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On the Foundations of Geometric Algebra

Filip Bár

Munich, 2011

Dedicated to Hermann Günther Grassmann
on the occasion of his 201th birthday

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1 Introduction

The ideas of Hermann G. Grassmann on the theory of extension in general and the construction of his exterior algebra in particular are widespread through all branches of mathematics. Besides the pure algebraic use of the exterior algebra in the theory of determinants of commutative rings there are, for example, the exterior algebra of differential forms in differential geometry, grassmannians as classifying spaces in K-theory or as schemes in algebraic geometry, $\bigwedge$-rings in representation theory or the Grassmann-Cayley algebra in combinatorics and invariant theory. It is for the last example that Grassmann's exterior algebra still enjoys some popularity as a "geometric algebra" in projective geometry.

On the other hand, there exist interesting and fruitful but far less popular applications of Grassmann's algebra as a "geometric algebra" to physics such as to screw theory and hence to the theory of rigid bodies. In these fields the term "geometric algebra" is rather associated with certain types of Clifford algebras than with Grassmann's algebra. This is mainly due to David Hestenes, who has been popularizing the use of the Clifford algebra of Euclidean vector space of three dimensions and the Clifford algebra of the four-dimensional Minkowski vector space as "geometric algebra" and "spacetime algebra" respectively, since around 1960.

Although we find many of Grassmann's ideas widespread or reinvented, it is in general only little known about Grassmann's original theory the "Ausdehnungslehre" (extension theory) and his motivations for developing it. One such motivation (compliant with G. W. Leibniz's wish for a "geometric analysis") was to develop a geometric calculus that combines the prospects of synthetic and analytic reasoning in geometry[1]. Broadly, the term "synthetic" refers to synthetic geometry, where one derives geometric theorems by axiomatical and logical methods dealing with the geometric objects of interest directly. The term "analytic" refers to analytic geometry, where one uses systems of coordinates to translate geometric problems into equa-

[1] Cf. [Gra47], [Rot14][section 1 in Einleitung], [Gra78][Anhang III, II.7, II.8]. Note that in [Gra78][II.7] Grassmann uses the terms "synthetic" and "analytic" in two related but different ways. When he refers to the methods, he means "synthetic" and "analytic" in the general but not the particular way both terms are used in philosophy (cf. the discussion of "analytic truth" in [Bos09]). When he refers to the operation, he means them in his own specific way introduced in [Gra78][§5].

tions between numbers to solve them. With his work on the "Ausdehnungslehre" Grassmann wanted to lay out the foundation for a new mathematical discipline, which he felt could, among other things, accomplish that goal.

At that time, in fact, many mathematicians conducted research in the spirit of Leibniz's idea of a "geometric analysis" and tried to develop geometrical calculi in form of particular geometric algebras. Some famous examples beside Grassmann are W. R. Hamilton with his theory of quaternions [Ham66], A. F. Möbius with his barycentric calculus [Möb44], W. K. Clifford with his work on biquaternions [Cli71] and, inspired by Grassmann, with his work on Clifford algebra [Cli78], as well as E. Study with his "Geometrie der Dynamen" [Stu04], which contains the application of dual quaternions to Euclidean motions[2].

An analysis of some of the main and guiding ideas of several of those mathematicians reveals a surprising depth that the author feels not being fully reflected within their particular results; at least not in the way we use those results today. The author even believes that given the technical possibilities of nowadays mathematics those mathematicians would arrive at much more powerful results today than it was possible for them in their time just from the sole possibility to exploit their ideas more thoroughly.

Let us take Leibniz's wish for a "geometric analysis" for example. From a nowadays point of view on geometry and with emphasize put on logic, one could state the respective task like this: search for algebraic structures, that represent the logic behind geometry and have a decent amount of computable and expressive power. When one puts emphasize on the algebra, it could sound like this: search for possibilities to model geometry algebraically that don't rely on coordinate methods alone. Such goals are indeed pursued in one way or another today. The first task, for instance, touches upon aspects of topos theory; the second task was and is still pursued in (abstract) algebraic geometry.

Grassmann's algebra does answer Leibniz's wish to a certain extent[3] and yields a way to algebraize (affine) geometry in the spirit of Leibniz's "geometric analysis". On the conceptual level, this way is the opposite of the approach of basic algebraic geometry of affine varieties[4].

From what we can learn from Grassmann about his abstract concepts in [Gra78] he considers invariants of certain basic geometric objects of interest instead of rings of "coordinate functions", and studies algebraic structures on those sets of invariants that reflect constructions with the underlying geometric objects to a reasonable de-

[2] This application can be also found in [Stu91][II, §2]. Note that dual quaternions are what Clifford calls "parabolic biquaternions" in [Cli71]. Study himself refers to dual quaternions as "System dualer Größen" (system of dual quantities) (cf. [Stu04][II, §23]).

[3] We take account of all the operations Grassmann developed and used in [Gra78] and [Gra62]. Besides the well-known outer or progressive product these are the regressive product, the "Ergänzung" and the inner product that is derived from the last two.

[4] We will explain this in chapter 4 in detail.

gree. The algebraic structures that allow to write down equations and functions reflect the analytical aspect. The invariants and their interrelationship with their underlying geometric objects reflect the synthetic aspect. (Here the interrelationship of the algebraic manipulation to the respective geometric construction is of particular interest.) We will demonstrate this kind of approach to the construction of Grassmann's exterior algebra as a geometric algebra in appendix A.

Of course, this kind of thinking is nothing seriously new when looked at from todays point of view. (For example, in topology and geometry it gives the intuition behind certain homology arguments.) What is essentially new, is that this kind of thinking arises on such a basic level like that of affine spaces. As a matter of fact, it does not only arise, but it is also fruitful.

Motivated by the above thoughts and ideas we want to follow Grassmann's line of thought and make a first account for the program of a systematical understanding and investigation of geometrical algebras. In this thesis we want to give a foundation for elementary geometric algebra in the sense of Leibniz, Grassman and many other mathematicians in the nineteenth century (cf. [Rot14][footnote 2, pp. 1282]). To the best knowledge of the author such an account doesn't exist, since only special algebras have been studied as geometric algebras so far, but only little has been done for the understanding of what a geometric algebra actually "is" in general. Regarding this we will see to which categories of algebras a geometric algebra should belong at least. On one hand, this will allow us to see well-known categories of algebras like Hopf-algebras, for example, in an elementary geometric context. On the other hand, it gives directions for constructions of new geometric algebras and, hopefully, for more applications.

The term "elementary" refers to affine spaces and we will be concerned with geometric algebras of affine space here. Indeed, Grassmann and the other mathematicians we mentioned above always begin their considerations with an underlying affine space of points. Yet in nearly all cases they fix an origin and change to vector spaces when they want to introduce the algebraic notions. This is true for all but Grassmann and Möbius in [Gra78] and [Möb44]. In fact, the exterior algebra that Grassmann considers in the second part of [Gra78] is an exterior algebra of an affine space. This is one reason for our choice of affine spaces in the first place. The other reason is for affine geometry being a prominent type of model geometry in physics, and because we conceive physics as the major field for applications of geometric algebra. We will elaborate on this in the subsequent section.

The foundations of geometric algebra will be developed with the help of category theory, since we consider category theory the most appropriate framework for this task. Our main goal is to find a reasonable category of algebras (in a wider sense) that is built from the category of affine spaces in a meaningful way. Moreover, it has to be designed in such a way that Grassmann's algebra of an affine space is an object in that category.

We will take the most natural approach to reach this goal. This approach amounts to the following: begin with the study of the category of affine spaces and study its monoidal structures, consider the category of internal monoids for any such reasonable structure, and search for equivalent categories whose objects are built from linear structures. It will turn out that the right monoidal structure is the one induced by the internal hom; we call it the affine tensor product structure. The equivalent category will turn out to be the category of augmented, unital, associative, linear algebras. Showing this constitutes the main line of this thesis.

However, we don't intend to run through it strictly, for this work is meant to be of fundamental character. Instead, this main line serves just for orientation and we will deviate from it most of the time. In doing so, we will study the properties of the category of affine spaces in rather detail. Thereby we will exhibit the geometric meaning of some constructions and properties, draw some attention to the application of the affine tensor product to the Fréchet derivative in analysis, and we will discuss the dual affine space functor from a geometric and conceptual point of view.

With the help of the barycentric calculus we shall clarify on the relationship between a projective space and an affine space by emphasizing the role of the respective barycentric calculus as an algebraic intermediate; we will in particular argue that a projective space with a given projective hyperplane does not yield an affine space in a natural way.

We will also study the structural relationship between vector spaces and affine spaces in rather detail. Since the category of affine spaces isn't linear, the category of vector spaces can't be equivalent to the category of affine spaces. So both structures have to be considered differently and the frequently encountered practice to identify an affine space with a vector space by fixing an origin arbitrarily is unjustified from the structural point of view. On the other hand, we will see the category of vector spaces to arise in form of natural categories built from the categories of affine spaces in many ways, each one of those having a particular algebraic and/or geometric meaning. For example, in chapter 4 we will see it as the category of Eilenberg-Moore algebras of the monad related to the functor that comes from the construction of the barycentric calculus. The algebro-geometric meaning of such an algebra is that of "precoordinatization" of its underlying affine space.

Furthermore, in chapter 5 we will see that the category of vector spaces and (the algebraic presentation of) the category of non-empty affine spaces together constitute the slice category of vector spaces over the base field. In fact, even more is true: the burnside rig of this slice category that is derived from its closed monoidal structure is augmented by $\mathbb{N}$ (as a rig) and the augmentation ideal corresponds to the subcategory of vector spaces.

There will be two other peculiarities pointing towards directions of further research. After establishing our main result we will observe that the affine tensor algebra, i.e., the free internal monoid in the category of augmented algebras, has a

natural structure of a differential graded (DG) algebra. This holds also for Grassmann's algebra of an affine space and we will see it also for the Clifford algebra of an affine space that we will introduce as a further example of a geometric algeba of an affine space in the last chapter. Therefore, we will give a first account for the relationships of affine spaces and (connected) DG algebras in chapter 6. We will see, in particular, how affine spaces may be considered as connected DG algebras and, vice versa, connected DG algebras as affine spaces. This will yield a further algebraic presentation of affine spaces that, however, does not come from a natural internal structure of the (geometrically defined) category of affine spaces like it is the case for our main result on augmented algebras.

The second peculiarity is the Clifford algebra of an affine space itself, and, in particular, its spin group that acts on the respective affine space by translations.

With the Clifford algebra and Grassmann algebra of affine space we will have arrived at the end of this work. For geometric algebra, however, this is in fact the true starting point. Indeed, what we will have presented so far should be called general or abstract geometric algebra. It serves to understand the general structures, to present the concepts with the basic geometric ideas and to suggest directions of future research in general as well as fields of applications in particular. Regarding a particular or more concrete topic of geometric algebra, besides the two main examples we will present nothing the author is aware of that could be referred to in this way. However, we will at least give some directions for future research. Regarding geometric algebras of affine spaces we want to point out two. The first one concerns itself with the construction of a reasonable *geometric algebra of an affine metric space*. The second one concerns itself with so called *dynamical algebras*. It takes up Lawvere's fundamental observations in [Law96] on how to make precise Grassmann's simple law of change and how to relate the Grassmann algebra of affine space to the internal dynamics of affine spaces. The simple law of change determines thereby the notion of an invariant of dynamical/geometric origin, which is encoded by the binary products of points of the Grassmann algebra of an affine space. In fact, in this way the invariant does not determine the Grassmann algebra only, but a whole class of algebras. The Clifford algebra of the affine space in question belongs to this class, too. Moreover, this yields a general way of how to define an invariant by a set of "symmetries" and a class of algebras that encode this invariant within their binary products.

Undoubtedly, the main goal of this work is to revive the idea of a "geometric analysis" of Leibniz to fuse analytic and synthetic reasoning for geometry within what we call a geometric calculus. This idea was pursued besides analytic, synthetic and algebraic geometry in the 19th and the beginning of the 20th century, but, unfortunately, it has been mainly forgotten since then.

We approach the "geometric analysis" by taking up and recalling the ideas of

Hermann Günther Grassmann, whose work on the "Ausdehnungslehre" provided the first and major step in this field. Like it was Grassmann's goal for the "Ausdehnungslehre" in [Gra78] we want to give the first attempt for a foundation of geometric algebra of affine spaces here. With such a goal set we have two main, but very different kinds of tasks. On the one hand, we have to convey the philosophical, conceptual and geometric ideas. On the other hand, we have to provide many technical results in a breadth that is not too wide and not too narrow and in a reasonable abstract setting. Certainly, this may be demanding for most of the readers, who want to grasp the ideas and key results quickly, but are not interested in every result or every technical detail. An orientation regarding the results and different approaches taken can be found in the last section of this introduction that provides an overview and a programmatic outline. The introductions of each chapter, and sometimes of the sections also, try to provide such an overview of ideas and results for this particular chapter/section, too. Moreover, to allow for a quick referencing and orientation we have linked the references and referenced very often. Finally, for readers who are interested in the geometric ideas and particular geometric algebras only, we suggest the reading of the introduction of chapter 3, the beginning of section 3.1, the introduction of chapter 4, section 4.1, the beginning of section 4.2, the introduction of chapter 5, section 5.1, section 6.6, chapter 7 and the appendix A.

1.1 On the Importance of a Geometric Calculus in Physics

> [...] and especially in electrodynamics, where we have to deal with a number of physical quantities, the relations of which to each other can be expressed far more simply by a few words of Hamilton's, than by the ordinary equations.
>
> *(J. C. Maxwell in his Treatise)*

David Hestenes and others have already demonstrated that geometric algebra in the sense of a Clifford algebra of a three-dimensional Euclidean vector space and a four-dimensional Minkowski vector space has fruitful applications to classical and quantum physics (in particular for the discussion of Maxwell electrodynamics and spin; cf. e.g. [Hes03], [Hes97], [Hes90]). Yet, what we feel unsatisfactory about this approach, is that it doesn't take account of affine geometry at all. This is in contrast with the fact that the basic physical model of space is a three-dimensional (affine) Euclidean space, the basic physical model of spacetime a four-dimensional (affine) Minkowski space and, regarding non-relativistic physics, a four-dimensional (affine) Galilei space[5]. In considering only vector spaces, Hestenes et al thus choose a preferred origin implicitly, and they have to discern position vectors and displacement vectors carefully.

But are there any interesting applications of the geometric algebras of the respective affine spaces that depend on the affine structure at all? Yes, there are. We know, for instance, of applications of the Grassmann algebra of a three-dimensional Euclidean space to screw theory (cf. [Hyd88] and section 6.6.1). Moreover, since one can consider a Clifford algebra of a metric affine space, too, i.e. an affine space whose vector space of displacements is equipped with a non-degenerate symmetric bilinear form, and since the Clifford algebra of the vector space of displacements can be embedded in the respective Clifford algebra of the metric affine space as an algebra, one can expect many results of Hestenes et al to carry over to the affine case[6]. Due to the richer structure of the affine case one can even expect that those results may be extended and new applications will emerge. However, this goes beyond the scope of the present thesis and will be discussed elsewhere.

[5] Following Arnold [Arn89][section 1.2 B] a Galilei space or Galilei spacetime is a four-dimensional affine space A over $\mathbb{R}$ whose associated vector space V_A is equipped with a linear form $\tau : V_A \to \mathbb{R}$ and $\ker \tau$ is a Euclidean vector space with scalar product g.

[6] A closer inspection of the most obvious approach to incorporate a metric in the Grassmann algebra and hence a Clifford algebra of a metric affine space (as it is for instance employed in [Bro09] explicitly, and in [Gra62] and thus in [Hyd88] implicitly) reveals that this approach amounts to a choice of a preferred origin, too. So, strictly speaking, the Grassmann and Clifford algebra of an affine metric space doesn't account for the affine structure as well. Up to know, however, the author does not know of any other method to include a metric in such a way that the resulting geometric algebra is promising for applications. (For more details we refer to the discussion in chapter 7.)

Instead of attempting to justify the importance of geometric algebras in physics by stressing particular applications, we will take a much more general point of view and argue for the importance of a geometric calculus in physics in general. Rather than that there are applications we shall clarify why there should be applications and how applications should arise. Furthermore, the subsequent discussion shall also clarify on the use of category theory and our search and use of universally constructed geometric algebras like the Grassmann algebra and the Clifford algebra.

The importance of a geometric calculus in physics can be derived from two different aspects. The first aspect is due to the general link of geometry and observable, empirical reality[7], which we just call "reality" from now on. This link is established through the archetypes of space, shape, length and movement[8]. Because of this link, a theory written in the language of geometry is accessible to an ontology[9] and hence to a physical interpretation of the theory, as well as to experimental tests in principle.

For it involves conceptual matters of a physical theory, the link of geometry and reality seems to be especially important for fundamental questions in physics. Although it deserves more attention, we won't discuss this topic any further here.

The second aspect refers to "calculus" in "geometric calculus". In physics one wants to do computations and predictions and, therefore, needs a calculus in one way or another. There are certain preferences though. The more comprehensive a calculus is, the better it is. Moreover, one wants to have a calculus that is adapted to the physical way of thinking as much as possible. With a well adapted calculus the transition from physical reasoning on reality to computations in the theoretical model and vice versa becomes smooth; the calculus hence becomes an efficient tool in developing and analyzing physical theories. A non geometrical example of one trend towards such a calculus is the calculus with Heaviside and delta functions employed by engineers in signal-processing and by physicist in field theory. Yet, without mathematical rigor in the sense of proper non-contradicitng rules for calculation, such a calculus suffers from arbitrariness which can lead to physical imprecision, especially if the physical argument is of computational nature[10].

[7] Here we understand by "geometry" geometry in a (reasonably) broader sense, not only restricted to two or three dimensions. But it is of course the two and three-dimensional (Riemannian) geometry which forms the basis for this link.

[8] An archetype refers on one hand to an "idea" or "form" in Plato's sense and on the other hand to C. G. Jung's use as a mental form imprinted in the mind of every human being since birth. For example, when we refer to the archetype of space we mean the basic and inuitive understanding of what space is and how to perceive space common to every human being. What we do not mean are notions of space in mathematics and physics. These are built upon the archetype of space, but they are in no way primary.

[9] In this context "ontology" means no more than the correspondence of certain objects from the theory to certain objects of the observable, empirical reality. Especially, we don't mean to imply by "ontology" any physical reality of the ontologized objects whatsoever.

[10] Of course, this rigor can be achieved by abiding to the computational rules of distribution

The physical way of thinking deals primarily with physical objects and not with their mathematical representations. For example, one tends to think about the electron as an entity of reality and not as a specific wave function, field resonance or tuple of coordinates in configuration space. Thus, when using any mathematical language for a physical theory, a calculus that manipulates the mathematical objects and structures related to reality directly is as well adapted to the physical way of thinking as the employed mathematical language and the employed ontology of the theory. In particular, when one uses a geometric language for a physical theory, one would want to have a calculus that manipulates the geometric objects and structures related to reality directly. In this strict sense one would need to develop either a separate calculus for each possible choice of ontology, or a universal one that is not dependent on the ontology at all. Considering the wealth of geometric objects and structures as well as the advancement of geometry as a mathematical discipline, the first approach is not feasible in general. As for the second approach, its independence of ontology would mean by the above argumentation that it is (if at all) only poorly adapted to the physical way of thinking. Of course, in its strict sense the second approach isn't feasible, too. (This is due to the same argument as for the first one.) However, if we consider it in a weaker sense and in the context of category theory, then the question on a universal calculus for a given "gros" category of geometric spaces or a geometric space modelled by a "petit" category is meaningful and can be made precise. Such a universal solution will still be poorly adapted to the physical way of thinking in general, but one can see it as a firm ground to develop specialized calculi dependent on the chosen ontology or just simply as a calculus depending only on the chosen geometric language for the physical theory. An example for the latter point of view would be a geometric calculus.

Indeed, a geometric calculus yields a partial synthesis of the ontology dependent and ontology independent notion of a calculus. This synthesis is implied by the first aspect of the usage of a geometric language for a physical theory: namely the link of geometry and reality. Even if one restricts oneself to ontologies utilizing the geometric language only then it may be not a complete synthesis though, for the geometric notions may not be enough to establish a physically reasonable ontology. To make this last point more obvious, we consider the example of a point particle in Newtonian mechanics. The language of unparametrized curves in a fixed three-dimensional affine space A is not able to express how the particle moves and where it is actually located at a given time. Because of that the language is not rich enough to support a reasonable ontology for the theory. A sufficiently strong language would be, for example, the language of parametrized curves, e.g. that of C^2-maps $\mathbb{R} \to A$.

All in all we can see that a geometric calculus seems to be a reasonable approach

theory and/or of Colombeau algebra. The second one becomes important when multiplying distributions, which is common practice in physics.

for a calculus employed in theoretical physics. It is as adapted to the physical way of thinking as is the chosen geometric language that the calculus encodes. Regarding the richness of geometric structures it can also be hoped for that there are particular geometric calculi that are comprehensive enough for physical computations. This hope is corroborated by the results of Grassmann in [Gra78], Whitehead in [Whi98], John Browne in [Bro09] and most notably by the work of Hestenes in applications of geometric algebra to physics.

It remains to answer for which geometric language we should begin our search for a geometric calculus. Once again physics gives a direction by its most simple and most vivid geometric models of reality: the three-dimensional (affine) Euclidean space, the four-dimensional (affine) Galilei space and the four-dimensional (affine) Minkowski space.

1.2 Overview and Programmatic Outline

In this section we want to give an overview and a programmatic outline of the whole thesis. We present the aims, the ideas together with the basic approaches and the main results. We will follow the order of the chapters for most of the time.

The first (second) chapter is concerned with preliminaries. We begin with the obligatory technical remarks, in which we fix the notation and introduce the basic categorical notions we shall need right from the beginning. These are categories, functors, natural transformations, some special morphisms like isomorphisms, monos and epis, products of categories and bifunctors, functor categories, dual categories, contravariant functors and the duality principle, comma categories, limits and colimits.

The overall approach we follow in this thesis is to consider categories within set theory. We choose this to be ZFC + Universe following [Mac98]. For we will define functors by universal properties several times (the most important example being the affine tensor product bifunctor) the axiom of choice will be important for us, but not essential.

We then proceed with the discussion of the category of vector spaces, where we put basic facts about vector spaces in a categorical context. This serves mainly for the convenience of the reader, since she/he will see the categorical notions in a direct connection with a familiar structure.

It is also in this section that we will introduce the concepts and notions of adjoints, Cartesian closedness, internal hom and a (closed) symmetric monoidal category. Although we basically mean by an internal hom the internal hom in the Eilenberg-Kelly sense, we will only use it as the concept "hom-set with structure" throughout this thesis (with one exception).

We also introduce several, and from the point of view of the standard literature on linear algebra untypical kinds of approaches to familiar structures like the linear

tensor product and bilinear maps, which we shall take up for similar constructions on affine spaces later on.

The rest of the preliminary chapter specializes more on concepts and notions related to monoidal structures. For us, the most important one is the notion of an internal monoid of a monoidal category and its relation with a monoidal functor and monoidal natural transformations; that is the lift of monoidal functors from monoidal categories to the respective categories of internal monoids. It will be important for one of our main results that gives an algebraic presentation of affine algebras in terms of linear structures.

Besides that we introduce monads as internal monoids and their Eilenberg-Moore algebras as modules of external left actions. We will need the notion of a monadic functor to formulate our type of duality for the dual affine space functor.

We also prepare for the examples of geometric algebras that we will encounter in the final chapter and that are the motivating examples for the whole thesis. For this purpose we introduce graded vector spaces and graded algebras from a functorial point of view. We consider $\mathbb{N}$-graded vector spaces as functors from the discrete category $\mathbb{N}$ to the category of vector spaces and introduce $\mathbb{N}$-graded algebras as internal monoids of a convolution-like monoidal structure on the functor category of $\mathbb{N}$-graded vector spaces and, equivalently, as the category of monoidal functors from the additive monoid $(\mathbb{N}, +, 0)$ to the category of vector spaces. In addition, we stress the fact that a free internal monoid is an $\mathbb{N}$-graded internal monoid typically and naturally. Such gradations and particularly $\mathbb{N}$-gradations are important for geometric algebras and can be seen, for example, in geometric applications of the Grassmann or exterior algebra.

The third chapter is concerned with the category of our main interest: the category $\mathrm{Aff_K}$ of affine spaces over a field K with affine maps as morphisms. We begin the chapter with a reflection on the geometric notion of an affine space and arrive at a definition of an affine space as a torsor of a vector space, where we distinguish between the point set and the vector space of displacement vectors. We define our category of affine spaces, respectively. Our approach doesn't restrict to consider affine spaces of finite dimensions only and also allows for an affine space which point set is empty.

We then proceed to study this category by the natural functors that exist to and from the two categories of sets Set and vector spaces $\mathrm{Vct_K}$ over a field K. These are basically the forgetful functor $U_S : \mathrm{Aff_K} \to \mathrm{Set}$, which forgets the simply transitive action of the vector space of displacement vectors on the set of points, the forgetful functor $U_V : \mathrm{Aff_K} \to \mathrm{Vct_K}$, which forgets the underlying set of points and the self-action functor $S : \mathrm{Vct_K} \to \mathrm{Aff_K}$, which forgets the zero of a vector space, i.e., which considers a vector space as an affine space over itself. In doing so, we establish many properties of $\mathrm{Aff_K}$ from Set and $\mathrm{Vct_K}$ like completeness, a characterization of monos and epis, a terminal one-point space to be a separator, an affine line to be a coseparator, etc. as well as abstract existence and impossibility results on adjoints

for functors to and from linear categories. Since it is relevant for applications in geometry and physics, special attention is paid to the full subcategory of finite dimensional affine spaces and whether and what the universal constructions mean geometrically. Regarding the similarity of Aff_K and Vct_K we learn that Aff_K is not a linear category and thus by no means equivalent to Vct_K.

From our abstract existence results we obtain the existence of a left adjoint for U_S and S. For U_S this left adjoint is easily identified as the "free" affine space functor F_S, which maps a set to the affine space freely generated by it. With the help of this functor we can deduce the existence of small coproducts and together with an affine version of the fundamental theorem on homomorphisms we can conclude the cocompleteness of Aff_K. The coproducts yield also the first universal construction which looks neither like the coproduct in Vct_K nor the coproduct in Set but is a "fusion" of both. A dimension formula for coproducts in the finite dimensional case allows us to see that Aff_K is not distributive. This implies that Aff_K also doesn't look like Set and is, in particular, not a topos.

The next topic of study concerns the monoidal structures. From the non-distributivity we know that Aff_K can't be Cartesian closed. Yet we can observe that each hom-set is an affine space in a natural way and so get an internal hom functor $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$. With the help of this internal hom functor and the theorem on parametrized adjunctions we can deduce an affine tensor product bifunctor and respective structure morphisms such that this yields a symmetric closed monoidal structure on Aff_K alltogether. This structure is determined up to a strong monoidal isomorphism by the internal hom functor. This is the only case, where we have to keep in mind the structure morphisms of an internal hom as defined by Eilenberg-Kelly in [EK65], for the preceding statement can't be made precise otherwise.

Since such an abstract description of an affine tensor product is clearly unsatisfactory from the geometric point of view, we proceed by studying the underlying point sets and maps. This is done with the help of the notion of a biaffine map. We follow our internal hom approach to define what a biaffine map is. This constitutes one of the instances of the untypical kind of approach we mentioned before, and it is probably the most untypical one. Nevertheless, this has the advantage that it yields the tensor product as a representing object for biaffine maps for free. We then characterize biaffine maps as the to be expected maps of two arguments that are affine in each argument, respectively. This being established, we can deduce the underlying set of points of an affine tensor product together with its action on maps, the unit and counit of the affine hom-tensor adjunction, and the underlying maps of the structure morphisms of the symmetric monoidal structure of the affine tensor product.

It is at this point that we can see the similarities and differences between the linear tensor product and the affine tensor product very clearly: a point of the affine tensor product uniquely determines and is uniquely determined by a pair of points together with a linear tensor of displacement vectors, which is not necessarily

pure. Moreover, in contrast to the linear tensor product, which is purely bilinear, an affine tensor has linear parts. Those linear parts depend on the expansion with respect to a chosen pair of points, however, and are not uniquely determined just by the point in the affine tensor. Restated for biaffine maps this says that biaffine maps have besides a uniquely determined bilinear part, linear parts which depend on the point one fixes for the other argument.

We present three applications for the affine tensor product and biaffine maps. The first application concernes affine quadrics, the second uses the affine tensor product with its hom-tensor adjunction to reformulate the (finite dimensional) Fréchet-derivative in analysis in terms of affine approximations, and the third shows that the generalized product rule of analysis is actually due to biaffine maps having linear parts. This shall illuminate how biaffine maps arise in analytical geometry all the time without us being aware of it.

We turn to the dual affine space functor D next. It maps each affine space A to the affine space of affine maps $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$. From the structural point of view, this is a very interesting object. Even though it resembles the dual vector space functor very much, it doesn't behave like it. It behaves like the contravariant power-set functor instead. Exploiting this analogy leads, amongst other things, to the respective notion of duality: the dual affine space functor D has its opposite D^{op} as a left adjoint and is monadic. This is a duality, since the assertion of monadicity implies that the dual category of affine spaces is equivalent to the category of Eilenberg-Moore algebras of the monad induced by D and D^{op}.

We then try to understand the functor D geometrically, conceptually and from the point of view of logic. Each one of these aspects is motivated by what we have before us: geometry, since $\mathrm{Aff}_{\mathrm{K}}$ is a category of geometric structures; conceptuallity, since D yields a duality; logic, since D behaves like the contravariant power-set functor.

The multiplicative closure of $D(A)$ in the set of maps from A to K with the pointwise ring structure turns out to be the ring of polynomial functions for a finite dimensional A. This presupposes that we introduce a Cartesian system of coordinates of course, but on the other hand it shows how D induces such a ring purely from geometrical data. Stated differently, the multiplicative closure of $D(A)$ is the geometrical notion of polynomial functions on A.

From the point of view of geometric algebra, the affine tensor structure and its relation to the linear tensor structure is very interesting, because it is to be expected that canonical geometric algebras of affine spaces arise as internal monoids for this structure. We call such internal monoids affine algebras. Yet, since geometric algebras will be very likely constructed as actual algebras, we need an algebraic presentation of affine algebras. Everything up from chapter 4 will bring us step by step towards such a presentation. We will state and prove it in the final chapter.

Chapter 4 itself is concerned with the Möbius-Grassmann barycentric calculus of points. This purely geometric construction that is in the spirit of Leibniz's geometrical calculus turns out to be a left adjoint $\mathcal{B}$ to the self-action functor S,

which we left out of the discussion up to now. Using $\mathcal{B}$ to represent an affine space algebraically will lead us to geometric algebras like the Grassmann algebra of affine space. However, using the ring induced by $D(A)$ will lead us to affine varieties of algebraic geometry instead (at least it will do so for a field K with characteristic zero). Here we can see in which sense both approaches to algebraize geometry, i.e., the one from algebraic geometry and the one from geometric algebra differ.

With the barycentric functor $\mathcal{B}$ at hand, we observe that although it doesn't preserve products it preserves those products as pullbacks. We then deduce that it preserves non-empty J-limits for a connected J, and thus any non-empty J-limit as the limit of the diagram of shape J that is connected to the terminal one-point space. With this the barycentric functor is seen "as continuous as it possibly can be". Its right adjoint, the self-action functor $\mathcal{S}$, preserves all J-colimits for a connected J, and any J-colimit as the colimit of the diagram of shape J connected to an initial trivial vector space. It is thus "as cocontinuous as it possibly can be". In particular, $\mathcal{S}$ is monadic, so the category of vector spaces is equivalent to the category of Eilenberg-Moore algebras of the monad $S\mathcal{B}$. As one would expect, each such algebra carries an algebro-geometric meaning: It can be considered as the "precoordinatization" of its underlying affine space by its associated vector space, i.e., the "multiplication" of this algebra yields a vector space structure on the underlying affine space.

Instead of an affine space, we can equally well use its barycentric calculus. Although it is a vector space, we can do so, because we can distinguish points and vectors. This distinction is possible due to the weight: vectors have weight zero and points have weight one. The structure behind the weight is a linear form $\omega : \mathcal{B}A \to K$, which we call the weight form and that arises as the action of $\mathcal{B}$ on the terminal map $A \to *$. This gives us a linear algebraic structure that is equivalent to the one of an affine space as we defined it. The main result of chapter 5 is the theorem that the category $\mathrm{Aff_K}$ is strong symmetric monoidally equivalent to the full subcategory of epis of the slice category $\mathrm{Vct_K}/K$. The monoidal structure on $\mathrm{Vct_K}/K$ and this subcategory is the natural one induced from the linear tensor structure on $\mathrm{Vct_K}$. We call the epis $\omega : V \to K$ non-degenerated augmented vector spaces.

Besides the non-degenerated augmented vector spaces the other objects in $\mathrm{Vct_K}/K$ are the zero maps $0_V : V \to K$. They form a full subcategory as well, and this subcategory can be seen to be isomorphic to the category $\mathrm{Vct_K}$ itself. The isomorphism becomes even strict monoidal when we take the zero map $0_V : K \to K$ as the identity object for the induced monoidal structure. But note that this is not the identity object of the monoidal structure on $\mathrm{Vct_K}/K$, since this is the identity map 1_K.

The zero maps arise as the associated vector spaces to the non-degenerate augmented vector spaces. In fact, when we want to describe all of the structure of an affine space A in terms for its barycentric calculus $\omega : \mathcal{B}A \to K$, this amounts to

write down the short exact sequence

$$V_A \hookrightarrow \mathcal{B}\,A \xrightarrow{\omega} K.$$

All in all the slice category $\mathrm{Vct_K}\,/K$ is seen to be built up from the two full subcategories of zero maps and epis. These subcategories intersect only on the initial zero maps $0 \to K$ which for the first category correspond to trivial vector spaces and for the second to the empty affine spaces. By our identifications of those categories with $\mathrm{Vct_K}$ and $\mathrm{Aff_K}$ respectively, we can say that the slice category is built up from affine spaces and vector spaces. This is one of the reasons why we consider $\mathrm{Vct_K}\,/K$ as the proper stage to study affine spaces in their algebraic presentation.

As already stated in the introduction, besides the main line of development we shall clarify on the relationship of projective spaces with affine spaces and barycentric calculi. This is done with the hope that a barycentric calculus won't be mistaken with homogeneous coordinates or a projective space anymore. Furthermore, we will observe the very interesting fact that the Burnside rig of $\mathrm{Vct_K}\,/K$ whose multiplicative structure is induced by the tensor-product is actually an augmented rig, i.e., there is a rig homomorphism to $\mathbb{N}$. The isomorphism classes of vector spaces constitute the kernel of this homomorphism and the preimage of 1 are the isomorphism classes of non-empty affine spaces. This should be compared to our main result, which comes next.

From the point of view of geometric algebras, the key result of chapter 4 is that the functor $\mathcal{B}$ is not only a left adjoint, but strong symmetric monoidal. The key result of chapter 5 is that the equivalence is strong symmetric monoidal. With this, it follows that the category of affine algebras, i.e. the category of internal monoids of $\mathrm{Aff_K}$, is equivalent to the category of internal monoids of $\mathrm{Vct_K}\,/K$. These internal monoids are determined to be augmented K-algebras, however. So, finally, we get the algebraic presentation of affine algebras in terms of linear structures we wished for: they are equivalent to augmented K-algebras.

This result has the following surprising consequence: any bialgebra can be considered as an affine algebra. We have, in particular, that any Hopf-algebra "is" an affine algebra and its antipode "is" an affine map. On one hand this gives us a large class of examples of affine algebras. On the other hand it yields a basic geometric interpretation of these algebraic structures.

A further surprise awaits us, when we consider the affine tensor algebra of A, which is the algebraic presentation of a free internal monoid of $\mathrm{Aff_K}$. Here the weight form will be seen to determine a unique graded derivation ∂ of degree -1 that squares to zero. In other words, the weight form determines a unique structure of a differential graded (DG) K-algebra and vice versa. We thus proceed to elaborate on the relationship between affine spaces and (connected) DG algebras in more depth. Thereby, we will see that the category of affine spaces may be regarded as a full coreflective subcategory of connected DG algebras, in which sense an affine space can generate DG algebras, and, in particular, determine the DG structure uniquely.

Finally, we study two examples of affine algebras, which both may be considered as geometric algebras of affine space. The first example is Grassmann's exterior algebra of an affine space A. Although it is an affine algebra, it has A not as an affine subspace. It inherits the differential from the affine tensor algebra and is thus a DG algebra itself.

The second example of an affine algebra is the Clifford algebra of an affine space A. It can be constructed either with the help of a quadratic form induced by the weight form of A, or by deforming the exterior product of the Grassmann algebra of A by its differential ∂. This time the algebra has A as an affine subspace of generators. It is also a DG algebra with the same differential as the Grassmann algebra, but the gradation is only a $\mathbb{Z}_2$-gradation. Indeed, this Clifford algebra of affine space is very similar to the Grassmann algebra of affine space from which it is derived from. Yet it has an interesting extra property: its spin group acts by translations on A.

As one would guess from the elementariness of our topic, the author is not the first one to derive such results. A decent research in the literature over the course of this work has revealed that particular results were already known to others way before the author. The equivalence of non-degenerated augmented vector spaces and affine spaces was already observed by Schanuel (cf. [Car89]). The author does, however, not know whether Schanuel considered also the monoidal structures and the monoidality of this equivalence. The use of the slice category $\mathrm{Vct}_\mathrm{K}/K$, the affine tensor product and the coproduct are also used by Lawvere in [Law96]. It was also Lawvere who pointed out that Grassmann's "Ausweichung" is a differential and the Grassmann algebra of affine space thus a DG algebra in that paper. This, together with Chevalley's book [Che56] gave the author the main lead to consider such DG structures in the more broader context of graded affine algebras and in particular for the affine tensor algebra. Biaffine maps were already considered in [MB99] and actually used for the discussion of affine quadrics. Yet, Mac Lane and Birkhoff don't mention the affine tensor product. However, as indicated in [EK65] it seems that Mac Lane was aware of it.

The Clifford algebra of affine space and the property of its spin group was discovered by the author at an early stage of his work, but without any knowledge of the augmentation and DG structure. It served as a motivating example for the general study of geometric algebras. Until now the author has not found any reference that points out this Clifford algebra construction and property of its spin group in the literature.

2 Preliminaries

This chapter serves a twofold purpose. Firstly, as we will make heavy use of basic category theory in the whole thesis, it is for the convenience of the reader and shall either recall or introduce him to basic notions, concepts, constructions and theorems of category theory in the context of categories of familiar structures like vector spaces, sets and algebras (over vector spaces). All of the notions and concepts introduced here will be used later on to study the categories of affine spaces, affine algebras and others. Secondly, as we will need it for the development of algebraic presentations of affine structures later, we will reformulate definitions and results of linear algebra within the language and concepts of category theory.

2.1 Technical Remarks

We begin with the obligatory technical remarks to fix the notation, terminology and basic definitions to be used in this thesis. Many of the abstract notions that are introduced here only formally will be explained in more detail and for the particular familiar structures mentioned above in the subsequent sections. Because of that we will in general refrain from providing examples in this section.

All in all our notations and definitions will be based primarily on [Mac98]; but we shall also use the notations, conventions and definitions of [MM92], [LS97] and [AHS04].

2.1.1 Categories in ZFC + Universe

Like in [Mac98] we will make use of axiomatic set theory as an underlying framework of category theory. We will thus not work axiomatically but with set-theoretic interpretations, i.e. models. For this reason we will give a definition of a category as a set-theoretic structure[1]. We choose this approach for it is more suited for our aim to study particular categories of set-theoretic structures like the category of affine spaces.

[1] For an axiomatization of category theory cf. [Law65].

Definition 2.1.1 (category)**.** A *category* C consists of a set $\operatorname{Obj} C$ of *objects*, a set $\operatorname{Mor} C$ of *morphisms* and the following maps

$$\begin{aligned} \operatorname{dom} : \operatorname{Mor} C \to \operatorname{Obj} C, &\quad f \mapsto \operatorname{dom} f, \\ \operatorname{cod} : \operatorname{Mor} C \to \operatorname{Obj} C, &\quad f \mapsto \operatorname{cod} f, \\ 1 : \operatorname{Mor} C \to \operatorname{Obj} C, &\quad c \mapsto 1_c, \end{aligned}$$

$$\begin{gathered} \circ : \{(f,g) \in \operatorname{Mor} C \times \operatorname{Mor} C \mid \operatorname{dom} f = \operatorname{cod} g\} \to \operatorname{Mor} C, \\ (f,g) \mapsto f \circ g. \end{gathered}$$

For a morphism $f \in \operatorname{Mor} C$ we call $\operatorname{dom} f$ its *domain* and $\operatorname{cod} f$ its *codomain*. The map $\circ$ is called *composition* and for an object $c \in \operatorname{Obj} C$ we call 1_c the *identity morphism* for c. The following identities are required to hold for all $f, g, h \in \operatorname{Mor} C$ and $c \in \operatorname{Obj} C$ such that $\operatorname{dom} f = \operatorname{cod} g$ and $\operatorname{dom} g = \operatorname{cod} h$:

$$\begin{aligned} &\operatorname{dom}(f \circ g) = \operatorname{dom} g, \qquad \operatorname{cod}(f \circ g) = \operatorname{cod} f, \qquad \operatorname{dom} 1_c = \operatorname{cod} 1_c = c; \\ &f \circ (g \circ h) = (f \circ g) \circ h. \qquad \text{(associativity law)} \end{aligned}$$

Finally, for all $f \in \operatorname{Mor} C$ we require

$$f \circ 1_{\operatorname{dom} f} = 1_{\operatorname{cod} f} \circ f = f. \qquad \text{(left- and right-unit law)}$$

Let C be a category. A pair of C-morphisms (f, g) will be called *composable* if $\operatorname{dom} f = \operatorname{cod} g$, i.e. if (f, g) lies in the domain of the composition map. For each pair of objects $c, d \in \operatorname{Obj} C$ we define the *hom-set*

$$C(c,d) := \hom(c,d) := \{f \in \operatorname{Mor} C \mid \operatorname{dom} f = c \text{ and } \operatorname{cod} f = d\}.$$

The underlying set theory we assume here is the one given by the Zermelo-Fraenkel axioms together with the axiom of choice, which is usually abbreviated as ZFC[2]. Additionally, to be able to deal with the size issues of category theoretical constructions in a convenient and effective way we assume the existence of a set $\mathcal{U}$, a so called *universe*, which has the following properties (cf. [Mac98, section I.6]):

$$\begin{aligned} &X \in Y \in \mathcal{U} \Rightarrow X \in \mathcal{U}, && (\mathcal{U}\,1) \\ &(X \in \mathcal{U} \text{ and } Y \in \mathcal{U}) \Rightarrow \{X,Y\}, (X,Y), X \times Y \in \mathcal{U}, && (\mathcal{U}\,2) \\ &X \in \mathcal{U} \Rightarrow \mathcal{P}\,X, \bigcup X \in \mathcal{U}, && (\mathcal{U}\,3) \\ &\omega \in \mathcal{U}, && (\mathcal{U}\,4) \\ &\text{if } X \in \mathcal{U},\ Y \subset \mathcal{U} \text{ and } f : X \to Y \text{ is a surjection, then } Y \in \mathcal{U}\,. && (\mathcal{U}\,5) \end{aligned}$$

[2]For an informal presentation of ZFC cf. [Mac98, section I.6].

Here $\{X, Y\}$ denotes the unordered pair, (X, Y) the ordered pair, $X \times Y$ the Cartesian product of sets, $\mathcal{P}X$ the power set of X, $\bigcup X$ the union of X, ω the set of all finite ordinals and $f : X \to Y$ a (set-theoretic) map from X to Y. The existence of all of these constructions is either given directly by the axioms of ZFC or it can be deduced from them, like e.g. the set-theoretic presentation of an ordered pair by the Kuratowski ordered pair and hence the existence of a Cartesian product of sets.

These properties of $\mathcal{U}$ ensure that we can do usual mathematics within the universe $\mathcal{U}$. For example, from $(\mathcal{U}\,1)$ and $(\mathcal{U}\,3)$ we can see any subset $Y \subset X$ for a set $X \in \mathcal{U}$ to lie in $\mathcal{U}$; this holds in particular for the empty set. Essentially due to $(\mathcal{U}\,4)$ we have the natural numbers in $\mathcal{U}$ and because we can perform the usual set theoretic constructions in $\mathcal{U}$ the integers, reals, etc. are seen to lie in $\mathcal{U}$ as well. When we work in $\mathcal{U}$, what we will do up from section 2.2 most of the time, we indicate this by speaking about the elements of $\mathcal{U}$ as *small sets*.

Due to our approach we can't define the category of all sets, but we can define the category Set of small sets, which has as the set of objects the universe $\mathcal{U}$ and as the set of morphisms the set of all (set-theoretic) maps $f : X \to Y$ for $X, Y \in \mathcal{U}$. Recall that in set theory a map $f : X \to Y$ is usually presented as a particular type of binary relation; that is a triple (X, G_f, Y) where $G_f \subset X \times Y$ is the graph of f and as a relation it has the properties of being left-total and right-unique. By $(\mathcal{U}\,2)$ and what we have concluded on subsets of a small sets any set-theoretic map $f : X \to Y$ between small sets X, Y is thus seen to be an element $\mathcal{U}$; we have in particular $\operatorname{Mor}\mathbf{Set} \subset \mathcal{U}$. The composition is the ordinary composition of maps, the identity morphisms are the identity maps and domain and codomain are the usual such notions for maps which can be obtained as the respective projections to the first and third component of the triples (X, G_f, Y). Obviously, all of the required identities are seen to hold, so Set is indeed a category.

To indicate the size of a category within this approach, we introduce the following terminology. A category C is said to be *small* when its set of objects and its set of morphisms are both small sets; C is said to have *small hom-sets*, when for all pairs of objects $(c, d) \in \operatorname{Obj} C \times \operatorname{Obj} C$ the hom-sets $C(c, d)$ are small. Each category which is small has small hom-sets. The converse is not true as can be seen from the category Set, which can't be small, for $\mathcal{U} \in \mathcal{U}$ would contradict the axiom of regularity. Although one may be inclined to think of size issues of categories as just being of foundational relevance but irrelevant to the rest of the theory, this is not necessarily the case (cf. [Mac98, proposition V.2.3]).

2.1.2 Commutative Diagrams

Definition 2.1.1 shows that we can consider a category as a directed (reflexive) graph with an additional compatible monoid-like algebraic structure. The set of objects is the set of vertices and the set of morphisms is the set of edges which are directed

from dom to cod. In fact, we will usually denote morphisms of C by $f : c \to d$, where $c = \operatorname{dom} f$ and $d = \operatorname{cod} f$, as we do with set-theoretic maps. This emphasizes the interpertation of morphisms as directed edges once more. We will use this insight for a graphical representation of algebraic identities and properties of a category with the help of (commutative) diagrams.

Given a category C a *diagram in* C is a directed graph, which set of vertices is a subset of $\operatorname{Obj} C$ and which set of edges is a subset of $\operatorname{Mor} C$ such that the edges are directed from dom to cod. A diagram in C is said to *commute* if for any pair of its vertices (c, d) and any directed path from c to d in the diagram the composition of the morphisms along these paths yield the same morphism $c \to d$. This condition is fulfilled trivially when there is only one or no such path from c to d. It is thus sufficient to consider pairs of vertices where at least two directed paths exist.

2.1.3 Special Morphisms

The algebraic monoid-like structure of a category C motivates the definition of the following algebraically distinguished types of morphisms. First of all, as we have identity morphisms that act like identity elements, we can ask for invertibility.

A morphism $f : c \to d \in \operatorname{Mor} C$ is said be a *retraction* when it has a *right-inverse*, i.e. when there is a C-morphism $g : d \to c$ such that $f \circ g = 1_d$; it is said to be a *section* when it has a *left-inverse*, i.e. when there is a C-morphism $h : d \to c$ such that $h \circ f = 1_c$. The morphism f is said to be an *isomorphism* when it is a section and retraction. In that case the left-inverse and the right-inverse are seen to coincide and we call it the *inverse* of f. It is denoted by f^{-1}. If for two objects c and d in C there exists an isomorphism $c \to d$, we call these objects *isomorphic* and denote it as $c \cong d$. The binary relation for C-objects to be isomorphic is an equivalence relation on $\operatorname{Obj} C$.

In algebra besides invertibility a weaker but equally useful property is the property to be cancellable. In the context of categories we have to distinguish between the properties of being left- and right-cancellable. A morphism $f : c \to d$ is said to be *epi* when it is right-cancellable, i.e. for any two C-morphisms $g, h : d \to d'$ the identity $gf = hf$ implies $g = h$; it is said to be *monic* when it is left-cancellable, i.e. for any two C-morphisms $g, h : c' \to c$ the identity $fg = fh$ implies $g = h$. Any section is monic and any retraction is epi. The converse of these assertions does not hold in general, but it may hold in particular categories. For example it does hold in Set.

2.1.4 Functors

In the spirit of category theory we can, at least formally, consider categories as objects and ask about a meaningful notion of a morphism. This leads to the notion of a functor.

Definition 2.1.2 (functor)**.** Let C, D be two categories. A functor $F : C \to D$ consists of two maps $F_O : \operatorname{Obj} C \to \operatorname{Obj} D$ and $F_M : \operatorname{Mor} C \to \operatorname{Mor} D$ which preserve the structure maps of categories. This amounts to the following identities:

$$\operatorname{dom}_D(F_M(f)) = F_O(\operatorname{dom}_C f), \quad \operatorname{cod}_D(F_M(f)) = F_O(\operatorname{cod}_C f), \quad F_M(1_c) = 1_{F_O(c)},$$

which are required to hold for any $f \in \operatorname{Mor} C$ and $c \in \operatorname{Obj} C$, and where the index for the domain and codomain maps indicates to which category they belong. Furthermore, for any composable pair (f, g) of C-morphisms we require

$$F_M(f \circ g) = F_M(f) \circ F_M(g),$$

where the composition on the left-hand side means the composition map in C and the composition on the right-hand side means the composition map in D.

We will in general not distinguish between the map F_M and F_O of a functor F and denote both by F. It is common practice and shouldn't lead to confusion. The action of a functor F on an object c or a morphism f will be usually denoted as Fc and Ff the parantheses omitted.

Given two functors $F : C' \to D$, $G : C \to C'$ we can define a composite functor $F \circ G : C \to D$ by the composite maps

$$(F \circ G)_O := F_O \circ G_O \qquad \text{and} \qquad (F \circ G)_M := F_M \circ G_M.$$

It is easy to see that this is well-defined and that the composition is associative. We will usually omit "$\circ$" and denote the composite as FG and we will do the same with the composite of morphisms.

For each category C the pair of identity maps $\mathrm{id}_{\operatorname{Obj} C}$ and $\mathrm{id}_{\operatorname{Mor} C}$ yields a functor $I_C : C \to C$, which acts under the previously defined composition of functors as a left- or right-identity, respectively. It is called the *identity functor of* C. We can thus formally copy the notion of isomorphism of morphisms and get the notion of an *isomorphism of categories* as a functor $F : C \to D$ which has a right- and a left-inverse. Once again both the right- and the left-inverse are seen to coincide and we call the functor $F^{-1} : D \to C$ the inverse of F. If there exists an isomorphism $F : C \to D$ the categories are said to be *isomorphic* and we denote this by $C \cong D$.

With a functor we have actual set-theoretic maps at hand, so we can use the notions of injectivity and surjectivity of set-theoretic maps. Of course, we need to account for the fact that a functor is given by two maps. For example, a functor $F : C \to D$ is an isomorphism if and only if it is bijective on objects and on morphisms, i.e. F_O and F_M are both bijections.

A functor $F : C \to D$ which is injective on objects and morphisms, i.e. F_O and F_M are injective maps, is said to be an *inclusion*. We then often speak about C as a *subcategory* of D and denote it by $C \leq D$ or $C \hookrightarrow D$. In fact, in this case the sets $\operatorname{im} F_O \subset \operatorname{Obj} C$ and $\operatorname{im} F_M \subset \operatorname{Mor} C$ yield a category when we restrict and

corestrict the maps dom, cod and 1 of C, respectively. If we denote this particular subcategory by $\operatorname{im} F$ then we see that F yields an isomorphism $C \cong \operatorname{im} F$. We will often identify C with $\operatorname{im} F$ in that case.

Usually functors are neither injective on morphisms nor on objects. The same applies for surjectivity. However, there are many cases where functors are injective or surjective on hom-sets. As the morphisms are the part of a category which carry the algebraic structure, such properties of a functor are of particular interest for the theory. A functor $F : C \to D$ is said to be *faithful* when for any pair $(c, d) \in \operatorname{Obj} C \times \operatorname{Obj} C$ the restriction and corestriction[3] of F_M to

$$F_{c,d} : C(c, d) \to D(Fc, Fd)$$

is injective; it is said to be *full* when every such $F_{c,d}$ is surjective. Any inclusion $C \hookrightarrow D$ is faithful, but it may not be full in general. When it is full, we speak about a *full subcategory*. When we identify C with its image under the inclusion, a full subcategory of C is seen to be determined by its subset of objects $\operatorname{Obj} C \subset \operatorname{Obj} D$, for given two of its objects c, c' we have $C(c, c') = D(c, c')$.

2.1.5 Natural Transformations

We can also formally consider functors as objects and ask about a meaningful notion of a morphism. In this case we arrive at the notion of a natural transformation.

Definition 2.1.3 (natural transformation)**.** Let C, D be two categories and $F, G : C \to D$ two functors. A natural transformation $\eta : F \dot{\to} G$ is a map

$$\eta : \operatorname{Obj} C \to \operatorname{Mor} D, \qquad c \mapsto \eta_c : Fc \to Gc,$$

such that for any $f : c \to d \in \operatorname{Mor} C$ we have $\eta_d \circ Ff = Gf \circ \eta_c$. In other words for each C-morphism $f : c \to d$ the following diagram has to commute

$$\begin{array}{ccc} Fc & \xrightarrow{\eta_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fd & \xrightarrow[\eta_d]{} & Gd. \end{array}$$

Even though we defined it as a mapping, we think of natural transformation η more as a particular family of morphisms indexed by the objects of C. Therefore, we don't speak of η_c as the image of c under η but as the *c-component* of η.

Let $F, G, H : C \to D$ be functors and $\eta : G \dot{\to} H$, $\varepsilon : F \dot{\to} G$ two natural transformations. We can define a composite $\eta \cdot \varepsilon : F \dot{\to} H$ by

$$(\eta \cdot \varepsilon)_c := \eta_c \circ \varepsilon_d.$$

[3] By a *corestriction* of a map we mean the restriction of its codomain.

With the help of the diagrammatic formulation this is easily seen to yield a natural transformation and it is already immanent in the definition that this kind of composition is associative.

For any functor $F : C \to D$ we have a natural transformation $1_F : F \dot{\to} F$ which maps an object $c \in \operatorname{Obj} C$ to the identity morphism 1_{Fc}. We call it the *identity transformation of* F and it yields a left- and right-identity for the previously defined composition, respectively. With this we can once more copy the notion of isomorphism for morphisms and get the notion of an *isomorphism of functors* as an invertible natural transformation. According to the definition of the composition the invertibility of $\eta : F \dot{\to} G$ is equivalent to the invertibility of each component η_c, $c \in \operatorname{Obj} C$. Stated differently, η is an isomorphism if and only if every component η_c, $c \in \operatorname{Obj} C$ is an isomorphism.

There is a way to define a composition of natural transformations between functors of different categories. Let $F, F' : C' \to D$ and $G, G' : C \to C'$ be functors and $\eta : F \dot{\to} F'$, $\varepsilon : G \dot{\to} G'$ natural transformations. We define a natural transformation

$$\eta \circ \varepsilon : FG \dot{\to} F'G', \qquad c \mapsto \eta_{G'c} \circ F\varepsilon_c : FGc \to F'G'c.$$

The so defined composition can be shown to be associative as well. It has as left- and right-identities the identity transformations of the respective identity functors.

To distinguish between the two compositions we denote the one from above by "$\cdot$" and call it *the vertical composition*, the one from here is denoted by "$\circ$" and we call it *the horizontal composition*. Both are interrelated by a so called *interchange law*

$$(\eta' \cdot \varepsilon') \circ (\eta \cdot \varepsilon) = (\eta' \circ \eta) \cdot (\varepsilon' \circ \varepsilon),$$

where $\eta' : G' \dot{\to} H'$, $\varepsilon' : F' \dot{\to} G'$, are natural transformations for the functors $F', G', H' : C' \to D$ and $\eta : G \dot{\to} H$, $\varepsilon : F \dot{\to} G$ are natural transformations for the functors $F, G, H : C \to C'$ (cf. [Mac98, section II.5, theorem II.5.1]). Although we will meet the horizontal composition on several occasions we won't explicitly state it. In fact, we shall make very little use of it. Basically, we will only use the vertical composition of natural transformations in this thesis.

One can think of natural transformations as algebraic presentations of the notion of a homotopy. With this in mind it is natural to formulate a "homotopy equivalence" of categories. A functor $F : C \to D$ is said to be an *equivalence of categories* when there is a functor $G : D \to C$ and natural transformations $\eta : I_C \cong GF$, $\varepsilon : FG \cong I_D$. If there exists an equivalence of categories for C and D, we say that these categories are *equivalent* and denote this by $C \simeq D$.

2.1.6 Some Constructions on Categories

Like with sets there are also basic constructions on categories. We shall present here only those we will need in this thesis.

We begin with the *(Cartesian) product of categories*, which we can carry over directly from set theory. Let C and D be categories. We define the product $C \times D$ of C and D as a category with the set of objects $\operatorname{Obj} C \times \operatorname{Obj} D$, the set of morphisms $\operatorname{Mor} C \times \operatorname{Mor} D$ and the following structure maps:

$$\operatorname{dom}_{C\times D} := \operatorname{dom}_C \times \operatorname{dom}_D, \quad \operatorname{cod}_{C\times D} := \operatorname{cod}_C \times \operatorname{cod}_D, \quad 1_{C\times D} := 1_C \times 1_D.$$

The composition map of $C \times D$ is defined componentwise. In more detail, we have for all $(f, g) \in \operatorname{Mor} C \times \operatorname{Mor} D$ and $(c, d) \in \operatorname{Obj} C \times \operatorname{Obj} D$

$$\begin{aligned} \operatorname{dom}_{C\times D}(f, g) &= (\operatorname{dom}_C f, \operatorname{dom}_D g), \\ \operatorname{cod}_{C\times D}(f, g) &= (\operatorname{cod}_C f, \operatorname{cod}_D g), \\ 1_{C\times D}(c, d) &= (1_c, 1_d); \end{aligned}$$

the pair $((f, g), (f', g'))$ of $C \times D$-morphisms is composable if and only if (f, f') and (g, g') are composable, and in that case the composition is given by

$$(f, g) \circ (f', g') = (f \circ f', g \circ g'),$$

where on the left-hand side $\circ$ denotes the composition map in $C \times D$ and on the right-hand side the respective composition map of C and D. Functors $F : C \times D \to B$ from (binary) products of categories will be called *bifunctors*.

The next construction motivated by set theory is that of a *functor category*. Let C and D be categories. We define the functor category D^C as a category with the set of functors from $C \to D$ as objects and with the set of natural transformations between any such functors as morphisms. The domain and codomain map are the obvious ones for natural transformations. The composition map is defined by the vertical composition of natural transformations and the identity morphisms are the identity transformations of the respective functors.

The next two constructions have no analogue in set theory anymore. The first one has an analogue in algebra, however. It is the construction of the *dual* or *opposite* category. Given a category C we define its dual category C^{op} as the category with the same set of objects, the same set of morphisms and the same identity morphisms, but the domain and codomain interchanged; that is we have $\operatorname{Obj} C^{op} = \operatorname{Obj} C$, $\operatorname{Mor} C^{op} = \operatorname{Mor} C$, $1^{op} = 1$, but $\operatorname{dom}^{op} := \operatorname{cod}$ and $\operatorname{cod}^{op} := \operatorname{dom}$. For this to yield a category we have to reverse the order of the arguments for the composition map

$$f \circ^{op} g := g \circ f,$$

where (f, g) is a composable pair of C^{op} and hence (g, f) a composable pair of C. With these settings it is easily seen that the required identities of a category translate accordingly, so C^{op} is indeed a category.

Behind the construction of the dual category is the *duality principle* of category theory. It states roughly that any statement for a category C which involves only

first-oder logic and the terminology of a category, i.e. objects, morphisms and the structure maps from definition 2.1.1, can be translated into a "dual" statement in the category C^{op} by interchanging domain and codomain of each arrow and reversing the order of composition. In the pictorial language of commutative diagrams this amounts to "reverse all arrows". For example, the dual of the statement "f is epi" is "f is monic". Moreover, if we have a proof of the original statement which uses only the identities of a category and first-order logic, then it translates by the same procedure to a proof of the dual statement in C^{op}. In this sense, a true statement in C translates to a true statement in C^{op} and vice versa.

In its core the duality principle is a syntactical type of duality. To be able to formulate and prove it rigorously requires an axiomatic formulation of category theory like the elementary theory of abstract categories (ETAC) and its extension by a functor. For ETAC the duality principle asserts that the axioms are self-dual: a syntactical translation of an axiom of ETAC according to the above mentioned rules yields an axiom of ETAC again. An informal treatment of ETAC together with the duality principle is given in [Mac98, section II.I]; a formal treatment of ETAC together with an axiomatic treatment of the category of all categories can be found in [Law65]. In this thesis we will follow the practice in [Mac98] and apply duality informally. This means that we won't state a dual statement or result explicitly, but just refer to it as the dual statement or dual result and use it freely.

Tightly related with the notion of a dual category is the notion of a *contravariant* functor. In contrast with an ordinary functor, which we refer to as *covariant* in this context, a contravariant functor $F : C \to D$ doesn't preserve domain and codomain, but interchanges it. In doing so it reverses the order of compositions of two morphisms: $F(f \circ g) = Fg \circ Ff$. However, any contravariant functor $F : C \to D$ with its two maps F_O and F_M is seen to yield with the same maps a covariant functor $F : C^{op} \to D$ or $F : C \to D^{op}$ and vice versa. We will follow the usual convention and identify contravariant functors $F : C \to D$ with ordinary (covariant) functors $F : C^{op} \to D$ or $F : C \to D^{op}$, whichever is more convenient in the given situation.

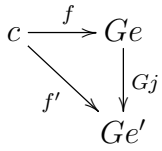
The last construction we will present here is the *comma category* construction. It is a very general construction which appropriately generalizes many important and useful constructions of categories of morphisms. We begin with the general case and specialize afterwards to the two cases which will be of importance to us. Let $F : D \to C$ and $G : E \to C$ be two functors. The comma category $(F \downarrow G)$ has as objects the triples (d, e, f) with $d \in \operatorname{Obj} D$, $e \in \operatorname{Obj} E$ and $f : Fd \to Ge \in \operatorname{Mor} C$. The morphisms $(d, e, f) \to (d', e', f')$ are pairs $(h, j) \in \operatorname{Mor} D \times \operatorname{Mor} E$ with $h : d \to d'$, $j : e \to e'$ such that the following diagram commutes

$$\begin{array}{ccc} Fd & \xrightarrow{Fh} & Fd' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Ge & \xrightarrow[Gj]{} & Ge'. \end{array}$$

The domain and codomain map are given as indicated by the definition of morphism, the identity morphisms are given by the pairs of the respective identity morphisms in D, and E and the composition is defined componentwise. Alltogether this is seen to satisfy the axioms of a category given in definition 2.1.1.

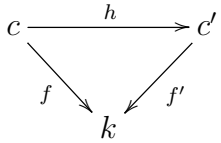
In situations like this, where the domain and codomain map as well as the identity morphisms and composition are clear from the definition of a morphism, we will henceforth not state them explicitly anymore but consider them as understood. Moreover, we will often omit any explicit statement of the data of a category if it is obvious from the context. For example, when we speak about the category of (small) groups it is understood implicitly that the objects are the (small) groups, the morphisms the group homomorphisms, domain and codomain map the one of set theoretic maps, the identity morphisms the identity maps and composition the composition of maps.

The first special case of the construction of a general comma category, which is of interest to us, is when we make the following special choice for the functor F. Consider the category $D = \underline{1}$ with precisely one object $\bullet$ and one morphism $1_\bullet$ the necessary identity morphism for that object. A functor $G : \underline{1} \to C$ is then the same thing as choosing a particular object c in C. The objects $(\bullet, e, f : c \to Ge)$ can then be identified with the pairs (e, f) and the morphisms $(e, f) \to (e', f')$ can be identified with morphisms $j : e \to e'$ such that



is rendered commutative. Such a comma category is denoted by $(c \downarrow G)$.

The second special case which is of interest to us is when we set $D = C$, $F = I_C$ and $E = \underline{1}$. Now the functor $G : \underline{1} \to C$ is the same thing as choosing a particular object in C, say k. With these choices the comma category is seen to have as objects $(c, \bullet, f : c \to k)$ which we can identify with the morphisms $f : c \to k$. The morphisms $(f : c \to k) \to (f' : c' \to k)$ can then be identified with C-morphisms $h : c \to c'$ such that



commutes. In the $\downarrow$-notation this kind of comma category is denoted as $(C \downarrow k)$. However, we will follow [MM92], denote it by C/k and call it the *slice category* of C over k in this thesis. We will encounter this construction in chapters 5 and 6. There the respective slice category will be seen to be of central importance for an overall structural understanding of the notion of an affine space.

2.1.7 Limits and Colimits

Finally, we shall introduce the notions of universality, limits and their respective duals, the colimits. In a certain sense, which, hopefully, will become more obvious in the course of discussion, universality is an extremal property of an object in a given category with respect to a certain apriori defined class of morphisms (i.e. a diagram in this category). We begin with the notion of an *initial object*. It is extremal with respect to the whole category in the following sense: an object $c \in C$ is said to be initial, when for any object $d \in \operatorname{Obj} C$ there is exactly one morphism $c \to d$. A direct consequence of this definition is that any two initial objects have to be isomorphic. The corresponding dual notion is that of a *terminal object*. Stated explicitly, an object $c \in \operatorname{Obj} C$ is said to be terminal when for any object $d \in \operatorname{Obj} C$ there is exactly one morphism $d \to c$. We can now define what we mean by a universal morphism.

Definition 2.1.4 (universal morphism)**.** Let $F : D \to C$ be a functor and $c \in \operatorname{Obj} C$. A *universal morphism from c to F* is defined to be an initial object in the comma category $(c \downarrow F)$. Stated differently, a universal morphism from c to F is a pair $(d, u : c \to Fd)$, $d \in \operatorname{Obj} D$, $u \in \operatorname{Mor} C$ with the following *universal property*: for any $d' \in \operatorname{Obj} D$ and C-morphism $f : c \to Sd'$ there is a unique D-morphism $h : d \to d'$ with $f = Fh \circ u$. In terms of commutative diagrams we express this as follows:

$$\begin{array}{ccc} c & \xrightarrow{u} & Fd \\ & \searrow^{f} & \downarrow{\scriptstyle Fh} \\ & & Fd', \end{array} \qquad \begin{array}{c} d \\ \downarrow{\scriptstyle \exists! h} \\ d'. \end{array}$$

The dual notion to a universal morphism from c to F is the universal morphism from F to c which is a terminal object in the comma category $(F \downarrow c)$. Here we have dualized the definition with respect to the categories C and D, but not with respect to the comma category involved.

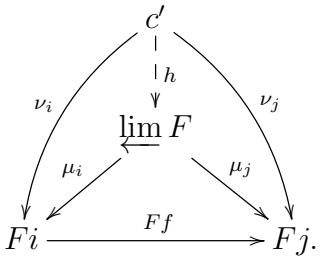
We proceed with the notion of a limit. Conceptually, in a given category C a limit of a (commutative) diagram of shape J in C is an object c in C which is structurally the "closest" one to this diagram. Here J means a category, but we think of it rather as a directed graph than a category in this context. A diagram of shape J is then the same thing as a functor $F : J \to C$. We need to make precise the idea of "structurally closest". To be able to do so, we first need to make precise how an object and a diagram of shape J can be structurally related. The most obvious way is to give for each vertex $j \in \operatorname{Obj} J$ a C-morphism $\mu_j : c \to Fj$, such that for any morphism $f : i \to j$ we have $Ff \circ \mu_i = \mu_j$. We call such an indexed family of C-morphisms $(\mu_j)_{j \in \operatorname{Obj} J}$ a *cone from c to F*. It is conveniently described as a natural transformation $\mu : \Delta c \dot{\to} F$, where Δc is the constant functor

$$\Delta c : J \to C, \qquad j \mapsto c, \qquad (f : i \to j) \mapsto 1_c.$$

The property of a limit to be "structurally closest" can then be understood as the universality of the cone μ. Indeed, we can understand μ as a universal morphism from Δ to F in the functor category C^J, because the assignment $c \mapsto \Delta c$ is seen to extend to a functor $\Delta : C \to C^J$ whose morphism map maps each morphism $f : c \to c'$ to the natural transformation $\Delta f : \Delta c \dot{\to} \Delta c'$ with components $(\Delta f)_j = f$, $j \in J$.

