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RESEARCH ARTICLE

Pseudogroups and their torsors

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Abstract
We provide a close analysis of the connections between pseudogroups, groupoids, and toposes. This analysis provides a topos perspective on both the localic germ groupoid of a pseudogroup defined by Resende, and the topological groupoid of a pseudogroup defined by Lawson and Lenz. In particular, we show how to analyze the topos of a pseudogroup using sheaf theory, leading to an examination of pseudogroup torsors. Consequently we obtain a concrete description of the category of points of an étendue.

Keywords Inverse semigroups · Pseudogroups · Poposes · Étendues · Torsors

1 Introduction

An *(abstract) pseudogroup* is an inverse semigroup in which each compatible subset of it admits a join, such that multiplication distributes over such joins. Classic examples of a pseudogroup arise by considering all homeomorphisms between open subsets of a space, or by taking all diffeomorphisms between open subsets of a smooth manifold. In fact, Mark Lawson [7,9] explains how pseudogroups have their origin in differential geometry going back to the work of Lie and Cartan.

By now there is a substantial amount of work on the interplay between inverse semigroups on the one hand, and categorical structures such as ordered groupoids and toposes on the other. For example, it is known how to associate with an inverse semigroup its so-called *classifying topos*. (We review this construction and other relevant background in some detail in Sect. 3.) This topos may be viewed as a category of

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representations of the inverse semigroup. Previous work by various authors analyzing the construction has exhibited a rich and fruitful interaction between the two areas.

The present paper continues the investigation into semigroups and toposes by analyzing the connections between pseudogroups and toposes. More concretely, using sheaf theory we present a description of the classifying topos of a pseudogroup and an account of the germ groupoid associated with a pseudogroup. Additionally, we give a complete description of the category of torsors for a pseudogroup. In particular, this reveals what an étendue classifies. From this point of view one may regard our work also as a contribution to the general question of what a localic groupoid classifies. Roughly, this question asks for a logical description of the category $\mathfrak{Top}(\mathscr{E}, \mathscr{B}(G))$ of geometric morphisms from a topos $\mathscr{E}$ to the classifying topos of a localic groupoid G; such a geometric morphism may be interpreted as specifying a model in $\mathscr{E}$ of a suitable theory associated with G. Indeed, Marta Bunge [1] provides a partial answer to this question, answering it for localic toposes $\mathscr{E}$ and isomorphisms between such models, thereby establishing an equivalence

$$\mathfrak{Top}(\mathsf{Sh}(X), \mathscr{B}(G)) \simeq \mathsf{Tor}(X; G)$$

where $\mathfrak{Top}(\mathsf{Sh}(X), \mathscr{B}(G)$ is the groupoid of geometric morphisms and natural isomorphisms between them, and $\mathsf{Tor}(X; G)$ is the groupoid of G-torsors over a locale X. The present paper provides another partial answer to the general question here admitting models in arbitrary toposes $\mathscr{E}$ and all morphisms between them, but confining our attention to étale groupoids G.

1.1 Overview

Pedro Resende [13] associates with a pseudogroup what he terms its *germ groupoid*. This groupoid is a *localic groupoid* in the sense that it is a groupoid internal to Loc, the category of locales. By definition, $\mathsf{Loc} = \mathsf{Frm}^{\mathrm{op}}$ is formally the opposite of the category Frm of frames and frame homomorphisms. A *frame* is a meet-semilattice admitting arbitrary joins, such that finite meets distribute over joins. A typical example of a frame is the lattice of open subsets of a topological space. Thinking of a locale X as an abstract topological space we often denote the corresponding frame by $\mathcal{O}(X)$. Likewise, we think of a locale morphism $f : X \to Y$ as a continuous map, and denote the corresponding frame homomorphism by $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

Definition 1 (*Germ Groupoid*) The localic groupoid $G(S)$ of a pseudogroup S has as its locale of objects $G(S)_0$ the locale corresponding to the frame $E \subseteq S$ of idempotents of S: $\mathcal{O}(G(S)_0) = E$. (The idempotents of a pseudogroup form a frame.) Its locale of morphisms $G(S)_1$ consists of the collection of downsets in S which are closed under suprema of compatible families of elements of S.

As it happens $G(S)$ is an example of an *étale groupoid* in the sense that its domain (and hence also codomain) map is a local homeomorphism.

One may associate with an arbitrary localic groupoid G (not necessarily étale) a topos $\mathscr{B}(G)$ of étale actions. As a matter of fact, a classic result in topos theory

states that every Grothendieck topos arises in this manner. When G happens to be étale the topos $\mathscr{B}(G)$ is an *étendue* in the sense that it has a globally supported object X (meaning that the unique morphism $X \to 1$ is an epimorphism) such that the slice $\mathscr{B}(G)/X$ is localic. Such an object X is called a *torsion-free generator* of the topos, and we think of it as a presentation, or coordinatization, of the topos. In fact, in the étale case one may take for X the domain map $G_1 \to G_0$, regarded in a canonical way as an object ∂ of $\mathscr{B}(G)$. It is known that every étendue $\mathscr{E}$ arises in this manner: if X is a torsion-free generator of $\mathscr{E}$, then there is an étale groupoid G and a coordinate-preserving equivalence $\mathscr{E} \simeq \mathscr{B}(G)$, meaning X and ∂ correspond across the equivalence. Combining the constructions we may thus associate with a pseudogroup S the topos $\mathscr{B}(S) = \mathscr{B}(G(S))$ of étale actions on the germ groupoid of S. As $G(S)$ is étale, the topos $\mathscr{B}(S)$ comes equipped with a canonical torsion-free generator ∂. Thus, each pseudogroup gives us not just an étendue but a coordinatized one.

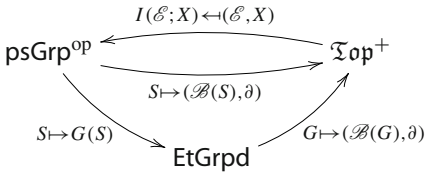
In the other direction suppose that X is an object of a topos $\mathscr{E}$. (In this paper, we only consider Grothendieck toposes.) Then let us say that *a bisection on* X is an equivalence class of pairs of monomorphisms

$$\begin{array}{ccc} U & \overset{m_2}{\rightarrowtail} & X \\ {\scriptstyle m_1}\downarrow & & \\ X & & \end{array}$$

in $\mathscr{E}$, where two such pairs (m_1, m_2) and (n_1, n_2) of monomorphisms are declared equivalent if there is an isomorphism i such that $m_1 i = n_1$ and $m_2 i = n_2$. Let $[m_1, m_2]$ denote the equivalence class of such a pair of monomorphisms; we think of such a class as a partial bijection from X to itself. (Such monomorphism pairs may themselves form a proper class, but there is just a set of isomorphism classes.) It follows that composition by pullback is a well-defined semigroup product of bisections on X, with a partial inverse

$$[m_1, m_2]^* = [m_2, m_1] \, .$$

In fact, this semigroup is an inverse semigroup, and even better it is an (abstract) pseudogroup which we shall denote $I(\mathscr{E}; X)$. Its lattice of idempotents is the subobject lattice $\mathrm{Sub}(X)$ of X. One of the noteworthy facts that we shall clarify in this paper is that up to coordinate-preserving Morita equivalence, every pseudogroup arises in this way. This means that for any pseudogroup S there is a pseudogroup of the form $I(\mathscr{E}; X)$ and a coordinate-preserving equivalence $\mathscr{B}(S) \simeq \mathscr{B}(I(\mathscr{E}; X))$. What is more, the pseudogroup $I(\mathscr{E}; X)$ is isomorphic to $\Gamma(I_{\mathscr{E}}(X))$, the global sections of a pseudogroup $I_{\mathscr{E}}(X)$ internal to $\mathscr{E}$, namely the (internal) symmetric inverse semigroup on X. (The latter can be constructed as an appropriate subobject of the object of relations from X to itself.) Thus, every pseudogroup is Morita equivalent to the global sections of a symmetric one. The following diagram summarizes the two constructions.



(1)

Here psGrp is the category of pseudogroups, EtGrpd the category of étale groupoids, and $\mathfrak{Top}^{+}$ denotes the category of pairs $(\mathscr{E}, X)$ where $\mathscr{E}$ is a topos and X is a globally supported object of it. Morphisms in these categories are explained below.

Our first goal in this paper is to describe directly, without recourse to étale groupoids, this composite functor from pseudogroups to toposes. This in turn depends on and is closely tied to what we may call the 'algebraic theory' of the underlying inverse semigroup. This theory associates with an inverse semigroup S a left-cancellative category $L(S)$, and thence a topos of presheaves on $L(S)$. However, pseudogroups take us naturally into sheaf theory: we work with a non-trivial (but relatively straightforward) Grothendieck (pre)topology on the category $L(S)$. We call this the $\bigvee$-topology on $L(S)$, since it is induced by the suprema in the pseudogroup S. Ultimately, we shall prove that the topos $\mathsf{Sh}(L(S), \bigvee)$ of $\bigvee$-sheaves on $L(S)$, which is obviously an étendue, is equivalent to the topos $\mathscr{B}(S)$ of étale actions associated with the germ groupoid $G(S)$. Our argumentation for this, although by no means trivial, uses known and straightforward techniques from topos theory and localic groupoids. Furthermore, the argumentation informs us how to explicitly identify a torsion-free sheaf on $L(S)$ whose étale groupoid coincides with the germ groupoid $G(S)$.

Thus, we may work with $\mathscr{B}(S)$ as a sheaf topos: henceforth, $\mathscr{B}(S)$ also denotes the sheaf topos $\mathsf{Sh}(L(S), \bigvee)$. Our analysis puts us in position to achieve our second goal, which is to determine what the topos of a pseudogroup classifies. Towards this goal we examine the passage of a pseudogroup homomorphism to a geometric morphism of toposes. In particular, we shall show that a pseudogroup homomorphism $\phi : S \to T$ is *hypercallitic* [9] precisely when it induces a geometric morphism of toposes

$$\phi^* \dashv \phi_* : \mathscr{B}(T) \to \mathscr{B}(S) \tag{2}$$

(note the change in direction), although we find it appropriate to use the topos term *flat* instead of hypercallitic. Moreover, the resulting geometric morphism preserves the canonical coordinate systems: there is a canonical isomorphism

$$m : \partial_T \to \phi^*(\partial_S) \ ,$$

where ∂_S denotes the domain object of $\mathscr{B}(S)$, and the same for T. Together, we have a pair

$$(\phi, m) : (\mathscr{B}(T), \partial_T) \to (\mathscr{B}(S), \partial_S)$$

constituting what we call a *Cartesian* morphism of coordinatized toposes.

Conversely, the $I(\mathscr{E}; X)$ construction is indeed a functor on such Cartesian morphisms

$$(\psi, m) : (\mathscr{E}, X) \twoheadrightarrow (\mathscr{F}, Y) \ .$$

Ultimately, we come to an equivalence

$$\frac{\text{Cartesian } (\psi, m) : (\mathscr{E}, X) \twoheadrightarrow (\mathscr{B}(S), \partial)}{\text{flat homomorphisms } S \twoheadrightarrow I(\mathscr{E}; X)} \tag{3}$$

for any topos $\mathscr{E}$, and pseudogroup S. For fixed $\mathscr{E}$ and S (but letting the globally supported X vary) these are the objects of a categorical equivalence explained below. Note that for $\mathscr{E} = \mathsf{Set}$ we have $I(\mathsf{Set}; X) = I(X)$, the full symmetric pseudogroup on X. Thus, a flat pseudogroup homomorphism $S \twoheadrightarrow I(\mathsf{Set}; X)$ may be regarded as a partial (left) action of S on the non-empty set X by partial bijections. The flatness condition then specifies that this action is transitive and locally free, making X an S-torsor, and $\bigvee$-preservation corresponds to the action being locally determined.

For a general topos $\mathscr{E}$, we find in particular that there is an equivalence of categories

$$\mathfrak{Top}(\mathscr{E}, \mathscr{B}(S)) \simeq \mathsf{Tor}(\mathscr{E}; S)$$

between the category of geometric morphisms from $\mathscr{E}$ to $\mathscr{B}(S)$ and the category of S-torsors internal to $\mathscr{E}$ (in general, these categories are not groupoids). This makes precise the sense in which $\mathscr{B}(S)$ classifies S-torsors.

Finally, our work reveals how pseudogroups may be organized in a bicategory whose morphisms have the form of a certain flavour of bimodule. As it happens flat pseudogroup homomorphisms between pseudogroups do not account for all geometric morphisms between their toposes. In order to account for all geometric morphisms (2) we must admit what we call flat (S, T)-bimodules. We shall see that the resulting bicategory of pseudogroups, flat bimodules and bimodule homomorphisms is weakly equivalent to the 2-category of étendues, geometric morphisms and geometric transformations.

