$*$-categories $\mathbb{A}$ and $\mathbb{B}$ is a type preserving function $f : \mathbb{A}_0 \to \mathbb{B}_0$ satisfying the inequality

$$\mathbb{A}(x, y) \leq \mathbb{B}(fx, fy) .$$

A functor $f : \mathbb{A} \to \mathbb{B}$ is **fully faithful** if

$$\mathbb{A}(x, y) = \mathbb{B}(fx, fy) .$$

The category of $\mathcal{Q}$-$*$-categories has $\mathcal{Q}$-$*$-categories as objects and *functors* between $\mathcal{Q}$-$*$-categories as morphisms and we denote it by $\mathsf{Cat}_*(\mathcal{Q})$. The units for the composition are of the form $\mathsf{id}_{\mathbb{A}} : \mathbb{A} \to \mathbb{A} : \ x \mapsto x$. Every functor $f : \mathbb{A} \to \mathbb{B}$ gives rise to a $*$-map (but the converse does not need to hold):

Proposition 4.3

1. *A functor $f : \mathbb{A} \to \mathbb{B}$ between $\mathcal{Q}$-$*$-categories induces a $*$-map $\phi_f : \mathbb{A} \multimap \mathbb{B}$ with*
$$\phi_f(y, x) = \mathbb{B}(y, fx).$$
2. *In this way we get a functor:*
$$\phi_- : \mathsf{Cat}_*(\mathcal{Q}) \to \underline{\mathsf{Dist}}_*(\mathcal{Q}) : \ \mathbb{A} \xrightarrow{f} \mathbb{B} \ \mapsto \ \mathbb{A} \overset{\phi_f}{\multimap} \mathbb{B}$$

Proof

1. $\phi_f \circ \mathbb{A} \leq \phi_f$:
$$\begin{aligned}(\phi_f \circ \mathbb{A})(y, x) &= \bigvee_{z \in \mathbb{A}_0} \mathbb{B}(y, fz) \circ \mathbb{A}(z, x) \\ &\leq \bigvee_{z \in \mathbb{A}_0} \mathbb{B}(y, fz) \circ \mathbb{B}(fz, fx) \\ &\leq \bigvee_{w \in \mathbb{B}_0} \mathbb{B}(y, w) \circ \mathbb{B}(w, fx) \\ &= \mathbb{B}(y, fx) \\ &= \phi_f(y, x)\end{aligned}$$

   $\mathbb{B} \circ \phi_f \leq \phi_f$:
$$\begin{aligned}(\mathbb{B} \circ \phi_f)(y, x) &= \bigvee_{z \in \mathbb{B}_0} \mathbb{B}(y, z) \circ \mathbb{B}(z, fx) \\ &= \mathbb{B}(y, fx) \\ &= \phi_f(y, x)\end{aligned}$$

   $\mathbb{A} \leq \phi_f^* \circ \phi_f$:
$$\begin{aligned}(\phi_f^* \circ \phi_f)(x, y) &= \bigvee_{z \in \mathbb{B}_0} \phi_f(z, y)^* \circ \phi_f(z, x) \\ &= \bigvee_{z \in \mathbb{B}_0} \mathbb{B}(fy, z) \circ \mathbb{B}(z, fx) \\ &= \mathbb{B}(fy, fx) \\ &\geq \mathbb{A}(y, x)\end{aligned}$$

$\phi_f \circ \phi_f^* \leq \mathbb{B}$:

$$\begin{aligned}(\phi_f \circ \phi_f^*)(y,x) &= \bigvee_{z\in\mathbb{A}_0} \phi_f(y,z) \circ \phi_f(x,z)^* \\ &= \bigvee_{z\in\mathbb{A}_0} \mathbb{B}(y,\ fz) \circ \mathbb{B}(fz,x) \\ &\leq \bigvee_{w\in\mathbb{B}_0} \mathbb{B}(y,w) \circ \mathbb{B}(w,x) \\ &= \mathbb{B}(y,x)\end{aligned}$$

2. $\phi_{\mathsf{id}_{\mathbb{A}}} = \mathbb{A} = \iota\delta_{\mathbb{A}}$

$$\begin{aligned}\phi_{g\circ f} &= \phi_g \circ \phi_f \ :\ \big(\textit{with}\ \mathbb{A} \xrightarrow{f} \mathbb{B} \xrightarrow{g} \mathbb{C}\big) \\ (\phi_g \circ \phi_f)(y,x) &= \bigvee_{z\in\mathbb{B}_0} \phi_g(y,z) \circ \phi_f(z,x) \\ &= \bigvee_{z\in\mathbb{B}_0} \mathbb{C}(y,gz) \circ \mathbb{B}(z,\ fx)\end{aligned}$$

$$\begin{array}{l|l}
\leq \bigvee_{z\in\mathbb{B}_0} \mathbb{C}(y,gz) \circ \mathbb{C}(gz,gfx) & \geq \mathbb{C}(y,gfx) \circ \mathbb{B}(fx,\ fx) \\
\leq \bigvee_{w\in\mathbb{C}_0} \mathbb{C}(y,w) \circ \mathbb{C}(w,gfx) & \geq \mathbb{C}(y,gfx) \circ \mathsf{id}_{tfx} \\
= \mathbb{C}(y,gfx) & = \mathbb{C}(y,gfx) \\
= \phi_{g\circ f}(y,x) & = \phi_{g\circ f}(y,x)
\end{array}$$

□

A distributor ϕ is **representable** if it is the image ϕ_f of a functor f, which is said to represent ϕ. We will use the notation $\mathsf{Func}(\mathbb{A},\mathbb{B}) := \mathsf{Cat}_*(\mathcal{Q})(\mathbb{A},\mathbb{B})$ and $\mathsf{Map}(\mathbb{A},\mathbb{B}) := \underline{\mathsf{Dist}}_*(\mathcal{Q})(\mathbb{A},\mathbb{B})$.

Example 4.4

1. **(Pre)sheaves:** The matrices $\mathbb{A}_F$ and $\mathbb{B}_F$ (from Example 3.2) are $\mathcal{Q}$-$*$-categories, for $\mathcal{Q}$ respectively $\mathsf{R}(\mathcal{C})$ and $\mathsf{R}_j(\mathcal{C})$. Moreover, for $F, G \in [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$ it is easy to verify that the natural transformations between F and G correspond one-to-one to functors between $\mathbb{A}_F$ and $\mathbb{A}_G$. We have in fact an embedding:

$$\big[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}\big] \ \hookrightarrow\ \mathsf{Cat}_*(\mathsf{R}(\mathcal{C}))\,.$$

2. **Ω-sets:** Let Ω be a locale. An Ω-set is a pair (A,d) consisting of a set A with a function $d : A \times A \to \Omega$, which is transitive and symmetric: $d(x,z) \wedge d(z,y) \leq d(x,y)$ and $d(x,y) = d(y,x)$ [4]. Ω-sets and $\mathsf{Pr}(\Omega)$-$*$-categories are the same thing: (A,d) can be equivalently presented as a $\mathsf{Pr}(\Omega)$-$*$-category $\mathbb{A}$, with $\mathbb{A}_0 = A$, $tx = d(x,x)$ and $\mathbb{A}(x,y) = d(x,y)$ (regarded as a morphism from ty to tx, see Example 2.6.3, which is well-defined because $d(x,y) \leq d(x,x) \wedge d(y,y)$). The reflexiveness of $\mathbb{A}$ does not require $\mathbb{A}(x,x)$ to be the top of Ω, we only have $\mathsf{id}_{tx} = \mathbb{A}(x,x)$. In particular, every presheaf $F : \Omega^{\mathsf{op}} \to \mathsf{Set}$ induces an Ω-set $\mathbb{B}_F$ (see Example 3.2).
That these Ω-sets should in fact be considered as $\mathsf{Pr}(\Omega)$-$*$-categories becomes apparent after a closer examination of the category $\mathsf{Set}(\Omega)$ of Ω-sets, defined to consist of the Ω-sets and what one calls *maps* between them, which can be identified with $\underline{\mathsf{Dist}}_*(\mathsf{Pr}(\Omega))$ (a short survey of Ω-sets can be found in the second volume of [7], pp.502–513). Moreover, in this form, the generalization of these notions from locales to sites becomes evident. It would have been more appropriate to identify the category of Ω-sets with $\mathsf{Cat}_*(\mathsf{Pr}(\Omega))$, because it is the

category in which the process of *completion* takes place, as we will see in the next section.

3. **Strict $\mathcal{Q}$-sets:** In [6], a $\mathcal{Q}$-set, with $\mathcal{Q}$ an involutive quantaloid, is a symmetric idempotent $\mathcal{Q}$-matrix $\mathbb{S}$, i.e., $\mathbb{S}^* = \mathbb{S}$ and $\mathbb{S} \circ \mathbb{S} = \mathbb{S}$ (which is a generalization of an Ω-set). However, the theory in [6] cannot be developed without imposing the condition of *strictness*: $\mathbb{S}(x, x) \circ \mathbb{S}(x, y) = \mathbb{S}(x, y)$ (which is a consequence of the other conditions if $\mathcal{Q}$ is a locale). But, due to symmetry, this is equivalent to asking that $\mathbb{S}(x, y)$ is a morphism from $\mathbb{S}(y, y)$ to $\mathbb{S}(x, x)$, in $\mathsf{Pr}(\mathcal{Q})$ (Example 2.6.3). Hence, strict $\mathcal{Q}$-sets are in fact $\mathsf{Pr}(\mathcal{Q})$-$*$-categories. In particular, from this perspective, strict $\mathcal{Q}$-sets *are* reflexive.