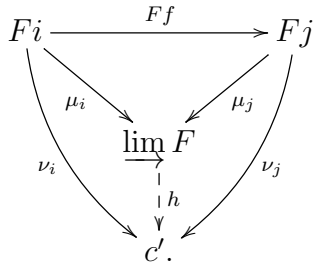
Definition 2.1.5 (limit and colimit)**.** Let $F : J \to C$ be a functor. An object c in C together with a cone $\mu : \Delta c \dot{\to} F$ is called *a limit of* F if μ is a universal morphism from Δ to F in C^J. In this case we usually write $c = \varprojlim F$ and call μ the *limiting cone*. The universal property is expressed by commutative diagrams as follows:

$$\exists! h \; \forall i, j, f : i \to j$$



The dual notion to a limit of F is that of a *colimit of* F; it is a $c \in \operatorname{Obj} C$ and a cone $\mu : F \dot{\to} \Delta c$ from F to c that is a universal morphism from F to Δ. In this case we usually write $c = \varinjlim F$ and call μ the *colimiting cone*. The universal property is expressed by commutative diagrams as follows:

$$\exists! h \; \forall i, j, f : i \to j$$



Limits and colimits can be regarded as the general notions behind all universal constructions. For example, an initial object in C is a limit of the identity functor I_C and the d-components of the cone are given by the unique C-morphisms $c \to d$. Dually, a terminal object can be considered as a colimit of the identity functor I_C; but it is also possible to see it as a limit of the unique functor $\underline{0} \to C$, where $\underline{0}$ is the empty category. This category has an empty set of objects, an empty set of morphisms, domain, codomain, the map 1 and the composition map all coincide (as maps) and are the one and only possible identity map of the empty set. As there is exactly one set-theoretic map from the empty set to any other set, we can see that there is exactly one functor from the empty category to any other category. Note $C^{\underline{0}} \cong \underline{1}$, so the functor Δ maps each $c \in \operatorname{Obj} C$ to the one and only identity

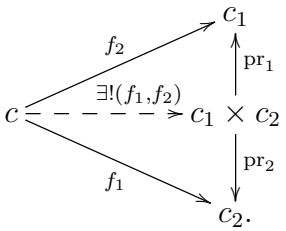
transformation. Hence, if a limit c exists for $\underline{0} \to C$ then each C-object d trivially yields a cone and by the universal property of a limit we have a unique morphism $d \to c$. This shows that c is terminal.

Another type of limit we will need right from the beginning is the *(Cartesian or categorical) product*. Products are limits of diagrams with a shape of a *discrete category*. This is a category D whose morphisms are the identity morphisms only. In terms of our structure maps this means that dom = cod and dom is the inverse of $1 : \operatorname{Obj} D \to \operatorname{Mor} D$. Conversely, for a given set X we can define a discrete category X by $\operatorname{Obj} X = X$, $\operatorname{Mor} X = X$ and $1 = \mathrm{id}_X$. Discrete categories are thus seen to be nothing more than sets considered as categories and we will freely identify both notions whenever necessary.

We distinguish between binary, finite and small products, depending on whether the discrete category is the two-element set, a finite set or a small set. We restate the universal property of limits for the case of binary products. Observe that a functor $\{1, 2\} \to C$ is the same thing as a pair of objects $(c_1, c_2) \in \operatorname{Obj} C \times \operatorname{Obj} C$ and a cone from an object c to this functor is a pair (f_1, f_2) of C-morphisms $f_1 : c \to c_1$, $f_2 : c \to c_2$. We denote the limit by $c_1 \times c_2$ and call the components of the limiting cone projections

$$\mathrm{pr}_1 : c_1 \times c_2 \to c_1, \qquad \mathrm{pr}_2 : c_1 \times c_2 \to c_2.$$

The universal property then reads that for any object c and any pair of morphisms $(f_1 : c \to c_1, f_2 : c \to c_2)$ there is a unique morphism $h : c \to c_1 \times c_2$ usually denoted by $(f_1, f_2) = h$, such that $\mathrm{pr}_i \circ h = f_i$ for $i = 1, 2$



The difference to finite and small products is just the cardinality of the set of factors (components) and respective projections. Note that by definition of products the product of an empty set of objects is a terminal object. The dual notion to products is that of *coproducts* to which we will occasionally refer to as *sums*.

In the course of this chapter we will encounter many other special types of limits and colimits that, like products and coproducts, describe important universal constructions within particular categories. From the structural point of view it is thus desirable to know for a particular category C which types of limits and colimits do exist. In this thesis we will most of the time only distinguish between the cardinality of the categories that define the shape of a diagram for the respective limit type. A category C is said to be *finitary (co)complete* if all *finite (co)limits* exist, that is for any category J with a finite set of objects and a finite set of morphisms any

functor $F : J \to C$ has a (co)limit in C. A category C is said to be *(co)complete* if all *small (co)limits* exist, that is for any small category J any functor $F : J \to C$ has a (co)limit in C.

From the definition 2.1.4 of a universal morphism c to F we can see that any two such universal morphisms $(d, f : c \to Fd)$, $(d', g : c \to Fd')$ are isomorphic in the comma category $(c \downarrow F)$. Conversely, given a universal morphism $(d, f : c \to Fd)$ and an isomorphism $h : (d, f : c \to Fd) \to (d', g : c \to Fd')$ then $(d', g : c \to Fd')$ is seen to be a universal morphism from c to F as well. In summary a universal morphism from c to F is only determined up to an isomorphism in $(c \downarrow F)$.

When we write this isomorphism out for the particular case of two limits c, d of a functor $F : J \to C$ with respective limiting cones μ and ν, then it is seen to be a C-isomorphism $h : c \to d$ that respects the cones, i.e. $\mu_j = \nu_j \circ h$ for each $j \in \operatorname{Obj} J$. Conversely, given an F-limit c with limiting cone μ and an C-isomorphism $h : c \to d$ we can make d an F-limit by setting $\nu_j := \mu_j \circ h^{-1}$. Hence, limits of a functor F are determined only up to C-isomorphisms by F.

With this fact we want to point out the following subtlety, that at the same time is the main reason for our need of the axiom of choice. We discuss it for the example of binary products. Assume C to have for each pair (c, d) of its objects a product. Given two pairs $(c, d), (c', d') \in \operatorname{Obj} C \times \operatorname{Obj} C$ and a pair of morphisms $(f : c \to c', g : c \to c')$ with the help of the universal property we obtain a unique morphism $f \times g : c \times d \to c' \times d'$ as indicated in the subsequent commutative diagram

$$\begin{array}{ccccc} c & \xleftarrow{\mathrm{pr}_c} & c \times d & \xrightarrow{\mathrm{pr}_d} & d \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \exists! f\times g} & & \downarrow{\scriptstyle g} \\ c' & \xleftarrow[\mathrm{pr}_{c'}]{} & c' \times d' & \xrightarrow[\mathrm{pr}_{d'}]{} & d'. \end{array}$$

From the universal property it also follows that

$$1_c \times 1_d = 1_{c\times d} \quad \text{and} \quad (f' \circ f) \times (g' \circ g) = f' \times g' \circ f \times g.$$

Under the condition that we have for each pair (c, d) a particular categorical product $c \times d$ together with its projections at hand, the binary products will yield a bifunctor

$$\times : C \times C \to C, \qquad (c, d) \mapsto c \times d, \qquad (f, g) \mapsto f \times g.$$

However, as we have already seen, the universality alone determines a binary product only up to C-isomorphism. Hence, to get a bifunctor from the sole fact that binary products exist in C we need to choose representants from the respective isomorphism classes of C. Any such choice will yield a bifunctor as indicated and the bifunctors for two different choices are isomorphic (as functors). This is a consequence of the universal property.

With this we have finally introduced all of the categorical notions which we will encounter from the beginning of our discussion. One very important notion, substantial for our investigation but still missing at this point, is the notion of *adjoint functors*. We will introduce it in section 2.2.2 later.

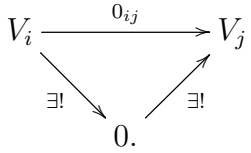
2.2 The Category of Vector Spaces

The goal of this section is to study categorical properties and constructions in the category of vector spaces over a field K. The results should be all well-known. Therefore, we shall use this section mainly to recall some basic categorical notions, concepts, constructions and theorems by way of example. We will also explain some standard arguments in detail and put results from linear algebra in the context of category theory. All this is done to prepare and to make the later approach to similar questions in the category of affine spaces and others better understandable.

We denote the category of small K-vector spaces with morphisms the linear maps by $\mathrm{Vct_K}$. For convenience the field K is assumed to have a characteristic of zero in the whole thesis when not stated otherwise; but many of our results actually don't depend on the characteristic and hold as well for arbitrary characteristic p with the minor restriction of $p \neq 2$ when dealing with Clifford-like geometric algebra constructions. First of all, $\mathrm{Vct_K}$ is a linear category. This means that the category has a zero object 0, binary products, binary coproducts and the canonical morphism

$$I : V_1 \times V_2 \to V_1 + V_2,$$

is an isomorphism. In a category an object is said to be a *zero object* when it is initial and terminal at the same time. The canonical morphism I is given by the universal properties of product and coproduct applied to the morphisms $I_{ij} : V_j \to V_i$ with $I_{ii} = 1_{V_i}$, $I_{ij} = 0_{ij}$ for $i \neq j$, where $0_{ij} : V_i \to V_j$ denotes the unique zero morphism



Using the natural associativity isomorphisms of products and coproducts this generalizes to finite products and coproducts respectively (cf. [Mac98, proposition III.5.1]). Thus, loosely speaking, to be linear means the identity-matrix is an isomorphism. Indeed, linear categories are the proper stage for a matrix calculus as can be seen in [LS97, session 26.3][4]. Note that there is no commonly accepted definition of a linear category.

[4]This point can be seen more directly when one defines a linear category C as a category with finite products enriched over abelian monoids. This is equivalent to our definition here when one defines $f + g := \delta_b \circ I \circ f \times g \circ \Delta_a$ for $f, g \in C(a, b)$ (cf. [Mac98, exercise VIII.2.4(a)]),

In the case of $\mathrm{Vct_K}$ we have all (small) products, namely the direct product $\prod_{i\in I} V_i$ together with the canonical projections $\mathrm{pr}_j : \prod_{i\in I} V_i \to V_j$, and all (small) coproducts, namely the (outer) direct sums $\bigoplus_{i\in I} V_i$ together with the canonical injections $\iota_j : V_j \to \bigoplus_{i\in I} V_i$ for an indexed family of vector spaces $(V_i)_{i\in I}$ by a small set I, i.e. an object of the category Set (cf. [KM03] or [Mey80, Satz 5.2.2] for details). If one constructs the direct sum explicitly as the vector subspace

$$\bigoplus_{i\in I} V_i := \{(v_i)_{i\in I} \in \prod_{i\in I} V_i \mid v_i = 0 \text{ for all but finitely many } i \in I\} \leq \prod_{i\in I} V_i,$$

as is for example done in [Mey80] then one can see directly that for finite index sets I product and coproduct coincide. A zero object in $\mathrm{Vct_K}$ is any trivial vector space $\{0\}$, so we have indeed a linear category where the identity matrix is the identity map.

As a category $\mathrm{Vct_K}$ enjoys many good properties. For example, it is well-powered, complete, cocomplete, separated, coseparated and abelian. It is not Cartesian closed, i.e. for a vector space V, the functor $(-)\times V : \mathrm{Vct_K} \to \mathrm{Vct_K}$, which assigns to a vector space W the vector space $W \times V$ and to a linear map $f : W \to W'$ the linear map $f \times 1_W$, $(w, v) \mapsto (f(w), v)$, has not a right adjoint in general. Indeed the only exponentiable object in $\mathrm{Vct_K}$, i.e. an object V for which a right adjoint for $(-)\times V : \mathrm{Vct_K} \to \mathrm{Vct_K}$ exists, is the zero object. Nevertheless, due to the tensor product, $\mathrm{Vct_K}$ can be equipped with a symmetric monoidal structure under which it is closed. It has internal hom-objects in particular. We will address and recall each of those notions and properties in the following three subsections.

2.2.1 Some Useful Properties of $\mathrm{Vct_K}$

For reference we collect the properties mentioned above in a proposition.

Proposition 2.2.1. *The category* $\mathrm{Vct_K}$ *of* K*-vector spaces is linear, well-powered, complete, cocomplete, separated, coseparated and abelian.*

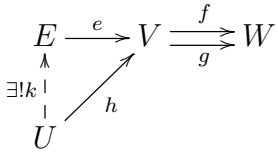
For a category C to be *well-powered* it means that the set of isomorphism classes of subobjects $\mathrm{Sub}\, a$ for each $a \in \mathrm{Obj}\, C$ is a small set. Recall that a *subobject* of a is a *mono* $f : b \to a$ in C, i.e. a morphism with the property of being left-cancellable, that is $fg = fh$ implies $g = h$. Two such subobjects $f : b \to c$, $g : d \to c$ are said to be equivalent or isomorphic when there is an isomorphism $h : b \to d$ such that $gh = f$. The idea behind this definition is that monos correspond to inclusion maps of subspaces and two inclusion maps are equivalent, i.e. they descrice the same subspace, if they have the same image. Because of that one usually resorts to the

where $\Delta_a : a \to a \times a$ is the diagonal morphism and $\delta_b : b + b \to b$ the folding morphism, both obtained by applying the respective universal properties of product and coproduct to the respective identity morphisms.

common abuse of notation to denote a subobject $f : a \to c$ only by its domain a, the morphism being understood. Note that there are many cases where the subobjects defined in the sense of category theory like this don't correspond in a one-to-one fashion to the usual notion of subspace. A prominent example is the category of smooth manifolds with the smooth maps as morphisms. In the case of Vct_K both notions are equivalent which is a consequence of proposition 2.2.4 and the notion of vector subspace.

The assertion of smallness of $\operatorname{Sub} V$ is clear for a V in Vct_K, because for a vector space V each isomorphism class $[U] \in \operatorname{Sub} V$ has a representant $U \hookrightarrow V$ with underlying domain U an actual subset of V. This U can be obtained from the isomorphism class by taking the image of an arbitrary representant. We explain this in more detail. The underlying set of the small vector space V is a small set by definition. We denote it by V again. By the axioms of the universe $\mathcal{U}$ the power set $\mathcal{P}(V)$ of the set V is again an element of $\mathcal{U}$ and hence so is the subset S of the power set $\mathcal{P}(V)$ that consists of the underlying sets of the subset-subvector spaces $U \subset V$ from above. With this we can construct a surjective map $S \to \operatorname{Sub} V$ by assigning the respective isomorphism class of subobjects to a subset-vector subspace $U \in S$. By the axioms of the universe $\mathcal{U}$ this yields that $\operatorname{Sub} V \in \mathcal{U}$ and proves the assertion.

In the following we will show Vct_K to be complete. Recall that a category C is said to be *complete* when it has all small limits, i.e. for each small category I and each functor $F : I \to C$, $\varprojlim F$ exists in C. Of course with the existence of $\varprojlim F$ we implicitly mean the existence of a limiting cone $\Delta \varprojlim F \dot{\to} F$. Small limits exist if and only if small products and equalizers of pairs exist (cf. [Mac98, theorem V.2.1]). We already have small products, so we just need to proof the existence of equalizers. Let $f, g : V \to W$ be two linear maps. An equalizer $e : E \to V$ of f, g satifies $fe = ge$ and is the universal such morphism, i.e. for each morphism $h : U \to V$ with $fh = gh$ there is a unique morphism $k : U \to E$ such that $ek = h$.



In Set the equalizer e is given by the inclusion of the subset $E := \{v \in V \mid f(v) = g(v)\}$. If we show this to be a vector subspace of V then e will be linear and the universal property from Set will also hold in Vct_K and yield for a linear $h : U \to V$ with $fh = gh$ a unique linear map again. This is because the unique map in Set is just the corestriction of h to E. We know from linear algebra that the hom-set $\mathrm{Vct}_\mathrm{K}(V, W)$ carries a K-vector space structure by pointwise operations. In particular $f - g$ is again a morphism. With this we obtain $\{v \in V \mid f(v) = g(v)\} = \ker(f - g)$ and in particular that E is a vector subspace as asserted.

Our argument gives evidence for the property of the forgetful functor $U : \mathrm{Vct_K} \to \mathrm{Set}$ to create equalizers. Recall that a functor $U : C \to D$ is said to *create limits* for $F : I \to C$ when for each limiting cone $\tau : \Delta d \dot{\to} UF$ there is exactly one cone $\sigma : \Delta c \to F$ with $Uc = d$ and $U\sigma_i = \tau_i$ for all $i \in \mathrm{Obj}\, I$ and moreover this cone σ is a limiting cone in C. Recall as well that products are limits of functors whose domain are *discrete categories*, i.e. categories which have only identity morphisms as morphisms, and that equalizers are limits of functors with domain category the free category of the directed graph $\bullet \rightrightarrows \bullet$ (cf. [Mac98, section II.7]). The construction of small products in $\mathrm{Vct_K}$ suggests the forgetful functor $U : \mathrm{Vct_K} \to \mathrm{Set}$ to create small products. Alltogether this suggests the following proposition, which together with the completeness of Set yields another way to show that $\mathrm{Vct_K}$ is complete (cf. [Mac98, theorem V.1.1, theorem V.4.2]).

Proposition 2.2.2. *The forgetful functor $U : \mathrm{Vct_K} \to \mathrm{Set}$ creates small limits.*

Proof. Let $F : I \to \mathrm{Vct_K}$ be a functor and I a small category.

(i) By theorem V.1.1 in [Mac98] we can represent the limit of UF by $\mathrm{Nat}(\Delta *, UF)$ the set of cones from the one-point set $*$ to F. Each such a cone $\sigma : \Delta * \dot{\to} UF$ represents a "coherent choice" of elements $\sigma_i(\bullet) \in UFi$ for $i \in \mathrm{Obj}\, I$, $* = \{\bullet\}$. First, observe that $\mathrm{Nat}(\Delta *, UF) \neq \emptyset$ because the assignment $i \mapsto 0_{Fi}$, where $0_{Fi} : * \to UFi$, $\bullet \mapsto 0$ selects the zero in Fi, yields a cone $\Delta * \dot{\to} UF$ due to the trivial vector space $\{0\}$ being initial in $\mathrm{Vct_K}$, i.e. for each vector space V in $\mathrm{Vct_K}$ there is only one linear map $\{0\} \to V$, namely the zero map. The limiting cone is given by evaluation, i.e.

$$\begin{aligned} \tau &: \Delta\, \mathrm{Nat}(\Delta *, UF) \dot{\to} UF, \quad \text{with} \\ \tau_i &: \mathrm{Nat}(\Delta *, UF) \to UFi, \quad \sigma \mapsto \sigma_i(\bullet), \qquad i \in \mathrm{Obj}\, I. \end{aligned}$$

We have to show that there is a unique K-vector space structure on $\mathrm{Nat}(\Delta *, UF)$ such that the τ_i are K-linear and this unique lift of τ, which we denote by $\tau : \Delta\, \mathrm{Nat}(\Delta *, UF) \dot{\to} F$ again, is a limiting cone for F in $\mathrm{Vct_K}$. Suppose we have such a K-vector space structure on $\mathrm{Nat}(\Delta *, UF)$. The linearity of the τ_i imply for $\sigma, \mu \in \mathrm{Nat}(\Delta *, UF)$ and $\lambda \in K$

$$\begin{aligned} (\sigma + \mu)_i(\bullet) = \tau_i(\sigma + \mu) = \tau_i(\sigma) + \tau_i(\mu) = \sigma_i(\bullet) + \mu_i(\bullet), \\ (\lambda\sigma)_i(\bullet) = \tau_i(\lambda\sigma) = \lambda\sigma_i(\bullet), \qquad \forall i \in \mathrm{Obj}\, I. \end{aligned}$$

This shows that the addition and scalar multiplication on $\mathrm{Nat}(\Delta *, UF)$ are defined in a pointwise manner, namely first in each index $i \in \mathrm{Obj}\, I$ and second in each such index on the only argument $\bullet$. On the other hand such a pointwise definition of the operations yields a K-vector space structure on $\mathrm{Nat}(\Delta *, UF)$ such that the τ_i are K-linear. Hereto one needs to check that the pointwise

sum $\sigma + \mu$ and the scalar multiple $\lambda\sigma$ of $\sigma, \mu \in \mathrm{Nat}(\Delta *, UF)$ are again cones $\Delta * \dot{\to} UF$. Let $\ell : i \to j$ be a morphism in I then $F\ell$ is a linear map, in particular we have

$$\begin{aligned}(UF\ell \circ (\sigma + \lambda\mu)_i)(\bullet) &= UF\ell(\sigma_i(\bullet) + \lambda\mu_i(\bullet)) \\ &= (UF\ell \circ \sigma_i)(\bullet) + \lambda(UF\ell \circ \mu_i)(\bullet) && (F\ell \text{ linear}) \\ &= \sigma_j(\bullet) + \lambda\mu_j(\bullet) && (\sigma,\ \mu \text{ cones}) \\ &= (\sigma + \lambda\mu)_j(\bullet).\end{aligned}$$

This shows that $\mathrm{Nat}(\Delta *, UF)$ is indeed closed under such pointwise operations. All in all this shows the existence and uniqueness of the K-vector space structure.

Next, we have to show that this unique lift of τ is a limiting cone for F. Let $\nu : \Delta V \dot{\to} F$ be given. If we map it down to Set by U and denote the underlying cone again by ν, the universality of τ in Set yields a unique $t : V \to \mathrm{Nat}(\Delta *, UF)$, such that $\nu_i = \tau_i \circ t$ for each object i of I. We have to show the linearity of t. This follows from an argument like above, employing the pointwise definitions comming from τ and the linearity of the ν_i

$$\begin{aligned}(t(v + \lambda w))_i(\bullet) = \tau_i \circ t(v + \lambda w) = \nu_i(v + \lambda v) &= \nu_i(v) + \lambda\nu_i(w) \\ = (t(v))_i(\bullet) + \lambda(t(w))_i(\bullet), &\qquad \forall i \in \mathrm{Obj}\, I,\end{aligned}$$

hence $t(v + \lambda w) = t(v) + \lambda t(w)$. We thus get a linear $t : V \to \mathrm{Nat}(\Delta *, UF)$ such that $\nu_i = \tau_i \circ t$ in Vct_K. Next we need to show the uniqueness of t. Suppose there is a linear $t' : V \to \mathrm{Nat}(\Delta *, UF)$ such that $\nu_i = \tau_i \circ t'$. Mapping this down to Set by U implies $Ut' = t$ by the uniqueness of t in Set. However U is faithful, i.e. for parallel morphisms $f, g : V \to W$, $Uf = Ug$ implies $f = g$. Therefore we obtain $t = t'$.

(ii) Next we have to show that for any limiting cone $\nu : \Delta X \dot{\to} UF$ there is a unique K-vector space structure on X such that the ν_i are K-linear. Hereto we use the result (i) and that U reflects identities, i.e. if $Uf = 1_X$ for $f : V \to W$ then $V = W$ and $f = 1_V$. In terms of vector spaces this means that if the vector spaces V, W have the same underlying set X, i.e. are different vector space structures on X and the underlying map of f is the identity, then the vector space structures have to be equal and f is the identity in Vct_K [5]. By assumption ν is a limiting cone for UF. By the universal property of a limiting cone in Set there exists a unique natural bijection $\theta : X \to \mathrm{Nat}(\Delta *, UF)$ such

[5] Note that it is indeed possible to have two different vector space structures on one set X. For example, take a non trivial vector space V and a permutation f of its underlying set, different from the identity. The vector space structure given by $(v, w) \mapsto f(v) + f(w)$ and $(\lambda, v) \mapsto \lambda f(v)$ is clearly different from the original one.

that $\tau_i \circ \theta = \nu_i$ for all $i \in \operatorname{Obj} I$. By (i) $\operatorname{Nat}(\Delta *, UF)$ has a unique vector space structure such that the τ_i are linear. We can transport this structure by θ^{-1} to X. Equipped with this vector space structure θ becomes a linear isomorphism and the ν_i become linear maps. In particular ν lifts to a limiting cone in $\mathrm{Vct}_{\mathrm{K}}$, because τ does by (i) and they are both linear isomorphic by construction, i.e. by θ. We denote this lift by X and ν again. Suppose that there is another vector space structure on X such that the ν_i are linear. We denote this lift by X' and $\nu' : \Delta X' \dot{\to} F$. By the universal property of the limiting cone ν there is a unique linear $t : X' \to X$ such that $\nu'_i = \nu_i \circ t$. However both lifts are mapped to X and $\nu : \Delta X \dot{\to} UF$ in Set by U. By the universality of ν in Set we get $Ut = 1_X$. But U reflects identities, so $X = X'$ and $t = 1_X$ in $\mathrm{Vct}_{\mathrm{K}}$. This shows the uniqueness of the respective vector space structure on X.

(qed)

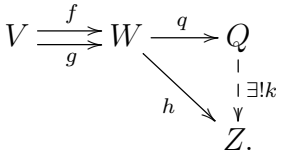
Remark 2.2.3.
The proof of proposition 2.2.2 consists of standard arguments which can be applied similarly to other situations where one wants to proof that a faithful functor $C \to \mathrm{Set}$ creates or lifts limits[6]. In these cases we will just name them and refer to this proof for the details. More concretely, given a faithful functor $U : C \to \mathrm{Set}$, for the lifting of limits we just have to show that on $\operatorname{Nat}(\Delta *, UF)$ for $F : I \to C$, I small, a pointwise structure can be defined such that it becomes a C-object and the τ_i C-morphisms. Next we need that the C-structures are U-transportable. This means that given a bijection $h : X \to Uc$ there is a U-lift x of X and a U-lift $g : x \to c$ of h such that g is an isomorphism. The rest, i.e. to show the universal property of a limiting cone to hold in C as well as there is a lift for any limiting cone of UF, follows (usually) by the standard arguments of pointwise constructions given in the proof.

A case of "standard arguments/constructions" suggests deeper structural connections. This is also the case here, namely in terms of internal structures in categories and functors which respect these. We will touch upon this topic when discussing internal monoids. However, for a rigorous argument employing internal structures for the cases here we would need to develop some amount of theory. Therefore, and because the abstract arguments wouldn't be more enlightning in our concrete case anyway, we refrain from employing it.

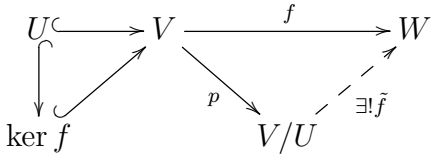
We turn to the asserted cocompleteness of $\mathrm{Vct}_{\mathrm{K}}$. The dual statement to the case of limits yields that a category C has all small colimits if and only if it has small coproducts and coequalizers of pairs. We already have the small coproducts, namely

[6] Recall that a functor $U : C \to D$ *lifts limits* for a functor $F : I \to C$, if for every limiting cone $\tau : \Delta d \dot{\to} UF$ there is a limiting cone $\sigma : \Delta c \dot{\to} F$ such that $U\sigma = \tau$. U is said to *lift small limits* when it lifts limits for each functor $F : I \to C$ with small domain category I.

the direct sums. Hence we just have to show the existence of coequalizers of pairs. Recall coequalizers to be the dual counterpart to equalizers. In more detail, given two parallel morphisms $f, g : V \to W$ a *coequalizer* $q : W \to Q$ satifies $qf = qg$ and is a universal such morphism. We express this in the following commutative diagram:



Coequalizers can be obtained from the fundamental theorem on homomorphisms in its vector space version (cf. e.g. [KM03, Folgerung 9.2.19]). It reads: given a vector subspace $U \leq V$ and a linear map $f : V \to W$ with $U \leq \ker f$ then there is a unique linear map $\tilde{f} : V/U \to W$ such that $f = \tilde{f}p$ for $p : V \to V/U$ the canonical quotient map.



(2.1)

To obtain a coequalizer of $f, g : V \to W$ from this we use again that $\mathrm{Vct_K}(V, W)$ has a vector space structure for which composition is bilinear and that the set-theoretic image of a linear map is a vector subspace. We set $Q := W/\operatorname{im}(f - g)$ and $q : W \to Q$ to the canonical quotient map. This indeed yields a coequalizer, because for any linear $h : W \to U$

$$hf = hg \quad \Leftrightarrow \quad h(f - g) = 0 \quad \Leftrightarrow \quad \operatorname{im}(f - g) \leq \ker h.$$

Next, we will show $\mathrm{Vct_K}$ to be separated and coseparated. For a category C to be *separated* it means that there is an object c such that for each pair of parallel morphisms $f, g : a \to b$ with $f \neq g$ there is a morphism $h : c \to a$ such that $fh \neq gh$. The idea behind this notion comes from Set and the fact that morphisms in Set are determined by their values on points. Points in a set X can be represented by maps $* \to X$ where $*$ denotes the one-point set. Hence for two maps $f, g : X \to Y$ to differ on a point $P \in X$ means that there is a map $p : * \to X$ such that $fp \neq gp$. The one-point set $*$ is the most basic but nontrivial set. In this sense one can see a seperator in a category (if it exists) as the most basic but nontrivial object in the category. This suggests a candidate for a separator in $\mathrm{Vct_K}$, namely a one-dimensional K-vector space, e.g. K regarded as a K-vector space over itself. Indeed, selecting a vector $v \in V$ for a vector space V in a linear way amounts to give a linear map $\nu : K \to V$ with e.g. $\nu(1) = v$ [7]. Because the forgetful functor $U : \mathrm{Vct_K} \to \mathrm{Set}$ is faithful, i.e. linear maps are determined by their set-theoretic

[7] One can indeed take any $\lambda \neq 0$ such that $\nu(\lambda) = v$. This is an instance of the following fact.

maps, different linear maps $f, g : V \to W$ have to differ on a vector $v \in V$. Let $\nu : K \to V$ be the linear map defined by $\nu(1) = v$, then $f\nu \neq g\nu$ holds. This shows that K is indeed a separator.

A coseparator is the dual notion to a separator, i.e. for a category C an object c is said to be a *coseparator* when for each pair of parallel morphisms $f, g : a \to b$ with $f \neq g$ there is a map $h : b \to c$ such that $hf \neq hg$. Here, the idea behind this notion comes again from Set and the assignment of attributes, e.g. truth values. In Set for two different maps $f, g : X \to Y$ there is a point $P \in X$ such that $f(P) \neq g(P)$. We can now "sample" Y in such a way that $f(P)$ and $g(P)$ have different attributes assigned. We need only two such attributes, so we take as the set of attributes the set $\{0, 1\}$ of two elements. A sampling of Y is then a map $h : Y \to \{0, 1\}$ which gives a partition of Y in two disjoint subsets[8]. Each of those subsets is the set of the objects with a specific attribute. We could for example define h to be 1 on $f(P)$ and 0 on $Y \setminus \{f(P)\}$. This yields $hf \neq hg$. $\{0, 1\}$ is thus a coseperator in Set and in fact the basic example of such. In $\mathrm{Vct_K}$ we can separate with the help of linear forms, i.e. linear maps $V \to K$. Let $f, g : V \to W$ be linear maps with $f \neq g$. Let $v \in V$ be a vector such that $f(v) \neq g(v)$. Set $w := f(v) - g(v)$. By construction $w \neq 0$. With the help of the basis extension theorem together with the structure theorem of vector spaces (cf. e.g. [KM03, Satz 2.3.14, Satz 2.3.15]) we can construct a linear form $\alpha : W \to K$ with $\alpha(w) = 1$. By linearity we get $\alpha(f(v)) - \alpha(g(v)) = 1$ and in particular $\alpha(f(v)) \neq \alpha(g(v))$ or $\alpha f \neq \alpha g$. This shows that K is a coseparator.

The last property stated in proposition 2.2.1 is the abelianess of $\mathrm{Vct_K}$. For a category to be *abelian* it means that it is a linear category which has kernels, cokernels and where every mono is a kernel and every epi a cokernel (cf. [Mac98, section VIII.3] or [Fre03, chapter 2]). A *kernel* of a morphism $f : a \to b$ can be described as the equalizer of f and the zero arrow $0 : a \to b$. A *cokernel* of f is in a dual manner a coequalizer of f and the zero arrow. Indeed, we have implicitly used the constructions for kernels and cokernels in our above constructions of equalizers and coequalizers. So a kernel of a linear map $f : V \to W$ is nothing else than the inclusion $\ker f \hookrightarrow V$ and a cokernel the quotient map $q : W \to W/\operatorname{im} f$.

Recall a morphism f to be called a mono if it is left cancellable, i.e. $fg = fh$ implies $g = h$ for any pair of parallel morphisms g, h composable with f from the

From an intrinsic point of view, i.e. within $\mathrm{Vct_K}$, the 1 of the field K is nowhere a distinguished element. We will see further instances of this later. Note that this fact doesn't mean that the $1 \in K$ is not important to the vector space structure. It just states that if we want to access the 1 by the means of linear maps only, we can't do so. Instead we have to do it externally, for example we have to consider the underlying sets of vector spaces, i.e. apply the forgetful functor $U : \mathrm{Vct_K} \to \mathrm{Set}$. This is tightly related with the fact that $\mathrm{Vct_K}$ has a zero object. Thus each vector space has only one point from the intrinsic point of view, namely the zero vector.

[8] We don't assert that h is surjective, i.e. one of these mentioned subsets may be empty.

right. The dual notion is that of an epi. f is called *epi* if it is right cancellable, i.e. $gf = hf$ implies $g = h$ for any pair of parallel morphisms g, h composable with f from the left. In Set the monos are precisely the injective maps and the epis are precisely the surjective maps. One can see this for example with the help of the separating one-point set $*$ and the coseperating set $\{0, 1\}$. In the case of a monic $f : X \to Y$ the defining property yields that $fp = fq$ implies $p = q$ for two points $p, q : * \to X$. This is equivalent to the defining property of injectivity, namely $f(P) = f(Q)$ implies $P = Q$ for $P, Q \in X$. Given an injective map $f : X \to Y$ and maps $g, h : Z \to X$ such that $fg = fh$ we obtain $h = g$ from the injectivity because $f(g(x)) = f(h(x))$ implies $h(x) = g(x)$ for all $x \in X$. In the case of an epi $f : X \to Y$ we can consider the two characteristic maps $g := \chi_{\operatorname{im} f} : Y \to \{0, 1\}$ and $h := \chi_Y : Y \to \{0, 1\}$. We get $gf = hf$ and so $h = g$ by the defining property of epis. This shows that $\operatorname{im} f = Y$, hence the surjectivity of f. For a surjective $f : X \to Y$ let $g, h : Y \to Z$ be maps such that $gf = hf$. By surjectivity for each $y \in Y$ there is an $x \in X$ with $f(x) = y$. This yields $g(y) = g(f(x)) = h(f(x)) = h(y)$ and hence $g = h$. In $\mathrm{Vct_K}$ a similar fact holds.

Proposition 2.2.4. *In* $\mathrm{Vct_K}$ *a linear map* $f : V \to W$ *is a monomorphism if and only if it is monic and it is an epimorphism if and only if it is an epi.*

Proof. First recall that a faithful functor U reflects monos and epis, i.e. if Uf is monic then f is already monic and similar for epis (cf. [Mac98, exercise I.5.9 and its dual]). Because a monomorphism is an injective linear map and an epimorphism is a surjective linear map its image under the forgetful functor $U : \mathrm{Vct_K} \to \mathrm{Set}$ is thus a mono, respectively an epi in Set by our result from above and hence a mono, respectively an epi in $\mathrm{Vct_K}$ by reflection. On the other hand we can use the separator and coseperator K in $\mathrm{Vct_K}$ to show that a mono, respectively an epi is an monomorphism, respectively an epimorphism in $\mathrm{Vct_K}$ like we did in Set. For monos being monomorphisms the argument doesn't even need a separator if one uses that f is a monomorphism if and only if $k : \ker f \hookrightarrow V$ is the zero map. By the definition of a kernel we have $f0 = fk$. By f being monic we have $k = 0$, hence a trivial kernel and thus f has to be a monomorphism. In the case of epis we can't use characteristic maps though. Instead of using $\chi_{\operatorname{im} f}$ we can argue indirectly and assume that for an epi $f : V \to W$ there is a $w \in W \setminus \operatorname{im} f$. We set $h : W \to K$ to be the zero map and choose a $g : W \to K$ with $g(w) = 1$. We have already seen the existence of such a g in the discussion of K being a coseperator. For g we obtain by construction $\operatorname{im} f \leq \ker g$, hence $gf = hf$ and finally the contradiction $g = h$ by the defining property of an epi.

(qed)

The property that every mono is a kernel means that for each monic $f : a \to b$ there is a morphism $g : b \to c$ such that f is a kernel of g. The idea behind

this is that each subobject can be described by the "zero set" of a morphism. In Vct_K this last statement holds literally and is in particular one consequence of the fundamental theorem on homomorphisms (2.1). Indeed by proposition 2.2.4 we know that a monic $f : V \to W$ is a monomorphism and hence an isomorphism when corestricted to its image. This yields that f is a kernel for $g : W \to W/\operatorname{im} f$ the quotient map.

Dually, the property that every epi is a cokernel means that for each epi $f : a \to b$ there is a morphism $g : c \to a$ such that f is a cokernel of g. This asserts, so to speak, that each epi is a "quotient map" of a subobject which can be described by the "image" of a morphism. In Vct_K this holds literally again. It is also a consequence of the fundamental theorem on homomorphisms. Given an epi $f : V \to W$ we know it to be an epimorphism as well. Hence f descends to an isomorphism $V/\ker f \cong W$ by the fundamental theorem on homomorphisms. This yields that f is a cokernel of its kernel $\ker f \hookrightarrow V$.

Indeed, both of these last two properties in the definition of an abelian category encode the essential features of the fundamental theorem on homomorphisms and have far reaching consequences. However, the axioms for abelian categories were designed for such categories to be like the categories Vct_K, Ab and categories of modules. That is exactly why we in particular won't employ the theory here. We have the fundamental theorem on homomorphisms, mono-epi factorisations, images, etc. directly at hand. We have just used them to recall some facts about monos and epis in the connection with Vct_K.

2.2.2 Cartesian Closedness, Internal Hom and the Linear Tensor Product

As we have already mentioned in the last section, Vct_K is not Cartesian closed. For a category C with finite products to be *Cartesian closed* it means that for each object c the endofunctor $(-) \times c : C \to C$ has a right adjoint $(-)^c : C \to C$, which is called the *exponential* of c. Recall that for a functor $F : C \to D$ to have a *right adjoint* $G : D \to C$ means that for each $c \in \operatorname{Obj} C$ and $d \in \operatorname{Obj} D$ there is a bijection $\varphi_{c,d}$

$$D(Fc, d) \overset{\varphi_{c,d}}{\cong} C(c, Gd),$$

natural in c and in d. Stated differently $\varphi : \hom(F(-), -) \dot{\to} \hom(-, G(-))$ is an isomorphism in the respective functor category[9]. This natural isomorphism φ induces two natural transformations, the so called *unit* $\eta : I_C \dot{\to} GF$ by $\eta_c := \varphi_{Fc,Fc}(1_{Fc})$ and the so called *counit* $\varepsilon : FG \dot{\to} I_D$ by $\varepsilon_d := \varphi^{-1}_{Gd,Gd}(1_{Gd})$. For each c, η_c is a universal morphism from G to c, i.e. an initial object in the comma category $(c \downarrow G)$, while for each d, ε_d is a universal morphism from F to d, i.e. a terminal

[9] Recall that for two categories C, D a functor category D^C has as objects the functors $F : C \to D$ and as morphisms the natural transformations between these functors with the composition given by the vertical, i.e. componentwise composition of natural transformations.

object in the comma category $(F \downarrow c)$[10]. Unit and counit satisfy the following *triangular identities* (cf. [Mac98, theorem IV.1.1]):

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & {\scriptstyle 1_F}\searrow & \downarrow{\scriptstyle \varepsilon F} \\ & & F, \end{array} \qquad \begin{array}{ccc} GFG & \xleftarrow{\eta G} & G \\ {\scriptstyle G\varepsilon}\downarrow & \swarrow{\scriptstyle 1_G} & \\ G. & & \end{array} \tag{2.2}$$

An *adjunction* $F \dashv G$, i.e. the data of two functors F, G together with the natural bijection φ like above, can be represented by the unit and counit satisfying the triangular identities as well (cf. [Mac98, theorem IV.1.2(v)]). The natural bijection $\varphi_{c,d}$ for the adjunction are then represented as follows

$$\begin{aligned} \varphi_{c,d}(f) &= Gf \circ \eta_c, & f &\in D(Fc, d), \\ \varphi^{-1}_{c,d}(g) &= \varepsilon_d \circ Fg, & g &\in C(c, Gd). \end{aligned} \tag{2.3}$$

Indeed, even less data is necessary to determine an adjunction (cf. [Mac98, theorem IV.1.2]). In an adjunction $F \dashv G$, F is called a left adjoint to G and G is called a right adjoint to F. Any two left adjoints for G are naturally isomorphic (cf. [Mac98, corollary IV.1.1]), dually any two right adjoints are naturally isomorphic. Note that the dual statement to $F \dashv G$ is $G \dashv F$, the dual to "η is a unit" is "η is a counit" and the dual to "ε is a counit" is "ε is a unit" respectively (cf. [Mac98, exercise IV.3.1]). Furthermore, note that a functor can have a left adjoint and a right adjoint which don't need to be equal. A prominent example is the forgetful functor $U : \mathrm{Top} \to \mathrm{Set}$ from the category Top of topological spaces with continuous maps as morphisms. Its left adjoint is the functor which assigns to a set X the topological space (X, τ_d) with discrete topology τ_d, its right adjoint is the functor which assigns to X the topological space (X, τ_i) with indiscrete topology τ_i.

If we write down the definition of adjointness for $(-) \times c : C \to C$ and its hypothetical right adjoint $(-)^c : C \to C$, we get

$$C(a \times c, b) \cong C(a, b^c).$$

This resembles "currying" in computer science, i.e. to see a map of two arguments as a map with only one argument but maps as values. Because of this the exponential b^c is usually regarded as an *internal hom-object* $\underline{\hom}(c, b)$, which can be made precise

[10] Recall that given two functors $F : C \to E$, $G : D \to E$ the *comma category* $(F \downarrow G)$ has as objects all triples (c, d, f) with $c \in \mathrm{Obj}\, C$, $d \in \mathrm{Obj}\, D$ and $f : Fc \to Gd$ a morphism in E; morphisms $(c, d, f) \to (c', d', f')$ are pairs (k, h) with $k : c \to c'$ a morphism in C, $h : d \to d'$ a morphism in D such that $f' \circ Fk = Gh \circ f$. The composition is the obvious one, which is defined componentwise. By denoting the second or first functor by an object one means the unique functor from the one-object category to the respective category D or C selecting this particular object. Confer also section II.6 in [Mac98] for further facts and examples of comma categories.

axiomatically (cf. the Eilenberg-Kelly axioms in [EK65, section I.2], which roughly state an internal hom to be compatible with the external hom and there to be an internal composition of morphisms with respective associativity and unit laws). To get a better understanding of this in the abstract setting, recall morphisms $* \to c$ to be considered as points or "elements" of the object $c \in \operatorname{Obj} C$, where $*$ denotes a terminal object in C; the interpretation of b^c as an internal hom is then supported by the following observation

$$C(a, b) \cong C(* \times a, b) \cong C(*, b^a),$$

where we have used the (natural) isomorphism $* \times a \cong a$. We refrain from giving explicitly the Eilenberg-Kelly axioms for an internal hom here. The reason is that we will use this notion either in connection with a (symmetric) monoidal structure, like for example the one given by the product and a terminal object as its identity object[11], or we will directly use the (external) hom-sets and equip them with an appropriate structure such that composition of morphisms is structure-preserving in a reasonable sense, like e.g. the hom-sets $\mathrm{Vct}_{\mathrm{K}}(V, W)$ with the vector space structure by pointwise operations. For us an internal hom is thus either the parametrized right adjoint to the endofunctor $(-) \otimes c$, $c \in \operatorname{Obj} C$, induced by the "tensor-product" bifunctor $\otimes : C \times C \to C$, which by the parameter theorem on adjunctions yields a bifunctor $\underline{\mathrm{hom}} : C^{op} \times C \to C$ (cf. [Mac98, theorem IV.7.3]), or, for the concrete categories C considered here[12], the bifunctor $\underline{\mathrm{hom}} : C^{op} \times C \to C$ as the actual external hom-functor $C(-, -)$, where each $C(a, b)$ is equipped with an appropriate structure such that it can be regarded as an object of C and the maps $C(f, g)$ can be regarded as morphisms of C. Of course an internal hom obtained from a monoidal structure satisfies the Eilenberg-Kelly axioms (cf. [EK65, sections I.2,II.3]). The internal homs from our second use of this notion belong to the type of internal homs which in fact were the case studies for the Eilenberg-Kelly axioms (cf. introduction of [EK65]), so, like in the case of abelianess of $\mathrm{Vct}_{\mathrm{K}}$, it doesn't seem meaningful to apply the general theory to them. A prototypical example of a Cartesian closed category is the category Set. The exponential X^Y is then indeed given by the set of maps $Y \to X$ and the counit of the adjunction is the evaluation map $\mathrm{ev}_X : X^Y \times Y \to X$, $(f, y) \mapsto f(y)$ (cf. [Mac98, section IV.6]).

An object c in a category C with binary products is said to be *exponentiable* when $(-) \times c : C \to C$ has a right adjoint. In $\mathrm{Vct}_{\mathrm{K}}$ we will show that the the only exponentiable object is the trivial vector space 0. The reason for 0 being exponentiable lies solely in the fact that it is an identity for the product in $\mathrm{Vct}_{\mathrm{K}}$, i.e. $V \oplus 0 \cong V$ naturally in V. Stated otherwise $(-) \oplus 0 \cong I$, where I denotes

[11] We will recall the general notion of a monoidal structure in more detail later.

[12] Recall that a concrete category (over Set) is a pair (C, U) with a category C and a faithful functor $U : C \to \mathrm{Set}$. One can think of the objects of a concrete category as being sets equipped with a further structure.

the identity functor. This yields that the identity functor is also a right adjoint for $(-) \oplus 0$ in particular.

For a non-trivial vector space V, $(-) \oplus V$ can't have a right adjoint. The reason is that a functor which has a right adjoint is cocontinuous as well (cf. dual of [Mac98, theorem V.5.1]). Recall that for a functor $F : C \to D$ to be *cocontinous* it means to preserve colimits, i.e. for any functor $J : I \to C$ which has a colimit $\varinjlim J$ in C together with the colimiting cone $\tau : J \dot{\to} \Delta \varinjlim J$, $F \varinjlim J$ is a colimit of FJ in D and $F\tau : FJ \dot{\to} \Delta F \varinjlim J$ is a colimiting cone. Dually, one gets the notion of *continuity* of F, or what it means for F to preserve limits, by replacing "colimit" by "limit" and "colimiting cone" by "limiting cone". Of course there are specialized versions of these definitions to preserve special type of limits or colimits, i.e. only limits or colimits for a particular category I and functors $J : I \to C$, as well. To be cocontinuous it means to preserve small coproducts in particular. But for $(-) \oplus V$ to preserve small coproducts it would mean

$$(\bigoplus_{i \in I} V_i) \oplus V \cong \bigoplus_{i \in I} (V_i \oplus V)$$

for any family of small vector spaces V_i with small index set I. This can be only the case for $V = 0$. To see this suppose $V \neq 0$. Set $V_i = 0$ for every $i \in I$. Since $\bigoplus_{i \in I} 0 \cong 0$ and $0 \oplus V \cong V$ the above isomorphism reads

$$V \cong (\bigoplus_{i \in I} 0) \oplus V \cong \bigoplus_{i \in I} (0 \oplus V) \cong \bigoplus_{i \in I} V.$$

However, since I may be any arbitrary small set and V is a small set this can't hold in general because the cardinalities of the right-hand side and the left-hand side may differ. We can consider the dimensions[13] for example. Let $B \subset V$ be a basis. Set I to be the power set of B. We have then $\dim V = \operatorname{card} B$ and $\dim \bigoplus_{i \in I} V = \operatorname{card}(I \times B)$. By Cantor's diagonal argument we have $\operatorname{card} B < \operatorname{card} I \leq \operatorname{card}(I \times B)$ and so $\dim V < \dim \bigoplus_{i \in I} V$. Since the dimension of a vector space is invariant under isomorphism, this shows that both vector spaces can't be isomorphic. With this we have finally established our assertion of a trivial vector space being the only exponentiable object in $\mathrm{Vct}_{\mathrm{K}}$.

Proposition 2.2.5. *In* $\mathrm{Vct}_{\mathrm{K}}$ *the only exponentiable object is a zero object.*

However, the failure of $\mathrm{Vct}_{\mathrm{K}}$ to not be Cartesian closed isn't due to an inability to represent linear maps between two vector spaces internally, i.e. to have an internal hom. Indeed, we have already used several times that the (external) hom-set

[13] We mean the general definition of the dimension $\dim V$ of an arbitrary vector space V as the cardinality of a basis for V.

$\mathrm{Vct_K}(V,W)$ carries a K-vector space structure by pointwise operations. Furthermore, due to the pointwise operations, one has

$$\begin{aligned}\mathrm{Vct_K}(f,g)(h+\lambda l) &= g\circ(h+\lambda l)\circ f = g\circ(h\circ f+\lambda l\circ f) = g\circ h\circ f+\lambda g\circ h\circ l\\ &= \mathrm{Vct_K}(f,g)h+\lambda\,\mathrm{Vct_K}(f,g)l, \qquad \lambda\in K.\end{aligned}$$

Thus, regarding $\mathrm{Vct_K}(V,W)$ as a vector space, $\underline{\mathrm{Vct}}_{\mathrm{K}}(V,W)$ will yield an internal hom. The failure of Cartesian closedness lies solely in the fact that linear maps $V\oplus W\to Z$ aren't the correct notion of linear maps of "two arguments". As is well known, the correct notion is that of bilinearity and bilinear maps are represented by the tensor product as linear maps $V\otimes W\to Z$ and not by the (Cartesian) product (cf. e.g. [KM03, section 10.1]).

One can obtain this in two ways. The first way is the usual approach of linear algebra to consider bilinear maps. In categorical terms this amounts to consider the functor $Bil(V,W;(-)):\mathrm{Vct_K}\to\mathrm{Set}$ and to show this functor representable. Here $Bil(V,W;(-))$ assigns to each vector space Z the set of bilinear maps $V\times W\to Z$ and to morphisms $f:Z\to Z'$ the covariant hom-map $f_*:Bil(V,W;Z)\to Bil(V,W;Z')$, $b\mapsto f\circ b$. Recall that for a functor $F:C\to\mathrm{Set}$ to be *representable* it means that there is an object $c\in C$ such that $C(c,-)\cong F$ and that (for locally small categories) universal morphisms and representability are equivalent notions (cf. [Mac98, section III.2])[14]. The representing object of $Bil(V,W;(-))$ is then indeed $V\otimes W$ which can be read off from the universal property of the tensor product as it is usually given in linear algebra.

The other way to obtain the tensor product is by decurrying, i.e. to consider linear maps from V with values in the linear maps $W\to Z$ and to ask how to present them as linear maps of two arguments. Again, in categorical terms this amounts to search for a natural bijection $\mathrm{Vct_K}(V\otimes W,Z)\cong\mathrm{Vct_K}(V,\underline{\mathrm{Vct}}_{\mathrm{K}}(W,Z))$, or in other words to search for a left adjoint for $\underline{\mathrm{Vct}}_{\mathrm{K}}(W,-)$ for each vector space W. The next proposition shows that we could indeed define the tensor product as a left adjoint in this way.

Proposition 2.2.6. *In* $\mathrm{Vct_K}$ *for each vector space* W *the internal hom-functor* $\underline{\mathrm{Vct}}_{\mathrm{K}}(W,-)$ *has a left adjoint* $F(-,W)\dashv\underline{\mathrm{Vct}}_{\mathrm{K}}(W,-)$. *By the parameter theorem on adjunctions there is a unique way to make* F *a bifunctor* $\mathrm{Vct_K}\times\mathrm{Vct_K}\to\mathrm{Vct_K}$. *Moreover as a bifunctor* F *is isomorphic to the tensor product of* K*-vector spaces from linear algebra.*

[14] If one considers categories which hom-sets are not small, one has to consider a bigger category of sets than Set. This category is demanded to have as hom-sets all maps $X\to Y$ between two of its objects X,Y. Representability, to be equivalent to the notion of universal morphisms, has to be considered in this larger setting then. Note the two notions to be only equivalent when one modells categories on set-theory. The notion of representability is bounded to sets whereas the notion of universal morphism can be formulated in a pure categorical language like ETAC, the elementary theory of abstract categories, extended by a functor (cf. [Mac98, section II.1], [Law65]).

In the proof we will use the following abstract argument which is due to the special adjoint functor theorem.

Lemma 2.2.7. *Given a category C with small hom-sets and a functor $F : \mathrm{Vct_K} \to C$, then F has a left adjoint if and only if it is continuous, i.e. if and only if it preserves small limits.*

Proof. (lemma) This follows from proposition 2.2.1 and the corollary to the special adjoint functor theorem in [Mac98, section V.8].

(qed)

Proof. (proposition) We show the continuity of $\underline{\mathrm{Vct}}_{\mathrm{K}}(W, -)$ and apply the lemma. This will yield F as a bifunctor by the dual of the parameter theorem on adjunctions (cf. dual of [Mac98, theorem IV.7.3]). Next, we will show the adjunction $(-)\otimes W \dashv \underline{\mathrm{Vct}}_{\mathrm{K}}(W, -)$ by an explicit construction. By the uniqueness of the left adjoint up to a natural isomorphism this will yield in particular that F and $\otimes$ are isomorphic as bifunctors.

(i) The continuity of $\underline{\mathrm{Vct}}_{\mathrm{K}}(W, -)$ can be concluded easily from $U\,\underline{\mathrm{Vct}}_{\mathrm{K}}(W, -) = \mathrm{Vct}_{\mathrm{K}}(W, -)$, for the forgetful functor $U : \mathrm{Vct_K} \to \mathrm{Set}$ creates limits by proposition 2.2.2 and the external hom $\mathrm{Vct}_{\mathrm{K}}(W, -)$ is continuous by theorem V.4.1 in [Mac98]. Indeed, let I be a small category and $F : I \to \mathrm{Vct_K}$ a functor. By the completeness of $\mathrm{Vct_K}$, $L := \varprojlim F$ exists in $\mathrm{Vct_K}$ anyway, so to require its existence is not necessary. Let $\tau : \Delta L \dot{\to} F$ be the limiting cone. By the continuity of $\mathrm{Vct}_{\mathrm{K}}(W, -)$ we have that $\mathrm{Vct}_{\mathrm{K}}(W, \tau) : \Delta\,\mathrm{Vct}_{\mathrm{K}}(W, L) \dot{\to} \mathrm{Vct}_{\mathrm{K}}(W, F)$ is a limiting cone, hence $U\,\underline{\mathrm{Vct}}_{\mathrm{K}}(W, \tau) : \Delta U\,\underline{\mathrm{Vct}}_{\mathrm{K}}(W, L) \dot{\to} U\,\underline{\mathrm{Vct}}_{\mathrm{K}}(W, F)$ is a limiting cone. By proposition 2.2.2 U creates limits. This means that there is a unique cone $\sigma : \Delta\,\underline{\mathrm{Vct}}_{\mathrm{K}}(W, L) \dot{\to} \underline{\mathrm{Vct}}_{\mathrm{K}}(W, F)$ such that $U\sigma = U\,\underline{\mathrm{Vct}}_{\mathrm{K}}(W, \tau)$ and moreover this σ is a limiting cone. Because $\underline{\mathrm{Vct}}_{\mathrm{K}}(W, \tau) : \Delta\,\underline{\mathrm{Vct}}_{\mathrm{K}}(W, L) \dot{\to} \underline{\mathrm{Vct}}_{\mathrm{K}}(W, F)$ is a cone which is obviously mapped to $U\,\underline{\mathrm{Vct}}_{\mathrm{K}}(W, \tau)$ by U, we obtain $\sigma = \underline{\mathrm{Vct}}_{\mathrm{K}}(W, \tau)$ by the uniqueness and in particular that $\underline{\mathrm{Vct}}_{\mathrm{K}}(W, \tau)$ is a limiting cone.

(ii) Now we want to show $(-) \otimes W \dashv \underline{\mathrm{Vct}}_{\mathrm{K}}(W, -)$. By theorem IV.1.2(iii) in [Mac98] to prove the adjointness it suffices to give a counit, i.e. a natural transformation $\varepsilon : \underline{\mathrm{Vct}}_{\mathrm{K}}(W, -) \otimes W \dot{\to} I$ to the identity functor I such that for each vector space V, ε_V is a universal morphism from $(-) \otimes W$ to V. For this purpose we choose the evaluation map ev_V uniquely determined by

$$\mathrm{ev}_V : \underline{\mathrm{Vct}}_{\mathrm{K}}(W, V) \otimes W \to V, \quad f \otimes v \mapsto fv \quad \text{on pure tensors.}$$

By the pointwise vector space structure of $\underline{\mathrm{Vct}}_{\mathrm{K}}(W, V)$ and due to $f \otimes v = g \otimes w$ if and only if $w = \lambda v$, $g = 1/\lambda\, f$ for a scalar $\lambda \neq 0$, this map is well-defined and linear. The naturality follows from the commutativity of the subsequent

diagram, because those pure tensors span the vector space of the linear tensor product and we know the underlying map of the linear tensor product of two linear maps from linear algebra (recall $h_* = \underline{\mathrm{Vct}}_{\mathrm{K}}(W,h)$):

$$\begin{array}{c} V \\ {\scriptstyle h}\downarrow \\ V', \end{array} \qquad \begin{array}{ccc} \underline{\mathrm{Vct}}_{\mathrm{K}}(W,V)\otimes W & \xrightarrow{\mathrm{ev}_V} & V \\ {\scriptstyle h_*\otimes 1_W}\downarrow & & \downarrow{\scriptstyle h} \\ \underline{\mathrm{Vct}}_{\mathrm{K}}(W,V')\otimes W & \xrightarrow{\mathrm{ev}_{V'}} & V', \end{array} \qquad \begin{array}{ccc} f\otimes w & \longmapsto & f(w) \\ \downarrow & & \downarrow \\ (h\circ f)\otimes w & \longmapsto & h(f(w)). \end{array}$$

As for the universality, let $\alpha : U \otimes W \to V$ be a linear map. Suppose there is a linear map $A : U \to \underline{\mathrm{Vct}}_{\mathrm{K}}(W,V)$ such that $\alpha = \mathrm{ev}_V \circ (A \otimes 1_W)$. In particular $A(u)$ is the linear map

$$\alpha(u\otimes(-)) : W \to V, \quad w \mapsto \alpha(u\otimes w).$$

On the other hand if we define a map $A : U \to \underline{\mathrm{Vct}}_{\mathrm{K}}(W,V)$ by $u \mapsto \alpha(u\otimes(-))$ then the bilinearity of the tensor-map $\otimes : U\times V \to U\otimes V$ yields the linearity of A. By construction $\alpha = \mathrm{ev}_V \circ (A\otimes 1_W)$ holds again. This shows the existence and uniqueness of such a linear A and hence ev_V to be a universal morphism as asserted.

$$\begin{array}{c} U \\ {\scriptstyle \exists! A}\downarrow \\ \underline{\mathrm{Vct}}_{\mathrm{K}}(V,W), \end{array} \qquad \begin{array}{ccc} U\otimes W & & \\ {\scriptstyle A\otimes 1_W}\downarrow & \searrow^{\alpha} & \\ \underline{\mathrm{Vct}}_{\mathrm{K}}(W,V)\otimes W & \xrightarrow[\mathrm{ev}_V]{} & V. \end{array}$$

(qed)

Remarks 2.2.8.

(a) In the proof of proposition 2.2.6 we have shown in particular that there is an adjunction $(-)\otimes W \dashv \underline{\mathrm{Vct}}_{\mathrm{K}}(W,-)$ with counit the evaluation map

$$\mathrm{ev}_Z : \underline{\mathrm{Vct}}_{\mathrm{K}}(W,Z)\otimes W \to Z, \quad f\otimes z \mapsto fz \quad \text{on pure tensors.}$$

One can moreover show the unit to be given by[15]

$$\eta_V : V \to \underline{\mathrm{Vct}}_{\mathrm{K}}(W, V\otimes W), \quad v \mapsto s_v, \quad s_v : w \mapsto v\otimes w.$$

Thus η_V assigns to each v its "slice" of $V\otimes W$ at v.

[15] One way of doing this is to check the identity $1_{V\otimes W} = \mathrm{ev}_{V\otimes W}\circ(\eta_V\otimes 1_W)$. Here we use the representation of the natural bijection φ of the adjunction by the counit as in (2.3) and the V-component of the unit to be the φ-image of $1_{V\otimes W}$.

(b) The nice thing about defining a tensor product as a left adjoint to the internal hom-functor is that we obtain two properties for free. First, we obtain it being a bifunctor by the parameter theorem on adjunctions as already stated in the proposition. Second, we obtain from its left adjointness that it is cocontinuous. In particular we get the distributivity laws concerning arbitrary direct sums. In general the cocontinuity would hold only in the left argument of the so defined tensor product, but because of its symmetry, i.e. $V \otimes W \cong W \otimes V$ natural in V, W, it holds in both.

(c) In the proof of proposition 2.2.6 we have argued $\underline{\mathrm{Vct}}_{\mathrm{K}}(W, -)$ to be continuous, for $U\,\underline{\mathrm{Vct}}_{\mathrm{K}}(W, -)$ is and U creates (small) limits. This is an instance of the general fact, that if U creates limits, it will reflect them as well (cf. [AHS04]). For a functor $U : C \to D$ to *reflect* small limits it means that given any functor $F : I \to C$ with small I and a cone $\tau : \Delta c \dot{\to} F$ such that $U\tau : \Delta Uc \dot{\to} UF$ is a limiting cone for UF, then τ is a limiting cone for F. This fact follows immediately from the definitions and is in essence the argument we gave in the proof.

(d) A further consequence of U creating limits and Set being complete is U to be continuous (cf. [Mac98, theorem V.4.2]). By lemma 2.2.7 U has a left adjoint $F : \mathrm{Set} \to \mathrm{Vct}_{\mathrm{K}}$. This functor assigns to each set X the free K-vector space $F(X)$ over X and each map $f : X \to Y$ its unique linear extension to $F(X) \to F(Y)$. We will see later in proposition 3.2.17 why this is the case.

2.2.3 Monoidal Structures on $\mathrm{Vct}_{\mathrm{K}}$ and Set

Like the Cartesian product in Set or the direct product in $\mathrm{Vct}_{\mathrm{K}}$, the tensor product in $\mathrm{Vct}_{\mathrm{K}}$ can also be understood as a "multiplication" of vector spaces. This idea of "multiplication" of spaces can be made precise by the means of the notion of a *monoidal structure* on a category. Recall that a *monoidal category* is a category C together with a bifunctor $\otimes : C \times C \to C$, an object e of C and three natural isomorphisms

$$\alpha : \otimes \circ (I \times \otimes) \dot{\to} \otimes \circ (\otimes \times I), \quad \rho : (-) \otimes e \dot{\to} I, \quad \lambda : e \otimes (-) \dot{\to} I,$$

where I is the identity functor of C and we identify $C \times (C \times C)$, $(C \times C) \times C$ with $C \times C \times C$ like we do it in the case of ordered triples. The following diagrams are

required to commute for all objects $a, b, c, d \in \operatorname{Obj} C$:

$$\begin{array}{ccccc} a \otimes (b \otimes (c \otimes d)) & \xrightarrow{\alpha_{a,b,c\otimes d}} & (a \otimes b) \otimes (c \otimes d) & \xrightarrow{\alpha_{a\otimes b,c,d}} & ((a \otimes b) \otimes c) \otimes d \\ \downarrow{\scriptstyle 1_a \otimes \alpha_{b,c,d}} & & & & \uparrow{\scriptstyle \alpha_{a,b,c} \otimes 1_d} \\ a \otimes ((b \otimes c) \otimes d) & & \xrightarrow[\alpha_{a,b\otimes c,d}]{} & & (a \otimes (b \otimes c)) \otimes d, \end{array}$$

(associativity)

$$\begin{array}{ccc} a \otimes (e \otimes c) & \xrightarrow{\alpha_{a,e,c}} & (a \otimes e) \otimes c \\ & {\scriptstyle 1_a \otimes \lambda_a} \searrow \quad \swarrow {\scriptstyle \rho_a \otimes 1_c} & \\ & a \otimes b. & \end{array}$$

(left- and right unit law)

Moreover, one imposes $\lambda_e = \rho_e : e \otimes e \to e$. A monoidal category is said to be *strict* when α, λ, ρ are the identity transformations, i.e. they are the identity morphism in each component. It is said to be *symmetric* when there is a natural isomorphism

$$\tau : \otimes \dot{\to} \otimes \circ T, \qquad \text{such that} \qquad (\tau \circ 1_T) \cdot \tau = 1_\otimes,$$

where "$\cdot$" denotes the vertical, i.e. componentwise composition and "$\circ$" denotes the horizontal composition of natural transformations (cf. [Mac98, section II.5]), $1_\otimes$ is the identity transformation of $\otimes$ and 1_T the identity transformation of the "twist" endofunctor $T : C \times C \to C \times C$ which maps (a, b) to (b, a) and (f, g) to (g, f). Stated differently, a *symmetric braiding* of C, which is also called *symmetry* or *symmetric commutativity constraint*, is for each pair of objects an isomorphism $\tau_{c,d} : c \otimes d \to d \otimes c$ natural in c, d and the following diagram is required to commute for all $c, d \in \operatorname{Obj} C$:

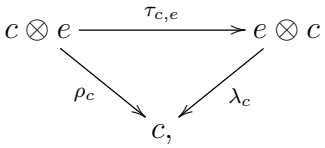
$$\begin{array}{ccc} c \otimes d & \xrightarrow{1_{c\otimes d}} & c \otimes d \\ & {\scriptstyle \tau_{c,d}} \searrow \quad \nearrow {\scriptstyle \tau_{d,c}} & \\ & d \otimes c & \end{array}$$

(symmetry I)

Furthermore, the following diagrams have to commute to ensure the compatibility to the rest of the structure morphisms:

$$\begin{array}{ccccc} a \otimes (b \otimes c) & \xrightarrow{\alpha_{a,b,c}} & (a \otimes b) \otimes c & \xrightarrow{\tau_{a\otimes b,c}} & c \otimes (a \otimes b) \\ \downarrow{\scriptstyle 1_a \otimes \tau_{b,c}} & & & & \downarrow{\scriptstyle \alpha_{c,a,b}} \\ a \otimes (c \otimes b) & \xrightarrow[\alpha_{a,c,b}]{} & (a \otimes c) \otimes b & \xrightarrow[\tau_{a,c}\otimes 1_b]{} & (c \otimes a) \otimes b, \end{array}$$

(symmetry II)

$\rho = \lambda \cdot \tau_{(-,e)}$; on components:  (symmetry III)

where $\tau_{(-,e)}$ stands for the natural isomorphism $\tau_{(-,e)} : (-) \otimes e \dot{\to} e \otimes (-)$ induced by τ (cf. [Mac98, proposition II.3.2]) and $a, b, c, d \in \operatorname{Obj} C$.

A monoidal category is the "objectification" of the idea of a monoid: take the axioms of a monoid, replace equality by natural isomorphisms and impose appropriate coherence axioms such that the associativity and identity laws up to isomorphism will be meaningful. The bifunctor is therefore often called a "*product*", e is called an "*identity*", α the "*associativity law*", λ the "*left-unit law*" and ρ the "*right-unit law*". As already suggested by the notation and because most of the products considered in this thesis resemble the tensor product from linear algebra, we will call the bifunctor preferably a "tensor-product" here. The *coherence axioms*, i.e. the postulated diagrams which have to commute, are necessary for applying the associativity and unit laws in the case of multiple products like we are used to. This is expressed in the so called "coherence theorems" (cf. [Mac98, corollary VII.2.1, theorem XI.1.1]).