1.2 Plan of the paper

Section 2 reviews the necessary concepts and results from the theory of localic groupoids and Grothendieck toposes. Lemma 2 deserves special mention as it plays a key role in our argumentation for Proposition 6. In Sect. 3 we review the 'algebraic' theory of an inverse semigroup and its topos. In particular, we discuss the classifying topos of an inverse semigroup in its various equivalent forms, describe the canonical torsion-free generator, and describe torsors for an inverse semigroup. With the necessary background in place, we turn in Sect. 4 to the topos associated with a pseudogroup. We define a Grothendieck topology on $L(S)$, and describe the resulting étendue of sheaves $\mathsf{Sh}(L(S), \bigvee)$. We also analyse the special case of a symmetric pseudogroup in preparation of our study of torsors. The main result in this section, Proposition 6, shows that $\mathsf{Sh}(L(S), \bigvee)$ coincides with the topos of étale actions on the germ groupoid of S. Finally, Sect. 5 studies the functoriality of the construction of a classifying topos. The main technical result states that flat homomorphisms of

pseudogroups induce geometric morphisms between their toposes. It also introduces the notion of torsor for a pseudogroup, and culminates in the second main result of the paper, Corollary 2, stating that $\mathscr{B}(S)$ classifies S-torsors.

2 Review of basic notions

We begin with a review of some notions, terminology, and technical facts from the theory of localic groupoids and from topos theory. The literature provides much more detail on the subject [5,10]. Throughout, Frm denotes the category of frames, whose formal opposite $\mathsf{Loc} = \mathsf{Frm}^{\mathrm{op}}$ is the category of locales. The category Frm is (pre)-order enriched, using the pointwise ordering on morphisms. For more on the subject we refer the reader to [4,12].

2.1 Toposes and étendues

When $\mathbb{C}$ is a small category we denote by $\mathscr{C} = \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}$ the topos of presheaves on $\mathbb{C}$. The Yoneda embedding $\mathbb{C} \to \mathscr{C}$ sends an object C of $\mathbb{C}$ to the representable presheaf $\widehat{C}$, defined by

$$\widehat{C} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set} ; \qquad \widehat{C}(D) = \mathbb{C}(D, C) .$$

Another perspective on presheaves is sometimes useful: a *discrete fibration* over $\mathbb{C}$ is a functor $F : \mathbb{D} \to \mathbb{C}$ with the property that any morphism $f : C \to F(D)$ of $\mathbb{C}$ lifts uniquely to one $E \to D$ of $\mathbb{D}$. The categories of presheaves on $\mathbb{C}$ and discrete fibrations over $\mathbb{C}$ are equivalent. For instance, a presheaf P on $\mathbb{C}$ is associated with a discrete fibration $\int_{\mathbb{C}} P \to \mathbb{C}$, where a typical object of $\int_{\mathbb{C}} P$ is a pair (C, x) such $x \in PC$, and a morphism $(C, x) \to (D, y)$ is a morphism $f : C \to D$ of $\mathbb{C}$ such that $P(f)(y) = x$.

In the theory of Grothendieck toposes, presheaf toposes play the role of free objects: $\mathscr{C}$ is freely generated by $\mathbb{C}$ under the formation of colimits. In order to study general toposes we use the notion of a site, which may be thought of as a presentation of a topos via generators and relations. A *Grothendieck pretopology* on a small category $\mathbb{C}$ specifies a collection $K(C)$ of sieves $R \rightarrowtail \widehat{C}$ on each object C of $\mathbb{C}$, said to be covering sieves. Covering sieves are required be pullback stable in the sense that for $m : C \to D$ and $R \in K(D)$ the sieve $m^*R = \{f : X \to C \mid fm \in R\}$ is again covering. We also require that the maximal sieve on any object C is a covering sieve. We refer to a small category $\mathbb{C}$ equipped with a pretopology K as a *site*, denoted $(\mathbb{C}, K)$. A presheaf F on $\mathbb{C}$ is said to satisfy the sheaf condition for a sieve $R \rightarrowtail \widehat{C}$ if it has the unique extension property depicted in the following diagram.

$$\begin{array}{ccc} R & \rightarrowtail & \widehat{C} \\ & \searrow^{\forall} & \downarrow{\scriptstyle \exists!} \\ & & F \end{array} \tag{4}$$

We say that a presheaf F on $\mathbb{C}$ is a *sheaf* for a pretopology K on $\mathbb{C}$ if it has the sheaf property for every covering sieve of K. Let

$$\mathsf{Sh}(\mathbb{C}, K) \underset{i}{\overset{a}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathscr{C} \tag{5}$$

denote the full subcategory of sheaves on a site $(\mathbb{C}, K)$. We refer to such a category as *a Grothendieck topos*, or just *topos*. The inclusion i of sheaves into presheaves has a left adjoint a (for associated sheaf) that preserves finite limits, making it an instance of a geometric morphism (7).

Example 1 (*Sheaves on a locale*) The frame $\mathcal{O}(X)$ associated with a locale X carries a canonical topology. If a sieve R on an element $U \in \mathcal{O}(X)$ is a downclosed subset below U, then we define a topology by declaring a sieve R on U to be covering when $\bigvee R = U$. We denote the corresponding topos of sheaves by $\mathsf{Sh}(X)$. Toposes of this form are called *localic*, characterized by the fact that the subobjects of the terminal object generate in the sense that every object is a colimit of subterminals.

A representable presheaf $\widehat{C}$ need not be a sheaf, but it may always be sheafified, meaning that we may apply the associated sheaf functor to it. If F is a sheaf, then morphisms $a(\widehat{C}) \to F$ correspond to elements of $F(C)$. A presheaf is said to be *separated* when extensions (4) are unique even if they do not always exist. A presheaf F is separated if and only if the unit $F \to ia(F)$ of the adjunction $a \dashv i$ is a monomorphism.

The associated sheaf functor a can be understood explicitly in terms of what is sometimes called the $+$ construction. For a presheaf F, define another presheaf

$$F^+(C) = \mathrm{Colim}_{R \in K(C)} \mathscr{C}(R, F) \ .$$

The colimit is taken over the partially ordered set of covering sieves, ordered by inclusion. We refer to an element of $F^+(C)$ as a *germ* of F at C. The presheaf F^+ is separated, and for a separated presheaf F, F^+ is a sheaf. Thus $a(F) \cong F^{++}$.

Example 2 Consider a preorder $\mathbb{P}$. Its *Alexandroff locale* $X_{\mathbb{P}}$ has as its frame the downclosed subsets in $\mathbb{P}$:

$$\mathcal{O}(X_{\mathbb{P}}) = \{U \subseteq \mathbb{P} \mid U \text{ downclosed } \} \ . \tag{6}$$

Suprema in $\mathcal{O}(X_{\mathbb{P}})$ are given by union. This assignment $\mathbb{P} \mapsto X_{\mathbb{P}}$ is in fact a functor from preorders to locales. The topos of sheaves on $X_{\mathbb{P}}$ is equivalent to presheaves on $\mathbb{P}$: $\mathsf{Sh}(X_{\mathbb{P}}) \simeq \mathsf{Set}^{\mathbb{P}^{\mathrm{op}}}$.

If $\mathscr{E}$ and $\mathscr{F}$ are toposes, then *a geometric morphism*

$$\phi : \mathscr{E} \to \mathscr{F} \ ; \qquad \mathscr{E} \underset{\phi_*}{\overset{\phi^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathscr{F} \tag{7}$$

is an adjoint pair $\phi^* \dashv \phi_*$, such that $\phi^* : \mathscr{F} \to \mathscr{E}$ preserves finite limits. The functor ϕ^* is called the inverse image functor, and ϕ_* the direct image functor of the geometric morphism.

If X is an object of $\mathscr{E}$, then the slice category $\mathscr{E}/X$ is again a topos. The functor $X^* : \mathscr{E} \to \mathscr{E}/X$, given by $A \mapsto (\pi_X : A \times X \to X)$, has both adjoints, giving in particular a geometric morphism $\mathscr{E}/X \to \mathscr{E}$, said to be *étale*.

Every topos $\mathscr{E}$ comes equipped with an essentially unique *structure geometric morphism* given by

$$\mathscr{E} \underset{\gamma_*}{\overset{\gamma^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Set} \;\; ; \quad \gamma_*(X) = \mathscr{E}(1, X) \; ; \qquad \gamma^* : \mathsf{Set} \to \mathscr{E} \; ; \quad \gamma^*(A) = \coprod_A 1 \; .$$

The functor γ_* is also called the *global sections functor* (often denoted Γ) while γ^* is called the *constant objects functor* (often denoted Δ).

Just as a topos may be presented by a site we also commonly work with geometric morphisms in terms of sites. A site morphism

$$F : (\mathbb{C}, K) \to (\mathbb{D}, L)$$

is a functor $F : \mathbb{C} \to \mathbb{D}$ that preserves covers, and is flat in the sense that the functor

$$\mathbb{C} \to \mathbb{D} \to \mathsf{Sh}(\mathbb{D}, L) \; ; \quad C \mapsto a(\widehat{F(C)}) \; , \tag{8}$$

is filtering. In turn, this is the case just when the left extension

$$F_! : \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}} \to \mathsf{Sh}(\mathbb{D}, L)$$

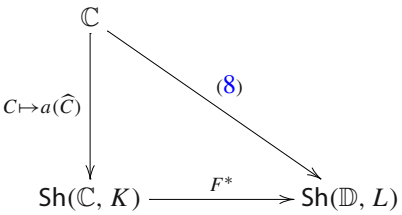
preserves finite limits. Also if F preserves covers, then composition with F sends sheaves to sheaves. If we denote this functor by F_*, then its left adjoint is $F^* = a \circ F_! \circ i$ making a geometric morphism

$$F^* \dashv F_* : \mathsf{Sh}(\mathbb{D}, L) \to \mathsf{Sh}(\mathbb{C}, K) \; .$$

Remark 1 In the situation described above, it also follows that $F^* \circ a \cong a \circ F_!$. In particular, for any object C we have

$$F^*(a(\widehat{C})) \cong a(\widehat{F(C)}) \; .$$

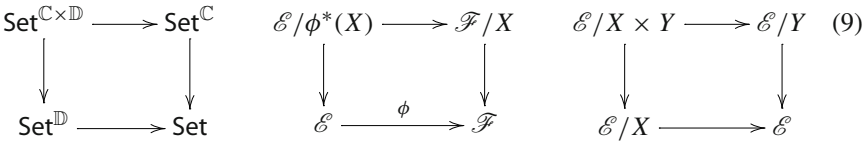
In other words, the diagram



commutes, exhibiting F^* as the left Kan extension of (8) along Yoneda.

Finally, there is a notion of 2-cell between geometric morphisms $\phi, \psi : \mathscr{E} \longrightarrow \mathscr{F}$: a *geometric transformation* $\alpha : \phi \Rightarrow \psi$ is a natural transformation $\alpha : \phi^* \longrightarrow \psi^*$ between the inverse image functors. $\mathfrak{Top}$ shall denote the 2-category of Grothendieck toposes, geometric morphisms and geometric transformations.

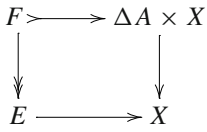
Remark 2 (*Topos pullbacks*) Topos pullbacks are a necessary and useful tool. Throughout, when we say that a diagram of toposes and geometric morphisms is a pullback we really mean that it has a 2-categorical universal property sometimes called a bipullback. Sometimes we may explicitly describe such a pullback as in the following specific cases.



(9)

In general it is not easy to describe pullbacks, but being able to is often not terribly important. What is important is the fact that pullbacks always exist, and that we may reason using their universal property.

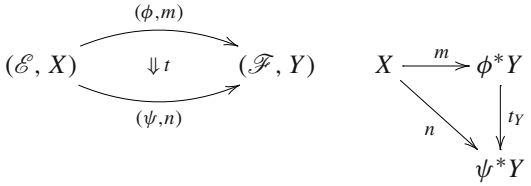
Following Kock and Moerdijk [6], we say that a morphism $E \twoheadrightarrow X$ of a topos $\mathscr{E}$ is *locally monic* if in $\mathscr{E}$ there is a diagram as follows.



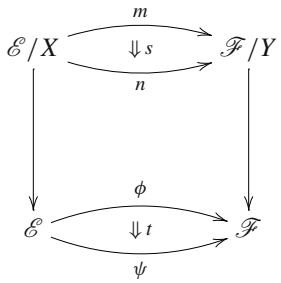
An object X of $\mathscr{E}$ is said to be *torsion-free* if every morphism $E \twoheadrightarrow X$ of $\mathscr{E}$ is locally monic. Equivalently, X is torsion-free if and only if $\mathscr{E}/X$ is localic. If $\mathscr{E}/X$ is localic, then it is equivalent to $\mathsf{Sh}(\mathsf{Sub}(X))$, where $\mathsf{Sub}(X)$ is the frame of subobjects of X. A topos $\mathscr{E}$ is termed *an étendue* when it has a globally supported torsion-free object X.

