

Residuation in Existential Graphs

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Abstract. Residuation has become an important concept in the study of algebraic structures and algebraic logic. Relation algebras, for example, are residuated Boolean algebras and residuation is now recognized as a key feature of substructural logics. Early work on residuation can be traced back to studies in the logic of relations by De Morgan, Peirce and Schröder. We know now that Peirce studied residuation enough to have listed equivalent forms that residuals may take and to have given a method for arriving at the different permutations. Here, we present for the first time a graphical treatment of residuation in Peirce's Beta part of Existential Graphs (EGs). Residuation is captured by pairing the ordinary transformations of rules of EGs—in particular those concerning the cuts—with simple topological deformations of lines of identity. We demonstrate the effectiveness and elegance of the graphical presentation with several examples. While there might have been speculation as to whether Peirce recognized the importance of residuation in his later work, or whether residuation in fact appears in his work on EGs, we can now put the matter to rest. We cite passages where Peirce emphasizes the importance of residuation and give examples of graphs Peirce drew of residuals. We conclude that EGs are an effective means of enlightening this concept.

Keywords: Residuation · Existential graphs · Charles Peirce · Cuts · Lines of identity

1 Introduction

As discussed by Pratt [19] and Maddux [13], De Morgan described the residuation laws in the form of Theorem K in 1860 [3,6]. The resulting equivalences state that given any three relations—$a, b,$ and c—and well known relation operations—relational composition (;), complement ($\bar{}$), converse ($\breve{}$) and relational containment/inclusion ($\sqsubseteq$)—the following are equivalently defined:

$$a; b \sqsubseteq c \iff \breve{a}; \bar{c} \sqsubseteq \bar{b} \iff \bar{c}; \breve{b} \sqsubseteq \bar{a} \,. \tag{1}$$

Residuation can broadly be thought of as an inverse operation, much like how division is inverse to multiplication and subtraction to addition. In the context

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of relations discussed here, residuation gives a remainder when relational composition is denied or converted (as in the equivalences above).

Another example is found in what is called the residuation property (RES), which shows how residuation also acts like implication:

$$\text{(RES)} \qquad p \wedge q \sqsubseteq r \iff q \sqsubseteq p \rightarrow r.$$

This is related to the deduction theorem (and to currying), and the property plays an important role in characterizations of a range of implications from classical to intuitionistic (Heyting algebra) implication. Residuation is now recognized as a key property of substructural logics [10,14] and, following Lambek, of categorical grammar [7,8].

Given the significance that has been placed on residuation in the study of relations since, it is perhaps curious that Charles S. Peirce, who studied De Morgan's work closely and who went on to make significant contributions to the algebra of relations (e.g. his 'dual' and 'general' algebras of relatives), seems to have placed little emphasis on it in his later work. Maddux even describes the omission of residuation (De Morgan's Theorem K) in Peirce's later work as "puzzling" [13, p. 435].

We now know that Peirce did emphasize the residuation property above in his characterization of propositional logic as occurs in his 1880 algebra of the copula paper, §4, which presents a calculus of the consequence relation [11]. In it, two meanings of the copula $\prec\!\!\!-$ (or $\prec$) are delineated by using two signs: (1) the consequence relation (Peirce's sign of illation) $\Rightarrow$; and (2) the material implication $\rightarrow$. An expression of the form $x \Rightarrow y$ is called a *sequent*, according to the proof-theoretic terminology. Then the calculus of the copula (a Boolean algebra) consists of the following axiom and rules:

1. Identity: $\quad \text{(Id)}\ x \Rightarrow x$
2. Peirce's Rule:
$$\frac{x \wedge y \Rightarrow z}{x \Rightarrow y \rightarrow z}\,\text{(PR)}$$
3. Rule of Transitivity:
$$\frac{x \Rightarrow y \quad y \Rightarrow z}{x \Rightarrow z}\,\text{(Tr)}$$

The double line in (PR) means that the lower sequent can be derived from the upper sequent and vice versa. The second rule, here renamed as *Peirce's Rule*, is probably the first formulation of the *law of residuation*: that the material implication is a right residual of conjunction.

We also now know that Peirce studied residuation enough to have listed equivalent forms residuals may take and to have given a method for arriving at the different permutations [16, CP 4.343] [19]. While this helps confirm Peirce's awareness of residuation and to assuage some doubts about the scope of his insights, it does not help explain why Peirce seems to have placed much less emphasis in his later work on a concept whose importance he had—and when looking back on it perhaps should have—so emphasized.

Pursuing Peirce's potential connection to residuation from another direction, it is equally curious that Peirce makes no direct mention of his earlier algebraic studies of residuation in his later presentation of Existential Graphs (EGs). Given that Peirce often cites EGs as the culmination of his earlier work on relations (with his algebraic studies of residuation, no doubt, as one), along with his insistence that EGs should be the "logic of the future" [18], it would be problematic if such a concept was left without representation.