Indeed, a monoid can be regarded as a discrete category equipped with a (strict) monoidal structure. It is (strictly) symmetric if and only if it is abelian. Conversely, a (symmetric) monoidal category C yields a (commutative) monoid by passing from C to its set of isomorphism classes. Particular constructions which involve this are known in the literature as the Burnside rig for distributive categories or the Grothendieck group in K-theory. Further basic examples of symmetric monoidal categories are Set equipped with the monoidal structure induced by the (Cartesian) product as the set of pairs, or $\mathrm{Vct_K}$ with the monoidal structure induced by the tensor product. In the case of Set the identity is a terminal object, i.e. a one-point set $*$, the natural isomorphisms are the usually ignored natural isomorphisms for associativity for the iterated Cartesian product of sets, as well as the left- and right-unit isomorphisms. In fact, any category C with finite products admits a structure of a symmetric monoidal category in this way (cf. [Mac98, section VII.1]), in particular $\mathrm{Vct_K}$ with the direct product and a trivial vector space as identity. We call such symmetric monoidal structures *Cartesian*. In the case of $\mathrm{Vct_K}$ and the tensor product the identity is a one-dimensional vector space, preferably one chooses the field K regarded as a one-dimensional vector space over itself. The associativity, symmetry and unit laws are the well-known one from linear algebra, namely the one given on pure tensors as

$$\begin{aligned} &\alpha_{V,W,Z} : \ v \otimes (w \otimes z) \mapsto (v \otimes w) \otimes z, \quad \rho_V : \ v \otimes \mu \mapsto \mu v, \quad \lambda_V : \ \mu \otimes v \mapsto \mu v, \\ &\tau_{V,W} : \ v \otimes w \mapsto w \otimes v, \end{aligned} \tag{2.4}$$

for $v \in V$, $w \in W$, $z \in Z$, $\mu \in K$ (cf. e.g. [KM03, section 10.1] and [Mac98, section VII.1]). One can verify the imposed diagrams to commute directly. In fact, there is a generalization of the uniqueness of the identity element in an ordinary monoid to monoidal categories: given the bifunctor and the associativity law the identity elements are determined up to a natural isomorphism (cf. remark 1.1.4 in [Sch92] and the references therein)[16]. So, in the case of Set the identity has to be a one-point set and in the case of $\mathrm{Vct_K}$ the identity has to be a one-dimensional vector space. In Set and $\mathrm{Vct_K}$ the symmetric braiding τ is determined by the other structure maps as well. We will explain this in more detail for $\mathrm{Vct_K}$ [17]. It is essentially due to the identity K being a separator in $\mathrm{Vct_K}$.

Firstly, from the two coherence axioms $\lambda_K = \rho_K$ together with $\rho = \lambda \cdot \tau'_{(-,K)}$ we obtain for any (symmetric) braiding τ' that $\tau'_{K,K} = 1_{K\otimes K}$, because all components of the structure morphisms are isomorphisms. Secondly, we use the naturality of τ' to intertwine $\tau'_{V,W}$ with $\tau'_{K,K}$

$$\begin{array}{ccccc}
K & \xrightarrow{\lambda_K^{-1}} & K\otimes K = \otimes(K,K) & \xrightarrow{v\otimes w=\otimes(v,w)} & V\otimes W = \otimes(V,W) \\
{\scriptstyle 1_K}\downarrow & & \downarrow{\scriptstyle \tau'_{K,K}=1_{K\otimes K}} & & \downarrow{\scriptstyle \tau'_{V,W}} \\
K & \xrightarrow[\lambda_K^{-1}]{} & K\otimes K = \otimes\circ T(K,K) & \xrightarrow[w\otimes v=\otimes\circ T(v,w)]{} & W\otimes V = \otimes\circ T(V,W).
\end{array}$$

Here $v \otimes w$ denotes the map $K \otimes K \to V \otimes W$, which maps $1 \otimes 1 \in K \otimes K$ to the element $v \otimes w \in V \otimes W$; as indicated in the diagram it is the linear tensor bifunctor applied to the pair (v, w) of linear maps $v : K \to V$ with $1 \mapsto v$ and $w : K \to W$ with $1 \mapsto w$. The diagrams commute for any $v \otimes w \in V \otimes W$. Because the set of such tensors linearly spans the tensor product, we can read off from these diagrams that $\tau'_{V,W}$ coincides with the $\tau_{V,W}$ introduced above.

There is a more abstract result for monoidal categories encompassing both of our special cases. It states that for any right-closed monoidal category, where the identity object is a separator, there can be at most one symmetric braiding for the respective monoidal structure (cf. proposition 2.2 in [KR85] or proposition III.6.1 in [EK65]). For a monoidal category the property of being "right-closed" is a direct generalization from the Cartesian case and will be explained in the following.

A symmetric monoidal category C is said to be *closed* when for each object c the endofunctor $(-) \otimes c : C \to C$, which assigns to an object a the object $a \otimes c$ and to a morphism f the morphism $f \otimes 1_c$, has a right adjoint. If the monoidal structure is not symmetric one speaks of a *right-closed* monoidal category and defines *left-closedness* in a similar way. In the symmetric case the symmetric

[16] Note the fact of two identity objects in a monoidal category to be isomorphic to follow from the uniqueness of the identity element in the induced monoid of isomorphism classes. This does not show the existence of a natural isomorphism, however.

[17] In the case of Set the argument is basically the same.

braiding τ yields $(-) \otimes c \cong c \otimes (-)$. In particular $(-) \otimes c$ has a right adjoint if and only if $c \otimes (-)$ has one, so right-closedness implies left-closedness and vice versa. The right adjoint is denoted by $\underline{\hom}(c, -)$ again and comprises an internal hom-functor. To understand this point of view take the identity e and the left-unit law λ to obtain $C(c, b) \cong C(e \otimes c, b) \cong C(e, \underline{\hom}(c, b))$ natural in c and b. This generalizes the result from Cartesian closedness. Moreover, with the help of the counit of this adjunction

$$\mathrm{ev} : \underline{\hom}(c, -) \otimes c \dot{\to} I_C,$$

which is again called "evaluation", one can define an internal composition of maps. This internal composition corresponds to the external composition (cf. [MM92, section IV.2] for the Cartesian case). In proposition 2.2.6 we see in particular the closedness of $\mathrm{Vct_K}$ under the tensor product of linear algebra. Here the internal hom was the external hom-set equipped with a K-vector space structure by pointwise operations. Because of that its internal composition has as underlying set-theoretic map the external composition. The fact of being an internal composition is expressed by the property that it is a bilinear map. Note that for the internal hom the symmetry of a symmetric monoidal category C is reflected in the isomorphism $C(a, \underline{\hom}(c, b)) \cong C(c, \underline{\hom}(a, b))$, natural in a, b, c. On the other hand, such a natural isomorphism for the internal hom-functor yields the symmetry of the (right- or) left-closed monoidal structure by the Yoneda lemma (cf. [Mac98, lemma III.2.1, corollary III.2.1]).

2.3 Monoidal Functors and Monoidal Natural Transformations

We want to recall the different notions of a monoidal functor next. Let C and C' be monoidal categories. The structure morphisms for C are denoted by $\otimes$, e, α, λ, ρ, the one for C' by $\otimes'$, e', λ', ρ'. A *monoidal functor* $F : C \to C'$ consists of the following data:

(I) A functor $F : C \to C'$ between the underlying categories;

(II) A natural transformation

$$F_2 : \otimes' \circ (F \times F) \dot{\to} F \circ \otimes,$$

i.e. for each pair of C-objects (c, d) a C'-morphism $F_2(c, d) : F(c) \otimes' F(d) \to F(c \otimes d)$, natural in c, d;

(III) A C'-morphism $F_0 : e' \to Fe$.

As for the compatibility conditions the following diagrams involving the structure morphisms of C' have to commute for all $a, b, c \in \operatorname{Obj} C$:

$$\begin{array}{ccc} F(a) \otimes' (F(b) \otimes' F(c)) & \xrightarrow{\alpha'_{F(a),F(b),F(c)}} & (F(a) \otimes' F(b)) \otimes' F(c) \\ \downarrow{\scriptstyle 1_{F(a)} \otimes' F_2(b,c)} & & \downarrow{\scriptstyle F_2(a,b) \otimes' 1_{F(c)}} \\ F(a) \otimes' F(b \otimes c) & & F(a \otimes b) \otimes' c \\ \downarrow{\scriptstyle F_2(a, b\otimes c)} & & \downarrow{\scriptstyle F_2(a\otimes b, c)} \\ F(a \otimes (b \otimes c)) & \xrightarrow[F(\alpha_{a,b,c})]{} & F((a \otimes b) \otimes c), \end{array} \tag{2.5}$$

$$\begin{array}{ccc} F(c) \otimes' e' & \xrightarrow{\rho'_{F(c)}} & F(c) \\ \downarrow{\scriptstyle 1_{F(c)} \otimes' F_0} & & \uparrow{\scriptstyle F(\rho_c)} \\ F(c) \otimes' F(e) & \xrightarrow[F_2(c,e)]{} & F(c \otimes e), \end{array} \qquad \begin{array}{ccc} e' \otimes' F(c) & \xrightarrow{\lambda'_{F(c)}} & F(c) \\ \downarrow{\scriptstyle F_0 \otimes' 1_{F(c)}} & & \uparrow{\scriptstyle F(\lambda_c)} \\ F(e) \otimes' F(c) & \xrightarrow[F_2(e,c)]{} & F(e \otimes c). \end{array} \tag{2.6}$$

A monoidal functor is said to be *strong* when F_2 and F_0 are isomorphisms, it is said to be *strict* when F_2 and F_0 are the respective identity morphisms. If we have symmetric monoidal structures with symmetries τ and τ' a monoidal functor is said to be *symmetric* when in addition the following diagram commutes for all $c, d \in \operatorname{Obj} C$:

$$\begin{array}{ccc} F(c) \otimes' F(d) & \xrightarrow{\tau'_{c,d}} & F(d) \otimes' F(c) \\ \downarrow{\scriptstyle F_2(c,d)} & & \downarrow{\scriptstyle F_2(d,c)} \\ F(c \otimes d) & \xrightarrow[F(\tau_{c,d})]{} & F(d \otimes c). \end{array} \tag{2.7}$$

For example, the forgetful functor $U : \mathrm{Vct_K} \to \mathrm{Set}$ can be extended to a symmetric monoidal functor between the respective symmetric monoidal categories when we set

$$U_2(V, W) := \otimes_{V,W} : V \times W \to V \otimes W, \qquad U_0 := 1 : * \to K.$$

Here we mean by 1 the point of K that is the identity element of the field K. As we have already done in the definition, we will follow the usual abuse of notation and denote a monoidal functor only by its underlying functor, the structure maps F_2, F_0 being understood. Moreover, in cases like the one in the example of U from above we will call a functor $F : C \to D$ between the underlying categories of (symmetric) monoidal categories (symmetric) monoidal if there are structure morphisms such that F can be extended to a (symmetric) monoidal functor. This shouldn't lead to confusions.

Finally, we shall recall the notion of a morphism between monoidal functors, namely a *monoidal natural transformation*. Given two monoidal functors F, G :

$C \to C'$ a monoidal transformation $\varphi : F \dot{\to} G$ is a natural transformation of the underlying functors F, G which connects the tensor- and unit diagrams of monoidal functors for all $c, d \in \operatorname{Obj} C$:

$$\begin{array}{ccc} F(c) \otimes' F(d) & \xrightarrow{F_2(c,d)} & F(c \otimes d) \\ {\scriptstyle \varphi_c \otimes' \varphi_d}\downarrow & & \downarrow{\scriptstyle \varphi_{c\otimes d}} \\ G(c) \otimes' G(d) & \xrightarrow[G_2(c,d)]{} & G(c \otimes d), \end{array} \qquad \begin{array}{ccccc} & & e' & & \\ & {\scriptstyle F_0}\swarrow & & \searrow{\scriptstyle G_0} & \\ F(e) & & \xrightarrow{\varphi_e} & & G(e). \end{array} \tag{2.8}$$

If C and C' are symmetric monoidal categories and F, G are symmetric monoidal functors, then we don't need a further condition to assure compatibility with the symmetric braidings for a respective monoidal natural transformation $\varphi : F \dot{\to} G$; it is given by the naturality. In other words, a symmetric monoidal natural transformation $\varphi : F \dot{\to} G$ is just a monoidal natural transformation.

We shall use monoidal categories, monoidal functors and monoidal natural transformations only for a proper terminology until chapter 6. In this particular chapter those notions will, however, become crucial for the discussion and results, because we will study the category of affine algebras, the natural stage for geometric algebras of affine space. Nonetheless, we will consider mainly cases that belong to the class of the leading examples to design the general definitions, so we won't be too explicit about chasing the coherence diagrams in our theorems.

2.4 The Category of Graded Vector Spaces

In this thesis, besides working with K-vector spaces we will have to deal with graded K-vector spaces and graded K-algebras in chapter 6. To prepare for that as well as to introduce some further categorical concepts we will employ in this thesis, we give a brief discussion on both structures in this and the subsequent section. We begin with graded vector spaces first.

For convenience we focus on $\mathbb{N}$-graded vector spaces only, but remark that nearly everything we will present for this particular case carries over to the case of G-graded vector spaces for G an abelian monoid and in particular for G an abelian group. The only such group G we will need later is the additive group of $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

Our aim is to understand the category of graded K-vector spaces as a functor category. We start with recalling this general notion and some of its important properties. Given two categories C, D the *functor category* D^C has as objects the functors $C \to D$ and as morphisms the natural transformations between such functors. The composition of natural transformations has to be necessarily the vertical one, i.e.

$$(\tau \cdot \sigma)_c = \tau_c \circ \sigma_c : Fc \to Hc, \qquad c \in \operatorname{Obj} C,\ \sigma : F \dot{\to} G,\ \tau : G \dot{\to} H,$$

where the composition $\tau_c \circ \sigma_c$ is the one in D (cf. section 2.1.6 or [Mac98, section II.5]). Note that the formation of a functor category can blow up the cardinality of the underlying set of objects or the hom-sets. A trivial example is the following. Consider the universe $\mathcal{U}$ and the two-element set 2 as discrete categories. The functor category $2^{\mathcal{U}}$ is a discrete category as well. It can be understood as the set of characteristic functions on the set $\mathcal{U}$. Its set of objects has thus the cardinality of the power set of the universe $\mathcal{U}$ and is hence of strictly greater cardinality than $\mathcal{U}$. It may also happen that the functor category built from categories with small hom-sets won't have small hom-sets anymore (cf. [Mac69], but Mac Lane gives no example). Nevertheless, if C is small and D has small hom-sets then D^C has small hom-sets, too. The functor categories we will consider in this thesis are of this kind.

The employed notation for functor categories suggests their involvement with exponentials and internal homs. They are indeed related. Consider for example the category of small categories with functors as morphisms. The product of categories becomes the Cartesian product, the functor category becomes the internal hom and they are related by the exponential adjunction, which says in plain English that bifunctors can be naturally understood as functors to functor categories and vice versa. Of course, the latter statement holds more generally, but can only be expressed in this fashion if there is a particular category of categories at hand. So, morally speaking, the functor category is the internal hom in the "category of all categories", although the latter doesn't exist within our set theory.

Recall that due to the respective universal properties the formation of limits or colimits is functorial in nature (cf. exercise V.2.3 in [Mac98]). Hence, if D has all C-limits, then a coherent choice of such limits yields a functor $\varprojlim : D^C \to C$ and different choices yield isomorphic functors. An analogous statement holds for $\varinjlim$ if D has all C-colimits. Such situations arise, for example, when C is small and D is complete or cocomplete.

We can recast these facts in the language of adjoints as well. Consider the diagonal functor $\Delta : D \to D^C$ which assigns to each object d the constant functor $\Delta d : C \to D$ with $c \mapsto d$ and $f \mapsto 1_d$. D has all C-limits if and only if Δ has a right adjoint, namely $\varprojlim$, and D has all C-colimits if and only if Δ has a left adjoint, namely $\varinjlim$. An immediate consequence of this is that the functor $\varprojlim$, if it exists, is continuous. This yields a proposition on the interchange of C-limits with other limits in D. It is strengthened by the fact that limits in a functor category are determined on "points". More precisely, consider a functor $J : I \to D^C$ and for each $c \in \operatorname{Obj} C$ the evaluation-at-c functor $E_c : D^C \to D$, $F \mapsto F(c)$, $\tau \mapsto \tau_c$. If for each $c \in \operatorname{Obj} C$ the pointwise limit L_c of $E_c \circ J$ with a limiting cone $\mu_c : \Delta_I L_c \dot{\to} E_c \circ J$ exists in D, then there is a functor $L : C \to D$ with object function $c \mapsto L_c$, which is a limit of J in D^C and has as limiting cone $\tau : \Delta_I L \dot{\to} J$ with $\tau_j = (c \mapsto \mu_c(j))$ (cf. [Mac98, theorem V.3.1])[18]. Of course, analogous results hold for colimits as well. The result

[18] Note that the two Δ_I functors for μ_c and τ have a different target category.

about pointwise limits and colimits has as a consequence a further very convenient property of functor categories; I-limits, respectively I-colimits which exist in D also exist in D^C. In particular, D^C is complete or cocomplete if D is.

We want to describe the category of $\mathbb{N}$-graded K-vector spaces by a suitable functor category next. Hereto we take the set of natural numbers $\mathbb{N}$ and consider it as a discrete category. Functors $F : \mathbb{N} \to \mathrm{Vct_K}$ are just sequences of vector spaces $(F(n))_{n\in\mathbb{N}}$ and natural transformations between such functors are sequences of linear maps. Because $\mathbb{N}$ is discrete, the colimit of such a functor F is the direct sum $\bigoplus_{n\in\mathbb{N}} F(n)$. In the language of graded vector spaces F corresponds to a *gradation* on the vector space $\varinjlim F = \bigoplus_{n\in\mathbb{N}} F(n)$, the $F(k)$ are the subvector spaces of *homogeneous elements of degree* k and a natural transformation $\tau : F \to G$ corresponds to the sequence of homogeneous components of the *homogeneous map* $\varinjlim \tau = \bigoplus_{n\in\mathbb{N}} \tau_n$ *of degree zero* from $\varinjlim F$ to $\varinjlim G$. From this viewpoint the $\mathbb{N}$-graded K-vector spaces with homogeneous maps of degree zero are thus best described by the functor $\varinjlim : {\mathrm{Vct_K}}^{\mathbb{N}} \to \mathrm{Vct_K}$. Given a vector space V the functors $F \in {\mathrm{Vct_K}}^{\mathbb{N}}$ with $\varinjlim F \cong V$ constitute the set of possible $\mathbb{N}$-gradations for V. They even form a subcategory[19] of ${\mathrm{Vct_K}}^{\mathbb{N}}$, namely as the preimage of the subcategory of $\mathrm{Vct_K}$ under $\varinjlim$, which consists of the isomorphism class of V as objects and the isomorphisms as morphisms. However, a particular $\mathbb{N}$-graded vector space means a particular gradation. In consequence we define the category of $\mathbb{N}$-graded K-vector spaces (with homogeneous maps of degree zero as morphisms) to be the functor category ${\mathrm{Vct_K}}^{\mathbb{N}}$. As an immediate consequence of the above discussion on functor categories together with $\mathrm{Vct_K}$ being complete and cocomplete, we obtain the category of $\mathbb{N}$-graded K-vector spaces to be complete and cocomplete as well. Moreover, it has many of the properties of $\mathrm{Vct_K}$: it is a linear category, it is well-powered, separated, coseparated and abelian. Each of these properties can be shown by pointwise constructions and the fact that they hold in $\mathrm{Vct_K}$ (cf. proposition 2.2.1).

$\mathbb{N}$-graded vector spaces admit several very different symmetric monoidal structures. The first monoidal structure comes form the (linear) Cartesian structure. Here everything is defined in a pointwise manner. For example, the product is the pointwise direct sum, i.e. $(F \oplus G)(n) = F(n) \oplus G(n)$ for $F, G \in {\mathrm{Vct_K}}^{\mathbb{N}}$, $n \in \mathbb{N}$, and the identity is $\Delta 0$, where 0 denotes a trivial vector space. One can also introduce a pointwise tensor-product. This will yield a symmetric monoidal structure which is closed. However, due to the distributivity law

$$(\bigoplus_{n\in\mathbb{N}} F(n)) \otimes (\bigoplus_{n\in\mathbb{N}} G(n)) \cong \bigoplus_{n\in\mathbb{N}} \bigoplus_{k+\ell=n} F(k) \otimes G(l), \qquad n \in \mathbb{N},\ F, G \in {\mathrm{Vct_K}}^{\mathbb{N}},$$

which in general isn't isomorphic to $\bigoplus_{n\in\mathbb{N}}(F(n)\otimes G(n))$, the functor $\varinjlim$ isn't strong monoidal. This property is nevertheless a desirable one, for it says that the notion of

[19] Recall that given a category C, a category S with $\mathrm{Obj}\, S \subset \mathrm{Obj}\, C$ and $\mathrm{Mor}\, S \subset \mathrm{Mor}\, C$ is said to be a *subcategory* of C.

multilinearity of graded vector spaces is compatible with the notion of multilinearity in $\mathrm{Vct_K}$ and enables us to consider graded algebras as monoids in $\mathrm{Vct_K}^{\mathbb{N}}$, which we will explain in the next section. For this to be possible we need to consider the following convolution-like monoidal structure on $\mathrm{Vct_K}^{\mathbb{N}}$. The tensor-product of two functors F, G is defined for each $n \in \mathbb{N}$ by

$$(F \otimes G)(n) := \bigoplus_{k+\ell=n} F(k) \otimes G(l).$$

The identity e_K is given by the functor with $e_K(0) = K$ and $e_K(n) = 0$, $n \in \mathbb{N}$. The associativity, left- and right-unit law are defined in the obvious way from the one of the direct sum and the tensor product in $\mathrm{Vct_K}$. We won't formulate them explicitly here. Observe that e isn't a separator in $\mathrm{Vct_K}^{\mathbb{N}}$. Hence, in contrast to the tensor structure in $\mathrm{Vct_K}$ the symmetric braiding isn't uniquely determined here. Indeed, there are two symmetric braidings of great importance for this monoidal structure. The first is again the obvious one, which deploys the symmetry of the direct sum and the tensor product in an obvious way. It is given on pure tensors as

$$\begin{aligned}(\tau^s_{F,G})_n : \bigoplus_{k+\ell=n} F(k) \otimes G(\ell) &\to \bigoplus_{k+\ell=n} G(k) \otimes F(\ell),\\ v_k \otimes w_\ell &\mapsto w_\ell \otimes v_k, \qquad v_k \in F(k),\ w_\ell \in G(\ell),\ n \in \mathbb{N}.\end{aligned}$$

The second one is refered to as *graded anticommutativity*. It is given on pure tensors as

$$\begin{aligned}(\tau^a_{F,G})_n : \bigoplus_{k+\ell=n} F(k) \otimes G(\ell) &\to \bigoplus_{k+\ell=n} G(k) \otimes F(\ell),\\ v_k \otimes w_\ell &\mapsto (-1)^{k\ell} w_\ell \otimes v_k, \qquad v_k \in F(k),\ w_\ell \in G(\ell),\ n \in \mathbb{N}.\end{aligned}$$

We refrain from explicitly demonstrating the commutativity of the respective diagrams of coherence involved. Some concluding remarks are in order.

Remarks 2.4.1.

(a) Graded K-vector spaces can be defined for any index set G. Considering G as a discrete category we can define the category of G-graded vector spaces to be $\mathrm{Vct_K}^G$. If we want to be able to define the convolution-like monoidal structure on $\mathrm{Vct_K}^G$, we need G to be a monoid. For the definition of τ^s we need G to be abelian. For the definition of τ^a we need the existence of an ideal $I < G$ (of a monoid) such that $G/I \cong \mathbb{Z}/2\mathbb{Z}$. This is ideal is needed to be able to define the alternating sign function. In the case of an abelian group G this amounts to have a subgroup of index 2 (cf. [Che56]). Of course, there are deeper structural reasons behind these constructions of monoidal structures in functor categories. We won't address these in this thesis though.

(b) The convolution-like monoidal structure is not closed in $\mathrm{Vct_K}^{\mathbb{N}}$, for the tensor-product doesn't distribute over the (pointwise) direct sum in $\mathrm{Vct_K}^{\mathbb{N}}$. To get a closed structure one needs to enlarge the hom-sets from the degree zero to homogeneous maps of arbitrary degree $k \in \mathbb{N}$ (cf. [EK65, section IV.6]).

(c) Instead of the above notation for the grade of a homogeneous element we will use the degree-notation, i.e. $\deg v = k$ if and only if $v \in F(k)$ for a $v \in \varinjlim F$.

From now on $\mathrm{Vct_K}^{\mathbb{N}}$ is to be understood as a symmetric monoidal category with the convolution-like monoidal structure. We will use both symmetric braidings. Which one determines the notion of commutativity in the concrete situation will be stated explicitly or will be obvious from the context and shouldn't lead to confusion.

2.5 Internal Monoids

In this section we will present the concept of an internal monoid of a monoidal category and apply it, amongst others, to $\mathrm{Vct_K}$ and $\mathrm{Vct_K}^{\mathbb{N}}$ to obtain the category of K-algebras with algebra maps as morphisms and the category of graded K-algebras with morphisms the homogeneous algebra maps of degree zero, respectively. Like in the previous section our intention here is to give the ideas and the definitions. Eventually we will sketch some details, but to keep this exposition brief we will omit the proofs.

2.5.1 The Cartesian Case

We start with the notion of an internal monoid of a Cartesian monoidal category. As usual the leading example will be Set. Consider the classical notion of a monoid M, i.e. a set M together with a binary operation $\mu : M \times M \to M$, an identity element $u \in M$ such that associativity and identity laws hold. We can express these properties with commutative diagrams:

$$\begin{array}{ccccc}
M \times (M \times M) & \xrightarrow{\cong} & (M \times M) \times M & \xrightarrow{\mu \times 1_M} & M \times M \\
\downarrow{\scriptstyle 1_M \times \mu} & & & & \downarrow{\scriptstyle \mu} \\
M \times M & & \xrightarrow[\mu]{} & & M,
\end{array}$$

$$\begin{array}{ccccc}
 & & M & & \\
 & \overset{\cong}{\searrow} & \downarrow{\scriptstyle \mu} & \overset{\cong}{\swarrow} & \\
* \times M & \xrightarrow{u \times 1_M} & M \times M & \xleftarrow{1_M \times u} & M \times *
\end{array}$$

where $*$ denotes a terminal one-point set and u is regarded as a point in the sense of category theory, that is a map $* \to M$ which maps the unique element of $*$ to

$u \in M$. Conversely, any set M together with maps $\mu : M \times M \to M$, $u : * \to M$, such that the above diagrams commute, is a monoid with μ as the binary operation and the image of u as an identity element. We can even express homomorphisms of monoids $f : M \to M'$ in terms of commutative diagrams as follows

$$\begin{array}{ccc} M \times M & \xrightarrow{\mu} & M \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ M' \times M' & \xrightarrow[\mu']{} & M', \end{array} \qquad \begin{array}{ccc} & * & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle u'} \\ M & \xrightarrow{f} & M'. \end{array}$$

The advantage of these diagramatical formulations is that they work in every Cartesian monoidal category, i.e. in every category which has finite (Cartesian) products and a terminal object. Indeed, we can define an internal monoid for any such category in terms of these commutative diagrams. In more detail, an *internal monoid* for a Cartesian monoidal category C is a triple (m, μ, u) with $\mu : m \times m \to m$, $u : * \to m$ such that the following diagrams commute

$$\begin{array}{ccccc} m \times (m \times m) & \xrightarrow{\alpha_{m,m,m}} & (m \times m) \times m & \xrightarrow{\mu\times 1_m} & m \times m \\ {\scriptstyle 1_m\times\mu}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ m \times m & & \xrightarrow[\mu]{} & & m, \end{array} \qquad \text{(associativity)}$$

$$\begin{array}{ccccc} * \times m & \xrightarrow{u\times 1_m} & m \times m & \xleftarrow{1_m\times u} & m \times * \\ & {\scriptstyle \lambda_m}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \rho_m} & \\ & & m. & & \end{array} \qquad \text{(unit law)}$$

Here α, ρ, λ are the Cartesian structure mophisms and $*$ a terminal object, which is an identity obejct for the Cartesian product. Additionally, we define morphisms $f : (m, \mu, u) \to (m', \mu', u')$ of internal monoids in terms of commutative diagrams like above, i.e. as C-morphisms $f : m \to m'$ such that the following diagrams commute

$$\begin{array}{ccc} m \times m & \xrightarrow{\mu} & m \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ m' \times m' & \xrightarrow[\mu']{} & m', \end{array} \qquad \begin{array}{ccc} & * & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle u'} \\ m & \xrightarrow{f} & m'. \end{array}$$

It is easy to see the identity morphism 1_m to be a monoid morphisms for each monoid (m, μ, u) and that the composition of two monoid morphisms is again a monoid morphism. With this notion of morphism the internal monoids of C form a category $\mathrm{Mon}(C)$. Note that this category is non-empty, since the terminal object is a monoid in a natural way and moreover an initial object in $\mathrm{Mon}(C)$.

We have already seen Mon(Set) to be the category of ordinary monoids. Further trivial examples are Mon(Top) and Mon(Cat). Top denotes the category of topological spaces with continuous maps as morphisms. The Cartesian product is given by the set-theoretic product equipped with the product topology and Mon(Top) is the category of topological monoids with continuous monoid morphisms. Cat denotes the category of small categories with functors as morphisms, the product is the product of categories and Mon(Cat) is the category of small strict monoidal categories with strict monoidal functors as morphisms. There are also non-trivial examples, e.g. the category of small categories with a fixed set of objects O and functors as morphisms is isomorphic to Mon(O-Grph), where O-Grph denotes the category of (small) directed graphs with a fixed set of vertices O and the respective morphism of directed graphs (cf. [Mac98, sections VII.3, II.7]). As a further example note $\mathrm{Mon}(\mathrm{Vct_K})$ to be isomorphic to $\mathrm{Vct_K}$, for the Cartesian product in $\mathrm{Vct_K}$ is also a coproduct and the left- and right-unit laws determine the binary operation to be the universal folding morphism[20], which is the vector addition $+ : V \oplus V \to V$ for each vector space V.

2.5.2 The General Case

Like we have generalized Cartesian structures to general monoidal structures, we can do this with internal monoids, too. Given a monoidal category M with tensor-product $\otimes$, identity e, associativity α, left-unit law λ, right-unit law ρ we just have to replace the Cartesian product $\times$ with $\otimes$, the terminal object $*$ with e and the structure morphisms in the above diagrams respectively to obtain the notion of an internal monoid of M as well as the respective notion of monoid morphisms. In the same manner as for Cartesian internal monoids we get a category $\mathrm{Mon}(M, \otimes, e)$, where the tensor-product and identity object indicate which monoidal structure is used. Our internal monoids of Cartesian structures reappear as $\mathrm{Mon}(C, \times, *)$. From now on we will in general suppress the structure morphisms and use the shorthand notation $(M, \otimes, e)$ to denote a monoidal category. Again, note $(e, \lambda_e, 1_e)$ to be an initial object in $\mathrm{Mon}(M, \otimes, e)$.

Two examples of categories of internal monoids of monoidal categories are $\mathrm{Mon}(\mathrm{Vct_K}, \otimes, K)$ and $\mathrm{Mon}(\mathrm{Vct_K}^{\mathbb{N}}, \otimes, e_K)$, where, as we have agreed on, the first tensor-product is the tensor product of vector spaces and the second tensor-product is the convolution-like tensor product for graded K-vector spaces. $\mathrm{Mon}(\mathrm{Vct_K}, \otimes, K)$ is the category of K-algebras with the respective morphisms and $\mathrm{Mon}(\mathrm{Vct_K}^{\mathbb{N}}, \otimes, e_K)$ is the category of $\mathbb{N}$-graded K-algebras with homogenous K-algebra maps of degree zero.

In contrast to the Cartesian product of sets the utilization of the tensor product to define the binary operations doesn't require to explicitly state the distributivity

[20] Recall the folding map $\delta : c + c \to c$ to be the morphism which is obtained by applying the universal property of the coproduct to the the pair of identity morphisms $(1_c, 1_c)$ (cf. [Mac98, section IV.2]).

laws, for they just state the bilinearity of the product. Note that in our two examples the identity of any such algebra as internal monoid chooses a unique element in the algebra and vice versa. Which particular element is chosen, is determined by the left- and right-unit laws of the monoidal structure. For a K-algebra A and $u : K \to A$ it is $u(1)$. For a graded K-algebra $A \in \mathrm{Vct_K}^{\mathbb{N}}$ and $u : e_K \dot{\to} A$ it is $e_0(1) \in A(0)$, which corresponds to the well-known fact that an identity element of a $\mathbb{N}$-graded algebra has to be a homogeneous element of degree zero. Another famous example of this kind is $\mathrm{Mon}(\mathrm{Ab}, \otimes, \mathbb{Z})$, where Ab denotes the category of abelian groups with group homomorphisms and $\otimes$ the tensor product of abelian groups. The category of internal monoids is the category of rings. In the language of algebras (over commutative rings) this is the statement that rings are precisely the $\mathbb{Z}$-algebras.

2.5.3 Monads and their Eilenberg-Moore Algebras

A further example, which we will encounter in this thesis and which is of a different kind than the previous ones, is that of monads. Given a category C consider the functor category C^C. With the composition of functors as tensor-product and the identity functor I_C as identity object it becomes a strict monoidal category. The internal monoids (T, μ, η) of this monoidal category are called *monads*. For the convenience of the reader we state the defining diagrams explicitly. The multiplication of a monad is a natural transformation $\mu : T^2 = T \circ T \dot{\to} T$, its identity is a natural transformation $\eta : I_C \dot{\to} T$ and the following diagrams are required to commute:

$$\begin{array}{ccc} T \circ (T \circ T) = (T \circ T) \circ T & \xrightarrow{1_T \circ \mu} & T \\ \downarrow{\scriptstyle \mu \circ 1_T} & & \downarrow{\scriptstyle \mu} \\ T \circ T & \xrightarrow[\mu]{} & T, \end{array} \qquad \begin{array}{ccccc} I_C \circ T & \xrightarrow{\eta \circ 1_T} & T \circ T & \xleftarrow{1_T \circ \eta} & T \circ I_C \\ & {\scriptstyle 1_T} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle 1_T} & \\ & & T. & & \end{array} \tag{2.9}$$

Here the composition of natural transformation by $\circ$ means the vertical composition of natural transformations (cf. [Mac98, section II.5]). One usually replaces 1_T with T in expressions like $1_T \circ \mu$ or $\mu \circ 1_T$. This has the advantage to indicate how to form the respective natural transformation explicitly, e.g. $(T \circ \mu)_c = T(\mu_c)$, $(\mu \circ T)_c = \mu_{Tc}$. Monads and their dual notion the comonads are very important structures ubuiquitos in mathematics. Each adjunction $F \dashv G$ gives rise to a monad GF with multiplication $\mu = G \circ \varepsilon \circ F$ and identity η. Here, ε denotes the counit and η denotes the unit of the adjunction. In fact the converse also holds; each monad comes from an adjunction (cf. [Mac98, theorem VI.2.1]). One important notion coming along with monads is that of their category of Eilenberg-Moore algebras. We will address this in the next paragraph.

With internal monoids one can also define *internal right* and *left actions* of these monoids on objects of the underlying monoidal category. As we won't need them

here, we refer only to section V.4 in [Mac98] for the obvious defining diagrams (cf. also (3.1) in section 3.1 for a diagrammatic definition of a right action of a monoid, i.e. an internal right action in the Cartesian monoidal category Set). For example, given a K-algebra A the category of internal right actions together with the respective morphisms, the intertwiners, is the category of right A-modules; given a graded K-algebra the respective category of internal left actions is the category of graded left A-modules and homogeneous module-homomorphisms of degree zero as morphisms. For monads one can also define a type of *external* left action by evaluation on objects[21]. This yields the category of Eilenberg-Moore algebras. In more detail, given a category C and a monad (T, μ, η) an *Eilenberg-Moore algebra* for T, or T-*algebra* for short, is a pair (c, h) with $c \in \operatorname{Obj} C$ and $h : Tc \to c$ such that the following diagrams commute

$$\begin{array}{ccc} T^2c & \xrightarrow{Th} & Tc \\ {\scriptstyle \mu_c}\downarrow & & \downarrow{\scriptstyle h} \\ Tc & \xrightarrow[\mu]{} & c, \end{array} \qquad \begin{array}{ccc} c & \xrightarrow{\eta_c} & Tc \\ & {\scriptstyle 1_c}\searrow & \downarrow{\scriptstyle h} \\ & & c. \end{array} \tag{2.10}$$

A Morphism $f : (c, h) \to (c', h')$ between T-algebras is defined to be a morphism $f : c \to c'$ in C which renders commutative the diagram

$$\begin{array}{ccc} Tc & \xrightarrow{h} & c \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f} \\ Tc' & \xrightarrow[h']{} & c'. \end{array} \tag{2.11}$$

The importance of T-algebras can be derived from Beck's theorem and its various applications in classification of structures[22]. We demonstrate the basic idea on a monoidal category M and its category of internal monoids $\operatorname{Mon}(M, \otimes, e)$. There is an obvious forgetful functor $U : \operatorname{Mon}(M, \otimes, e) \to M$. Suppose U has a left adjoint F. F assigns to any m the free internal monoid over m as can be seen by writing out the universality of the unit of the adjunction for the object m. From $F \dashv U$ we obtain the monad $T = UF : M \to M$ and its category of T-algebras M^T. T can be also written as $T = U_T F_T$, where $U_T : M^T \to M$ is the obvious forgetful functor and $F_T : M \to M^T$ is its left adjoint, the free T-algebra functor. From these two decompositions of T one can construct a natural "comparison" functor $\operatorname{Mon}(M, \otimes, e) \to M^T$. In many situations, namely the one given in Beck's theorem and its variants, this comparison functor is an isomorphism or equivalence of categories. In this case one says that the functor U is monadic. One can then

[21] In fact, this external left action can be considered also as internal by embedding C via the diagonal functor $\Delta : C \mapsto C^C$ as a full and faithful subcategory of C.

[22] Cf. [Mac98, theorem VI.7.1] for a strict version of Beck's theorem and the theorems VI.8.1 and VI.9.1 in [Mac98] for examples of applications.

consider the structure of the internal monoid of M as being determined solely by the two-fold process of forgetting and freely generating the respective structure. In this thesis we will employ a monadic approach together with Beck's theorem in theorem 3.5.3 and corollary 4.4.2. In general we say that a functor $F : C \to D$ is *monadic* when it has a left adjoint $G : D \to C$ and the comparison functor $K_T : C \to D^T$ is an equivalence of categories for the respective monad $T = FG$.

2.5.4 Free Monoids as Graded Internal Monoids

There is a very convenient constructive result about the existence of free monoids. If a monoidal category $(M, \otimes, e)$ has denumerable coproducts and for each $m \in \operatorname{Obj} M$ the functors $m \otimes (-)$, $(-) \otimes m$ preserve these denumerable coproducts, then the forgetful functor $U : \operatorname{Mon}(M, \otimes, e) \to M$ has a left adjoint $F : M \to \operatorname{Mon}(M, \otimes, e)$. The proof can be found in section VII.3 in [Mac98]. In particular, in every cocomplete closed symmetric monoidal category we have a free monoid functor. For convenience we assume M to be of that kind. For us it is of interest how the free monoid $F(m)$ generated by an object m looks like. As one would probably guess, it is of the form

$$F(m) = \coprod_{n \in \mathbb{N}} m^{\otimes n} \tag{2.12}$$

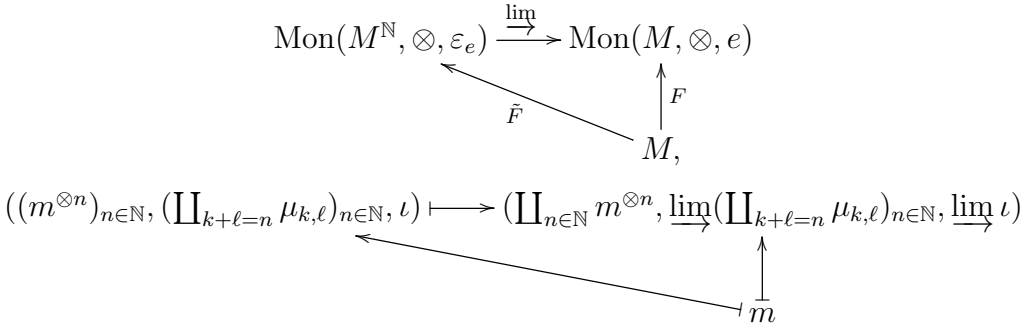
where $m^{\otimes n}$ denotes the n-fold tensor-product of m with itself, $m^{\otimes 0} := e$ and $\coprod$ the coproduct in M. Loosely speaking, the multiplication μ is defined by "juxtaposition" of tensor-words. This can be formalized with the help of the isomorphisms

$$\mu_{k,\ell} : m^{\otimes k} \otimes m^{\otimes \ell} \to m^{\otimes (k+\ell)},$$

which arise by coherence from the associativity law of the monoidal structure. with the presupposed distributivity

$$\coprod_{n \in \mathbb{N}} m^{\otimes n} \otimes \coprod_{n \in \mathbb{N}} m^{\otimes n} \cong \coprod_{n \in \mathbb{N}} \coprod_{k+\ell=n} m^{\otimes k} \otimes m^{\otimes \ell}$$

we can then define the multiplication μ by applying the universal property of the coproduct to the morphisms $\mu_{k,\ell}$. The identity is given by the "inclusion" morphism $\iota_e : e \to \coprod_{n \in \mathbb{N}} m^{\otimes n}$ of the coproduct. These constructions show that every free internal monoid is graded. Stated differently, we have the following factorization of F through $\operatorname{Mon}(M^{\mathbb{N}}, \otimes, \varepsilon_e)$



(2.13)

where $\mu = \varinjlim(\coprod_{k+\ell=n} \mu_{k,\ell})_{n\in\mathbb{N}}$ and $\iota_e = \varinjlim \iota$. The functor category $(M^{\mathbb{N}}, \otimes, \varepsilon_e)$ is to be understood as the category of $\mathbb{N}$-graded M-objects and the tensor-product $\otimes$ is the convolution-like tensor-product constructed from the tensor-product in M in the same way it was done for the particular case of $M = \mathrm{Vct}_{\mathrm{K}}$; the identity ε_e is the functor which assigns e to 0 and a fixed initial object i in M to any other $k \in \mathbb{N}$[23]. The sequence $(m^{\otimes n})_{n\in\mathbb{N}}$ denotes a functor $\mathbb{N} \to M$ which maps a $k \in \mathbb{N}$ to $m^{\otimes k}$ and the natural transformation $\iota : \varepsilon_e \dot{\to} (m^{\otimes n})_{n\in\mathbb{N}}$ is defined componentwise by $\iota_0 = \iota_e$ and ι_k the unique morphism $i \to m^{\otimes k}$ for each $k \in \mathbb{N} \setminus \{0\}$.

The functor $\tilde{F}$ has a right adjoint. It is the lifted evaluation-at-1 functor

$$E_1 : \mathrm{Mon}(M^{\mathbb{N}}, \otimes, \varepsilon_e) \to M, \qquad (F, \mu, u) \mapsto F(1), \quad \tau \mapsto \tau_1. \tag{2.14}$$

The adjunction is an immediate consequence of the universal property of the free monoid functor. It is a particular instance of this adjunction together with the above factorization we use implicitly when speaking about a vector space to generate a (graded) algebra. Indeed, setting $M = \mathrm{Vct}_{\mathrm{K}}$ yields F to be the tensor algebra functor and $\tilde{F}$ considers the tensor algebra as a graded algebra with the natural gradation given by wordlength. A further example of this type will be the free affine algebra of an affine space. In fact, it is the factorization through graded algebras which is of importance for universal geometric algebras like the Grassmann algebra, since the gradation has a definite geometric meaning there.

2.5.5 Internal Commutative Monoids

Suppose $(M, \otimes, e, \tau)$ is a symmetric monoidal category with symmetric braid τ. Besides $\mathrm{Mon}(M, \otimes, e)$ we can define the subcategory of internal commutative monoids, where an internal monoid (m, μ, u) is said to be *commutative* when the following obvious diagram commutes

[23] Observe an initial object i in M to act as a zero for the tensor-product in M, i.e. $m \otimes i \cong i$ for any $m \in \mathrm{Obj}\, M$. This follows from the assumed closedness of $(M, \otimes, e)$, for the respective hom-tensor adjunction $M(m \otimes i, c) \cong M(i, \underline{M}(m, c))$ shows $M(m \otimes i, c)$ to be a one-point space for any $c \in \mathrm{Obj}\, M$ and thus $m \otimes i$ to be initial.

$$\begin{array}{ccc} m \otimes m & \xrightarrow{\mu} & m \\ {\scriptstyle \tau_{m,m}}\downarrow & \nearrow_{\mu} & \\ m \otimes m. & & \end{array} \tag{2.15}$$

From the coherence axioms for a symmetric braid one can read off the initial monoid $(e, \lambda_e, 1_e)$ to be commutative; the category of internal commutative monoids is thus not empty. Note the notion of commutativity to be defined by the particular symmetric braid. For example, "commutative" in $\mathrm{Mon}(\mathrm{Vct_K}^{\mathbb{N}}, \otimes, e_K, \tau^s)$ is the commutativity in the usual sense of algebra, whereas "commutative" in $\mathrm{Mon}(\mathrm{Vct_K}^{\mathbb{N}}, \otimes, e_K, \tau^a)$ is the graded anticommutativity. This will be important to keep in mind in the subsequent construction as well.

The symmetric monoidal structure on M induces a symmetric monoidal structure on $\mathrm{Mon}(M, \otimes, e)$. For two monoids (m, μ, u), (m', μ', u') their tensor-product is an internal monoid $(m \otimes m', \mu_{m\otimes m'}, u_{m\otimes m'})$, with its structure morphisms defined as

$$\begin{array}{ccc} (m \otimes m') \otimes (m \otimes m') & & \\ {\scriptstyle \cong}\downarrow & \searrow^{\mu_{m\otimes m'}} & \\ (m \otimes m) \otimes (m' \otimes m') & \xrightarrow[\mu\otimes\mu']{} & m \otimes m', \end{array} \qquad \begin{array}{ccc} e & & \\ {\scriptstyle \lambda_e^{-1}}\downarrow & \searrow^{u_{m\otimes m'}} & \\ e \otimes e & \xrightarrow[u\otimes u']{} & m \otimes m'. \end{array} \tag{2.16}$$

Here the natural isomorphism $(m \otimes m') \otimes (m \otimes m') \cong (m \otimes m) \otimes (m' \otimes m')$ is obtained by coherence from the associativity law and the symmetric braid. The unit law can be seen rather easily from the defining diagrams, but to show the associativity law to hold by means of commutative diagrams is a tedious exercise. The identity object for this tensor-product is the initial monoid $(e, \lambda_e, 1_e)$ and the respective structure morphisms are constructed from the ones of $(M, \otimes, e)$. For this monoidal structure the forgetful functor $\mathrm{Mon}(M, \otimes, e) \to M$ becomes a strict monoidal functor.

With a monoidal structure at hand we can form the category of internal monoids of the monoidal category $(\mathrm{Mon}(M, \otimes, e), \otimes, (e, \lambda_e, 1_e))$, i.e. $\mathrm{Mon}(\mathrm{Mon}(M, \otimes, e), \otimes, (e, \lambda_e, 1_e))$. The question arises as of what kind of structures these objects are. They are precisely the commutative monoids. To show this in full generality is a tedious exercise in writing down commutative diagrams and we refrain from doing it here. Nonetheless, the only reason for this fact is the algebraic version of the Eckmann-Hilton argument (cf. e.g. [Mac98, exercise II.5.5]).

This settled, one can iterate this construction one step further and consider internal monoids of the symmetric monoidal category of internal commutative monoids of M. This will however yield the same category of internal commutative monoids again. The reason is as follows. The tensor-product of M becomes the coproduct in the category of internal commutative monoids and for a general category C with

finite coproducts, denoted by $+$, and an initial object 0 we have $C \cong \mathrm{Mon}(C, +, 0)$. We encountered a particular instance of this fact for $\mathrm{Vct_K}$ when we argued that there is an isomorphism of categories $\mathrm{Vct_K} \cong \mathrm{Mon}(\mathrm{Vct_K}, \oplus, 0)$. The abstract argument is in essence just the restatement of the particular argument from there: any morphism $\mu : c + c \to c$, such that the left- and right-unit laws hold for μ, is necessarily the folding morphism, for the left- and right-unit laws are determined uniquely due to 0 being initial and yield $i_c \circ \mu = 1_c = j_c \circ \mu$ for the two natural "inclusions" $i_c, j_c : c \to c + c$.

2.5.6 Lifts of Monoidal Functors to Categories of Internal Monoids

We turn to monoidal functors next. Given two monoidal categories $(M, \otimes, e)$, $(M', \otimes', e')$ and a monoidal functor $F : M \to M'$ then F lifts to a functor $F : \mathrm{Mon}(M, \otimes, e) \to \mathrm{Mon}(M', \otimes', e')$. We indicate this by the following diagrams, which define the structure maps of $F(m, \mu, u) := (F(m), \mu_{F(m)}, u_{F(m)})$ for an internal monoid (m, μ, u) of $(M, \otimes, e)$.

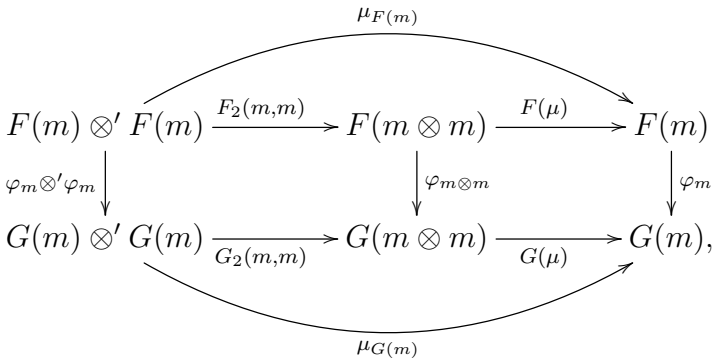
$$\begin{array}{ccc} F(m) \otimes' F(m) & & \\ {\scriptstyle F_2(m,m)}\downarrow & \searrow^{\mu_{F(m)}} & \\ F(m \otimes m) & \xrightarrow[F(\mu)]{} & F(m), \end{array} \qquad \begin{array}{ccc} e' & & \\ {\scriptstyle F_0}\downarrow & \searrow^{u_{F(m)}} & \\ F(e) & \xrightarrow[F(u)]{} & F(m). \end{array} \tag{2.17}$$

Indeed, a closer inspection of the defining diagrams of a monoidal functor shows them to be the weakest reasonable assumptions for such a lift to be possible. For the forgetful functor $U : \mathrm{Vct_K} \to \mathrm{Set}$, which was argued to be monoidal in section 2.2.3, the lift says that K-algebras are also set-theoretical monoids. The convolution-like tensor product on $\mathrm{Vct_K}^{\mathbb{N}}$ was designed so that the $\varinjlim$-functor becomes strong monoidal. In that case the lift says that the monoids in $\mathrm{Vct_K}^{\mathbb{N}}$ are indeed K-algebras. For Cartesian monoidal categories C the strong monoidal functors are precisely the functors which preserve finite products[24]; they are ubiquitous. If the category C has small hom-sets, then a particular example is the covariant hom-functor $C(a, -)$ for an arbitrary $a \in \mathrm{Obj}\, C$. The strong monoidality means then, that for any internal monoid (m, μ, u), $(C(a, m), \mu_*, u_*)$ is an internal monoid of Set, i.e. a usual monoid. Writing out the structure maps explicitly, one can see this to be the abstract formulation of what is commonly known as a pointwise structure. Indeed, this example is the abstract reason behind algebraic pointwise constructions and their naturality.

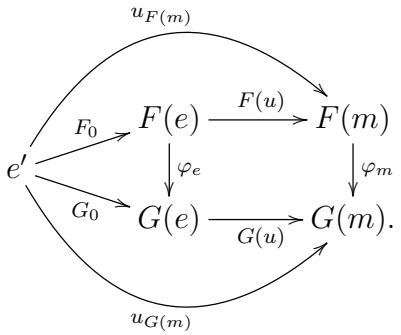
[24] Note this to include the preservation of terminal objects as well, for a terminal object in C can be considered as a limit for the unique functor $\underline{0} \to C$, where $\underline{0}$ denotes the category with an empty set of objects and an empty set of morphisms. Note the empty category to be finite, in particular.

One has to carefully distinct between monoidal and strong monoidal functors. It may happen that a monoidal functor is an isomorphism, but the lift to internal monoids is not. It will lift to an isomorphism if and only if it is strong monoidal. We will encounter such an example of a monoidal functor which isn't strong in the identity functor on the category of affine spaces in the next chapter (cf. proposition 3.4.17). One also has to be careful with symmetric braids. If the monoidal functor is symmetric its lift to internal monoids will descend to a functor between the respective categories of internal commutative monoids, otherwise it will not.

Besides monoidal functors we introduced the notion of a monoidal transformation between such functors. We can see such natural transformations to lift to natural transformations between the lifts of the monoidal functors. To make this more obvious consider two monoidal functors $F, G : (M, \otimes, e) \to (M', \otimes', e')$ and a monoidal natural transformation $\varphi : F \dot{\to} G$. We have to show that for an internal monoid (m, μ, u) of $(M, \otimes, e)$ the morphism $\varphi_m : F(m) \to G(m)$ lifts to a monoid morphism $\varphi_m : (F(m), \mu_{F(m)}, u_{F(m)}) \to (G(m), \mu_{G(m)}, u_{G(m)})$. To see this we write out the respective structure morphism as defined in (2.17) and apply the defining diagram (2.8) of the monoidality of φ together with the naturality of φ:



(2.18)



For a last application of monoidal functors we consider graded K-algebras again. We have argued that they are the internal monoids of $\mathrm{Vct_K}^{\mathbb{N}}$. One can describe them in terms of monoidal functors as well. Consider the additive monoid $(\mathbb{N}, +, 0)$ as a discrete strict monoidal category. Besides the functor category $\mathrm{Vct_K}^{\mathbb{N}}$ there is the category of monoidal functors with monoidal natural transformations as morphisms.

This category is isomorphic to the one of graded K-algebras. To see this, recall how a graded K-algebra A can be described in terms of its subspaces $A(n)$ of homogeneous elements $n \in \mathbb{N}$. For each pair $(k, \ell) \in \mathbb{N} \times \mathbb{N}$ we need linear maps

$$A(k) \otimes A(\ell) \to A(k + \ell)$$

and we have to specify an identity element in $A(0)$, i.e. we have a linear map $K \to A(0)$. As $\mathbb{N}$ is discrete such data determines already a monoidal functor and vice versa. The rest, i.e. that monoidal natural transformations correspond to homogeneous algebra maps of degree zero and vice versa, follows then from the defining diagrams immediately.

We conclude this section and the chapter with remarks about differential graded (DG) vector spaces and their respective algebras, since we will encounter them in this thesis as well[25]. As long as the morphisms are homogeneous maps of zero degree, DG vector spaces can be understood as a subcategory of the functor category $\mathrm{Vct_K}^{(\mathbb{N},\geq)}$. By $(\mathbb{N}, \geq)$ we mean $\mathbb{N}$ as an ordered space, regarded as a category. This means for two natural numbers $n, m \in \mathbb{N}$ that there is a morphism $n \to m$ if and only if $n \geq m$. The category of $\mathbb{N}$-graded K-vector spaces can be embedded in $\mathrm{Vct_K}^{(\mathbb{N},\geq)}$ as the full subcategory[26] of functors which map every morphism to the zero map. The category of DG vector spaces can be embedded as the full subcategory of functors which map the morphisms $n + 1 \to n$ to the linear boundary map d_{n+1}. For this case we have adopted the homological or chain complex convention. For the cochain complex convention one would need to consider contravariant functors instead of covariant ones; these are the functors $(\mathbb{N}, \leq) \to \mathrm{Vct_K}$.

We already know of two ways as how to describe graded K-algebras: either as internal monoids of the convolution-like monoidal structure or as monoidal functors in $\mathrm{Vct_K}^{\mathbb{N}}$. The approach with monoidal functors carries over to $\mathrm{Vct_K}^{(\mathbb{N},\geq)}$ without any difficulties, for $(\mathbb{N}, +, \geq)$ is an ordered monoid and this is in particular equivalent to $(\mathbb{N}, \geq)$ being a strict monoidal category; the monoidal structure is the one induced from $(\mathbb{N}, +, 0)$. However, DG algebras don't arise as the subcategory of monoidal functors, which underlying functor is a DG vector space in this approach. This is because for a monoidal functor $F : (\mathbb{N}, \geq) \to \mathrm{Vct_K}$ its natural transformation F_2 doesn't express the graded Leibniz rule; but it is the graded Leibniz rule we need to hold so that the boundary maps become graded derivations of degree -1.

Regarding the approach with the convolution-like monoidal structure, in addition to what we have for the tensor-product of $\mathbb{N}$-graded vector spaces already, we need to define the action of $F \otimes G$ on morphisms. For this approach to yield the subcategory

[25] Another and more common term for DG vector spaces is that of chain complexes over vector spaces.

[26] Recall that a subcategory $S \leq C$ is said to be *full* when for each pair of objects $a, b \in \mathrm{Obj}\, S$ each C-morphism $f : a \to b$ is a S-morphism; in other words the hom-sets $S(a, b)$ and $C(a, b)$ are equal.

of DG vector spaces as a strict monoidal subcategory[27] we need to do it in such a way that the tensor product of two DG vector spaces becomes a DG vector space. This amounts to extend the definition on boundary maps as it is given in [May99, section 12.3] or [EK65, section IV.6]. Doing it like this will indeed yield DG algebras as internal monoids, but since in this thesis DG algebras will arise rather as DG structures on graded algebras than internal monoids of DG vector spaces, we won't need this kind of approach here and thus don't go into the details. We just wanted to indicate the difficulties.

[27] By a "strict monoidal subcategory" we mean the respective inclusion functor to be strict monoidal.

3 The Category of Affine Spaces

In this chapter we will consider the category of our main interest. It is the category Aff_K of (small) affine spaces over a field K with affine maps as morphisms. We will define it in a geometrical way.

This category is an interesting example in many ways. From the structural point of view it is related with the linear category of vector spaces over K and the distributive category Set, the latter being the basic and motivating example of a topos (cf. [LR03, section 6.1], [MM92, prologue], [LS97, session 33.2])[1].

Loosely speaking, both of these categories are of an essentially different "homotopy-type". Where a linear category like Vct_K is star-shaped with the zero object as its center resembling topologically an open ball, Set resembles a double cone, i.e. a sphere suspendend between an initial and terminal object, which is of the homotopy-type of a sphere of one dimension higher. To some extent Aff_K interpolates between linear and distributive categories. We will see that it is neither linear nor a distributive category. It has the "homotopy-type" of Set because it has an initial and terminal object different from each other. It is complete and cocomplete. In particular, it has products and coproducts, but it is not Cartesian closed. Nonetheless, it has a closed monoidal structure given by the affine tensor product, which very much resembles the linear tensor product of vector spaces.

Like the linear tensor product the affine tensor product yields an internal representation of "bimorphisms". In the case of the linear tensor product we saw these bimorphisms to be the bilinear maps. In the case of the affine tensor product we will see these to be the biaffine maps, i.e. maps of two arguments that are affine in each argument. We shall see that biaffine maps are used in analysis and physics unknowingly all the time; they are just mistaken for bilinear maps.

In consequence of the closed monoidal structure the coproduct of affine spaces distributes over the affine tensor product, but it does not distribute over the (Cartesian) product. The affine tensor product and the Cartesian product are related though, since the identity functor of Aff_K can be regarded as a symmetric monoidal func-

[1] For a category with finite products and coproducts to be distributive it means, loosely speaking, the coproduct to distribute over the (Cartesian) product and an initial object to act as a zero for the product. We will give a more precise definition later on.

tor from the Cartesian symmetric monoidal structure to the symmetric monoidal structure induced by the tensor product.

From the structural point of view, the dual affine space functor is another interesting object. It maps each affine space A to the affine space of affine maps from A to K. Even though it resembles the dual vector space functor it does not behave like it. In fact, it behaves more like the (contravariant) power set functor. For example, as there is no set which is isomorphic to its power set, there is no affine space which is ismorphic to its dual. The (contravariant) power set functor is known to be monadic and we will show the dual affine space functor space to be monadic, too.

From the geometric viewpoint the notion of affine space is the structure related to a homogeneous geometry which serves as a prominent type of model geometry. This prominence is not due to it being the most basic type of homogeneous geometry[2], but due to its deep connection with our observable reality. Of course the connection with our directly observable reality is in particular given by the more special Euclidean geometry with its notion of orthogonality. However, affine geometry is the proper stage for a unified view on different geometries for which "homogeneity of space" is an inherent concept. This concept is of importance because it is one of our basic assumptions about the physical space and spacetime. Therefore affine spaces yield the stage to discuss such physically relevant model geometries like the Euclidean geometry, Galilean geometry of spacetime and the hyperbolic Minkowskian geometry of spacetime. This said, it is interesting to see whether and in which way the categorical constructions in the "gros" category Aff_K are related to and in particular reflect affine geometry.

3.1 Definition and Elementary Facts about Affine spaces

Before we will start with our formal definitions, we should reflect on our own way of thinking like it was done by Grassmann. This will help us to get a clear understanding of what we are going to do, what not and which point of view we will assume implicitly.

As stated in the introduction our primary definition of an affine space is motivated by geometry. To be even more precise, it is motivated by the homogeneity of space of our observable reality. We perceive this homogeneity as follows. To any object of our observable reality we can assign the quality of a location and we can, at least in principle, move any object from one location to another. The most natural such displacement is, as we perceive it, to move the object along an oriented straight line segment. To describe the displacement of a particular object this line segment has to be located in space as well. However, the notion of the displacement itself is perceived to be independent of a particular location. It is completely described by

[2] In the sense of F. Klein's Erlangen program this would be the projective geometry.

the non-located oriented line segment, i.e. by its direction and length. This is what we understand as the homogeneity of space.

In geometry the quality of location is captured by the notion of a point in space and a space itself is modelled by the set of its locations, i.e. as a set of its points.In our considerations we have additionally a set of displacements which operate on the set of points. This operation corresponds to the motion along oriented line segments as described above and can be understood in geometric terms as follows. Take a non-located oriented straight line segment, place it in such a way that its starting point coincides with the point in space we like to displace, then its endpoint marks the displaced point in space. Dynamically, one can regard this oriented line segment as a geometric object representing a rule on how to move naturally in space in a given time. The displacement is then to be described like we did in the explanation of the homogeneity of space: we move a point from the starting point along our line segment to its endpoint.

Note that we have used two different notions of a point here, which have to be distinguished carefully. The first notion is the point in space, which describes a location. The second notion is that of a moving point. It is a parametrized family of representations of an object by points in space. To make this more obvious think about points in space as locations in our observable reality and a moving point as the representation of a physical body by the locations of its center of mass in time.

Like in our previous consideration we can define the equality of displacements operationally: two displacements are equal if they operate in the same way. Hence a displacement considered as a geometric object is a class of parallel line segments which have the same orientation and length. Due to its dynamical meaning W. R. Hamilton coined the name "*vector*" from the latin "*vehere*" ("to carry") for this type of geometric object (cf. [Ham66][Book I, section I.1]). However, because nowadays the term "vector" is associated mainly with the algebraic structure of a vector space, we will use the term "*displacement vector*" instead.

The dynamical interpretation of the action of displacement vectors together with their geometric meaning leads to the well-known algebraic structure of a vector space on the set of displacements. The addition and multiplication by scalars are both derived from the respective geometric constructions with oriented line segments as depicted in the figure 3.1 for $K = \mathbb{R}$.

The action then becomes an action of a vector space. It has the property that any ordered pair of points in space determines a displacement vector which moves the first point to the second. Moreover, the displacement vector is determined uniquely by this pair. Recall such a type of action to be called *simply transitive*.

To obtain the notion of an affine space we can now abstract from the geometric meaning to the underlying structure. An *affine space* A with associated K-vector space V_A is a V_A-torsor, i.e. a set A together with a simply transitive action $A \times V_A \to A$ of V_A on A from the right. Here the action of V_A is understood as the group action of the underlying abelian group $(V_A, +, 0)$ of the vector space V_A. Although

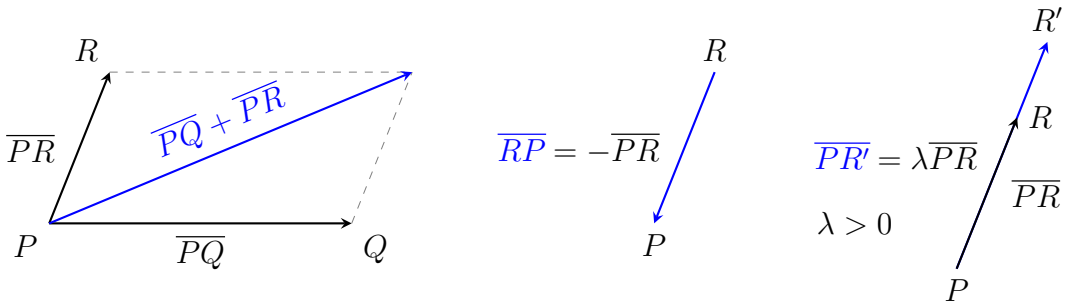


Figure 3.1: Geometric addition and scalar multiplication of oriented line segments in the affine plane over $\mathbb{R}$.

this general notion as it stands now is devoid of geometric content, we make it geometrical by associating to it the geometric picture we presented above. The underlying set of an affine space A will be considered as a set of points and the associated K-vector space V_A as the vector space of displacement vectors.

When we consider a (small) group G as an internal monoid (G, μ, u) of the Cartesian monoidal category Set, then the defining axioms of a right group action $\alpha : A \times G \to A$ can be expressed diagrammatically by the following two commutative diagrams:

$$\begin{array}{ccccc} A \times (G \times G) & \xrightarrow{\cong} & (A \times G) \times G & \xrightarrow{\alpha \times 1_G} & A \times G \\ {\scriptstyle 1_A \times \mu}\downarrow & & & & \downarrow{\scriptstyle \alpha} \\ A \times G & & \xrightarrow[\alpha]{} & & A, \end{array} \qquad \begin{array}{ccc} A \times * & \xrightarrow{1_A \times u} & A \times G \\ & \searrow^{\cong} & \downarrow{\scriptstyle \alpha} \\ & & A. \end{array} \tag{3.1}$$

The simply transitiveness of such an action can be formulated diagrammatically, too. First of all note that a group action $\alpha : A \times G \to A$ together with the projection $\mathrm{pr}_A : A \times G \to A$ onto the first factor yields a map $\theta = (\mathrm{pr}_A, \alpha) : A \times G \to A \times A$ by the universal property of the product

$$\begin{array}{ccc} & & A \\ & {\scriptstyle \mathrm{pr}_A} \nearrow & \uparrow{\scriptstyle \mathrm{pr}_1} \\ A \times G & \overset{\exists! \theta}{\dashrightarrow} & A \times A \\ & {\scriptstyle \alpha} \searrow & \downarrow{\scriptstyle \mathrm{pr}_2} \\ & & A, \end{array} \qquad \theta : (P, g) \mapsto (P, \alpha(P, g)). \tag{3.2}$$

This θ is surjective if and only if the action is transitive, it is injective if and only if the action is free, i.e. for $P, Q \in A$ there is at most one $g \in G$, such that $\alpha(P, g) = Q$. Hence it is a bijection if and only if the action is simply transitive (compare this with the discussion of principal G-bundles in [MM92, section II.4]).

Because the groups we will consider here are abelian, we don't need to distinguish between left and right group actions. We will nevertheless keep the notation of right actions for affine spaces. Moreover, we follow the custom to omit naming the action map and to denote the action in an operational manner using infix notation. In particular, for an affine space A the action of a vector $v \in V_A$ on a point $P \in A$ will be denoted by $P + v$. Furthermore, we will follow the common abuse of notation and won't distinguish between the underlying set of an affine space and the space as the whole structure itself within the notation.