2.2 The category $\mathfrak{Top}^+$

Definition 2 ($\mathfrak{Top}^+$) The 2-category $\mathfrak{Top}^+$ shall have objects $(\mathscr{E}, X)$ consisting of a topos $\mathscr{E}$ paired with a globally supported object X of it. A morphism (or 1-cell) in this category is a geometric morphism $\phi : \mathscr{E} \to \mathscr{F}$ and a morphism $m : X \to \phi^* Y$ of $\mathscr{E}$, which we usually write as a pair (ϕ, m). A 2-cell $(\phi, m) \Rightarrow (\psi, n)$ of $\mathfrak{Top}^+$ is a geometric transformation $t : \phi \Rightarrow \psi$ such that $t_Y m = n$.



Remark 3 For any two objects X and Y of toposes $\mathscr{E}$ and respectively $\mathscr{F}$ the category $\mathfrak{Top}^+((\mathscr{E}, X), (\mathscr{F}, Y))$ and the category of commutative 'cylinder' diagrams



of geometric morphisms and transformations are equivalent. Such a geometric transformation s exists if and only if $t_Y m = n$, in which case t uniquely determines s.

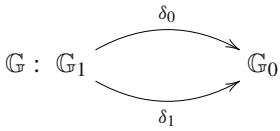
As explained in the introduction, when X is a generator we think of the pair $(\mathscr{E}, X)$ as a presentation of $\mathscr{E}$, or as a topos with coordinates. For example, if X is globally supported and torsion-free (so that $\mathscr{E}$ is an étendue), then X is a generator.

Definition 3 We shall say that a morphism (ϕ, m) of $\mathfrak{Top}^+$ is *Cartesian* if m is an isomorphism.

Remark 4 The terminology "Cartesian" is justified by the fact that the forgetful 2-functor $\mathfrak{Top}^+ \to \mathfrak{Top}$ is a 2-fibration.

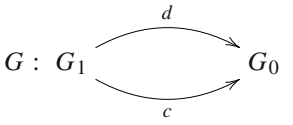
2.3 Ordered groupoids

An ordered groupoid



is a groupoid object in the category of posets (as such it is a kind of double category) with the property that its domain map δ_0 is a discrete fibration of posets. Explicitly, this means that whenever $x \leq \delta_0(m)$ there exists a unique morphism $m \cdot x \leq m$ for which $\delta_0(m \cdot x) = x$. One may think of $m \cdot x$ as restricting m to the smaller domain x. Of course, an ordered groupoid carries more structure: composition, identities, and inverses, but we suppress mentioning this data unless confusion is likely to arise. However, we mention that we write composition in functional order, meaning that gh means first h, then g.

Applying the Alexandroff functor (Example 2) to $\mathbb{G}$ gives a groupoid in the category Loc of locales, denoted



.

The frame $\mathcal{O}(G_0)$ is the frame of downclosed subsets of the poset $\mathbb{G}_0$, and likewise for the locale G_1. The frame morphism d^* is the inverse image function ${\delta_0}^{-1}$. Both d and c are local homeomorphisms, whence $\mathbb{G}$ is an *étale localic groupoid*. We have

$$\mathsf{Sh}(G_0) \simeq \mathsf{Set}^{{\mathbb{G}_0}^{\mathrm{op}}} .$$

Ordered groupoids and their functors form a category denoted OGrpd.

With each ordered groupoid $\mathbb{G}$ we may associate a topos $\mathscr{B}(\mathbb{G})$. A typical object of it is a discrete fibration $p : X \twoheadrightarrow \mathbb{G}_0$ of posets that carries a unital and associative (right) action

$$X \times_{\mathbb{G}_0} \mathbb{G}_1 \to X .$$

The action is fibered over $\mathbb{G}_0$ where the pullback object above means pullback along codomain and compose with domain.

For an ordered groupoid $\mathbb{G}$, the topos $\mathscr{B}(G)$ is in fact a presheaf topos. To see this let $L(\mathbb{G})$ denote the category whose objects are the elements of $\mathbb{G}_0$ and whose morphisms $g : y \twoheadrightarrow x$ are triples (y, g, x) such that $g \in \mathbb{G}_1$, $d(g) = y$ and $c(g) \leq x$, where d, c denote the domain and codomain map of $\mathbb{G}$. Since y is determined as $d(g) = y$, we may simplify notation and write (g, x) instead of (y, g, x). Composition in $L(\mathbb{G})$ is accomplished by using the unique lifting property of d as it is a discrete fibration (by assumption). We call $L(\mathbb{G})$ *the left-cancellative category* of the ordered groupoid $\mathbb{G}$ [8] as every morphism of it is a monomorphism. Then we have

$$\mathscr{B}(G) \simeq \mathsf{Set}^{L(\mathbb{G})^{\mathrm{op}}} . \tag{10}$$

Of course, the equivalent presheaf formulation using $L(\mathbb{G})$ in (10) may serve as an equally viable definition of $\mathscr{B}(G)$ in this case.

2.4 Localic groupoids

A localic groupoid

$$G: \; G_1 \underset{c}{\overset{d}{\rightrightarrows}} G_0 \tag{11}$$

is by definition a groupoid internal to the category Loc of locales. Thus G_0, G_1 are locales, and the structure maps c, d, as well as the composition, inverse, and identities morphism, are locale morphisms. Any localic groupoid has associated with it a topos $\mathscr{B}(G)$ of étale (right) actions: an object of $\mathscr{B}(G)$ is a local homeomorphism $X \to G_0$, equipped with an associative and unital action $X \times_{G_0} G_1 \to X$ over G_0. In this paper we are mostly interested in the étale case where d, c are local homeomorphisms. In this case, $\mathscr{B}(G)$ is known to be an étendue.

An important role is played by the domain map d regarded as an object of $\mathscr{B}(G)$ by virtue of its canonical right action given by composition. (This is a *right* action because composition is written in the functional order.) We shall write ∂ for this object. It is a torsion-free generator of $\mathscr{B}(G)$, and we have $\mathscr{B}(G)/\partial \simeq \mathsf{Sh}(G_0)$. Similarly, we have $\mathscr{B}(G)/\partial \times \partial \simeq \mathsf{Sh}(G_1)$. In fact, the étale groupoid G is recovered from $\mathscr{B}(G)$ as the localic groupoid of subobjects

$$\mathrm{Sub}(\partial) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Sub}(\partial \times \partial) \quad . \tag{12}$$

The frames $\mathcal{O}(G_0)$ and $\mathrm{Sub}(\partial)$ are isomorphic, as are $\mathcal{O}(G_1)$ and $\mathrm{Sub}(\partial \times \partial)$. The frame morphism p_1^* denotes pullback along a projection p_1 in $\mathscr{B}(G)$. In fact, the following two squares of toposes and geometric morphisms are canonically equivalent for any étale groupoid G.

$$\begin{array}{ccc} \mathscr{B}(G)/\partial \times \partial & \xrightarrow{p_1} & \mathscr{B}(G)/\partial \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ \mathscr{B}(G)/\partial & \longrightarrow & \mathscr{B}(G) \end{array} \qquad \begin{array}{ccc} \mathsf{Sh}(G_1) & \xrightarrow{d} & \mathsf{Sh}(G_0) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathsf{Sh}(G_0) & \xrightarrow{\varphi} & \mathscr{B}(G) \end{array} \tag{13}$$

The inverse image functor φ^* simply forgets the action by G that an object $X \to G_0$ of $\mathscr{B}(G)$ carries. The left square is a topos pullback, so that the right one is also, which says that an étale groupoid is *étale complete* in the sense of Moerdijk [11].

A localic functor

$$F = (F_0, F_1) : G \to H \tag{14}$$

of localic groupoids induces a geometric morphism

$$\mathscr{B}(F) = F^* \dashv F_* : \mathscr{B}(G) \to \mathscr{B}(H) \tag{15}$$

whose inverse image functor F^* is given by pullback along F_0.

Example 3 The geometric morphism φ in (13) is isomorphic to the geometric morphism induced by the localic functor $G_0 \to G$ given by the identities map $G_0 \to G_1$.

A localic functor F is said to be *full and faithful* if

$$\begin{array}{ccc} G_1 & \xrightarrow{F_1} & H_1 \\ \downarrow{\scriptstyle (d,c)} & & \downarrow{\scriptstyle (d,c)} \\ G_0 \times G_0 & \xrightarrow{F_0 \times F_0} & H_0 \times H_0 \end{array}$$

is a pullback (in the category of locales). If a full and faithful localic functor of localic groupoids also has the property that F_0 is an inclusion, then we shall say that F is *a sublocalic full subgroupoid*. In this case, F_1 is also an inclusion.

We have now two lemmas the second of which will be used in Sect. 4.5 to prove crucially that the classifying topos $\mathscr{B}(S)$ of a pseudogroup S does indeed coincide with the topos of étale actions on the germ groupoid of S. In turn, the proof of Lemma 2 depends on the following fact about geometric inclusions.

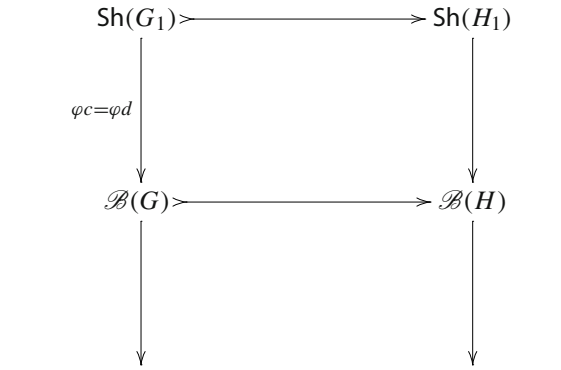
Lemma 1 *If a geometric morphism $\phi : \mathscr{E} \to \mathscr{F}$ is an inclusion* (in the sense that ϕ_* is full and faithful)*, then the diagonals of $\mathscr{E}$ and $\mathscr{F}$ form a pullback with ϕ.*

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\phi} & \mathscr{F} \\ \downarrow & & \downarrow \\ \mathscr{E} \times \mathscr{E} & \xrightarrow{\phi \times \phi} & \mathscr{F} \times \mathscr{F} \end{array}$$

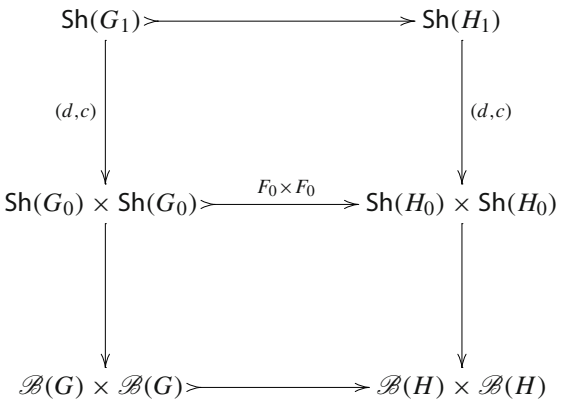
Proof In general, for any ϕ the above pullback is equivalent to the (bi)kernel pair of ϕ. The diagonal into the kernel pair of an inclusion is an equivalence ([5], Prop. 3.3.8, (iii)). □

Lemma 2 *Let G and H denote étale localic groupoids. Suppose that $F : G \to H$ is a localic functor whose induced geometric morphism* (15) *is a Cartesian inclusion. Then F is a sublocalic full subgroupoid.*

Proof By the Cartesian hypothesis F_0 is an inclusion as inclusions are pullback stable. To verify that F is full and faithful consider diagrams (16) and (17).

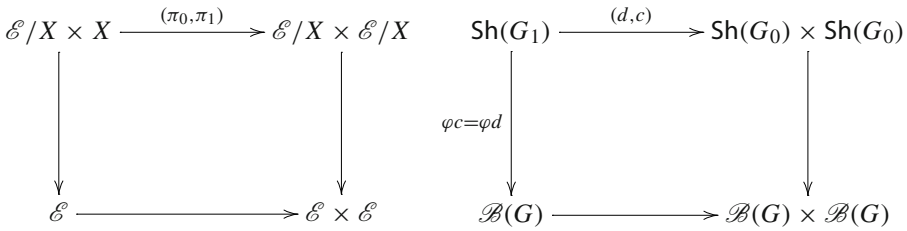


(16)



(17)

We claim that the two outer rectangles in (16) and (17) are equivalent in the sense that their vertical composites are isomorphic. In other words, the square below (right) with G commutes, and the same for H.



This is so because for any object X of a topos $\mathscr{E}$ the square above (left) commutes, and because $\mathsf{Sh}(G_1) \simeq \mathscr{B}(G)/\partial \times \partial$. The bottom square of (16) is a pullback by Lemma 1. Its top square is a pullback again by the Cartesian assumption, and by étale completeness. Therefore, the outer rectangle in (16) is a pullback, whence so also is

the equivalent outer rectangle in (17). The bottom square in (17) is a pullback, so that its top square is also one. □

Remark 5 Let us say that a localic functor $F : G \twoheadrightarrow H$ is a discrete opfibration if

$$\begin{array}{ccc} G_1 & \xrightarrow{F_1} & H_1 \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ G_0 & \xrightarrow{F_0} & H_0 \end{array}$$

is a pullback. When G and H are étale, F is a discrete opfibration if and only if the geometric morphism induced by F is Cartesian vis-a-vis the objects ∂_G and ∂_H (Definition 3), which in turn is equivalent to the condition that

$$\begin{array}{ccc} \mathsf{Sh}(G_0) & \xrightarrow{F_0} & \mathsf{Sh}(H_0) \\ \downarrow & & \downarrow \\ \mathscr{B}(G) & \xrightarrow{\mathscr{B}(F)} & \mathscr{B}(H) \end{array}$$

is a pullback. To paraphrase Lemma 2: if a discrete opfibration F has the property that $\mathscr{B}(F)$ is an inclusion, then F is a sublocalic full subgroupoid.