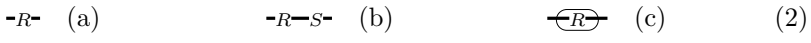
Of course this is not the case. We remedy the seeming omission here by showing how residuation is naturally presented in EGs. Given the relatively sparse syntax of the graphs and that residuation can easily be represented without any changes to the syntax or transformation (i.e., inference) rules, it would seem rather that Peirce's supposed omission might be due to a belief that the other rules of EGs suffice to enlighten the concept. The presentation of residuation in EGs given here is the first of its kind—in particular, the first for its quantificational Beta extension that includes lines of identity.

This paper presents residuation in the context of relations and relational operations and sets aside for the time the functional characterization in terms of Galois connections.[1] The aim is to help situate Peirce's work in the development of residuation (following, in particular, the work of Maddux and Pratt cited above), to present the beginnings of a graphical presentation of residuation, and to address the connection between residuation and Peirce's work on Existential Graphs.

2 Beta Graphs and Relational Operations

We begin with a short introduction to EGs and the diagrammatic presentation of the operations needed to represent residuation in a logic of relations. We assume basic familiarity with the interpretation and transformation rules of EGs. Helpful introductions to Peirce's EGs can be found in [18,20]. The richer algebraic/categorical framework upon which this work relies can be found in [5]. We save an extended treatment of residuation in the latter context for subsequent work. Here, we stick rather to the perspective from relation algebra, leveraging the more traditional notation for relation algebras found for example in [1,4,21]. Though it predates Peirce's EGs, a helpful introduction along these lines is given by Peirce in his "Note B" [15]. What follows can be seen as a graphical treatment *á la* the later EGs of the algebraic work given in this note.

A general binary relation is scribed on the sheet with an ingoing 'wire' serving as a placeholder for the domain of the relation and an outgoing wire signalling the codomain. These wires represent the collection of individuals who might satisfy/stand in the relation presented.

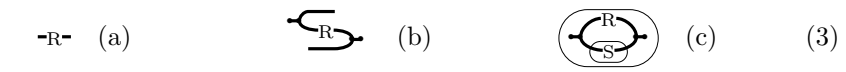


(2)

[1] On the history of residuation (adjunctions) and its relation to Galois connections, see [2].

Operations from relation algebra and their corresponding EGs are given in (2), where (a) is a general relation R, (b) is relational composition $R;S$, and (c) is complement $\bar{R}$. Given the -loves- and -benefits- relations we can for example express 'lovers of benefactors' as -loves—benefits- and 'lovers of non-benefactors' as -loves-(benefits)-. .

Relations have a definite order such that reversing the domain and codomain gives a different relation. Changing the domain and codomain of the "x loves y" relation, for example, forms the converse relation "y is loved by x".[2] In [5] it is shown that lines of identity in fact obey the equations of a special Frobenius algebra. Graphically, this involves the addition of 'cups' and 'caps' (,) that serve as markers for keeping explicit track of the bending of wires and the respective domain and codomain for each relation. For example, EGs of (a) a relation R, (b) its converse $\breve{R}$, and (c) relation inclusion/containment $R \sqsubseteq S$ are given below in (3).



(3)

The addition of cups and caps are important since the initial presentations of residuation in De Morgan's and Peirce's works depend on tracking the converse (and other) relations. Importantly, whereas the single cut represents complement/negation, a nested cut represents inclusion/containment relation (Peirce's "scroll"). The use of 'cups' and 'caps' as endcaps in this context is to show that the domain of R is preserved in the domain of S and that the codomain of R is preserved in the codomain of S.

More discussion on relation algebras can be found in [4]. For a detailed translation of the EG syntax into first-order logic, relation algebras, and a discussion of the transformation rules in this context, see [5]. With relational composition, complement, converse, and inclusion expressible graphically in the syntax, we have the relational operations needed to present residuation in EGs.

3 Residuation in Existential Graphs

Given relational composition (;), left and right residuals take the form of division.

$$a \sqsubseteq c/b \quad \Longleftrightarrow \quad a;b \sqsubseteq c \quad \Longleftrightarrow \quad b \sqsubseteq a\backslash c. \tag{4}$$

Peirce enumerated several equivalent forms residuals may take (Schröder lists many more in [22]). The list depends on which operations are taken as primitive.

[2] While Peirce emphasizes this ordering in his algebraic work, he says little about reading such an ordering off the graphs. The few such places on the ordering of the 'hooks' around the relation terms appear in the early drafts of EGs from late 1886, in which Peirce notices how the connections of the lines to the relations should be read "clockwise" or "counterclockwise" (their converses) "beginning at the left/right" of the relation term; see [18, pp.220,263,295,302,303]. A further advantage of the notation in [5] is that the ordering is always explicit.