For now we define $\mathrm{Aff}_{\mathrm{K}}$ to be the category of small affine spaces with affine maps as morphisms. An *affine map* $f : A \to A'$ between two affine spaces A, A' is a pair of maps $(f, \hat{f})$ with a set-theoretic map $f : A \to A'$ between the underlying sets and a linear map $\hat{f} : V_A \to V_{A'}$ between the associated vector spaces, such that it is compatible with the actions, i.e. the following diagram commutes:

$$\begin{array}{ccc} A \times V_A & \xrightarrow{f \times \hat{f}} & A' \times V_{A'} \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & A'. \end{array} \tag{3.3}$$

The composition of affine maps is the obvious one $(f, \hat{f}) \circ (g, \hat{g}) := (f \circ g, \hat{f} \circ \hat{g})$ as are the identities $(1_A, 1_{V_A})$. Alltogether this indeed satisfies the axioms of a category, so $\mathrm{Aff}_{\mathrm{K}}$ is well-defined. Several remarks are in order.

Remarks 3.1.1.

In the following A and A' are affine spaces with associated vector spaces V_A and $V_{A'}$, their underlying sets of points are assumed to be non-empty.

(a) For an affine space A to be small it means that its underlying set of points and its associated vector space V_A are small.

(b) Let $f : A \to A'$ be an affine map with associated linear map $\hat{f} : V_A \to V_{A'}$. In terms of the infix-notation of the actions we can express the commutativity of (3.3) on points and vectors as

$$f(P + v) = f(P) + \hat{f}(v), \qquad P \in A,\ v \in V_A.$$

(c) For technical reasons, which will become obvious later, we allow for an *empty affine space* $\emptyset$. Its associated vector space is defined to be a trivial vector space 0. This is compatible with our definition in terms of actions, because $\emptyset \times X = \emptyset$ for any set X and there is a unique map $\emptyset \to \emptyset$, namely the identity $1_\emptyset$ [3]. Indeed, the empty set together with the trivial vector space render commutative the

[3] In this and similar situations it is helpful to think about the empty set as an inital object of Set rather than a set of no elements.

defining diagrams (3.1). Furthermore, we get $\theta = 1_\emptyset$ which is an isomorphism. This is all due to the fact that the empty set is an initial object in Set together with $\emptyset \times X = \emptyset$ and would work for every other vector space V. However, only with the trivial vector space the empty space is convenient for us in Aff_K. It will provide an initial object. The other variants of empty spaces with non-trivial associated vector spaces are inconvenient, for they oppose our geometric picture of affine spaces and because for them the set-theoretic map and the associated linear map are unrelated, contrary to all the other affine spaces (cf. remarks (g), (h) and proposition 3.1.3). We exclude them from Aff_K. Later in section 5, when we will describe affine spaces purely algebraically as a subcategory of the slice category $\mathrm{Vct}_\mathrm{K}/K$, we will see that such objects with an empty set of points but a non-trivial vector space of displacements correspond precisely to vector spaces. This is why they appear naturally within our structures, although they are devoid of geometric meaning.

(d) The property of the action of V_A on A to be simply transitive is equivalent to the existence of a unique map

$$A \times A \to V_A, \quad (P, Q) \mapsto \overrightarrow{PQ}, \quad \text{such that} \quad Q = P + \overrightarrow{PQ},$$

which we will call the *difference map*. It can be obtained as $\mathrm{pr}_2 \circ \theta^{-1}$ where $\theta : A \times V_A \to A \times A$ is the map from (3.2). Note the pure existence of such a map to be equivalent to the action being transitive (e.g. as a section of θ from above followed by the projection on the second factor). In this case it doesn't need to be surjective[4]. Nonetheless it is surjective in the simply transitive case. Furthermore, for each $O \in A$ it induces a bijection by restriction

$$\kappa_O : A \hookrightarrow \{O\} \times A \to V_A, \qquad P \mapsto (O, P) \mapsto \overrightarrow{OP},$$

whose inverse is given by the respective restriction of the action as

$$\kappa_O^{-1} : V_A \hookrightarrow \{O\} \times V_A \to A, \qquad v \mapsto (O, v) \mapsto O + v.$$

Note that the associativity law of the action translates to the following rule for the difference map

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}, \qquad P, Q, R \in A.$$

Futhermore we have $\overrightarrow{PQ} = -\overrightarrow{QP}$ and $\overrightarrow{PP} = 0$, $P, Q \in A$ by the simply transitiveness.

[4] For example consider the action of the vector space $V \oplus W$, $W \neq 0$ on V by $(P, v + w) \mapsto P + v$ and the map $V \times V \to V \hookrightarrow V \oplus W$, $(P, Q) \mapsto Q - P$.

(e) Given a vector space V_A the structure of an affine space on a set $A \neq \emptyset$ can be equivalently defined in terms of a suitable "difference map"

$$A \times A \to V_A, \quad (P, Q) \mapsto \overrightarrow{PQ},$$

such that for each $P \in A$, κ_P defined like in the preceding remark is a bijection and $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$, for all $P, Q, R \in A$ (compare with [KM03, definition 8.1.1]).

From the preceding remark we know that a simply transitive action yields such a difference map. Conversely, given such a difference map we can define an action by

$$A \times V_A \to A, \quad (P, v) \mapsto \kappa_P^{-1}(v).$$

The second property yields by $\overrightarrow{PP} + \overrightarrow{PP} = \overrightarrow{PP}$ the identity $\overrightarrow{PP} = 0$. This implies $(P, 0) \mapsto P$ and hence the unit law of the action. It implies the associativity law for the action as well. Let $v, w \in V_A$. For $Q := \kappa_P^{-1}(v)$, $R := \kappa_Q^{-1}(w)$ we have $v = \overrightarrow{PQ}$, $w = \overrightarrow{QR}$ and hence $v + w = \overrightarrow{PR}$ by the second property. This is equivalent to $\kappa_P^{-1}(v + w) = R$ and yields $P + (v + w) = R = (P + v) + w$. Finally, by the first property we have that the action defined in this way is simply transitive.

Another way to show this would be to use the θ-map description of simply transitive actions. This amounts to constructing the map $\tilde{\theta} : A \times A \to A \times V_A$ from the given difference map and the projection pr_1 to the first factor, showing its bijectivity and finally, showing that $\mathrm{pr}_2 \circ \tilde{\theta}^{-1}$ is an action.

(f) The property of a set-theoretic map $f : A \to A'$ together with a linear map $\hat{f} : V_A \to V_{A'}$ to yield an affine map $f : A \to A'$ can be restated in terms of the difference map from remark (d) as the commutativity of the diagram (3.4) below. Indeed, an affine map $f : A \to A'$ renders this diagram commutative

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & A' \times A' \\ \downarrow & & \downarrow \\ V_A & \xrightarrow[\hat{f}]{} & V_{A'}, \end{array} \qquad \begin{array}{ccc} (P, Q) & \longmapsto & (f(P), f(Q)) \\ \downarrow & & \downarrow \\ \overrightarrow{PQ} & \longmapsto & \hat{f}(\overrightarrow{PQ}), \end{array} \tag{3.4}$$

for we have $f(Q) = f(P + \overrightarrow{PQ}) = f(P) + \hat{f}(\overrightarrow{PQ})$ by remark (b) and thus $\overrightarrow{f(P)f(Q)} = \hat{f}(\overrightarrow{PQ})$ by the simple transitiveness of the action. Here $\hat{f}$ denotes the associated linear map of f. On the other hand, given a map $f : A \to A'$ of sets and a linear map $\hat{f} : V_A \to V_{A'}$ such that the above diagram commutes,

$(f, \hat{f})$ yields an affine map. To see this observe that the commutative diagram (3.4) can be extended to the following commutative diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & A' \times A' \\ {\scriptstyle (\mathrm{pr}_1, d)}\downarrow & & \downarrow{\scriptstyle (\mathrm{pr}'_1, d')} \\ A \times V_A & \xrightarrow[f \times \hat{f}]{} & A' \times V_{A'}, \end{array}$$

where $\mathrm{pr}_1 : A \times A \to A$, $\mathrm{pr}'_1 : A' \times A' \to A'$ are the projections onto the first factor, $d : A \times A \to V_A$, $d' : A' \times A' \to V_{A'}$ are the difference maps and (pr_1, d), (pr'_1, d') are the respective maps given by the universal property of the product. From the definition of the difference map by the θ-map we can see $\theta^{-1} = (\mathrm{pr}_1, d)$ and $\theta'^{-1} = (\mathrm{pr}'_1, d')$. With this, the respective projections $\mathrm{pr}_2 : A \times A \to A$, $\mathrm{pr}'_2 : A \times A \to A$ onto the second factor and the previous commutative diagram we get the commutative diagram which concludes our assertion

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle \mathrm{pr}_2}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}'_2} \\ A \times A & \xrightarrow{f \times f} & A' \times A' \\ {\scriptstyle \theta}\uparrow & & \uparrow{\scriptstyle \theta'} \\ A \times V_A & \xrightarrow[f \times \hat{f}]{} & A' \times V_{A'}, \end{array}$$

for $\mathrm{pr}_2 \circ \theta$ and $\mathrm{pr}'_2 \circ \theta'$ are the respective action maps as can be seen from the definition of the θ-map in (3.2). Employing points one can write the commutativity condition of (3.4) as follows (cf. [KM03, definition 8.2.1])

$$\overrightarrow{f(P)f(Q)} = \hat{f}(\overrightarrow{PQ}), \qquad P, Q \in A.$$

(g) As already suggested by the notation for an affine map $f : A \to A'$, the associated linear map $\hat{f} : V_A \to V_{A'}$ is uniquely determined by the set-theoretic map $f : A \to A'$. This is due to the simply transitive action and can be obtained from the surjectivity of the difference map and the preceding remark. In more detail, by the last remark the property of being affine is equivalent to the commutativity of diagram (3.4). Given any linear $\hat{g} : V_A \to V_{A'}$ such that (3.4) commutes with $\hat{g}$ in place of $\hat{f}$ yields $\hat{g}d = \hat{f}d$ and hence $\hat{g} = \hat{f}$ by the surjectivity of the difference map d [5].

[5] Recall that, as we explained in 2.2, a surjective map is an epi in Set and vice versa. Note that we have implicitly applied the forgetful functor $U : \mathrm{Vct}_{\mathrm{K}} \to \mathrm{Set}$ and used its faithfulness. As we have done now, we will omit such details when applying standard arguments like that in the future.

This result justifies to proceed for affine maps in the same way as for affine spaces to not distinguish between them and their set-theoretic maps within the notation.

(h) A linear map $\hat{f} : V_A \to V_{A'}$ determines an affine map $f : A \to A'$ only up to a translation; i.e. for two affine maps $f, g : A \to A'$ with equal associated linear maps there is a unique translation $\tau : A' \to A'$ such that $f = \tau \circ g$. An affine endomap $A' \to A'$ is said to be a *translation* when its associated linear map is the identity. Such translations τ are uniquely determined by the vector $\overrightarrow{P\tau(P)}$ for a $P \in A'$. Because of $\hat{\tau} = 1_{V_{A'}}$ this vector is independent of $P \in A'$. All in all translations of A' are precisely the affine endomaps of the form $P \mapsto P + v$, for an arbitrary but fixed $v \in V_{A'}$.

With this we can proof our assertion from above. Suppose $\hat{f} = \hat{g}$. Fix a $P \in A$. By the remarks (b) and (d) this yields

$$\begin{aligned} f(Q) &= f(P + \overrightarrow{PQ}) = f(P) + \hat{f}(\overrightarrow{PQ}) \\ &= f(P) + \hat{g}(\overrightarrow{PQ}) = g(P) + \overrightarrow{g(P)f(P)} + \hat{g}(\overrightarrow{PQ}) \\ &= g(Q) + \overrightarrow{g(P)f(P)}, \qquad Q \in A. \end{aligned}$$

Set $v := \overrightarrow{g(P)f(P)}$ and consider the corresponding translation $\tau_v : A' \to A'$, $P \mapsto P + v$, then $f = \tau_v \circ g$. Furthermore a translation τ_w associated to the vector $w \in V_{A'}$ with $f = \tau_w \circ g$ satisfies $f(P) = g(P) + w$, hence $w = v$ and in particular $\tau_w = \tau_v$ by the property of the action to be simply transitive.

(i) Even though we are concerned mainly about geometry, we haven't restricted our categories of affine spaces and vector spaces to finite dimensional such objects. An affine space is said to be of finite dimension if its associated vector space has that property. The reason is a purely technical one. As one can already see in the case of vector spaces the subcategory of finite dimensional vector spaces is neither complete nor cocomplete. In this matter it is only finitary complete and finitary cocomplete; i.e. one can form only finite limits and finite colimits but in general not the one coming from small categories. Although this is sufficient in many situations, it is inconvenient for strong abstract arguments like the adjoint functor theorem. A similar situation will occur for affine spaces as well. Because of that we consider spaces of arbitrary dimension and develop our theory there. To account for finite dimensionality, which will be of interest for applications, we will study whether a given construction descends to the finite dimensional case or not and give dimension formulas whenever possible.

We can give the final definition of the category $\mathrm{Aff}_{\mathrm{K}}$ now.

Definition 3.1.2. (1) Aff_K is the category of small affine spaces such that if an affine space has as underlying set the empty set $\emptyset$ then its associated vector space is trivial. The morphisms are the affine maps with composition and identities given above. Spaces with underlying set the empty set are called empty spaces.

(2) ${\mathrm{Aff}_K}^\times$ is the full subcategory of Aff_K with objects the affine spaces that are not empty spaces[6].

(3) FAff_K is the full subcategory of Aff_K of those affine spaces which are of finite dimension. The dimension $\dim A$ of an affine space A is understood as the dimension $\dim V_A$ of its associated vector space V_A.

Note that we can't strictly speak about *the* empty space, because the trivial vector space is defined only up to a natural isomorphism. In particular, we can take two different one-point sets $*$ and $*'$ and define a trivial vector space structure on them. They are not equal (as sets) but isomorphic as vector spaces. This is why we have formulated the definition of Aff_K in the above manner. However, there is only one empty set, because we work in the realm of axiomatic set-theory of ZFC+Universe, where equality of sets is given by the principle of extensionality.

The remarks 3.1.1(g) and (h) lead to the following characterization of affine maps.

Proposition 3.1.3. *Let A, A' be affine spaces with associated vector spaces V_A, respectively $V_{A'}$ whose underlying sets aren't empty. An affine map $f : A \to A'$ is completely and uniquely determined by either one of the following data*

(i) A map $f : A \to A'$ of the underlying sets such that there is an $O \in A$ and the map $\hat{f} : V_A \to V_{A'}$ defined by $\kappa_{f(O)} \circ f \circ \kappa_O^{-1}$ is linear. The maps κ_O, $\kappa_{f(O)}$ are the bijections from remark 3.1.1(d) and $\hat{f}$ is independent of $O \in A$, i.e. for any other $P \in A$, $\kappa_{f(P)} \circ f \circ \kappa_P^{-1} = \hat{f}$.

(ii) A linear map $\hat{f} : V_A \to V_{A'}$ and a pair of points $(O, O') \in A \times A'$ such that $f(O) = O'$.

Proof. (i) Let $f : A \to A'$ be an affine map. By remark 3.1.1(b) and (d) we can verify $\kappa_{f(O)} \circ f \circ \kappa_O^{-1} = \hat{f}$ on vectors $v \in V_A$:

$$\kappa_{f(O)} \circ f \circ \kappa_O^{-1}(v) = \kappa_{f(O)} \circ f(O + v) = \kappa_{f(O)}(f(O) + \hat{f}v) = \hat{f}(v),$$

where the last equality is due to the action being simply transitive. Additionally, we can see this argument to be independent of the chosen point $O \in A$.

[6] Recall that a functor $F : C \to D$ is said to be *full* when for each D-morphism $f : Uc \to Uc'$ there is a C-morphism $g : c \to c'$ such that $Ug = f$. A subcategory is full if and only if its inclusion functor is full.

As for the other direction, suppose that there is an $O \in A$ such that $\hat{f} := \kappa_{f(O)} \circ f \circ \kappa_O^{-1}$ is linear for a map $f : A \to A'$. In particular, we have $\kappa_{f(O)}^{-1} \circ \hat{f} = f \circ \kappa_O^{-1}$ which is just a reformulation of $f(O) + \hat{f}(v) = f(O + v)$, $v \in V_A$. From this and the linearity of $\hat{f}$ we obtain (cf. remark 3.1.1(d))

$$\begin{aligned} f(P + v) &= f(O + (\overrightarrow{OP} + v)) = f(O) + \hat{f}(\overrightarrow{OP} + v) \\ &= (f(O) + \hat{f}(\overrightarrow{OP})) + \hat{f}(v) \\ &= f(P) + \hat{f}(v), \qquad P \in A,\ v \in V_A. \end{aligned}$$

By remark 3.1.1(b) this shows f to be affine with associated linear map $\hat{f}$. The rest of the assertion follows from remark 3.1.1(g) and the repetition of our previous argument.

(ii) We define $f : A \to A'$ by $P \mapsto O' + \hat{f}(\overrightarrow{OP})$. This is easily seen to be an affine map which satisfies $f(O) = O'$, e.g. by the remarks 3.1.1(b), (d) from above

$$\begin{aligned} f(P + v) &= O' + \hat{f}(\overrightarrow{O(P + v)}) = O' + \hat{f}(\overrightarrow{OP} + v) = O' + \hat{f}(\overrightarrow{OP}) + \hat{f}v \\ &= f(P) + \hat{f}(v), \qquad P \in A,\ v \in V_A. \end{aligned}$$

This yields the existence of an affine map. As for the uniqueness: by remark 3.1.1(h), two affine maps $h, g : A \to A'$ that have $\hat{f}$ as associated linear map differ by a unique translation $\tau_v : A' \to A'$, where $v = \overrightarrow{h(O)g(O)}$. With this any other such affine map g that maps O to O' is equal to the f constructed above.

(qed)

Corollary 3.1.4. *For A, A' in $\mathrm{Aff_K}$ and an affine map $f : A \to A'$ the following statements are equivalent*

(1) $f : A \to A'$ is an isomorphism in $\mathrm{Aff_K}$;

(2) $f : A \to A'$ is a bijection.

If A, A' are both not empty spaces then (1) is equivalent to

(3) $\hat{f} : V_A \to V_{A'}$ is an isomorphism.

Proof.

(1)⇒(2),(3) Let f^{-1} be the inverse. By the composition law in $\mathrm{Aff_K}$ the set-theoretic part of f^{-1} is an inverse to the set-theoretic part of f and the associated linear map $\widehat{f^{-1}}$ is an inverse to $\hat{f}$.

(2)⇒(1) By the assumption of bijectivity of f if A or A' has as underlying set the empty set then the underlying set of the other space is empty too and $f = 1_\emptyset$. The associated vector spaces are trivial and $\hat{f}$ is the unique isomorphism. In the non-empty case by proposition 3.1.3(i) $\hat{f}$ is a bijection and thus a linear isomorphism. In both cases the affine map $(f, \hat{f})$ is an isomorphism by the composition law in Aff_K.

(3)⇒(1) Fix an arbitrary point $O \in A$. Again, by proposition 3.1.3(i) and the fact that κ_O and $\kappa_{f(O)}$ are bijections, one can see that the set-theoretic f is a bijection and hence the affine map f is an isomorphism due to the composition law in Aff_K.

(qed)

Note that for (3) to be equivalent to (1) the assumption of non-empty spaces is necessary. Consider for example a one-point space A' and an empty space A. Both have as associated vector spaces a trivial vector space. There is a unique affine map $f : A \to A'$ for which $\hat{f}$ is an isomorphism, but the set-theoretic map f is not a bijection.

3.2 Affine Spaces as Structures between Vector Spaces and Sets

We explore the properties of the category Aff_K by exploiting our point of view that Aff_K describes structures situated between Vct_K and Set . This suggests to begin with the consideration of the evident functors between these categories. These functors are the forgetful functor $U_S : \mathrm{Aff}_\mathrm{K} \to \mathrm{Set}$, the self-action functor $S : \mathrm{Vct}_\mathrm{K} \to \mathrm{Aff}_\mathrm{K}$ and the forgetful functor $U_V : \mathrm{Aff}_\mathrm{K} \to \mathrm{Vct}_\mathrm{K}$. U_S assigns to each affine space its underlying set and to each affine map its underlying set-theoretic map. S assigns to each vector space V itself viewed as a V-torsor by the right self-action induced from the addition-map $+ : V \times V \to V$ and each linear map $f : V \to W$ the affine map induced by it, i.e. (f, f) with the underlying map of f as set-theoretic map and f itself as linear map. Finally, U_V assigns to each affine space its associated vector space and each affine map its associated linear map. Loosely speaking, U_S forgets the linear structure involved, S forgets the neutral element and U_V forgets the set of points.

In 2.2 we have already considered the forgetful functor $U : \mathrm{Vct}_\mathrm{K} \to \mathrm{Set}$. We have noted that it is faithful, reflects identities, creates limits and has a left adjoint, which in the particular case of Vct_K is equivalent to being continuous (cf. lemma 2.2.7). Moreover, it reflects isomorphisms, i.e. if Uf is a bijection of sets then f is an

isomorphism of vector spaces. This is just the categoric formulation of the well-known fact from linear algebra that the inverse of a a bijective linear map is linear as well. We will show that U_S has similar properties.

Proposition 3.2.1. *The forgetful functor $U_S : \mathrm{Aff}_K \to \mathrm{Set}$ is faithful, reflects isomorphisms and lifts small limits. Its restriction $U_S^\times : {\mathrm{Aff}_K}^\times \to \mathrm{Set}$ is faithful, reflects isomorphisms and moreover reflects identities. It doesn't lift small limits though.*

Proof. (i) For spaces which aren't empty spaces the faithfulness and the reflection of identities are a direct consequence of remark 3.1.1(g) or proposition 3.1.3. By remark 3.1.1(g) any two affine maps, which have the same set-theoretic map, have the same associated linear map and are hence equal. If in particular the set-theoretic map is the identity, then the associated linear map has to be the identity and hence the affine map is the identity. Again, this can be seen by the same remark or directly by proposition 3.1.3(i). In the case of one space being the empty space there is at most one map, so the condition for U_S being faithful is satisfied trivially. The reflection of isomorphisms is a direct consequence of corollary 3.1.4.

(ii) For the lifting of small limits with the faithfulness from (i) we can deploy our standard arguments from proposition 2.2.2 (cf. also remark 2.2.3). Recall that for a functor $G : C \to D$ to lift small limits it means that for each limiting cone τ of GF for $F : I \to C$, I small, there is a limiting cone σ of F such that $G\sigma = \tau$. Let $F : I \to \mathrm{Aff}_K$ be a functor and I a small category. To begin with, we need to check that we can define an affine structure in a pointwise manner on $\mathrm{Nat}(\Delta *, U_S F)$ such that the evaluation maps $\tau_i : \mathrm{Nat}(\Delta *, U_S F) \to U_S F i$, $\sigma \mapsto \sigma_i(\bullet)$ for $\bullet \in *$ become affine maps. This will yield that τ can be lifted to a cone in Aff_K to F. For that case we assume w.l.o.g. that $\mathrm{Nat}(\Delta *, U_S F)$ is not empty, because otherwise it has a U_S-lift as an empty space and there is exactly one affine map from an empty space to an affine space given by $(1_\emptyset, 0)$ with the respective zero map 0. Consider the vector space $V := \mathrm{Nat}(\Delta *, U U_V F)$, whose K-vector space structure is given by pointwise operations like in proposition 2.2.2(i) and define the action in a pointwise manner

$$\begin{aligned} &\mathrm{Nat}(\Delta *, U_S F) \times V \to \mathrm{Nat}(\Delta *, U_S F), \\ &\quad (\sigma, v) \mapsto \sigma + v, \quad \text{with} \quad (\sigma + v)_i(\bullet) = \sigma_i(\bullet) + v_i(\bullet), \quad i \in \mathrm{Obj}\, I. \end{aligned}$$

By the construction of V this action is well-defined. Indeed, let $\ell : i \to j$ be a

morphism in I, then $F\ell$ is an affine map and we have

$$\begin{aligned}(U_S F\ell \circ (\sigma + v)_i)(\bullet) &= U_S F\ell(\sigma_i(\bullet) + v_i(\bullet)) \\ &= (U_S F\ell \circ \sigma_i)(\bullet) + (\widehat{F\ell} \circ v_i)(\bullet) && (F\ell \text{ affine}) \\ &= (U_S F\ell \circ \sigma_i)(\bullet) + (UU_V F\ell \circ v_i)(\bullet) && (\widehat{F\ell} = U_V F\ell) \\ &= \sigma_j(\bullet) + v_j(\bullet) && (\sigma,\, v \text{ cones}) \\ &= (\sigma + v)_j(\bullet).\end{aligned}$$

For the simply transitiveness let $\sigma, \mu \in \mathrm{Nat}(\Delta *, U_S F)$. By the simply transitive action of $V_{Fi} = U_V Fi$ on Fi there is only one $v_i \in V_{Fi}$ such that $\mu_i(\bullet) = \sigma_i(\bullet) + v_i$ for each $i \in \mathrm{Obj}\, I$. Define $\nu_i : * \to UU_V Fi$ by $\bullet \mapsto v_i$. We have to show that this yields a cone $\nu : \Delta * \dot{\to} UU_V F$. Again, let $\ell : i \to j$ be a morphism in I. For the naturality of ν consider the following calculation:

$$\begin{aligned}\mu_j(\bullet) &= (U_S F\ell \circ \mu_i)(\bullet) && (\mu \text{ cone}) \\ &= (U_S F\ell \circ (\sigma + \nu)_i)(\bullet) \\ &= (U_S F\ell \circ \sigma_i)(\bullet) + (\widehat{F\ell} \circ \nu_i)(\bullet) && (F\ell \text{ affine}) \\ &= (U_S F\ell \circ \sigma_i)(\bullet) + (UU_V F\ell \circ \nu_i)(\bullet).\end{aligned}$$

On the other hand we have $\mu_j(\bullet) = \sigma_j(\bullet) + v_j$. The uniqueness of v_j yields

$$(UU_V F\ell \circ \nu_i)(\bullet) = v_j = \nu_j(\bullet), \qquad \text{hence} \qquad UU_V F\ell \circ \nu_i = \nu_j,$$

which at last shows ν to be a cone and the action from above to be simply transitive. The τ_i are affine maps by the pointwise construction. Their associated linear maps $\hat{\tau}_i : V \to U_V Fi$ form the limiting cone

$$\begin{aligned}&\hat{\tau} : \Delta\, \mathrm{Nat}(\Delta *, UU_V F) \dot{\to} U_V F, \quad \text{with} \\ &\hat{\tau}_i : \mathrm{Nat}(\Delta *, UU_V F) \to U_V Fi, \quad \nu \mapsto \nu_i(\bullet), \qquad i \in \mathrm{Obj}\, I,\end{aligned}$$

which is just the unique lift of the limiting cone $U\hat{\tau}$ in Set by proposition 2.2.2 and its proof.

To show that τ is a limiting cone we employ a similar argument like in part (i) of the proof of proposition 2.2.2 for the case of linear maps. Let $\nu : \Delta A \dot{\to} F$ be a cone in $\mathrm{Aff}_{\mathrm{K}}$ and $t : U_S A \to \mathrm{Nat}(\Delta *, U_S F)$ the unique map in Set such that $U_S \nu_i = \tau_i \circ t$ given by the universal property of limiting cones like in (i) of the proof of proposition 2.2.2. If $\mathrm{Nat}(\Delta *, U_S F) = \emptyset$ then $U_S A = \emptyset$ because there is no map from a non-empty set to the empty set. In that case $t = 1_\emptyset$ and t has a unique U_S-lift $A \to \mathrm{Nat}(\Delta *, U_S F)$ in $\mathrm{Aff}_{\mathrm{K}}$ as $(1_\emptyset, 0)$ by the definition of $\mathrm{Aff}_{\mathrm{K}}$. For the case of $\mathrm{Nat}(\Delta *, U_S F)$ not being the empty space note that $\hat{\nu} = U_V \nu$ is a cone from $V_A = U_V A$ to $U_V F$ in $\mathrm{Vct}_{\mathrm{K}}$, hence there is a

unique linear map $\hat{t} : U_V A \to \mathrm{Nat}(\Delta *, UU_V F)$ such that $\hat{\nu}_i = \hat{\tau}_i \circ \hat{t}$. As already suggested by the notation we want to see $\hat{t}$ as the associated linear map to t. The following standard argument together with the explicit representation of the $\hat{\tau}_i$ shows that this is indeed the case

$$\begin{aligned}(t(P+v))_i(\bullet) &= \tau_i \circ t(P+v) = \nu_i(P+v) = \nu_i(v) + \hat{\nu}_i(v) \\ &= (t(P))_i(\bullet) + (\hat{t}(v))_i(\bullet), \qquad \forall i \in \mathrm{Obj}\, I.\end{aligned}$$

We have to check the transportability of affine structures by bijections next. Let A be an affine space with associated vector space V_A and $f : A \to X$ a bijection of sets. W.l.o.g. A isn't an empty space. To make X an affine space we just need to transport the action by f, i.e.

$$\begin{aligned}X \times V_A \longrightarrow A \times V_A \longrightarrow\!\!\!\!\!\!\rightarrow\ A \longrightarrow\!\!\!\!\!\!\rightarrow\ X,\\ (x, v) \longmapsto (f^{-1}(x), v) \mapsto f^{-1}(x) + v \mapsto f(f^{-1}(x) + v).\end{aligned}$$

This yields a simply transitive action and f becomes an affine isomorpism with associated linear map the identity. With this we can extend the lifting of τ to any other limiting cone $\nu : \Delta X \dot{\to} U_S F$ like in (ii) of the proof of proposition 2.2.2.

(iii) $U_S^\times$ can't lift small limits, for a limit can result in an empty space. Take for example the inclusions of the affine line as two parallel but disjoint lines in the affine plane regarded as sets. Their equalizer in Set is their intersection and hence the empty set, which has not a $U_S^\times$-lift. Note that if we repeat this argument in $\mathrm{Aff_K}^\times$, this shows that $\mathrm{Aff_K}^\times$ can't be complete.

(qed)

Remarks 3.2.2.

(a) U_S neither creates limits nor lifts limits uniquely. This is due to the fact that for an affine space A one can define different isomorphic affine structures on the underlying set A by taking a vector space V isomorphic to the associated vector space V_A and transport the action map $A \times V_A \to A$ by the respective linear isomorphism to a simply transitive action $A \times V \to A$. In the terms of [AHS04] this shows that U_S is not amnestic (cf. also [AHS04, proposition 13.21]). Loosely speaking, this just expresses the fact that an affine space consists of two pieces of interrelated data, namely points and vectors, but the points aren't determined uniquely by the vectors alone and vice versa.

(b) U_S doesn't reflect identities because there are several different empty spaces and for each affine map from an empty space the underlying set-theoretic map is $1_\emptyset$. This is a consequence of the fact that there is only one empty set in Set but several trivial vector spaces in $\mathrm{Vct_K}$.

Although U_S doesn't create limits like the forgetful functor U, the lifting of limits is sufficient to obtain the completeness of Aff_K from the completeness of Set as well as the continuity of U_S.

Theorem 3.2.3. Aff_K *is complete, i.e. has all small limits and* U_S *is continuous, i.e. preserves small limits.*

The following lemma is a slightly stronger version of theorem V.4.2 in [Mac98] which says that a functor that creates particular limits preserves these limits if they exist in the target category.

Lemma 3.2.4. *If* $G : C \to D$ *is a functor which lifts limits for* $F : I \to C$*,* I *small, and* GF *has a limit in* D*, then* G *preserves the limit of* F*. In particular, if* G *lifts small limits and* D *is complete then* G *is continuous.*

Proof. (lemma) This fact follows from the same arguments like the one given in the proof of theorem V.4.2 in [Mac98] (cf. also [AHS04, remark 13.20(2)]). However, for the sake of completeness we will give the argument explicitly. Let $\tau : \Delta c \dot{\to} F$ be a limiting cone for F. By assumption GF has a limiting cone $\sigma : \Delta d \dot{\to} GF$. As G lifts the limits of F there is a G-lift $\sigma' : \Delta c' \dot{\to} F$ of σ in C. The universal property of limiting cones yields a unique isomorphism $t : c \to c'$ in C such that $\tau_i = \sigma'_i \circ t$ for all $i \in \mathrm{Obj}\, I$, in particular $G\tau_i = G\sigma'_i \circ Gt = \sigma_i \circ Gt$ and Gt is an isomorphism. This shows that $G\tau$ is a limiting cone in D.

(qed)

Proof. (theorem) The completeness and the continuity follow directly from $U_S : \mathrm{Aff}_K \to \mathrm{Set}$ lifting limits (cf. proposition 3.2.1) and the completeness of Set together with the lemma from above.

(qed)

Remarks 3.2.5 (finite dimensions)**.**

(a) Recall that a product of an I-indexed family of affine spaces $(A_i)_{i\in I}$ can be described as the limit of a functor $F : I \to \mathrm{Aff}_K$ with $Fi = A_i$, where I is regarded as a discrete category. The universal "projections" pr_i, $i \in I$ are then the components of the limiting cone $\mathrm{pr} : \Delta \varinjlim F \dot{\to} F$. By reconsidering the sets of natural transformations employed in the proofs of proposition 3.2.1 and proposition 2.2.2 for this particular type of functor F we get that the product of an I-indexed family of affine spaces $(A_i)_{i\in I}$ can be obtained as the set-theoretic Cartesian product $\prod_{i\in I} A_i$ with associated vector space the product $\prod_{i\in I} V_{A_i}$ in Vct_K with an componentwise action and the canonical projection maps $\mathrm{pr}_i : \prod_{j\in I} A_j \to A_i$ with associated linear maps the linear projections $\prod_{j\in I} V_{A_j} \to$

V_{A_i}. If the A_i are finite dimensional and I is finite, $\prod_{i\in I} A_i$ has as dimension the sum of the dimensions $\dim A_i$ and is in particular finite dimensional.

$$\dim(A_1 \times A_2) = \dim A_1 + \dim A_2, \qquad A_1, A_2 \in \operatorname{Obj} \mathrm{Aff}_{\mathrm{K}}\,. \tag{3.5}$$

Because subvector spaces of finite dimensional vector spaces are finite dimensional, this fact holds for subspaces of affine spaces as well. The latter are defined for an affine space A in the obvious way as a subset $U \subset A$ together with a vector subspace $V_U \leq V_A$, so that the action restricts to a simply transitive action $U \times V_U \to U$. Recall that limits can be obtained as equalizers of products, and these equalizers in turn can be obtained as subspaces of these products. Consequently, $\mathrm{FAff}_{\mathrm{K}}$ has all finite limits.

(b) Once again by proposition 3.2.1 a terminal object in $\mathrm{Aff}_{\mathrm{K}}$ is a one-point space $*$. Its associated vector space is a trivial vector space. It is thus zero-dimensional and in particular a terminal object in $\mathrm{FAff}_{\mathrm{K}}$. Contrary to the trivial vector space in $\mathrm{Vct}_{\mathrm{K}}$ it is not initial. For an affine space A, affine maps $* \to A$ correspond one-to-one to the points of A. We have thus the same situation as in Set and like there one can see easily that $*$ is a separator. This shows in particular $\mathrm{Aff}_{\mathrm{K}}$ and $\mathrm{FAff}_{\mathrm{K}}$ to be well-pointed, i.e. the terminal objects are separators. This terminology is due to the interpretation of points as maps from the terminal objects in category theory. Being well-pointed means that morphisms can be compared by comparing their values on points. Note that in this sense each vector space has only one point, namely the zero vector and $\mathrm{Vct}_{\mathrm{K}}$ is so to speak the complete opposite of being well-pointed. Although this might seem odd at first sight, the categorical notion of a point is a very important concept for the intrinsic point of view on structures.

(c) The inclusion functor $\iota : \mathrm{FAff}_{\mathrm{K}} \hookrightarrow \mathrm{Aff}_{\mathrm{K}}$ creates finite limits and preserves finite limits, because $\mathrm{Aff}_{\mathrm{K}}$ has these. The creation follows from remark (a) and ι being full, faithful and injective on objects. In more detail, by remark (a) for each limiting cone $\tau : \Delta A \to \iota F$ in $\mathrm{Aff}_{\mathrm{K}}$ for a functor $F : I \to \mathrm{FAff}_{\mathrm{K}}$, I finite, A has to be finite-dimensional. Since ι is faithful, full and injective on objects there is only one cone from A to F in $\mathrm{FAff}_{\mathrm{K}}$ that is mapped to τ by ι and because ι is faithfull and full again this cone is also a limiting cone. The preservation of finite limits follows from theorem V.4.2 in [Mac98] or lemma 3.2.4.

(d) If we restrict U_S to $\mathrm{FAff}_{\mathrm{K}}$ then it will stay faithfull, reflect isomorphisms, lift finite limits and preserve them. This is due to the inclusion functor $\iota : \mathrm{FAff}_{\mathrm{K}} \hookrightarrow \mathrm{Aff}_{\mathrm{K}}$ and U_S having these properties.

We observe that the self-action functor S and the forgetful functors U and U_S are interrelated in the following way:

$$U = U_S \circ S.$$

With this we can conclude some properties of S from the properties of U and U_S.

Proposition 3.2.6. *The self-action functor $S : \mathrm{Vct_K} \to \mathrm{Aff_K}$ is faithful, reflects identities, reflects isomorphisms, creates small limits and is continuous.*

Proof. The first three properties are an immediate consequence of the definition of S. They can also be obtained from straightforward arguments using $U = U_S \circ S$, the respective properties of U, U_S and functoriality. Lemma 3.2.4 together with the completeness of Set yields the continuity of S from the creation and hence the lifting of limits. We thus only need to show the creation of limits. Let $F : I \to \mathrm{Vct_K}$ be a functor, I small and $\tau : \Delta A \dot{\to} SF$ a limiting cone from A to SF. By theorem 3.2.3 $U_S\tau$ is a limiting cone for $U_SSF = UF$ in Set. By proposition 2.2.2 U creates limits. Hence there is a unique cone σ in $\mathrm{Vct_K}$ for F such that $U\sigma = U_S\tau$ and moreover σ is a limiting cone. In particular we have $U_SS\sigma_i = U_S\tau_i$ for all $i \in \mathrm{Obj}\, I$. By U_S being faithful we obtain $\tau_i = S\sigma_i$ and hence $S\sigma = \tau$. This cone σ is also the unique one with this property. Any other cone σ' in $\mathrm{Vct_K}$ that is mapped to τ by S is further mapped to $U_S\tau = U\sigma'$ by U_S. Hence $\sigma' = \sigma$ by U creating limits.

One can obtain the creation of limits from another more general abstract result, namely exercise V.6.3 in [Mac98] for $A' = X' = \mathrm{Vct_K}$, $A = \mathrm{Aff_K}$, $X = \mathrm{Set}$, $G' = 1_{\mathrm{Vct_K}}$, $G = U_S$, $H' = S$ and $H = U$. The fact that $\mathrm{Vct_K}$ is the pullback of U and U_S in the category of large categories with small hom-sets is obvious from $U = U_S \circ S$[7].

(qed)

Remark 3.2.7 (finite dimensions)**.**

The self-action functor can be restricted to finite dimensions, i.e. to $S : \mathrm{FVct_K} \to \mathrm{FAff_K}$, where $\mathrm{FVct_K} \hookrightarrow \mathrm{Vct_K}$ is the full subcategory of finite-dimensional vector spaces, without loosing any of its properties. It creates and preserves finite limits, in particular. With the restriction to finite limits this follows from the following diagram

$$\begin{array}{ccc} \mathrm{FVct_K} & \xrightarrow{S} & \mathrm{FAff_K} \\ \downarrow & & \downarrow \\ \mathrm{Vct_K} & \xrightarrow[S]{} & \mathrm{Aff_K}, \end{array}$$

because the inclusion functors have all these properties. The reflection properties are a direct consequence of the inclusion functors being injective on objects and

[7] Note that the category of large categories with small hom-sets can indeed be formed within the universe approach. Nonetheless, we can not form the category of all categories with small hom-sets, because an arbitrary set can be regarded as a discrete category which in particular has small hom-sets. As a consequence the category of all categories with small hom-sets would contain its set of objects regarded as a discrete category. This contradicts the axiom of regularity.

injective on morphisms. The creation of finite limits can be shown for $\mathrm{FVct}_\mathrm{K} \hookrightarrow \mathrm{Vct}_\mathrm{K}$ in the same manner as it was shown for $\mathrm{FAff}_\mathrm{K} \hookrightarrow \mathrm{Aff}_\mathrm{K}$ in remark 3.2.5(c). Alternatively, the creation of finite limits can be obtained by exercise V.6.3 in [Mac98], for the above diagram is a pullback in the category of large categories with small hom-sets. The fact that $S : \mathrm{FVct}_\mathrm{K} \to \mathrm{FAff}_\mathrm{K}$ even creates and preserves small limits is due to the definition of the dimension of an affine space by its associated vector space and because $S : \mathrm{Vct}_\mathrm{K} \to \mathrm{Aff}_\mathrm{K}$ has these properties.

Next we turn to the forgetful functor U_V. By our results in section 3.1 we can't expect the properties of U_V to be as strong as the properties of U_S or U. Indeed, we will loose nearly everything. U_V is neither faithful, nor does it reflect isomorphisms or identities and it doesn't preserve limits.

The failure to be faithful lies in the fact that an associated linear map doesn't uniquely determine an affine map as one can for example see for two different translations $\tau_v, \tau_w : A \to A$, $v, w \in V_A$, because their associated linear maps $U_V\tau_v$, $U_V\tau_w$ are the identity maps 1_{V_A} (cf. remark 3.1.1(h)). Translations also yield counterexamples for the reflection of identities and the preservation of limits. For the latter consider the equalizer of the translations τ_v, τ_w from above. It is an empty space with its unique map to A. Nonetheless, on the level of the associated linear maps the equalizer of $1_{V_A}, 1_{V_A} : V_A \to V_A$ is 1_{V_A} again and not the zero map $0 \to V_A$.

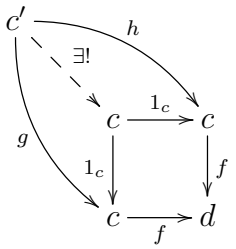
The failure to reflect isomorphisms is a subtle one. As one can see from corollary 3.1.4 and its following discussion U_V fails to reflect isomorphisms only because of the existence of empty spaces, which have the same associated vector space like the one-point spaces. In particular, the restriction $U_V^\times : {\mathrm{Aff}_\mathrm{K}}^\times \to \mathrm{Vct}_\mathrm{K}$ does reflect isomorphisms by corollary 3.1.4.

U_V preserves monos and epis. It reflects monos, but it doesn't reflect epis in general. However, only the epis with a trivial vector space as codomain aren't reflected properly, so $U_V^\times$ does reflect epis at least. To see this, observe first that given an affine map $f : A \to A'$, where $U_S A$ and $U_S A'$ are not empty, the linear map $\hat{f} = U_V f$ is a mono (epi) if and only if $U_S f$ is a mono (epi). This is due to proposition 3.1.3(i) together with the characterization of monos and epis by injective and surjective maps in Vct_K and Set (cf. proposition 2.2.4 and its analogue in Set). In the case of at least one of the affine spaces A, A' being an empty space, the asserted equivalence still holds for monos. This is because there is no set-theoretic map into the empty set, so A has to be the empty space and for the same reason f as well as its underlying set-thereotic map $U_S f$ is monic. In this case $U_V f$ is the unique map from the trivial vector space, which is also monic. For epis we have only that $U_S f$ being epi implies $U_V f$ being epi. A counterexample regarding the reverse implication is given by the affine map $f : \emptyset \to *$, which also yields a counterexample for U_V to reflect epis. This is due to $U_V f$ being the natural isomorphism between trivial vector spaces, but neither f nor $U_S f$ being epi. As for the asserted direction, the set-theoretic map $U_S f$ is epi if and only if A' is an empty space as well. This

is a consequence of the empty set being initial in Set and two initial objects being naturally isomorphic. In more detail, the fact that $U_S A$ is initial in Set together with the right-cancellation property of $U_S f$ show that $U_S A'$ is initial. In particular, this implies that A' is the empty space and f is an isomorphism. $U_V f$ is then an isomorphism too, namely the one between two trivial vector spaces.

With this we can show the assertion as follows. As for the respective reflection properties, this follows from U_S and its restriction $U_S^\times$ being faithful, since faithful functors reflect monos and epis. As for the respective preservation properties, we show that U_S preserves monos and epis.

Recall that a continuous functor preserves monos, because for any morphism $f : c \to d$ in any category C the condition of being monic can be expressed as a pullback diagram (cf. dual of [Mac98, exercise III.4.4])



This shows that U_S preserves monos. For the preservation of epis we use that Aff_K has the affine line L, i.e. a one-dimensional affine space, as a coseparator. This will be proved in theorem 3.2.12 and the proof will be independent of this particular result of course. With a coseperator we can in essence repeat the argument given in proposition 2.2.4; although we will do it in a more categorical fashion. Suppose f is epi and there is a point $P \in A$ with $P \notin \operatorname{im} f$, where $\operatorname{im} f$ is the affine subspace of A' with underlying set the set-theoretic image of f and associated vector space the image of $\hat{f}$. Fix a $Q \in \operatorname{im} f \leq A'$ and consider the affine maps $P, Q : * \to A'$ representing those points, where $*$ is a one-point space. Because L is a coseparator there is an affine map $\alpha : A' \to L$ with $\alpha P \neq \alpha Q$. Define a second affine map $\chi : A' \to L$ as the composite of the unique map $A' \to *$ with the point $\alpha Q : * \to L$. By construction we have $\alpha f = \chi f$ and hence $\alpha = \chi$ by the defining property of epis. However, this leads to the contradiction $\chi P = \alpha Q \neq \alpha P$. Hence $P \in \operatorname{im} f$ and the set-theoretic map is surjective. This concludes the proof of our assertion.

Finally, we want to show that U_V preserves small products. This amounts to show the following: for any product of an I-indexed family of affine spaces $(A_i)_{i \in I}$ in Aff_K (with I being small), that is an affine space C together with affine maps the "projections" $p_i : C \to A_i$, $i \in I$ such that the respective universal property holds in Aff_K, the vector space $U_V C$ together with $U_V p_i$, $i \in I$ is a product in Vct_K. Since functors preserve isomorphisms, it suffices to show this for a particular choice of a product, which we will take from remark 3.2.5(a); but with this choice the assertion becomes obvious.

We summarize the obtained results in the following proposition.

Proposition 3.2.8. *The forgetful functor $U_V : \mathrm{Aff_K} \to \mathrm{Vct_K}$ preserves small products, monos and epis. It reflects monos, but it is neither faithful, nor continuous, nor does it reflect epis, isomorphisms or identities. Its restriction $U_V^\times : {\mathrm{Aff_K}}^\times \to \mathrm{Vct_K}$ does reflect epis and isomorphisms, however.*

Remark 3.2.9 (finite dimensions)**.**
U_V can be restricted to finite dimensions. Its restriction $U_V : \mathrm{FAff_K} \to \mathrm{FVct_K}$ preserves finite products, monos, epis and reflects monos, too. One can see this by a similar type of argument as for S in remark 3.2.7, which employs the respective creation, reflection and preservation properties of the inclusion functors.

Since we have already shown that U_S preserves and reflects monos and epis, this result together with proposition 3.2.8 yield a characterization of monos and epis in $\mathrm{Aff_K}$ for free.

Theorem 3.2.10. *In $\mathrm{Aff_K}$ an affine map $f : A \to A'$ is a mono if and only if its underlying set-theoretic map is injective, which is the case if and only if its associated linear map is injective. The affine map f is an epi if and only if its underlying set-theoretic map is surjective. If A' isn't an empty space, f is an epi if and only if its associated linear map is surjective.*

Proof. (equivalence)

"⇒" This is a consequence of U_S and U_V to preserve monos and epis together with proposition 2.2.4 and its analogue in Set.

"⇐" Together with proposition 2.2.4 and its analogue in Set this is a consequence of U_S to reflect monos and epis and U_V to reflect monos as well as $U_V^\times$ to reflect epis, for A can't be an empty space when A' is not and f is an epi. The latter is due to the same argument with initial objects as it was used in the proof of proposition 3.2.8.

(qed)

Remark 3.2.11 (finite dimensions)**.**
The above results holds for $\mathrm{FAff_K}$ as well, for the inclusion $\mathrm{FAff_K} \hookrightarrow \mathrm{Aff_K}$ preserves and reflects monos and epis which follows from remark 3.2.5(c) by the same argument as for U_S given in the proof of proposition 3.2.8.

Before we study the existence of adjoints for our functors, we list some further properties of $\mathrm{Aff_K}$ and $\mathrm{FAff_K}$.

Theorem 3.2.12. Aff_K *and* FAff_K *are well-powered, well-pointed, the affine line is a coseparator for both categories and both have a strict initial object. Moreover, the initial objects are precisely the empty spaces.*

Proof. (i) Aff_K is well-powered because U_S preserves monics. In more detail, the fact that U_S preserves monics yields a map $\mathrm{Sub}\, A \to \mathrm{Sub}\, U_S A$ between the sets of isomorphism classes of subobjects. As for the existence of images of affine maps as particular subsets this map is seen to be injective, which implies the existence of a surjection $\mathrm{Sub}\, U_S A \to \mathrm{Sub}\, A$. The axioms for the universe then yield $\mathrm{Sub}\, A$ to be a small set, because $\mathrm{Sub}\, U_S A$ is. With this result we can repeat the same argument for the inclusion functor $\mathrm{FAff}_K \hookrightarrow \mathrm{Aff}_K$ to see that FAff_K is well-powered, too.

(ii) By remark 3.2.5 Aff_K and FAff_K are well-pointed, i.e. the terminal object is a separator. For the coseparator let L denote an affine line, i.e. a one-dimensional affine space, e.g. K regarded as an affine space. Fix two distinct points $O, I \in L$. We will imitate the construction of the coseperator in Vct_K. Let $f, g : A' \to A$ be affine maps such that $f \neq g$. Then there is a point $P \in A'$ such that $f(P) \neq g(P)$. Consider a linear map $\hat{\alpha} : V_A \to V_L$ which maps $\overrightarrow{g(P)f(P)}$ to $\overrightarrow{OI}$. Recall that such a map exists by the basis extension theorem and the structure theorem of vector spaces of linear algebra. With this and the pair of points $(g(P), O)$ proposition 3.1.3(ii) yields a unique affine map $\alpha : A \to L$ such that $\hat{\alpha}$ is its associated linear map and $\alpha(g(P)) = O$. By construction we get

$$\alpha(f(P)) = \alpha(g(P) + \overrightarrow{g(P)f(P)}) = O + \overrightarrow{OI} = I,$$

hence $\alpha g \neq \alpha f$. This proofs L to be a coseparator.

(iii) Recall that an initial object i in a category C is said to be strict when each morphism $f : c \to i$ is an isomorphism. For example, $\emptyset$ is a strict initial object in Set. That an empty space is an initial object was used several times and is obvious, because an empty space has $\emptyset$ as underlying set, which is initial in Set, and as associated vector space it has a trivial vector space 0, which is initial in Vct_K and FVct_K. Furthermore, the dimension of the empty space is zero by definition, so it belongs to FAff_K, too. Again, because of the strictness of $\emptyset$ an empty space is strict as well. In particular, each initial object in Aff_K or FAff_K has a morphism to an empty space and hence has to be the empty space itself.

(qed)

In geometric terms the coseparating property of an affine line L means that for two distinct points $P, Q \in A$ there is a pair of parallel but disjoint hyperplanes

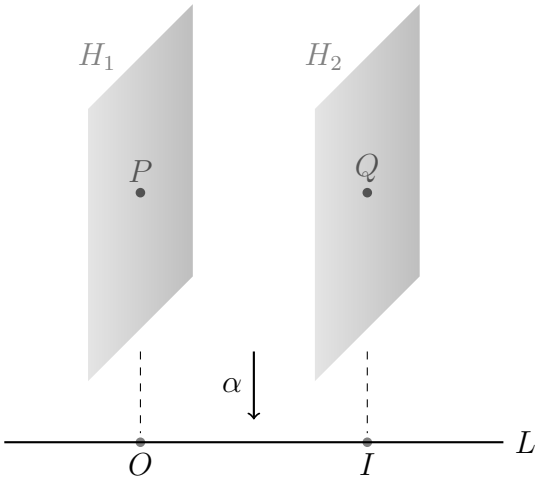


Figure 3.2: The coseparator property of an affine line L geometrically. The distinct points $P, Q \in A$ are separated by the hyperplanes H_1 and H_2, which are the respective preimages of O and I under the affine map $\alpha : A \to L$.

$H_1, H_2 \subset A$ with $P \in H_1$ and $Q \in H_2$ as depicted in figure 3.2. In terms of the coseparating affine map $\alpha : A \to L$ with $\alpha(P) = O$, $\alpha(Q) = I$, we have $H_1 = \alpha^{-1}(O) = P + \ker \hat{\alpha}$ and $H_2 = \alpha^{-1}(I) = Q + \ker \hat{\alpha}$, where the latter notations stand for the respective images under the action maps. We can turn to the discussion of adjoints now. First, from the above theorem and the special adjoint functor theorem we get the same powerful criterion for the existence of left adjoints like in lemma 2.2.7 for Vct_K.

Theorem 3.2.13. *Let C be a category with small hom-sets and $F : \mathrm{Aff}_\mathrm{K} \to C$ be a functor. Then F has a left adjoint if and only if it is continuous.*

From this criterion together with proposition 3.2.1, proposition 3.2.6 and lemma 2.2.7 we obtain the existence of the left adjoints.

Corollary 3.2.14. *U_S and S have left adjoints.*

Concerning U_V, the fact that it isn't continuous yields already that it can't have a left adjoint (cf. [Mac98, theorem V.5.1]). However, regarding this, there is a far stronger result, which can be understood as a consequence of the different "homotopy-types" of linear catgegories like Vct_K and categories like Aff_K and Set with strict initial objects.

Theorem 3.2.15. *Let C be a linear category. Then the only cocontinuous, i.e. small-colimit preserving functors $C \to \mathrm{Aff}_\mathrm{K}$ are, up to isomorphism, the constant functors $\Delta\emptyset$, $c \mapsto \emptyset$, $f \mapsto 1_\emptyset$, where $\emptyset$ denotes an empty space.*

Proof. Let 0 be a zero object of C. Suppose $F : C \to \mathrm{Aff}_\mathrm{K}$ is cocontinuous, then F preserves initial objects. Hence $F0$ is an empty space by theorem 3.2.12. Because 0 is also a terminal object in C, for each $c \in \mathrm{Obj}\, C$ there is a unique morphism $c \to 0$ and hence $Fc \to F0$. The strictness of $F0$ implies Fc to be an empty space. Thus F maps each object to an initial object in Aff_K and each morphism to the unique isomorphism between the respective initial objects. In particular. F is an initial object in the functor category[8] $\mathrm{Aff}_\mathrm{K}{}^C$ and hence isomorphic to $\Delta\emptyset$ for an empty space $\emptyset$.

(qed)

This impossibility result yields that the self-action functor can't be cocontinuous and hence by the dual of theorem V.5.1 in [Mac98], S can't have a right adjoint. Of course, theorem 3.2.15 yields once more the impossibility of a left adjoint for U_V. Moreover, there is again a far stronger result subsuming these two statements, namely that there is no continuous functor $\mathrm{Aff}_\mathrm{K} \to \mathrm{Vct}_\mathrm{K}$ except the one isomorphic to $\Delta 0$ for a trivial vector space 0.

Theorem 3.2.16. *Let C be a linear category with small hom-sets, then the only continuous functors $\mathrm{Aff}_\mathrm{K} \to C$ are the right adjoints to the constant functor $\Delta\emptyset : C \to \mathrm{Aff}_\mathrm{K}$ for an empty space $\emptyset$, i.e. those functors isomorphic to $\Delta 0$, $A \mapsto 0$, $f \mapsto 1_0$, for a zero object $0 \in \mathrm{Obj}\, C$.*

Proof. First, observe that $\Delta\emptyset \dashv \Delta 0$, because

$$\mathrm{Aff}_\mathrm{K}(\emptyset, A) = \mathrm{Aff}_\mathrm{K}(\Delta\emptyset(c), A) \cong C(c, 0) = C(c, \Delta 0(A)),$$

natural in A and c, which is due to both sets being the one-point set for any pair of arguments; stated differently, this is due to both bifunctors $\mathrm{Aff}_\mathrm{K}(\Delta\emptyset(-), -)$, $C(-, \Delta 0(-))$ being terminal objects in the functor category $\mathrm{Set}^{C^{op} \times \mathrm{Aff}_\mathrm{K}}$, where C^{op} denotes the dual or opposite category of C, i.e. the category with objects the ones of C and morphisms the ones of C with domain and codomain interchanged (cf. section 2.1.6). Let $F : \mathrm{Aff}_\mathrm{K} \to C$ be a continuous functor. By theorem 3.2.13 it has a right adjoint, which by the dual of theorem V.5.1 in [Mac98] is cocontinuous and hence isomorphic to $\Delta\emptyset$ by theorem 3.2.15.

(qed)

From the next abstract result we will obtain that the left adjoints F_S of the forgetful functor $U_S : \mathrm{Aff}_\mathrm{K} \to \mathrm{Set}$ and F of the forgetful functor $U : \mathrm{Vct}_\mathrm{K} \to \mathrm{Set}$ will be the "free" functors, i.e. those functors which assign to a set the affine space or vector space freely generated by this set. Such free functors are very

[8] Recall that for two categories C, D a functor category D^C has as objects the functors $F : C \to D$ and as morphisms the natural transformations between these functors with the composition given by the vertical, i.e. componentwise composition of natural transformations.

closely related to copowers of separators. Recall that for a set X, a category C and an object $b \in C$, the Xth *copower* of b, which is denoted by $X \cdot b$, is defined to be the representing object of the exponential $C(b,-)^X = \mathrm{Set}(X, C(b,-))$, i.e. $C(X \cdot b, -) \cong \mathrm{Set}(X, C(b,-))$. Equivalently, one can understand $X \cdot b$ as an X-fold coproduct of b with itself (cf. [Mac98, section III.3]). This point of view suggests some examples like the free K-vector space generated by the set X as $X \cdot K = \bigoplus_{x \in X} K$ or that each set X can be regarded as $X \cdot *$, the Xth copower of the one-point set. However, our defining property has the advantage of suggesting the possible adjunction involved, namely $(-) \cdot b \dashv C(b,-)$ if C has small hom-sets and all small copowers of b.

Proposition 3.2.17. *Let C be a category with small hom-sets and let $U : C \to \mathrm{Set}$ be a faithful functor, i.e. let (C,U) be a concrete category, such that U has a left adjoint. The following holds:*

(i) U is representable and the representing object $u \in \mathrm{Obj}\, C$ is a separator;

(ii) In C all small copowers of u exist and a left adjoint of U is given by a copower functor $(-) \cdot u : \mathrm{Set} \to C$.

Proof. (i) Recall that a functor $F : C \to \mathrm{Set}$ is said to be *representable* if there is an object $b \in \mathrm{Obj}\, C$ such that $F \cong C(b,-)$. By the assumption U has a left adjoint $F : \mathrm{Set} \to C$. Together with $\mathrm{Set}(*, U(-)) \cong U$ for the one-point set $*$ this yields

$$C(F*, -) \cong \mathrm{Set}(*, U(-)) \cong U,$$

hence the representability of U by $u := F*$. The separator property for u is a consequence of U being faithful. Let $f, g : a \to b$ be morphisms in C with $f \neq g$. Because U is faithful, $C(u,-)$ is a faithful functor, so with $C(u,f) = f_*$ and $C(u,g) = g_*$ we have $f_* \neq g_*$. Hence there is a morphism $h : u \to a$ such that $f_*h \neq g_*h$. Yet $f_*h = fh$ and $g_*h = gh$, so $f_*h \neq g_*h$ shows u to be a separator. In fact, reversing the argumentation shows $C(u,-)$ to be faithful if u is a separator. Alltogether this yields that a representable functor is faithful if and only if its representing object is a separator (cf. [AHS04, proposition 7.12]).

(ii) From $U \cong C(u,-)$ and the adjunction $F \dashv U$ we get $F \dashv C(u,-)$ and hence the existence of small copowers of u, namely $X \cdot u = FX$. In particular this shows that each left adjoint for U can be regarded as a copower functor. Vice versa, the defining property of copowers of u together with corollary IV.1.2 in [Mac98] imply that any copower functor is a left adjoint for $C(u,-)$ and hence for U.

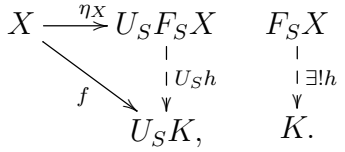
(qed)

This abstract result is the reason behind the constructions of free structures like the free abelian groups $\bigoplus_{x\in X}\mathbb{Z}$, free K-vector spaces $\bigoplus_{x\in X}K$ (cf. remark 2.2.8(d)) and in particular free affine spaces, which by the proposition are X-fold coproducts of the one-point space $*$ for a particular small set X. Note that in the case of K-vector spaces the structure theorem from linear algebra states that each K-vector space is free, i.e. isomorphic to a free vector space generated from a particular set. So the free functor $F : \mathrm{Set} \to \mathrm{Vct_K}$ is isomorphism-dense[9]. We will see later that the same result holds for $F_S : \mathrm{Set} \to \mathrm{Aff_K}$.

At last, we know that the free vector space functor F is faithful. This can be understood as a categorical expression of the well-known fact from linear algebra that a linear map is uniquely determined by giving its values on a basis. The same fact holds for F_S.

Proposition 3.2.18. *Each component $\eta_X : X \to U_S F_S X$ of the unit η of the adjunction $F_S \dashv U_S$ is monic and F_S is faithful.*

Proof. By the dual of theorem IV.3.1 in [Mac98] F_S is faithful if and only if each component η_X of the unit η is monic, so we just need to show one of the two proposed statements. We will show the injectivity of η_X, which was seen in section 2.2 to be equivalent to η_X being monic. Let $P, Q \in X$ be distinct elements of X. Consider the field K as an affine line. There is a map $f : X \to K$ with $f(P) = 0$ and $f(Q) = 1$ in Set. By the universality of η_X there is a unique affine map $h : F_S X \to K$ with $h\eta_X = f$. In particular $\eta_X(P) \neq \eta_X(Q)$. Being more careful in the notation the universal property reads as follows:



(qed)

The left adjoint of S will be discussed in section 4.2.

3.3 The Cocompleteness of $\mathrm{Aff_K}$

Due to theorem 3.2.15 we can't study colimits in $\mathrm{Aff_K}$ by the means of S or any other reasonable functor from $\mathrm{Vct_K}$. Regarding our viewpoint on $\mathrm{Aff_K}$ to describe structures between $\mathrm{Vct_K}$ and Set this suggests that colimits in $\mathrm{Aff_K}$ may "look more like the colimits in Set". We have already seen this for the initial objects in $\mathrm{Aff_K}$ which were the empty spaces. If at all, we thus should study colimits by means of

[9] Recall that a functor $F : C \to D$ is said to be *isomorphism-dense* when to each object $d \in \mathrm{Obj}\, D$ there is an object $c \in \mathrm{Obj}\, C$ such that $Fc \cong d$.

cocontinuous functors $\mathrm{Set} \to \mathrm{Aff}_{\mathbb{K}}$. The only such functor we have at hand is the free functor $F_S : \mathrm{Set} \to \mathrm{Aff}_{\mathbb{K}}$ which is a left adjoint to the forgetful functor U_S. However, contrary to U_S, we lack an explicit description of F_S. The only thing we know is F_S to be given by copowers of the one-point space. In particular, $F_S\emptyset$ is an empty space and $F_S* $ is a one-point space $*$. What is $F_S\{0,1\}$? To answer this question we reconsider the universality of $\eta_{\{0,1\}} : \{0,1\} \to F_S\{0,1\}$

$$\begin{array}{ccc} \{0,1\} \xrightarrow{\eta_{\{0,1\}}} U_S F_S\{0,1\} & & F_S\{0,1\} \\ \quad\searrow^{f} \quad\quad\downarrow{\scriptstyle U_S h} & & \quad\downarrow{\scriptstyle \exists! h} \\ \quad\quad U_S A, & & \quad A. \end{array}$$

This states that for each affine space A and two distinct points $P := f(0)$, $Q := f(1)$ in A there is a unique affine map $h : F\{0,1\} \to A$ such that $h(\eta_{\{0,1\}}(0)) = P$ and $h(\eta_{\{0,1\}}(1)) = Q$. In terms of affine geometry the object defined uniquely by two distinct points is the affine line. Indeed, $F_S\{0,1\}$ has to be a one-dimensional affine space if the universal property is to hold: firstly, by proposition 3.2.18 the injectivity of $\eta_{\{0,1\}}$ shows that $F_S\{0,1\}$ has two distinct points $O := \eta_{\{0,1\}}(0)$ and $I := \eta_{\{0,1\}}(1)$, so $\dim F_S\{0,1\} \geq 1$. Secondly, if there would be a vector $v \in V_{F_S\{0,1\}}$ linearly independent from $\overrightarrow{OI}$, then the uniqueness of h in the universal property wouldn't be given just from the data of two points in A. This shows $\dim F_S\{0,1\} \leq 1$.

One can generalize this argument for a set X of $n \geq 1$ elements and show FX to be an $n-1$-dimensional affine space. We can thus view $F\{0\}$ as the abstract point, $F\{0,1\}$ as the abstract affine line, $F\{0,1,2\}$ as the abstract affine plane, etc. and affine maps $h : F\{0,\ldots,n\} \to A$ as conrete (but possibly degenerate) realizations of these abstract geometric objects in a concrete affine space A. This extends the idea of the categorical concept of point as a morphism from the terminal object to higherdimensional geometric objects of affine geometry.

Like the terminal object each of these abstract higherdimensional objects is determined up to isomorphism. However, with the exception of the affine line and contrary to the terminal object the model spaces of the higherdimensional objects are not determined by intrinsic mapping properties in $\mathrm{Aff}_{\mathbb{K}}$. This gives the point and the affine line sort of a special status from the structural point of view. This special status is supported by geometry, e.g. if one thinks of the geometric idea behind affine coordinates and its use in analytic geometry or structurally even more involved the geometric construction of the Grassmann algebra (cf. appendix A and [Gra78]).

Remarks 3.3.1.

(a) One can even extend the argument for $\{0,1\}$ to an arbitrary (small) set X by using the Cantor-Bernstein-Schröder theorem, i.e. that the cardinality relation

is antisymmetric. This yields $\dim F_S X = \operatorname{card} X^*$ for $X^* = X \setminus \{P\}$ for a $P \in X$ in the general case, if $X \neq \emptyset$.

(b) Let A be a non-empty affine space. Fix an $O \in A$ and consider the map $\kappa_O : A \to V_A$ from remark 3.1.1(d). Until now we have considered it only as a bijection, but κ_O can be regarded as an affine map $A \to SU_V A$ with associated linear map the identity $1_{V_A} : V_A \to V_A$ and hence as an isomorpism in $\mathrm{Aff_K}$, too. This follows directly from its definition and the simple transitiveness of the action, for we have $P + v = O + \overrightarrow{OP} + v$:

$$\kappa_O(P + v) = \overrightarrow{O(P + v)} = \overrightarrow{OP} + v = \kappa_O(P) + v.$$

(c) From the free vector space functor $F : \mathrm{Set} \to \mathrm{Vct_K}$ being isomorphism-dense we obtain F_S to be isomorphism-dense, too. Let A be an affine space from $\mathrm{Aff_K}$ and V_A its associated vector space. W.l.o.g. assume A not to be an empty space, otherwise A would be isomorphic to $F_S\emptyset$ and we would be done. F is isomorphism-dense, so there is a small set X^*, such that $FX^* \cong V_A$. Consider the set $X := X^* + *$, where the plus denotes the coproduct in Set, i.e. the disjoint union. By $\dim F_S X = \operatorname{card} X^* = \dim FX^*$ from remark (a) we have $U_V F_S X = V_{F_S X} \cong FX^*$ in $\mathrm{Vct_K}$. Thus $SU_V F_S X \cong SFX^* \cong SU_V A$ holds. Finally, with this isomorphism and the one from the preceding remark we get $F_S X \cong U_V F_S X \cong SU_V A \cong A$.

Remark 3.3.2 (finite dimensions)**.**
F_S can be restricted to $F_S : \mathrm{FSet} \to \mathrm{FAff_K}$, where $\mathrm{FSet} \hookrightarrow \mathrm{Set}$ is the full subcategory of finite sets. However, U_S and hence the adjunction can't be restricted in this way, because an n-dimensional affine space with $n \geq 1$ has a set of points with a greater or equal cardinality than $\mathbb{N}$ [10].