3 The topos of an inverse semigroup

In this section we review the 'algebraic theory' of an inverse semigroup as our subsequent work with pseudogroups depends on it. We present the topos of an inverse semigroup, the geometric morphism associated with a prehomomorphism, and torsors.

3.1 Étale actions versus presheaves

One approach to the topos of an inverse semigroup is that Ehresmann's classical *inductive groupoid*

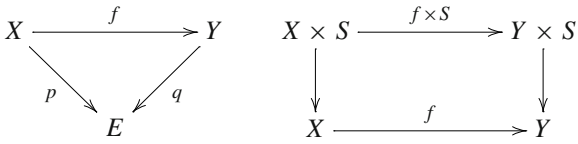
$$\mathbb{G}(S) : \; S \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} E \quad ; \; \delta_0(s) = s^* s \; , \; \delta_1(s) = s s^* \; , \tag{18}$$

associated with an inverse semigroup S may be regarded as an étale localic groupoid via the Alexandroff functor (Example 2). Like any localic groupoid the ordered groupoid $\mathbb{G}(S)$ therefore has a topos of étale actions.

Let us work out both sides of (10) in the case of $\mathbb{G}(S)$. We simplify the notation somewhat writing $\mathscr{B}(S)$ for $\mathscr{B}(\mathbb{G}(S))$, and $L(S)$ for $L(\mathbb{G}(S))$, so that (10) reads

$$\mathscr{B}(S) \simeq \mathsf{Set}^{L(S)^{\mathrm{op}}} . \tag{19}$$

On the left side of (19) a typical object is called an *étale S-set* [3]: this is a set X equipped with an associative (right) action by S and a function $p : X \to E$, which we sometimes call the projection to E associated with X, such that $p(xs) = s^* p(x)s$ and $xp(x) = x$. A morphism f of étale S-sets is a function over E (20) (left), which is S-equivariant in the sense that (20) (right) commutes.



(20)

For example, the terminal étale S-set is the identity map $E \to E$ where E carries the conjugation action by S: $e \cdot s = s^* e s$.

As for the right side of (19), an object of $L(S)$ is an idempotent of S and a morphism is a pair (s, e) such that $ss^* \leq e$ (equivalently, $s = es$). The domain of (s, e) is s^*s, and the codomain is e. We depict such a morphism as follows.

$$s : d \to e \ ; \qquad d = s^*s, \ ss^* \leq e \ . \tag{21}$$

As with any presheaf topos we may consider representable objects of $\mathscr{B}(S)$: up to isomorphism, these are presheaves of the form

$$\widehat{e} = L(S)(_, e) \ ; \ \ \widehat{e}(d) = \{s \in S \mid s^*s = d, \ es = s\} \ ,$$

where d, e are idempotents of S. If we let

$$eS = \{es \mid s \in S\} = \{s \mid s = es\} = \{s \mid ss^* \leq e\} \ ,$$

then the étale S-set corresponding to $\widehat{e}$ is

$$eS \to E \ ; \ \ s \mapsto s^*s \ .$$

We refer to the presheaf

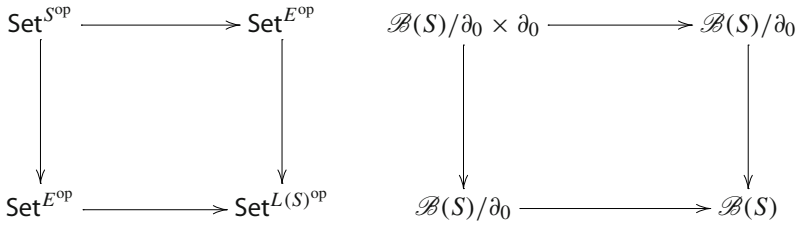
$$\partial_0(e) = \{t \in S \mid t^*t = e\}$$

as *the domain object* of $\mathscr{B}(S)$: the set $\partial_0(e)$ consists of all elements of S representing a morphism of $L(S)$ with domain e. Transition in ∂_0 is given by $\partial_0(s)(t) = ts \in \partial_0(s^*s)$. The étale S-set corresponding to ∂_0 is the domain map δ_0 (18), whose action by S is given by right multiplication. We emphasize that generally ∂_0 is not representable, unless S has a global unit u, in which case $\partial_0 = \widehat{u}$.

The domain object ∂_0 exhibits $\mathscr{B}(S)$ as an étendue: it is a globally supported torsion-free object. In fact, we have

$$\mathscr{B}(S)/\partial_0 \simeq \mathsf{Set}^{E^{\mathrm{op}}} \ .$$

Moreover, the general theory (13) specializes, showing that two topos squares



are equivalent, and they are pullbacks, where S is regarded as a poset in the usual way.

3.2 Prehomomorphisms to geometric morphisms

A *prehomomorphism* $\phi : S \rightarrowtail T$ of inverse semigroups is a function ϕ satisfying

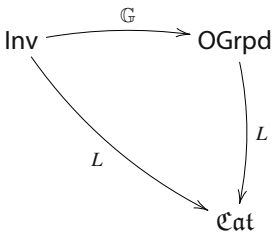
$$\forall s, t \in S \ , \ \ \phi(st) \leq \phi(s)\phi(t) \ . \tag{22}$$

A prehomomorphism is a *homomorphism* when (22) is an equality for all $s, t \in S$. Inverse semigroups and prehomomorphisms form a category Inv.

A prehomomorphism $\phi : S \rightarrowtail T$ induces a functor

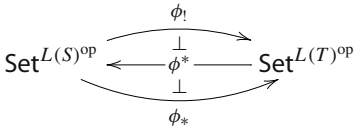
$$L(\phi) : L(S) \rightarrowtail L(T) \ .$$

Moreover, if ϕ is a homomorphism, then $L(\phi)$ preserves pullbacks (and conversely for inverse monoids). This assignment is part of a functor L depicted in the following diagram.



The top horizontal depicts Ehresmann's functor to ordered groupoids (18), sending an inverse semigroup to its inductive groupoid.

Thus, as with any functor of small categories there is a geometric morphism



(23)

associated with the functor $L(\phi)$; we simplify the notation writing ϕ^* instead of $L(\phi)^*$ and so on.

Example 4 Like any object of a category, an idempotent e of an inverse semigroup S has associated with it a geometric morphism $\overline{e} : \mathsf{Set} \to \mathscr{B}(S)$, which we call a representable point. It has inverse image functor

$$\overline{e}^* : \mathscr{B}(S) \to \mathsf{Set} ; \qquad \overline{e}^* F = F(e) .$$

In terms of étale S-sets, $\overline{e}^*$ assigns to $p : X \to E$ the fiber $p^{-1}(e)$.

3.3 Semigroup torsors

Throughout and as usual, S denotes an inverse semigroup. Every point $\mathsf{Set} \to \mathscr{B}(S)$ is a filtered colimit of its representable points $\overline{e}$ (Example 4). Following [2] we can account for these in terms of actions of S on a set X. Let $I(X) = I(\mathsf{Set}; X)$ denote the symmetric inverse monoid on a set X.

Definition 4 (*S-set*) A (left) S-set is a set X together with a prehomomorphism

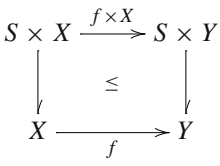
$$\mu : S \to I(X) .$$

An S-set (X, μ) is called *strict* when μ is a homomorphism.

Transposing $\mu \to I(X)$ we obtain a partial function $S \times X \rightharpoonup X$, written $(s, x) \mapsto sx$, such that sx is defined precisely when $\mu(s)(x)$ is defined.

Remark 6 We work with *left* S-actions here as opposed to the right actions used to define the objects of $\mathscr{B}(S)$. Of course some objects carry both a left and a right action: notably the canonical partial left action by S on the domain object ∂_0 makes it an S-torsor internal to $\mathscr{B}(S)$ (Eg. 6).

Definition 5 (*Morphism of S-sets*) A morphism $f : (X, \mu) \to (Y, \nu)$ of S-sets is a function $f : X \to Y$ such that the diagram of partial functions



commutes up to inequality, meaning that if sx is defined, then so is $sf(x)$ and $sf(x) = f(sx)$. If the diagram commutes strictly then we say that f a *strict* morphism.

If (X, μ) is an S-set, then for any $e \in E(S) = E$, the element $\mu(e) \in I(X)$ is an idempotent thus corresponding to a subset of X, which we denote eX. The diagram

$$\begin{array}{ccc} eX & \xrightarrow{\mu(s)} & X \\ {\scriptstyle\subseteq}\downarrow & & \\ X & & \end{array}$$

depicts the element $\mu(s) \in I(X)$ associated with an $s \in S$, where $e = s^*s$.

We say that an element $x \in X$ is *supported* by $e \in E$ when $x \in eX$, and that (X, μ) is *well-supported* (not to be confused the term globally supported) when every element of X is supported by some idempotent:

$$X = \bigcup_E \mu(e) = \bigcup_E eX \ . \tag{24}$$

Example 5 The so-called Wagner–Preston representation is the S-set $\mu : S \to I(S)$ assigning to $s \in S$ the partial map

$$\begin{array}{ccc} s^*sS & \xrightarrow{\mu(s)} & S \\ \downarrow & & \\ S & & \end{array}$$

such that $\mu(s)(t) = st$ for $t = s^*st$. This S-set is strict and well-supported.

Definition 6 (*S-torsor*) Let S be an inverse semigroup. An *S-torsor* is an S-set (X, μ) such that the following conditions hold:

1. X is non-empty;
2. the action is transitive: for all $x, y \in X$ there is $s \in S$ such that $x = sy$;
3. the action is locally free: for any $x \in X$ and $s, t \in S$, if $sx = tx$, then there exists $e \in E$ with $x \in eX$ and $se = te$.

A torsor is necessarily a well-supported strict action. Torsors and their morphisms form a category denoted $\mathsf{Tor}(S)$. Unlike in the group case $\mathsf{Tor}(S)$ is not necessarily a groupoid. However, each morphism is an epimorphism, and a morphism is strict if and only it is an isomorphism.

Example 6 For any $e \in E$, the subset $\mathbf{S}_e = \partial_0(e) = \{t \in S \mid t^*t = e\} \subseteq S$ is an S-set

$$S \times \mathbf{S}_e \rightharpoonup \mathbf{S}_e \ ; \quad (s, t) \mapsto \begin{cases} st & \text{if } s^*st = t \\ \text{undefined} & \text{otherwise} \, . \end{cases}$$

The S-set $\mathbf{S}_e$ is an S-torsor, said to be *principal*, corresponding to the point $\overline{e}$ (Example 4). Any S-torsor is a filtered colimit of such principal S-torsors.

The following proposition characterizes the torsor conditions in terms of properties of the prehomomorphism $S \to I(X)$.

Proposition 1 *Let S be an inverse semigroup. A (left) S-set $\mu : S \to I(X)$ is:*

1. *Transitive if and only if for every $m \in I(X)$ we have*

$$m = \bigvee_S m \wedge \mu(s); \quad \text{as in (42)} \tag{25}$$

2. *Locally free if and only if μ preserves binary infima* (assuming that S has binary infima).

Proof In the pseudogroup $I(X)$ the supremum of a compatible family of partial injections is its union, and the infimum of two partial injections is their largest common agreement. For 1, suppose that μ is transitive, and suppose that $m(x)$ is defined. There is s such that $sx = m(x)$, so $m \wedge \mu(s)(x)$ is defined, establishing (25). Conversely, assume that (25) holds for every m, and let $x, y \in X$. Define a partial map m such that $m(x) = y$. By (25) there is s such that $m \wedge \mu(s)(x)$ is defined. This means there is s such that sx is defined and $sx = m(x) = y$. For 2, assume that μ is locally free, and let $s, t \in S$. We have $\mu(s \wedge t) \leq \mu(s) \wedge \mu(t)$. Suppose that sx and tx are defined, and $sx = tx$. Then there exists $e \in E$ with $x \in eX$ and $se = te$. Then $r = se = te \leq s \wedge t$ and rx is defined. Therefore, $(s \wedge t)(x)$ is defined. Conversely, suppose that μ preserves binary infima, and suppose that sx and tx are defined, and $sx = tx$. We conclude that if $r = s \wedge t$, then rx is defined. Take $e = r^* r$ so that $se = r = te$. □

When $p : \mathsf{Set} \to \mathscr{B}(S)$ is a point of $\mathscr{B}(S)$, then $p^*(\partial_0)$ is a torsor, where as usual ∂_0 is the domain object. This is part of an equivalence of categories

$$\mathfrak{Top}(\mathsf{Set}, \mathscr{B}(S)) \simeq \mathsf{Tor}(S) .$$

This may be extended to arbitrary Grothendieck toposes. An S-torsor in a topos $\mathscr{E}$ is a globally supported object X of $\mathscr{E}$ with a transitive and locally free left action by $\Delta(S)$. (Recall that $\Delta : \mathsf{Set} \to \mathscr{E}$ is the constant objects functor, and therefore $\Delta(S)$ is a semigroup object in $\mathscr{E}$.) Notably the domain object ∂_0, which is an object of $\mathscr{B}(S)$ by virtue of its canonical right action, is a torsor internal to $\mathscr{B}(S)$ by virtue of its canonical (partial) left action. We have a category $\mathsf{Tor}(\mathscr{E}; S)$ of S-torsors in $\mathscr{E}$. Then there is an equivalence of categories

$$\mathfrak{Top}(\mathscr{E}, \mathscr{B}(S)) \simeq \mathsf{Tor}(\mathscr{E}; S) .$$

This expresses the sense in which $\mathscr{B}(S)$ classifies S-torsors.