5 Singletons and Completion

One has the canonical embeddings:

$$\mathsf{Bim}_*(\mathcal{Q}) \hookrightarrow \mathsf{Dist}_*(\mathcal{Q}) \qquad \underline{\mathsf{Bim}}_*(\mathcal{Q}) \hookrightarrow \underline{\mathsf{Dist}}_*(\mathcal{Q})$$

The embeddings are given by the following observation: we can regard the objects of $\mathsf{Bim}_*(\mathcal{Q})$ as one-object $\mathcal{Q}$-$*$-categories. Suppose $e \in \mathsf{Bim}_*(\mathcal{Q})_0$ an equivalence relation, then we define the matrix $*_e$ by the $\mathcal{Q}_0$-typed set $\{*\} \to \mathcal{Q}_0 : * \mapsto \mathsf{dom}\, e$ and the single value $*_e(*, *) = e$. It is easy to see that $*_e$ is an object of $\mathsf{Dist}_*(\mathcal{Q})$, being a one-object $\mathcal{Q}$-$*$-category. Note that the extent $\mathsf{E}x$ of an element x of a $\mathcal{Q}$-$*$-category $\mathbb{A}$ has the properties of an equivalence relation. Each object C of $\mathcal{Q}$ corresponds to a special matrix $*_C := *_{\mathsf{id}_C}$. The bimodules/$*$-maps between the objects of $\mathsf{Bim}_*(\mathcal{Q})$ can be seen as distributors/$*$-maps between the corresponding $\mathcal{Q}$-$*$-categories.

The notion of a *singleton* of an Ω-set from [4] can be easily generalized to $\mathcal{Q}$-$*$-categories:

Definition 5.1 A **singleton** of a $\mathcal{Q}$-$*$-category $\mathbb{A}$ is a $*$-map

$$\phi : *_C \multimap \mathbb{A}$$

with C an object of $\mathcal{Q}$.

We can characterize a singleton $\phi : *_C \multimap \mathbb{A}$ in the following way:

1. A function $\phi : \mathbb{A}_0 \to \mathcal{Q}_1$ such that $\phi(x) \in \mathcal{Q}(C, tx)$. Because $*_C$ has only one element, we write $\phi(x)$ instead of $\phi(x, *)$.
2. $\mathbb{A} \circ \phi = \phi$:
$$\bigvee_{z \in \mathbb{A}_0} \mathbb{A}(x, z) \circ \phi(z) = \phi(x)$$
3. $*_C \leq \phi^* \circ \phi$ and $\phi \circ \phi^* \leq \mathbb{A}$:
$$\mathsf{id}_C \leq \bigvee_{x \in \mathbb{A}_0} \phi(x)^* \circ \phi(x) \qquad \phi(y) \circ \phi(x)^* \leq \mathbb{A}(y, x)$$

$\phi \circ *_C = \phi$ is always the case because $*_C(*, *) = \mathsf{id}_C$.

The composition $\phi^* \otimes \phi$ (which we can identify with $(\phi^* \otimes \phi)(*, *)$) is an equivalence relation:

$$\mathsf{id}_C \leq \phi^* \otimes \phi \qquad \phi^* \otimes \phi \otimes \phi^* \otimes \phi = \phi^* \otimes \phi \qquad (\phi^* \otimes \phi)^* = \phi^* \otimes \phi$$

We define the **extent of a singleton** ϕ to be

$$\mathsf{E}\phi := \phi^* \otimes \phi = \bigvee_{x \in \mathbb{A}_0} \phi(x) \circ \phi(x)^*$$

Note that a $*$-map $D \xrightarrow{a} C$ in $\mathcal{Q}$ can be regarded as a singleton a of $*_C$:

$$*_D \xrightarrow{a} *_C \qquad \mathsf{E}a = a^* \circ a$$

Lemma 5.2 *Every element x of a $\mathcal{Q}$-$*$-category $\mathbb{A}$ induces a functor*

$$\overline{x} : *_{tx} \to \mathbb{A} : * \mapsto x$$

and therefore we have the representable singleton $\widetilde{x} := \phi_{\overline{x}}$:

$$\widetilde{x} : *_{tx} \multimap \mathbb{A} \qquad \textit{with} \qquad \widetilde{x}(y) = \mathbb{A}(y, x)\,.$$

Proof $\overline{x}$ is a functor because $*_{tx}(*, *) = \mathsf{id}_{tx} \leq \mathbb{A}(x, x) = \mathbb{A}(\overline{x}(*), \overline{x}(*))$. By Proposition 4.3 the functor $\overline{x}$ induces a $*$-map $\phi_{\overline{x}} : *_{tx} \multimap \mathbb{A}$, a singleton in this case, with $\phi_{\overline{x}}(y, *) = \mathbb{A}(y, \overline{x}(*)) = \mathbb{A}(y, x)$. □

Proposition 5.3 *We list the following properties, for $\mathcal{Q}$-$*$-categories $\mathbb{A}$ and $\mathbb{B}$, elements $a, x, y \in \mathbb{A}_0, b \in \mathbb{B}_0$ and singletons ϕ and ψ of $\mathbb{A}$:*

1. *If $\mathbb{A} \xrightarrow{\mathbb{M}} \mathbb{B}$ is a distributor then:*

$$\mathbb{M}(b, a) = \widetilde{b}^* \otimes \mathbb{M} \otimes \widetilde{a} \qquad \mathbb{M}(-, a) = \mathbb{M} \otimes \widetilde{a} \qquad \mathbb{M}(b, -) = \widetilde{b}^* \otimes \mathbb{M}$$

2. $\psi^* \otimes \phi = \psi \to \phi$ (*with $\psi \to \phi$ calculated in* $\mathsf{Dist}_*(\mathcal{Q})$)
3. $\phi \leq \psi \quad \Rightarrow \quad \phi = \psi$

4. $\widetilde{x}^* \otimes \phi = \phi(x)$
5. $\widetilde{y}^* \otimes \widetilde{x} = \mathbb{A}(y, x) \qquad \mathsf{E}\widetilde{x} = \mathsf{E}x$
6. $\mathbb{A}(y, x) \circ \phi(x) \leq \phi(y)$
7. $\phi(x) \circ \mathsf{E}\phi = \phi(x) = \mathsf{E}x \circ \phi(x)$

Proof

1. Let $\mathbb{A} \overset{\mathbb{M}}{\multimap} \mathbb{B}$ be a distributor.

$$\begin{aligned}\mathbb{M}(b\,, a) &= (\mathbb{B} \otimes \mathbb{M} \otimes \mathbb{A})(b, a)\\ &= \textstyle\bigvee_{d \in \mathbb{B}_0} \mathbb{B}(b, d) \circ \left[\bigvee_{c \in \mathbb{A}_0} \mathbb{M}(d, c) \circ \mathbb{A}(c, a)\right]\\ &= \textstyle\bigvee_{d \in \mathbb{B}_0} \widetilde{b}\,(d)^* \circ (\mathbb{M} \circ \widetilde{a})(d)\\ &= \widetilde{b}^* \otimes \mathbb{M} \otimes \widetilde{a}\end{aligned}$$

2. Because ψ is a $*$-map, one has the equality $\psi^* \otimes \phi = \psi \to \phi$ (by Proposition 2.9).
3. Also follows from Proposition 2.9 .
4.

$$\begin{aligned}\widetilde{x}^* \otimes \phi &= \textstyle\bigvee_{y \in \mathbb{A}_0} \mathbb{A}(x, y) \circ \phi(y)\\ &= (\mathbb{A} \otimes \phi)(*, x)\\ &= \phi(x)\end{aligned}$$

5. $\widetilde{y}^* \otimes \widetilde{x} \overset{(4)}{=} \widetilde{x}(y) = \mathbb{A}(y, x)$. In particular $\mathsf{E}\widetilde{x} = \widetilde{x}^* \otimes \widetilde{x} = \mathbb{A}(x, x) = \mathsf{E}x$.
6. Follows from $\mathbb{A} \circ \phi \leq \phi$.
7. $\phi(x) \circ \mathsf{E}\phi = \phi(x)$ follows from the regularity of $*$-maps: $\mathsf{E}\phi \otimes \phi = \phi \otimes \phi^* \otimes \phi = \phi$. Moreover:

$$\begin{aligned}\phi(x) \quad &\leq \mathsf{E}x \circ \phi(x) \text{ because } \mathsf{id}_{tx} \leq \mathsf{E}x\\ \mathsf{E}x \circ \phi(x) \leq \quad &\phi(x) \quad \text{by (6)}\end{aligned}$$

□

The set of all singletons of $\mathbb{A}$ is represented by $\widetilde{\mathbb{A}}_0$. It can be given the structure of a $\mathcal{Q}$-$*$-category $\widetilde{\mathbb{A}}$:

$$t : \widetilde{\mathbb{A}}_0 \to \mathcal{Q}_0 : \ \phi \ \mapsto \ C \ \ if \ \ \phi : *_C \multimap \mathbb{A}$$

$$\widetilde{\mathbb{A}}(\psi, \phi) := \psi^* \otimes \phi = \bigvee\nolimits_{x \in \mathbb{A}_0} \psi(x)^* \circ \phi(x)$$

Lemma 5.4

1. $\widetilde{\mathbb{A}}$ *is a* $\mathcal{Q}$-$*$-*category.*
2. $\widetilde{\mathbb{A}}(\psi, \phi) = \psi \to \phi \qquad \widetilde{\mathbb{A}}(\widetilde{x}, \phi) = \phi(x) \qquad \widetilde{\mathbb{A}}(\widetilde{y}, \widetilde{x}) = \mathbb{A}(y, x)$
3. $\eta_{\mathbb{A}} : \mathbb{A} \to \widetilde{\mathbb{A}} : x \mapsto \widetilde{x}$ *is a fully faithful functor:* $A(x, y) = \widetilde{\mathbb{A}}(\widetilde{x}, \widetilde{y})$.
4. $\epsilon_{\mathbb{A}} := \phi^*_{\eta_{\mathbb{A}}} : \widetilde{\mathbb{A}} \multimap \mathbb{A}$ *is an isomorphism in* $\underline{\mathsf{Dist}}_*(\mathcal{Q})$.