In Set we have $\{0, 1\} \cong * + *$. Moreover, we have $F_S * = *$, where the latter denotes the one-point space. From the point of view on copowers as coproducts together with the cocontinuity of F_S we know that $F_S(* + *) = * + *$, i.e. the coproduct of one-point spaces in $\mathrm{Aff_K}$ exists and is an affine line which contains the two one-point spaces as distinct points $i_1, i_2 : * \to * + *$. This suggests a way to construct the coproducts for two affine spaces A_1, A_2: "connect" A_1 with A_2 by an affine line like depicted in figure 3.3 for the special cases of points and lines, and for the universal property "interpolate" between affine maps $f_1 : A_1 \to C$, $f_2 : A_2 \to C$ along this connecting line.

[10] This depends on the cardinality of the field K. Recall that we have implicitly assumed a characterisitc zero, so $\operatorname{card} K \geq \operatorname{card} \mathbb{N}$. However, for finite fields, finite-dimensional affine spaces have a finite set of points. In that case a restriction of U_S would be possible.

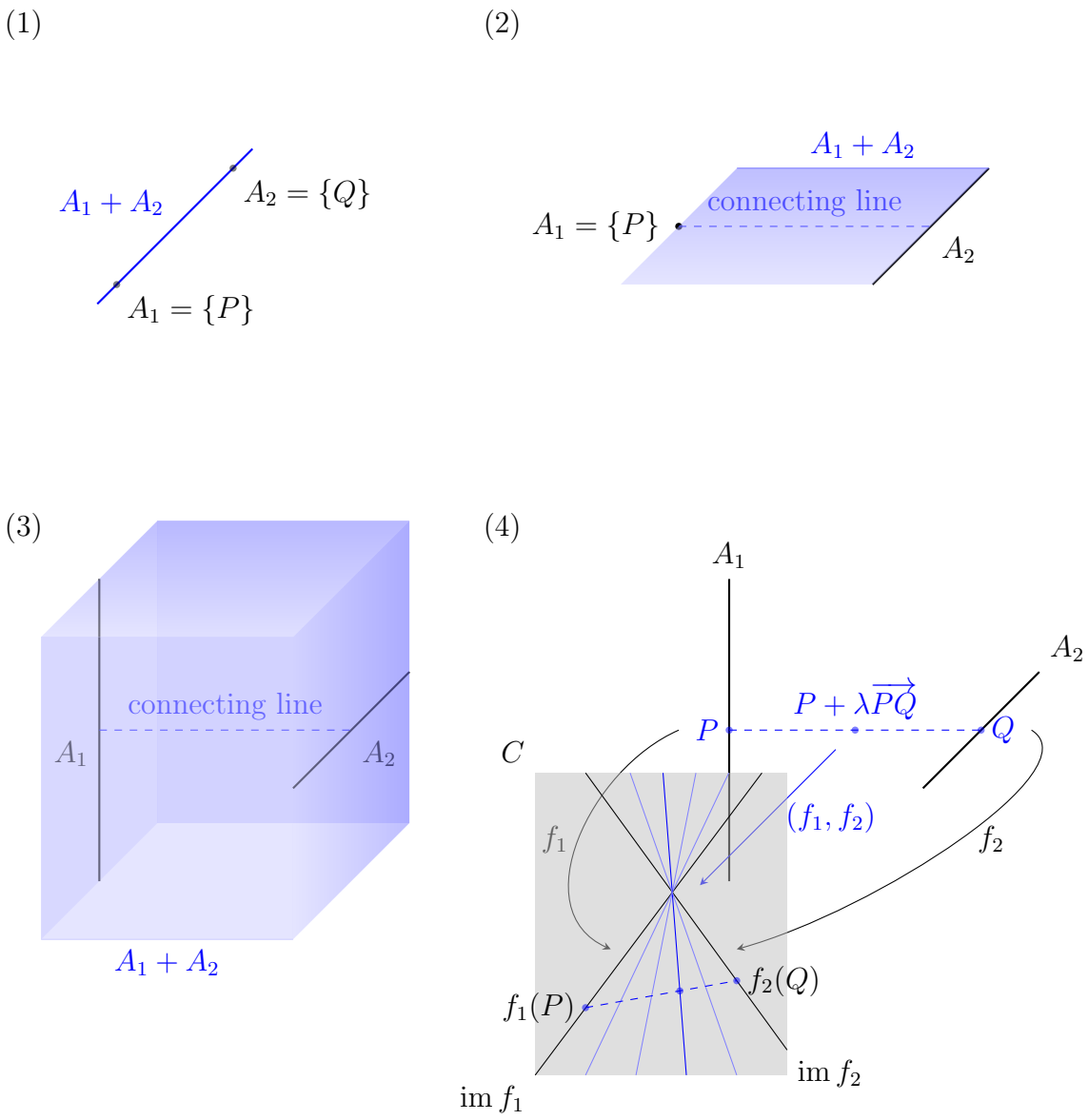


Figure 3.3: (1) Coproduct of two one-point spaces. (2) Coproduct of a one-point space and an affine line. (3) Coproduct of two affine lines. (4) The universal property of the coproduct of two affine lines geometrically. The sum $(f_1, f_2) : A_1 + A_2 \to C$ of $f_1 : A_1 \to C$ and $f_2 : A_2 \to C$ is obtained by interpolation along connecting lines. The blue lines in C indicate the images of (f_1, f_2) for a constant value of λ and varying points from A_1 (or A_2). Note that we have restricted ourselves only to some particular values of λ in the picture. For $K = \mathbb{R}$ all these satisfy $0 < \lambda < 1$. However, for negative values of λ or values greater than one the images would be located left from im f_1 or right from im f_2, respectively. Note as well that except for (1) the connecting lines are not determined by the coproduct alone, but involve a particular but arbitrary choice of points in A_1 and A_2.

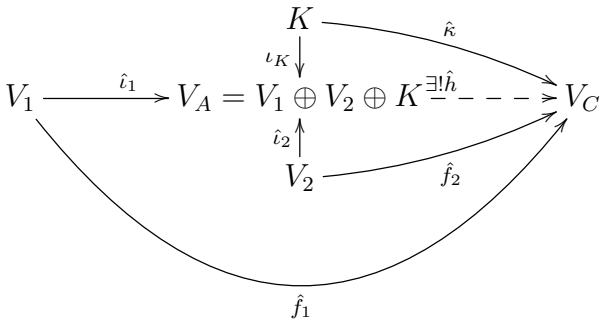
Lemma 3.3.3. Aff_K *has small coproducts.*

Proof. (i) We will first construct binary coproducts by exploiting the idea given above. Let A_1 and A_2 be objects from Aff_K with associated vector spaces V_1 and V_2. W.l.o.g. both spaces are not empty; otherwise the coproduct $A_1 + A_2$ is either the empty space when both A_i are empty, or the A_i which is not empty due to the empty space being initial (cf. dual of proposition III.5.1 in [Mac98]). Consider the vector space $V := V_1 \oplus V_2 \oplus K$, where we see the field K as a vector space over itself. By proposition 2.2.1, this coproduct exists. Set $A := SV$. Fix $O_i \in A_i$ and define

$$\iota_1 : A_1 \to A, \quad P \mapsto (\overrightarrow{O_1P}, 0, 0), \qquad \iota_2 : A_2 \to A, \quad Q \mapsto (0, \overrightarrow{O_2Q}, 1).$$

Note in particular the identities $\mathrm{pr}_1 \iota_1 = \kappa_{O_1}$ and $\mathrm{pr}_2 \iota_2 = \kappa_{O_2}$, where the κ_{O_i} are the induced bijections from the difference map (cf. remark 3.1.1(d)). By remark 3.3.1(b) we know that they are affine maps with associated linear maps the identity. With this it is easy to see that the ι_i are affine as well as that the the $\hat{\iota}_i$ are the canonical inclusions $V_i \hookrightarrow V$.

We have to show the universal property of the coproduct for A. Let C be a further affine space in Aff_K and $f_1 : A_1 \to C$, $f_2 : A_2 \to C$ affine maps. Define a linear map $\hat{\kappa} : K \to V_C$ by $\hat{\kappa}(1) = \overrightarrow{f_1(O_1)f_2(O_2)}$. By the universal property of the coproduct $V_1 \oplus V_2 \oplus K = V_A$ in Vct_K, there is a unique linear map $\hat{h} : V_A \to V_C$ such that $\hat{h}\hat{\iota}_i = \hat{f}_i$, $i \in \{1, 2\}$ and $\hat{h}\iota_K = \hat{\kappa}$, where $\iota_K : K \hookrightarrow V_A$ is the canonical inclusion



With this and $h(0) = f_1(O_1)$ we obtain by proposition 3.1.3(ii) a unique affine map $h : A \to C$ such that its associated linear map is our $\hat{h}$

$$h : A \to C, \quad (v, w, \lambda) \mapsto f_1(O_1) + \hat{h}(v, w, \lambda).$$

In particular, we have

$$\begin{aligned} h\iota_1(O_1) = f_1(O_1), \qquad h\iota_2(O_2) &= f_1(O_1) + \hat{h}(0, 0, 1) \\ &= f_1(O_1) + \overrightarrow{f_1(O_1)f_2(O_2)} \\ &= f_2(O_2), \end{aligned}$$

and

$$\begin{aligned} h\iota_1(P) &= h\iota_1(O_1 + \overrightarrow{O_1P}) = f_1(O_1) + \hat{h}\hat{\iota}_1(\overrightarrow{O_1P}) \\ &= f_1(O_1) + \hat{f}_1(\overrightarrow{O_1P}) \\ &= f_1(P), && P \in A_1, \\ h\iota_2(Q) &= h\iota_2(O_2 + \overrightarrow{O_2Q}) = f_2(O_2) + \hat{h}\hat{\iota}_2(\overrightarrow{O_2Q}) \\ &= f_2(O_2) + \hat{f}_2(\overrightarrow{O_2Q}) \\ &= f_2(Q), && Q \in A_2. \end{aligned}$$

This shows $h\iota_1 = f_1$ and $h\iota_2 = f_2$. For the uniqueness of h as an affine map with these properties ,it suffices to show that any affine map $g : A \to C$ with $g\iota_1 = f_1$, $g\iota_2 = f_2$ has $\hat{h}$ as its associated linear map. This is due to $g(0) = g\iota_1(O_1) = f_1(O_1) = h(0)$ (cf. proposition 3.1.3(ii) or remark 3.1.1(h)). Because of the universal property of V_A we just need to show $\hat{g}\iota_K(1) = \hat{g}(0,0,1) = \overrightarrow{f_1(O_1)f_2(O_2)}$, for the equality $\hat{g}\hat{\iota}_i = \hat{f}_i$ follows from $g\iota_i = f_i$, $i \in \{1,2\}$, and $\hat{h} : V_A \to V_B$ is by this universal property of V_A the unique linear map with these three properties. This assertion, however, follows by the simple transitiveness of the action from the following identity

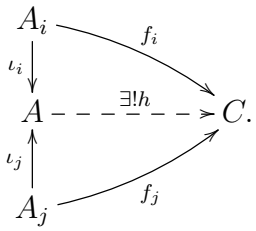
$$f_2(O_2) = g\iota_2(Q) = g(0,0,1) = g(0,0,0) + \hat{g}(0,0,1) = f_1(O_1) + \hat{g}(0,0,1).$$

(ii) Now, let I be a small set and $(A_i)_{i\in I}$ an indexed family of affine spaces A_i from $\mathrm{Aff_K}$ with associated vector spaces V_i. By proposition 2.2.1 $\mathrm{Vct_K}$ has small coproducts. Because of this we can extend the construction from above to this case in a straightforward manner. W.l.o.g. $I \neq \emptyset$, otherwise the coproduct has to be an initial element, which we know to be an empty space in $\mathrm{Aff_K}$. We can further assume that for no $i \in I$, A_i is an empty space; otherwise it would act as an identity for the coproduct of the family indexed by $I \setminus \{i\}$ and we could consider this reduced family instead. Fix $i_0 \in I$, set $I^* := I \setminus \{i_0\}$, $V := \bigoplus_{i\in I} A_i \oplus (I^* \cdot K)$ and $A := SV$. For each $i \in I$ fix a point $O_i \in A_i$ and define the inclusion maps $\iota_j : A_j \to A$ componentwise[11] by $\mathrm{pr}_j\, \iota_j = \kappa_{O_j}$, $\mathrm{pr}_i\, \iota_j = 0$, for $i \neq j$, $i, j \in I$, $\mathrm{pr}_{I^*\cdot K}\, \iota_j(P) = (\delta_{ji})_{i\in I^*}$, where 0 denotes the respective zero map and δ_{ij} the Kronecker delta. These ι_j are again easily seen to be affine with associated linear maps the canonical inclusions $V_i \hookrightarrow V_A$.

[11] Here we use the explicit set-theoretic construction of a direct sum of vector spaces as a vector subspace of the direct product with almost all components being zero (cf. section 2.2). This is why as set-theoretic maps the ι_j are uniquely determined by giving the components, i.e. the compositions of ι_j with the respective projections, due to the universal property of products in Set. As one can see, these components are almost all the zero maps, so ι_j indeed corestricts to the direct sum.

Together with A they yield the universal property of the coproduct for a given I-indexed family of affine maps $f_i : A_i \to B$

$\forall i, j \in I$



This can be seen in the same way like in (i). First, use the universal property of V_A in Vct_K for $\hat{f}_i$, $i \in I$ and $\hat{\kappa}_i : K \to V_C$, $1 \mapsto \overrightarrow{f_{i_0}(O_{i_0})f_i(O_i)}$, $i \in I^*$ to obtain a respective linear map $\hat{h} : V_A \to V_C$. This linear map and $h(0) = f_{i_0}(O_{i_0})$ yield by proposition 3.1.3(ii) a respective affine map $h : A \to C$ for which $h\iota_i = f_i$, $i \in I$ can be seen to hold like above. For the uniqueness of h one can repeat the argument from (i), too, because for an affine map $g : A \to C$ with $g\iota_i = f_i$, $i \in I$ and the canonical ith inclusion $\iota_{K,i} : K \hookrightarrow V_A$, $i \in I^*$ one gets

$$f_i(O_i) = g((\delta_{ij})_{j\in I}) = g(0) + \hat{g}((\delta_{ij})_{j\in I}) = f_{i_0}(O_{i_0}) + \hat{g}\iota_{K,i}(1), \qquad i \in I^*.$$

(qed)

The explicit construction of the coproduct in the proof of lemma 3.3.3 shows that neither U_S nor U_V preserve coproducts. In particular, they cannot be cocontinuous. Nonetheless, for both functors we have natural inclusions

$$U_S A_1 + U_S A_2 \hookrightarrow U_S(A_1 + A_2), \qquad U_V A_1 + U_V A_2 \hookrightarrow U_V(A_1 + A_2).$$

Because of this we cannot say that the coproducts in Aff_K look more like the ones in Set. Instead, the coproducts in Aff_K are a fusion of coproducts in Vct_K and Set; this underpins our viewpoint of affine spaces to be structures between Vct_K and Set.

It may seem odd to construct the coproducts via vector spaces and arbitrary choices of points. There is another way, which is more intrinsic from the viewpoint of affine geometry. To fully apprehend this, one needs to introduce a further equivalent description of the structure of an affine space, namely the one by the notion of affine combinations. We postpone this until chapter 4 (cf. remark 4.2.2(i)).

Remarks 3.3.4 (finite dimensions)**.**

(a) The explicit construction of the coproduct $A_1 + A_2$ gives us a dimension formula in the case when A_1 and A_2 are finite-dimensional and at least one of them isn't the empty space, namely

$$\dim(A_1 + A_2) = \dim A_1 + \dim A_2 + 1. \tag{3.6}$$

In particular, this shows that $\mathrm{FAff}_{\mathrm{K}}$ has finite coproducts. Comparing this formula to the one for coproducts of vector spaces we have a difference of 1. In geometric terms one can see this as an account for the degree of freedom of choosing basis points. Contrary to the case of vector spaces, where the morphisms are bound to respect the one and only point which each vector space has, i.e. the zero vector, affine maps are not. This is because there is in general more than one point in an affine space. We will see more instances of this extra degree of freedom to account for points in subsequent dimension formulas, too.

(b) From the construction of coproducts we can read off that the inclusion functor $\mathrm{FAff}_{\mathrm{K}} \hookrightarrow \mathrm{Aff}_{\mathrm{K}}$ creates and hence preserves finite coproducts. The existence and uniqueness assertions for the creation property follow like in remark 3.2.5(c) from the inclusion functor being full, faithful and injective on objects. The preservation of finite coproducts is a consequence of the dual of theorem V.4.2 in [Mac98].

With the dimension formula we can conclude the non-distributivity of $\mathrm{Aff}_{\mathrm{K}}$ immediatly. Recall that a category C with finite products and coproducts is said to be *distributive* when the natural maps $i \to c \times i$ for $i \in \operatorname{Obj} C$ an initial object as well as $c \times e + d \times e \to (c + d) \times e$, which is given by the respective universal properties of product and coproduct applied to the respective combinations of inclusions and projections, are isomorphisms for all $c, d, e \in \operatorname{Obj} C$.

Theorem 3.3.5. $\mathrm{FAff}_{\mathrm{K}}$ *and* $\mathrm{Aff}_{\mathrm{K}}$ *are not distributive categories.*

Proof. Let A, C, D be objects in $\mathrm{FAff}_{\mathrm{K}}$, none of them being an empty space. By the dimension formula for coproducts and products (cf. remark 3.3.4(a), remark 3.2.5(a)) we have

$$\begin{aligned} \dim(A + C) \times D &= \dim A + \dim C + 1 + \dim D, \\ \dim(A \times D + C \times D) &= \dim A + \dim D + 1 + \dim C + \dim D + 1, \end{aligned}$$

hence $\dim(A + C) \times D \neq \dim(A \times D + C \times D)$. This shows that there is no isomorphism $(A + C) \times D \cong A \times D + C \times D$ in $\mathrm{Aff}_{\mathrm{K}}$; if there would be one, then applying U_V would yield the associated vector spaces to be isomorphic, which contradicts the different dimensions of these spaces. This shows the non-distributivity of $\mathrm{FAff}_{\mathrm{K}}$. By remark 3.3.4(b) and remark 3.2.5(c) the inclusion functor $\mathrm{FAff}_{\mathrm{K}} \hookrightarrow \mathrm{Aff}_{\mathrm{K}}$ preserves products and coproducts, so $\mathrm{Aff}_{\mathrm{K}}$ can't be distributive either.

(qed)

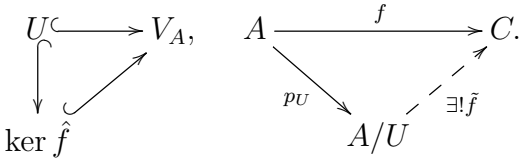
Remarks 3.3.6.

(a) The non-distributivity in particular implies that Aff_K and FAff_K can't be Cartesian closed or a topos like Set[12]. We will address the topic of closedness once again in section 3.4.

(b) Even though the distributivity law does not hold, we still have $\emptyset \times A \cong \emptyset$, i.e. that an empty space $\emptyset$ acts like a zero for the product in Aff_K and FAff_K. This follows, for example, from U_S preserving limits as well as it preserving and reflecting the initial objects, which is a direct consequence of their characterization in theorem 3.2.12 and our definition of Aff_K. This is because the empty set acts like a zero for the Cartesian product in Set. Furthermore, the inclusion $\mathrm{FAff}_\mathrm{K} \hookrightarrow \mathrm{Aff}_\mathrm{K}$ preserves finite limits and preserves and reflects the initial objects, too, so this property of $\emptyset$ in Aff_K carries over to FAff_K. The assertion for the empty set in Set is due to the distributivity of Set; the distributivity itself is an immediate consequence of the Cartesian closedness, since for arbitrary small sets X, Y we have $\mathrm{Set}(\emptyset \times X, Y) \cong \mathrm{Set}(\emptyset, \mathrm{Set}(X, Y))$ and the set $\mathrm{Set}(\emptyset, \mathrm{Set}(X, Y))$ is the one-point set because $\emptyset$ is initial (cf. the discussion of Cartesian closedness in section 2.2.2 and the dual of theorem V.5.1 in [Mac98]).

With the existence of small coproducts, to prove the existence of arbitrary small colimits we only need to show the existence of coequalizers of pairs of maps (cf. the dual of theorem V.2.1 in [Mac98]). Like in Vct_K we will construct coequalizers from a fundamental theorem on homomorphisms.

Theorem 3.3.7 (fundamental theorem on homomorphisms)**.** *Let A be an affine space in Aff_K with associated vector space V_A and $U \leq V_A$ a vector subspace. There is an affine space A/U with associated vector space the quotient space V_A/U and an affine map $p_U : A \to A/U$ in Aff_K such that for each affine map $f : A \to C$ in Aff_K with $U \leq \ker \hat{f}$ there is a unique affine $\tilde{f} : A/U \to C$ such that $f = \tilde{f} p_U$*



(3.7)

Proof. (i) We can assume w.l.o.g. A to be non-empty, otherwise the only vector subspace is the trivial vector space V_A itself and the statement becomes trivial. Set

$$A/U := \{P + U \mid P \in A\},$$

[12] One can also show that neither Aff_K nor FAff_K have a subobject classifier. The proof is similar to the proof of the impossibility of subobject classifiers for Vct_K or Ab, the category of abelian groups with group homomorphisms as morphisms.

where $P+U$ denotes the image of $\{P\} \times U$ under the action of V_A on A. We have

$$P + U = Q + U \quad \Longleftrightarrow \quad \overrightarrow{PQ} \in U, \tag{$*$}$$

for $P + w = Q + v$ holds if and only if $P + (w - v) = Q$ holds, which by the action being simply transitive holds if and only if we have $w - v = \overrightarrow{PQ}$. Hence, the following map is well-defined

$$A/U \times V_A/U \to A/U, \quad (P + U, v + U) \mapsto (P + v) + U.$$

Here $v+U$ denotes the equivalence class of v in V_A/U. It is easily seen to be a simply transitive action because of the respective properties of the action map of the action of V_A on A and ($*$). For example, the simple transitiveness can be seen as follows: given $P+U$ and $Q+U$, $\overrightarrow{PQ}+U$ is the unique vector in V_A/U such that $(P + U) + (\overrightarrow{PQ} + U) = Q + U$ holds, because for each $v \in V_A$ with $(P+v)+U = Q+U$, $\overrightarrow{(P+v)Q}$ lies in U by ($*$) and from $\overrightarrow{(P+v)Q} = \overrightarrow{PQ} - v$ we obtain $\overrightarrow{PQ} + U = v + U$.

(ii) Let $\hat{p}_U : V_A \to V_A/U$, $v \mapsto v + U$ be the canonical quotient map in $\mathrm{Vct}_{\mathrm{K}}$. Define

$$p_U : A \to A/U, \quad P \mapsto P + U.$$

With the definition of our action on A/U this is seen to yield an affine map with associated linear map the quotient map $\hat{p}_U$ by

$$p_U(P + v) = (P + v) + U = (P + U) + (v + U) = p_U(P) + \hat{p}_U(v).$$

In particular, p_U is surjective and hence epi in $\mathrm{Aff}_{\mathrm{K}}$ by theorem 3.2.10.

(iii) Finally, we show the asserted universal property. Let $f : A \to C$ be an affine map. Because p_U is epi, we only need to show the existence of the asserted $\tilde{f}$. We define it to be the obvious map

$$\tilde{f} : A/U \to C, \quad P + U \mapsto f(P).$$

The assumption $U \leq \ker \hat{f}$ yields $f(P + v) = f(P) + \hat{f}(v) = f(P)$ for $v \in U$, so $\tilde{f}$ is well-defined. Moreover, it is an affine map with associated linear map $\hat{\tilde{f}} : V_A/U \to V_C$, which is the unique linear map such that $\hat{f} = \hat{\tilde{f}}\hat{p}_U$ and is given by the fundamental theorem on homomorphisms in $\mathrm{Vct}_{\mathrm{K}}$ (cf. (2.1) in section 2.2). That the map $\hat{\tilde{f}}$ is the associated linear map can be seen as follows

$$\tilde{f}((P + U) + (v + U)) = f(P + v) = f(P) + \hat{f}(v) = \tilde{f}(P + U) + \hat{\tilde{f}}(v + U).$$

(qed)

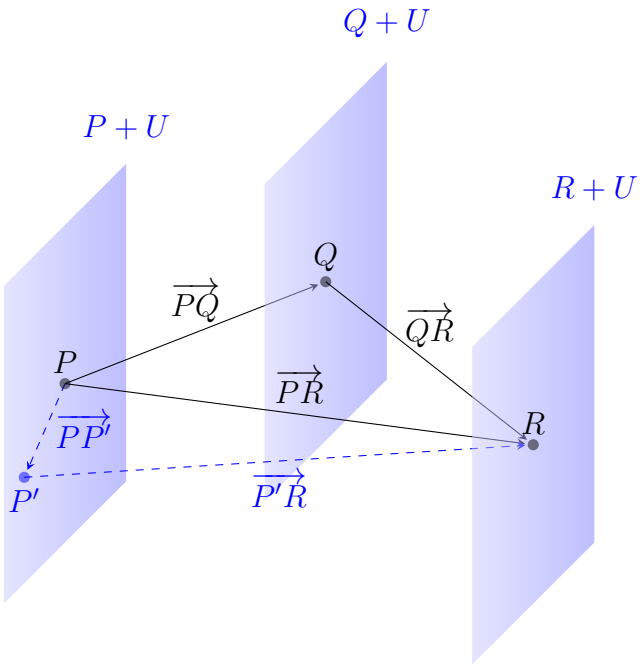


Figure 3.4: The quotient A/U geometrically. The points of A/U are parallel disjoint affine subspaces of A with U as the vector space of displacements. They are indicated by the blue shaded rectangles. Moving transversally to the subspaces defines an action of V_A. As it is indicated by $\overrightarrow{P'R}$ there are several different ways to move from one subspace to another transversally. For a simply transitive action we have to identify those, which amounts to passing to the action of the quotient V_A/U.

In geometric terms by passing from A to A/U we collaps affine subspaces $P+U$ to points. Thus the points in A/U are parallel disjoint affine subspaces which extend in the directions of U as it is depicted in figure 3.3. The action of V_A/U comes from the fact that we can move between different parallel subspaces if we move transversally to them. However, there are different possibilities to get from one subspace to another. This is covered by passing from V_A to V_A/U.

We can prove the existence of coequalizers of pairs and hence the existence of small colimits now.

Theorem 3.3.8. Aff_K *is cocomplete, i.e., has all small colimits.*

Proof. We show the existence of coequalizers of pairs. Let $f, g : A \to C$ be two affine maps in Aff_K. Consider the vector subspace

$$U := \mathrm{span}\{\overrightarrow{f(Q)g(Q)} \mid Q \in A\} \leq V_C$$

and the quotient map $p_U : C \to C/U$ from theorem 3.3.7. The latter is the candidate for the desired coequalizer. First of all, by remark 3.1.1(f) and the construction of U we have

$$\overrightarrow{p_U f(Q) p_U g(Q)} = \hat{p}_U(\overrightarrow{f(Q)g(Q)}) = 0, \qquad Q \in A.$$

This yields $p_U f = p_U g$. For the universal property, let $h : C \to D$ be a further affine map in Aff_K with $hf = hg$. Once again by remark 3.1.1(f), this yields $\hat{h}(\overrightarrow{f(Q)g(Q)}) = 0$ for all $Q \in A$. In particular, this shows $U \leq \ker \hat{h}$. The fundamental theorem on homomorphisms thus yields a unique $\tilde{h}$ in Aff_K such that $h = \tilde{h} p_U$.

This concludes the existence of coequalizers of pairs in Aff_K and with lemma 3.3.3 and the dual of theorem V.2.1 in [Mac98] we get the existence of all small colimits.

(qed)

Remarks 3.3.9 (finite dimensions)**.**

(a) Let A be an affine space in FAff_K. As is well-known from linear algebra each vector subspace $U \leq V_A$ is of finite dimension in this case. Because it resorts to a quotient construction, we have the following dimension formula for the fundamental theorem on homomorphismsm in FAff_K (cf. Satz 9.2.21 in [KM03])

$$\dim A/U = \dim A - \dim U. \tag{3.8}$$

(b) $\mathrm{FAff}_K \hookrightarrow \mathrm{Aff}_K$ creates finite colimits and preserves them. This follows from remark 3.3.4(b), the explicit construction of coequalizers of pairs in the proof of theorem 3.3.8 and the preceding remark, because a finite colimit can be obtained as a coequalizer of pairs from a finite coproduct. The existence and uniqueness assertions for the creation of finite colimits are a consequence of the inclusion functor being full, faithful and injective on objects (cf. remark 3.2.5(c) for this type of argument). The preservation follows from the dual of lemma 3.2.4 or the dual of theorem V.4.2 in [Mac98], for Aff_K has all finite colimits by theorem 3.3.8.

(c) FAff_K is finitary cocomplete, i.e. has all finite colimits by the preceding remark and theorem 3.3.8.

From a geometric viewpoint the construction of the coequalizer of two affine maps f, g is obtained by dividing out the smallest vector subspace U generated by vectors connecting the parametrized subspaces $\operatorname{im} f, \operatorname{im} g$. This means for each affine subspace that has a vector subspace of U as its vector space of displacements: if it intersects $\operatorname{im} f$ in $f(Q)$, then it intersects $\operatorname{im} g$ in $g(Q)$. This vector subspace U does not have to be the one associated to the affine subspace that is generated by the images of the maps (cf. figure 3.3).

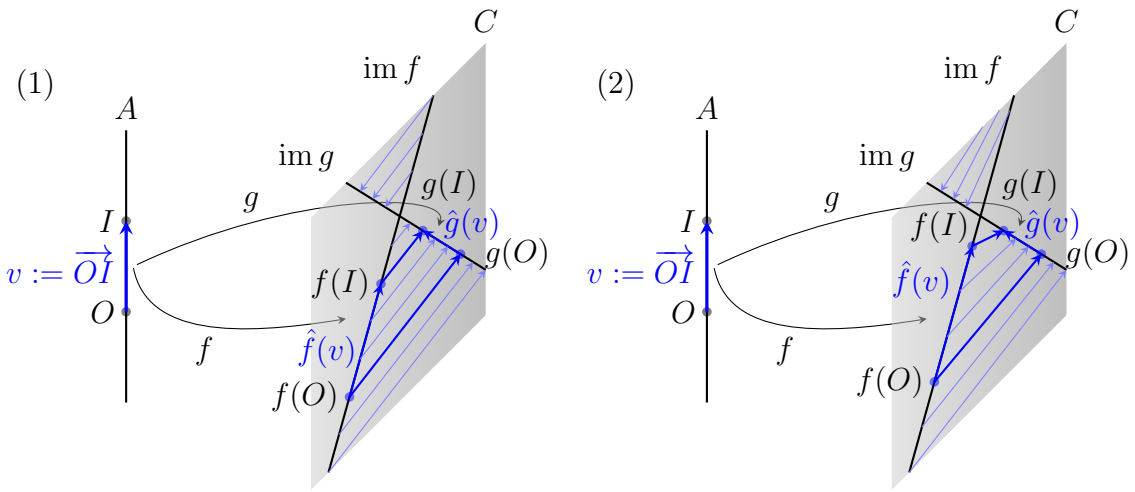


Figure 3.5: Coequalizer of f and g geometrically. (1) $\overrightarrow{f(O)g(O)}$ and $\overrightarrow{f(I)g(I)}$ are parallel. The coequalizer of f and g is thus the one-dimensional affine space of affine lines $C/K\overrightarrow{f(O)g(O)}$. The vector space $K\overrightarrow{f(O)g(O)}$ does not coincide with the associated vector space of the affine space generated by $\operatorname{im} f$ and $\operatorname{im} g$, which is V_C. (2) $\overrightarrow{f(O)g(O)}$ and $\overrightarrow{f(I)g(I)}$ are not parallel. The coequalizer of f and g is thus the one point space C/U with $U := \operatorname{span}\{\overrightarrow{f(O)g(O)}, \overrightarrow{f(I)g(I)}\}$. In this case we have $U = V_C$.

Now that we have the existence of small colimits and a separator, we have the dual of the special adjoint functor theorem at hand (cf. dual of theorem V.8.2 and in particular the dual of its corollary in [Mac98]). With this we obtain the dual version of theorem 3.2.13, which is our criterion for the existence of left adjoints for functors $F : \mathrm{Aff}_K \to C$ for a category C with small hom-sets.

Theorem 3.3.10. *Let C be a category with small hom-sets and $F : \mathrm{Aff}_K \to C$ a functor. F has a right adjoint if and only if it is cocontinuous, i.e. if it preserves small colimits.*

Proof. To be able to conclude the assertion from the dual of the corollary of the theorem V.8.2 in [Mac98] the only minor thing we need to show is that Aff_K is *co-well-powered*. This is a dual notion of well-powerdness. It means that for each object A in Aff_K the set $\mathrm{Quot}(A)$ of isomorphism classes of quotient objects is small. A *quotient object* of A is the dual notion of a subobject of A, i.e. an epi $A \to C$, and two quotient objects $f : A \to C$, $g : A \to D$ are said to be equivalent when there is an isomorphism $h : C \to D$ such that $hf = g$. The idea behind this is that epis can be seen as quotient maps to quotient spaces of A. In Aff_K this is indeed the case due to the fundamental theorem on homomorphisms and the fact that epis

in Aff_K are precisely the surjective affine maps. The fact that $\mathrm{Quot}(A)$ is a small set actually follows from that.

In more detail assume w.l.o.g. that A is not an empty space, because otherwise $\mathrm{Quot}(A)$ is the one-point set, which is small. Given a quotient object $f : A \to C$ the fundamental theorem on homomorphisms yields that f is isomorphic to $p_{\ker \hat{f}}$ as a quotient object with the isomorphism given by $\tilde{f}$. $\tilde{f}$ is injective, because we divide out $\ker \hat{f}$ and it is surjective, because $p_{\ker \hat{f}}$ and f are. Hence it is bijective and thus an isomorphism by corollary 3.1.4. As a consequence the isomorphism classes of quotient objects correspond one-to-one to those subspaces of V_A which are kernels of linear maps. But due to the fundamental theorem on homomorphisms (2.1) these are all subspaces of V_A in Vct_K. V_A is a small set by assumption. Thus so is the set of subspaces of V_A, which can be seen by the same type of argument presented in Vct_K for the well-poweredness of Vct_K or directly from the well-poweredness of Vct_K.

Alternatively, one can obtain the smallness of $\mathrm{Quot}(A)$ by the type of argument presented for the well-poweredness of Aff_K in theorem 3.2.12 due to U_S preserving epis, which we have already shown in the proof of proposition 3.2.8 and which is a restatement of the part of theorem 3.2.10 concerning epis.

(qed)

We proved the cocompleteness of Aff_K from coproducts and coequalizers like we did for Vct_K. We did it in this way to be able to interpret the constructions geometrically. However, from the structural point of view there is a more interesting way to obtain this result. One can employ a duality induced by the internal hom together with the affine line K as a "dualizing" element. We will demonstrate this in section 3.5. The idea behind this alternative proof comes from the proof of the existence of finite colimits for topoi as presented in [MM92]. From a geometric point of view and with the point-hyperplane duality of projective geometry in mind it is interesting to discuss the geometric meaning of this duality. This will be done in section 3.5 in fair detail, too. However, we will see that the duality is more of conceptual nature than geometric.

3.4 On Monoidal Structures on Aff_K

From the completeness and cocompleteness of Aff_K together with proposition III.5.1 in [Mac98] and its dual we obtain two symmetric monoidal structures on Aff_K, namely the one induced by a particular Cartesian product with a terminal one-point set as an identity and the one induced by a particular coproduct with an initial empty space as an identity. The respective natural isomorphisms for associativity, left- and right-unit laws and symmetric braiding are induced by the respective

universal properties[13]. For such (co)cartesian monoidal structures to exist only the existence of finite (co)products is required. Thus the finitary complete and finitary cocomplete FAff_K admits these structures as well. In particular, both structures can be chosen in such a way that the inclusion functor $\mathrm{FAff}_\mathrm{K} \hookrightarrow \mathrm{Aff}_\mathrm{K}$ is a strict symmetric monoidal functor[14]. Note that for each particular choice of (co)cartesian structure on Aff_K or on FAff_K there are respective structure morphisms such that the respective identity functor becomes strong symmetric monoidal. These morphisms exist once again due to the universality of product and coproduct.

We consider our functors U, U_S, S and U_V and fix a particular Cartesian and cocartesian structures on Vct_K, Aff_K and Set, where by the linearity of Vct_K Cartesian and cocartesian is the same. The continuity of U, U_S and S yields that they are all strong symmetric monoidal with respect to the Cartesian structures. Of course, we only need the preservation of finite products for this, so U_V is strong symmetric monoidal with respect to the Cartesian structures as well. Loosely speaking, the property of creating limits, which holds for U and U_S, shows that Vct_K, Aff_K and Set agree on the notion of limit and in particular on the notion of Cartesian product. The strong monoidality of U, U_S and U_V is thus something one would expect, because it expresses the far weaker property of the products in the three categories to be compatible and thus to be strongly preserved in the language of monoidal categories.

With regard to this point of view the more interesting structures are the cocartesian ones. We have already noted that the coproduct in Aff_K is a fusion of the coproduct in Vct_K and the one in Set. In the terminology of monoidal categories this can be made precise: U_S and U_V are symmetric monoidal with respect to the cocartesian structures, but not strong. The structure morphisms are the natural inclusions we have already mentioned in the previous section

$$U_S A_1 + U_S A_2 \hookrightarrow U_S(A_1 + A_2), \qquad U_V A_1 + U_V A_2 \hookrightarrow U_V(A_1 + A_2), \tag{3.9}$$

where the first $+$ denotes the coproduct in Set, the second and fourth $+$ the coproduct in Aff_K and the third $+$ the direct sum in Vct_K. For reference we collect all these results in the following proposition.

[13] Cf. section 2.2.3 for the respective definitions and [Mac98, section III.5] for details up to the symmetry.

[14] Recall a product and a coproduct to be defined only up to a natural isomorphism, but monoidal structures need explicit bifunctors for the tensor-products. To meet this requirement one has to choose explicit representants of the isomorphism classes. In principle, one could choose differently for Aff_K and Aff_K. Either way, the inclusion functor is a strong symmetric monoidal functor.

Proposition 3.4.1. *For any choice of Cartesian structures on* Vct_K*, on* Aff_K *and on* Set*, the forgetful functors* $U : \mathrm{Vct}_\mathrm{K} \to \mathrm{Set}$*,* $U_S : \mathrm{Aff}_\mathrm{K} \to \mathrm{Set}$*,* $U_V : \mathrm{Aff}_\mathrm{K} \to \mathrm{Vct}_\mathrm{K}$ *and the self-action functor* $S : \mathrm{Vct}_\mathrm{K} \to \mathrm{Aff}_\mathrm{K}$ *are strong symmetric monoidal. For any choice of cocartesian structures on the above categories* $U_S : \mathrm{Aff}_\mathrm{K} \to \mathrm{Set}$ *and* $U_V : \mathrm{Aff}_\mathrm{K} \to \mathrm{Vct}_\mathrm{K}$ *are symmetric monoidal.*

Proof. As we have already stated in section 2.2.3, we will only give the main ideas and omit the details of the proof, for this only amounts to showing the defining diagrams commutative, which is a tedious formal exercise not enlightening in any way.

To begin with, given two Cartesian or cocartesian structures on a category C one shows the identity functor to be strong monoidal, respectively, where the structure morphisms are the natural isomorphisms that result from the respective universal property. One then fixes the canonical product structures in Vct_K and Aff_K created by U_S and U from the one in Set, respectively. This shows U, U_S and U_V to be strong monoidal for this particular choice of Cartesian monoidal structure already. For the coproduct structure in Aff_K, as we have not given an intrinsic description yet, one needs to bear with the construction given in the proof of lemma 3.3.3 by employing S and the direct sum of vector spaces. Although one has to fix basepoints for non-empty affine spaces for such a coproduct bifunctor, this still has the advantage that one can give the inclusions in (3.9) explicitly. We obtain the monoidality of U_V and S for the particular cocartesian structures from this. Because compositions of (strong) monoidal functors yield (strong) monoidal functors, we can combine these particular results on (strong) monoidality with the previously obtained strong monoidality of the identity functor to yield the assertion for an arbitrary Cartesian or cocartesian structure.

(qed)

Remark 3.4.2 (finite dimensions)**.**

The above proposition still holds if we replace Vct_K by FVct_K and Aff_K by FAff_K, which can be seen for example from the strong monoidality of the inclusion functors as well as their property of reflecting products and coproducts in connection with the above result.

3.4.1 The Internal Hom

The next natural thing to ask in this context is whether a Cartesian or cocartesian structure is closed in Aff_K. As we know from the non-distributivity of Aff_K (cf. theorem 3.3.5) a Cartesian structure isn't closed. A cocartesian structure can't be closed, for the coproduct isn't distributive over itself. This is due to the coproduct being non-trivial, i.e. $A+A$ is not isomorphic to A in general; take for example for A

a one-point space. With this we just need to apply the hypothesized distributivity law to the identity object to get a contradiction:

$$A \cong A + (\emptyset + \emptyset) \cong (A + \emptyset) + (A + \emptyset) \cong A + A.$$

One can also see the non-distributivity of the coproduct over itself from finite dimensional counterexamples by our dimension formulas (3.5) and (3.6). The latter even yields that for a non-empty A in FAff_K the endofunctor $A + (-)$ doesn't preserve products and thus can't be continuous, so even to consider a dual notion to closedness does not work.

The failure of Cartesian closedness leads to the question if there are "mapping objects" in Aff_K at all, i.e. if there is an internal hom in the sense of Eilenberg-Kelly. In our particular (concrete) category Aff_K one can ask much more concretely whether the external hom-set $\mathrm{Aff}_K(A, C)$ can be equipped with a structure of an affine space in a natural way. The answer to this is affirmative and this internalization of hom-sets $\mathrm{Aff}_K(A, C)$ yields indeed an internal hom $\underline{\mathrm{Aff}}_K(-,-)$.

Theorem 3.4.3 (internal hom)**.** *Each hom-set* $\mathrm{Aff}_K(A, C)$ *has a* U_S*-lift* $\underline{\mathrm{Aff}}_K(A, C)$ *with associated vector space* $\mathrm{Aff}_K(A, SU_VC)$*, where the latter carries a* K*-vector space structure by pointwise operations and the action is defined in a pointwise manner as well. Moreover, this pointwise* U_S*-lift extends to an* U_S*-lift of the external hom-bifunctor* $\mathrm{Aff}_K(-,-) : {\mathrm{Aff}_K}^{op} \times \mathrm{Aff}_K \to \mathrm{Set}$ *and yields an internal hom* $\underline{\mathrm{Aff}}_K : {\mathrm{Aff}_K}^{op} \times \mathrm{Aff}_K \to \mathrm{Aff}_K$.

Proof. (i) To begin with, consider the set of set-theoretic maps $\mathrm{Set}(U_SA, UV_C)$. It is well-known that this set has a K-vector space structure by pointwise operations, for V_C is a vector space (cf. e.g. [AEI][4.12]). Recall the identities $U = U_SS$ and $U_VC = V_C$. This yields $\mathrm{Aff}_K(A, SU_VC) \hookrightarrow \mathrm{Set}(U_SA, UV_C)$. We just have to show that $\mathrm{Aff}_K(A, SU_VC)$ is closed under this vector space structure, i.e. for $f, g \in \mathrm{Aff}_K(A, SU_VC)$, $f + g$ and λf, $\lambda \in K$ are affine maps again. This can be verified on points

$$\begin{aligned}(f + \lambda g)(P + v) &= f(P + v) + \lambda g(P + v) = f(P) + \hat{f}(v) + \lambda g(P) + \lambda \hat{g}(v) \\ &= (f + \lambda g)(P) + (\hat{f} + \lambda \hat{g})(v), \qquad P \in A,\ v \in V_A,\end{aligned}$$

because $\hat{f} + \lambda\hat{g}$ is again a linear map by the pointwise K-vector space structure of $\mathrm{Vct}_K(V_A, V_C)$. Here we have assumed w.l.o.g. A to be non-empty; since otherwise $\mathrm{Set}(U_SA, U_VC)$ is a trivial vector space and the assertion becomes trivial.

(ii) Now we will show that a pointwise action of $\mathrm{Aff}_K(A, SU_VC)$ on $\mathrm{Aff}_K(A, C)$ can indeed be defined and that it is simply transitive. W.l.o.g. we assume once again A to be non-empty; since otherwise $\mathrm{Aff}_K(A, C)$ is a one-point set,

$\mathrm{Aff}_{\mathrm{K}}(A, SU_V C)$ is a trivial vector space and the assertion becomes trivial. Let $f \in \mathrm{Aff}_{\mathrm{K}}(A, C)$ and $h \in \mathrm{Aff}_{\mathrm{K}}(A, SU_V C)$. Due to the action of V_C on C we can define $f + h$ in a pointwise manner to be $P \mapsto f(P) + h(P)$. This map is also seen to be affine with associated linear map $\hat{f} + \hat{h} : V_A \to V_C$ by

$$\begin{aligned}(f+h)(P+v) &= f(P+v) + h(P+v) = f(P) + \hat{f}(v) + h(P) + \hat{h}(v) \\ &= (f+h)(P) + (\hat{f} + \hat{h})(v), \qquad P \in A,\ v \in V_A.\end{aligned}$$

Hence, the pointwise action

$$\mathrm{Aff}_{\mathrm{K}}(A, C) \times \mathrm{Aff}_{\mathrm{K}}(A, SU_V C) \to \mathrm{Aff}_{\mathrm{K}}(A, C), \quad (f, h) \mapsto f + h,$$

is well-defined. Due to the action of V_C on C being simply transitive, it is simply transitive as well. Hereto, let $f, g \in \mathrm{Aff}_{\mathrm{K}}(A, C)$ be given. Define $h : A \to SU_V C$ by $P \mapsto \overrightarrow{f(P)g(P)}$, then we have $f + h = g$ and h is the unique map with this property. The only thing we need to show is h to be affine. We do this again by evaluation on points

$$\begin{aligned}h(P+v) &= \overrightarrow{f(P+v)g(P+v)} = \overrightarrow{(f(P) + \hat{f}(v))(g(P) + \hat{g}(v))} \\ &= \overrightarrow{f(P)g(P)} + \hat{g}(v) - \hat{f}(v) = h(P) + (\hat{g} - \hat{f})(v), \quad P \in A,\ v \in V_A.\end{aligned}$$

We denote the so obtained affine space by $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, C)$.

(iii) At last, we have to show that for an affine map $f : A \to D$, the maps

$$\begin{aligned}f^* &:= \mathrm{Aff}_{\mathrm{K}}(f, C) : \mathrm{Aff}_{\mathrm{K}}(D, C) \to \mathrm{Aff}_{\mathrm{K}}(A, C) \qquad \text{and} \\ f_* &:= \mathrm{Aff}_{\mathrm{K}}(C, f) : \mathrm{Aff}_{\mathrm{K}}(C, A) \to \mathrm{Aff}_{\mathrm{K}}(C, D)\end{aligned}$$

are actually affine, i.e. there is a U_S-lift to affine maps

$$\begin{aligned}&\underline{\mathrm{Aff}}_{\mathrm{K}}(f, C) : \underline{\mathrm{Aff}}_{\mathrm{K}}(C, D) \to \underline{\mathrm{Aff}}_{\mathrm{K}}(C, A) \qquad \text{and} \\ &\underline{\mathrm{Aff}}_{\mathrm{K}}(C, f) : \underline{\mathrm{Aff}}_{\mathrm{K}}(C, A) \to \underline{\mathrm{Aff}}_{\mathrm{K}}(C, D),\end{aligned}$$

for an arbitrary C in $\mathrm{Aff}_{\mathrm{K}}$. This indeed shows the bifunctoriality of $\underline{\mathrm{Aff}}_{\mathrm{K}}(-,-)$ due to proposition II.3.1 in [Mac98]. Let $g \in \mathrm{Aff}_{\mathrm{K}}(D, C)$ and $h \in \mathrm{Aff}_{\mathrm{K}}(D, SU_V C)$. Recall the definition $f^* g = gf$, so we have $f^* h \in \mathrm{Aff}_{\mathrm{K}}(A, SU_V C)$ and

$$f^* = \mathrm{Aff}_{\mathrm{K}}(f, SU_V C) : \mathrm{Aff}_{\mathrm{K}}(D, SU_V C) \to \mathrm{Aff}_{\mathrm{K}}(A, SU_V C)$$

is a linear map due to the operations being defined in a pointwise manner. The same argument shows that f^* respects the pointwise action. Hence, $\underline{\mathrm{Aff}}_{\mathrm{K}}(f, C) = \mathrm{Aff}_{\mathrm{K}}(f, C)$ is affine with associated linear map $\widehat{\underline{\mathrm{Aff}}_{\mathrm{K}}}(f, C) =$

$\mathrm{Aff}_K(f, SU_V C)$. We turn to f_* next. Recall the definitions $f_* g = fg$ and $\hat{f} = U_V f$. We can evaluate $f_*(g+h)$ at $P \in A$ and obtain

$$f_*(g+h)(P) = f(g(P) + h(P)) = f(g(P)) + \hat{f}(h(P)) = (f_* g + \hat{f}_* h)(P),$$

where we have used $h(P) \in V_C$. With this we only need to show

$$\hat{f}_* = \mathrm{Aff}_K(C, SU_V f) : \mathrm{Aff}_K(C, SU_V A) \to \mathrm{Aff}_K(C, SU_V D)$$

a linear map. This follows, however, from the linearity of $\hat{f}$ and the pointwise vector space structure. Alltogether $\underline{\mathrm{Aff}}_K(C, f) = \mathrm{Aff}_K(C, f)$ is affine with associated linear map $\widehat{\underline{\mathrm{Aff}}_K}(C, f) = \mathrm{Aff}_K(C, SU_V f)$.

(qed)

Note that we have only needed the property of f to be be affine to show f_* to be affine. For f^* the argument would hold for any map $f : A \to D$. There is a deeper categoric reason behind this, which is connected to the notion of internal structures in Cartesian monoidal categories and pointwise constructions. Concerning the pointwise constructions we have indicated it at the end of section 2.5 already.

Corollary 3.4.4. *For each affine space A in Aff_K, $\underline{\mathrm{Aff}}_K(A, -)$ and $\underline{\mathrm{Aff}}_K(-, A)$ are continuous. In particular, $\underline{\mathrm{Aff}}_K(A, -)$ preserves small limits and $\underline{\mathrm{Aff}}_K(-, A)$ maps small colimits to small limits.*

Proof. The hom-functors $\mathrm{Aff}_K(A, -)$ and $\mathrm{Aff}_K(-, A)$ are known to be continuous (cf. theorem V.4.1 in [Mac98])[15]. By proposition 3.2.1 U_S creates small limits. The assertion then follows from $U_S \underline{\mathrm{Aff}}_K(-, -) = \mathrm{Aff}_K(-, -)$ (cf. remark 2.2.8(c) and the proof of proposition 2.2.6).

(qed)

3.4.2 The Affine Tensor Product

With these results we can now perform our decurrying-procedure, i.e. to consider affine maps $C \to \underline{\mathrm{Aff}}_K(A, D)$ and ask the category Aff_K itself for the correct notion of an affine map $C \otimes A \to D$ of "two arguments".

Theorem 3.4.5 (affine tensor product)**.** *There is a bifunctor*

$$\otimes : \mathrm{Aff}_K \times \mathrm{Aff}_K \to \mathrm{Aff}_K,$$

which we will call the affine tensor product*, with the following properties:*

[15] Recall that $\mathrm{Aff}_K(-, A) = \mathrm{Aff}_K{}^{op}(A, -)$ and that colimits in C are limits in C^{op} and vice versa, where C^{op} denotes the dual or opposite category (cf. section 2.1 for the respective definitions).

(i) For each affine space A in Aff_K, $(-)\otimes A \dashv \underline{\mathrm{Aff}}_\mathrm{K}(A,-)$;

(ii) (Symmetric braiding.) There is a natural isomorphism $\tau : \otimes \dot{\to} \otimes \circ T$ with $\tau_{A,C}^{-1} = \tau_{C,A}$ for all affine spaces C, A in Aff_K, where

$$T : \mathrm{Aff}_\mathrm{K} \times \mathrm{Aff}_\mathrm{K} \to \mathrm{Aff}_\mathrm{K} \times \mathrm{Aff}_\mathrm{K}$$

is the twist endofunctor;

(iii) (Associativity.) There is a natural isomorphism

$$\alpha : \otimes \circ (I_{\mathrm{Aff}_\mathrm{K}} \times \otimes) \dot{\to} \otimes \circ (\otimes \times I_{\mathrm{Aff}_\mathrm{K}});$$

(iv) (Identity object.) A one-point space $*$ is an identity, i.e. there is a natural isomorphism $\rho : (-)\otimes * \dot{\to} I_{\mathrm{Aff}_\mathrm{K}}$, where $I_{\mathrm{Aff}_\mathrm{K}}$ denotes the identity functor of Aff_K;

(v) (Distributivity.) For affine spaces C, D in Aff_K, $(C+D)\otimes A \cong C\otimes A + D\otimes A$ and we have $\emptyset \otimes A \cong \emptyset$ for an empty space $\emptyset$.

Proof. (i) (Existence.) By corollary 3.4.4 the endofunctor $\underline{\mathrm{Aff}}_\mathrm{K}(A,-)$ is continuous, hence it has a left adjoint $(-)\otimes A$ by theorem 3.2.13. By the parameter theorem on adjunctions (cf. the dual of theorem IV.7.3 in [Mac98]) together with the bifunctoriality of $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$ this yields the existence of a unique bifunctor $\otimes$ up to a natural isomorphism.

(ii) For the symmetric braiding and the subsequent two properties we will show the respective translated properties for $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$, which in our case, that is due to $\mathrm{Aff}_\mathrm{K}(A,C) \hookrightarrow \mathrm{Set}(U_S A, U_S C)$, natural in A, C, resorts to show that the respective natural bijections for set-theoretic maps restrict to affine maps. From this, any particular adjuntion from (i), which we think of being fixed from now on for each A in Aff_K, and the fully faithful *Yoneda embedding*

$$y : {\mathrm{Aff}_\mathrm{K}}^{op} \to \mathrm{Set}^{\mathrm{Aff}_\mathrm{K}}, \quad A \mapsto \mathrm{Aff}_\mathrm{K}(A,-), \quad f \mapsto \mathrm{Aff}_\mathrm{K}(f,-),$$

we get the respective natural isomorphisms for the tensor product.

As already noted in section 2.2.3, the symmetry of the tensor product translates to a bijection $\mathrm{Aff}_\mathrm{K}(A, \underline{\mathrm{Aff}}_\mathrm{K}(C,D)) \cong \mathrm{Aff}_\mathrm{K}(C, \underline{\mathrm{Aff}}_\mathrm{K}(A,D))$ natural in A, C, D. On the level of sets this bijection is the natural

$$\tilde{\tau}_{A,C,D} : \mathrm{Set}(A, \mathrm{Set}(C,D)) \to \mathrm{Set}(C, \mathrm{Set}(A,D)), \quad f \mapsto (P \mapsto f(-)(P)),$$

which is easily seen to be natural in all three arguments and where we have omitted the forgetful functor U_S. For the subsequent argument we will omit the notation of the index for $\tilde{\tau}$, too and assume w.l.o.g. C, D, A to be not

empty; otherwise we would have to deal with one-point or empty spaces for which the assertion becomes trivial. As explained above, we only have to show that $\tilde{\tau}$ restricts to a bijection of the respective subsets of affine maps. In particular, we only need to show for an affine map $f : A \to \underline{\mathrm{Aff}}_{\mathrm{K}}(C, D)$ that $\tilde{f} := \tilde{\tau}(f) : C \to \underline{\mathrm{Aff}}_{\mathrm{K}}(A, D)$ is an affine map for arbitrary affine spaces A, C, D in $\mathrm{Aff}_{\mathrm{K}}$; because the identity $\tilde{\tau}^{-1}_{A,C,D} = \tilde{\tau}_{C,A,D}$ then already yields the asserted bijection for affine maps. To begin with, we note that for $P \in C$, $\tilde{f}(P)$ is an affine map $A \to D$ with associated linear map $\hat{\tilde{f}}(P) := \hat{f}(-)(P) : V_A \to V_D$, where $\hat{f} : V_A \to \mathrm{Aff}_{\mathrm{K}}(C, SU_V D)$ is the associated linear map to f

$$\begin{aligned}\tilde{f}(P)(Q+v) &= f(Q+v)(P) = (f(Q) + \hat{f}(v))(P)\\ &= f(Q)(P) + \hat{f}(v)(P)\\ &= \tilde{f}(P)(Q) + \hat{\tilde{f}}(P)(v), \qquad\qquad Q \in A,\ v \in V_A.\end{aligned}$$

We have used the pointwise operations in $\underline{\mathrm{Aff}}_{\mathrm{K}}(C, D)$ here. This shows that $\tilde{f}$ is indeed a map from C to $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, D)$. In a similar way we will obtain $\tilde{f}$ to be affine with associated linear map

$$\hat{\tilde{f}} : V_C \to \mathrm{Aff}_{\mathrm{K}}(A, SU_V D), \quad v \mapsto (Q \mapsto \widehat{f(Q)}(v)),$$

where $\widehat{f(Q)} : V_C \to V_D$ is the associated linear map to $f(Q) : C \to D$. For this we have to check two things: for each $v \in V_C$ the map $Q \mapsto \widehat{f(Q)}(v)$ is affine and $\hat{\tilde{f}}$ is linear. The first assertion will follow from

$$\widehat{f(Q) + \hat{f}(w)} = \widehat{f(Q)} + \widehat{\hat{f}(w)}, \qquad Q \in A,\ w \in V_A, \qquad (*)$$

where the second sum is the pointwise sum in $\mathrm{Vct}_{\mathrm{K}}(V_C, V_D)$, applied to the pointwise action in $\underline{\mathrm{Aff}}_{\mathrm{K}}(C, D)$:

$$\begin{aligned}\widehat{f(Q+w)}(v) &= (\widehat{f(Q) + \hat{f}(w)})(v) = (\widehat{f(Q)} + \widehat{\hat{f}(w)})(v)\\ &= \widehat{f(Q)}(v) + \widehat{\hat{f}(w)}(v), \qquad\qquad Q \in A,\ w \in V_A.\end{aligned}$$

The second assertion is due to the pointwise operations in $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, SU_V D)$ and the linearity of $\widehat{f(Q)}$ for each $Q \in A$:

$$\begin{aligned}\hat{\tilde{f}}(v + \lambda w)(Q) &= \widehat{f(Q)}(v + \lambda w)\\ &= \widehat{f(Q)}(v) + \lambda \widehat{f(Q)}(w), \qquad v, w \in V_C,\ \lambda \in K.\end{aligned}$$

It remains to show (∗). Once again due to the pointwise operations in $\underline{\mathrm{Aff}}_\mathrm{K}(C, D)$ we can apply first $f(Q + w)$ and then f being affine to obtain the identity

$$\begin{aligned} f(Q+w)(P+v) &= f(Q+w)(P) + \widehat{f(Q+w)}(v) \\ &= (f(Q) + \hat{f}(w))(P) + \widehat{(f(Q) + \hat{f}(w))}(v), \end{aligned}$$

or we can first apply f and then $f(Q)$ as well as $\hat{f}(w)$ being affine to obtain the identity

$$\begin{aligned} f(Q+w)(P+v) &= (f(Q) + \hat{f}(w))(P+v) = f(Q)(P+v) + \hat{f}(w)(P+v) \\ &= f(Q)(P) + \widehat{f(Q)}(v) + \hat{f}(w)(P) + \widehat{\hat{f}(w)}(v) \\ &= (f(Q) + \hat{f}(w))(P) + (\widehat{f(Q)} + \widehat{\hat{f}(w)})(v), \end{aligned}$$

for all $v \in V_C$ and $P \in C$. By the simply transitiveness of the action this yields (∗). Finally, with the natural isomorphism $\mathrm{Aff}_\mathrm{K}(A, \underline{\mathrm{Aff}}_\mathrm{K}(C, D)) \cong \mathrm{Aff}_\mathrm{K}(C, \underline{\mathrm{Aff}}_\mathrm{K}(A, D))$ and the definition of adjoints in terms of hom-sets we can form the following chain of natural isomorphisms

$$\begin{aligned} \mathrm{Aff}_\mathrm{K}(A \otimes C, D) &\cong \mathrm{Aff}_\mathrm{K}(A, \underline{\mathrm{Aff}}_\mathrm{K}(C, D)) \\ &\cong \mathrm{Aff}_\mathrm{K}(C, \underline{\mathrm{Aff}}_\mathrm{K}(A, D)) \cong \mathrm{Aff}_\mathrm{K}(C \otimes A, D). \end{aligned} \tag{3.10}$$

The naturality in D yields $\mathrm{Aff}_\mathrm{K}(A \otimes C, -) \cong \mathrm{Aff}_\mathrm{K}(C \otimes A, -)$ natural in A, C. By the Yoneda embedding being full and faithful this yields a unique ${\mathrm{Aff}_\mathrm{K}}^{op}$-isomorphism $\tau^{op}_{C,A} : A \otimes C \cong C \otimes A$ natural in A, C. Translated to Aff_K this becomes a natural isomorphism $\tau_{C,A} : C \otimes A \to A \otimes C$. Due to $\tilde{\tau}^{-1}_{A,C,D} = \tilde{\tau}_{C,A,D}$ we have $\tau^{-1}_{C,A} = \tau_{A,C}$ as asserted.

(iii) The asserted associativity isomorphisms for the tensor products translate for the internal hom as follows

$$\underline{\mathrm{Aff}}_\mathrm{K}(C \otimes D, E) \cong \underline{\mathrm{Aff}}_\mathrm{K}(C, \underline{\mathrm{Aff}}_\mathrm{K}(D, E)),$$

natural in C, D, E. In other words an adjunction of (i) can be internalized; i.e. the bijection of the external hom-sets has a U_S-lift as isomorphism of the internal homs. This can be seen by representing the bijection, call it φ, in terms of the unit $\eta : I_{\mathrm{Aff}_\mathrm{K}} \dot{\to} \underline{\mathrm{Aff}}_\mathrm{K}(D, (-) \otimes D)$ as

$$\varphi_{C,E} : \mathrm{Aff}_\mathrm{K}(C \otimes D, E) \to \mathrm{Aff}_\mathrm{K}(C, \underline{\mathrm{Aff}}_\mathrm{K}(D, E)), \quad f \mapsto \underline{\mathrm{Aff}}_\mathrm{K}(D, f) \circ \eta_C.$$

Confer (2.3) in section 2.2.2 or theorem IV.1.1 in [Mac98] for the general context. For an explicit representation of η in the context of the linear tensor

product we refer to remark 2.2.8(a). We will give a very similar representation in corollary 3.4.12 later; but we won't need it here. In particular, we have

$$\varphi_{C,E} = \eta_C^* \circ \underline{\mathrm{Aff}}_{\mathrm{K}}(D, -) = \underline{\mathrm{Aff}}_{\mathrm{K}}(\eta_C, \underline{\mathrm{Aff}}_{\mathrm{K}}(D, E)) \circ \underline{\mathrm{Aff}}_{\mathrm{K}}(D, -),$$

so due to η_C^* being affine by theorem 3.4.3 we only need to show

$$\underline{\mathrm{Aff}}_{\mathrm{K}}(D, -) : \underline{\mathrm{Aff}}_{\mathrm{K}}(C \otimes D, E) \to \mathrm{Aff}_{\mathrm{K}}(\underline{\mathrm{Aff}}_{\mathrm{K}}(D, C \otimes D), \underline{\mathrm{Aff}}_{\mathrm{K}}(D, E))$$

to be actually an affine map

$$\underline{\mathrm{Aff}}_{\mathrm{K}}(D, -) : \underline{\mathrm{Aff}}_{\mathrm{K}}(C \otimes D, E) \to \underline{\mathrm{Aff}}_{\mathrm{K}}(\underline{\mathrm{Aff}}_{\mathrm{K}}(D, C \otimes D), \underline{\mathrm{Aff}}_{\mathrm{K}}(D, E)).$$

As usual, we do this on points and assume w.l.o.g. that the affine spaces involved are not empty; otherwise by (v), which is independent of the argument given here, we would have to deal with either one-point spaces or empty spaces for which the assertion becomes trivial. Let $f \in \underline{\mathrm{Aff}}_{\mathrm{K}}(C \otimes D, E)$ and $g \in \mathrm{Aff}_{\mathrm{K}}(C \otimes D, SU_V E)$

$$\begin{aligned}
\underline{\mathrm{Aff}}_{\mathrm{K}}(D, f+g)h &= (f+g)h \\
&= \underline{\mathrm{Aff}}_{\mathrm{K}}(h, E)(f+g) \\
&= \underline{\mathrm{Aff}}_{\mathrm{K}}(h, E)f + \widehat{\underline{\mathrm{Aff}}_{\mathrm{K}}}(h, E)g \\
&= fh + gh \\
&= \underline{\mathrm{Aff}}_{\mathrm{K}}(D, f)h + \underline{\mathrm{Aff}}_{\mathrm{K}}(D, g)h \\
&= (\underline{\mathrm{Aff}}_{\mathrm{K}}(D, f) + \underline{\mathrm{Aff}}_{\mathrm{K}}(D, g))h,
\end{aligned}$$

for all $h \in \underline{\mathrm{Aff}}_{\mathrm{K}}(D, C \otimes D)$, where we have used the explicit bifunctoriality of the internal hom from the proof of theorem 3.4.3. A similar argument shows the linearity of

$$\underline{\mathrm{Aff}}_{\mathrm{K}}(D, -) : \underline{\mathrm{Aff}}_{\mathrm{K}}(C \otimes D, SU_V E) \to \mathrm{Aff}_{\mathrm{K}}(\underline{\mathrm{Aff}}_{\mathrm{K}}(D, C \otimes D), SU_V \underline{\mathrm{Aff}}_{\mathrm{K}}(D, E)),$$

where we have $SU_V \underline{\mathrm{Aff}}_{\mathrm{K}}(D, E) = S \underline{\mathrm{Aff}}_{\mathrm{K}}(D, SU_V E)$, $\underline{\mathrm{Aff}}_{\mathrm{K}}(D, SU_V E)$ is understood as a vector space and the target hom-set carries a pointwise vector space structure induced by it. Hereto, it is helpful to recall a fact mentioned in the proof of theorem 3.4.3 about f^*, which more broadly states that contravariant homming with any (set-theoretic) map preserves pointwise algebraic structures:

$$\underline{\mathrm{Aff}}_{\mathrm{K}}(D, f + \lambda g)h = (f + \lambda g)h = fh + \lambda gh = (\underline{\mathrm{Aff}}_{\mathrm{K}}(D, f) + \lambda \underline{\mathrm{Aff}}_{\mathrm{K}}(D, g))h,$$

which holds for all $h \in \underline{\mathrm{Aff}}_{\mathrm{K}}(D, C \otimes D)$. At last, we can once again use our result for the internal hom to write down a chain of natural isomorphisms

$$\begin{aligned}
\mathrm{Aff}_{\mathrm{K}}((A \otimes C) \otimes D, E) &\cong \mathrm{Aff}_{\mathrm{K}}(A \otimes C, \underline{\mathrm{Aff}}_{\mathrm{K}}(D, E)) \\
&\cong \mathrm{Aff}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C, \underline{\mathrm{Aff}}_{\mathrm{K}}(D, E))) \\
&\overset{\varphi_*^{-1}}{\cong} \mathrm{Aff}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C \otimes D, E)) \\
&\cong \mathrm{Aff}_{\mathrm{K}}(A \otimes (C \otimes D), E),
\end{aligned} \tag{3.11}$$

natural in A, C, D, E and apply our Yoneda argument to obtain the desired natural isomorphism first as $\alpha^{op}_{A,C,D} : (A \otimes C) \otimes D \cong A \otimes (C \otimes D)$ in $\mathrm{Aff}_\mathrm{K}{}^{op}$ and finally as $\alpha_{A,C,D} : A \otimes (C \otimes D) \cong (A \otimes C) \otimes D$ in Aff_K.