4 The topos of a pseudogroup

We are prepared to examine the topos of a pseudogroup. After reviewing basic notions concerning pseudogroups we introduce and study sheaves on the left-cancellative category $L(S)$ of a pseudogroup S. In particular, we give a characterization of sheaves and describe the torsion-free generator. Ultimately, we shall show that the topos (27) of $\bigvee$-sheaves on $L(S)$ coincides with the topos $\mathscr{B}(S) = \mathscr{B}(G(S))$ of étale actions on the germ groupoid $G(S)$ (Definition 1).

4.1 Pseudogroups

Let us introduce some standard terminology and review the definitions of a pseudogroup and of a pseudogroup morphism [7,9].

For any two elements s, t of an inverse semigroup S we have $st^*t = ts^*s$ precisely when st^* is an idempotent. Indeed, if the former holds, then the codomain idempotent of st^*t is

$$st^*t(st^*t)^* = st^*t(ts^*s)^* = st^*ts^*st^* = ss^*st^*tt^* = st^* .$$

On the other hand, if st^* is an idempotent, then $st^* = (st^*)^* = ts^*$ so that we have

$$ts^*s = tt^*ts^*s = (ts^*s)t^*t = (st^*s)t^*t = (st^*st^*)t = st^*t .$$

In this case the common restriction $st^*t = ts^*s$ is the infimum $s \wedge t$ of s and t in the usual ordering of S. The codomain of $s \wedge t$ is the idempotent st^*. The elements s and t are said to be *left compatible*. The right compatible condition is $tt^*s = ss^*t$, which is so if and only if t^*s is an idempotent. In this case $tt^*s = ss^*t$ is again the infimum $s \wedge t$ with domain t^*s. Two elements s and t that are both left and right compatible are said to be *compatible*, in which case all four restrictions equal the infimum $s \wedge t$. A subset of an inverse semigroup is *(left) compatible* if every pair of elements of it are (left) compatible. (Compatibility is necessary for joins: in an inverse semigroup if a subset of it has a join, then that subset must be compatible.)

Definition 7 A *pseudogroup* is an inverse semigroup S in which every compatible subset of it $A \subseteq S$ has a join $\bigvee A$. Moreover, multiplication must distribute over joins in the sense that

$$s(\bigvee A) = \bigvee sA .$$

The other distributive law $(\bigvee A)s = \bigvee As$ holds by inversion. Since any two idempotents of a pseudogroup are compatible any subset of idempotents has a join in the pseudogroup, and the join is necessarily again an idempotent. Thus, the poset of idempotents of a pseudogroup forms a frame. (Conversely, a frame is a pseudogroup in which every element is idempotent.) Any two elements s, t of a pseudogroup (not necessarily left or right compatible) have an infimum we denote $s \wedge t$ as above. Multiplication distributes over infima: $r(s \wedge t) = rs \wedge rt$.

If A and B are subsets of an inverse semigroup S, then let

$$AB = \{ab \mid a \in A\,,\ b \in B\}\,; \quad A^* = \{a^* \mid a \in A\}\,.$$

Accordingly, $AA^* = \{st^* \mid s, t \in A\}$. If A is a left compatible subset of a pseudogroup, so that every element of AA^* is an idempotent, then

$$\bigvee AA^* = \bigvee \{ss^* \mid s \in A\}$$

because $st^* \leq ss^*$ for any left compatible s and t.

Lemma 3 *Suppose that $A \subseteq B$ are left compatible subsets of a pseudogroup S. Suppose also that*

$$\bigvee \{s^*s \mid s \in A\} = \bigvee \{s^*s \mid s \in B\}\,.$$

Then $\bigvee AA^* = \bigvee BB^*$.

Proof We have $\bigvee AA^* \leq \bigvee BB^*$ because $A \subseteq B$. For the reverse inequality, let $t \in B$. Then

$$\begin{aligned} tt^* = tt^*tt^* \leq t\left(\bigvee \{s^*s \mid s \in B\}\right)t^* &= t\left(\bigvee \{s^*s \mid s \in A\}\right)t^* \\ &= \bigvee \{ts^*st^* \mid s \in A\} \leq \bigvee AA^*\,, \end{aligned}$$

where the last inequality holds because $ts^*st^* = st^* \leq ss^*$ for every $s \in A$. This proves $\bigvee BB^* \leq \bigvee AA^*$. □

A homomorphism $\phi : S \to T$ of inverse semigroups takes a compatible subset to a compatible subset. If the inverse semigroups are pseudogroups, then we always have

$$\bigvee \phi(A) \leq \phi(\bigvee A) \tag{26}$$

for any compatible subset $A \subseteq S$.

Definition 8 A *pseudogroup homomorphism* $\phi : S \to T$ is an inverse semigroup homomorphism that moreover satisfies:

1. $\phi(u_S) = u_T$, where u denotes the global unit;
2. ϕ preserves compatible joins in the sense that for each compatible subset $A \subseteq S$ the inequality (26) is an equality.

As it happens preserving compatible joins is equivalent to preserving suprema of idempotents. Thus, an inverse semigroup homomorphism of pseudogroups is a pseudogroup homomorphism if and only if its restriction to idempotents is a frame morphism. However, a pseudogroup homomorphism does not necessarily preserve binary infima; when it does we call it a $\wedge$-*homomorphism* [9].

4.2 Sheaves

We turn to sheaves on $L(S)$, the left-cancellative category associated with S. Throughout, we fix an inverse semigroup S with idempotent set $E(S) = E$.

If e is an idempotent with associated representable presheaf $\widehat{e}$, then we may identify a sieve $R \subseteq \widehat{e}$ with a consistent collection of subsets

$$R(d) \subseteq L(S)(d, e) \ .$$

As subsets of S the $R(d)$ are disjoint. Their union

$$\bigcup_d R(d) \subseteq eS$$

is a right ideal of S, which we refer to as *a right ideal on* e. Moreover, a sieve R on e is generated by the idempotents in it in the sense that $R = \{dt \mid d \in R\ ,\ t \in S\}$, so that right ideals on e are in one-to-one correspondence with downclosed subsets of idempotents under e.

Consider a presheaf F on $L(S)$. We shall sometimes write $y = xs$ to mean that transition in F along a morphism $s : d \rightarrow\!\!\!\!\rightarrow e$ of $L(S)$ takes $x \in F(e)$ to $y \in F(d)$. We shall say that two elements $x \in F(d)$ and $y \in F(e)$ *are left compatible* when their mutual restrictions to $F(de)$ are equal. Likewise, a family of elements $\{x_d \in F(d)\}$ is said to be left compatible when every two elements of it are left compatible.

Lemma 4 *A presheaf F on $L(S)$ has the sheaf property for a right ideal R on e if and only if for every left compatible family $\{x_d \in F(d) \mid d \in R\}$ there is a unique $x \in F(e)$ such that $x \mapsto_F x_d$ for all $d \in R$.*

Proof A left compatible family $\{x_d \in F(d) \mid d \in R\}$ corresponds to a natural transformation $m : R \rightarrow\!\!\!\!\rightarrow F$. In fact, for any $s \in R(e)$ we have $ss^* \in R$, so that we may define $m(s) = x_{ss^*}s$. It follows that m so defined is natural. □

Suppose now that S is a pseudogroup, so that E is a frame. We still write $L(S)$ to mean the left-cancellative category associated with the underlying inverse semigroup of a pseudogroup S. However, in the case of a pseudogroup $L(S)$ naturally carries a Grothendieck (pre)topology, which we call its supremum topology, or $\bigvee$-topology.

Definition 9 ($\bigvee$-sheaves) A sieve R on e is $\bigvee$-covering if $\bigvee R = \bigvee\{d \in R\} = e$. A presheaf on $L(S)$ is a $\bigvee$-*sheaf* if it has the sheaf property for every $\bigvee$-covering sieve. The subtopos of $\bigvee$-sheaves is denoted

$$\mathsf{Sh}(L(S), \textstyle\bigvee) = \mathsf{Sh}(S) \ . \tag{27}$$

Example 7 A pseudogroup has a bottom idempotent denoted 0. The empty sieve $\emptyset$ on 0 is a covering sieve because of course $\bigvee \emptyset = 0$. A $\bigvee$-sheaf F therefore must satisfy $F(0) = 1$.

The following lemma slightly simplifies the sheaf condition. It is immediate from the fact that sieves are determined by downclosed sets of idempotents and from Lemma 4.

Lemma 5 *A presheaf F on $L(S)$ is a $\bigvee$-sheaf if and only if for every subset $A \subseteq E$ of idempotents and every left compatible family $\{x_a \in F(a) \mid a \in A\}$ there is a unique element $x \in F(\bigvee A)$ such that $x \mapsto_F x_a$ for all $a \in A$.*

Remark 7 We may recognize $\mathsf{Sh}(S)$ as a subcategory of étale S-sets. Indeed, by Lemma 5 an étale S-set $p : X \twoheadrightarrow E$ on a pseudogroup S is a $\bigvee$-sheaf if and only if for every subset $A \subseteq E$ and every family $\{x_a \in X \mid p(x_a) = a\}$, which is left compatible in the sense that $\forall a, b \in A,\ x_a b = x_b a$, there is a unique $x \in X$ such that $p(x) = \bigvee A$ and $x_a = xa$ for every $a \in A$.

Example 8 The idempotent centralizer presheaf

$$\delta(e) = \{s \in S \mid s^* s = e = ss^*\ ,\ \forall d \in E\ sd = ds\}$$

is a $\bigvee$-sheaf. To see this use the fact that an element of the centralizer is left compatible if and only if it is compatible.

4.3 $\mathsf{Sh}(S)$ as an étendue

We turn to finding a canonical torsion-free generator of $\mathsf{Sh}(S)$ for a pseudogroup S. As usual, ∂_0 denotes the domain presheaf on $L(S)$. In the case of an inverse monoid, such as a pseudogroup, we have $\partial_0 = \widehat{u}$, where u denotes the multiplicative unit or top idempotent. (But generally u is not terminal in $L(S)$.)

Lemma 6 *If R is a sieve on e, then a natural transformation $R \twoheadrightarrow \partial_0$ corresponds to a left compatible family $\{s_d \in \partial_0(d) \mid d \in R\}$.*

Proof A natural transformation $R \twoheadrightarrow \partial_0$ corresponds to a left compatible family $\{s_d \in \partial_0(d) \mid d \in R\}$ in the presheaf sense (for the presheaf ∂_0), which is just the same as a left compatible family in the semigroup sense explained in Sect. 4.1. □

In general, ∂_0 is not a $\bigvee$-sheaf. For instance, it is not in the case of a symmetric pseudogroup $I(X)$ (Remark 9).

Proposition 2 *If S is a pseudogroup, then ∂_0 is a $\bigvee$-separated presheaf.*

Proof An extension $s \in \partial_0(e)$ (if it exists) of a left compatible family

$$\{s_d \in \partial_0(d) \mid d \in R\}$$

over a downclosed subset $R \subseteq {\downarrow}e$ of idempotents, for which $\bigvee R = e$, is uniquely identified as the supremum of the family: $s = \bigvee s_d$. Note that the given left compatible family $\{s_d\}$ is necessarily compatible in this case. □

Remark 8 A subobject of a separated object is separated. Thus, a representable presheaf $\widehat{e}$ on $L(S)$ is $\bigvee$-separated as it is a subobject of ∂_0.

As ∂_0 is separated a single application of the $+$ construction produces its associated $\bigvee$-sheaf. Let $\partial = {\partial_0}^+ = \widehat{u}^+$ denote this sheaf. The unit of sheafification

$$\partial_0 \rightarrowtail \partial \tag{28}$$

is therefore injective. The sheaf ∂ is given by germs in the following sense.

Definition 10 Let e be an idempotent, and write G_e for the collection

$$G_e = \{A \subseteq S \mid A \text{ is left compatible and } \forall s \in A\ s^*s \leq e\}\ .$$

Let $\sim$ denote the equivalence relation on G_e generated by the relation $A \subseteq {\downarrow}B$. Then a *germ on an idempotent e* is an equivalence class $[A]$.