$$\epsilon_{\mathbb{A}}(x, \phi) = \phi(x) \qquad \epsilon^*_{\mathbb{A}}(\phi, x) = \phi(x)^* \qquad \epsilon^*_{\mathbb{A}} = \epsilon^{-1}_{\mathbb{A}}$$

$$\epsilon_{\mathbb{A}} \otimes \widetilde{\phi} = \phi$$

Proof

1. $\widetilde{\mathbb{A}}(\phi,\phi) = \phi^* \otimes \phi \geq \mathsf{id}_{t\phi}$
   $\widetilde{\mathbb{A}}(\chi,\psi) \circ \widetilde{\mathbb{A}}(\psi,\phi) = \chi^* \otimes \psi \otimes \psi^* \otimes \phi \leq \chi^* \otimes \mathbb{A} \otimes \phi \leq \chi^* \otimes \phi = \widetilde{\mathbb{A}}(\chi,\phi)$
   $\widetilde{\mathbb{A}}^*(\psi,\phi) = \widetilde{\mathbb{A}}(\phi,\psi)^* = (\phi^* \otimes \psi)^* = \psi^* \otimes \phi = \widetilde{\mathbb{A}}(\psi,\phi)$
2. These equalities follow from Proposition 5.3 (2, 4 and 5).
3. Evident.
4. $\epsilon_{\mathbb{A}}(x,\phi) = \phi^*_{\eta_{\mathbb{A}}}(x,\phi) = \phi_{\eta_{\mathbb{A}}}(\phi,x)^* = \widetilde{\mathbb{A}}\big(\phi,\widetilde{x}\big)^* = \widetilde{\mathbb{A}}\big(\widetilde{x},\phi\big) = \phi(x)$
   $\epsilon_{\mathbb{A}} \otimes \epsilon^*_{\mathbb{A}} = \iota\delta_{\mathbb{A}}$:

$$\begin{aligned}(\epsilon_{\mathbb{A}} \otimes \epsilon^*_{\mathbb{A}})(y,x) &= \bigvee_{\phi \in \widetilde{\mathbb{A}}_0} \epsilon_{\mathbb{A}}(y,\phi) \circ \epsilon^*_{\mathbb{A}}(\phi,x)\\ &= \bigvee_{\phi \in \widetilde{\mathbb{A}}_0} \phi(y) \circ \phi(x)^*\\ &= \bigvee_{\phi \in \widetilde{\mathbb{A}}_0} \widetilde{\mathbb{A}}\big(\widetilde{y},\phi\big) \circ \widetilde{\mathbb{A}}\big(\phi,\widetilde{x}\big)\\ &= \widetilde{\mathbb{A}}\big(\widetilde{y},\widetilde{x}\big)\\ &= \mathbb{A}(y,x)\end{aligned}$$

   $\epsilon^*_{\mathbb{A}} \otimes \epsilon_{\mathbb{A}} = \iota\delta_{\widetilde{\mathbb{A}}}$:

$$\begin{aligned}(\epsilon^*_{\mathbb{A}} \otimes \epsilon_{\mathbb{A}})(\psi,\phi) &= \bigvee_{x \in \mathbb{A}_0} \epsilon^*_{\mathbb{A}}(\psi,x) \circ \epsilon_{\mathbb{A}}(x,\phi)\\ &= \bigvee_{x \in \mathbb{A}_0} \psi(x)^* \circ \phi(x)\\ &= \psi^* \circ \phi\\ &= \widetilde{\mathbb{A}}(\psi,\phi)\end{aligned}$$

   $\epsilon_{\mathbb{A}} \otimes \widetilde{\phi} = \epsilon_{\mathbb{A}}(-,\phi) = \phi$ (by Proposition 5.3.1).

□

Definition 5.5 A $\mathcal{Q}$-$*$-category $\mathbb{A}$ is **skeletal** if $\eta_{\mathbb{A}}$ is injective, **weakly $*$-complete** if $\eta_{\mathbb{A}}$ is surjective and **$*$-complete** if $\eta_{\mathbb{A}}$ is bijective.

In the context of [16], *Cauchy completeness* for symmetric B-categories is equivalent to weakly $*$-completeness, because $\mathsf{Dist}_*(B)$ satisfies Freyd's law of modularity, such that $*$-maps and maps (left adjoint morphisms) are the same.

Proposition 5.6

1. *If x and y are elements of a $\mathcal{Q}$-$*$-category $\mathbb{A}$ then:*

$$\widetilde{x} \;=\; \widetilde{y} \qquad \Leftrightarrow \qquad \mathsf{id}_{tx} \;\leq\; \mathbb{A}(x,y)$$

2. *A $\mathcal{Q}$-$*$-category $\mathbb{A}$ is skeletal iff*

$$\forall\, f,g : \mathbb{B} \to \mathbb{A}\ (\textit{functors}) : \quad \phi_f = \phi_g \quad \Rightarrow \quad f = g$$

3. *A $\mathcal{Q}$-$*$-category $\mathbb{A}$ is weakly $*$-complete iff all $*$-maps into $\mathbb{A}$ are representable.*
4. *A $\mathcal{Q}$-$*$-category $\mathbb{A}$ is $*$-complete iff all $*$-maps into $\mathbb{A}$ are* uniquely *representable*
5. *$\widetilde{\mathbb{A}}$ is $*$-complete (for a $\mathcal{Q}$-$*$-category $\mathbb{A}$).*

Proof

1. If $\widetilde{x} = \widetilde{y}$ then $\mathsf{id}_{tx} \leq \mathbb{A}(x,x) = \mathbb{A}(x,y)$. If $\mathsf{id}_{tx} \leq \mathbb{A}(x,y)$ then $\mathbb{A}(z,x) \leq \mathbb{A}(z,x) \circ \mathbb{A}(x,y) \leq \mathbb{A}(z,y)$ so, because of symmetry, $\mathbb{A}(z,x) = \mathbb{A}(z,y)$ (for all $z \in \mathbb{A}_0$). Hence, $\widetilde{x} = \widetilde{y}$.
2. ($\Rightarrow$) Suppose $\phi_f = \phi_g : \mathbb{B} \multimap \mathbb{A}$, with $\mathbb{A}$ skeletal. Let b be an element of $\mathbb{B}_0$.

$$\begin{aligned} \phi_f = \phi_g &\Rightarrow \forall x \in \mathbb{A}_0 : \phi_f(x,b) = \phi_g(x,b) \\ &\Leftrightarrow \forall x \in \mathbb{A}_0 : \mathbb{A}(x, fb) = \mathbb{A}(x, gb) \\ &\Leftrightarrow \widetilde{fb} = \widetilde{gb} \\ &\Rightarrow fb = gb \qquad \text{(because } \mathbb{A} \text{ is skeletal)} \end{aligned}$$

   ($\Leftarrow$) This is an easy consequence of the fact that a singleton $\widetilde{x} \in \widetilde{\mathbb{A}}_0$ is induced by the functor $\overline{x} : \widetilde{x} = \phi_{\overline{x}}$ (see Lemma 5.2).
3. ($\Rightarrow$) Suppose that $\eta_{\mathbb{A}}$ is surjective, i.e., $\forall \phi \in \widetilde{\mathbb{A}} : \exists x \in \mathbb{A}_0 : \phi = \widetilde{x}$. Let $\phi : \mathbb{B} \multimap \mathbb{A}$ be a $*$-map from $\mathcal{Q}$-$*$-category $\mathbb{B}$ to $\mathbb{A}$. For every $b \in \mathbb{B}_0$ we can compose $\widetilde{b} : *_{tb} \multimap \mathbb{B}$ with ϕ, resulting in the singleton:

$$\phi \otimes \widetilde{b} = \phi(-, b) : *_{tb} \multimap \mathbb{A} \qquad \text{(by Proposition 5.3.1).}$$

   Hence there exists an $x_b \in \mathbb{A}_0$ with: $\widetilde{x}_b = \phi \otimes \widetilde{b}$. We use this to build the following function:

$$f : \mathbb{B}_0 \to \mathbb{A}_0 : b \mapsto fb := x_b$$

   f induces a functor from $\mathbb{B}$ to $\mathbb{A}$:

$$\mathbb{A}(fc, fb) = \widetilde{fc}^* \otimes \widetilde{fb} = (\phi \otimes \widetilde{c})^* \otimes \phi \otimes \widetilde{b} = \widetilde{c}^* \otimes \phi^* \otimes \phi \otimes \widetilde{b} \geq \widetilde{c}^* \otimes \widetilde{b} = \mathbb{B}(c,b)$$

   and $\phi_f = \phi$:

$$\phi_f(a,b) = \mathbb{A}(a, fb) = \widetilde{a}^* \otimes \phi \otimes \widetilde{b} = \phi(a,b) \qquad \text{(by Proposition 5.3.1).}$$

   ($\Leftarrow$) Suppose that all $*$-maps into $\mathbb{A}$ are representable. Then for every singleton $\phi \in \widetilde{\mathbb{A}}_0$ there exists a functor $f : *_{t\phi} \to \mathbb{A}$ with $\phi_f = \phi$. This functor is of the form $\overline{x}$, hence: $\phi = \widetilde{x}$, making $\eta_{\mathbb{A}}$ surjective.
4. This follows from (2) and (3).
5. Let $\mathbb{A}$ be a $\mathcal{Q}$-$*$-category. Then $\widetilde{\mathbb{A}}$ is skeletal: if $\widetilde{\phi} = \widetilde{\psi}$ ($\phi, \psi \in \widetilde{\mathbb{A}}_0$), then for every $x \in \mathbb{A}_0$ we have:

$$\phi(x) = \widetilde{\mathbb{A}}\big(\widetilde{x}, \phi\big) = \widetilde{\mathbb{A}}\big(\widetilde{x}, \psi\big) = \psi(x) \qquad \text{(by Lemma 5.4.2)}$$

   so we can conclude that $\phi = \psi$. To prove the weakly $*$-completeness of $\widetilde{\mathbb{A}}$ let γ be a singleton of $\widetilde{\mathbb{A}}$. Then we can compose γ with the isomorphism $\epsilon_{\mathbb{A}}$ (from Lemma 5.4.4):

$$*_{t\gamma} \overset{\gamma}{\multimap} \widetilde{\mathbb{A}} \overset{\epsilon_{\mathbb{A}}}{\multimap} \mathbb{A}$$