(iv) For the right-unit law we will show $\underline{\mathrm{Aff}}_\mathrm{K}(*, -) \cong I_{\mathrm{Aff}_\mathrm{K}}$. With this we get

$$\mathrm{Aff}_\mathrm{K}(A \otimes *, C) \cong \mathrm{Aff}_\mathrm{K}(A, \underline{\mathrm{Aff}}_\mathrm{K}(*, C)) \cong \mathrm{Aff}_\mathrm{K}(A, C), \tag{3.12}$$

natural in A, C and hence an isomorphism $\rho_A : A \otimes * \cong A$ natural in A by our Yoneda argument. Recall that a one-point space $*$ is a separator in Aff_K and that $U_S \cong \mathrm{Aff}_\mathrm{K}(*, -)$. $\underline{\mathrm{Aff}}_\mathrm{K}(*, -) \cong I_{\mathrm{Aff}_\mathrm{K}}$ is just the obvious internalization of this. More explicitly, we only need to show the evaluation map

$$\mathrm{ev}_A : \underline{\mathrm{Aff}}_\mathrm{K}(*, A) \to A, \quad P \mapsto P(\bullet), \qquad \bullet \in *$$

to be affine and natural in A, since it is bijective by the separator property of $*$ and hence an isomorphism by corollary 3.1.4. Here we have assumed once again and w.l.o.g. A to be not empty, since $\underline{\mathrm{Aff}}_\mathrm{K}(*, \emptyset)$ is the empty space by definition and $\mathrm{ev}_\emptyset$ is an affine isomorphism in a trivial way. For the first assertion of being affine observe the chain of isomorphisms $\underline{\mathrm{Aff}}_\mathrm{K}(*, SU_V A) \cong \mathrm{Set}(*, V_A) \cong V_A$ as vector spaces; i.e. each affine map $* \to SU_V A$ corresponds to one particular displacement vector of A and vice versa. Together with the pointwise structure of $\underline{\mathrm{Aff}}_\mathrm{K}(*, A)$ this yields the assertion. The second assertion is a consequence of the commutativity of the following diagram

$$\begin{array}{ccc} A & \underline{\mathrm{Aff}}_\mathrm{K}(*, A) & \xrightarrow{\mathrm{ev}_A} A \\ {\scriptstyle f}\downarrow & {\scriptstyle f_*}\downarrow & \downarrow{\scriptstyle f} \\ C, & \underline{\mathrm{Aff}}_\mathrm{K}(*, C) & \xrightarrow{\mathrm{ev}_C} A, \end{array} \qquad \begin{array}{ccc} P & \longmapsto & P(\bullet) \\ \downarrow & & \downarrow \\ f \circ P & \longmapsto & f(P(\bullet)). \end{array}$$

(v) The asserted first part of distributivity is a consequence of $(-) \otimes A$ to preserve coproducts, which is due to (i) and the dual of theorem V.5.1 in [Mac98]. An empty space $\emptyset$ to act as a zero is a consequence of the hom-tensor adjunction in (i) and $\emptyset$ being initial in Aff_K. In more detail, we need to show for any affine spaces C in Aff_K that $\mathrm{Aff}_\mathrm{K}(\emptyset \otimes A, C)$ is the one-point set; but this is an immediate consequence of

$$\mathrm{Aff}_\mathrm{K}(\emptyset \otimes A, C) \cong \mathrm{Aff}_\mathrm{K}(\emptyset, \underline{\mathrm{Aff}}_\mathrm{K}(A, C)),$$

and $\emptyset$ being initial in Aff_K.

(qed)

Note that to get particular structure morphisms in theorem 3.4.5 we have to fix a particular hom-tensor adjunction in (i) for each A in Aff_K. This will yield a particular bifunctor $\otimes$ as well as the other structure morphisms from the respective constructions given in the proof. As is to be expected these structure morphisms are then compatible in such a way that Aff_K admits a monoidal structure $(\otimes, *, \alpha, \rho \circ \tau_{(*,-)}, \rho)$. We will call this monoidal structure an *affine tensor structure*. A different choice of hom-tensor adjunctions in (i) will give an isomorphic structure.

Corollary 3.4.6. Aff_K *admits a closed symmetric monoidal structure with* $\otimes$, $*$, α, $\rho \circ \tau_{(*,-)}$, ρ *as structure morphisms, where* $\tau_{(*,-)} : * \otimes (-) \dot{\to} (-) \otimes *$ *is the natural isomorphism induced from* τ. *It is determined up to natural isomorphism by the internal hom* $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$*; i.e. for any other closed symmetric monoidal structure with tensor-product* $\tilde{\otimes}$ *such that* $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$ *is the internal hom*[16]*, the identity functor* $I_{\mathrm{Aff}_\mathrm{K}}$ *lifts to a strong symmetric monoidal functor between these two symmetric monoidal categories.*

Proof. Basically, to show the coherence axioms of a symmetric monoidal category to hold we need to translate them formally into commutative diagrams for the internal hom for a given closed monoidal structure, show them to hold for our particular $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$ and then apply our Yoneda argument to get the respective result for the tensor-product bifunctor, like we did in the proof of the previous theorem several times. Indeed, due to this proof we have already formulated the symmetric braid as well as the associativity and unit maps for our internal hom explicitly. We only need to show the respective diagrams to commute, which may be verified on points. However, we refrain from doing so due to the same reasons we have stated at the end of section 2.3 and in the proof of proposition 3.4.1. Another way to show our assertion is to apply theorem II.5.3 in [EK65], for its assumptions are met due to the existence of a hom-tensor adjunction by (i) of the previous theorem. To be completely rigorous, we would need to show the axioms of an internal hom for $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$. Once again we refrain from doing so. It would be just another tedious formal exercise in chasing diagrams, which were designed in such a way as to capture the essential properties of examples like our $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$ (cf. introduction of [EK65] and the discussion in section 2.2.3).

Given another closed symmetric monoidal structure, such that $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$ is the internal hom with tensor-product $\tilde{\otimes}$, the structure morphisms, which lift the identity functor to a strong monoidal one, are obtained as the natural isomorphism of

[16] This is the only occasion in this thesis, where "internal hom" has to be understood not only as a particular bifunctor related to the external hom but with all its structure morphisms as defined in [EK65, section I.2], and the term "closed monoidal" has to be understood as in [EK65, section II.2] asserting the compatibility between the structure morphisms of the internal hom and the structure morphisms of the monoidal structure. Basically, this amounts to say that given an adjunction $(-)\tilde{\otimes}A \dashv \underline{\mathrm{Aff}}_\mathrm{K}(A,-)$ for each A in Aff_K the structure maps of the monoidal structure can be obtained from the translated maps for $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$ like in the proof of theorem 3.4.5.

the bifunctors $\otimes$ and $\tilde{\otimes}$ from the respective hom-tensor adjunction with left adjoint $\underline{\mathrm{Aff}}_\mathrm{K}(A,-)$ for each A in Aff_K (cf. the proof of theorem 3.4.5(i)). For this second closed monoidal structure to have $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$ as an internal hom it means that the monoidal structure morphisms translate to the same respective maps for $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$ as the structure morphisms of our first monoidal structure did in the proof of theorem 3.4.5. In fact, this shows that the structure morphisms for the identity functor are indeed compatible with the respective monoidal structure morphisms and hence that the identity functor is strong monoidal, accordingly. The argument for this is essentially the same as the first one to show the coherence axioms for a symmetric monoidal category.

(qed)

Remarks 3.4.7 (finite dimensions)**.**
Let A, C be non-empty affine spaces in FAff_K.

(a) For the internal hom $\underline{\mathrm{Aff}}_\mathrm{K}(A,C)$ the following dimension formula holds

$$\dim \underline{\mathrm{Aff}}_\mathrm{K}(A,C) = (\dim A + 1)\dim C. \tag{3.13}$$

Compare this with $\dim \underline{\mathrm{Vct}}_\mathrm{K}(V_A,V_C) = \dim V_A \dim V_C$ from linear algebra. As we have already mentioned in remark 3.3.4(a), the additional $1\cdot\dim C$ represents the degree of freedom to choose different basis points. Indeed, $\dim \underline{\mathrm{Aff}}_\mathrm{K}(A,C) = \dim \mathrm{Aff}_\mathrm{K}(A, SU_VC)$ holds by definition. Fix a point $P\in A$. By proposition 3.1.3(ii) each affine map $f: A\to SU_VC$ can be uniquely represented as $Q\mapsto f(P)+\hat{f}(\overrightarrow{PQ})$ where $\hat{f}=U_Vf: V_A\to V_C$ is the associated linear map to f. With this we can obtain a basis for $\mathrm{Aff}_\mathrm{K}(A,SU_VC)$ by choosing a basis $(\ell_i)_{i\in I}$ of $\underline{\mathrm{Vct}}_\mathrm{K}(V_A,V_C)$ and a basis $(v_j)_{j\in J}$ of V_C and define constant affine maps $\nu_j: A\to SU_VC$, $P\mapsto v_j$, $j\in J$. There are unique scalars $(\lambda_j)_{j\in J}$ and $(\mu_i)_{i\in I}$ such that $f(P)=\sum_j \lambda_j v_j$ and $\hat{f}=\sum_i \mu_i\ell_i$ hold, hence there are unique scalars such that the following identity holds:

$$f = \sum_j \lambda_j\nu_j + \sum_i \mu_i\ell_i.$$

Due to $\operatorname{card} I = \dim A\dim C$ and $\operatorname{card} J = \dim C$ this shows the asserted dimension formula.

(b) The previous remark shows in particular the internal hom $\underline{\mathrm{Aff}}_\mathrm{K}(-,-)$ to restrict to $\mathrm{FAff}_\mathrm{K}{}^{op}\times\mathrm{FAff}_\mathrm{K}\to\mathrm{FAff}_\mathrm{K}$. We will denote this restriction by $\underline{\mathrm{Aff}}_\mathrm{K}$ again.

(c) With remark (a) we obtain the following dimension formula for the affine tensor product $A\otimes C$

$$\dim A\otimes C = (\dim A+1)(\dim C+1)-1 = \dim A\dim C+\dim A+\dim C. \tag{3.14}$$

This can be for example read off from the internalized adjunction isomorphism

$$\underline{\mathrm{Aff}}_{\mathrm{K}}(A \otimes C, L) \cong \underline{\mathrm{Aff}}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C, L))$$

and the dimension formula for the internal hom, where L denotes an affine line[17]. The dimension formula shows in particular that the affine tensor product and the whole hom-tensor adjunction can be restricted to $\mathrm{FAff}_{\mathrm{K}}$. Thus the affine tensor structure restricts to a closed symmetric monoidal structure on $\mathrm{FAff}_{\mathrm{K}}$.

3.4.3 Biaffine Maps

Until now we have had only an abstract description of our affine tensor product in terms of a hom-tensor adjunction from theorem 3.4.5 at hand. From the categorical perspective this may be sufficient, but from a geometric perspective it clearly isn't. A remedy for this deficiency is the notion of a biaffine map $A \times C \to E$. If we compare theorem 3.4.5 with the situation for $\mathrm{Vct}_{\mathrm{K}}$ in section 2.2.2, then what we find to be missing is a corresponding notion to bilinear maps for which the tensor product is a representing object. Instead of directly defining the notion of biaffine maps and then showing the representability, we keep our decurrying-approach and let the categories itselves give the correct notion. The representability will be a byproduct of our approach.

First of all we need to understand how something like bilinear or biaffine maps can be related to the respective tensor product. We show this for the affine case. The linear case can be obtained in a similar way. Fix A, C in $\mathrm{Aff}_{\mathrm{K}}$ and consider the Set-valued functor $\mathrm{Aff}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C, -))$. For each affine space E in $\mathrm{Aff}_{\mathrm{K}}$ due to the forgetful functor U_S being faithful we have the following inclusions

$$\mathrm{Aff}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C, E)) \hookrightarrow \mathrm{Set}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C, E)) \hookrightarrow \mathrm{Set}(A, \mathrm{Set}(C, E)),$$

natural in A, C, E, where we have omitted the respective use of the forgetful functor U_S in the notation. In particular we have the inclusion

$$\mathrm{Aff}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C, -)) \hookrightarrow \mathrm{Set}(A, \mathrm{Set}(C, -)) \tag{3.15}$$

of functors and one often says that $\mathrm{Aff}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C, -))$ is a subfunctor of $\mathrm{Set}(A, \mathrm{Set}(C, -))$. The Cartesian closedness of sets yields the natural isomorphism

$$\varphi : \mathrm{Set}(A, \mathrm{Set}(C, -)) \dot{\to} \mathrm{Set}(A \times C, -). \tag{3.16}$$

With this we can consider the φ-image of the subfunctor $\mathrm{Aff}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C, -))$; i.e. the subfunctor of $\mathrm{Set}(A \times C, -)$ which assigns to each E the φ_E image of the

[17] Take for L e.g. the field K regarded as an affine space.

set $\mathrm{Aff}_\mathrm{K}(A, \underline{\mathrm{Aff}}_\mathrm{K}(C,E))$ and to each affine map $f : E \to E'$ the restriction of $f_* = \mathrm{Set}(A \times C, f)$ to

$$f_* : \varphi(\mathrm{Aff}_\mathrm{K}(A, \underline{\mathrm{Aff}}_\mathrm{K}(C,E))) \to \varphi(\mathrm{Aff}_\mathrm{K}(A, \underline{\mathrm{Aff}}_\mathrm{K}(C,E'))).$$

That this is indeed well-defined follows essentially from the naturality of φ[18]:

$$\varphi_{E'} f_* = f_* \varphi_E.$$

We denote this φ-image by $\mathrm{Biaff}_\mathrm{K}(A,C;-) : \mathrm{Aff}_\mathrm{K} \to \mathrm{Set}$. For each E it is our respective set of biaffine maps $A \times C \to E$. The assertion of representability is then an immediate consequence of our construction and the hom-tensor adjunction in Aff_K

$$\mathrm{Aff}_\mathrm{K}(A \otimes C, E) \cong \mathrm{Aff}_\mathrm{K}(A, \underline{\mathrm{Aff}}_\mathrm{K}(C,E)) \cong \mathrm{Biaff}_\mathrm{K}(A,C;E), \tag{3.17}$$

natural in A, C, E; here the representability just needs the naturality in E. We also get a tensor map $\otimes : A \times C \to A \otimes C$, which is the image of $1_{A\otimes C}$ under the above natural isomorphism and the expected universal property (cf. proposition II.2.2 in [Mac98] for the general relation between representability and universal morphisms)

$$\begin{array}{ccc} A \times C & \xrightarrow{\otimes} & A \otimes C \\ & \searrow^{f} & \downarrow{\scriptstyle \exists! \tilde{f}} \\ & & E, \end{array} \tag{3.18}$$

where the map f is biaffine and the map $\tilde{f}$ is affine. Eventually, our construction of $\mathrm{Biaff}_\mathrm{K}(A,C;E)$ from $\mathrm{Aff}_\mathrm{K}(A, \underline{\mathrm{Aff}}_\mathrm{K}(C,E))$ tells us how (non-trivial) biaffine maps actually look like. It is what one would expect from the beginning.

Proposition 3.4.8. *Let A, C, E be non-empty affine spaces in Aff_K. A (set-theoretic) map*

$$f : A \times C \to E$$

is biaffine, i.e. $f \in \mathrm{Biaff}_\mathrm{K}(A,C;E)$, if and only if the map f is affine in each of its argument; i.e. for each $P \in A$, $f(P,-) : C \to E$ is affine and for each $Q \in C$, $f(-,Q) : A \to E$ is affine. Moreover, there is a bilinear map $\hat{f} : V_A \times V_C \to V_E$, such that for each $P \in A$, $v \in V_A$, $Q \in C$, $w \in V_C$,

$$\hat{f}(P+v,w) = \hat{f}(P,w) + \hat{f}(v,w) \qquad \text{and} \qquad \hat{f}(v,Q+w) = \hat{f}(v,Q) + \hat{f}(v,w),$$

where $\hat{f}(P,-) : V_C \to V_E$, $\hat{f}(-,Q) : V_A \to V_E$ are the associated linear maps to $f(P,-)$, $f(-,Q)$ respectively.

[18] One can define the direct image in purely categorical terms as a certain functor between the sets of subobjects regarded as preorders, i.e. categories with at most one arrow between two objects. Cf. e.g. [MM92] for the case of topoi.

Proof. We have to show that given an affine map $g \in \mathrm{Aff}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C,E))$ its φ_E-image f fullfils the asserted conditions and conversely that for each map $f : A \times C \to E$, for which the above conditions hold, there is a $g \in \mathrm{Aff}_{\mathrm{K}}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C,E))$ such that $f = \varphi_E g$.

"⇒" Employing the notation and results from the proof of theorem 3.4.5(ii) we have

$$f(P+v, Q+w) = g(P+v)(Q+w) = g(P)(Q) + \widehat{g(P)}(w) + \hat{g}(v)(Q) + \widehat{\hat{g}(v)}(w).$$

Here, like in the proof mentioned above, we have first applied g to be affine and then $g(P)$, $\hat{g}(v)$ to be affine. This suggests to set

$$\hat{f}(P,-) := \widehat{g(P)}, \quad \hat{f}(-,Q) := \widehat{g(-)}(Q), \quad \hat{f} := (v,w) \mapsto \widehat{\hat{g}(v)}(w).$$

Recall the pointwise operations in $\underline{\mathrm{Aff}}_{\mathrm{K}}(C,E)$ and in particular the pointwise operations in its associated vector space $\underline{\mathrm{Aff}}_{\mathrm{K}}(C, SU_V E)$ as well as in $\underline{\mathrm{Vct}}_{\mathrm{K}}(V_C, V_E)$. Then the first and second map are linear by definition, the bilinearity of the last is a consequence of the linearity of $\hat{g}$ and $\widehat{\hat{g}(v)}$ for each $v \in V_A$. In more detail, the linearity of the latter in the first argument can be obtained from

$$\hat{g}(v + \lambda v')(P + w) = \hat{g}(v + \lambda v')(P) + \hat{g}(\widehat{v + \lambda v'})(w),$$

and

$$\begin{aligned} \hat{g}(v + \lambda v')(P + w) &= \hat{g}(v)(P + w) + \lambda \hat{g}(v')(P + w) \\ &= \hat{g}(v + \lambda v')(P) + \widehat{\hat{g}(v)}(w) + \lambda \widehat{\hat{g}(v')}(w), \end{aligned}$$

for all $w \in V_C$, $v, v' \in V_A$ and $\lambda \in K$ by the simply transitiveness of the action of V_E on E. This shows the asserted first implication together with the existence of the asserted bilinear map and its properties.

"⇐" Conversely, let $f : A \times C \to E$ be a map such that for each $P \in A$, $f(P,-) : C \to E$ is affine and for each $Q \in C$, $f(-,Q) : A \to E$ is affine. We need to show

$$g : A \to \underline{\mathrm{Aff}}_{\mathrm{K}}(C,E), \quad P \mapsto f(P,-),$$

to be an affine map. To begin with, it is well-defined as a set-theoretic map by our first assumption. Our second assumption together with the pointwise structure of $\underline{\mathrm{Aff}}_{\mathrm{K}}(C,E)$ yields the map $\hat{g} : v \mapsto (Q \mapsto \hat{f}(v,Q))$ to be linear as well as

$$g(P + v)(Q) = f(P + v, Q) = f(P,Q) + \hat{f}(v,Q), \qquad Q \in C,\ v \in V_A,$$

i.e. $g(P+v) = g(P) + \hat{f}(v,-)$. For g to be affine with associated linear map $\hat{g}$ we have to show the map $\hat{f}(v,-) : C \to SU_V E$ affine. Assume the existence of the asserted bilinear map $\hat{f}$. With this $\hat{f}(v,-)$ to be affine follows immediately from

$$\hat{f}(v, Q+w) = \hat{f}(v,Q) + \hat{f}(v,w), \qquad Q \in C,\ w \in V_C.$$

Finally, we will show the existence of the asserted bilinear map $\hat{f}$ from the weaker assumption of f being affine in each of its arguments. Fix two arbitrary points $P \in A$, $Q \in C$. We can write out $f(P+v, Q+w)$ for any $v \in V_A$, $w \in V_C$ in two ways:

$$\begin{aligned} f(P+v,Q+w) &= f(P+v,Q) + \hat{f}(P+v,w) \\ &= f(P,Q) + \hat{f}(v,Q) + \hat{f}(P+v,w), \\ f(P+v,Q+w) &= f(P,Q+w) + \hat{f}(v,Q+w) \\ &= f(P,Q) + \hat{f}(P,w) + \hat{f}(v,Q+w). \end{aligned}$$

The simply transitiveness of the action in E yields

$$\begin{aligned} & \hat{f}(v,Q) + \hat{f}(P+v,w) = \hat{f}(P,w) + \hat{f}(v,Q+w) \\ \iff\quad & \hat{f}(P+v,w) - \hat{f}(P,w) = \hat{f}(v,Q+w) - \hat{f}(v,Q). \end{aligned}$$

Define $\hat{f} : V_A \times V_C \to V_E$, $(v,w) \mapsto \hat{f}(P+v,w) - \hat{f}(P,w)$. The last identity together with f being affine in each of its arguments yield the bilinearity of $\hat{f}$. Moreover, since Q doesn't depend on P, we have for any $P' \in A$

$$\hat{f}(P'+v,w) - \hat{f}(P',w) = \hat{f}(v,Q+w) - \hat{f}(v,Q) = \hat{f}(P+v,w) - \hat{f}(P,w).$$

This shows $\hat{f}$ to be independent of the particular point P, i.e. $\hat{f}(v,w) = \hat{f}(P'+v,w) - \hat{f}(P',w)$ for any $P' \in A$. The asserted properties of $\hat{f}$ follow immediately from this.

(qed)

Remarks 3.4.9.

Let $f : C \times D \to E$ be a biaffine map like in the proposition.

(a) We can write the conditions out like for bilinear maps

$$f(P+v,Q+w) = f(P,Q) + \hat{f}(P,w) + \hat{f}(v,Q) + \hat{f}(v,w)$$

with $P \in A$, $Q \in C$, $v \in V_A$, $w \in V_C$.

(b) The bilinear map $\hat{f} : V_A \times V_C \to V_E$ is uniquely determined by any two of the linear maps $\hat{f}(-,Q)$, $\hat{f}(-,Q')$, $Q, Q \in C$, $Q \neq Q'$ or $\hat{f}(P,-)$, $\hat{f}(P',-)$, $P, P' \in A$, $P' \neq P$. This can be seen for example by the simply transitive action of V_E on E together with the properties of $\hat{f}$ given in the proposition. We will refer to the bilinear map $\hat{f}$ as the *associated bilinear map* of f.

3.4.4 The Underlying Set of Points of an Affine Tensor Product

If we apply the first remark of remark 3.4.9 to our tensor map $\otimes : A \times C \to A \otimes C$ we get a glimpse on how points and the action in $A \otimes C$ look like

$$(P+v)\otimes(Q+w) = P\otimes Q+P\otimes w+v\otimes Q+v\otimes w, \quad P \in A,\ Q \in C,\ v \in V_A,\ w \in V_C.$$

Here $P \otimes w + v \otimes Q + v \otimes w$ is a vector in $V_{A\otimes C}$ and we have omitted the distinction between the set-theoretic, linear and bilinear maps in the notation. In $P \otimes w$ and $v \otimes Q$ we have a notion of "displacement vectors bound to points". With regard to the geometric motivation of an affine structure, where the displacement vectors are understood geometrically as classes of parallel directed line segments with the same direction and length, this may recall the idea of a a geometric notion of boundary. We will see particular examples that go in this direction like the (generalized) product rule of differential calculus and how it is connected to the affine tensor product as well as the natural DG structures on our geometric algebras of affine space (cf. chapter 6).

Given any action on $P \otimes Q$ in $A \otimes C$ by a finite sum of vectors

$$P \otimes Q + \sum_i P_i \otimes w_i + \sum_j v_j \otimes Q_j + \sum_\ell v_\ell \otimes w_\ell,$$

we can use the biaffinity of the tensor map to reduce this expression to the following form

$$P' \otimes Q' + \sum_k v'_k \otimes w'_k,$$

where the latter sum is finite again. Consider hereto a vector of the form $P_i \otimes w_i$. First of all we can use $P_i = P + \overrightarrow{PP_i}$ and represent it as

$$P_i \otimes w_i = P \otimes w_i + \overrightarrow{PP_i} \otimes w_i.$$

After that we can add the second term to $P\otimes Q$ and obtain $P\otimes(Q+w_i)$. Repeating this procedure for each $P_i \otimes w_i$ we reduce the above expression to

$$P \otimes Q' + \sum_j v_j \otimes Q_j + (\sum_i \overrightarrow{PP_i} \otimes w_i + \sum_\ell v_\ell \otimes w_\ell).$$

Proceeding in complete analogy with each $v_j \otimes Q_j$ results in our asserted expression. This line of thought results in the following representation of the affine tensor product $A \otimes C$ and its elements.

Theorem 3.4.10. *Let A, C be non-empty affine spaces in $\mathrm{Aff}_{\mathrm{K}}$.*

(i) Fix $(P, Q) \in A \times C$, then there is a (natural) isomorphism $S(V_C \oplus V_A \oplus V_A \otimes V_C) \to A\otimes C$ which maps 0 to $P\otimes Q$. Here the tensor product on the left hand side denotes the linear tensor product, the tensor product on the right hand side denotes the affine tensor product and $V_C\oplus V_A\oplus V_A\otimes V_C = V_C\oplus V_A\oplus(V_A\otimes V_C)$.

(ii) Each $R \in A \otimes C$ can be represented as

$$R = P_R \otimes Q_R + \sum_{i \in I} v_i \otimes w_i \quad \text{for a } P_R \in A,\ Q_R \in C,\ v_i \in V_A,\ w_i \in V_C,\ i \in I,$$

and a finite index-set I. The points P_R, Q_R are uniquely determined by R.

Proof. (i) Consider the following set-theoretic map

$$t : A \times C \to V_C \oplus V_A \oplus V_A \otimes V_C, \quad (U, R) \mapsto \overrightarrow{QR} + \overrightarrow{PU} + \overrightarrow{PU} \otimes \overrightarrow{QR}.$$

We will show this map a universal morphism for biaffine maps; stated differently, we will show $S(V_C \oplus V_A \oplus V_A \otimes V_C)$ to be a representing object for $\mathrm{Biaff}_{\mathrm{K}}(A, C; -)$. The asserted isomorphism is then a consequence of the universal property of such a representing object. In fact, comming from a universal property the asserted isomorphism is natural and respects the tensor maps t and $\otimes$. Hence, the following diagram commutes

$$\begin{array}{ccc} A \times C & \xrightarrow{t} & S(V_C \oplus V_A \oplus V_A \otimes V_C) \\ & \searrow^{\otimes} & \downarrow \cong \\ & & A \otimes C, \end{array}$$

and we have in particular

$$\begin{array}{ccc} (P, Q) & \longmapsto & 0 \\ & \searrow & \downarrow \\ & & P \otimes Q. \end{array}$$

This shows 0 to be mapped to $P \otimes Q$ as asserted. It thus remains to show the universal property for t and $S(V_C \oplus V_A \oplus V_A \otimes V_C)$, which in our case amounts to show for each biaffine map $f : A \times C \to E$ there to be a unique affine map $g : S(V_C \oplus V_A \oplus V_A \otimes V_C) \to E$, such that $gt = f$. We will show the uniqueness of g first and then its existence.

Suppose we have an affine map $g : S(V_C \oplus V_A \oplus V_A \otimes V_C) \to E$, such that $f = gt$. By remark 3.4.9(a) we have then for $v \in V_A$, $w \in V_C$

$$f(P + v, Q + w) = f(P, Q) + \hat{f}(P, w) + \hat{f}(v, Q) + \hat{f}(v, w),$$

and

$$gt(P + v, Q + w) = g(0 + w + v + v \otimes w) = g(0) + \hat{g}(w) + \hat{g}(v) + \hat{g}(v \otimes w).$$

Together with the simply transitiveness of the action of V_E on E this yields the following identities: for $v = w = 0$ we get $g(0) = f(P, Q)$, for $v = 0$ we get $\hat{g}(w) = \hat{f}(P, w)$ and we get $\hat{g}(v) = \hat{f}(v, Q)$ for $w = 0$. For both vectors different from zero the previous identities yield $\hat{g}(v \otimes w) = \hat{f}(v, w)$. Recall the pure tensors $v \otimes w$, $v \in V_A$, $w \in V_C$ to span $V_A \otimes V_C$ as a K-vector space, hence $\hat{g}$ is uniquely determined by $\hat{f}$ on $V_A \otimes V_C$ as well. Alltogether this shows g to be uniquely determined by f by proposition 3.1.3(ii).

For the existence of g we use the previous result and set

$$g : S(V_C \oplus V_A \oplus V_A \otimes V_C) \to E,$$
$$w + v + \sum_{i \in I} v_i \otimes w_i \mapsto f(P, Q) + \hat{f}(P, w) + \hat{f}(v, Q) + \sum_{i \in I} \hat{f}(v_i, w_i).$$

Due to the linearity of $\hat{f}(P, -)$ and $\hat{f}(-, Q)$ as well as the bilinearity of $\hat{f}$ the map $v + w + \sum_{i \in I} v_i \otimes w_i \mapsto \hat{f}(P, w) + \hat{f}(v, Q) + \sum_{i \in I} \hat{f}(v_i, w_i)$ is linear; this can be either verified directly or obtained from the explicit descriptions of the maps given by the respective universal properties of the direct sum and the linear tensor product. With this and proposition 3.1.3(ii) the map g is shown to be affine and the identity $gt = f$ is then a trivial consequence of its definition.

Note that our discussion of the uniqueness and the existence of g shows how the asserted isomorphism actually looks like

$$S(V_C \oplus V_A \oplus V_A \otimes V_C) \to A \otimes C,$$
$$w + v + \sum_j v_j \otimes w_j \mapsto P \otimes Q + P \otimes w + v \otimes Q + \sum_j v_j \otimes w_j.$$

(ii) Let $R \in A \otimes C$. Fix $(P, Q) \in A \times C$ and consider the isomorphism from (i). This yields the following representation of R

$$P \otimes Q + P \otimes w + v \otimes Q + \sum_j v_j \otimes w_j,$$

for unique $v \in V_A$ and $w \in V_C$, because these components are uniquely determined by an element of $V_C \oplus V_A \oplus V_A \otimes V_C$. Accordingly, this shows the existence and uniqueness of $Q_R = Q + w$, $P_R = P + v$ such that

$$R = P_R \otimes Q_R + v \otimes \overrightarrow{Q_R Q} + \sum_j v_j \otimes w_j = P_R \otimes Q_R + \sum_{i \in I} v_i \otimes w_i,$$

for a respective index Set I and respective pure tensors $v_i \otimes w_i$. In more detail, the uniqueness of v, w implies the uniqueness of P_R, Q_R as follows. Suppose

there is a $(P', Q') \in A \times C$ such that

$$\begin{aligned} R &= P' \otimes Q' + \sum_{i \in I'} v'_i \otimes w'_i \\ &= P \otimes Q + P \otimes \overrightarrow{QQ'} + \overrightarrow{PP'} \otimes Q + \overrightarrow{PP'} \otimes \overrightarrow{QQ'} + \sum_{i \in I'} v'_i \otimes w'_i, \end{aligned}$$

where for the last equals sign we have applied the biaffinity of the tensor map $\otimes$ and $P' = P + \overrightarrow{PP'}$, $Q' = Q + \overrightarrow{QQ'}$. Our isomorphism from (i) yields the identities

$$\overrightarrow{PP'} = v, \ \overrightarrow{QQ'} = w \qquad \text{and} \qquad \overrightarrow{PP'} \otimes \overrightarrow{QQ'} + \sum_{i \in I'} v'_i \otimes w'_i = \sum_j v_j \otimes w_j.$$

Hence $P' = P_R$, $Q' = Q_R$ hold by the simple transitiveness of the respective actions.

(qed)

Remarks 3.4.11.

(a) Statement (i) of the preceding theorem justifies our negligence in the notation of the bilinear part of an affine tensor map.

(b) Although the points P_R, Q_R are uniquely determined by R, the representation as a whole is not unique. This is due to the representation of mixed tensors by pure ones not being unique, unless one fixes a basis of pure tensors. Accordingly, part (ii) of the preceding theorem says that each affine tensor $R \in A \otimes C$ can be uniquely represented as a tensor of points and a linear tensor from $V_A \otimes V_C$.

(c) Given two affine maps $f : A \to C$, $g : A' \to C'$ in Aff_K then with theorem 3.4.10 we obtain an explicit description of the affine map $f \otimes g : A \otimes A' \to C \otimes C'$ which denotes the action of the bifunctor $\otimes$ on the morphism (f, g). It is again the map one would expect

$$f \otimes g(P \otimes Q + \sum_i v_i \otimes w_i) = f(P) \otimes g(Q) + \sum_i \hat{f}(v_i) \otimes \hat{g}(w_i),$$

with $P \in A$, $Q \in A'$, $v_i \in V_A$, $w_i \in V_{A'}$. Be aware that "$\otimes$" stands for different affine and linear tensor maps here.

To show this we first apply the universal property (3.18) of the affine tensor product as a representing object for bilinear maps to the biaffine map $\otimes_{C,C'} \circ f \times g$

$$\begin{array}{ccc} A \times A' & \xrightarrow{\otimes_{A,A'}} & A \otimes A' \\ {\scriptstyle f\times g}\downarrow & & \downarrow{\scriptstyle \exists! f\otimes g} \\ C \times C' & \xrightarrow[\otimes_{C,C'}]{} & C \otimes C'. \end{array}$$

Next, we use the naturality of (3.17) in the first two arguments A and C, with C replaced by A'. This yields the unique affine map $f \otimes g$ in the above diagram to be indeed the action of the bifunctor $\otimes$ on the morphism (f, g). To see this, besides (3.17) one needs to understand how the bifunctoriality of the affine tensor product is actually obtained from the parametrized adjunction in theorem 3.4.5(i). The details can be found in [Mac98, section IV.7]. Finally, we apply theorem 3.4.10(ii) and evaluate the above diagram on points. This yields the asserted representation.

Equipped with an explicit representation of affine tensors we can represent the counit and unit of an adjunction $(-) \otimes A \dashv \underline{\mathrm{Aff}}_{\mathrm{K}}(A, -)$ from theorem 3.4.5(i) explicitly. As one would presumably guess they are a kind of evaluation and slice map similar to the one presented in remark 2.2.8(a).

Corollary 3.4.12. *For an adjunction $(-) \otimes A \dashv \underline{\mathrm{Aff}}_{\mathrm{K}}(A, -)$ and $C \in \mathrm{Obj}\,\mathrm{Aff}_{\mathrm{K}}$ the C-components of the counit $\varepsilon : \underline{\mathrm{Aff}}_{\mathrm{K}}(A, -) \otimes A \dot{\to} I_{\mathrm{Aff}_{\mathrm{K}}}$ is the evaluation map*

$$\begin{aligned} \varepsilon_C = \mathrm{ev}_C &: \underline{\mathrm{Aff}}_{\mathrm{K}}(A, C) \otimes A \to C, \\ f \otimes P + \sum_i \nu_i \otimes v_i &\mapsto f(P) + \sum_i \hat{\nu}_i(v_i), \end{aligned}$$

where $\nu_i \in \mathrm{Aff}_{\mathrm{K}}(A, SU_V C)$, $v_i \in V_C$ and $\hat{\nu}_i = U_V \nu_i \in \mathrm{Vct}_{\mathrm{K}}(V_A, V_C)$ for each $i \in I$; the C-component for the unit $\eta : I_{\mathrm{Aff}_{\mathrm{K}}} \dot{\to} \underline{\mathrm{Aff}}_{\mathrm{K}}(A, (-) \otimes A)$ is the slice map

$$\begin{aligned} \eta_C = \sigma_C &: C \to \underline{\mathrm{Aff}}_{\mathrm{K}}(A, C \otimes A), \\ P \mapsto s_P, \quad &\textit{with} \quad s_P : Q \mapsto P \otimes Q. \end{aligned}$$

Proof. (i) Once again there is either the possibility to show ev to be a counit explicitly or to use more abstract arguments. The first approach amounts to show ev to be a natural transformation together with the respective universal property like we have done in the proof of proposition 2.2.6 for the counit there. However, we will follow the second approach here, for it fits nicely in our general approach to study $\mathrm{Aff}_{\mathrm{K}}$ by its natural functors to and from Set and $\mathrm{Vct}_{\mathrm{K}}$.

Recall the counit ε_C to be the *left adjunct* of $1_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,C)}$; i.e. the image of $1_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,C)}$ under the inverse of the natural isomorphism from our hom-tensor

adjunction $(-)\otimes A \dashv \underline{\mathrm{Aff}}_{\mathrm{K}}(A,-)$. Recall furthermore that the representation of ε_C by a biaffine map is $\varphi_{\mathrm{Aff}_{\mathrm{K}}(A,C)}(1_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,C)})$, where

$$\varphi_{\mathrm{Aff}_{\mathrm{K}}(A,C)} : \mathrm{Set}(\mathrm{Aff}_{\mathrm{K}}(A,C), \mathrm{Set}(U_S A, U_S C)) \cong \mathrm{Set}(\mathrm{Aff}_{\mathrm{K}}(A,C)\times U_S A, U_S C)$$

is the natural bijection from the hom-product adjunction in Set and

$$j : \mathrm{Aff}_{\mathrm{K}}(\underline{\mathrm{Aff}}_{\mathrm{K}}(A,C), \underline{\mathrm{Aff}}_{\mathrm{K}}(A,C)) \hookrightarrow \mathrm{Set}(\mathrm{Aff}_{\mathrm{K}}(A,C), \mathrm{Set}(U_S A, U_S C))$$

is considered as a subfunctor[19]. Like before, we will omit the forgetful functor U_S in the notation from now on. Recall the evaluation map to be the counit for the hom-product adjunction in Set. The inclusion $\iota : \mathrm{Aff}_{\mathrm{K}}(A,C) \hookrightarrow \mathrm{Set}(A,C)$ together with the naturality of φ

$$\begin{array}{ccc}
\mathrm{Set}(\mathrm{Set}(A,C), \mathrm{Set}(A,C)) & \xrightarrow{\varphi_{\mathrm{Set}(A,C)}} & \mathrm{Set}(\mathrm{Set}(A,C)\times A, C) \\
\downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle (\iota\times 1_A)^*} \\
\mathrm{Set}(\mathrm{Aff}_{\mathrm{K}}(A,C), \mathrm{Set}(A,C)) & \xrightarrow{\varphi_{\mathrm{Aff}_{\mathrm{K}}(A,C)}} & \mathrm{Set}(\mathrm{Aff}_{\mathrm{K}}(A,C)\times A, C), \\
1_{\mathrm{Set}(A,C)} & \longmapsto & \mathrm{ev}_C \\
\downarrow & & \downarrow \\
1_{\mathrm{Aff}_{\mathrm{K}}(A,C)} & \longmapsto & \epsilon_C .
\end{array}$$

yield then the explicit description of ε_C in terms of its biaffine map[20] $\epsilon_C := \varphi_{\mathrm{Aff}_{\mathrm{K}}(A,C)}(1_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,C)})$ to be the restriction of the evaluation map ev_C

$$\epsilon_C : \underline{\mathrm{Aff}}_{\mathrm{K}}(A,C)\times A \to C, \quad (f,P)\mapsto f(P).$$

Due to the pointwise affine structure of $\underline{\mathrm{Aff}}_{\mathrm{K}}(A,C)$ we have for any $\nu \in \mathrm{Aff}_{\mathrm{K}}(A, SU_V C)$ and $v \in V_A$

$$\begin{aligned}
\epsilon_C(f+\nu, P+v) &= (f+\nu)(P+v) = f(P+v)+\nu(P+v) \\
&= f(P)+\hat{f}(v)+\nu(P)+\hat{\nu}(v).
\end{aligned}$$

This combined with the usual simply transitiveness argument shows the bilinear map $\hat{\epsilon}_C$ to be the evaluation map

$$\hat{\epsilon}_C : \mathrm{Aff}_{\mathrm{K}}(A, SU_V C)\times V_A \to V_C, \quad (\nu, v)\mapsto \hat{\nu}(v).$$

Finally, this result and the relation of the biaffine ϵ_C to ε_C via the tensor map $\otimes$ as $\epsilon_C = \varepsilon_C \circ \otimes$ together with theorem 3.4.10(ii) yield the assertion for the counit.

[19] Recall the identity $U_S \underline{\mathrm{Aff}}_{\mathrm{K}}(-,-) = \mathrm{Aff}_{\mathrm{K}}(-,-)$ which was implicitly used here. Confer also (3.15), (3.16), (3.17).

[20] Note that $1_{\mathrm{Aff}_{\mathrm{K}}(A,C)} = j(1_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,C)})$.

(ii) To show the assertion for the unit it is enough to show the left adjunct of the slice map σ_C to be $1_{A\otimes C}$; for recall from section 2.2.2 that the unit η_C is the *right adjunct* of $1_{A\otimes C}$, i.e. its image under the respective natural isomorphism from our hom-tensor adjunction $(-)\otimes A \dashv \underline{\mathrm{Aff}}_{\mathrm{K}}(A,-)$. We will use our result in (i) and write the left adjunct in terms of the evaluation map (cf. (2.3) in section 2.2.2). With theorem 3.4.10(ii) and remark 3.4.11(c) this yields for $P\in C$, $Q\in A$, $v_i\in V_C$, $w_i\in V_A$

$$\begin{aligned}
&\mathrm{ev}_{C\otimes A}\circ(\sigma_C\otimes 1_A)(P\otimes Q+\sum_i v_i\otimes w_i)\\
&=\mathrm{ev}_{C\otimes A}(\sigma_C(P)\otimes Q+\sum_i \hat{\sigma}_C(v_i)\otimes w_i)\\
&=s_P(Q)+\sum_i \widehat{\hat{\sigma}_C(v_i)}(w_i).
\end{aligned}$$

Furthermore, we have $s_P(Q)=P\otimes Q$ by definition, $\hat{\sigma}_C(v_i)=(Q\mapsto v_i\otimes Q)$ by the usual simply transitiveness argument combined with the pointwise structure of the internal hom and thus $\widehat{\hat{\sigma}_C(v_i)}=(w\mapsto v_i\otimes w)$. Alltogether this indeed shows the asserted identity

$$\mathrm{ev}_{C\otimes A}\circ(\sigma_C\otimes 1_A)=1_{A\otimes C}.$$

(qed)

Corollary 3.4.13. *Let α, λ, ρ, τ be the structure morphisms of an affine tensor structure $(\mathrm{Aff}_{\mathrm{K}},\otimes,*)$ as given by corollary 3.4.6. We have*

$$\begin{aligned}
&\alpha_{A,C,D}: A\otimes(C\otimes D)\to(A\otimes C)\otimes D,\\
&\quad P\otimes(Q\otimes R)+\sum_i P\otimes(v_i\otimes w_i)+\sum_j u_j\otimes(v'_j\otimes w'_j)\\
&\quad\mapsto(P\otimes Q)\otimes R+\sum_i(P\otimes v_i)\otimes w_i+\sum_j(u_j\otimes v'_j)\otimes w'_j,\\
&\rho_A: A\otimes *\to A,\qquad P\otimes\bullet\mapsto P,\\
&\lambda_A: *\otimes A\to A,\qquad \bullet\otimes P\mapsto P,\\
&\tau_{A,C}: A\otimes C\to C\otimes A,\qquad P\otimes Q+\sum_i u_i\otimes v_i\mapsto Q\otimes P+\sum_i v_i\otimes u_i.
\end{aligned}$$

Proof. This time we will proceed the other way around and argue that when we read the above maps as definitions, then they yield together with the affine tensor bifunctor and one-point space $*$ a symmetric monoidal structure on $\mathrm{Aff}_{\mathrm{K}}$. If this is established then, by corollary 3.4.6, we just need to show that the respective translated maps for the internal hom $\underline{\mathrm{Aff}}_{\mathrm{K}}(-,-)$ are the one we used in the proof

of theorem 3.4.5. So we have to "reverse" the respective Yoneda arguments from that proof in a certain sense.

To begin with, observe with the help of theorem 3.4.10(ii) that each map is affine and an isomorphism. With remark 3.4.11(c) in mind it is obvious from the definitions that the required coherence diagrams of a symmetric monoidal category are rendered commutative by these affine maps. Finally, we need to go back to the proof of theorem 3.4.5. There we have to use the above maps and rearrange the chains of natural isomorphisms in (3.10) for τ, (3.11) for α and (3.12) for ρ in such a way that the respective maps we used for the internal hom in that proof are expressed by the hom-tensor adjunctions and our maps. Then we need to show that the reexpressed maps are indeed the ones we used in the proof to define the structure morphisms. This, however, is seen to hold. Since only our maps and left and right adjuncts under the hom-tensor adjunction are needed to verify this, whose underlying set-theoretic maps we know by (2.3) in section 2.2.2 together with corollary 3.4.12 and remark 3.4.11(c), this is just a tedious exercise in expanding definitions and evaluating maps so we will leave out the details.

Our above left-unit law λ and right-unit law ρ are interrelated by our symmetry τ. This follows directly from the definition of the maps. It is also in this way we introduce the left-unit law in corollary 3.4.6. Alltogether we can thus conclude our assertion from corollary 3.4.6.

(qed)

Remarks 3.4.14.

(a) (Symmetric monoidality of U_S). We can state the preceding corollary equivalently as follows: the forgetful functor U_S together with

$$(U_S)_1(A,C) = \otimes_{A,C} : U_S A \times U_S C \to U_S(A \otimes C), \qquad (U_S) = 1_{U_S *},$$

is symmetric monoidal.

(b) For each triple (A, C, D) of affine spaces the associated trilinear map $\hat{\alpha}_{A,C,D}$ of the affine associativity law $\alpha_{A,C,D}$ from the preceding corollary is the respective associativity law of the linear tensor structure for (V_A, V_C, V_D) as given in (2.4) in section 2.2.3. Here the affine associativity law is considered as a triaffine map. By this we mean the obvious extension of the notion of biaffine maps to multiaffine maps. The same correspondence holds for the affine symmetric braiding $\tau_{A,C}$, its associated bilinear map and the linear symmetric braiding. Both assertions can be read off directly from the assertion of the corollary.

(c) Theorem 3.4.10(ii) yields for any affine tensor product $A \otimes C$ a linear embedding

$$(U_V)_1(A,C) : V_A \otimes V_C \hookrightarrow V_{A \otimes C}.$$

The existence of the associated bilinear map of a biaffine map implies the naturality of $(U_V)_1(A, C)$ in A and C. In fact, in consequence of the previous remark one can even show the coherence diagram for the associativity law (2.5) and symmetric braiding (2.7) of a monoidal functor to be rendered commutative by $(U_V)_1$. This can be verified by evaluation directly. However, U_V is not a symmetric monoidal functor. What is missing is a linear map $K \to U_V *$ which renders commutative the coherence diagrams (2.6). Indeed, since $U_V * = 0$ there is only one map $K \to U_V *$, namely the terminal zero map and this map doesn't and can't work, because the right- and left-unit law in Vct_K are not zero maps.

Examples 3.4.15.

(a) Theorem 3.4.10 suggests a big class of examples of biaffine maps, namely bilinear maps on a vector space V, if V is considered as an affine space over itself, i.e. as SV for the self-action functor S. Indeed, let U, V, W be vector spaces in Vct_K and $\beta : U \times V \to W$ a bilinear map, then β can be regarded as a biaffine map $SU \times SV \to SW$. Such a situation is typical for analysis or geometry where one uses $\mathbb{R}^n$ or other vector spaces at one time as a vector space and at the other time as an affine space.

(b) Another example of a biaffine map is the composition of affine maps

$$\underline{\mathrm{Aff}}_\mathrm{K}(C, D) \times \underline{\mathrm{Aff}}_\mathrm{K}(A, C) \to \underline{\mathrm{Aff}}_\mathrm{K}(A, D), \qquad (f, g) \mapsto f \circ g.$$

Although one can check this explicitly, the reason for this is a purely categorical one: for any closed symmetric monoidal category we can define an internal composition for the internal hom with the help of the counit of the respective hom-tensor adjunction. The idea behind this is the formalization of the observation that a composite of set-theoretic maps can be understood as a sequence of evaluations. In our case this amounts to the affine map as defined by the following diagram

$$\begin{array}{ccccc} (\underline{\mathrm{Aff}}_\mathrm{K}(C, D) \otimes \underline{\mathrm{Aff}}_\mathrm{K}(A, C)) \otimes A & & & & \\ \cong \downarrow & & & \searrow & \\ \underline{\mathrm{Aff}}_\mathrm{K}(C, D) \otimes (\underline{\mathrm{Aff}}_\mathrm{K}(A, C) \otimes A) & \xrightarrow[1_{\underline{\mathrm{Aff}}_\mathrm{K}(C,D)} \otimes \mathrm{ev}_C]{} & \underline{\mathrm{Aff}}_\mathrm{K}(C, D) \otimes C & \xrightarrow[\mathrm{ev}_D]{} & D. \end{array}$$

The right adjunct of this map is then the internal composition $\underline{\mathrm{Aff}}_\mathrm{K}(C, D) \otimes \underline{\mathrm{Aff}}_\mathrm{K}(A, C) \to \underline{\mathrm{Aff}}_\mathrm{K}(A, D)$. With our explicit representation of the evaluation map in the pervious corollary and theorem 3.4.10 it is not difficult to see the respective biaffine map to be our example from above; just evaluate the above diagram on points $f \otimes g$.

Remarks 3.4.16.

(a) The example (a) may suggest that there is a correspondence between biaffine maps and bilinear maps like the one given proposition 3.1.3(ii) for affine maps; i.e. that each biaffine map $f : A \times C \to E$ is completely and uniquely determined by a bilinear map $V_A \times V_C \to V_E$ and a value $f(P, Q)$ for a fixed point $(P, Q) \in A \times C$. This is however not the case as one can see from the biaffine tensor map $t : A \times C \to S(V_C \oplus V_A \oplus V_A \otimes V_C)$ with $t(P, Q) = 0$, which is given explicitly in the proof of theorem 3.4.10. Consider e.g. the biaffine map

$$t' : A \times C \to S(V_C \oplus V_A \oplus V_A \otimes V_C), \quad (U, R) \mapsto \overrightarrow{PU} \otimes \overrightarrow{QR},$$

then we have $\hat{t} = \hat{t'}$ for the bilinear maps, $\hat{t}(-, Q) = U \mapsto \overrightarrow{PU}$ with $\overrightarrow{PU} \neq 0$ for $P \neq U$ but $\hat{t'}(-, Q) = 0$.

(b) The argument from the previous remark shows that the converse of remark 3.4.9(b) does not hold; i.e. the associated bilinear map of a biaffine map doesn't uniquely determine the componentwise associated linear maps.

(c) (Affine quadrics and quadratic forms.) With the notion of a biaffine map one can define a notion of an affine quadratic form which extends the notion of an ordinary quadratic form in the same way as the notion of biaffinity extends bilinearity. Let A be a non-empty affine space in FAff_K and regard K as an affine line. A map $q : A \to K$ is said to be an affine quadratic form when there is a biaffine map $f : A \times A \to K$, such that $q = f \circ \Delta$, where $\Delta : A \to A \times A$, $P \mapsto (P, P)$ denotes the diagonal map. Each affine quadric $Q \subset A$ is then the locus of zeros of a respective affine quadratic form $q : A \to K$ and the quadratic form can be chosen in such a way that the biaffine f is symmetric, i.e. $f(P, Q) = f(Q, P)$ for all $Q, P \in A$. Indeed, choosing an affine coordinate system, i.e. a basis point in A together with an ordered basis of V_A, shows that affine quadratic forms $q : A \to K$ correspond one-to-one to polynomials in $K[X_1, \ldots, X_n]$ of degree atmost 2 for $n = \dim A$. A particular example of an affine quadratic form is the one induced by the biaffine inertia tensor in rigid body mechanics.

With the notion of an affine quadratic form we have the possibility to treat affine quadrics in the same coordinate free manner like it is possible for projective quadrics with ordinary quadratic forms. This was for example done in [MB99, appendix, sections 5 and 7]. For an exhaustive classification of quadrics we refer to [KM03].

(d) (Affine Fréchet derivative). An application of the closed affine tensor structure to analysis can be given with the reinterpretation of the Fréchet derivative as an affine concept. Let E, F be normed K-vector spaces, $K \in \{\mathbb{R}, \mathbb{C}\}$ and

$U \subset E$ open. For convenience we will assume the normed vector spaces to be finite dimensional here. Recall that a map $f : U \to F$ is said to be (Fréchet) differentiable in $x_0 \in U$ when there is a continuous linear map $\partial f(x_0) : E \to F$, the derivation of f at x_0, such that (cf. [AE99b, VII.2.2(a)])

$$f(x_0 + h) = f(x_0) + \partial f(x_0)h + o(\|h\|) \qquad (h \to 0).$$

The map f is said to be (Fréchet) differentiable when it is differentiable in every $x_0 \in U$. Fréchet differentiability and the Fréchet derivative depend only on the norm-topologies of the vector spaces and not on the particular norms (cf. [AE99b, VII.2.2(c)]). As for finite dimensional K-vector spaces all norms are equivalent, i.e. there is a unique norm-topology on each vector space, we can abandon the norms whatsoever. Moreover, in finite dimensions each linear map is continuous.

In analysis the notion of Fréchet derivative builds on the idea of linear approximation. A closer look at the definition shows it to be an affine approximation, however. Indeed, we are precisely in the situation mentioned in example (a). The structurally correct setting isn't that of finite dimensional K-vector spaces but of finite dimensional affine spaces over K instead. More precisely, we need affine spaces which associated K-vector spaces carry its unique norm-topology; but due to the uniqueness of this topology the respective category is isomorphic to FAff_K. Hence, like with the norms, we don't need to bother with. The interesting part is now about higher derivatives. Given a differentiable $f : U \to E$ we can ask about the differentiability of its first derivative $\partial f : U \to \underline{\mathrm{Vct}}_\mathrm{K}(E, F)$. If this map is differentiable we get a map $\partial^2 f : U \to \underline{\mathrm{Vct}}_\mathrm{K}(E, \underline{\mathrm{Vct}}_\mathrm{K}(E, F)) \cong \underline{\mathrm{Vct}}_\mathrm{K}(E \otimes E, F) \cong Bilin_K(E, E; F)$ with the respective natural linear isomorphisms and $E \otimes F$ the linear tensor product. If we consider affine spaces instead and omit the functor S in the notation, then the first affine derivative is a function $df : U \to \underline{\mathrm{Aff}}_\mathrm{K}(E, F)$. It assigns to each point $x_0 \in U$ the affine map

$$df(x_0) : E \to F, \quad x \mapsto f(x_0) + \partial f(x_0)\overrightarrow{x_0 x}.$$

Its derivative, i.e. the second affine derivative of f is a function

$$d^2 f : U \to \underline{\mathrm{Aff}}_\mathrm{K}(E, \underline{\mathrm{Aff}}_\mathrm{K}(E, F)) \cong \underline{\mathrm{Aff}}_\mathrm{K}(E \otimes E, F) \cong \mathrm{Biaff}_\mathrm{K}(E, E; F)$$

which assigns to each point $x_0 \in U$ the biaffine map

$$d^2 f(x_0) : E \times E \to F, \quad (x, y) \mapsto f(x_0) + \partial f(x_0)\overrightarrow{x_0 x} + \partial^2 f(x_0)[\overrightarrow{x_0 x}, \overrightarrow{x_0 y}].$$

Its associated bilinear map is $\partial^2 f(x_0)$; but $d^2 f(x_0)$ isn't just a bilinear map considered as a biaffine one, for we have

$$\widehat{d^2 f(x_0)}(x, v) = \partial f(x_0)v + \partial^2 f(x_0)[\overrightarrow{x_0 x}, v], \qquad v \in V_E = E.$$

Suppose $F = K = \mathbb{R}$, then to give a critical point of f amounts to find a point x_0, such that $d^2 f(x_0)$ is purely quadratic, i.e. bilinear with respect to x_0. Given the affine quadratic form induced by $d^2 f(x_0)$ such points with respect to which $d^2 f(x_0)$ becomes purely quadratic are called center points in [MB99]; the terminology is borrowed from the fact that the center point of an affine quadratic form which has spheres as quadrics is indeed the center of those spheres.

Observe the second order affine derivative to approximate up to order two and not only by order two, like the second order linear derivative does. Such a behaviour can also be observed with the second order tangent map in differential geometry. To be able to formulate the second order tangent map, one needs to construct the tangent bundle of the tangent bundle. This iterated tangent bundle has the first tangent bundle as a vertical subbundle and the second order tangent map, i.e. the tangent of the tangent map is seen to contain the first order tangent map as well, namely by its restriction to this vertical subbundle.

As is well-known the notion of Fréchet derivative carries over to infinite dimensional normed vector spaces without difficulties. In particular, we can consider higher derivatives as continuous multilinear maps and we have a hom-tensor adjunction, where the internal hom is the vector space of continuous linear operators together with the operator norm and the tensor product is the linear (algebraic) tensor product equipped with the so called projective norm (cf. [AE99b, VII.4.3, VII.5] and [Rya02, sections 2.1, 2.2] for details[21]). This doesn't apply to the affine notions though. We can't just replace affine spaces with affine spaces which associated vector spaces are normed and affine maps with continuous affine maps. This is due to the associated vector space $\mathrm{Aff}_{\mathrm{K}}(A, SU_V C)$ of the internal hom of continuous affine maps $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, C)$ to not carry a natural norm for normed V_A and V_C in general. Nevertheless, it carries a natural norm-topology[22] which would be sufficient to formulate Fréchet differentiability. The approach would be then to consider the category of affine spaces which associated vector spaces carry a norm-topology with continuous affine maps as morphisms instead. However, as this isn't in the scope of this thesis we won't pursue this matter any further here and postpone it to a later work.

(e) (Generalized product rule.) A further conceptual application of biaffine maps to analysis is the explanation of the generalized product rule. The generalized product rule is the consequence of the chain rule and the explicit Fréchet derivative of a (continuous) bilinear map (cf. [AE99b, VII.4.8(b)]). Let $b : E \times E' \to F$

[21] In these texts the normed vector spaces are additionally required to be complete; this isn't necessary for the cited results, however.

[22] It is the norm-topology induced by the norm $f \mapsto \|f(P)\|_{V_C} + \|\hat{f}\|$ for an arbitrary but fixed $P \in A$, where the second norm denotes the operator norm of $\hat{f}$. The topology is independent of P, since choosing a different P yields an equivalent norm.

be a bilinear map and E, E', F finite dimensional K-vector spaces equipped with their unique norm-topology, $K \in \{\mathbb{R}, \mathbb{C}\}$. The bilinear map b is then differentiable with derivative

$$\partial b(x_0, y_0) : E \times E' \to F, \quad (v, w) \mapsto b(x_0, w) + b(v, y_0).$$

This may seem odd, for recalling linear approximation as the idea behind the Fréchet derivative it seems strange to linearly aproximate a map which is thought of and is actually used as a map with a second order term only. However, this result can be perfectly understood with the notion of a biaffine map. As explained in the previous remark Fréchet differentiability belongs structurally to affine spaces and not to linear one. In fact, to derive the bilinear b we have considered it as a biaffine map. As such it does indeed have a linear part at each point (x_0, y_0), namely the (pointwise) sum of the associated linear maps of $b(x_0, -)$ and $b(-, y_0)$ (cf. remark 3.4.9(a)); but this is precisely the derivative given above. The biaffine map b is only of second order with respect to its center points and one particular such point is $(0, 0)$, which we recall to be the only (intrinsic) point of a vector space.

3.4.5 The Identity Functor as a Symmetric Monoidal Functor

All in all we have three types of symmetric monoidal structures: the Cartesian, the cocartesian and the affine tensor structure. An interesting observation is the Cartesian and the tensor structure to have the same type of identity object, namely a one-point space $*$. A one-point space is moreover a terminal object. With both facts we obtain for each A, C in Aff_K an affine map $p_{A,C} : A \otimes C \to A \times C$ by the universal property of the product as follows ($!_A : A \to *$ and $!_C : C \to *$ denote the unique terminal affine maps)

$$\begin{array}{ccc} A \otimes * & \xrightarrow{\cong} & A \\ {\scriptstyle 1_A \otimes !_C}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}_A} \\ A \otimes C & \overset{\exists! p_{A,C}}{\dashrightarrow} & A \times C \\ {\scriptstyle !_A \otimes 1_C}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_C} \\ * \otimes C & \xrightarrow[\cong]{} & C. \end{array}$$

This map is natural in A and C, because the projections of the product as well as the left- and right-unit map for the affine tensor structure are. With theorem 3.4.10(ii), remark 3.4.11(c) and corollary 3.4.13 we can give the underlying set-theoretic map $p_{A,C}$ explicitly by evaluating the above diagram on points

$$\begin{array}{ccc} P \otimes \bullet & \longmapsto & P \\ \uparrow & & \uparrow \\ P \otimes Q + \sum_i v_i \otimes w_i & \longmapsto & (P, Q) \\ \downarrow & & \downarrow \\ Q \otimes \bullet & \longmapsto & Q. \end{array}$$

This yields $p_{A,C} : A \otimes C \to A \times C$, $P \otimes Q + \sum_i v_i \otimes w_i \mapsto (P, Q)$. With this the affine maps $p_{A,C}$ can itselves be considered as projections of affine tensors to their pure point parts. Furthermore, from the explicit description one can see easily the natural transformation $p : \otimes \dot{\to} \times$ to yield the necessary structure morphism for I_{Aff_K} to become a symmetric monoidal functor from the Cartesian structure to the affine tensor structure.

Theorem 3.4.17. *With the natural transformation $p : \otimes \dot{\to} \times$ the identity functor I_{Aff_K} becomes a symmetric monoidal functor $(\mathrm{Aff}_K, \times, *) \to (\mathrm{Aff}_K, \otimes, *)$, which is not strong monoidal.*

Proof. With corollary 3.4.13 and the underlying map of each component of p it is straight forward to show the coherence diagrams commutative. Like on previous occasions we omit verifying this explicitly. The fact that I_{Aff_K} is not strong follows for example from the dimension formulas for the product and tensor product for finite dimensional affine spaces (cf. remark 3.2.5(a) and remark 3.4.7(c)), because finite dimensional affine spaces of different dimension can't be isomorphic by corollary 3.1.4 and the respective fact for vector spaces in linear algebra.

(qed)

Remark 3.4.18 (finite dimensions)**.**
Obviously, the above result restricts to FAff_K: with $p : \otimes \dot{\to} \times$ the identity functor I_{FAff_K} becomes a symmetric monoidal functor $(\mathrm{FAff}_K, \times, *) \to (\mathrm{FAff}_K, \otimes, *)$, which is not strong monoidal.

With this monoidality of the identity functor the affine tensor product $A \otimes C$ can be regarded as an enlargement of the Cartesian product $A \times C$ in Aff_K. We support this line of thought with the following example.

Example 3.4.19.
Consider the additive monoid $(K, +, 0)$ and the multiplicative monoid $(K, \cdot, 1)$, where K denotes our base field as usual. When we regard K as an affine line the additive monoid becomes an internal monoid of $(\mathrm{Aff}_K, \times, *)$, whereas the multiplicative monoid becomes an internal monoid of $(\mathrm{Aff}_K, \otimes, *)$. The multiplicative monoid isn't an internal monoid of $(\mathrm{Aff}_K, \times, *)$, but due to theorem 3.4.17 the additive monoid can be regarded as an internal monoid of $(\mathrm{Aff}_K, \otimes, *)$. Hereto,

recall a monoidal functor to lift to a functor between the respective categories of internal monoids as explained in section 2.5; $I_{\mathrm{Aff_K}}$ lifts to a (faithful) inclusion $\mathrm{Mon}(\mathrm{Aff_K}, \times, *) \hookrightarrow \mathrm{Mon}(\mathrm{Aff_K}, \otimes, *)$, in particular. The explicit embedding of $(K, +, 0)$ as an internal monoid in $(\mathrm{Aff_K}, \otimes, *)$ looks like this

$$K \otimes K \xrightarrow{p_{K,K}} K \times K \xrightarrow{+} K, \qquad * \xrightarrow{0} K.$$

The embedding $\mathrm{Mon}(\mathrm{Aff_K}, \times, *) \hookrightarrow \mathrm{Mon}(\mathrm{Aff_K}, \otimes, *)$ shows the affine tensor product to capture the internal monoids of the Cartesian structure. In this sense it enlarges the Cartesian product. This enlargement can also be observed directly on points by the affine map p, for points in an affine tensor product $A \otimes C$ can be understood by theorem 3.4.10(ii) as the points of $A \times C$ with additional bilinear information.

Internal monoids in $(\mathrm{Aff_K}, \otimes, *)$, which we will call *affine K-algebras* borrowing on K-algebras as internal monoids in $(\mathrm{Vct_K}, \otimes, K)$, will be considered in chapter 6 in more detail.

Remark 3.4.20.
Many of our results, in particular example 3.4.15(a), theorem 3.4.10, corollary 3.4.12 and corollary 3.4.13 show that the linear tensor product and the affine tensor product are strongly interrelated. From our viewpoint on $\mathrm{Aff_K}$ to describe structures between $\mathrm{Vct_K}$ and Set this comes of no suprise. We will see later that both categories $\mathrm{Aff_K}$ and $\mathrm{Vct_K}$ can be regarded as parts of one category, namely the slice category $\mathrm{Vct_K}/K$ and both tensor products can be obtained from one tensor product on $\mathrm{Vct_K}/K$ by restriction. This will constitute the algebraic viewpoint on affine spaces, for their structure will be given only in terms of linear algebraic structures, completely freed of a geometric meaning.

3.5 A Notion of Duality

In this section we will consider the functor $D : \mathrm{Aff_K}^{op} \to \mathrm{Aff_K}$ with $D := \underline{\mathrm{Aff}}_{\mathrm{K}}(-, K)$ as a dual space functor. We will show its opposite functor $D^{op} : \mathrm{Aff_K} \to \mathrm{Aff_K}^{op}$ to be its left adjoint and D to be monadic[23]. It is this latter fact which justifies to speak about a duality here for it shows the opposite category $\mathrm{Aff_K}^{op}$ to be equivalent to the category of Eilenberg-Moore algebras for the monad induced by D and D^{op}. Although Eilenberg-Moore algebras are still rather abstract structures and this result should thus be considered only as a first step towards the discovery of a more particular structure which respective category is equivalent to $\mathrm{Aff_K}^{op}$, we

[23] For the definition of the opposite category and opposite functor we refer to section 2.1; for the definition of monads, monadicity of a functor and Eilenberg-Moore algebras we refer to section 2.5.

won't pursue such a characterization any further here. Nevertheless, we will give an interesting application of this duality. Mimicking the approach for topoi in [MM92, section IV.5] we will derive the cocompleteness of Aff_K from its completeness.

The most famous example of a duality comes from the dual vector space construction in Vct_K. Given a K-vector space V its dual V^* is constructed via the internal hom using the field K as a so called dualizing object, i.e. $V^* = \underline{\mathrm{Vct}}_\mathrm{K}(V, K)$ (cf. the Chu-construction in [Bar79] for the structural relevance of this). It is well-known that for finite dimensional vector spaces we have $V \cong V^*$, but the isomorphism isn't natural. When we consider the double dual $V^{**} = (V^*)^*$ instead, we get a natural isomorphism, however. In categorical terms we consider the functors $\underline{\mathrm{Vct}}_\mathrm{K}(-, K) : \mathrm{FVct}_\mathrm{K}{}^{op} \to \mathrm{FVct}_\mathrm{K}$ and $\underline{\mathrm{Vct}}_\mathrm{K}(-, K)^{op} : \mathrm{FVct}_\mathrm{K} \to \mathrm{FVct}_\mathrm{K}{}^{op}$, which constitute an equivalence of categories and are adjoint to each other. Of course, we can consider those functors on Vct_K and $\mathrm{Vct}_\mathrm{K}{}^{op}$ as well, but then they won't constitute an equivalence anymore. They will still be adjoint to each other though. This is an instance of the fact that for infinite dimensional vector spaces the natural linear map $V^* \hookrightarrow V^{**}$ is in general not an epimorphism anymore.