We begin by adding to (4) the residuals in terms of complement ($\bar{\cdot}$) and converse ($\breve{\cdot}$) relations. This allows us to fairly directly convert the residuation laws into a form amenable to the syntax of the EGs. We also use ($\dagger$) to represent what Peirce calls *relative sum*, which is the dual to relative composition.

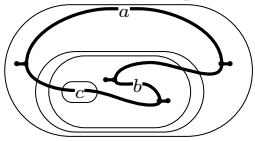
$$a \sqsubseteq c/b \iff a;b \sqsubseteq c \iff b \sqsubseteq a\backslash c \quad (4)$$

$$a \sqsubseteq (\bar{c};\breve{b})\bar{} \iff a;b \sqsubseteq c \iff b \sqsubseteq (\breve{a};\bar{c})\bar{} \quad (5)$$

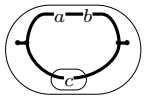
$$a \sqsubseteq c \dagger \bar{\breve{b}} \iff a;b \sqsubseteq c \iff b \sqsubseteq \bar{\breve{a}} \dagger c \quad (6)$$

$$\ldots \qquad \ldots \qquad \ldots$$

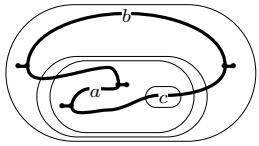
Example 1 (Residuation laws in EGs). Let us begin with EGs of (4) that correspond to the row of equations in (5).



$a \sqsubseteq c/b \quad \left[= \; a \sqsubseteq (\bar{c};\breve{b})\bar{} \right]$



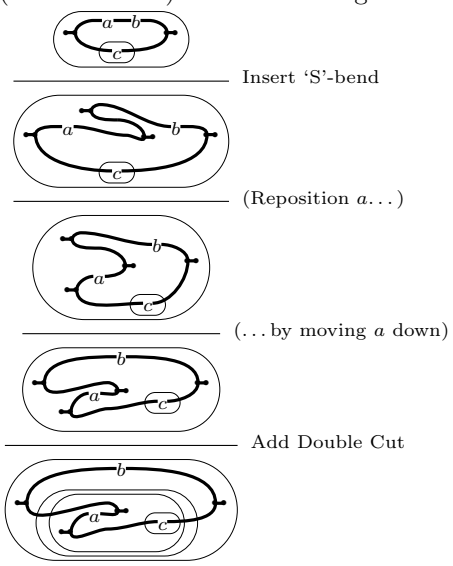
$a;b \sqsubseteq c$



$b \sqsubseteq a\backslash c \quad [= \; b \sqsubseteq (\breve{a};\bar{c})]$

Relational composition (;) is represented in the graphs by connecting, via a line of identity, the respective outgoing and ingoing wire for the relations a and b. Relational inclusion/containment is captured by nested cuts, i.e. Peirce's scroll, and complement and converse are likewise represented as discussed in Sect. 2.

Only two graphical transformations are needed to represent the residuals in the side columns of (5), depicted by the left and right graphs above. One transformation is to add an S- or Z-shaped bend to a line of identity (cf. 'cups' and 'caps' producing converses). The other is to add a double cut around subgraphs. This has the effect of changing the consequent in the newly directed implication (see Remark 2). Both are straightforward transformation rules in EGs.

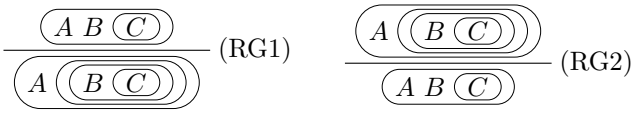


The right residual is derived as follows. The transformation to the final graph begins by inserting an S-shaped bend in the line of identity (employing 'cups' and 'caps') between the composition of a and b. We then grab a and pull it down to the left. Finally, add a double cut around the bottom subgraph to form the consequent of the new implication. Notice that in the process we switch the endcap that preserves the codomain of the relation. This topographical deformation yields the converse on a. Performing similar operations (now with a Z-shaped bend) on b yields the graph on the left

in Example 1. The simple transformations of bending wires to reposition subgraph relations and adding/removing a double cut are sufficient to capture the long list of equivalences given by Peirce in [16, CP 3.341].

Remark 1. Peirce's Rule of 1880 (residuation) is of particular importance, as the full distributivity laws can be deduced from it and the standard lattice rules [9]. The rules of Modus Ponens, LEM and *Ex Falso* are deducible similarly.