In this way we get the singleton $\phi := \epsilon_{\mathbb{A}} \otimes \gamma \in \widetilde{\mathbb{A}}_0$ with $\gamma = \widetilde{\phi}$, because:

$$\gamma(\psi) = \widetilde{\psi}^* \otimes \gamma = \widetilde{\psi}^* \otimes \epsilon_{\mathbb{A}}^* \otimes \epsilon_{\mathbb{A}} \otimes \gamma = \left(\epsilon_{\mathbb{A}} \otimes \widetilde{\psi}\right)^* \otimes (\epsilon_{\mathbb{A}} \otimes \gamma) = \psi^* \otimes \phi = \widetilde{\phi}(\psi)$$

□

The functor $\eta_{\mathbb{A}} : \mathbb{A} \to \widetilde{\mathbb{A}} : x \mapsto \widetilde{x}$ can be considered as the **completion** of $\mathbb{A}$. Let us make this more specific:

Proposition 5.7 *The functor*

$$\phi_- : \mathsf{Cat}_*(\mathcal{Q}) \to \underline{\mathsf{Dist}}_*(\mathcal{Q}) : \quad \mathbb{A} \xrightarrow{g} \mathbb{B} \;\mapsto\; \mathbb{A} \overset{\phi_g}{\multimap} \mathbb{B}$$

$$\phi_g(y, x) = \mathbb{B}(y, gx)$$

has a right adjoint $\phi \dashv f$:

$$f_- : \underline{\mathsf{Dist}}_*(\mathcal{Q}) \to \mathsf{Cat}_*(\mathcal{Q}) : \quad \mathbb{A} \overset{\psi}{\multimap} \mathbb{B} \;\mapsto\; \widetilde{\mathbb{A}} \xrightarrow{f_\psi} \widetilde{\mathbb{B}}$$

$$f_\psi(\sigma) := \psi \otimes \sigma$$

so we have the natural equivalence

$$\mathsf{Map}(\mathbb{A}, \mathbb{B}) \cong \mathsf{Func}(\mathbb{A}, \widetilde{\mathbb{B}}) .$$

The unit and counit of this adjunction are respectively:

$$\mathbb{A} \xrightarrow{\eta_{\mathbb{A}}} \widetilde{\mathbb{A}} \qquad \widetilde{\mathbb{B}} \overset{\epsilon_{\mathbb{B}}}{\multimap} \mathbb{B}$$

Proof We just prove that the functor ϕ has a right adjoint. It suffices to show that if $\psi : \mathbb{A} \multimap \mathbb{B}$ is a $*$-map between $\mathcal{Q}$-$*$-categories, then there exists a unique functor $g : \mathbb{A} \to \widetilde{\mathbb{B}}$ such that $\epsilon_{\mathbb{B}} \otimes \phi_g = \psi$. Because $\epsilon_{\mathbb{B}} : \widetilde{\mathbb{B}} \multimap \mathbb{B}$ is an isomorphism, there is only one possibility: $\phi_g = \epsilon_{\mathbb{B}}^* \otimes \psi : \mathbb{A} \multimap \widetilde{\mathbb{B}}$. By the $*$-completeness of $\widetilde{\mathbb{B}}$, there exists a unique functor $g : \mathbb{A} \to \widetilde{\mathbb{B}}$ such that $\phi_g = \epsilon_{\mathbb{B}}^* \otimes \psi$. This setting yields the following natural equivalence, with f the functor as described in the proposition:

$$\begin{array}{rccc} \theta : & \mathsf{Map}(\mathbb{A}, \mathbb{B}) & \longrightarrow & \mathsf{Func}\left(\mathbb{A}, \widetilde{\mathbb{B}}\right) \\ & \| & & \| \\ & \underline{\mathsf{Dist}}_*(\mathcal{Q})(\phi\mathbb{A}, \mathbb{B}) & & \mathsf{Cat}_*(\mathcal{Q})(\mathbb{A},\ f\mathbb{B}) \end{array}$$

with $\theta(\psi)(a) = \psi(-, a)$ and $\theta^{-1}(g)(b, a) = g(a)(b)$. □

Corollary 5.8 $\mathsf{C} := f \circ \phi$ *is a monad on* $\mathsf{Cat}_*(\mathcal{Q})$. *The category of* C*-algebra's corresponds to the full subcategory of $*$-complete $\mathcal{Q}$-$*$-categories and functors between them, denoted by* $\mathsf{CCat}_*(\mathcal{Q})$. *This gives the equivalence of categories:*

$$\underline{\mathsf{Dist}}_*(\mathcal{Q}) \;\simeq\; \mathsf{CCat}_*(\mathcal{Q})$$

The embedding i of $\mathsf{CCat}_*(\mathcal{Q})$ *in* $\mathsf{Cat}_*(\mathcal{Q})$ *has* C *as left adjoint:*

$$\mathsf{Cat}_*(\mathcal{Q}) \xrightarrow{\mathsf{C}} \mathsf{CCat}_*(\mathcal{Q}) \quad \dashv \quad \mathsf{CCat}_*(\mathcal{Q}) \xrightarrow{i} \mathsf{Cat}_*(\mathcal{Q})$$

Proof Because the counit ϵ is an isomorphism the functor f is fully faithful, so $\underline{\mathsf{Dist}}_*(\mathcal{Q})$ is equivalent to the full subcategory of C-algebra's. □

Example 5.9

1. **Complete Ω-sets:** Remember that one can identify $\mathsf{Set}(\Omega)$ with $\underline{\mathsf{Dist}}_*(\mathsf{Pr}(\Omega))$ (Example 4.4). The complete Ω-sets, in our terminology the $*$-complete $\mathsf{Pr}(\Omega)$-$*$-categories, and $*$-maps between them form by definition the category $\mathsf{CSet}(\Omega)$ (see, e.g., [7] II, pp.502-513). Because every $*$-map between complete Ω-sets is uniquely representable by a functor, $\mathsf{CSet}(\Omega)$ is isomorphic to $\mathsf{CCat}_*(\mathsf{Pr}(\Omega))$, so by the corollary above we have the misleading equivalence:

   $$\mathsf{Set}(\Omega) \simeq \mathsf{CSet}(\Omega).$$

   However, this is due to the fact that the morphisms of $\mathsf{Set}(\Omega)$ are defined to be $*$-maps (*distributors*), which are unable to distinguish between Ω-sets and their completions. This equivalence disappears when one takes *functors* between Ω-sets as morphisms, such that we have the proper embedding $\mathsf{CCat}_*(\mathsf{Pr}(\Omega)) \hookrightarrow \mathsf{Cat}_*(\mathsf{Pr}(\Omega))$.
2. **Sheaves on sites:** Using our terminology, in [16] it is shown that, for a site $(\mathcal{C}, \mathfrak{F})$, the weakly $*$-complete $\mathsf{R}_j(\mathcal{C})$-$*$-categories and functors between them are biequivalent to $\mathsf{Sh}(\mathcal{C}, \mathfrak{F})$ by the correspondence $F \mapsto \mathbb{A}_F$ (Example 3.2), the bicategorical aspect stemming from the equivalence relation on functors induced by the distributors $f \mapsto \phi_f$ they represent. However, without losing any information, restricting to the (stronger) $*$-complete $\mathsf{R}_j(\mathcal{C})$-$*$-categories gives an *isomorphism* of categories:

   $$\mathsf{Sh}(\mathcal{C}, \mathfrak{F}) \cong \mathsf{CCat}_*(\mathsf{R}_j(\mathcal{C})).$$

   In particular, for the trivial Grothendieck topology, which corresponds to the nucleus $j = Id_{\mathcal{Q}}$, we get the remarkable result:

   $$[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}] \cong \mathsf{CCat}_*(\mathsf{R}(\mathcal{C})).$$

   If we take a locale Ω as site, then the isomorphism results in:

   $$\mathsf{Sh}(\Omega) \cong \mathsf{CCat}_*(\mathsf{Pr}(\Omega)) \cong \mathsf{CSet}(\Omega).$$

   This correspondence is elaborated systematically in the next two sections.

6 Pseudo-presheaves

In this section, the *restriction* of elements of $\mathcal{Q}$-$*$-categories is analysed, in the form of the *specification* of relations by functions. We will define relational presheaves in a slightly different but equivalent way compared to [13]. Firstly, to make calculations easier, we characterize them in a covariant way, which is possible because the codomain Rel is self-dual. Secondly, the reversal of the local order is made explicit in the formalism by the use of $\mathcal{Q}^{\mathsf{co}}$.

Definition 6.1 A **relational presheaf** on a quantaloid $\mathcal{Q}$ is a lax 2-functor:

$$F : \mathcal{Q}^{\mathsf{co}} \to \mathsf{Rel}$$

meaning:

$$Fa \circ Fb \subseteq F(a \circ b) \qquad \Delta_{FC} \subseteq F(\mathsf{id}_C) \qquad a \leq b \;\Rightarrow\; F(a) \supseteq F(b)$$

F is **continuous** if

$$F\left(\bigvee_i a_i\right) = \bigcap_i Fa_i.$$

F is **separated** if $F(\mathsf{id}_C) = \Delta_{FC}$ and in case $\mathcal{Q}$ has an involution, F is **involutive** if $F(a^*) = F(a)^*$. We call a continuous involutive relational presheaf a $\mathcal{Q}$-$*$-**presheaf**. A **specification** of a relational presheaf F is a presheaf (in the usual sense) on the category $\underline{\mathcal{Q}}$:

$$\underline{F} : \underline{\mathcal{Q}}^{\mathsf{op}} \to \mathsf{Set}$$

satisfying: $\underline{F} = F$ on objects and $(x, \underline{Fa}(x)) \in Fa$.