For the following lemma, whose proof is immediate from the definition of the $+$ construction, we note that the principal downset ${\downarrow}s$ of an element s is a compatible subset of S.

Lemma 7 *For any idempotent e we have:*

$$\partial(e) = \{\, \textit{germs on } e \,\}\ .$$

The unit (28) *maps an element s to the principal germ* $[{\downarrow}s]$.

More generally, the associated sheaf $\widehat{d}^+$ associated with an idempotent d may also be expressed in terms of germs:

$$\widehat{d}^+(e) = \{\, \text{germs } [A] : e \dashrightarrow d \,\}\ , \tag{29}$$

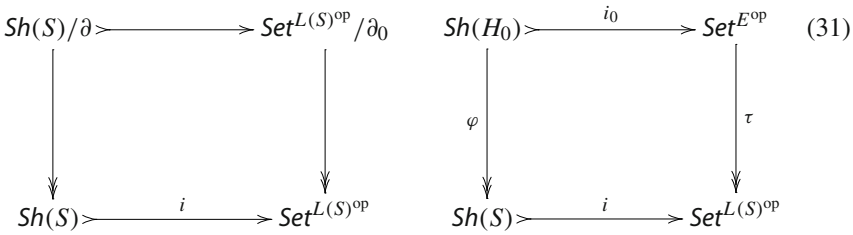
where by this notation we mean that $[A]$ is a germ on e, so that $\bigvee\{s^*s \mid s \in A\} = e$, and $\bigvee AA^* \leq d$ (well-defined by Lemma 3). Thus, a morphism $\widehat{e}^+ \rightarrowtail \widehat{d}^+$ of $\mathsf{Sh}(S)$ may be identified as a germ $e \dashrightarrow d$.

As a locale we shall denote the frame E of idempotents of a pseudogroup S by $H_0 = H_0(S)$, so that $\mathcal{O}(H_0) = E$. Let

$$\mathsf{Sh}(H_0) = \mathsf{Sh}(E, \textstyle\bigvee) \tag{30}$$

denote the topos of sheaves on the frame E.

Proposition 3 *For any pseudogroup S, the toposes* $\mathsf{Sh}(S)/\partial$ *and* $\mathsf{Sh}(H_0)$ *are equivalent: the equivalence identifies the sheaf on* H_0 *associated with an idempotent d and the subobject* $\widehat{d}^+ \subseteq \partial$ *of* $\mathsf{Sh}(S)/\partial$. *Moreover, the two squares*



(31)

are equivalent, and each is a pullback. Here τ denotes the geometric morphism engendered by the functor $E \to L(S)$. The inverse image functor φ^* simply restricts to E the action of a sheaf on $L(S)$.

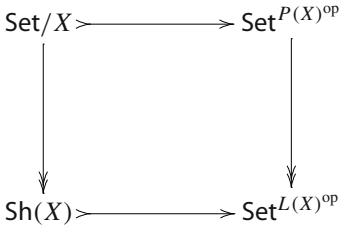
Proof The left square is a pullback, and the other square is also seen to be a pullback by an examination of the supremum topology that defines $\mathsf{Sh}(S)$. The upper right corners of the two squares are equivalent, and since both squares are pullbacks the conclusion follows. □

We see from the right-hand square in (31) that we may form the topos $\mathscr{B}(S)$ of a pseudogroup S in another equivalent way: it is the middle topos in the surjection-inclusion factorization of the diagonal in diagram (31). The surjection factor φ is étale.

4.4 The topos of a symmetric pseudogroup

We have already encountered the symmetric inverse semigroup $I(X) = I(\mathsf{Set}; X)$ in the theory of torsors for an inverse semigroup (Sect. 3.3). Naturally since $I(X)$ is a pseudogroup it likewise plays a similar role in our explanation of pseudogroup torsors (Sect. 5.3).

Let $L(X) = L(I(X))$ denote the left-cancellative category of $I(X)$. This is simply the category of subsets of X and injective maps between them. The frame of idempotents of $I(X)$ equals the poset $P(X)$ of subsets of X, and the topos of sheaves on this frame is Set/X. The topos $\mathsf{Sh}(X) = \mathsf{Sh}(I(X))$ of the pseudogroup $I(X)$ thus fits into the following pullback, instantiating diagram (31).



(32)

In order to explicitly determine $\mathsf{Sh}(X)$ consider that if X is non-empty, then the topos of presheaves on $L(X)$ has a generic point

$$p : \mathsf{Set} \to \mathsf{Set}^{L(X)^{\mathrm{op}}} \tag{33}$$

corresponding to the filtering functor $U : L(X) \twoheadrightarrow \mathsf{Set}$, such that $U(A) = A$ for any subset $A \subseteq X$. The point p corresponds to the $I(X)$-torsor X, which is the identity map $I(X) \twoheadrightarrow I(X)$. According to this correspondence we have the following topos pullback.

$$\begin{array}{ccc} \mathsf{Set}/X & \rightarrowtail & \mathsf{Set}^{P(X)^{\mathrm{op}}} \\ \downarrow & & \downarrow \\ \mathsf{Set} & \xrightarrow{p} & \mathsf{Set}^{L(X)^{\mathrm{op}}} \end{array} \tag{34}$$

Lemma 8 *For any non-empty set X the generic point p (33) is an inclusion.*

Proof We verify that the counit $p^* p_*(Y) \twoheadrightarrow Y$ is an isomorphism for any set Y. For any $A \subseteq X$ we have $p_*(Y)(A) = \mathsf{Set}(U(A), Y) = Y^A$. For any presheaf F on $L(X)$, the set $p^* F$ is the colimit of the functor

$$\mathbb{F} \twoheadrightarrow L(X) \twoheadrightarrow \mathsf{Set}\ ,$$

where $\mathbb{F}$ is the category of elements of F. Thus, $p^* p_*(Y)$ equals the colimit of the functor

$$\mathbb{Y} \twoheadrightarrow L(X) \twoheadrightarrow \mathsf{Set}\ , \tag{35}$$

where $\mathbb{Y}$ is the category of elements of $p_*(Y)$: a typical object of $\mathbb{Y}$ is a partial map

$$\begin{array}{ccc} A & \longrightarrow & Y \\ {\scriptstyle\subseteq}\downarrow & & \\ X & & \end{array}$$

and the functor (35) sends it to A. If X is non-empty, then the colimit of (35) is the set Y. □

Proposition 4 *If X is a non-empty set, then $\mathsf{Sh}(X) \simeq \mathsf{Set}$, identifying the subtopos $\mathsf{Sh}(X) \hookrightarrow \mathsf{Set}^{L(X)^{\mathrm{op}}}$ with the generic point p.*

Proof By Lemma 8, diagrams (32) and (34) are equivalent as the surjection-inclusion factorization of a geometric morphism is essentially unique. □

Remark 9 For $I(X)$ the domain presheaf ∂_0 assigns to a subset $A \subseteq X$ the set of all injective maps $A \twoheadrightarrow X$. Its associated $\bigvee$-sheaf ∂ therefore assigns to A the set of all maps $A \twoheadrightarrow X$. Under the equivalence $\mathsf{Sh}(X) \simeq \mathsf{Set}$ the sheaf ∂ corresponds to the set X. Thus, in diagram (1) the pseudogroup $I(X)$ passes to the coordinatized topos (Set, X).

Remark 10 When $X = \emptyset$ the category $L(X)$ consists of a single object and a single morphism. There is no generic point as the functor U is not filtering in this case. Nevertheless, the topos $\mathsf{Sh}(\emptyset)$ exists as it is the trivial one-object topos.

4.5 The germ groupoid

Resende [13] associates with a pseudogroup S an étale localic groupoid $G(S)$, whose spatial version is studied in [7,9]. On the other hand, we have shown how to associate with S an étendue $\mathsf{Sh}(S)$. Thus, we may recover in the usual way an étale localic groupoid we shall (temporarily) denote

$$H(S) = (H_0, H_1) \ ,$$

so that

$$\mathscr{B}(H(S)) \simeq \mathsf{Sh}(S) \ . \tag{36}$$

We argue now that the localic groupoids $G(S)$ and $H(S)$ are equivalent.

We have already introduced in (30) the locale H_0 whose frame $O(H_0)$ is the idempotent semilattice E of a given pseudogroup S. Then we define the locale H_1 as the kernel pair pullback of the étale geometric morphism φ (Prop. 3), depicted in the following pullback square.

$$\begin{array}{ccc} \mathsf{Sh}(H_1) & \xrightarrow{\ d\ } & \mathsf{Sh}(H_0) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathsf{Sh}(H_0) & \xrightarrow{\ \varphi\ } & \mathsf{Sh}(S) \end{array} \tag{37}$$

Thus, $H(S)$ so defined is an étale localic groupoid such that (36) holds.

We wish to identify the locale H_1 directly in terms of the pseudogroup S. Let X_E denote the locale whose frame $O(X_E)$ is the frame of downclosed subsets of the poset E. Likewise, X_S denotes the locale associated with the poset S. Then $X_{\mathbb{G}} = (X_E, X_S)$ is an étale localic groupoid where the domain map $d : X_S \to X_E$ is the locale map whose corresponding frame map is

$$d^* : O(X_E) \to O(X_S) \ ; \ d^*(\downarrow e) = \{s \mid s^* s \leq e\} \ ,$$

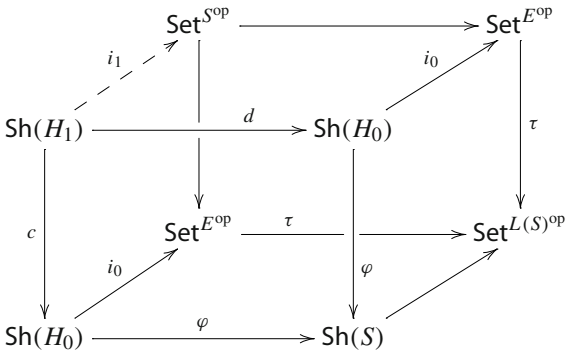
and likewise for the codomain c. The groupoid $X_{\mathbb{G}}$ is precisely the inductive groupoid $\mathbb{G}(S)$ (§ 2.3) associated with (the underlying inverse semigroup of) S, but regarded as a localic groupoid (by the Alexandroff functor). Moreover, we have

$$\mathsf{Sh}(X_E) \simeq \mathsf{Set}^{E^{\mathrm{op}}} \ , \qquad \mathsf{Sh}(X_S) \simeq \mathsf{Set}^{S^{\mathrm{op}}} \ , \qquad \mathscr{B}(X_{\mathbb{G}}) \simeq \mathsf{Set}^{L(S)^{\mathrm{op}}} \ .$$

We have a localic functor

$$i = (i_0, i_1) : H(S) \twoheadrightarrow X_{\mathbb{G}} \ .$$

The objects map $i_0 : H_0 \twoheadrightarrow X_E$ is already given in diagram (31). For the morphisms map i_1 consider the following diagram.



Front, bottom, back and right-hand sides are pullbacks, inducing i_1. Since this is a morphism between localic toposes, it is itself induced by a locale morphism, and this is the one we call i_1. The functor (i_0, i_1) induces the geometric morphism $\mathsf{Sh}(S) \twoheadrightarrow \mathsf{Set}^{L(S)^{\mathrm{op}}}$, which is a Cartesian inclusion. By Lemma 2, i is therefore a sublocalic full subgroupoid. The full and faithful condition says that

$$\begin{array}{ccc} H_1 & \xrightarrow{\ i_1\ } & X_S \\ \downarrow{\scriptstyle (d,c)} & & \downarrow{\scriptstyle (d,c)} \\ H_0 \times H_0 & \xrightarrow{\ i_0 \times i_0\ } & X_E \times X_E \end{array} \tag{38}$$

is a locale pullback (frame pushout). Moreover, we know that i_0 is the inclusion of H_0 as a sublocale into X_E, determined by the supremum topology on X_E.

Define a $\bigvee$-*ideal* to be a downclosed subset $D \subseteq S$ with the property that it contains the join of any compatible subset contained in D. For instance, a principal downset $\downarrow s$ is a $\bigvee$-ideal, and moreover $\downarrow s$ is a compatible subset. Conversely, if a $\bigvee$-ideal is itself a compatible subset, then it must be principal.

Proposition 5 *The frame $\mathcal{O}(H_1)$ may be identified with the collection of $\bigvee$-ideals, and the right adjoint i_{1*} in (38) with inclusion $\mathcal{O}(H_1) \hookrightarrow \mathcal{O}(X_S)$.*

Proof We shall identify the generating covering sieves $R \subseteq \downarrow s$ for the pullback locale (38). This pullback is generated by covering sieves

$$(A, B) = \{t \leq s \mid t^*t \in A \ , \ tt^* \in B \} \ ,$$

where $A, B \subseteq E$ are downclosed, such that $\bigvee A = s^*s$ and $\bigvee B = ss^*$. Then (A, B) is a downclosed subset, which is compatible as it is a subset of $\downarrow s$, such that $\bigvee(A, B) = s$. On the other hand, any compatible downclosed subset R with join $\bigvee R = s$ equals (A, B) for $A = \{t^*t \mid t \in R\}$ and $B = \{tt^* \mid t \in R\}$. Thus, the compatible downclosed subsets of S, which are multiplicatively stable, generate the pullback topology for H_1. Therefore, $\mathcal{O}(H_1)$ is the frame of $\bigvee$-ideals. □

We have thus shown that the locale H_1 associated with a pseudogroup S is none other than the locale of morphisms of the germ groupoid $G(S)$ defined by Resende (Definition 1). Thus we have recovered the germ groupoid $G(S)$ using sheaves.