An interesting sideremark is in order here. Considered the other way around, the adjunction $\underline{\mathrm{Vct}}_\mathrm{K}(-, K)^{op} \dashv \underline{\mathrm{Vct}}_\mathrm{K}(-, K)$ restricts to an equivalence of FVct_K and $\mathrm{FVct}_\mathrm{K}{}^{op}$. Such a behaviour of an adjunction to restrict to an equivalence can be observed mainly for adjunctions for posets (partial orders) or preorders, the so called Galois connections[24]. In terms of the kernel and hull construction, which appear as the comonad and monad of the respective Galois connection, the equivalence becomes a bijection between the invariant objects for the kernel and hull operator respectively. Those invariant objects one usually wishes to characterize. Trivial examples are the open kernel and closure operator on the power set of a topological space. The open sets constitute the invariants for the first and the closed sets constitute the invariants for the second operator. Note the open kernel and closure operator to be not the kernel and hull operator of one Galois connection, however. The probably most famous example is that of Galois correspondence, which is an (antitone) Galois connection between the poset of subgroups of the Galois group of a field extension L/K and the poset of intermediate fields of L/K. The fundamental theorem of Galois theory says then that for finite Galois extensions L/K, the respective sets of invariants are the whole posets. Another famous example comes from algebraic geometry. It is the (antitone) Galois connection between the poset of affine subvarieties of K^n and the poset of ideals of the polynomial ring $K[X_1, \ldots, X_n]$, whereby K is an algebraically closed field. The characterization of the invariants on the poset of ideals is given by Hilbert's Nullstellensatz. They are the radical ideals. On the poset of subvarieties the invariants are the Zariski

[24] By this term we mean both the covariant or monotone and the contravariant or antitone Galois connections. For a definition of antitone Galois connection in terms of an adjunction for a preorder we refer to [Mac98, section IV.6].

closed subvarieties. These examples are only few; there are many more. However, besides the dual vector space functor there is also an important example in the general setting of adjunctions, namely the equivalence of étale spaces and sheaves on a topological space, which can be found in [MM92, section II.6].

3.5.1 The Monadicity of the Dual Space Functor D

We return to our previous discussion. If we consider K as an affine line, we can imitate the dual vector space construction in $\mathrm{Aff_K}$. Like for vector spaces this dual affine space functor turns out to be its own left adjoint modulo the contravariance.

Theorem 3.5.1. *The dual space functor $D : {\mathrm{Aff_K}}^{op} \to \mathrm{Aff_K}$, $D = \underline{\mathrm{Aff}}_{\mathrm{K}}(-, K)$ has its opposite functor $D^{op} : \mathrm{Aff_K} \to {\mathrm{Aff_K}}^{op}$ as a left adjoint. A unit $\eta : I_{\mathrm{Aff_K}} \dot{\to} DD^{op}$ and counit $\varepsilon : D^{op}D \dot{\to} I_{{\mathrm{Aff_K}}^{op}}$ are given by*

$$\begin{aligned} \eta_A : A \to \underline{\mathrm{Aff}}_{\mathrm{K}}(\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K), K), \quad P \mapsto \mathrm{ev}_P, \quad \textit{with} \\ \mathrm{ev}_P : \underline{\mathrm{Aff}}_{\mathrm{K}}(A, K) \to K, \quad f \mapsto f(P), \end{aligned}$$

and $\varepsilon_A = (\eta_A)^{op}$, for each non-empty A in $\mathrm{Aff_K}$. Moreover each η_A is monic and each ε_A is epi.

Proof. (i) From corollary 3.4.4 and the special adjoint functor theorem [Mac98, corollary V.8.2] we can conclude at once that D has a left adjoint. However, the special adjoint functor theorem needs the completeness of ${\mathrm{Aff_K}}^{op}$ and hence the cocompleteness of $\mathrm{Aff_K}$ which we want to deduce from D and its properties in corollary 3.5.6 independently. We thus show D^{op} to be a left adjoint explicitly.

An adjunction $D^{op} \dashv D$ can be obtained in a natural way from the hom-tensor adjunction and symmetric braiding τ given in theorem 3.4.5 as follows

$$\begin{aligned} {\mathrm{Aff_K}}^{op}(D^{op}A, C) &= \mathrm{Aff_K}(C, \underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)) \cong \mathrm{Aff_K}(C \otimes A, K) \\ &\overset{\tau^*_{C,A}}{\cong} \mathrm{Aff_K}(A \otimes C, K) \cong \mathrm{Aff_K}(A, \underline{\mathrm{Aff}}_{\mathrm{K}}(C, K)) \\ &= \mathrm{Aff_K}(A, DC), \end{aligned} \tag{$*$}$$

where all bijections are natural in A and C. To get the asserted map η_A for a non-empty affine space A in $\mathrm{Aff_K}$ one has to consider the left adjunct of $1_{D^{op}A} \in {\mathrm{Aff_K}}^{op}(D^{op}A, D^{op}A)$ for this adjunction. This amounts to give the image of $1_{D^{op}A} = 1_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,K)}$ under the above chain of bijections for $C = D^{op}A$. Let $\varphi_{A,K} : \mathrm{Aff_K}(C \otimes A, K) \cong \mathrm{Aff_K}(C, \underline{\mathrm{Aff}}_{\mathrm{K}}(A, K))$ denote the natural bijection of the hom-tensor adjunction in $\mathrm{Aff_K}$ and σ, ev its unit and counit. We can then write η_A as follows

$$\begin{aligned} \eta_A &= \varphi_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,K),K}(\varphi^{-1}_{A,K}(1_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,K)}) \circ \tau_{A,C}) \\ &= ((\mathrm{ev}_A \circ 1_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,K)} \otimes 1_A) \circ \tau_{A,C})_* \circ s_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,K)} \\ &= (\mathrm{ev}_A \circ \tau_{A,C})_* \circ s_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,K)}. \end{aligned}$$

Here we have used the representation of φ in terms of the unit and φ^{-1} in terms of the counit as given by (2.3) in section 2.2.2. The lower $*$-index means the covariant homming with the respective affine map and is used here as a shorthand notation for the action of the respective covariant internal hom functor on affine maps. For a point $P \in A$ this yields

$$\eta_A(P) = \mathrm{ev}_A \circ \tau_{A,C} \circ (s_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,K)}(P)),$$

which we have to show to be the evaluation-at-P map. Since the underlying (set-theoretic) maps of the affine maps on the right-hand side are given by corollary 3.4.12 and corollary 3.4.13 this can be seen by evaluation on points:

$$\begin{aligned} \mathrm{ev}_A \circ \tau_{A,C} \circ (s_{\underline{\mathrm{Aff}}_{\mathrm{K}}(A,K)}(P))(f) &= \mathrm{ev}_A \circ \tau_{A,C}(P \otimes f) \\ &= \mathrm{ev}_A(f \otimes P) \\ &= f(P), \qquad\qquad f \in \underline{\mathrm{Aff}}_{\mathrm{K}}(A,K),\ P \in A. \end{aligned}$$

The components ε_A of the counit follow from this as η_A^{op}, for going forward or backward in $(*)$ for $C = \underline{\mathrm{Aff}}_{\mathrm{K}}(A,K) = DA = D^{op}A$ yields the same result up to dualization.

(ii) Finally, we need to show η_A monic; for this in particular implies $\varepsilon_A = (\eta_A)^{op}$ to be epi by categorical duality. Hereto recall the affine line K to be a coseparator by theorem 3.2.12. Hence it is a separator in $\mathrm{Aff}_{\mathrm{K}}{}^{op}$ and as explained in the proof of proposition 3.2.17(i) the hom-functor $\mathrm{Aff}_{\mathrm{K}}{}^{op}(K,-) = \mathrm{Aff}_{\mathrm{K}}(-,K)$ is faithful. Due to $U_S\, \underline{\mathrm{Aff}}_{\mathrm{K}}(-,K) = \mathrm{Aff}_{\mathrm{K}}(-,K)$ and U_S being faithful this shows $\underline{\mathrm{Aff}}_{\mathrm{K}}(-,K)$ to be faithful, too. The dual of theorem IV.3.1 in [Mac98] then shows η_A monic.

(qed)

Remarks 3.5.2.

(a) Observe that the only result where it was important to have an affine line as a dualizing object was to proof D to be faithful and equivalently the unit η_A being monic for each A in $\mathrm{Aff}_{\mathrm{K}}$.

(b) Although we said D to be its own left adjoint modulo contravariance, it isn't its own right adjoint. Indeed, D can't have a right adjoint for it doesn't preserve initial objects and thus can't be cocontinuous (cf. dual of [Mac98, theorem V.5.1]). In more detail, recall the initial object in $\mathrm{Aff}_{\mathrm{K}}{}^{op}$ to be a one-point space $*$; but $\underline{\mathrm{Aff}}_{\mathrm{K}}(*,K) \cong K$ has at least two points. In particular, $\underline{\mathrm{Aff}}_{\mathrm{K}}(*,K)$ can't be an empty space and hence an initial object by theorem 3.2.12. A similar argument shows that D^{op} can't have a left adjoint.

(c) Contrary to the case of vector spaces an affine space in FAff_K is never isomorphic to its dual space. This follows from the the dimension formula (3.13), for it yields

$$\dim DA = (\dim A + 1)\dim K = \dim A + 1, \qquad A \in \operatorname{Obj}\mathrm{FAff}_\mathrm{K}\,.$$

In fact, for each affine space A in Aff_K the natural embedding $\eta_A : A \hookrightarrow DD^{op}A$ is never epi. To see this, recall $K \cong 0 + 1$ in Aff_K where 0 and 1 are different one-point spaces and thus different points in K. Consider the constant affine map $\tilde{0} : \underline{\mathrm{Aff}}_\mathrm{K}(A, K) \to K$ which sends every map to 0. Suppose there is a $P \in A$ such that $\mathrm{ev}_P = \tilde{0}$, then we have $0 = \mathrm{ev}_P(f) = f(P)$ for all $f \in \mathrm{Aff}_\mathrm{K}(A, K)$; but there is a constant map $\tilde{1} \in \mathrm{Aff}_\mathrm{K}(A, K)$ which sends each $Q \in A$ to 1. In particular, we get the contradiction $0 = \mathrm{ev}_P(\tilde{1}) = 1$. This shows the asserted $P \in A$ to not exist and thus η_A not being surjective. By theorem 3.2.10 this is equivalent to η_A not being epi.

Although the notion of a dual affine space was motivated by the notion of a dual vector space, the last remark shows that the affine dual space functor behaves in a very different way than the dual vector space functor. In particular, we don't have the type of duality for affine spaces like we have for vector spaces. According to our general approach to see affine spaces as structures between vector spaces and sets, we will try to compare D to a duality functor in Set which is of the same conceptual type as D. This approach will prove very fruitful.

Recall the coseparator in Set to be a two point set $\{0, 1\}$; so a duality functor in Set of the same conceptual type as D is $\mathrm{Set}(-, \{0, 1\})$. It is a particular representation of the contravariant power set functor, namely the one by characteristic functions. Now remark 3.5.2(c) turns out to show the first similarity between these two; because there is no bijection between a set and its power set by Cantor's diagonal argument, the natural embedding of a set in its double dual is injective but never surjective, too.

We know a lot about the contravariant power set functor. A first fact, which also holds in the general context of topoi (cf. [MM92, theorem IV.5.1]), is $\mathrm{Set}(-, \{0, 1\})$ to be monadic. The very same result holds for D as well.

Theorem 3.5.3. *The dual space functor D is monadic.*

Proof. Due to Aff_K being complete it has equalizers. By categorical duality ${\mathrm{Aff}_\mathrm{K}}^{op}$ has coequalizers. If we can show that D preserves coequalizers of reflexive pairs and reflects isomorphisms then we can apply a version of Beck's theorem to conclude the assertion (cf. [MM92, theorem IV.4.2]). Here two morphisms $f, g : c \to d$ in a category C are said to be a *reflexive pair* when they have a common section $s : d \to c$, i.e. $fs = gs = 1_d$.

The dual space functor D is faithful (cf. remark 3.5.2(a)), hence it reflects monos and epis. If $Df = f^* : DA \to DC$ is an isomorphism in Aff_K, it is in particular

monic and epi. So $f : C \to A$ is monic and epi as well. By theorem 3.2.10 it is bijective and by corollary 3.1.4 an isomorphism in Aff_K. This shows D to reflect isomorphisms.

Next we will show D to preserve coequalizers of reflexive pairs. By categorical duality this amounts to show that given an equalizer $e : E \to A$ of two affine maps $f, g : A \to C$ in Aff_K, such that there is an affine $s : C \to A$ with $sf = 1_A$, $sg = 1_A$, $e^* : DA \to DE$ is a coequalizer of $f^*, g^* : DC \to DA$. We will do this in two steps. First of all we will show e^* to be surjective. Looking at points of DE, i.e. affine maps $E \to K$, this is equivalent to say that each affine $\omega : E \to K$ has an extension to $\tilde{\omega} : A \to K$. Then we can show that given an $h : DA \to E'$, such that $hf^* = hg^*$, h maps each two such extensions of ω to the same point. With this we will construct a unique affine $t : DE \to E'$, such that $te^* = h$. This concludes our proof.

(i) We assume w.l.o.g. E to be non empty; since otherwise DE is the one-point set which shows $e^* : DA \to DE$ surjective in a trivial way. Recall that an equalizer $e : E \to A$ is in general monic and thus injective in our case. Accordingly, it is meaningful to speak about extensions of $\omega : E \to K$ to A. In the case of ω being a constant map set $\tilde{\omega} : A \to K$ to be the constant map with the same value as ω; otherwise fix two distinct points $P, Q \in K$ and consider the preimages $\omega^{-1}(P)$, $\omega^{-1}(Q)$. These are non-empty affine subspaces of E with associated vector space $\ker \hat{\omega}$, for ω is surjective if it isn't a constant map. More concretely both are parallel disjoint hyperplanes of E. Due to the geometric interpretation of the coseparator property of K we know that there are two parallel and disjoint hyperplanes $H_P, H_Q \leq A$ such that $e(\omega^{-1}(P)) \leq H_P$ and $e(\omega^{-1}(Q)) \leq H_Q$ (cf. the explanation given after the proof of theorem 3.2.12). To see this consider $e(\omega^{-1}(P))$ and $e(\omega^{-1}(Q))$ as images of two realizations of an appropriate abstract affine space $i_P, i_Q : F_S X \to A$. Such an $F_S X$ exists due to F_S being isomorphism-dense (cf. remark 3.3.1(c)) and $e(\omega^{-1}(Q)) \cong e(\omega^{-1}(Q))$ in Aff_K. By the coseparator property there is an $\alpha : A \to K$ such that $\alpha i_P \neq \alpha i_Q$. Fix an $O \in F_S X$ and set $H_P := \alpha^{-1}(\alpha i_P(O))$, $H_Q := \alpha^{-1}(\alpha i_Q(O))$.

Define an affine $\tilde{\omega} : A \to K$ by $\tilde{\omega}(R) = P$ for $R \in H_P$ and $\tilde{\omega}(R) = Q$ for $R \in H_Q$, then by construction we have $\tilde{\omega} e = \omega$. This is due to an affine map $E \to K$ to be defined uniquely by two parallel disjoint hyperplanes where it restricts to constant maps and by its respective values on that hyperplanes. In more technical terms consider for example the quotient map $p_V : A \to A/V$ for $V = U_V H_P = U_V H_Q = \ker \hat{\alpha}$. Theorem 3.3.7 applied to α then shows $A/V \cong K$ to be an affine line with $p_V(H_P)$ and $p_V(H_Q)$ as distinct points. It is a free affine space over $\{p_V(H_P), p_V(H_Q)\}$, so the set-theoretic map

$$\{p_V(H_P), p_V(H_Q)\} \to K, \qquad p_V(H_P) \mapsto P, \quad p_V(H_Q) \mapsto Q$$

yields a unique affine map $\beta : A/V \to K$ with $\beta \circ p_V(R) = P$ for $R \in H_P$ and $\beta \circ p_V(R) = Q$, $R \in H_Q$. Set $\tilde{\omega} := \beta \circ p_V$. For the second assertion let $\mu : E \to K$ be any affine map with $\mu|H_P$ being the constant map with value P and $\mu|H_Q$ being the constant map with value Q, then $\ker \hat{\mu} = \ker \hat{\omega}$. By theorem 3.3.7 μ and ω factor through $E/\ker \hat{\omega} \cong K$ as $\tilde{\mu} \circ p_{\ker \hat{\omega}}$ and $\tilde{\omega} \circ p_{\ker \hat{\omega}}$ with uniquely determined $\tilde{\mu}$ and $\tilde{\omega}$; but like before $E/\ker \hat{\omega}$ is free over $\{p_{\ker \hat{\omega}}(H_P), p_{\ker \hat{\omega}}(H_Q)\}$ on which both $\tilde{\mu}$ and $\tilde{\omega}$ agree, so they have to be equal. We conclude $\omega = \mu$.

(ii) Let $h : DA \to E'$ be an affine map in $\mathrm{Aff_K}$ which equalizes f^* and g^*. Consider $\alpha, \beta : A \to K$ such that $e^*\alpha = e^*\beta$, then we have $f^*s^*\alpha = (sf)^*\alpha = \alpha$ and $g^*s^*\beta = \beta$ for the common retraction $s : C \to A$ from above. Accordingly, we obtain

$$h(\alpha) = hf^*(s^*\alpha) = hg^*(s^*\beta) = h(\beta).$$

With this we can define a set-theoretic map $t : DE \to E'$ by $\omega \mapsto h(\tilde{\omega})$ where $\tilde{\omega} \in (e^*)^{-1}(\omega)$ is an arbitrary extension of ω from (i). By construction we have $te^* = h$ as set-theoretic maps and t is unique with this property due to e^* being surjective by (i) and thus epi in Set. We just need to show t affine, because the faithfullness of U_S will yield its uniqueness as an affine map with the property $te^* = h$ as well. Recall $U_V \underline{\mathrm{Aff}}_{\mathrm{K}}(E, K) = \mathrm{Aff_K}(E, K)$ with its pointwise K-vector space structure and $\widehat{l^*} = l^*$ for any affine map l in $\mathrm{Aff_K}$. We can repeat the previous argument for $\hat{h} : \mathrm{Aff_K}(A, K) \to V_{E'}$ and $\alpha, \beta : A \to K$ such that $e^*\alpha = e^*\beta$ to get $\hat{h}(\alpha) = \hat{h}(\beta)$. Therefore, on the level of associated vector spaces we can define a set-theoretic map $\hat{t} : \mathrm{Aff_K}(E, K) \to V_{E'}$ by $\nu \mapsto h(\tilde{\nu})$ with $\tilde{\nu} \in (e^*)^{-1}(\nu)$ as well. We will show $\hat{t}$ linear. Let $\omega, \nu \in \mathrm{Aff_K}(E, K)$ and $\lambda, \mu \in K$. Furthermore, let $\tilde{\omega}$ and $\tilde{\nu}$ be respective extensions of ω and ν, then due to the pointwise operations one can see $\lambda\tilde{\omega} + \mu\tilde{\nu}$ to be an extension of $\lambda\omega + \mu\nu$. From this we obtain $\hat{t}$ to be linear:

$$\hat{t}(\lambda\omega + \mu\nu) = \hat{h}(\lambda\tilde{\omega} + \mu\tilde{\nu}) = \lambda\hat{h}(\tilde{\omega}) + \mu\hat{h}(\tilde{\nu}) = \lambda\hat{t}(\omega) + \mu\hat{t}(\nu).$$

Basically, the same argument shows t to be affine with associated linear map $\hat{t}$

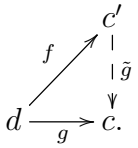
$$t(\omega + \nu) = h(\tilde{\omega} + \tilde{\nu}) = h(\tilde{\omega}) + \hat{h}(\tilde{\nu}) = t(\omega) + \hat{t}(\nu).$$

(qed)

Remarks 3.5.4.

(a) Part (i) of the previous proof together with theorem 3.2.10 show D to preserve epis. To see this recall an epi in $\mathrm{Aff_K}^{op}$ to be a mono in $\mathrm{Aff_K}$ and hence an injective map by theorem 3.2.10. Since in part (i) of the proof only the injectivity of the affine map $e : E \to A$ has been used, the argument holds for any mono e in $\mathrm{Aff_K}$ and shows e^* to be surjective, i.e. to be epi in $\mathrm{Aff_K}$ by theorem 3.2.10.

(b) The property of D to preserve epis is equivalent to the affine line K being an injective object. Recall that in a category C an object c is said to be *injective* when for any mono $f : d \to c'$ and any morphism $g : d \to c$ there is a morphism $\tilde{g} : c' \to c$ such that $\tilde{g}f = g$.



The idea behind this notion is that of an extension property. It was also such an extension property which we saw to hold for affine maps with codomain K in the proof of theorem 3.5.3. The fact that the injectivity of K translates to D preserving epis holds in more generality for categories with small hom-sets: c is injective if and only if $C(-, c) : C^{op} \to \mathrm{Set}$ preserves epis.

(c) A careful analysis of the proof of theorem 3.5.3 reveals that we only needed the affine map $e : E \to A$ to be monic and to equalize $f, g : A \to C$, but we never used the universal property of e to be an equalizer. This may appear strange at first sight, yet it indicates the following fact. In Aff_K each mono is an equalizer of two maps.

To see this let $f : A \to C$ be monic in Aff_K and consider the quotient map $p : C \to C/\operatorname{im} \hat{f}$ together with the constant map $g : C \to C/\operatorname{im} \hat{f}$ with value $p(f(Q))$ for an arbitrary $Q \in A$. Here we have assumed w.l.o.g. A to be non-empty; since otherwise f is the equalizer of the two natural inclusion maps $C \to C + C$. We will show f to be the equalizer of p and g. By theorem 3.2.10 we can think of f as the inclusion of the affine subspace $\operatorname{im} f \leq A$. The quotient map p is constructed in such a way that it sends $\operatorname{im} f$ to $p(f(Q))$. This shows f to equalize p and g. Suppose there is now an affine $h : E \to C$ in Aff_K which equalizes p and g. Due to $p^{-1}(p(f(Q))) = \operatorname{im} f$ we have then $\operatorname{im} h \subset \operatorname{im} f$. This shows f to be an equalizer indeed.

In the general categorical setting a mono which is also an equalizer is called *regular* and in a given category not every mono needs to be regular in general (cf. [AHS04, remark 7.60]). The above result then states that in Aff_K every mono is regular.

Remark 3.5.5 (finite dimensions)**.**
Consider the restriction and corestriction of the dual space functor D to $D : \mathrm{FAff}_K{}^{op} \to \mathrm{FAff}_K$. Then $D^{op} : \mathrm{FAff}_K{}^{op} \to \mathrm{FAff}_K$ and the adjunction from theorem 3.5.1 restricts to finite dimensions. As one can see from the proof of theorem 3.5.3 the monadicity of D holds for this D as well.

For the duality functor in Set we know a much more concrete structure than that of an Eilenberg-Moore algebra which respective category is equivalent to Set^{op}: it is the category CABool of (small) complete atomic Boolean algebras with morphism the Boolean algebra homomorphisms which preserve arbitrary meets (infima) and joins (suprema). The term "complete" refers to "complete lattice", which means the Boolean algebra to have meets and joins for an arbitrary subset. The term "atomic" is a further order-theoretic notion. It means for a Boolean algebra that for any non-zero element $b > 0$ there is an atom a such that $0 < a \leq b$, where for a to be an atom it means that it is a minimal element with the property to be greater than zero.

To make this more obvious consider the two-point set as the field $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. This makes each $\mathrm{Set}(X, \mathbb{Z}_2)$ into a $\mathbb{Z}_2$-vector space. Additionally, consider the bijection $\mathbb{Z}_2 \to \mathbb{Z}_2$, $x \mapsto 1 - x$. With this as a complement $\mathbb{Z}_2$ becomes a complete atomic Boolean algebra 2 and so does $\mathrm{Set}(X, 2)$. The latter is the Boolean algebra of characteristic functions and a representation of the well-known set-theoretic Boolean algebra structure on the power set of X; the supremum of an arbitrary subset (of the power set) is given by its union, the infimum by its intersection and the atoms are precisely the one-element subsets of X. This lifts our duality functor to a functor $\mathrm{Set}^{op} \to \mathrm{CABool}$, for recall $f^* = \mathrm{Set}(f, 2)$ for a map $f : X \to Y$ to preserve pointwise structures. From the explicit knowledge of what atoms are in a power set it becomes obvious how to recover a set from a complete atomic Boolean algebra, namely as the set of its atoms[25]. However, the morphisms in CABool don't need to map atoms to atoms, so this construction isn't functorial. Nonetheless, with the result of monadicity of $\mathrm{Set}(-, \{0, 1\})$ one can use this insight to show the respective category of Eilenberg-Moore algebras to be equivalent to CABool. This is for example done in [Joh82, proposition VI.4.3].

For affine spaces we know $DA = \underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ to be a K-vector space with pointwise structure. The abstract reason behind this is $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, -)$ to preserve products and thus it being a strong symmetric monoidal functor for a Cartesian monoidal structure of $\mathrm{Aff}_{\mathrm{K}}$. Furthermore, in example 3.4.19 $(K, +, 0)$ was seen to be an internal monoid of $(\mathrm{Aff}_{\mathrm{K}}, \times, *)$, hence $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ becomes an internal monoid, too and it is not difficult to see that $\mathrm{Mon}(\mathrm{Aff}_{\mathrm{K}}, \times, *)$ is equivalent to $\mathrm{Vct}_{\mathrm{K}}$; indeed,

[25] From the duality approach alone it would be natural to consider the functor $\mathrm{CABool}(-, 2) : \mathrm{CABool} \to \mathrm{Set}^{op}$. However, together with $\mathrm{Set}(-, 2)$ this wouldn't yield an equivalence of categories. Nevertheless, this functor is of equal importance for the representation theory of Boolean algebras. The set $\mathrm{CABool}(A, 2)$ can be understood as the set of maximal ideals of A. Since for Boolean algebras maximal ideals and prime ideals coincide it can be equivalently understood as the prime spectrum of the Boolean algebra A. Enriched with the so called Stone- or kernel-hull topology it becomes a totally disconnected compact Hausdorff space, the Stone space of A. For the bigger category of Boolean algebras this construction is part of Stone's representation theorem for Boolean algebras. Note the analogies with Gelfand's representation theory of commutative Banach algebras and the duality between affine schemes and commutative (unital) rings.

any pointed affine space becomes a vector space in a natural way. So far, this matches with the duality functor in Set. Another structure on K is the multiplication. In example 3.4.19 we have seen the multiplicative monoid $(K, \cdot, 1)$ to be an internal monoid of the affine tensor structure $(\mathrm{Aff}_K, \otimes, *)$. However, $\underline{\mathrm{Aff}}_K(A, -)$ isn't monoidal for the affine tensor structure, so the multiplicative structure isn't preserved by D. Of course, one can check the pointwise product of affine maps to not being affine in general explicitly. As vector spaces alone doesn't seem to be sufficient to characterize the Eilenberg-Moore algebras of DD^{op} one needs to study the structure of dual affine spaces in more depth. In particular, one has to deal with the difficulty that contrary to the situation for sets an affine space A itself is not represented in DA in an obvious way.

Although it is an interesting problem, we won't pursue this any further here and postpone it to a later work. Instead, we will apply our results about the dual space functor D and give a second proof of the the cocompleteness of Aff_K and the finitary cocompleteness of Aff_K. The interesting fact is that via duality we can conclude both properties solely from the completeness of Aff_K and finitary completeness of FAff_K, respectively.

Corollary 3.5.6. Aff_K *is cocomplete and* FAff_K *is finitary cocomplete.*

Proof. Essentially, we can repeat the argument in [MM92, corollary IV.5.4] for our case. We will do it for Aff_K. For FAff_K one can repeat the argument with Aff_K replaced by FAff_K, D replaced by its restriciton to FAff_K and "completeness", "small limits" with "finitary completeness", "finite limits". This is due to remark 3.5.5.

Consider the monad $T = DD^{op}$ and the category of the respective Eilenberg-Moore algebras ${\mathrm{Aff}_K}^T$. It is well-known that the forgetful functor ${\mathrm{Aff}_K}^T \to \mathrm{Aff}_K$ creates (arbitrary) limits (cf. [Mac98, exercise VI.2.2]), in particular it creates small limits. Due to the completeness of Aff_K and lemma 3.2.4, ${\mathrm{Aff}_K}^T$ is complete, too. By theorem 3.5.3 the comparison functor yields an equivalence of categories ${\mathrm{Aff}_K}^{op}$ and ${\mathrm{Aff}_K}^T$. This shows ${\mathrm{Aff}_K}^{op}$ to be complete, but the completeness of ${\mathrm{Aff}_K}^{op}$ is equivalent to the cocompleteness of Aff_K by categorical duality.

(qed)

3.5.2 On the Geometric Meaning of a Dual Affine Space

Until now we have developed the notion and properties of a dual affine space by a conceptual analysis based on our general approach to see affine spaces as structures between vector spaces and sets. The last result shows that they are of significance for the structure of affine spaces. As our viewpoint on affine spaces is a geometric one here, we should ask about the geometric meaning of a dual affine space.

To begin with we will consider the dual vector space V^* once again. As it stands it is of course just an algebraic construction devoid of any geometric meaning.

Nonetheless, it is related to a certain geometric duality: the point-hyperplane duality in projective geometry. If one models a projective space on a K-vector space, that is as the set $P(V)$ of one-dimensional subspaces of V with an incidence structure induced by the linear structure of V (cf. [KM03, sections 8.4f] for the details), then in finite dimensions $V \cong V^*$ yields a point-hyperplane duality of projective geometry. This duality can only be made precise in the framework of synthetic projective geometry and is like categorical duality a syntactical type of duality.

To see this in more detail, note that for $\alpha \in V^* \setminus \{0\}$ the projective space $P(\ker \alpha)$ can be considered as a projective hyperplane of $P(V)$. Given $\alpha, \beta \in V^*$ by the fundamental theorem on homomorphisms for vector spaces (2.1) we have $\ker \alpha = \ker \beta$ if and only if $\alpha = \lambda\beta$ for a $\lambda \in K$. This allows us to see $P(V^*)$ as to have projective hyperplanes of $P(V)$ as points. Any isomorphism $V \cong V^*$ yields a projectivity $P(V) \cong P(V^*)$. As projectivities preserve incidence relations, which, loosely speaking, projective geometry is all about, we can transform any statement of projective geometry about points to respective statements about projective hyperplanes and vice versa. Note that as long as one doesn't use properties special to the underlying vector spaces such a transformation is independent of the particular projectivity $P(V) \cong P(V^*)$; but to make this truly rigorous one needs to formulate projective geometry axiomatically.

In the affine case one can try to understand affine maps $A \to K$ as hyperplanes as well. Of course, this works only for the affine maps that aren't constant. Moreover, to be able to speak about a particular affine hyperplane associated to such an affine map $\omega : A \to K$ we need to consider the affine line as a pointed affine space $0 : * \to K$. We can then set $H_\omega := \omega^{-1}(0) < A$ which becomes an affine subspace by the obvious simply transitive action of $\ker \hat{\omega}$. For $\omega, \omega' \in \underline{\mathrm{Aff}}_K(A, K)$ we have $H_\omega = H_{\omega'}$ if and only if $\hat{\omega} = \lambda\hat{\omega}'$ and $\omega'(P) = 0$ for one $P \in H_\omega$. This can be deduced from the fundamental theorem on homomorphisms (cf. theorem 3.3.7). Because the affine maps $A \to K$ are the points of $\underline{\mathrm{Aff}}_K(A, K)$, we thus can't speak about points being hyperplanes here.

To be determined uniquely a non-constant affine map $\omega : A \to K$ needs a bipointed affine line $0, 1 : * \rightrightarrows K$ with $0 \neq 1$ and the data of two parallel disjoint hyperplanes $H_0 = \omega^{-1}(0)$, $H_1 = \omega^{-1}(1)$ (cf. the proof of theorem 3.5.3). Accordingly, we may consider non-constant affine maps geometrically as a pair of disjoint parallel hyperplanes. If one doesn't want to consider the affine line as a bipointed object, then one can consider non-constant affine maps $A \to K$ as foliations of A by hyperplanes. Anyhow, this doesn't apply to constant maps and we can't thus consider all points of $\underline{\mathrm{Aff}}_K(A, K)$ like that. Moreover, in contrast to the constant affine maps non-constant affine maps do not form an affine subspace of $\underline{\mathrm{Aff}}_K(A, K)$.

A slightly different point of view would be to regard affine maps $A \to K$ as bundles. From the inherent dynamical viewpoint one may see such a map as a "space through time", where the affine line K is considered as the timeline and the fiber over each $t \in K$ is the space at the instant t. This has the advantage that

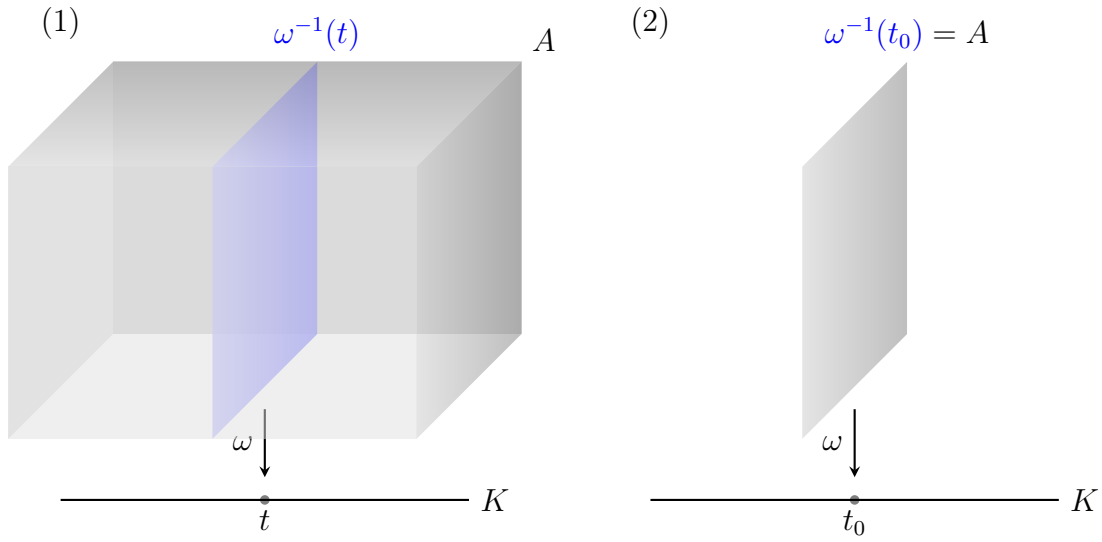


Figure 3.6: (1) A non-constant affine map $\omega : A \to K$ as a "space through time". The space at the instant t is given by $\omega^{-1}(t)$. (2) A constant map $\omega : A \to K$ with value $t_0 \in K$ as a "space through time". Since $\omega^{-1}(t) \neq \emptyset$ if and only if $t = t_0$ there is only one space A at the instant t_0.

it stays meaningful for constant maps as well; each one represents a space that exist only in one instant of time (cf. figure 3.5.2). But the clear disadvantage of this viewpoint is its focus on the affine spaces which appear as the fibers, whereas the "space through time" that are points in $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ emphasize the total space. Such an interpretation for $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ thus seems only meaningful when A can be understood as a spacetime. However, in that case the constant maps won't fit in again, because their fiber at the respective instant is the total space and not a hyperplane.

3.5.3 The Functor D as the Intensive Quantity-type of Affine Measurement Devices

The difficulty to find a consistent geometric interpretation of the whole space $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ suggests to reconsider its structure and to look for other different but related viewpoints. One such viewpoint arises with the observation that $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ is a K-vector space, provided we consider K as the additive monoid $(K, +, 0)$. The dual space functor D then becomes a functor $\mathrm{Aff}_{\mathrm{K}}{}^{op} \to \mathrm{Vct}_{\mathrm{K}}$.

Following Lawvere in [Law92] we can see D to resemble an intensive quantity-type for the category of affine spaces. To make this understandable, we will recall Lawvere's basic notions. He considers distributive categories as categories of

space and linear categories as categories of quantity. Covariant functors from categories of space to categories of quantity that preserve sums (coproducts) are regarded as extensive quantity-types, whereas contravariant functors from categories of space to categories of quantity that map sums (coproducts) to products and whose values have a multiplicative structure are considered as intensive quantity-types. The terms "extensive" and "intensive" are borrowed from thermodynamics. For a quantity-type to be extensive it means that if one joins (thermodynamic) systems to a big system (coproduct!) the quantities add, whereas to be intensive it means that subdividing a system into subsystems the quantity for each subsystem will be the same as for the composite system. Volume, mass and entropy are examples of extensive quantity-types; pressure, mass density and temperature are examples of intensive quantity-types.

Strictly speaking, D is not an intensive-quantity type in the sense of Lawvere. This is for two reasons. First, Aff_K isn't a distributive category and thus not a category of space as Lawvere considers it. Second, $\underline{\mathrm{Aff}}_\mathrm{K}(A, K)$ has not a natural multiplicative structure. Yet D is a contravariant functor and as the functor $D : {\mathrm{Aff}_\mathrm{K}}^{op} \to \mathrm{Vct}_\mathrm{K}$ it preserves small products, i.e. D maps small coproducts to products which can be seen by the continuity of $\underline{\mathrm{Aff}}_\mathrm{K}(-, K)$ and the forgetful functor U_V to preserve products (cf. proof of corollary 3.4.4 and proposition 3.2.8). We will thus continue to consider affine spaces as spaces and D as assigning to an affine space a vector space of intensive affine quantities varying over that particular space. Here the constant maps are the constant quantities. They originate from the embedding $K \cong \underline{\mathrm{Aff}}_\mathrm{K}(*, K) \hookrightarrow \underline{\mathrm{Aff}}_\mathrm{K}(A, K)$ induced by the unique map $A \to *$; that is to say that particular values of K can be considered as "variable" quantities with one particular value. On the other hand, points, that are maps $P : * \to A$, induce restriction maps $P^* : \underline{\mathrm{Aff}}_\mathrm{K}(A, K) \to \underline{\mathrm{Aff}}_\mathrm{K}(A, K) \cong K$ which are the evaluation on the respective point P; they assign a quantity its particular value at that point.

That it is indeed meaningful to consider affine maps $A \to K$ as intensive quantities and D as an intensive quantity-type can be seen as follows. Let $A = A_1 + A_2$ then D maps the natural injections $A_i \hookrightarrow A_1 + A_2$ to the linear restriction maps $\underline{\mathrm{Aff}}_\mathrm{K}(A_1 + A_2, K) \to \underline{\mathrm{Aff}}_\mathrm{K}(A_i, K)$. To restrict a map to A_i corresponds to considering the variable quantity on a subspace and as one can see this doesn't alter the quantity in any way. On the other hand, $\underline{\mathrm{Aff}}_\mathrm{K}(A_1 + A_2, K)$ together with the two restriction maps being a product says that we can join two quantities on the subspaces A_i to a quantity on the whole space $A_1 + A_2$ such that the quantities on each subspace aren't altered.

Now that we have clarified the conceptual background we will present a more specific approach. Along the same lines as in [Ozi86] we can utilize a physical point of view and regard an affine map $\omega : A \to K$ as representing a kind of affine "measurement device" for A, the evaluation at a point $P \in A$ being understood as performing a measurement at P with ω. Here the pointed space $0 : * \to K$ is considered as the gauged affine line of possible measurement results. In fact, any

pointed affine space is naturally a vector space, so the vector space structure of $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ can be understood as arising solely from the fact that we gauge the affine line of values. When we adopt the point of view of affine maps $C \to A$ to be generalized C-elements in A, which can be understood in a representation theoretic sense and generalizes the categoric notion of a point $* \to A$ as a $*$-element (cf. [MM92, section VI.1]), then the fact of K being a coseperator says that there are enough affine measuring devices to distinguish between two different C-elements in A for each C in $\mathrm{Aff}_{\mathrm{K}}$. Stated differently, each affine space can be faithfully represented by its intensive quantity type of affine measurement devices.

What do these affine measurement devices actually measure? The answer to this question depends on what the affine space under consideration shall represent. With respect to our implicit and general geometric viewpoint we will argue that it may be regarded as to measure either a current density or a length. To make this more obvious, consider a particular non-constant affine $\omega : A \to K$ for a non-empty A. Recall a displacement vector $v \in V_A$ to be a geometric object which represents a class of parallel line segments with same length and orientation. One can think of it as a linear flux in A, where each line segment represents a truncated flow line; that is the way of a test particle that it would mark out following the flux for one unit of time. This time unit is defined implicitly by the action of V_A on A. One can thus think of the algebraic vector $v \in A$ as being the velocity of its flux. Our ω defines a reference hyperplane $\omega^{-1}(0)$. For any test particle we place in the flux represented by v at the reference hyperplane, say at $P \in \omega^{-1}(0)$, after one time unit it will reach $P + v$ and ω will give us a value $\omega(P + v) = \hat{\omega}(v)$. This value measures which amount of hyperplanes parallel to the reference hyperplane have been penetrated by the test particle in one time unit. It can thus be considered as measuring the strength of the current of the flux at that particular point and is seen to be independent of this particular point P as long as that point lies in the reference hyperplane at least. Indeed, everything of this applies only with respect to the foliation by the hyperplanes of ω. This is why ω can be regarded as to measure a current density of the respective flux.

In fact, with respect to this point of view v should be considered as the velocity of a current density of its flux and ω as to measure one particular component of v at a given location in space. This is because we are given not only a foliation by hyperplanes but a time unit as well. However, as the time unit is given by the action of V_A on A and thus in contrast to the foliation it is independent of the particular ω, we will consider ω to measure current densities in space. One has to be very careful about this subtle point of the time unit within likewise physical interpretations; when v is considered as an element of V_A apriori to an action on A it has to be perceived as a respective quantity per unit of time, whereas aposteriori to an action it represents solely the resepective quantity. Another example besides current density and velocity of a current density is the distinction between a (linear) trajectory and velocity.

Concerning the role of $\hat{\omega}$ we have a similar situation like in the discussion of located and non-located line segments at the beginning of section 3.1: the linear map $\hat{\omega}$ can be understood as the particular measurement device itself, whereas ω is this measurement device located in space; an $\omega' : A \to K$ with $\hat{\omega}' = \hat{\omega}$ is this same measurement device located elsewhere.

To be able to interpret affine maps $A \to K$ as to measure lengths, we will take up this last fact of a vector to be regarded as a non-located oriented line segment. The map $\hat{\omega}$ can then be considered as to measure a length in a certain direction. We think of it as a ruler which is not located in space, but which is fixed in a certain direction. Passing to ω means to fix such a ruler in space. To be able to measure actual lengths of line segments with this one needs several such "rulers" ω_i, for each "degree of freedom" of A one, and in such a way that their reference hyperplanes intersect in a single point O, which becomes the reference point of measurement. In finite dimensions this is equivalent to say that the ω_i, $1 \leq i \leq n$ are the components of an isomorphism $A \to K^n$ in Aff_K with $n = \dim A$ such that the common intersection point is mapped to $(0, \ldots, 0)$; in other words we introduce a Cartesian system of coordinates for A.

3.5.4 Vectors as Flux-like or Line-like Quantities and their Relation to Self-dualities of the Grassmann Exterior Algebra

With the preceding discussion we want to make the following point. The fact that algebraic vectors may be used to describe flux-like or line-like quantities is already inherent in the geometric notion of a displacement vector to represent a class of parallel line segments of same length and orientation.

In physics a particular instance of this fact arises in the geometric formulation of electrodynamics in three dimensions, where Maxwell's distinction between field strengths and flux densities has to be taken seriously (cf. [HO03]). Although this physical and geometric distinction was already explained by Maxwell himself in his Treatise [Max73, articles 10ff] it seems to have been mainly forgotten. Forgetting this fact results in a confusion about there being two different electric and magnetic fields.

Structurally, the geometric fact of vectors to be able to describe flux-like or line-like quantities is reflected within the two subsequent classes of self-dualities for a Grassmann exterior algebra[26] $\bigwedge V = \bigoplus_{k=1}^{n} \bigwedge^k V$ of a finite-dimensional vector space V with $\dim V = n$. This notion of duality is related but different from the type of duality discussed before. Instead of identifying a dual category via an equivalence with another category, that is to identify structures in general, we shall identify objects with their duals in particular. In the context of finite dimensional vector spaces this means for a particular vector space V to give a vector space W and an

[26] The basic facts about the algebraic construction of the exterior algebra can be found, for example, in [KM03, section 10.4] and in appendix A.

isomorphism $W \cong V^*$, which by the linear hom-tensor adjunction is equivalent to a non-degenerate bilinear form[27] $W \otimes V \to K$. A self-duality is then a particular isomorphism $V \cong V^*$; therefore, by a self-duality of the exterior algebra we shall mean only a particular (linear) isomorphism $\bigwedge V \cong (\bigwedge V)^*$.

Both classes of self-dualities will utilize a natural isomorphism $(\bigwedge V)^* \cong \bigwedge V^*$. The class of self-dualities related to line-like quantities shall then be seen to respect degrees, i.e. we will have $\bigwedge^k V \cong \bigwedge^k V^*$ for each $0 \leq k \leq n$; the class of self-dualities related to flux-like quantities shall then be seen to flip degrees, i.e. we will have $\bigwedge^k V \cong \bigwedge^{n-k} V^*$ for each $0 \leq k \leq n$. These are the only two possible classes of self-dualities that restrict to isomorphisms $\bigwedge^k V \cong \bigwedge^\ell V^*$ between the subspaces of homogeneous elements of a fixed degree (cf. the dimension formula (3.20)).

To begin with we will construct the above mentioned natural isomorphism $(\bigwedge V)^* \cong \bigwedge V^*$. Despite the fact that within our terminology this is a duality itself, we will see this natural isomorphism as an identification and not as a duality. This is due to its naturality. In contrast, the other dualities that will be constructed constructed won't be natural; each one will involve a choice of particular extra structure.

We use the property of determinants to be non-degenerate alternating multilinear maps and introduce for each $2 \leq k \leq n$ a non-degenerate bilinear form

$$\langle -, - \rangle_k : \bigwedge^k V^* \otimes \bigwedge^k V \to K,$$

$$(\alpha_1 \wedge \ldots \wedge \alpha_k) \otimes (v_1 \wedge \ldots \wedge v_k) \mapsto \det \begin{pmatrix} \alpha_1(v_1) & \ldots & \alpha_k(v_1) \\ \vdots & \ddots & \vdots \\ \alpha_1(v_k) & \ldots & \alpha_k(v_k) \end{pmatrix} \text{ (on pure tensors).}$$

To see this map well-defined and non-degenerate recall the following facts (cf. appendix A and [Zad94, §43(7)]). Firstly, we have $v_1 \wedge \ldots \wedge v_k = 0$ if and only if $\{v_1, \ldots, v_k\} \subset V$ is a linearly dependent set. Secondly, suppose $v_1 \wedge \ldots \wedge v_k \neq 0$ we have $v_1 \wedge \ldots \wedge v_k = w_1 \wedge \ldots \wedge w_k$ if and only if $\operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{span}\{w_1, \ldots, w_k\}$ and the change of basis matrix has determinant one. This second fact shows the bilinear maps to be well defined, for replacing the linear forms or the vectors in the matrix $(\alpha_j(v_i))_{ij}$ by the respective ones such that the exterior products aren't altered is the same as multiplying the matrix $(\alpha_j(v_i))_{ij}$ from the left and the right with the respective change of basis matrices; since the determinant is a homomorphism for the matrix ring and both change of basis matrices have determinant one, the determinant of the new matrix equals $\det(\alpha_j(v_i))_{ij}$.

The non-degeneracy can be seen from the first fact as follows. Given $v_1 \wedge \ldots \wedge v_k \neq 0$ we can extend $(v_1, \ldots, v_k)$ to an ordered basis of V and define α_i as the respective

[27] From now on we will usually follow the custom to not distinguish between a linear or affine map from the respective tensor product of two spaces and its respective representation as a bilinear or biaffine map.

duals to v_i for $1 \leq i \leq k$, i.e. $\alpha_i(v_j) = \delta_{ij}$. This yields

$$\langle \alpha_1 \wedge \ldots \wedge \alpha_k, v_1 \wedge \ldots \wedge v_k \rangle_k = 1.$$

On the other hand, given $\alpha_1 \wedge \ldots \wedge \alpha_k \neq 0$ we do the same with $(\alpha_1, \ldots, \alpha_k)$ and use the natural isomorphism $V \cong V^{**}$ to get the respective v_i, $1 \leq i \leq k$.

Applying the linear hom-tensor adjunction to each bilinear map from above yields $\bigwedge^k V^* \cong (\bigwedge^k V)^*$ for each $k \geq 2$. For the other two k the isomorphisms are a consequence of the definitions $\bigwedge^0 V^* = K \cong K^*$ and $\bigwedge^1 V^* = V^*$. Due to the coincidence of finite products and coproducts in $\mathrm{FVct_K}$ we get the asserted isomorphism of K-vector spaces

$$(\bigwedge V)^* = (\bigoplus_{k=0}^{n} \bigwedge^k V)^* = \prod_{k=0}^{n} (\bigwedge^k V)^* = \bigoplus_{k=0}^{n} (\bigwedge^k V)^* \cong \bigoplus_{k=0}^{n} \bigwedge^k V^* = \bigwedge V^*.$$

Once again by the linear hom-tensor adjunction this yields a non-degenerate bilinear form

$$\begin{gathered} \langle -, - \rangle : \bigwedge V^* \otimes \bigwedge V \to K, \\ \alpha \otimes v \mapsto \begin{cases} \langle \alpha, v \rangle_k, & \text{for } \deg \alpha = \deg v = k, \\ 0, & \text{else} \end{cases} \quad (\text{for homogeneous } \alpha,\ v), \end{gathered} \tag{3.19}$$

where we recall the terminology of graded vector spaces that a $w \in \bigwedge V$ is said to be homogeneous of degree $\deg w = k$ when it lies in $\bigwedge^k V$.

We will now discuss the self-dualities that belong to the first class. Fix hereto an isomorphism $V \cong V^*$. This yields an isomorphism of finite dimensional $\mathbb{N}$-graded (unital) K-algebras $\bigwedge V \cong \bigwedge V^*$, for the universal property of the exterior algebra shows the whole construction to be functorial. Indeed, we can make it a functor

$$\bigwedge : \mathrm{FVct_K} \to \mathrm{Mon}(\mathrm{FVct_K}^{\mathbb{N}}, \otimes, e_K).$$

Here $\mathrm{Mon}(\mathrm{FVct_K}^{\mathbb{N}}, \otimes, e_K)$ is the category of internal monoids of the functor category $\mathrm{FVct_K}^{\mathbb{N}}$ equipped with the convolution-like tensor product and these internal monoids are precisely the finite dimensional $\mathbb{N}$-graded K-algebras[28]. The isomorphism $V \cong V^*$ reflects the geometric interpretation of a displacement vector as a non-located line segment, for, geometrically, such isomorphisms are induced by non-degenerate symmetric bilinear forms, which stem from quadratic forms by polarization[29]; the quadratic forms itselves are used for length measurement of line segments and originate from the geometric Pythagorean theorem for right-angled

[28] Confer the discussion of graded K-algebras in section 2.5.

[29] We recall our assumption of $\mathrm{char}\, K = 0$, but as is well-known $\mathrm{char}\, K \neq 2$ would be sufficient here.

triangles and the notion of orthogonality. We emphasize this self-duality to respect the gradation and that any other such self-duality singles out the extra structure of a non-degenerate bilinear form by the linear hom-tensor adjunction and $V = \bigwedge^1 V \cong \bigwedge^1 V^* = V^*$. Nonetheless, we don't mean to imply that each self-duality that belongs to the first class is induced by this isomorphism and the functoriality of $\bigwedge$. This is only the case when the isomorphism $\bigwedge V \cong \bigwedge V^*$ is that of $\mathbb{N}$-graded (unital) K-algebras, for the functor $\bigwedge$ is seen to be fully faithful by [Mac98, theorem IV.3.1]; one can also see this explicitly from the explicit construction of the components $\bigwedge^k f$ of the algebra homomorphism $\bigwedge f$ for a linear $f : V \to W$.

The second class of self dualities exploits the fact that for $0 \leq k \leq n$ we have

$$\dim \bigwedge^k V = \binom{n}{k} = \binom{n-k}{k} = \dim \bigwedge^{n-k} V. \tag{3.20}$$

In particular, $\bigwedge^n V$ and $\bigwedge^n V^*$ are one-dimensional vector spaces. Fix a $vol \in \bigwedge^n V^* \setminus \{0\}$ then this yields an isomorphism $\bigwedge^n V^* \cong K$ by $vol \mapsto 1$. Considered as a linear form vol can be understood as a volume measure and its dual via (3.19), that is the element $v \in \bigwedge^n V$ with $\langle vol, v\rangle = 1$, can be understood as an (oriented) unit of volume. For this interpretation to become more obvious recall the geometric ideas behind the exterior algebra (cf. also appendix A). For each $w \in \bigwedge V$ let $w \wedge (-) : \bigwedge V \to \bigwedge V$ be the linear map given by left multiplication by w. We can then consider its adjoint linear map $(-) \lrcorner w : \bigwedge V^* \to \bigwedge V^*$ with respect to the pairing in (3.19)

$$\langle \alpha \lrcorner w, v\rangle = \langle \alpha, w \wedge v\rangle, \qquad v \in \bigwedge V,\ \alpha \in \bigwedge V^*.$$

This adjoint $(-) \lrcorner w$ is commonly refered to as the right contraction by w. Its existence follows from the natural isomorphism $\bigwedge V^* \cong (\bigwedge V)^*$ induced by (3.19) and the following commutative diagram

$$\begin{array}{ccc} (\bigwedge V)^* & \xrightarrow{(w\wedge(-))^*} & (\bigwedge V)^* \\ {\scriptstyle\cong}\uparrow & & \downarrow{\scriptstyle\cong} \\ \bigwedge V^* & \xrightarrow[(-)\lrcorner w]{} & \bigwedge V^*. \end{array}$$

It is determined uniquely by $w\wedge(-)$. This is a consequence of $(-)^* = \underline{\mathrm{Vct}}_{\mathrm{K}}(-, K) : \mathrm{FVct}_{\mathrm{K}}{}^{op} \to \mathrm{FVct}_{\mathrm{K}}$ being faithful, which is due to K being a coseperator in $\mathrm{FVct}_{\mathrm{K}}$. The bilinearity of the exterior product together with the uniqueness of the adjoints $(-) \lrcorner w$ for each $w \in \bigwedge V$ show $\lrcorner$ to be bilinear. In particular, we can consider the linear map

$$vol \lrcorner (-) : \bigwedge V \to \bigwedge V^*.$$

Suppose w is a homogeneous element of degree k, then $vol \llcorner w$ is a homogeneous element of degree $n-k$. To show this we will use the adjunction and (3.19). Given any $v \in \bigwedge V$ we can write it as $\sum_{\ell=0}^{n} v_\ell$ with unique $v_\ell \in \bigwedge^\ell V$. Due to $\deg vol = n$, $\deg(w \wedge v_\ell) = k + \ell$ and (3.19) we have

$$\langle vol \llcorner w, v\rangle = \langle vol, w \wedge v\rangle = \sum_{\ell=0}^{n} \langle vol, w \wedge v_\ell\rangle = \langle vol, w \wedge v_{n-k}\rangle_n = \langle vol \llcorner w, v_{n-k}\rangle,$$

so by (3.19) $vol \llcorner w$ can have a non-trivial component only in $\bigwedge^{n-k} V^*$. This shows $vol \llcorner (-)$ to restrict to linear maps $\bigwedge^k V \to \bigwedge^{n-k} V^*$, $0 \leq k \leq n$.

We show it to be injective next, which will be a slightly more involved argument. Let $w \in \bigwedge V$ be represented as the sum of its uniquely determined homogeneous components $w_k \in \bigwedge^k V$, then $vol \llcorner w = \sum_{k=0}^{n} vol \llcorner w_k$ and for an equally represented $v \in \bigwedge V$ as $v = \sum_{\ell=0}^{n} v_\ell$ (3.19) implies $\langle vol \llcorner w, v\rangle = \sum_{k=0}^{n} \langle vol \llcorner w_k, v_{n-k}\rangle_{n-k}$. Since v is arbitrary we have $vol \llcorner w = 0$ if and only if $vol \llcorner w_k = 0$ for each $0 \leq k \leq n$. Therefore, it suffices to show the injectivity of each restriction $\bigwedge^k V \to \bigwedge^{n-k} V^*$. For $k = 0$ this follows from $\bigwedge^0 V = K$, $vol \llcorner 1 = vol$ and $vol \neq 0$; for $k = n$ this follows from $\dim \bigwedge^n V = 1$ and $vol \llcorner v = 1$, where $v \in \bigwedge^n V$ is the dual of vol. For $1 \leq k \leq n-1$ we will first show for a pure w_k the identity $vol \llcorner w_k = 0$ to imply $w_k = 0$ and then the same for an arbitrary $w_k \in \bigwedge^k V$. Since w_k is assumed pure it can be represented as $w_k = v_1 \wedge \ldots \wedge v_k$. Suppose we have $w_k \neq 0$, then as before $\{v_1, \ldots, v_k\}$ is a linearly independent subset. We can extend it to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V. This in turn implies $v_1 \wedge \ldots \wedge v_n \in \bigwedge^n V \setminus \{0\}$. Our adjunction then yields

$$0 = \langle vol \llcorner w_k, v_{k+1} \wedge \ldots \wedge v_n\rangle = \langle vol, v_1 \wedge \ldots \wedge v_n\rangle,$$

which is a contradiction to $vol \neq 0$. Now suppose $w \in \bigwedge^k V \setminus \{0\}$ to be not pure. It can then be represented as a finite sum of pure $w_i \in \bigwedge^k V$, $1 \leq i \leq \ell$. Moreover, for w to be not pure being possible we need to have $k \geq 2$. If we can show that there is a $v \in \bigwedge^{n-k} V$ such that $w \wedge v \neq 0$, then we can repeat the above argument to get a contradiction to $vol \neq 0$. For this we define for any $v \in \bigwedge V$ its *support* as the vector subspace

$$\operatorname{supp} v := \{x \in V \mid v \wedge x = 0\} \leq V.$$

For pure $v = v_1 \wedge \ldots \wedge v_k$ this vector subspace is seen to be the linear span of $v_1, \ldots, v_k$. Moreover, any $x \in \operatorname{supp} v \setminus \{0\}$ can be extended to an ordered basis $(x, x_2, \ldots, x_k)$ of $\operatorname{supp} v$ such that $v = x \wedge x_2 \wedge \ldots \wedge x_k$. In particular, pure $v, v' \in \bigwedge^k V$ are linearly dependent if and only if their supports are equal.

Suppose the w_i to be linearly independent. We will show the existence of a $v \in \bigwedge^{n-k} V$ such that $w \wedge v \neq 0$ by studying the intersection $I := \cap_{i=1}^{\ell} \operatorname{supp} w_i$ and the internal sum $\Sigma := \sum_{i=1}^{\ell} \operatorname{supp} w_i$ of the supports of the w_i. Recall the internal sum

of subvector spaces of a vector space to be the smallest vector subspace containing each of the subvector spaces, i.e. the supremum in the lattice of subvector spaces; it is the vector subspace generated by finite sums of vectors from the respective subvector spaces.

First of all, suppose the intersection I to be trivial. The sum Σ is then a direct sum. Therefore, given representations $w_i = v_1^i \wedge \ldots \wedge v_k^i$ the set $\{v_j^i \mid 1 \leq j \leq k,\ 1 \leq i \leq \ell\}$ is linearly independent and can be extended by $n - k\ell$ vectors $x_1, \ldots, x_{n-k\ell} \in V$ to a basis of V. Set

$$v := v_1^2 \wedge \ldots \wedge v_k^2 \wedge v_1^3 \wedge \ldots \wedge v_k^3 \wedge \ldots \wedge v_k^\ell \wedge x_1 \wedge \ldots \wedge x_{n-k\ell},$$

then $v \in \bigwedge^{n-k} V \setminus \{0\}$ and $w \wedge v = w_1 \wedge v \neq 0$. Suppose the intersection I to be non-trivial next and set $d := \dim I$. We have then $d \leq k-2$, for in the case of $\dim I = k$ the supports and hence the w_i would coincide up to scalars, which is in contradiction to the assumed linear independence; in the case of $\dim I = k-1$ each w_i can be written as $w_i = v_1 \wedge \ldots \wedge v_k \wedge v^i$, which by the distributivity of the $\wedge$-product yields their sum w to be pure, which is once again in contradiction to our assumption. Let $(v_1^0, \ldots, v_d^0)$ be an ordered basis of I. Each w_i can then be written as

$$w_i = v_1^0 \wedge \ldots \wedge v_d^0 \wedge v_1^i \wedge \ldots \wedge v_{k-d}^i,$$

which yields $w = v_1^0 \wedge \ldots \wedge v_d^0 \wedge (\sum_{i=1}^\ell w_i')$ with $w_i' := v_1^i \wedge \ldots \wedge v_{k-d}^i$. By construction the intersection $\cap_{i=1}^\ell \operatorname{supp} w_i'$ has to be a trivial vector space. Consequently, the following set is linearly independent:

$$\{v_1^0, \ldots, v_d^0\} \cup \{v_j^i \mid 1 \leq i \leq \ell,\ 1 \leq j \leq k-d\}.$$

We extend it by $m := n - (d + (k-d)\ell)$ vectors $x_1, \ldots, x_m \in V$ to a basis of V and set

$$v := v_1^2 \wedge \ldots \wedge v_{k-d}^2 \wedge v_1^3 \wedge \ldots \wedge v_{k-d}^3 \wedge \ldots \wedge v_{k-d}^\ell \wedge x_1 \wedge \ldots \wedge x_m.$$

Like in the previous case we have then $v \in \bigwedge^{n-k} V \setminus \{0\}$ and $w \wedge v = w_1 \wedge v \neq 0$, which is due to $w_i' \wedge v = 0$ for each $2 \leq i \leq \ell$. With this we have established the injectivity of each restriction of $vol \lrcorner (-)$ to $\bigwedge^k V$ and hence the injectivity of $vol \lrcorner (-)$ itself at last. Due to $\dim \bigwedge V = \dim \bigwedge V^*$, $vol \lrcorner (-)$ is also surjective and hence an isomorphism. Moreover, because of $\dim \bigwedge^k V = \dim \bigwedge^{n-k} V^*$ the same applies to $vol \lrcorner (-) : \bigwedge^k V \to \bigwedge^{n-k} V^*$. It is this isomorphism $vol \lrcorner (-)$ that yields a self-duality of the second class. In turn, any duality that belongs to the second class singles out a $vol \in \bigwedge^n V^*$ and a $v \in \bigwedge^n V$ by the isomorphism $K \cong \bigwedge^n V^*$, $\bigwedge^n V \cong K$ as the respective image of 1. However, like it was the case for the first class of self-dualities this doesn't imply that this isomorphism has to be $vol \lrcorner (-)$.

The relation of this class of self-dualities with the fact of vectors to represent flux-like quantities can be seen as follows. By the linear hom-tensor adjunction the

isomorphism $V \cong \bigwedge^{n-1} V^*$ yields a non-degenerate bilinear form $b : V \otimes \bigwedge^n V \to K$. Each element of $\bigwedge^n V$ is pure, which is due to the same argument as for $d \leq k-2$ in the previous proof. In fact, this argument actually shows that the sum $v + w$ of pure $v, w \in \bigwedge^k V$ is pure if $\dim(\operatorname{supp} w \cap \operatorname{supp} v) \geq k-1$; in the case of $k = n-1$ we have necessarily $\dim(\operatorname{supp} w \cap \operatorname{supp} v) \geq n-2$ (cf. e.g. the dimension theorem of linear algebra [KM03, satz 2.2.16]). Given an $h \in \bigwedge^n V$ we can consider it as a hyperplane with an (oriented) "area" unit, for as a pure element its support $\operatorname{supp} h$ is a hyperplane and due to $h = v_1 \wedge \ldots \wedge v_{n-1}$, $v_i \in \operatorname{supp} h$ it becomes a volume unit for that hyperplane, i.e. we may consider it as to lie in $\bigwedge^{n-1} \operatorname{supp} h \hookrightarrow \bigwedge^{n-1} V$; the term "area" is taken from the three-dimensional case. Regarding a vector $v \in V$ as a current density of a linear flux we can consider $b(v, h)$ as to give the current through the given hyperplane "area" h. This becomes even more suggestive when the isomorphism is of the form $vol \lrcorner (-)$. In this case $b(v, h) = \langle vol, v \wedge h \rangle$ measures the oriented volume of the "prism" with h as base and v as height. In three dimensions this agrees with the intuitive geometric picture.

There is a second way to exhibit the relation with vectors to represent flux-like quantities. We consider hereto the isomorphism $\bigwedge^{n-1} V \cong V^*$. Keeping our interpretation from above it assigns to an oriented hyperplane "area" h a linear form α. In the case that the isomorphism is of the form $vol \lrcorner (-)$, then we have the representation $\alpha = \langle vol, h \wedge (-) \rangle$. This shows $\operatorname{supp} h = \ker \alpha$. Recall our deduction that a non-constant affine $\omega : A \to K$ measures current densities with respect to its reference hyperplane. Then $\alpha = \langle vol, h \wedge (-) \rangle$ confirms that argument structurally and agrees with the intuitive geoemtric picture mentioned before.

3.5.5 Remarks on Regressive Products

The self-duality given by $vol \lrcorner (-)$ defines a second product $\vee$ on $\bigwedge V$ by

$$v \vee w := (vol \lrcorner (-))^{-1}((vol \lrcorner v) \wedge (vol \lrcorner w)), \qquad v, w \in \bigwedge V.$$

It is called a "regressives Produkt" in [Gra62] and "eingewandtes Produkt" in [Gra78] by Grassman. We shall adopt the common name "regressive product" for $\vee$. As one would probably guess from the definition and the geometric interpretation from above, it is a kind of intersection product. Indeed, if we have two pure elements $v, w \in \bigwedge V$ whose degrees add up to a number greater or equal than n and

$$\dim(\operatorname{supp} v \cap \operatorname{supp} w) = \deg v + \deg w - n,$$

then this regressive product yields a pure element whose support is indeed the intersection of the supports of v and w (cf. [Zad94][§50]).

The regressive product can be seen to be "dual" to the exterior product. Roughly, this means that by turning the exterior algebra upside down, i.e. flipping degrees and regarding $\bigwedge^n V$ as the field K with identity element the dual of vol, we can see

the regressive product to behave exactly as the exterior product. Both products are related by an identity often referred to as the "law of the common factor", or "law of the middle factor". The original form of this identity can be found in [Gra78, §132]. We shall, however, adopt the version given as the "Common Factor Axiom" in [Bro09]:

$$(v \wedge x) \vee (w \wedge x) = (v \wedge w \wedge x) \vee x,$$
$$v \in \bigwedge^k V,\ w \in \bigwedge^\ell V,\ x \in \bigwedge^m V,\ k+\ell+m = n = \dim V,\ v, w, x \text{ pure}.$$

Applying our particular definition of a regressive product yields

$$(v \wedge x) \vee (w \wedge x) = \langle vol, v \wedge w \wedge x \rangle x.$$

Notice $\operatorname{supp}(v \wedge x) = \operatorname{supp} v + \operatorname{supp} x$ provided $v \wedge x \neq 0$. When one thus replaces the "extensors" x, $v \wedge x$, $w \wedge x$, $v \wedge w \wedge x$ by their respective supports then stated like this the law of the common factor says that the regressive product can be indeed considered an intersection product with $\langle vol, v \wedge w \wedge x \rangle$ being the "intersection number".

With this made clear the condition $k + \ell + m = n = \dim V$ for the degrees should recall the dimension theorem of linear algebra. In fact, on a structural level one can consider the pure extensors to correspond to subvector spaces, the exterior product to correspond to the join and the regressive product to the meet of subvector spaces. Of course, due to obvious reasons this holds only in very special cases; for example, for the exterior product to correspond to a join the subvector spaces have to intersect trivially. Nevertheless, with respect to this point of view the law of the common factor is seen to correspond to the modularity law for the (modular) lattice of subvector spaces.

The definition of a regressive product we gave above is an approach going back to Bourbaki and is used excessively in [Zad94]. It is a generalization and structural clarification of Grassmann's basis oriented approach to regressive products by his unary operation of "Ergänzung" and the exterior product in [Gra62]. We would refer to this "Ergänzung" as Hodge-duality today. It is a degree-flipping (linear) automorphism $\bigwedge V \to \bigwedge V$ that can be obtained by combining both self-dualities, the one induced by a non-zero $vol \in \bigwedge^n V$ and the one induced by a non-degenerate bilinear form $g : V \otimes V \to K$; the g is usually taken to be symmetric and vol is usually chosen in such a way that its dual can be written as an exterior product of a g-orthonormal basis. This approach of Grassmann to obtain a regressive product was often criticized later, for the regressive product is more than an orthogonal invariant; it is a unimodular invariant (cf. e.g. the discussion in [BT96] and the definition given above, which only utilizes vol and the exterior product both being invariant under the special linear group). However, Grassmann's original definition of regressive products given in [Gra78] doesn't employ this duality in any way; the

regressive products and their properties are deduced solely by a conceptual and geometric analysis there. His original approach is an axiomatical one. This kind of approach is also pursued in [Bro09], where the notion of regressive product is given axiomatically via the above mentioned duality and the law of the common factor. Yet in Browne's approach there is no explicit account for the fact that there is not one but several regressive products. Before we explain this in more detail, we shall mention Rota's computational and combinatorial approach to the regressive product. It utilizes the coproduct of the natural Hopf-algebraic structure on $\bigwedge V$ and a "bracket" that can be basically understood as *vol* regarded as a the respective alternating multilinear form (cf. [BT96]). Rota's approach has the advantage that the regressive product of two extensors can be given by an explict combinatorial and computable formula.