We use the conventions $x \langle a \rangle y \;\equiv\; (x, y) \in F(a)$ and $x \circ a \;\equiv\; \underline{Fa}(x)$, such that the condition of a specification becomes $x \langle a \rangle x \circ a$.

Definition 6.2 A **lax morphism** between relational presheaves F and G is a lax natural transformation, consisting of a $\mathcal{Q}_0$-indexed family of set functions:

$$\left(FC \xrightarrow{\tau_C} GC\right)_{C \in \mathcal{Q}_0}$$

such that for all $a \in \mathcal{Q}(D, C)$ we have $\langle \tau_C \rangle \circ Fa \;\subseteq\; Ga \circ \langle \tau_D \rangle$, with $\langle \tau_C \rangle$ and $\langle \tau_D \rangle$ the relations induced by τ_C and τ_D. The category of $\mathcal{Q}$-$*$-presheaves and lax morphisms is denoted by $[\mathcal{Q}^{\mathsf{co}}, \mathsf{Rel}]_*$.

Theorem 6.3

1. *We have an isomorphism:* $[\mathcal{Q}^{\mathsf{co}}, \mathsf{Rel}]_* \;\cong\; \mathsf{Cat}_*(\mathcal{Q})$.
2. *For corresponding objects F and $\mathbb{A}$ under this isomorphism we have:*

$$y \langle q \rangle x \;\Leftrightarrow\; q \leq \mathbb{A}(y, x) \;\Leftrightarrow\; \widetilde{y} \otimes q \leq \widetilde{x}.$$

If q is a $*$-map then the last inequality becomes $\widetilde{y} \otimes q = \widetilde{x}$. Moreover, F is separated iff $\mathbb{A}$ is skeletal.

Proof

1. We first describe the construction of a $\mathcal{Q}$-$*$-presheaf from a $\mathcal{Q}$-$*$-category and vice versa.
   - For $\mathbb{A}$ a $\mathcal{Q}$-$*$-category we take $F_{\mathbb{A}}(C) := \mathbb{A}_C$ and $F_{\mathbb{A}}(a) := \langle a \rangle_{\mathbb{A}}$, with $y \langle a \rangle_{\mathbb{A}} x \;\Leftrightarrow\; a \leq \mathbb{A}(y, x)$. It is easy to verify that this gives a $\mathcal{Q}$-$*$-presheaf $F_{\mathbb{A}}$. Evidently, we have:

$$\mathbb{A}(y, x) = \bigvee \{a \in \mathcal{Q}(tx, ty) \mid y \langle a \rangle_{\mathbb{A}} x\} \tag{2}$$

– For F a $\mathcal{Q}$-$*$-presheaf we make a $\mathcal{Q}$-matrix by setting $(\mathbb{A}_F)_C := FC$ and $\mathbb{A}_F(y,x) := \bigvee \{a \in \mathcal{Q}(tx,ty) \mid y \langle a \rangle x\}$. We leave the verification that $\mathbb{A}_F$ is a $\mathcal{Q}$-$*$-category to the reader. Because $y \bigcap \{\langle b \rangle \mid y \langle b \rangle x\} x$ and $\bigcap \{\langle b \rangle \mid y \langle b \rangle x\} = \left\langle \bigvee \{b \mid y \langle b \rangle x\} \right\rangle$, we have $y \langle \mathbb{A}_F(y,x) \rangle x$. If $y \langle a \rangle x$ then of course $a \leq \mathbb{A}_F(y,x)$, conversely: if $a \leq \mathbb{A}_F(y,x)$ then $\langle a \rangle \supseteq \langle \mathbb{A}_F(y,x) \rangle$ so $y \langle \mathbb{A}_F(y,x) \rangle x$ yields $y \langle a \rangle x$, resulting in:

$$y \langle a \rangle x \qquad \Leftrightarrow \qquad a \leq \mathbb{A}_F(y,x) \tag{3}$$

The correspondence between functors and lax morphisms is not hard to construct: one takes the same underlying (type preserving) functions, and verifies the conditions in question. It is easy to see, based on equations (2) and (3), that these constructions result in an isomorphism of categories.

2. This follows from the first part and:

$$\begin{aligned} q \leq \mathbb{A}(y,x) &\Leftrightarrow q \leq \widetilde{y}^* \otimes \widetilde{x} \\ &\Leftrightarrow \widetilde{y} \otimes q \leq \widetilde{x} \\ &\Leftrightarrow \widetilde{y} \otimes q = \widetilde{x} \text{ if } q \text{ is a } *\text{-map} \end{aligned}$$

The last equivalence holds because the order on $*$-maps is discrete (if q is a $*$-map, then $\widetilde{y} \otimes q$ is a $*$-map, being the composition of $*$-maps). The correspondence between separated $\mathcal{Q}$-$*$-presheaves and skeletal $\mathcal{Q}$-$*$-categories is straightforward.

□

Definition 6.4 A **pseudo-presheaf** $(F, \underline{F})$ on an involutive quantaloid $\mathcal{Q}$ is a $\mathcal{Q}$-$*$-presheaf F on $\mathcal{Q}$ together with a specification $\underline{F}$:

$$F : \mathcal{Q}^{\mathsf{co}} \to \mathsf{Rel} \qquad \underline{F} : \underline{\mathcal{Q}}^{\mathsf{op}} \to \mathsf{Set}$$

We will also speak of a pseudo-presheaf $\mathbb{A}$ (with $\mathbb{A}$ the $\mathcal{Q}$-$*$-category that corresponds to F), without mentioning the action of $\underline{\mathcal{Q}}$ given by $\underline{F}$.

Proposition 6.5

1. *If $F \leftrightarrow \mathbb{A}$ is a pseudo-presheaf then:*

$$\widetilde{x} \otimes a = \widetilde{x \circ a} \qquad \text{and} \qquad \mathbb{A}(y \circ b, x \circ a) = b^* \circ \mathbb{A}(y,x) \circ a$$

with $x, y \in \mathbb{A}_0$, $a, b \in \underline{\mathcal{Q}}_1$, $\mathsf{cod}\, a = tx$ and $\mathsf{cod}\, b = ty$. Moreover, if F is separated, then $\underline{F}$ is uniquely determined by F.

2. *Every $*$-complete $\mathcal{Q}$-$*$-category $\mathbb{A}$ has a unique pseudo-presheaf structure.*

Proof

1. This is an easy consequence of the second part of Theorem 6.3.
2. Let $\mathbb{A}$ be a $*$-complete $\mathcal{Q}$-$*$-category. If $x \in \mathbb{A}_C$ and $a \in \underline{\mathcal{Q}}(D,C)$ then we can define $x \circ a$ as the unique element y of $\mathbb{A}_D$ such that $\widetilde{y} = \widetilde{x} \otimes a$. This operation has the necessary properties. □

Hence, a skeletal $\mathcal{Q}$-$*$-category has at most one pseudo-presheaf structure.

Example 6.6

1. **Tensors:** Let $\mathbb{A}$ be a $\mathcal{Q}$-$*$-category, with $x \in \mathbb{A}_C$ and $a \in \underline{\mathcal{Q}}(D, C)$. In enriched category theory, a representant of the singleton $\widetilde{x} \otimes a$ is called the *colimit* of the diagram (a, x), or more specifically the *tensor* of x and a, written as $x \otimes a$. More on the topic of colimits for $\mathcal{Q}$-categories can be found in [15].
2. **Presheaves on a category:** for a presheaf $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$, with $\mathcal{C}$ a small category, the $\mathsf{R}(\mathcal{C})$-$*$-category $\mathbb{A}_F$ from Example 3.2 is evidently skeletal and has a canonical specification which is isomorphic to F itself (remember that $\mathcal{C} \cong \underline{\mathsf{R}}(\mathcal{C})$).
3. **Presheaves on a locale:** Suppose that Ω is a locale, and set $\mathcal{Q} := \mathsf{Pr}(\Omega) \cong \mathsf{R}_j(\Omega)$ (Example 2.14.2). Then the Ω-sets are the $\mathcal{Q}$-$*$-categories, equivalently described as $\mathcal{Q}$-$*$-presheaves. Observe that $\Omega \cong \underline{\mathcal{Q}}$, so the pseudo-presheaves on $\mathcal{Q}$ are Ω-sets $\mathbb{A}$ with a presheaf $F : \Omega^{\mathsf{op}} \to \mathsf{Set}$, defining *restrictions* of the elements of $\mathbb{A}$ (the condition of specification justifies this term). It is well known that there are plenty of Ω-sets for which no specification exists, however, the completions of these Ω-sets have canonical specifications. On the other hand, every presheaf $F : \Omega^{\mathsf{op}} \to \mathsf{Set}$ is a specification of the Ω-set $\mathbb{B}_F$ it induces.
4. **Gylys' restrictable pairs:** The idea of a specification is inspired by the notion of a *restriction* for a strict $\mathcal{Q}$-set (Example 4.4) in [6], which in turn is a rather technical adaptation of [4] to the context of involutive quantaloids. Gylys uses the concept of *restrictable pairs* for a strict $\mathcal{Q}$-set $\mathbb{S}$ to define a partial action, called restriction, $(-_{|-}) : \mathbb{S}_0 \times \mathcal{Q}_1 \to \mathbb{S}_0$ on the elements of $\mathbb{S}$: a restrictable pair is a pair $(x, p) \in \mathbb{S}_0 \times \mathcal{Q}_1$ with $tx = \mathsf{cod}\, p$, such that $\mathsf{E}x \circ p \circ p^* \circ \mathsf{E}x \leq \mathsf{E}x$ and $\mathsf{E}x \circ p \leq \mathsf{E}x \circ p \circ p^* \circ \mathsf{E}x \circ p$, with $\mathsf{E}x := \mathbb{S}(x, x)$. We prefer to rewrite this in the form: $q \circ q^* \leq \mathsf{E}x$ and $q \leq q \circ q^* \circ q$, with $q := \mathsf{E}x \circ p$, which just states that q can be considered as a map in $\mathsf{Pr}(\mathcal{Q})$ with codomain $\mathsf{E}x$. Because of the axioms $x_{|r \circ s} = (x_{|r})_{|s}$ and $x_{|\mathsf{E}x} = x$, we have $x_{|p} = x_{|q}$, such that no information is lost by only using pairs of the form (x, q). This observation together with Theorem 6.3 should be sufficient for the interested reader to compare the pseudo-presheaves on $\mathsf{Pr}(\mathcal{Q})$ with Gylys' presheaves on $\mathcal{Q}$ (which are strict $\mathcal{Q}$-sets with restrictions).