Proposition 6 *For a pseudogroup S, the étale localic groupoid $H(S)$ coincides with its localic germ groupoid $G(S)$.*

Theorem 1 *For a pseudogroup S, the toposes $\mathscr{B}(S)$ and $\mathsf{Sh}(S)$ are equivalent, by an equivalence that identifies the germ presheaf ∂ with the domain object of $G(S)$.*

5 Classifying torsors

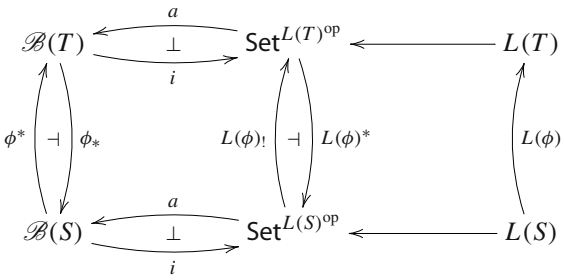
So far we have shown how to associate with a pseudogroup S a topos regarded either as the topos of étale actions on the germ groupoid $G(S)$ of the pseudogroup, or equivalently as a topos of sheaves on the left-cancellative category $L(S)$. We denote this topos $\mathscr{B}(S)$ (dropping $\mathsf{Sh}(S)$). In the present section we analyze how a (flat) pseudogroup morphism passes to a geometric morphism of toposes. This passage is contravariant as opposed to the covariant 'algebraic' direction for inverse semigroups. Ultimately, we use this to show that $\mathscr{B}(S)$ classifies pseudogroup torsors.

5.1 Pseudogroup morphisms to geometric morphisms

Our analysis of how a pseudogroup homomorphism passes to a geometric morphism is based on how a morphism of sites passes to a geometric morphism. Let us fix a pseudogroup homomorphism $\phi : S \to T$, and consider the functor

$$L(S) \xrightarrow{L(\phi)} L(T) \longrightarrow \mathsf{Set}^{L(T)^{\mathrm{op}}} \xrightarrow{a} \mathscr{B}(T) \quad ; \quad \Phi(e) = \widehat{\phi(e)}^{+} . \tag{39}$$

Here a denotes the associate sheaf functor. Since representables are separated we have $a(\widehat{\phi(e)}) = \widehat{\phi(e)}^{+}$ (Remark 8). The composite functor Φ is part of the following diagram:



(40)

The functor $L(\phi)^*$ depicted in (40) is defined by

$$L(\phi)^*(F) = F \circ L(\phi)^{\mathrm{op}} \; ; \quad L(\phi)^*(F)(e) = F(\phi(e)) \; .$$

(Notice that we used to write ϕ^* for this functor (23); however, we now wish to reserve the notation ϕ^*, ϕ_* for the induced functors between the classifying toposes of S and T qua pseudogroups.) Because $L(\phi)$ preserves covers, $L(\phi)^*(F)$ is a sheaf when F is. The direct image functor ϕ_* thus equals $L(\phi)^*$ restricted to sheaves: for any sheaf F on $L(S)$ we have

$$\phi_*(F) = F \circ L(\phi)^{\mathrm{op}} \; .$$

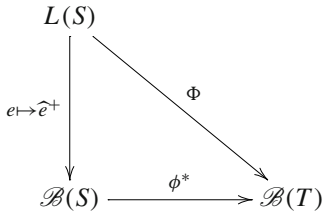
In general, $L(\phi)_!$ does not carry a sheaf to a sheaf: we must sheafify to get the inverse image functor ϕ^*. In any case, the composite

$$\phi^* = a \circ L(\phi)_! \circ i$$

is left adjoint to ϕ_*. Furthermore, Remark 1 explains that we have $\phi^* \circ a \cong a \circ L(\phi)_!$. For instance, for any $e \in E(S)$ we have

$$\phi^*(\widehat{e^+}) \cong a(L(\phi)_!\widehat{e}) \cong a(\widehat{\phi(e)}) \cong \widehat{\phi(e)}^+ = \Phi(e) \; . \tag{41}$$

In other words, the diagram



commutes and exhibits ϕ^* as the left Kan extension of Φ along $L(S) \longrightarrow \mathscr{B}(S)$. We emphasize that this left extension does not necessarily preserve finite limits, so that generally a geometric morphism is not obtained.

Lemma 9 *For a pseudogroup homomorphism $\phi : S \rightarrowtail T$, with induced $\phi^* \dashv \phi_*$ as above, we have $\phi^*(\partial_S) \cong \partial_T$.*

Proof Let u_S, u_T denote the maximal idempotents of S and T, respectively. We have $\phi(u_S) = u_T$ (by definition of pseudogroup homomorphism), and since $\partial_0 = \widehat{u}$ we have

$$\phi^*(\partial_S) = \phi^*(\widehat{u_S}^+) \cong \widehat{\phi(u_S)}^+ = \widehat{u_T}^+ = \partial_T \ . \qquad \square$$

Lawson and Lenz [9] refer to a pseudogroup homomorphism $\phi : S \rightarrowtail T$ as *hypercallitic* if it is $\wedge$-preserving, and if for every $t \in T$ we have

$$t = \bigvee \{t \wedge \phi(s) \mid s \in S\} = \bigvee_S t \wedge \phi(s) \ . \tag{42}$$

Note that the supremum does indeed make sense as the subset $\{t \wedge \phi(s) \mid s \in S\}$ is compatible. For example, a pseudogroup prehomomorphism $S \rightarrowtail I(X)$ satisfies (42) if and only if it is transitive (Proposition 1). We prefer and shall use the topos term *flat* instead of hypercallitic.

As before $\mathcal{O}(X_S)$ denotes the poset of downclosed subsets of S, and $\mathcal{O}(G(S)_1)$ denotes the frame of the germ groupoid of S, consisting of those downsets of S that are closed under compatible joins. A homomorphism $\phi : S \rightarrowtail T$ induces a direct image map

$$\phi[-] : \mathcal{O}(X_S) \rightarrowtail \mathcal{O}(X_T) \ ; \qquad \phi[D] = \{t \in T \mid \exists s \in D . t \leq \phi(s)\} \ .$$

Thus, $\phi[D]$ is the downclosure of the image of D under ϕ. For a subset $A \subseteq T$, let $\overline{A}$ denote the closure under compatible joins:

$$\overline{A} = \{ \bigvee A' \mid A' \subseteq A, A' \text{ is compatible } \} \ .$$

It is easy to see that $\overline{A}$ is downclosed when A is. Thus, $A \mapsto \overline{A}$ is a well-defined operation from $\mathcal{O}(X_T)$ to $\mathcal{O}(G(T)_1)$. Moreover, it preserves binary intersections: $\overline{D \cap D'} = \overline{D} \cap \overline{D'}$. (Of course, this simply expresses $G(T)_1$ as a sublocale of X_T.)

Lawson and Lenz [9] attribute the following observation to Pedro Resende. For the reader's convenience we include a sketch of an argument.

Lemma 10 *A pseudogroup homomorphism $\phi : S \rightarrowtail T$ satisfies (42) if and only if the $\bigvee$-extension*

$$\begin{array}{ccc} S & & \\ {\scriptstyle s \mapsto \downarrow s}\downarrow & \searrow^{s \mapsto \downarrow \phi(s)} & \\ \mathcal{O}(X_S) & \xrightarrow[D \mapsto \overline{\phi(D)}]{} & \mathcal{O}(G(T)_1) \end{array}$$

preserves the top element. Furthermore, the $\bigvee$-extension preserves binary infima if and only if ϕ does. Thus, a pseudogroup homomorphism is flat if and only if its corresponding $\bigvee$-extension depicted above is a frame morphism.

Proof The top element of $\mathcal{O}(X_S)$ is S itself, and likewise the top element of $\mathcal{O}(G(T)_1)$ is T. Thus to say that the $\bigvee$-extension preserves the top element is to say that $\overline{\phi(S)} = T$. If ϕ satisfies (42), then any $t \in T$ is the supremum of the compatible subset $\{t \wedge \phi(s) \mid s \in S\}$, giving $t \in \overline{\phi(S)}$. The converse is similar. For the statement regarding binary infima first note that for any downclosed subsets D, D' of S we have

$$\begin{aligned} a \in \phi[D \cap D'] &\Leftrightarrow \exists s \in D, t \in D'.a \leq \phi(s \wedge t) \\ &\Leftrightarrow \exists s \in D, t \in D'.a \leq \phi(s) \wedge \phi(t) \\ &\Leftrightarrow a \in \phi[D] \cap \phi[D'] \end{aligned}$$

where the middle step holds precisely when ϕ preserves binary infima. The result is now immediate from the fact that $D \mapsto \overline{D}$ always preserves binary infima. □

Proposition 7 *A pseudogroup homomorphism $\phi : S \to\!\!\!\!\rightarrow T$ is flat if and only if the left Kan extension ϕ^* preserves finite limits.*

Proof The direct approach, showing that Φ (39) is filtering if and only if ϕ is flat, is somewhat complicated and difficult (although not impossible). However, there is an easier indirect approach: if ϕ is flat, then by Lemma 10 we have a localic functor $G(T) \to\!\!\!\!\rightarrow G(S)$ that engenders a geometric morphism whose inverse image functor is the left extension ϕ^*. Conversely, if the left Kan extension ϕ^* preserves finite limits, then using Lemma 9 (which uses $\phi(u_S) = u_T$) there is a geometric morphism

$$\mathscr{B}(T)/\partial_T \times \partial_T \to\!\!\!\!\rightarrow \mathscr{B}(S)/\partial_S \times \partial_S \ ,$$

which corresponds to a locale morphism $G(T)_1 \to\!\!\!\!\rightarrow G(S)_1$. Its frame morphism is the $\bigvee$-extension in Lemma 10. The latter is therefore a frame morphism, so that the homomorphism ϕ is flat. □

Summing up, we have seen that a flat pseudogroup homomorphism $\phi : S \to\!\!\!\!\rightarrow T$ passes to a geometric morphism

$$\phi^* \dashv \phi_* : \mathscr{B}(T) \to\!\!\!\!\rightarrow \mathscr{B}(S) \ . \tag{43}$$

Moreover, the isomorphism from Lemma 9 tells us that ϕ is Cartesian (Definition 3). Thus, we have a (contravariant) functor, depicted in diagram (1),

$$S \mapsto (\mathscr{B}(S), \partial) \ ,$$

passing a pseudogroup S to an étendue paired with a globally supported torsion-free object, and passing a flat homomorphism to a Cartesian geometric morphism.

5.2 Toposes to pseudogroups

We have already mentioned in Sect. 1.1 how to pass from a coordinatized topos $(\mathscr{E}, X)$ to a pseudogroup $I(\mathscr{E}; X)$ of bisections on X. We claim that this construction is functorial with respect to Cartesian morphisms. Indeed, if

$$(\psi, m) : (\mathscr{F}, Y) \to (\mathscr{E}, X)$$

is Cartesian in the sense that m is an isomorphism, then we obtain a function

$$\phi = I(\psi, m) : I(\mathscr{E}; X) \to I(\mathscr{F}; Y)$$

defined by $\phi[m_1, m_2] = [m^{-1}\psi^* m_1, m^{-1}\psi^* m_2]$. This is easily seen to be well-defined on equivalence classes. Moreover, the following holds.

Proposition 8 *For a Cartesian morphism (ψ, m), the function $I(\psi, m)$ is a flat morphism of pseudogroups.*

Proof It suffices to prove this for the case where m is the identity. Then the comparison map $I(\mathscr{E}; X) \to I(\mathscr{F}; \psi^* Y)$ simply applies the inverse image functor ψ^* to an equivalence class $[m_1, m_2]$. Since ψ^* preserves pullbacks, this operation preserves the semigroup structure. Trivially, it also preserves the involution and the top idempotent. To see that it preserves suprema of compatible subsets, note first that applying ψ^* to a compatible family results in a compatible family. Moreover, since ψ^* preserves suprema of subobjects, it preserves suprema of compatible families. Finally, binary intersections are preserved for much the same reason. □

Let us denote by $\mathfrak{Top}^+_{Cart}$ the subcategory of $\mathfrak{Top}^+$ determined by the Cartesian morphisms, and by psGrp_{flat} the subcategory of psGrp determined by the flat homomorphisms.