Remark 2. Peirce's Rule of 1880 has a particularly clear representation in EGs. Graphical representation of logical constants brings out the relation between logical connectives as *adjunctions*. The following rules are provable in EGs (Alpha):



The rule (RG1) is immediate from the observation that it is permissible to add a double cut, namely the scroll with blank areas. The other direction (RG2) follows from the observation that permits removing that scroll. Hence Peirce's Rule is justified by the observational element that entitles the addition/removal of scrolls with blank areas. Indeed according to Peirce, "every copula is so closely connected with a conjunction that the notation should show the connection" [18, p. 428], concluding that "copulas are nothing but conjunctions" [18, p. 426].

Remark 3. On one loose manuscript leaf (RS 104, c.1903) Peirce formulated residuation as the pair of $A\,B \prec C$ and $A \prec C \Psi \overline{B}$. Here his algebra of logic notations Ψ (aggregate) and $\overline{\ }$ (vinculum) correspond to logical disjunction and negation, respectively. Above and below these two consequence relations he wrote in the language of EGs: "AB C" and "A $\overline{C}\ B$", respectively. Taking the blank to mean the derivation along the consequence relation, one can move between these two graphs solely by an addition (top-down) or erasure (bottom up) of a blank scroll.

Example 2. We give Peirce's rule for the cyclic permutation of terms [19] in graphs. In Peirce's words the rule is: "the three letters may be cyclically advanced one place in the order of writing, those which are carried from one side of the copula to the other being both negatived and converted" [16, CP 3.341].



$a; b \sqsubseteq c$ $\qquad$ $\breve{\breve{c}}; a \sqsubseteq \breve{\breve{b}}$

Again the equivalent expression is captured by the transformation rules in EGs—in this case rotating the subgraphs clockwise by pulling b down and around now also raising ⓒ around and up.

It is worth comparing the topological transformations used in the derivations above with an equivalent derivation using first-order logic.[3] When these moves are put into graphical notation many steps are found to be either roundabout or to have little content, such as to introduce, label, re-label, and eliminate excess variables.

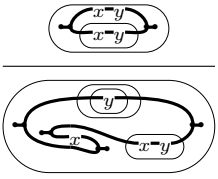
4 Two Further Examples of Residuation in EGs

Two further examples highlight the efficiencies gained by the graphical treatment of residuation for derivations and for further thought.

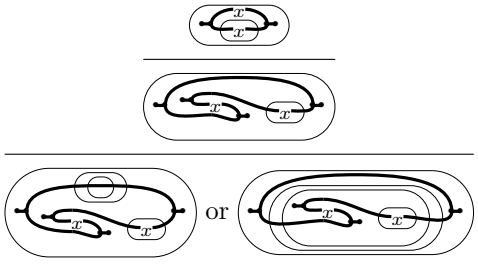
It is known that in axiomizations of relation algebras, such as by [23], Theorem K can replace the axioms governing the rules between involution and distribution over Boolean join [12, p. 25]. One version of the key axiom is $\breve{x}; \overline{x; y} \sqsubseteq \bar{y}$.

In "Note B" [15] Peirce also presents the following important equations: $\mathbb{I} \sqsubseteq \overline{\breve{x}; \bar{y}}$ and $x; \overline{\breve{x}} \sqsubseteq \bar{\mathbb{I}}$, where $\mathbb{I}$ is the identity relation for relational composition. These equations correspond to the *linear negation operation* in linear logic [19].

Example 3. We show that these equations follow from simple identities on x and the topographical moves described in Sect. 3.



Graphical transformation of $\breve{x}; \overline{x; y} \sqsubseteq \bar{y}$

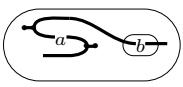


Graphical transformation of $\mathbb{I} \sqsubseteq \overline{\breve{x}; \bar{y}}$ or $x; \overline{\breve{x}} \sqsubseteq \bar{\mathbb{I}}$

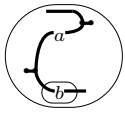
The equations for the right residual, $a \backslash b$, are given by the following [19]:

$$a \backslash b \quad = \quad \bar{\breve{a}} \dagger b \quad = \quad \overline{(\breve{a}; \bar{b})} \tag{7}$$

Example 4. The rightmost equation is expressed by the EG below left. We clearly see that the residual has the form of an implication.



$\overline{(\breve{a}; \bar{b})}$



[3] One such derivation is [21, p. 42] which we forego here due to space limitations. Schmidt speculates that equational reasoning using predicate logic results in derivations that are six times longer than the corresponding algebraic handling of relations [p. xi].

5 Concluding Remarks

This last example brings us back to a common place where Peirce draws specific attention to residuation. He singles out the graph on the right, equivalent to the one on the left and which has a very nice vertical symmetry, to represent the key feature of necessary reasoning. In this sense, residuation is a general logical principle that has a *maximum level of abstractness*. Being maximally abstract means that such principles add nothing to the premises of the inference which they govern [17, NEM 4:175, 1898].

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