7 Presheaves and Sheaves

The aim of the last section is a description of sheaves on a site, *and their relation to presheaves*, in the language of enriched category theory, i.e., in terms of $\mathcal{Q}$-$*$-categories. In order to do this, we consider the base quantaloid $\mathcal{Q}$ to be an abstraction of $\mathsf{R}(\mathcal{C})$ or $\mathsf{R}(\Omega)$, while the idea of coverages is represented by left exact $*$-nuclei on $\mathcal{Q}$ (Example 2.12). In this way, the different Grothendieck topologies and their sheaves can be compared to each other, especially to the trivial Grothendieck topology, for which sheaves and presheaves coincide (see Example 5.9.2), so basically, we like to introduce contextuality. On the other hand, Gylys, in terms of enriched categories (see Example 4.4), considers the base quantaloid to be an abstraction of $\mathsf{Pr}(\Omega)$ by replacing Ω by $\mathcal{Q}$, which in our opinion is based on some advantages of locales that

do not apply to their generalizations: sites. We first need some results concerning nuclei:

Proposition 7.1 *Let $\mathcal{Q}$ be $*$-quantaloid and j a $*$-nucleus on $\mathcal{Q}$.*

1. *Every $\mathcal{Q}$-$*$-category $\mathbb{A}$ gives rise to a $\mathcal{Q}_j$-$*$-category $\mathbb{A}_j$ with:*

$$t' : (\mathbb{A}_j)_0 := \mathbb{A}_0 \to (\mathcal{Q}_j)_0 : \quad x \mapsto tx$$
$$\mathbb{A}_j(x, y) := j\mathbb{A}(x, y)$$

Conversely, every $\mathcal{Q}_j$-$*$-category can be considered as a $\mathcal{Q}$-$*$-category.

2. *We have the following two adjunctions:*

(a) $\mathsf{Cat}_*(\mathcal{Q}) \xrightarrow{j} \mathsf{Cat}_*(\mathcal{Q}_j) \quad \dashv \quad \mathsf{Cat}_*(\mathcal{Q}_j) \xhookrightarrow{\mathsf{U}} \mathsf{Cat}_*(\mathcal{Q})$

$\mathbb{A} \xrightarrow{f} \mathbb{B} \mapsto \mathbb{A}_j \xrightarrow{f_j} \mathbb{B}_j$ *with* $f_j(x) = x$. U *is the forgetful functor that sends a $\mathcal{Q}_j$-$*$-category to itself regarded as a $\mathcal{Q}$-$*$-category.*

(b) $\mathsf{CCat}_*(\mathcal{Q}) \xrightarrow{\mathsf{a}} \mathsf{CCat}_*(\mathcal{Q}_j) \quad \dashv \quad \mathsf{CCat}_*(\mathcal{Q}_j) \xhookrightarrow{\mathsf{b}} \mathsf{CCat}_*(\mathcal{Q})$;

$\mathsf{a} = \mathsf{C}' \circ j \circ i$, *with i the embedding of* $\mathsf{CCat}_*(\mathcal{Q})$ *in* $\mathsf{Cat}_*(\mathcal{Q})$ *and* C' *the completion functor.* b *is the forgetful functor.*

U *and* b *are fully faithful, so one can respectively consider* $\mathsf{Cat}_*(\mathcal{Q}_j)$ *and* $\mathsf{CCat}_*(\mathcal{Q}_j)$ *as reflexive subcategories of* $\mathsf{Cat}_*(\mathcal{Q})$ *and* $\mathsf{CCat}_*(\mathcal{Q})$.

Proof

1. This is an easy verification. We just remark:

$$\mathbb{A}(x, z) \circ \mathbb{A}(z, y) \leq \mathbb{A}(x, y) \Rightarrow j[\mathbb{A}(x, z) \circ \mathbb{A}(z, y)] \leq j\mathbb{A}(x, y)$$
$$\Leftrightarrow \mathbb{A}_j(x, z) \circ_j \mathbb{A}_j(z, y) \leq \mathbb{A}_j(x, y)$$

2. (a) Let $f : \mathbb{A} \to \mathbb{B}$ be a functor between $\mathcal{Q}$-$*$-categories. Then f_j is a functor:

$$\mathbb{A}(x, y) \leq \mathbb{B}(fx, fy) \quad \Rightarrow \quad \mathbb{A}_j(x, y) \leq \mathbb{B}_j(f_j x, f_j y)$$

We have the natural bijection

$$\mathsf{Func}(\mathbb{A}, \mathsf{U}\mathbb{B}) \cong \mathsf{Func}(j\mathbb{A}, \mathbb{B})$$

because: $\mathbb{A}(x, y) \leq \mathbb{B}(x', y') \Leftrightarrow j\mathbb{A}(x, y) \leq \mathbb{B}(x', y')$.
It is evident that U is fully faithful: $\mathsf{Func}(\mathbb{C}, \mathbb{D}) \cong \mathsf{Func}(\mathsf{U}\mathbb{C}, \mathsf{U}\mathbb{D})$.

(b) Because $j \dashv \mathsf{U}$ and $\mathsf{C}' \dashv i'$ (i' the evident embedding), we have: $\mathsf{C}' \circ j \dashv \mathsf{U} \circ i'$. We only have to remark that a $*$-complete $\mathcal{Q}_j$-$*$-category $\mathbb{B}$ is also a $*$-complete $\mathcal{Q}$-$*$-category. Let ϕ be a singleton of $\mathbb{B}$ (as a $\mathcal{Q}$-$*$-category). ϕ is a $*$-map and $\mathbb{B}$ is closed, so $\mathbb{B} \otimes \phi = \phi$ is also closed (see Lemma 2.15). Hence ϕ can be considered as a singleton of $\mathbb{B}$ as a $*$-complete $\mathcal{Q}_j$-$*$-category, thus there exists a unique $b \in \mathbb{B}_0$ with $\widetilde{b} = \phi$, resulting in the $*$-completeness of $\mathbb{B}$. □

A $*$-complete $\mathcal{Q}_j$-$*$-category $\mathbb{A}$, also being a $*$-complete $\mathcal{Q}$-$*$-category, can be seen as a pseudo-presheaf for $\mathcal{Q}_j$ and as a pseudo-presheaf for $\mathcal{Q}$ (see Proposition 6.5.2). The action of $\underline{\mathcal{Q}}_1$ on $\mathbb{A}$ coincides with the action of $\underline{\mathcal{Q}}_1$ 'modulo j' on $\mathbb{A}$. If $x \in \mathbb{A}_0$ and $\phi = \widetilde{x}$ and if a is a $*$-map of $\mathcal{Q}$ such that $\mathsf{cod}\, a = tx$ then the action of a on x results in the element y of $\mathbb{A}$ corresponding to $\phi \otimes a$. The action of ja, a $*$-map in $\mathcal{Q}_j$, on x is obtained by the element of $\mathbb{A}$ corresponding to $\phi \otimes_j ja = j(\phi \otimes a) = \phi \otimes a$, which is the same $y \in \mathbb{A}_0$.

Definition 7.2 A (generalized) **site** is a pair $(\mathcal{Q}, j)$, consisting of a $*$-quantaloid $\mathcal{Q}$ and a left-exact $*$-nucleus j on $\mathcal{Q}$. The category of **presheaves** on $\mathcal{Q}$ is defined as follows:

$$\mathsf{Ps}(\mathcal{Q}) := \mathsf{CCat}_*(\mathcal{Q})$$

We say a $\mathcal{Q}$-$*$-category $\mathbb{A}$ is **j-closed** if $j\mathbb{A} = \mathbb{A}$. A **j-singleton** of a $\mathcal{Q}$-$*$-category $\mathbb{A}$ is a singleton of $\mathbb{A}_j$ as a $\mathcal{Q}_j$-$*$-category. A $\mathcal{Q}$-$*$-category $\mathbb{A}$ is a **sheaf** if it is j-closed and $*$-complete as a $\mathcal{Q}_j$-$*$-category. The category of sheaves on $(\mathcal{Q}, j)$ is:

$$\mathsf{Sh}(\mathcal{Q}, j) := \mathsf{CCat}_*(\mathcal{Q}_j)$$

In the light of these definitions, the functor $\mathsf{a} : \mathsf{Ps}(\mathcal{Q}) \to \mathsf{Sh}(\mathcal{Q}, j)$ from above can be considered as **sheafification**.