Proposition 9 *The assignment $(\mathscr{E}, X) \mapsto I(\mathscr{E}; X)$ underlies a functor*

$$I : \mathfrak{Top}^+_{Cart} \to psGrp^{\mathrm{op}}_{flat} .$$

Combined with the observations from Sect. 5.1 we thus have functors

$$\mathsf{psGrp}^{\mathrm{op}}_{flat} \underset{\mathscr{B}}{\overset{I}{\rightleftarrows}} \mathfrak{Top}^+_{Cart}$$

as in diagram (1). Since $\mathscr{B}$ takes values in étendues we may restrict I to the subcategory of $\mathfrak{Top}^+_{Cart}$ whose objects are étendues. While every étendue is realized as the classifying topos of a pseudogroup, it is not the case that every morphism of étendues is realized as a pseudogroup homomorphism. We further clarify this matter in Sect. 5.3.

5.3 Pseudogroup torsors

We are in a position to study the notion of torsor for a pseudogroup. To begin we revisit Definition 4 assuming now that S is a pseudogroup.

Definition 11 An S-set (X, μ) is *a local action* when

$$\mu : S \longrightarrow I(X) \tag{44}$$

is a pseudogroup homomorphism (Sect. 4.1).

A local action by a pseudogroup is implicitly strict (as it is a homomorphism) in the sense that $(st)x$ is defined if and only if tx and $s(tx)$ are defined, in which case we have $(st)x = s(tx)$. Because a pseudogroup homomorphism preserves suprema a pseudogroup action is consequently *locally determined* in the sense that if $s = \bigvee s_i$ and sx is defined, then there is s_i (depending on x) such that $sx = s_i x$.

Definition 12 A *pseudogroup torsor* is a non-empty local action (44) that is transitive and locally free in the same sense as Definition 6. (Such an action is necessarily well-supported.)

Thus, a pseudogroup torsor (X, μ) is the same thing as a non-empty set X equipped with a flat pseudogroup homomorphism $\mu : S \longrightarrow I(X)$ (Proposition 7). Let $\mathsf{Tor}(S)$ denote the full subcategory of S-sets determined by the pseudogroup torsors. (Note that we use the same notation as for inverse semigroup torsors.)

A pseudogroup torsor (X, μ) therefore passes to a Cartesian geometric morphism

$$\mu : (\mathsf{Set}, X) \longrightarrow (\mathscr{B}(S), \partial) \ ,$$

keeping in mind Proposition 4. This passage is simply the usual way of passing a flat pseudogroup homomorphism, in this case $\mu : S \longrightarrow I(X)$, to its Cartesian geometric morphism. In particular, every pseudogroup torsor for S gives a point $\mu : \mathsf{Set} \longrightarrow \mathscr{B}(S)$.

Proposition 10 *For a pseudogroup S, we have an equivalence*

$$\mathfrak{Top}(\mathsf{Set}, \mathscr{B}(S)) \simeq \mathsf{Tor}(S)$$

that associates with a point $p : \mathsf{Set} \longrightarrow \mathscr{B}(S)$ the torsor $p^*(\partial)$, and to a torsor (X, μ) the geometric morphism induced by $\mu : S \longrightarrow I(X)$. Moreover, the square

$$\begin{array}{ccc} \mathsf{Set}/X & \longrightarrow & \mathscr{B}(S)/\partial \\ \downarrow & & \downarrow \\ \mathsf{Set} & \xrightarrow{\mu} & \mathscr{B}(S) \end{array} \tag{45}$$

is a topos pullback: $X \cong \mu^*(\partial)$.

Proof We know that points $\mathsf{Set} \to \mathsf{Set}^{L(S)^{\mathrm{op}}}$ correspond to torsors for S *qua inverse semigroup*. It thus suffices to verify that such a point p factors through $\mathscr{B}(S) \hookrightarrow \mathsf{Set}^{L(S)^{\mathrm{op}}}$ precisely when the corresponding (X, μ) is a local action. This follows from the analysis of the correspondence between flat homomorphisms and geometric morphisms. □

It is known that every étendue has a site whose underlying category is left-cancellative. In fact, we have the following.

Proposition 11 *If X is a torsion-free generator of an étendue $\mathscr{E}$, then $\mathscr{E}$ is equivalent to $\mathscr{B}(I(\mathscr{E}; X))$, where $I(\mathscr{E}; X)$ denotes the pseudogroup of bisections on X.*

Proof The site $L(I(\mathscr{E}; X))$ for $\mathscr{B}(I(\mathscr{E}; X))$ is precisely the site of subobjects of X and monomorphisms between them given by Kock and Moerdijk [6]. □

As a consequence we have the following.

Corollary 1 *The category of points of an étendue is equivalent to $\mathsf{Tor}(S)$ for some pseudogroup S.*

Remark 11 In a sense we have come full circle as further examination reveals that $\mathsf{Tor}(S)$ with isomorphisms only is an essentially small groupoid that as a matter of fact coincides with the underlying groupoid of the topological germ groupoid studied by Lawson and Lenz [9]. Thus, we have not only recovered the localic germ groupoid by Resende, but also the topological version.

To conclude our analysis of the classifying topos of a pseudogroup we consider generalized points $\mathscr{E} \to \mathscr{B}(S)$. To this end we define an S-torsor in $\mathscr{E}$ as a globally supported object X that carries a transitive, locally free action $\mu : \Delta(S) \to I_{\mathscr{E}}(X)$ that furthermore is a local action. (These conditions can be expressed in geometric logic so that they make sense in any topos.) S-torsors internal to $\mathscr{E}$ and their morphisms form a category denoted $\mathsf{Tor}(\mathscr{E}; S)$. By transposing across the adjunction $\Delta \dashv \Gamma$, an S-torsor (X, μ) is equivalently given as a flat pseudogroup homomorphism $S \to I(\mathscr{E}; X) = \Gamma(I_{\mathscr{E}}(X))$. In particular, when X is a globally supported object of a topos $\mathscr{E}$, then X is canonically an $I(\mathscr{E}; X)$-torsor. Indeed, the action map $\Delta(I(\mathscr{E}; X)) \to I_{\mathscr{E}}(X)$ is simply the transpose of the identity on $I(\mathscr{E}; X)$. This canonical $I(\mathscr{E}; X)$-torsor structure on X corresponds to a geometric morphism

$$\eta : \mathscr{E} \to \mathscr{B}(I(\mathscr{E}; X)) \ .$$

Proposition 12 *For any globally supported object X of a topos $\mathscr{E}$, there is a pullback square*

$$\begin{array}{ccc} \mathscr{E}/X & \longrightarrow & \mathscr{B}(I(\mathscr{E}; X)/\partial \\ \downarrow & & \downarrow \\ \mathscr{E} & \xrightarrow[\eta]{} & \mathscr{B}(I(\mathscr{E}; X)) \end{array}$$

In particular, η underlies a Cartesian morphism $(\mathscr{E}, X) \to (\mathscr{B}(I(\mathscr{E}; X)), \partial)$. Also $\mathscr{B}(I(\mathscr{E}; X)/\partial$ is the localic reflection of $\mathscr{E}/X$.

Proof It suffices to note that $X = \eta^*(\partial)$ by virtue of the fact that η is by construction the unique geometric morphism with that property. □

Note that, according to Proposition 11, the map η is an equivalence precisely when the globally supported X is torsion-free.

Now suppose that $\phi : S \to I(\mathscr{E}; X)$ is a flat pseudogroup homomorphism. Applying $\mathscr{B}$ and composing with the morphism η gives a Cartesian geometric morphism:

$$(\mathscr{E}, X) \xrightarrow{\eta} (\mathscr{B}(I(\mathscr{E}; X)), \partial) \xrightarrow{\phi} (\mathscr{B}(S), \partial_S) \quad .$$

Conversely, a Cartesian geometric morphism $(\phi, m) : (\mathscr{E}, X) \to (\mathscr{B}(S), \partial)$ gives rise to a torsor $\phi^*(\partial)$ in $\mathscr{E}$ given by the flat homomorphism

$$S \xrightarrow{\epsilon} I(\mathscr{B}(S); \partial) \xrightarrow{I(\phi, m)} I(\mathscr{E}; \phi^*\partial)$$

where ϵ is the canonical counit morphism: this is the pseudogroup version of the Wagner–Preston representation of S. It maps $s \in S$ to the equivalence class of the two monomorphisms

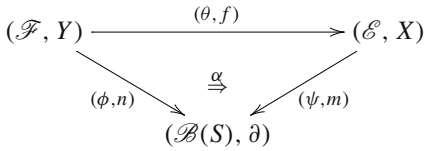
$$\begin{array}{ccc} \widehat{s^* s} & \xrightarrow{\ \widehat{s}\ } & \partial \\ \downarrow & & \\ \partial & & \end{array}$$

in $\mathscr{B}(S)$, where the natural transformation $\widehat{s}$ is given by (post) composition with s. This map is seen to be injective and bijective on idempotents, but generally not surjective: given two idempotents d, e of S there can be germs $[A] : d \to e$ which are not generated by a single element of S. However, the map ϵ is a weak equivalence in the sense that it induces a Cartesian equivalence of toposes $\mathscr{B}(I(\mathscr{B}(S); \partial)) \to \mathscr{B}(S)$ inverse to the unit map η. One may also verify that ϵ induces an isomorphism of germ groupoids $G(S) \cong G(I(\mathscr{B}(S); \partial))$.

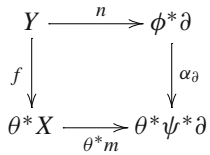
Thus, the two above constructions are mutually inverse (up to isomorphism), yielding the following correspondence:

$$\frac{\text{Cartesian } (\psi, m) : (\mathscr{E}, X) \to (\mathscr{B}(S), \partial)}{\text{flat homomorphisms } S \to I(\mathscr{E}; X) \ .}$$

Let us explain how to turn this correspondence into a categorical equivalence. Consider the lax slice category $\mathfrak{Top}^+ /\!/ (\mathscr{B}(S), \partial)$ on the Cartesian maps. Objects are the Cartesian maps $(\psi, m) : (\mathscr{E}, X) \to (\mathscr{B}(S), \partial)$, while a morphism is a 2-cell



in $\mathfrak{Top}^+$: this data determines a commutative square as follows.



Since m and n are isomorphisms, α is completely determined by f. Note that the torsor map f need not be an isomorphism, so that (θ, f) is not necessarily Cartesian. On the other hand we have for each topos $\mathscr{E}$ a category $\mathsf{Tor}(\mathscr{E}; S)$. Moreover, this is contravariantly functorial in $\mathscr{E}$: a geometric morphism $\theta : \mathscr{F} \to \mathscr{E}$ induces a functor

$$\theta^* : \mathsf{Tor}(\mathscr{E}; S) \to \mathsf{Tor}(\mathscr{F}; S) \ ,$$

sending an S-torsor X in $\mathscr{E}$ to the one θ^*X in $\mathscr{F}$. We thus have a functor

$$\mathsf{Tor}(-; S) : \mathfrak{Top}^{\mathrm{op}} \to \mathfrak{Cat} \ .$$

We write $\mathsf{TOR}(S)$ for its total category. A morphism

$$(\theta, f) : (\mathscr{F}, Y) \to (\mathscr{E}, X)$$

of $\mathsf{TOR}(S)$ is then a geometric morphism $\theta : \mathscr{F} \to \mathscr{E}$ together with a morphism of torsors $f : Y \to \theta^*X$. We have the following.

Theorem 2 *For a pseudogroup S there is an equivalence of categories*

$$\mathfrak{Top}^+ /\!/ (\mathscr{B}(S), \partial) \simeq \mathsf{TOR}(S) \ .$$

Corollary 2 *For any pseudogroup S and topos $\mathscr{E}$ there is an equivalence*

$$\mathfrak{Top}(\mathscr{E}, \mathscr{B}(S)) \simeq \mathsf{Tor}(\mathscr{E}; S)$$

that is natural in $\mathscr{E}$.

Finally, we note that the analysis presented implicitly gives a notion of bimodule between pseudogroups. As already remarked, not every geometric morphism between toposes $\mathscr{B}(T) \to \mathscr{B}(S)$ is induced by a morphism of pseudogroups $S \to T$; however, we may define the following.

Definition 13 Let S, T be pseudogroups. Then a *flat* (S, T)*-bimodule* is an étale T-set X that is a sheaf (Remark 7) equipped with a left action $\mu : S \to\!\!\!\!\rightarrow I(\mathscr{B}(T); X)$ that is flat (i.e., a flat pseudogroup homomorphism μ). A *morphism of bimodules* $f : X \to\!\!\!\!\rightarrow Y$ is a morphism of étale T-sets that is equivariant with respect to the left S-actions.

Thus a flat (S, T)-bimodule X is an object X of $\mathscr{B}(T)$ equipped with a $\Delta(S)$-torsor structure. By Corollary 2 this corresponds to a geometric morphism $\mathscr{B}(T) \to\!\!\!\!\rightarrow \mathscr{B}(S)$. Moreover, morphisms of bimodules $X \to\!\!\!\!\rightarrow Y$ correspond to geometric transformations. Thus we find that there is a bicategory whose objects are pseudogroups, whose 1-cells are flat bimodules, and whose 2-cells are bimodule morphisms. Furthermore, this bicategory is bi-equivalent to the 2-category of étendues, geometric morphisms, and geometric transformations.

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