Remark 7.3

1. The remarks from Example 5.9.2 justify our definition of (generalized) sites and sheaves. In this case we have the isomorphic embeddings:

$$\begin{array}{ccc} \mathsf{Sh}(\mathcal{C}, \mathfrak{F}) & \hookrightarrow & [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}] \\ \| & & \| \\ \mathsf{Sh}(\mathsf{R}(\mathcal{C}), j) & \hookrightarrow & \mathsf{Ps}(\mathsf{R}(\mathcal{C})) \end{array}$$

   This correspondence can be easily proved after we manage to characterize the sheaves on a (generalized) site $(\mathcal{Q}, j)$ by means of 'covers' and 'compatible elements'.
2. A presheaf is in particular a pseudo-presheaf (Proposition 6.5). We remark that a sheaf is a relative concept: the objects of $\mathsf{Sh}(\mathcal{Q}, j) = \mathsf{CCat}_*(\mathcal{Q}_j)$ get their meaning as sheaves within the context of the site $(\mathcal{Q}, j)$ and the embedding $\mathsf{Sh}(\mathcal{Q}, j) \hookrightarrow \mathsf{Ps}(\mathcal{Q})$. It might be tempting to forget about the site $(\mathcal{Q}, j)$ and to define a presheaf as a pseudo-presheaf for $\mathcal{Q}$, being a sheaf when it is a $*$-complete $\mathcal{Q}$-$*$-category, but then a presheaf would not be a generalization of the usual concept of a presheaf on a category.

Definition 7.4 Let be a $(\mathcal{Q}, j)$ site and $\mathbb{A}$ a $\mathcal{Q}$-$*$-category.

1. A **j-cover of an object** $C \in \mathcal{Q}_0$ is a family of $*$-maps $(C_i \xrightarrow{a_i} C)_{i \in I}$ in $\mathcal{Q}$ such that:

$$\mathsf{id}_C \sim_j \bigvee_{i \in I} a_i \circ a_i^* \tag{4}$$

2. $\mathbb{A}_\bullet := \mathbb{A}_0 \times_{\mathcal{Q}_0} \underline{\mathcal{Q}}_1$ is the collection of **elements in context**:

$$\mathbb{A}_0 \times_{\mathcal{Q}_0} \underline{\mathcal{Q}}_1 := \left\{(c_0, c_1) \in \mathbb{A}_0 \times \underline{\mathcal{Q}}_1 \mid tc_0 = \mathsf{dom}\, c_1\right\}.$$

A pair $c = (c_0, c_1) \in \mathbb{A}_\bullet$ is an **element in context** $C \in \mathcal{Q}_0$ if $\mathsf{cod}\, c_1 = C$.
$\mathbb{A}|_C$ is the set of all elements in context C.

3. A j-singleton ϕ of $\mathbb{A}$ is **locally representable** if there exists a family $(x_i\,, a_i)_{i\in I}$ of elements in context $t\phi$ such that $(a_i)_{i\in I}$ is a j-cover of $t\phi$ and if

$$\phi \otimes a_i \;\sim_j\; \widetilde{x}_i\,.$$

4. (a) Two elements $s, t \in \mathbb{A}|_C$ ($C \in \mathcal{Q}_0$) are **j-compatible** if:

$$s_1^* \circ t_1 \;\leq\; j\mathbb{A}(s_0, t_0)$$

(b) A **j-compatible family of elements over** C is a family $(x_i\,, a_i)_{i\in I}$ of pairwise j-compatible elements in context C such that $(a_i)_{i\in I}$ is a j-cover of C, i.e.:

$$\forall\, i, k \in I: \quad a_i^* \circ a_k \;\leq\; j\mathbb{A}(x_i\,, x_k) \quad \text{and} \quad \mathsf{id}_C \;\sim_j\; \bigvee_{i\in I} a_i \circ a_i^*$$

Note that Eq. 4 is equivalent to: $\mathsf{id}_C \;\leq\; j\bigvee_{i\in I} a_i \circ a_i^*$.

Example 7.5 Let $(\mathcal{C}, \mathfrak{F})$ be a site, j the nucleus on $\mathsf{R}(\mathcal{C})$ that corresponds to the Grothendieck topology $\mathfrak{F}$ (Example 2.12), $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ a presheaf and consider the embedding from Example 4.4.1. The following observations support the definitions above:

1. $\mathbb{A}_F$ is j-closed iff F is separated (w.r.t. $\mathfrak{F}$).
2. A family $(D_i \xrightarrow{s_i} C)_{i\in I}$ of morphisms in $\mathcal{C}$ is a cover (w.r.t. $\mathfrak{F}$) iff $(\underline{s_i})_{i\in I}$ is a j-cover (remember the isomorphism $f \mapsto \underline{f}$ from Example 2.8.2).
3. $(x, \underline{s})$ and $(y, \underline{t})$ are (Id-)compatible elements of $\mathbb{A}_F$ iff for all $f, g \in \mathcal{C}_1$ with $s \circ f = t \circ g$ we have $Ff(x) = Fg(y)$, which means that x and y are compatible elements of F in the classical sense. $(x, \underline{s})$ and $(y, \underline{t})$ are j-compatible iff $\overline{x}$ and $\overline{y}$ are compatible elements of the presheaf $\overline{F}$ obtained by separating F (w.r.t. $\mathfrak{F}$).
4. Every j-singleton ϕ of $\mathbb{A}_F$ is locally representable (from $\Delta_{t\phi} \leq j(\phi^* \circ \phi)$ it follows that there exists an $S \in \mathfrak{F}(t\phi)$ such that $R_S \subseteq \phi^* \circ \phi$, resulting in the existence of relations $(f_s, s) \in \phi(x_s)$, giving rise to a family $(Ff_s(x_s)\,, \underline{s}\,)_{s\in S}$ that locally represents ϕ).

Lemma 7.6 *Let $(\mathcal{Q}, j)$ be a site, $\mathbb{A}$ a $\mathcal{Q}$-$*$-category and $\alpha = (x_i\,, a_i)_{i\in I}$ a j-compatible family over an object $C \in \mathcal{Q}$. In this case α induces a j-singleton of $\mathbb{A}$:*

$$\phi_\alpha :\; *_C \multimap \mathbb{A}_j \qquad \textit{with} \qquad \phi_\alpha(x) := j\bigvee_{i\in I} \mathbb{A}(x, x_i) \circ a_i^*$$

Moreover:

$$\phi_\alpha \otimes a_i \;\sim_j\; \widetilde{x}_i$$

Proof

$$
\begin{aligned}
j(\mathsf{id}_C) &\leq \phi_\alpha^* \circ_j \phi_\alpha \ : \\
\phi_\alpha^* \circ_j \phi_\alpha &= \textstyle\bigvee^j_{x \in \mathbb{A}_0} \phi_\alpha^*(x) \circ_j \phi_\alpha(x) \\
&= j \textstyle\bigvee_{x \in \mathbb{A}_0} \phi_\alpha^*(x) \circ \phi_\alpha(x) \\
&= j \textstyle\bigvee_{x \in \mathbb{A}_0} \left[j \bigvee_{k \in I} \mathbb{A}(x, x_k) \circ a_k^* \right]^* \circ \left[j \bigvee_{i \in I} \mathbb{A}(x, x_i) \circ a_i^* \right] \\
&= j \textstyle\bigvee_{x \in \mathbb{A}_0} \left[\bigvee_{k \in I} a_k \circ \mathbb{A}(x_k, x) \right] \circ \left[\bigvee_{i \in I} \mathbb{A}(x, x_i) \circ a_i^* \right] \\
&= j \textstyle\bigvee_{i,k \in I} a_k \circ [\bigvee_{x \in \mathbb{A}_0} \mathbb{A}(x_k, x) \circ \mathbb{A}(x, x_i)] \circ a_i^* \\
&= j \textstyle\bigvee_{i,k \in I} a_k \circ \mathbb{A}(x_k, x_i) \circ a_i^* \\
&\geq j \textstyle\bigvee_{i,k \in I} a_k \circ a_k^* \circ a_i \circ a_i^* \\
&\geq j \left[\left(\bigvee_{k \in I} a_k \circ a_k^* \right) \circ \left(\bigvee_{i \in I} a_i \circ a_i^* \right) \right] \\
&= j(\mathsf{id}_C)
\end{aligned}
$$

$$
\begin{aligned}
\phi_\alpha \circ_j \phi_\alpha^* &\leq \mathbb{A}_j \ : \\
\phi_\alpha(y) \circ_j \phi_\alpha^*(x) &= j \left[\textstyle\bigvee_{k \in I} \mathbb{A}(y, x_k) \circ a_k^* \circ \bigvee_{i \in I} a_i \circ \mathbb{A}(x_i, x) \right] \\
&= j \left[\textstyle\bigvee_{i,k \in I} \mathbb{A}(y, x_k) \circ a_k^* \circ a_i \circ \mathbb{A}(x_i, x) \right] \\
&\leq j \left[\textstyle\bigvee_{i,k \in I} \mathbb{A}(y, x_k) \circ \mathbb{A}(x_k, x_i) \circ \mathbb{A}(x_i, x) \right] \\
&\leq j \mathbb{A}(y, x)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{A}_j \circ_j \phi_\alpha &= \phi_\alpha \ : \\
(\mathbb{A}_j \circ_j \phi_\alpha)(x) &= j \textstyle\bigvee_{y \in \mathbb{A}_0} \mathbb{A}(x, y) \circ \phi_\alpha(y) \\
&= j \textstyle\bigvee_{y \in \mathbb{A}_0} \mathbb{A}(x, y) \circ \left[\bigvee_{i \in I} \mathbb{A}(y, x_i) \circ a_i^* \right] \\
&= j \textstyle\bigvee_{i \in I} \left[\bigvee_{y \in \mathbb{A}_0} \mathbb{A}(x, y) \circ \mathbb{A}(y, x_i) \right] \circ a_i^* \\
&= j \textstyle\bigvee_{i \in I} \mathbb{A}(x, x_i) \circ a_i^* \\
&= \phi_\alpha(x)
\end{aligned}
$$

$$
\begin{aligned}
\phi_\alpha \circ a_i &\sim_j \widetilde{x}_i \ : \\
j[\phi_\alpha(x) \circ a_i] &= j \left[\textstyle\bigvee_{k \in I} \mathbb{A}(x, x_k) \circ a_k^* \circ a_i \right] \\
&\leq j \left[\textstyle\bigvee_{k \in I} \mathbb{A}(x, x_k) \circ \mathbb{A}(x_k, x_i) \right] \qquad \geq j \left[\mathbb{A}(x, x_i) \circ a_i^* \circ a_i \right] \\
&\leq j \mathbb{A}(x, x_i) \qquad\qquad\qquad\qquad \geq j \left[\mathbb{A}(x, x_i) \circ \mathsf{id}_{\mathsf{dom}\, a_i} \right] \\
&= j \widetilde{x}_i(x) \qquad\qquad\qquad\qquad = j \widetilde{x}_i(x)
\end{aligned}
$$

□

Lemma 7.7 *Let $(\mathcal{Q}, j)$ be a site and $\mathbb{A}$ a $\mathcal{Q}$-$*$-category. If ϕ is a locally representable j-singleton of $\mathbb{A}$, by the family $\alpha = (x_i, a_i)_{i \in I}$, then α is a j-compatible family over $t\phi$ and $\phi = \phi_\alpha$.*

Proof

1. $\phi = \phi_\alpha$:

$$\begin{aligned}\phi(x) = \ \phi(x) \circ_j j\mathsf{id}_{t\phi} \ &= \ j\bigvee_{i\in I} \phi(x) \circ a_i \circ a_i^* \\ &= \ j\bigvee_{i\in I} \mathbb{A}(x, x_i) \circ a_i^* \ = \ \phi_\alpha(x)\end{aligned}$$

2. $a_k^* \circ a_i \leq j\mathbb{A}(x_k, x_i)$:

Because $\phi(x_k) \circ \phi(x_i)^* \leq j\mathbb{A}(x_k, x_i)$ it suffices to show that $a_i^* \leq \phi(x_i)$:

$$\begin{aligned}\phi(x_i) = \ j\bigvee_{k\in I} \mathbb{A}(x_i, x_k) \circ a_k^* \ &\geq \ j\left[\mathbb{A}(x_i, x_i) \circ a_i^*\right] \\ \geq \ j\left[\mathsf{id}_{\mathsf{dom}\, a_i} \circ a_i^*\right] \ &\geq \ a_i^*\end{aligned}$$

Proposition 7.8 *For a site* $(\mathcal{Q}, j)$ *we have the following:*

1. *A pseudo-presheaf* $\mathbb{A}$ *is a sheaf iff*
   (a) $\mathbb{A}$ *is* j*-closed,*
   (b) *every* j*-singleton of* $\mathbb{A}$ *is locally representable,*
   (c) *for every* j*-compatible family* $(x_i\,, a_i)_{i\in I}$ *over an object* C *there exists a unique element* $x \in \mathbb{A}_C$ *such that* $x \circ a_i = x_i$ *.*

Proof

($\Rightarrow$) If $\mathbb{A}$ is a sheaf then it is of course j-closed and locally representable: if ϕ is a j-singleton of $\mathbb{A}$ then there exists an $x \in \mathbb{A}_0$ with $\widetilde{x} = \phi$ thus ϕ is representable by $(x, \mathsf{E}x)$. Let $(x_i\,, a_i)_{i\in I}$ be a j-compatible family over an object C. By Lemma (7.6) we can construct a j-singleton ϕ_α over C in the following way:

$$\phi_\alpha(x) := j\bigvee_{i\in I} \mathbb{A}(x, x_i) \circ a_i^*$$

Because $\mathbb{A}$ is a sheaf there exists a unique $x \in \mathbb{A}_0$ such that $\widetilde{x} = \phi_\alpha$. From $\phi_\alpha \circ a_i \ \sim_j \ \widetilde{x}_i$ (Lemma 7.6) and the j-closedness of $\mathbb{A}$ it follows that:

$$\widetilde{x} \otimes a_i \ = \ \widetilde{x}_i \qquad \text{or:} \qquad x \circ a_i \ = \ x_i$$

Of course x is unique with this property: a y with the same property would also correspond to ϕ_α (by Lemma 7.7).

($\Leftarrow$) Let $\mathbb{A}$ be a pseudo-presheaf with properties (a), (b) and (c). If ϕ is a j-singleton of $\mathbb{A}$ over an object C, then it is locally representable by (b), so there exists a family $\alpha = (x_i\,, a_i)_{i\in I}$ of elements in context C such that $(a_i)_{i\in I}$ is a j-cover of C and $\phi \otimes a_i \sim_j \widetilde{x}_i$. Because $\mathbb{A}$ is j-closed we can conclude that $\phi \otimes a_i = \widetilde{x}_i$. From Lemma (7.7) it follows that α is a j-compatible family over C so by property (c) there exists an $x \in \mathbb{A}_C$ such that $x \circ a_i = x_i$. By Proposition (6.5) we conclude that $\widetilde{x} \otimes a_i = \widetilde{x}_i$, so $\widetilde{x} = \phi_\alpha = \phi$ (Lemma 7.7).
Finally we show that $\mathbb{A}$ is skeletal (such that in particular the element x above is unique in inducing ϕ). If we supposed that $\mathbb{A}$ is a presheaf (= a $*$-complete $\mathcal{Q}$-$*$-category) then it automatically follows that $\mathbb{A}$ is skeletal. In the more general case, if $x, y \in \mathbb{A}$ and $\widetilde{x} = \widetilde{y}$ then $\beta := \{(x, \mathsf{E}x); (y, \mathsf{E}y)\}$ is a j-compatible family over $\mathsf{E}x\,(= \mathsf{E}y)$, because $\mathsf{E}x \circ \mathsf{E}y^* = \mathsf{E}x = \mathbb{A}(x, x) = \mathbb{A}(x, y)$. Of course $\phi_\beta = \widetilde{x} = \widetilde{y}$. By (c) there is a $z \in \mathbb{A}$ with $\mathsf{E}z = \mathsf{E}x = \mathsf{E}y$ and $y = z \circ \mathsf{E}y = z = z \circ \mathsf{E}x = x$. □

Remark 7.9 Considering Example 5.9.2, it is straightforward that, for a presheaf F, $\mathbb{A}_F$ (which has F as canonical specification) is a sheaf w.r.t. j iff F is a sheaf w.r.t. $\mathfrak{F}$, after noticing that (a) and (b) are consequences of (c), for $\mathbb{A}_F$. In this way, we obtain the isomorphism between $\mathsf{Sh}(\mathcal{C}, \mathfrak{F})$ and $\mathsf{CCat}_*(\mathsf{R}_j(\mathcal{C})) = \mathsf{Sh}(\mathsf{R}(\mathcal{C}), j)$. Moreover, the sheafification of a presheaf F, identified with $\mathbb{A}_F$, becomes very elegant in terms of singletons: the sheaf $\widetilde{F}$ corresponds to the complete $\mathcal{Q}$-$*$-category $\mathsf{a}\mathbb{A}_F = \widetilde{j\mathbb{A}_F}$.

References

1. Borceux, F., Van den Bossche, G.: Quantales and their sheaves. Order **3**, 61–87 (1986)
2. Betti, R., Carboni, A.: Notion of topology for bicategories. Cahiers Topologie Géom. Différentille Catég. **XXIV-1**, 19–22 (1983)
3. Betti, R., Carboni, A., Street, R., Walters, R.: Variation through enrichment. J. Pure Appl. Algebra **29**, 109–127 (1983)
4. Fourman, M.P., Scott, D.S.: Sheaves and logic, in applications of sheaves. Lecture Notes in Math. **753**, 302–401 (1979)
5. Freyd, P.J., Scedrov, A.: Categories, Allegories. North-Holland Mathematical Library, vol. 39 (1990)
6. Gylys, R.P.: Sheaves on involutive quantaloids. Lithuanian Math. J. **41**(1), 35–53 (2001)
7. Johnstone, P.T.: Sketches of an Elephant: A Topos Theory Compendium (I–II), Oxford Logic Guides, pp. 43–44. Oxford University Press (2002)
8. Kelly, G.M.: Basic Concepts of Enriched Category Theory. London Mathematical Society Lecture Note Series, vol. 64. Cambridge University Press (1982)
9. Lawvere, F.W.: Metric spaces, generalized logic and closed categories. Rend. Sem. Mat. Fis. Milano **43**, 135–166 (1973)
10. Mac Lane, S., Moerdijk, I.: Sheaves in Geometry and Logic: A First Introduction to Topos Theory Universitext. Springer-Verlag New York (1992)
11. Mulvey, C.J., Nawaz, M.: Quantales: Quantal Sets. Non-Classical Logics and their Application to Fuzzy Subsets: A Handbook of the Mathematical Foundations of Fuzzy Set Theory, pp. 159–217. Kluwer (1995)
12. Rosenthal, K.I.: Quantales and Their Applications. Pitman Research Notes in Mathematics Series, vol. 234. Longman, Harlow (1990)
13. Rosenthal, K.I.: The Theory of Quantaloids. Pitman Research Notes in Mathematics Series, vol. 348. Longman, Harlow (1996)
14. Stubbe, I.: Categorical structures enriched in a quantaloid: categories, distributors, functors. Theory Appl. Categ. **14**, 1–45 (2005)
15. Stubbe, I.: Categorical structures enriched in a quantaloid: tensored and cotensored categories. Theory Appl. Categ. **16**, 283–306 (2006)
16. Walters, R.F.C.: Sheaves on sites as cauchy complete categories. J. Pure Appl. Algebra **24**, 95–102 (1982)