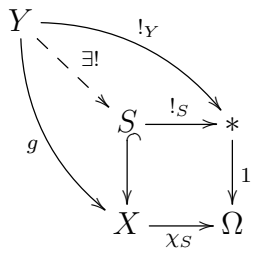
We have related a regressive product with intersections. Building on that we note the following peculiarity. Suppose we have linearly independent $v, w, x \in V$ and $\dim V = 3$, then by the law of the common factor we get $(v \wedge x) \vee (w \wedge x) = \lambda x$, where $\lambda \in K \setminus \{0\}$ is the intersection number and depends on the chosen $vol \in \bigwedge^3 V$. Suppose next that V is itself a vector subspace of a five-dimensional vector space V', then $(v \wedge x) \vee (w \wedge x) = 0$ and this is independent of the chosen $vol' \in \bigwedge^5 V'$. To be able to calculate the intersection of two pure $v, w \in \bigwedge V$ one would need a regressive product on $\operatorname{supp} v + \operatorname{supp} w$, provided v and w are linearly independent, that is $\operatorname{supp} v \neq \operatorname{supp} w$. One thus needs a regressive product for each vector subspace of dimension greater or equal than 2 to be able to calculate intersections freely. Each such regressive product on a particular vector subspace will be an interesection product only for those pure extensors that are linearly independent and for which the vector subspace is the sum of their supports. This was in effect already clear to Grassmann by the time he wrote the first Ausdehnunglehre [Gra78]. It seems that Grassmann even tried to develop one formal regressive product that comprises all those particular regressive products, but he gave up on that with the statement that it was unfruitful (cf. [Gra78, §125]).

3.5.6 On Connections to Logic

We conclude this section and chapter by studying connections to logic with the help of the duality functor $\mathrm{Set}(-, \{0,1\})$. Indeed, until now we have used only the dual vector space functor and the inherent geometric dynamical point of view on affine spaces to obtain a geometric and conceptual understanding of the dual space functor. Nonetheless, we have seen before that $\mathrm{Set}(-, \{0,1\})$ and D bear a greater resemblance than D and $\underline{\mathrm{Vct}}_K(-, K)$. Furthermore, it was $\mathrm{Set}(-, \{0,1\})$ that has lead us to the structurally important properties of D. We shall take account of that now.

Recall that for any (small) set X, $\mathrm{Set}(X, \{0,1\})$ can be considered as the Boolean algebra $\mathrm{Set}(X, 2)$ of characteristic functions on X. The set $\Omega := \{0,1\}$ is considered

as the set of truth values, where 0 shall correspond to "false" and 1 shall correspond to "true". As these are the only two truth values one speaks about a two-valued logic. Each characteristic function $\chi : X \to \Omega$ can then be considered as a predicate or property on X with the preimage $\chi^{-1}(1)$ as the subset of X whose elements (points) have that particular property. Recast into categorical language this says that for each subobject S of X represented by a mono $S \hookrightarrow X$ there is a unique morphism $\chi_S : X \to \Omega$ such that S is the pullback of $1 : * \to \Omega$ by χ_S



In this context Ω is said to be a subobject classifier the terminology being borrowed from classifying spaces in topology. The operations in $\mathrm{Set}(X, \Omega)$ considered as a Boolean algebra correspond to the basic connectives in first-order logic: the meet corresponds to conjunction, the join to disjunction and the complement to negation. This shows the logic to be Boolean. Moreover, as there is more than one set in Set which may serve as a universe of discourse the logic is many-sorted.

The set-theoretic maps interrelate this different universes of discourse and the functoriality of $\mathrm{Set}(-, 2)$ shows that this induces an interrelation of the respective Boolean algebras of propositions who respect the above mentioned logical connectives. In fact, we know for any $f : X \to Y$ that $f^* = \mathrm{Set}(f, 2)$ preserves arbitrary meets and joins. This can be seen again when we translate characteristic functions to subsets and use $f^*\chi_S = \chi_s \circ f = \chi_{f^{-1}(S)}$, for this shows f^* to correspond to taking preimages under f that has these asserted properties (cf. [AE99a, I.3.8]). When we consider the posets $\mathrm{Set}(X, 2)$ and $\mathrm{Set}(Y, 2)$ as categories, then f^* becomes a functor, the meet becomes the product, the join the coproduct, the constant map 0 the initial object and the constant map 1 the terminal object. The existence of arbitrary joins and meets, that is their completeness as a lattice, becomes the property of completeness and cocompleteness as a category. The functor f^* is then continuous and cocontinuous and by the special adjoint functor theorem [Mac98, corollary V.8.2] and its dual it has a left adjoint $\exists_f : \mathrm{Set}(X, 2) \to \mathrm{Set}(Y, 2)$ and a right adjoint $\forall_f : \mathrm{Set}(X, 2) \to \mathrm{Set}(Y, 2)$. On subsets $\exists_f$ corresponds to taking the image, i.e. $\exists_f(S) = f(S) = \{y \in Y \mid \exists x \in X : \ x \in S \wedge f(x) = y\}$, whereas $\forall_f$ corresponds to a restricted but uniform image $\forall_f(S) = \{y \in Y \mid \forall x \in X : \ f(x) = y \Rightarrow x \in S\}$ (cf. [MM92, theorem I.9.2]). It shall be those two adjoints that we will consider as the indicated quantifiers. That this is indeed a meaningful definition can be seen as follows. The unique map $! : X \to *$ induces a map $\Omega \cong \mathrm{Set}(*, \Omega) \to \mathrm{Set}(X, \Omega)$ and the quantifiers $\exists_!$, $\forall_!$ become maps $\mathrm{Set}(X, \Omega) \to \Omega$. The existential quantifier

$\exists_!$ will map a characteristic function χ to 1 if and only if χ is not the constant 0 function, i.e. if there is at least one $x \in X$ for which the property χ holds. The universal quantifier $\forall_!$ will map a χ to 1 if and only if it is the constant 1 function, i.e. if all $x \in X$ have the respective property. To sum it up it can be said that the internal logic of Set is a many-sorted two-valued Boolean first-order logic.

We consider $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ next. To begin with we compare Ω with K the latter being considered as a bipointed affine line. We have already seen in section 3.3 that $K \cong F_S\Omega$ and that both objects are the respective sums (coproducts) of two points, i.e. $K = 0 + 1$ in $\mathrm{Aff}_{\mathrm{K}}$ and $\Omega = 0 + 1$ in Set. However, in contrast to Ω the affine line has a multitude of points. From the point of view of logic this would mean to have a multitude of truth values. Indeed, our implicit assumption of $\operatorname{char} K = 0$ shows that we have at least countable infinitely many truth values.

In this context it is advisable to drop the restriction on the characteristic for a moment and to take a glimpse on affine spaces over finite fields in particular. The field bearing the greatest resemblence to our Boolean algebra 2 is $\mathbb{Z}_2$. An affine space over $\mathbb{Z}_2$ can be considered as the state space of a two-state system, e.g. a two-state model of a spin network of an atomic lattice in a solid state. Each point corresponds to a state of the system, that is a particular configuration of spins in our example, and each vector to a transition between states. The simply transitiveness then means that there is a unique transition between any two states. Similarly, affine spaces over finite fields can be considered as finite state systems. In arbitrary fields with non-zero characteristic, the characteristic may be regarded as an inherent periodicity; for any given state and transition vector v the characteristic is the smallest number of time units that are needed to arrive at the given state by changing state according to v.

Recall that the Boolean algebra structure on Ω was induced by the field $\mathbb{Z}_2$ and the map $x \mapsto 1 - x$ as the complement. This is due to $\mathbb{Z}_2$ being a Boolean ring, i.e. a ring that consists only of idempotent elements. Although the map $x \mapsto 1 - x$ can be defined on any field K and is an affine map (it is the point-reflection on 1) the only Boolean ring that is also a field is $\mathbb{Z}_2$. In fact, for a ring multiplication to be the meet of a lattice the field has to be a Boolean ring; this can be seen from the identity $x \wedge x = x$, which holds in an arbitrary lattice. As a consequence we will loose the whole lattice structure for any other field than $\mathbb{Z}_2$, in particular for our field K with $\operatorname{char} K = 0$. Because of that there isn't a naturally induced lattice structure on $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ as well. A next attempt to obtain such a lattice structure would be to restrict ourselves to ordered fields. From the geometric dynamical point of view this is a very reasonable thing to do, for we perceive the time line as being directed. It is true that an ordered field will make $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ into a poset by a pointwise (partial) order, yet $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ won't be a lattice in general. To see this consider for example $A = K^2$ and the canonical projections as affine maps $A \to K$. The meet (infimum) for those maps doesn't exist in $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$. Roughly, this is due to the fact that for two non-constant affine maps $f, g : A \to K$ being comparable

implies $\ker \hat{f} = \ker \hat{g}$; otherwise there would be a vector $v \in \ker \hat{f} \setminus \ker \hat{g}$ with which we could alter the value of g arbitrarily but leave f constant.

The next property Ω has but K has not is the property of being a subobject classifier. As we have argued above for the case of Set, to be a subobject classifier amounts to a (natural) bijection between the set of subobjects and morphisms to the subobject classifier, which may be regarded as the "characteristic functions" (cf. [MM92, proposition I.3.1]). We know affine maps $A \to K$ to correspond at best to hyperplanes, so K can't be a subobject classifier. Nonetheless it is advisable to develop a better understanding of this failure of K. One particular reason is that contrary to $\mathrm{Set}(A, \mathbb{Z}_2)$ or $\mathrm{Set}(A, K)$, $\mathrm{Aff}_{\mathrm{K}}(A, K) \subset \mathrm{Set}(A, K)$ isn't closed under pointwise multiplication. When we replace $\mathrm{Aff}_{\mathrm{K}}(A, K)$ by such a closure $R(A)$ then one can see the maps in $R(A)$ to correspond to polynomial functions on A. Indeed, when we assume A finite dimensional with $n = \dim A$ and introduce a Cartesian system of coordinates then affine maps $A \to K$ are seen to correspond to polynomials $\sum_{i=1}^{n} a_i X_i + b \in K[X_1, \ldots, X_n]$ and because we have $\mathrm{char}\, K = 0$ this correspondence is seen to be a bijection. As a consequence we can conclude the K-algebra $R(A)$ to be isomorphic to $K[X_1, \ldots, X_n]$ as a (unital) K-algebra. With this natural extension of $\underline{\mathrm{Aff}}_{\mathrm{K}}(A, K)$ to the (unital) K-algebra $R(A)$ we can additionally "classify" subsets of A that can be described as zero sets of polynomial equations; affine quadrics are a particular example. Although such subsets are not affine subspaces in general, it is still an interesting result from the viewpoint of geometry, because it yields a coordinate independent approach to affine varieties in algebraic geometry.

To sum it up the lacking lattice structure and the fact that K is not a subobject classifier indicate clearly that we can't expect any strong connections to logic, at least not in the way as it is considered in Set. Yet we will still encounter with one indicator for such a connection to exist at all. It will be the Grassmann algebra of affine space. Like in the case of vector spaces we will see particular pure "extensors" to correspond to affine subspaces and the join of two "transversal" affine subspaces to correspond to the exterior product of the respective extensors.

4 The Barycentric Calculus of Möbius and Grassmann

The first and crucial step towards a geometric calculus of an affine space is a geometric calculus of points that is able to encode the affine structure. We will construct and discuss such a calculus in form of the barycentric calculus of Möbius and Grassmann.

As indicated by the term "calculus" a geometric calculus is considered an algebraic object. In fact, our introductory discussion of geometric calculi and their relation with physics in section 1.1 suggests to consider a geometric calculus as an object of a category of quantities, i.e. as an object in a linear category. Hence given an affine space A, a geometric calculus for A should be an object in a linear category. We have already explained that we will preferably search for an universal such object assigned to A, because it will yield the most natural and basic choice that only depends on the geometric language chosen, which is the chosen "gros" category of geometric spaces. Recall that if such a universal element exists for every affine space, then a choice of representants of the isomorphism classes given by universality yields a functorial assignment. When we choose the target category accordingly, this yields even more, namely a pair of adjoint functors.

For the category of affine spaces Aff_K the most reasonable and basic linear category to consider is the category Vct_K of vector spaces over K. The next conceptual question we have to ask ourselves is which type of functor from affine spaces to vector spaces we want to consider. Shall it be a contravariant functor, that is an intensive quantity-type, or shall it be a covariant functor, that is an extensive quantity-type? When we would decide on an intensive quantity-type, then the dual space functor $D : \mathrm{Aff}_K{}^{op} \to \mathrm{Vct}_K$ and his lift to K-algebras as the functor $R : \mathrm{Aff}_K{}^{op} \to \mathrm{Mon}(\mathrm{Vct}_K, \otimes, K)$, which was discussed in the last part of section 3.5, would be a natural choice. Recall that for finite dimensional A and any choice of Cartesian coordinates $R(A)$ is isomorphic to the polynomial ring in $\dim A$ variables over K. So with this choice we would basically arrive at a coordinate free approach to affine varieties in classical algebraic geometry.

Note that the algebraic representation of a space by an intensive quantity-type is a common approach in nowadays mathematics. In algebraic and differential

geometry one can see this, for example, in the notion of a (locally) ringed spaces, where the geometric space is represented by a a sheaf of commutative rings over its underlying topological space of points. Here we consider a sheaf of commutative rings as a functor $\tau(X)^{op} \to \mathrm{CRng}$ with an additional gluing-property (cf. [MM92, section II.1]); the category $\tau(X)$ denotes the open subsets of the space X, i.e. its topology regarded as a poset and CRng is the category of commutative (unital) rings with the respective ring-homomorphisms as morphisms.

We will, however, consider the extensive quantity-type here. It is this choice that shall lead us to geometric calculi developed in the 19th century like the barycentric calculus of Möbius and Grassmann and Grassmann's algebra of affine space. It will also lead us to a geometric algebra that, so it seems, has been overlooked so far: it is the Clifford Algebra of affine space.

According to our choice we need to search for a functor $\mathcal{B} : \mathrm{Aff}_{\mathrm{K}} \to \mathrm{Vct}_{\mathrm{K}}$ that is part of an adjunction. From our previous studies we have exactly one candidate for $\mathcal{B}$, namely the left adjoint to the self-action functor S (cf. corollary 3.2.14). As for the forgetful functor $U_V : \mathrm{Aff}_{\mathrm{K}} \to \mathrm{Vct}_{\mathrm{K}}$ it can't have neither a left adjoint nor a right adjoint. The first fact is a consequence of theorem 3.2.16 and the second fact is due to U_V to not preserve coproducts (cf. the explicit construction of coproducts in the proof of lemma 3.3.3 and the dual of [Mac98, theorem V.5.1]).

This time we will consider the geometric construction first and show that this yields a left adjoint afterwards. We will then proceed with some further relevant properties of the functor $\mathcal{B}$. The most important one will be its strong monoidality, which we will take this up in the last chapter. This property will be the essential ingredient for an algebraic presentation of the internal monoids of $\mathrm{Aff}_{\mathrm{K}}$. These internal monoids are interesting for us, since we expect (and will see actually) that geometric algebras of affine space arise as such structures.

In fact, the barycentric functor is of central importance for both subsequent chapters, since both are concerned with an algebraic presentation of the geometric notions in terms of linear structures.

4.1 The Barycentric Calculus

Given an affine space A over $\mathbb{R}$ a construction of a vector space $\mathcal{B}\,A$ that satisfies our requirements has in essence already been given by Möbius in [Möb44] and in more generality by Grassmann in [Gra78]. We will give a different construction here, but first of all we shall give the (geometric) ideas.

The question on how to add points has been approached by mathematicians in the 18th and 19th century. Most notably are W. R. Hamilton and A. F. Möbius. Hamilton noticed that it doesn't make sense to speak of sums of points but it makes sense to speak about their differences. He used the term "vector" for such a difference. Möbius observed that one can make sense of a sum of points when one

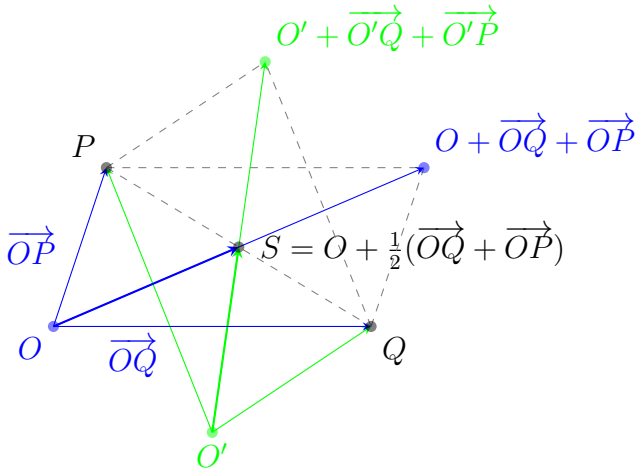


Figure 4.1: The barycentric sum of P and Q geometrically. We see that the central point S is independent of the choice of origin O or O' used to construct it.

allows for the additional datum of a weight. Given two points $P, Q \in A$, we can set $P + Q = 2S$ where S is the central point of P and Q. If one fixes an origin $O \in A$ one can represent S by its position vector

$$\overrightarrow{OS} = \frac{1}{2}\overrightarrow{OP} + \frac{1}{2}\overrightarrow{OQ} \qquad \text{or} \qquad 2\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{OQ}.$$

Indeed, this algebraic expression reflects the following fact: in geometric terms, the weight expresses the ratio of the directed line segment that one obtains as the geometric sum of the directed line segment $\overline{OP}$ and $\overline{OQ}$ to the directed line segment $\overline{OS}$ like depicted in the figure 4.1. Moreover, this weight is independent of the chosen origin and depends thus only on the points P, Q. The particular choice of P as the origin yields then the point S as the center of P and Q explicitly.

It was this geometric insight that was exploited by Möbius in [Möb44] and Grassmann in [Gra78] for the development of a geometric calculus of points[1]. However, we won't develop it in this form here. Note that although S is independent of the choice of the origin O, the point $O + \overrightarrow{OP} + \overrightarrow{OQ}$ does depend on O.

By bookkeeping the weights as an additional datum one can induce a geometric addition of points from the geometric addition of directed line segments. Indeed, the weight is a necessary datum to make such an addition associative. For three non-colinear points $P, Q, R \in A$ one can see this either pictorially in the figure 4.1 for three non-colinear points $P, Q, R \in A$, or in algebraic terms by chosing an origin

[1] Möbius developed his barycentric calculus before [Möb44]. In [Möb44] he realized that he can derive it from the geometric addition of directed line segments. In his first approach he used Cartesian coordinates.

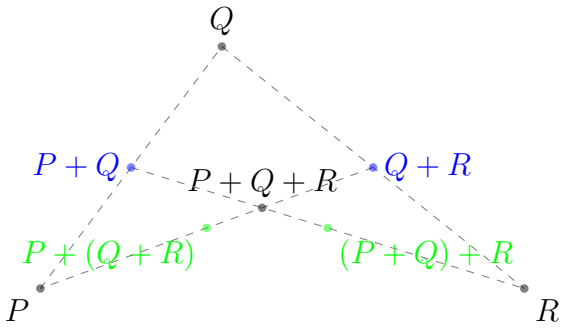


Figure 4.2: If we identify the sums of points with the resulting center point leaving out the weight, then the sum becomes non-associative.

$O \in A$ as follows: define $P + Q$ to be the central point S of P and Q without the weight. We get

$$O + \frac{1}{4}(\overrightarrow{OP} + \overrightarrow{OQ}) + \frac{1}{2}\overrightarrow{OR} \quad \text{for the point} \quad (P + Q) + R, \text{ and}$$
$$O + \frac{1}{2}\overrightarrow{OP} + \frac{1}{4}(\overrightarrow{OQ} + \overrightarrow{OR}) \quad \text{for the point} \quad P + (Q + R).$$

The points differ by the vector $1/4\ (\overrightarrow{OP} - \overrightarrow{OR})$, which isn't zero for $P \neq R$. Bookkeeping of the weights will on the contrary result in an associative (and commutative) addition.

The rule for adding weighted points that total weight is not zero can be derived by geometric arguments using ratios of the form above as well (cf. [Möb44], [Gra78, §98]). It results in the "center of mass formula", i.e. $\lambda P + \mu Q = (\lambda + \mu)S$, $\lambda, \mu \in \mathbb{R}$ for $\lambda + \mu \neq 0$, where S is the center of mass and can be represented for a choice of origin $O \in A$ by the position vector

$$\overrightarrow{OS} = \frac{\lambda\overrightarrow{OP} + \mu\overrightarrow{OQ}}{\lambda + \mu}.$$

The point S is seen to be independent of the chosen origin once again. This center of mass construction is the reason why this calculus bears the name "barycentric calculus".

Note that the weights need not to be positive. They can sum up to zero in particular. In this situation one cannot apply the center of mass construction. Therefore, we still have to define what the sum of two weighted points shall be in this case. We define it to be a vector. The idea behind this definition is based on regarding vectors as differences of points as mentioned above. Indeed, because of the simply transitive action of V_A on A, we have for each pair of points P, Q a

unique vector $\overrightarrow{PQ}$ such that $P + \overrightarrow{PQ} = Q$, where the plus is meant in terms of the action. This notation of the action already suggests to set $Q - P := \overrightarrow{PQ}$. One can now try to understand $Q - P$ as $Q + (-P)$ in terms of weights. This suggests the definition for weighted points whose total weight is zero to be

$$\lambda Q + (-\lambda)P = \lambda\overrightarrow{PQ}, \qquad \lambda \in \mathbb{R}.$$

By introducing vectors in the barycentric calculus, one is forced to define what the sum of a weighted point and a vector should be. There are two key observations that will result in a reasonable definition. The first observation is that in the case of weight one we have already such type of sum in terms of the action of V_A on A. The second observation is that up to now we have a "conservation of mass" in our barycentric calculus; i.e. in an equation the total weight of the right hand side and the total weight of the left hand side are equal. Our formula from above yields that vectors have a weight of zero. By our first observation, the sum of a weighted point λP and a vector $v \in V_A$ should result in a weighted point λQ of the same weight as P. The second observation together with our barycentric sum of points, which is induced by the action of V_A on A, suggests that the sum of λP and v should be induced by this action once again. Altogether this leads to the following definition:

$$\lambda P + v = \lambda(P + \frac{1}{\lambda}v), \qquad \lambda \in \mathbb{R},\ \lambda \neq 0.$$

From the aspect of a displacement vector to encode a translation and hence movement one can interpret this definition in form of an inertia principle combined with the conservation of energy. A vector $v := \overrightarrow{PQ}$ has the energy to move a pointmass of mass 1 from point P to Q. If we want to move a pointmass of mass $m > 0$, then the energy of v is only sufficient to move it from P to $Q' := P + 1/m\ v$ due to inertia. This resembles the idea of the "vis viva" of G. W. Leibniz. We won't use this interpretation here, however.

To sum it up our barycentric calculus of A is a set of objects the weighted points λP, $\lambda \in \mathbb{R} \setminus \{0\}$, $P \in A$ and the vectors $v \in V_A$ with an $\mathbb{R}$-vector space structure induced from the simply transitive action of V_A on A. The addition is the casewise defined addition from above. The scalar multiplication is the obvious multiplication of weights, where we set $0 \cdot P$ to the zero vector of V_A for all $P \in A$. We won't give a proof of these assertions here, since they will follow explicitly from our construction of the barycentric functor $\mathcal{B}$ later on.

A final remark is in order. Usually, the barycentric calculus of Möbius as well as the one of Grassmann is affiliated with and often identified with homogeneous coordinates in analytic projective geometry (cf. e.g. [Cra09]). It is indeed the case that the barycentric calculus, or, more precisely, the barycentric coordinates of Möbius were the predecessors for the notion of homogeneous coordinates of analytic projective geometry. It is also true that the barycentric calculus is tightly related

with the representation of affine spaces in projective geometry. It is, however, not the case that the barycentric calculus is the same as the introduction of homogeneous coordinates.

As we have argued in the introduction and as we will actually see in theorem 4.2.1 the barycentric calculus is the universal geometric calculus of points for an affine space. This point of view is corroborated by our above discussion and by the geometric construction of Möbius in [Möb44]. Indeed, as we have already mentioned before, in this work Möbius derived his barycentric calculus from the geometric addition of directed line segments and used the weights as universal ratios independent from a chosen origin. This is a construction in the spirit of a geometric calculus to algebraize synthetic reasoning. It doesn't use or rely on coordinate methods in any way.

4.2 The Barycentric Functor as a Left Adjoint to the Self-action Functor

We turn now to a formal construction of a barycentric calculus $\mathcal{B}\,A$ for an affine space A from our category $\mathrm{Aff_K}$ of (small) affine spaces over a field K of characteristic zero. Although the geometric construction in the previous section may suggest it, the construction doesn't actually depend on the characteristic. Recall that the self-action functor $S : \mathrm{Vct_K} \to \mathrm{Aff_K}$ assigns to a vector space V, V as a V-torsor and the forgetful functor $U_V : \mathrm{Aff_K} \to \mathrm{Vct_K}$ assigns to an affine space A its associated vector space V_A. Recall as well that $\mathrm{Aff_K}(A, SU_V A)$ has a K-vector space structure by pointwise operations.

We define $\mathcal{B}\,A$ as the vector subspace $\mathcal{B}\,A \leq \mathrm{Aff_K}(A, SU_V A)$ generated by the following set of affine maps

$$\{\sigma_P : A \to SU_V A \mid P \in A\}, \quad \text{with} \quad \sigma_P : Q \mapsto \overrightarrow{QP}.$$

By applying the identities of the difference map $A \times A \to V_A$, $(P, Q) \mapsto \overrightarrow{PQ}$ given in remark 3.1.1(d) it is easy to see the σ_P to be affine with associated linear map -1_{V_A}. For that matter it suffices to note $\sigma_P = -\kappa_P$, where the κ_P are the induced isomorphisms by the difference map, which had been shown affine in remark 3.3.1(b) already.

As we pointed out in the preceding exposition of the geometric ideas behind a barycentric calculus, the addition of weighted points can be derived from the geometric addition of (located) directed line segments with an arbitrary chosen origin. As we have argued in the beginning of section 3.1, those line segments are represented by displacement vectors, which may be also understood algebraically as their algebraic "invariants". These ideas are reflected in our construction as well: a point $P \in A$ is represented by the map σ_P that also represents all possible line segments ending in P. Thus the sum $\sigma_P + \sigma_Q$ can be understood as to sum any

two line segments ending in P or Q respectively and $\lambda\sigma_P$ as scaling all possible line segments ending in P with the weight λ simultaneously. $\mathcal{B}\,A$ is the smallest vector space containing all our (representations of) points $P \in A$. We just have to show that $\mathcal{B}\,A$ consists of "weighted points" and "vectors from V_A" only, that addition of points is the barycentric addition of weighted points and that the addition of points and vectors corresponds to the one defined in 4.1 in terms of the action.

To begin with, note the set-theoretic map $A \to \mathcal{B}\,A$, $P \mapsto \sigma_P$ to be injective, which is a consequence of the simple transitiveness of the action of V_A on A. It is thus meaningful to speak about the σ_P representing our points of A and $\lambda\sigma_P$, $\lambda \in K$ the weighted points. In the following we will omit the "representing" and just speak of σ_P and $\lambda\sigma_P$ as points respectively as weighted points. From linear algebra we know that each vector in $\mathcal{B}\,A$ can be represented as a (finite) linear combination of the generators

$$\sum_i \lambda_i \sigma_{P_i}, \qquad \lambda_i \in K.$$

For the case of the total weight $\sum_i \lambda_i$ not being zero we have to show that this yields a weighted point satisfying the center of mass formula and a vector of V_A in the case $\sum_i \lambda_i = 0$. Here a "vector of V_A" is understood in the terms of a faithful linear representation of V_A in $\mathcal{B}\,A$, which is in particular given by the injective linear map

$$\nu : V_A \to \mathrm{Aff}_K(A, SU_V A), \qquad v \mapsto (\nu_v : P \mapsto v). \tag{4.1}$$

Due to associativity we only need to consider linear combinations of two points $\lambda\sigma_P$, $\mu\sigma_Q$, $\lambda, \mu \in K$. Suppose $\lambda + \mu \neq 0$ and consider the center of mass

$$S = O + \frac{\lambda\overrightarrow{OP} + \mu\overrightarrow{OQ}}{\lambda + \mu},$$

for an arbitrary but fixed $O \in A$. We need to show $\lambda\sigma_P + \mu\sigma_Q = (\lambda + \mu)\sigma_S$. This can be verified by evaluation on points using the summation rule for the difference map in remark 3.1.1(d)

$$\begin{aligned}
(\lambda + \mu)\sigma_S(R) &= (\lambda + \mu)\overrightarrow{RS} = (\lambda + \mu)(\overrightarrow{RO} + \overrightarrow{OS}) \\
&= (\lambda + \mu)(\overrightarrow{RO} + \frac{\lambda\overrightarrow{OP} + \mu\overrightarrow{OQ}}{\lambda + \mu}) \\
&= \lambda\overrightarrow{RO} + \mu\overrightarrow{RO} + \lambda\overrightarrow{OP} + \mu\overrightarrow{OQ} = \lambda\overrightarrow{RP} + \mu\overrightarrow{RQ} \\
&= (\lambda\sigma_P + \mu\sigma_Q)(R), \qquad\qquad R \in A.
\end{aligned}$$

For the case of $\lambda = -\mu$ the rules of computation for the difference map yield

$$(\lambda\sigma_P + \mu\sigma_Q)(R) = \mu(-\overrightarrow{RP} + \overrightarrow{RQ}) = \mu(\overrightarrow{PR} + \overrightarrow{RQ}) = \mu\overrightarrow{PQ} = \mu\nu_{\overrightarrow{PQ}}(R), \qquad R \in A.$$

On one hand this shows that such a sum of points yields a vector in V_A, on the other hand it also implies the addition of weighted points and vectors to be given in terms of the action like it was defined for a barycentric calculus in 4.1

$$(\lambda\sigma_P + \nu_v)(R) = \lambda(\overrightarrow{RP} + \frac{1}{\lambda}v) = \lambda\sigma_{P+1/\lambda v}(R), \qquad R \in A,\ \lambda \in K \setminus \{0\},\ v \in V_A.$$

All in all we can indeed regard $\mathcal{B}\,A$ as a realization of a barycentric calculus. Next we will see that the assignment $A \mapsto \mathcal{B}\,A$ is functorial and the so obtained barycentric functor $\mathcal{B} : \mathrm{Aff_K} \to \mathrm{Vct_K}$ turns out to be a left adjoint to our self-action functor S.

Theorem 4.2.1. *There is a barycentric functor $\mathcal{B} : \mathrm{Aff_K} \to \mathrm{Vct_K}$ that assigns to each affine space A the barycentric calculus $\mathcal{B}\,A$. The self-action functor $S : \mathrm{Vct_K} \to \mathrm{Aff_K}$ is a right adjoint to $\mathcal{B}$. The unit $\eta : I_{\mathrm{Aff_K}} \dot{\to} S\,\mathcal{B}$ of this adjunction is for each A in $\mathrm{Aff_K}$ the "inclusion of points"*

$$\eta_A : A \to S\,\mathcal{B}\,A, \quad P \mapsto \sigma_P.$$

Proof. By theorem IV.1.2(ii) in [Mac98] both assertions of functoriality of $\mathcal{B}$ and $\mathcal{B} \dashv S$ follow from η_A being universal (from A to S) for each A in $\mathrm{Aff_K}$. For example, the functoriality of $\mathcal{B}$ is obtained by applying the universal property to $\eta_{A'} f$ for an $f : A \to A'$. This yields a unique linear $\mathcal{B}\,f : \mathcal{B}\,A \to \mathcal{B}\,A'$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & S\,\mathcal{B}\,A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle S\,\mathcal{B}\,f} \\ A' & \xrightarrow[\eta_{A'}]{} & S\,\mathcal{B}\,A', \end{array} \qquad \begin{array}{c} A \\ \downarrow \\ \downarrow{\scriptstyle \exists!\,\mathcal{B}\,f} \\ A'. \end{array}$$

Moreover, the uniqueness of each $\mathcal{B}\,f$ yields $\mathcal{B}(gh) = \mathcal{B}\,g \circ \mathcal{B}\,h$ and $\mathcal{B}\,1_A = 1_{\mathcal{B}\,A}$ as well. We just need to show the universal property. Suppose A is an empty space, then the generating set of points is empty and $\mathcal{B}\,A$ is a trivial vector space. In this case the assertion becomes trivial, because both objects are initial in their respective categories. We can thus assume A to be non-empty.

First of all, we will show η_A to be affine with associated linear map $\nu : V_A \to \mathcal{B}\,A$, which is the corestriction of the $\nu : V_A \to \mathrm{Aff_K}(A, SU_V A)$ defined in (4.1). Using the pointwise operations in $\mathcal{B}\,A$ yields

$$\eta_A(P + v)(R) = \sigma_{P+v}(R) = \overrightarrow{R(P+v)} = \overrightarrow{RP} + v = (\sigma_P + \nu_v)(R), \qquad R \in A.$$

Recall that $\overrightarrow{R(P+v)} = \overrightarrow{RP} + v$ holds due to the simply transitive action of V_A on A (cf. e.g. remark 3.1.1(d)). For the universal property let $f : A \to SV$ be an affine map. Suppose we have a linear $g : \mathcal{B}\,A \to V$ such that $f = Sg \circ \eta_A$. This yields $g(\sigma_P) = f(P)$ for $P \in A$. By the construction of $\mathcal{B}\,A$ as the linear span of the σ_P,

$P \in A$, g is thus uniquely determined by f. On the other hand define $g : \mathcal{B}A \to V$ as the linear extension of the map $\sigma_P \mapsto f(P)$. For this g to be well-defined, we need to show $\sum_i \lambda_i g(\sigma_{P_i}) = g(\sigma_S)$ for $\sigma_S = \sum_i \lambda_i \sigma_{P_i}$, because this is how the generators σ_P of $\mathcal{B}A$ are interrelated. This amounts to show $\sum_i \lambda_i f(P_i) = f(S)$. To see this we fix an origin $O \in A$ and use the center of mass representation. Due to the total weight $\sum_i \lambda_i = 1$ it takes the following form

$$S = O + \sum_i \lambda_i \overrightarrow{OP_i}.$$

We apply the affine map f and use the fact that we map into a vector space to obtain the asserted identity

$$\begin{aligned} f(S) &= f(O) + \lambda_i \hat{f}(\overrightarrow{OP_i}) = (\sum_i \lambda_i) f(O) + \sum_i \lambda_i \hat{f}(\overrightarrow{OP_i}) \\ &= \sum_i \lambda_i (f(O) + \hat{f}(\overrightarrow{OP_i})) = \sum_i \lambda_i f(P_i). \end{aligned}$$

(qed)

Remarks 4.2.2.

(a) (*Weight form*). The mapping $\omega : \mathcal{B}A \to K$, which assigns to each weighted point its weight, is a non-zero linear form on $\mathcal{B}A$ with $\ker\omega$ the ($\hat{\eta}_A$-embedded) vector space V_A and $\omega^{-1}(1)$ the (η_A-embeded) point set A. One can obtain it for example from the adjunction of the constant map $A \to SK$, which sends each point to $1 \in K$. Alternatively, one can consider the $\mathcal{B}$-image of the terminal map $A \to *$, where $*$ denotes a one-point space. Indeed, $\mathcal{B}* = K\sigma_*$ is a one-dimensional vector space with a distinguished non-zero vector σ_* given by $\eta_* : * \to SK\sigma_*$; hence it is naturally isomorphic to K by linear extension of $\sigma_* \mapsto 1$. This weight form ω will be the key to many constructions to come; in particular it will be the key to the construction of a geometric algebra of an affine space.

(b) The previous remark together with the fundamental theorem on homomorphisms (2.1) in $\mathrm{Vct_K}$ yield $\mathcal{B}A \cong V_A \oplus KP$ for any $P \in A$. In more detail the fundamental theorem on homomorphisms yields $\mathcal{B}A \cong \ker\omega \oplus K$[2]. We can use the isomorphism $\ker\omega = \hat{\eta}_A(V_A) \cong V_A$ and identify K with the vector space KP, because each $P \in A$, or more precisely σ_P is mapped to $1 \in K$ by ω.

(c) The barycentric functor $\mathcal{B}$ is cocontinuous, i.e. preserves small colimits. This is a consequence of the adjunction $\mathcal{B} \dashv S$ (cf. e.g. the dual of theorem V.5.1 in [Mac98]).

[2] In more sophisticated terms of abelian categories and exact sequences one says that we have the short exact sequence $0 \to \ker\omega \to \mathcal{B}A \xrightarrow{\omega} K \to 0$ that splits.

(d) The barycentric functor $\mathcal{B}$ is faithful and as such reflects monic and epis. This follows from each η_A being injective and hence a monic in $\mathrm{Aff_K}$ by the dual of theorem IV.3.1 in [Mac98].

(e) The components of the counit $\varepsilon_V : \mathcal{B}\,SV \to V$ are the restrictions of the evaluation-at-zero maps $\mathrm{ev}_{0,V} : \mathrm{Aff_K}(SV, SU_VSV) = \mathrm{Aff_K}(SV, SV) \to V,\ f \mapsto f(0)$ to $\mathcal{B}\,SV$. Here we have used $U_VS = 1_{\mathrm{Vct_K}}$ and that the zero vector $0 \in V$ is a distinguished point in SV. To prove the assertion we need to show $\mathrm{ev}_{0,V}\,|_{\mathcal{B}\,SV}$ the left adjunct of 1_{SV}, or, equivalently, 1_{SV} to be the right adjunct of $\mathrm{ev}_{0,V}\,|_{\mathcal{B}\,SV}$. Utilizing the unit this amounts to show the identity $1_{SV} = S(\mathrm{ev}_{0,V}\,|_{\mathcal{B}\,SV}) \circ \eta_{SV}$ (cf. (2.3) in section 2.2.2), which is seen to hold by evaluation on points

$$S(\mathrm{ev}_{0,V}\,|_{\mathcal{B}\,SV}) \circ \eta_{SV}(v) = \mathrm{ev}_{0,V}(\sigma_v) = \sigma_v(0) = \overrightarrow{0v} = v - 0 = v, \qquad v \in SV.$$

Note that the linear map ε_V restricted to $V \hookrightarrow \mathcal{B}\,SV$ becomes the identity 1_V. This can be seen either directly by evaluation with $\mathrm{ev}_{0,V}\,|_{\mathcal{B}\,SV}$ or more abstractly with the U_V-image of the respective triangular identity (cf. (2.2) in section 2.2.2), because the linear map embedding V is $\hat{\eta}_{SV}$.

(f) $\mathcal{B}$ preserves dual objects in the following sense: $(\mathcal{B}\,A)^* \cong DA$ in $\mathrm{Vct_K}$ and natural in A. Due to $(\mathcal{B}\,A)^* = \underline{\mathrm{Vct}}_{\mathrm{K}}(\mathcal{B}\,A, K)$ and $DA = \underline{\mathrm{Aff}}_{\mathrm{K}}(A, K) = \underline{\mathrm{Aff}}_{\mathrm{K}}(A, SK)$ one just needs to realize that the adjunction $\mathcal{B} \dashv S$ can be internalized in $\mathrm{Vct_K}$. This can be seen in the same way as in the the proof of theorem 3.4.5(iii) and is a consequence of the pointwise vector space structures involved.

(g) (*Affine combinations*) When we have a finite set of points $P_i \in A$ and scalars $\lambda_i \in K$, $1 \leq i \leq n$, such that $\sum_{i=1}^n \lambda_i = 1$ then in $\mathcal{B}\,A$ regarded as a realisation of a barycentric calculus the linear combination $\sum_{i=1}^n \lambda_i P_i$ will yield a point $P \in A$ again. Such linear combinations are termed *affine combinations*. One can use affine combinations to give a further equivalent definition of the structure of an affine space. Hereto one needs to define them formally. This amounts to give a set A and for every $n \in \mathbb{N} \setminus \{0\}$ and n-tuple $\lambda \in K^n$ with $\sum_{i=1}^n \lambda_i = 1$ an n-ary operation $a_\lambda : A^n \to A$, which is understood as to yield for any n points the resepective affine combination. The a_λ have to be compatible with each other in the sense of respective distributivity laws. Furthermore, we need commutation and cancellation laws as well as an identity law. Everything has to be defined in such a way that we can formally manipulate the so defined affine combinations as if they were linear combinations in a vector space. We shall make this more precise by the following set of axioms. In each axiom the respective n-tuples in K^n are supposed to be *partitions of unity*, i.e. for each n-tuple the sum of its components yields $1 \in K$.

(I) (Distributivity.) Let $\lambda \in K^n$ and $\mu^i \in K^{m_i}$ for $1 \leq i \leq n$, then together

they yield a $\nu \in K^m$ with $m = \sum_{i=1}^n m_i$ as

$$(\lambda_1\mu_1^1, \ldots, \lambda_1\mu_{m_1}^1, \lambda_2\mu_1^2, \ldots, \lambda_2\mu_{m_2}^2, \ldots, \lambda_n\mu_{m_n}^n).$$

The following identity holds:

$$\begin{aligned} & a_\lambda(a_{\mu^1}(P_1^1, \ldots, P_{m_1}^1), \ldots, a_{\mu^n}(P_1^n, \ldots, P_{m_n}^n)) \\ =& a_\nu(P_1^1, \ldots, P_{m_1}^1, \ldots, P_{m_n}^n), \qquad\qquad P_j^i \in A. \end{aligned}$$

(II) (Commutativity.) There is an obvious action of the permutation group S_n on K^n, namely by permuting the components of each element. Lets denote the action of a $\sigma \in S_n$ on a $\lambda \in K^n$ by $\sigma\lambda$, then $a_\lambda = a_{\sigma\lambda}$ holds for each $\sigma \in S_n$ and partition of unity $\lambda \in K^n$.

(III) (Cancellation.) The identity $a_{(\lambda,1,-\lambda)}(P, Q, P) = Q$ holds for any $P, Q \in A$ and $\lambda \in K$.

(IV) (Identity.) It is $a_1 = 1_A$.

For any affine space A defined in terms of a V_A-torsor of a K-vector space V_A we can define the operations a_λ with the help of the barycentric calculus $\mathcal{B}A$ as the respective linear combinations. It is straight forward to see that they satisfy the axioms from above. On the other hand, for a set A together with operations a_λ, which satisfy the above axioms, we can define a K-vector space V_A together with an a simply transitive action of this V_A on A as follows.

The idea is to see a vector $\overrightarrow{PQ}$ as an equivalence class of formal differences $Q - P$. Such formal differences appear in and are accessible by the cancellation law. W.l.o.g. we assume A to be non-empty. Define V_A as the quotient set $A \times A/\sim$, where $\sim$ denotes the equivalence relation

$$(P, Q) \sim (R, S) \quad :\Longleftrightarrow \quad a_{(1,1,-1)}(P, S, R) = Q.$$

We will prove $\sim$ an equivalence relation first. Observe that $a_{(1,1,-1)}(P, S, R) = Q$ implies $a_{(1,1,-1)}(R, Q, P) = S$:

$$\begin{aligned} a_{(1,1,-1)}(R, Q, P) &= a_{(1,1,-1)}(R, a_{(1,1,-1)}(P, S, R), P) \\ &\overset{\text{I+IV}}{=} a_{(1,1,-1,1,-1)}(R, P, S, R, P) \\ &\overset{\text{II}}{=} a_{(1,1,-1,1,-1)}(R, S, R, P, P) \\ &\overset{\text{I+IV}}{=} a_{(1,1,-1)}(a_{(1,1,-1)}(R, S, R), P, P) \\ &\overset{\text{III}}{=} a_{(1,1,-1)}(S, P, P) \\ &\overset{\text{II}}{=} a_{(1,1,-1)}(P, S, P) \\ &\overset{\text{III}}{=} S. \end{aligned}$$

The same argument with the roles of the respective points interchanged shows $a_{(1,1,-1)}(R,Q,P) = S$ to imply $a_{(1,1,-1)}(P,S,R) = Q$. This shows the symmetry of $\sim$. The transitivity can be shown by a similar type of argument and the reflexivity is an immediate consequence of axiom (III).

We will define a vector space structure on V_A next. Let $[(P,Q)], [(R,S)] \in V_A$ denote the equivalence classes of (P,Q) and (R,S) respectively. Define the multiplication by scalars $\lambda \in K$ as $\lambda[(P,Q)] = [(P, a_{(1,\lambda,-\lambda)}(P,Q,P))]$. This is well-defined, because for $(R,S) \sim (P,Q)$, that is $a_{(1,1,-1)}(R,Q,P) = S$, we have

$$\begin{aligned} a_{(1,1,-1)}(P, a_{(1,\lambda,-\lambda)}(R,S,R), R) &\overset{\text{I+IV}}{=} a_{(1,1,\lambda,-\lambda,-1)}(P,R,S,R,R) \\ &\overset{\text{I+IV}}{=} a_{(1,1,\lambda,\lambda,-\lambda,-\lambda,-1)}(P,R,R,Q,P,R,R) \\ &\overset{\text{II,I+IV,III}}{=} a_{(1,\lambda,\lambda,-\lambda,-\lambda,)}(P,R,Q,P,R) \\ &\overset{\text{II,I+IV,III}}{=} a_{(1,\lambda,-\lambda)}(P,Q,P). \end{aligned}$$

The identity $1 \cdot [(P,Q)] = [(P,Q)]$ follows from (III) and the identity

$$\mu(\lambda[(P,Q)]) = (\mu\lambda)[(P,Q)]$$

follows from

$$\begin{aligned} a_{(1,\mu,-\mu)}(P, a_{(1,\lambda,-\lambda)}(P,Q,P), P) &\overset{\text{I+IV}}{=} a_{(1,\mu,\mu\lambda,-\mu\lambda,-\mu)}(P,P,Q,P,P) \\ &\overset{\text{II,I+IV,III}}{=} a_{(1,\mu\lambda,-\mu\lambda)}(P,Q,P). \end{aligned}$$

For the operation of addition we set $[(P,Q)] + [(R,S)] = [(P, a_{(1,1,-1)}(Q,S,R))]$. This is again seen to be well-defined; for $(R',S') \sim (R,S)$ we can apply the same type of argument as for the symmetry of $\sim$ and for $(P',Q') \sim (P,Q)$ we can apply the same type of argument as for the well-definedness of the multiplication by scalars. $(V_A, +, [(P,P)])$ is seen to be a commutative monoid; the associativity follows from the distributivity law (I) and the commutativity law (II); the commutativity of $+$ can be seen in the same way as the well-definedness of the scalar-multiplication, and the neutral element is a consequence of the cancellation law (III), which yields it to be a left-neutral element, together with the previously established commutativity of $+$. Moreover, the distributivity-law for the scalar-multiplication holds and multiplication by -1 yields an inverse. As for the distributivity, we have on one hand

$$\begin{aligned} \lambda([(P,Q)] + [(R,S)]) &= \lambda[(P, a_{(1,1,-1)}(Q,S,R))] \\ &= [(P, a_{(1,\lambda,-\lambda)}(P, a_{(1,1,-1)}(Q,S,R), P))], \end{aligned}$$

and on the other hand

$$\begin{aligned} \lambda[(P,Q)] + \lambda[(R,S)]) &= [(P, a_{(1,\lambda,-\lambda)}(P,Q,P))] + [(R, a_{(1,\lambda,-\lambda)}(R,S,R))] \\ &= [(P, a_{(1,1,-1)}(a_{(1,\lambda,-\lambda)}(P,Q,P), a_{(1,\lambda,-\lambda)}(R,S,R), R))]. \end{aligned}$$

The identity

$$a_{(1,\lambda,-\lambda)}(R,S,R) = a_{(1,1,-1)}(a_{(1,\lambda,-\lambda)}(P,Q,P), a_{(1,\lambda,-\lambda)}(R,S,R), R)$$

can be seen by applying (IV) and (I) to each term and observing the cancellation of the last R with the first R in $a_{(1,\lambda,-\lambda)}(R,S,R)$ in the second term. As for the inverse, multiplying by -1 yields $-[(P,Q)] = [(P, a_{(1,-1,1)}(P,Q,P))]$. When we then form the sum of $[(P,Q)]$ and $-[(P,Q)]$, we get

$$[(P,Q)] + (-[(P,Q)]) = [(P, a_{(1,1,-1)}(Q, a_{(1,-1,1)}(P,Q,P), P))] \overset{\text{IV+I,III}}{=} [(P,P)].$$

With this we have finally established the asserted K-vector space structure of V_A. An action of V_A on A is then defined by the cancellation law as

$$A \times V_A \to A, \qquad (P, [(R,S)]) \mapsto a_{(1,1,-1)}(P,S,R).$$

That this is a well-defined map and a simply transitive action follows directly from our definition of $\sim$ and the vector space structure of V_A. We omit the details, as they trace back to the type of arguments we have already presented above. Finally, note that given an affine space as a V_A-torsor, if we define a vector space V'_A and a respective action on A from the induced affine combination operators a_λ like above, then the identity map $A \to A$ becomes an affine isomorphism with associated linear map $V'_A \to V_A$, $[(P,Q)] \mapsto \overrightarrow{PQ}$. This follows from $a_{(1,1,-1)}(P,Q,R) = P + \overrightarrow{RQ}$, the definition of the linear structure in V'_A together with the simply transitiveness of the action of V_A on A and corollary 3.1.4.

(h) When we describe an affine space in terms of affine combinations as in the pervious remark then the respective notion of an affine map would be a map that preserves affine combinations, i.e., which respects all the operators a_λ for any partition of unity $\lambda \in K^n$, $n \geq 1$. Each such a map f is seen to be an affine map in our original sense, for

$$\begin{aligned} f(P + \overrightarrow{PQ}) &= f(a_{(1,1,-1)}(P,Q,P)) = a_{(1,1-,1)}(f(P), f(Q), f(P)) \\ &= f(P) + \overrightarrow{f(P)f(Q)}, \end{aligned}$$

and $\overrightarrow{PQ} = [(P,Q)] \mapsto \overrightarrow{f(P)f(Q)} = [(f(P), f(Q))]$ is a linear map by our definition of the vector space structure on V_A in terms of affine combinations together with f respecting $a_{(1,\lambda,-\lambda)}$ for each $\lambda \in K$; note also the last statement in the previous remark. In the last step of the proof of the preceding theorem we have actually proved the converse, that is affine maps to preserve affine combinations.

(i) With the notion of affine combinations at hand we can give a representation of the coproduct $A_1 + A_2$ for two affine spaces A_i in Aff_K without choosing basis points (cf. with the actual construction in lemma 3.3.3). We can represent $A_1 + A_2$ as the affine space generated by (formal) affine combinations of $(P, Q) \in A_1 \times A_2$ with $1 \cdot P + 0 \cdot Q = P$ and $0 \cdot P + 1 \cdot Q = Q$, i.e. $A_1 + A_2$ has as a set of points

$$\begin{aligned}\{\lambda P + \mu Q \mid \lambda + \mu = 1,\ \lambda, \mu \in K,\ P \in A_1,\ Q \in A_2\}\\ \cup\{v + Q \mid v \in V_{A_1},\ Q \in A_2\}\\ \cup\{P + w \mid w \in V_{A_2},\ P \in A_1\}.\end{aligned}$$

Instead of defining this formally we will use the barycentric functor $\mathcal{B}$ to see this. In remark (c) we have mentioned its cocontinuity; in particular, we have $\mathcal{B}(A_1 + A_2) = \mathcal{B}A_1 \oplus \mathcal{B}A_2$ and the inclusion maps $\iota_i : A_i \to A_1 + A_2$ are mapped to inclusion maps $\mathcal{B}A_i \to \mathcal{B}A_1 \oplus \mathcal{B}A_2$. In remark (a) we have introduced the weight form $\omega_A : \mathcal{B}A \to K$ of an affine space A as the $\mathcal{B}$-image of the unique map $A \to *$. As $A_1 + A_2$ is a coproduct, the unique map $A_1 + A_2 \to *$ is the coproduct of the unique maps $A_1 \to *$ and $A_2 \to *$. Let $\omega_i : \mathcal{B}A_i \to K$ be the weight forms for the A_i, then this aforementioned fact yields the weight form $\omega : \mathcal{B}A_1 \oplus \mathcal{B}A_2 \to K$ to be the coproduct of ω_1 and ω_2. With this and $A_1 + A_2 = \omega^{-1}(1)$ we obtain the asserted representation, because $1 = \omega(v, w) = \omega_1(v) + \omega_2(w)$ for $(v, w) \in \mathcal{B}A_1 \oplus \mathcal{B}A_2$ holds precisely for the following three cases: $(v, w) \in A_1 \times V_{A_2}$, $(v, w) \in V_{A_2} \times A_2$, $(v, w) = (\lambda P, \mu Q)$ with $\lambda + \mu = 1$ and $(P, Q) \in A_1 \times A_2$.

Compare this representation of the coproduct of affine spaces with the "join" of two topological spaces as defined for example in [Hat02, chapter 0].

Remarks 4.2.3 (finite dimensions)**.**

(a) For A in FAff_K the following dimension-formula holds

$$\dim \mathcal{B}A = \dim A + 1. \tag{4.2}$$

This can be seen either directly from the explicit construction of $\mathcal{B}A$ or deduced from more abstract arguments. The explicit way just observes that we need a point together with $n = \dim A = \dim V_A$ linear independent vectors to span all the vectors and weighted points of $\mathcal{B}A$, and that a point is linear independent of any set of vectors from V_A in $\mathcal{B}A$. This just expresses the geometric fact that for an affine coordinate system, besides fixing a basis of V_A, we need to fix an origin. Hence the additional dimension can be interpreted once again as an account for this extra degree of freedom of choosing points.

As for an abstract argument we can either use the representation $\mathcal{B}A \cong V_A \oplus K$ given in remark 4.2.2(b) or remark 3.3.1(c), i.e. the free functor $F_S : \mathrm{Set} \to \mathrm{Aff}_K$

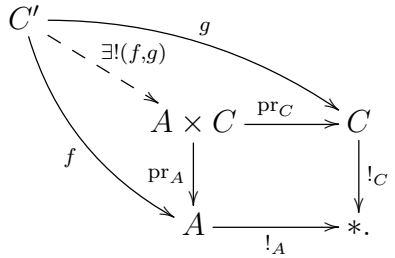
being isomorphism-dense. Because the first argument is obvious, we explain the second one in more detail. From F_S being isomorphism-dense we get $A \cong F_S\{0,\ldots,n\}$. Furthermore, we know from the discussion of free functors in section 3.3 that $F_S\{0,\ldots,n-1\} \cong F* + \ldots + F*$, where the latter coproduct is taken $n+1$ times. By remark 4.2.2(c) $\mathcal{B}$ is cocontinuous as well, hence $\mathcal{B}\,A \cong \bigoplus_{i=1}^{n+1} \mathcal{B}\,F_S*$. We have $\mathcal{B}\,F_S* = \mathcal{B}* \cong K$, where $F_S* = *$ is a consequence of the representation of F_S by the copower functor $(-)\cdot*$ (cf. proposition 3.2.17) and $\mathcal{B}* \cong K$ was shown in remark 4.2.2(a). According to the dimension formula for the direct sum in Vct_K this yields the dimension formula for $\mathcal{B}\,A$. Note that $\mathcal{B}\,F_S* \cong K \cong F*$ for the free functor $F : \mathrm{Set} \to \mathrm{Vct}_\mathrm{K}$ is a special instance of $\mathcal{B}\,F_S \cong F$, which is a consequence of $U_S S = U$ for the forgetful functors $U : \mathrm{Vct}_\mathrm{K} \to \mathrm{Set}$, $U_S : \mathrm{Aff}_\mathrm{K} \to \mathrm{Set}$, the composition of adjunctions $\mathcal{B} \dashv S$, $F_S \dashv U_S$ to $\mathcal{B}\,F_S \dashv U_S S$ (cf. section IV.8 in [Mac98]) and left adjoints for a functor to be determined up to a natural isomorphism.

(b) The adjunction $\mathcal{B} \dashv S$ holds for the restriction of $\mathcal{B}$ and S to finite dimensions, i.e. for $\mathcal{B} : \mathrm{FAff}_\mathrm{K} \to \mathrm{FVct}_\mathrm{K}$ and $S : \mathrm{FVct}_\mathrm{K} \to \mathrm{FAff}_\mathrm{K}$. This is a consequence of the above dimension formula and the fact that both of those categories are full subcategories of Aff_K and Vct_K respectively.

4.3 Connected Limits and Colimits

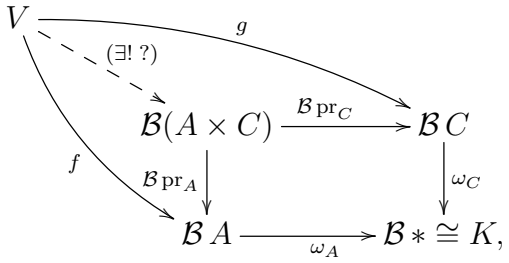
We proceed in studying further preservation properties of $\mathcal{B}$ and S. From our impossibility results theorem 3.2.15, theorem 3.2.16 we already know that S can't be cocontinuous and $\mathcal{B}$ can't be continuous. Indeed, from the dimension formula (4.2) in remark 4.2.3(a) we can already read of that for finite dimensional, non-empty $A, C \in \mathrm{Aff}_\mathrm{K}$ the barycentric functor $\mathcal{B}$ doesn't preserve the product $A \times C$. ISuprisingly, it does preserve $A \times C$ as a pullback.

To make this more obvious, recall that in a category with terminal object each binary product can be represented as a pullback of terminal maps. In our case, the product $A \times C$ for arbitrary $A, C \in \mathrm{Aff}_\mathrm{K}$ is the pullback



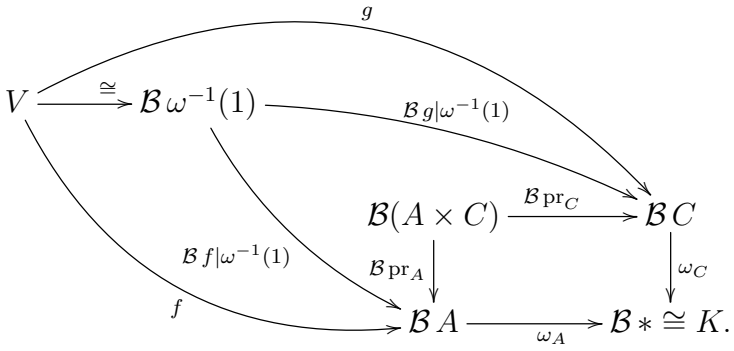
If A and C are not empty spaces, then $\mathcal{B}$ is seen to preserve this pullback. To see

this we consider an arbitrary $V \in \mathrm{Vct}_{\mathrm{K}}$ and the commutative diagram



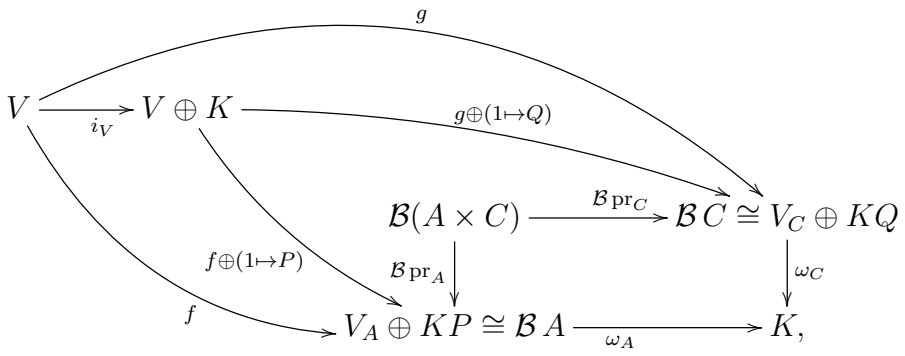
(∗)

where we have used that the $\mathcal{B}$-image of the terminal map is the respective weight form (cf. remark 4.2.2(a)). The composite $\omega := \omega_A \circ f = \omega_C \circ g$ defines a linear form on V. Suppose that ω is not the zero map. It is then surjective, and reverting to remark 4.2.2(a) we can identify V with the $\mathcal{B}$-image of the affine space $\omega^{-1}(1)$ with associated vector space $\ker\omega$. Moreover, by the definition of ω the restrictions $f|\omega^{-1}(1)$ and $g|\omega^{-1}(1)$ are seen to be affine maps with associated linear maps $f|\ker\omega$ and $g|\ker\omega$. Hence, with respect to our identification f and g are identified with the $\mathcal{B}$-images of these affine maps. Applying our identification we can rewrite the previous diagram as follows



By the universal property of the pullback in $\mathrm{Aff}_{\mathrm{K}}$ there is a unique affine map $h : \omega^{-1}(1) \to A \times C$ such that the respective diagram commutes. This yields the existence of a linear map $V \to \mathcal{B}(A \times C)$ such that the $\mathcal{B}$-image of the pullback diagram (∗) commutes in $\mathrm{Vct}_{\mathrm{K}}$. The uniqueness of such a map follows from $\mathcal{B}$ being faithful and any linear map $\mathcal{B}\,\omega^{-1}(1) \to \mathcal{B}(A \times C)$ that renders the diagram commutative being the $\mathcal{B}$-image of an affine $h : \omega^{-1}(1) \to A \times C$. This is once again due to the fact that any such linear map respects the weight forms.

Now suppose that ω is the zero map. In this case f factors through $\ker\omega_A \cong V_A$ and g factors through $\ker\omega_C \cong V_C$. We will reduce this case to the previous one by considering $V \oplus K$ with the "weight form" $\mathrm{pr}_K : V \oplus K \to K$. Since $A \times C$ is not the empty space we can fix a point $(P, Q) \in A \times C$ and use the representations from remark 4.2.2(b): $\mathcal{B}(A \times C) \cong V_{A\times C} \oplus K(P, Q)$, $BA \cong V_A \oplus KP$, $BC \cong V_C \oplus KQ$. This yields the following commutative diagram



(**)

whereby $(1 \mapsto P)$ and $(1 \mapsto Q)$ denote the linear maps $K \to KP$ with $1 \mapsto P$ and $K \to KQ$ with $1 \mapsto Q$. Since $\omega_A \circ f \oplus (1 \mapsto P) = \mathrm{pr}_K$ is not a zero map we can apply our previous result and obtain a unique linear map $h' : V \oplus K \to \mathcal{B}(A \times C)$ rendering the diagram commutative. The linear map $h' \circ i_V : V \to \mathcal{B}(A \times C)$ thus renders commutative our pullback diagram (∗). This yields the existence. The uniqueness is obtained as follows. Let $h : V \to \mathcal{B}(A \times C)$ be a linear map such that (∗) commutes with h. Since $\omega_{A\times C} = \omega_A \circ \mathcal{B}\,\mathrm{pr}_A$ and, therefore, $\omega_{A\times C} \circ h = \omega_A \circ f = 0$, h factors through $\ker \omega_{A\times C} \cong V_{A\times C}$. We define a map

$$h \oplus (1 \mapsto (P, Q)) : V \oplus K \to V_{A\times C} \oplus K(P, Q) \cong \mathcal{B}(A \times C),$$

which renders the inner diagram of (∗∗) commutative when inserted. This yields $h \oplus (1 \mapsto (P, Q)) = h'$ and so $h = h' \circ i_V$, which shows the uniqueness.

This observation raises the question of what kind of limits are preserved by $\mathcal{B}$. The following theorem gives an answer.

Theorem 4.3.1. *Let J be a small, non-empty and connected category*[3]*, then $\mathcal{B}$ preserves non-empty J-limits. That is for each functor $F : J \to \mathrm{Aff_K}$ and a limiting cone $\mu : \Delta \varprojlim F \dot{\to} F$ with non-empty $\varprojlim F$, $\mathcal{B}\,\mu : \Delta\,\mathcal{B}(\varprojlim F) \dot{\to} \mathcal{B}\,F$ is a limiting cone in $\mathrm{Vct_K}$. In particular, $\mathcal{B}$ preserves non-empty pullbacks and non-empty equalizers.*

Proof. To prove the general case we just need to generalize the arguments from the special case of $\mathcal{B}$ preserving non-empty binary products as pullbacks from above. We do this in two steps. In the first step we will show that we can extend the diagram $\mathcal{B}\,F : J \to \mathrm{Vct_K}$ by connecting each vertex to K with the weight form without altering the limit. More precisely, we will construct a category J_t by adjoinig a

[3]Recall that a category is said to be *connected* when it is connected as an undirecetd graph, i.e., each two objects can be connected by a path in the graph, where the orientation of the edges doesn't matter.

terminal object t to J, extend F to $F_t : J_t \to \mathrm{Aff}_K$ by mapping t to a terminal one-point space $*$ and show the inlusion $I : J \hookrightarrow J_t$ to be an initial functor[4]. This step was not necessary in our special case, since the weight forms were already inherent in our pullback diagram. For more general diagrams this step becomes necessary to be able to apply the same types of arguments like in the preceding discussion of binary products. In particular, if we can show $\mathcal{B}$ to preserve the limit of F_t, then by I being an initial functor $\mathcal{B}$ has to preserve the limit of F, too.

In the second step we will show $\mathcal{B}\,\mu^t : \Delta\,\mathcal{B}\varprojlim F_t \dot{\to} \mathcal{B}\,F_t$ a limiting cone, whereby $\mu^t : \Delta \varprojlim F_t \dot{\to} F_t$ denotes the limiting cone of F_t. We will do this in constructing a unique $h : V \to \mathcal{B}\varprojlim F_t$ with $\mathcal{B}\,\mu^t \circ h = \nu$ for a cone $\nu : \Delta V \dot{\to} \mathcal{B}\,F_t$. For the construction of h we will consider the cases of $\omega_{\mathcal{B}\,F_t j} \circ \nu_j \neq 0$ for one (and hence for any) $j \in J_t$ and $\omega_{\mathcal{B}\,F_t j} \circ \nu_j = 0$ for one (and hence for any) $j \in J_t$ separately. The arguments here will be very similar to the special pullback case from above. Therefore, we will just state the main steps and omit the technical details.

(i) Define a category J_t by $\mathrm{Obj}\,J_t := \mathrm{Obj}\,J \dot{\cup} \{t\}$ and $\mathrm{Mor}\,J_t$ by adjoing to $\mathrm{Mor}\,J$ for each $j \in J$ a unique morphism $!_j : j \to t$ as well as an identity morphism 1_t. The composition of morphisms on J_t is then defined by the usual composition of morphisms of J, 1_t to act as an identity morphism and $f \circ !_j = !_{j'}$, $!_{j'} \circ f = !_j$ for any $f : j \to j'$. With this J_t is readily seen to be a category with J as a full subcategory and t a terminal object.

The inclusion functor $I : J \hookrightarrow J_t$ is an initial functor. This can be seen as follows. For any $j \in \mathrm{Obj}\,J \subset \mathrm{Obj}\,J_t$, $1_j \in I/j$ is a terminal object in I/j. This shows I/j to be non empty and connected. For the remaining $t \in J_t$ we have $I/t \cong J$ by our definition of J_t. Since by assumption J is non-empty and connected I/t shares those properties, too. This shows I to be initial.

We extend our diagram $F : J \to \mathrm{Aff}_K$ to $F_t : J_t \to \mathrm{Aff}_K$ by setting $F_t \circ I := F$ and $F_t(t) := *$, $F_t(1_t) := 1_*$ as well as $F_t(!_j) = !_{Fj}$. By I being initial F_t and F have both the same limit in Aff_K (cf. the dual of [Mac98][theorem IX.3]). Of course, in this case this assertion is rather obvious for passing to F_t amounts to connect each vertex Fj via the unique terminal map $!_{Fj}$ with $*$. As for $\mathcal{B}\,F_t$ we can see it to extend $\mathcal{B}\,F$ by connecting each $\mathcal{B}\,Fj$ with $K \cong \mathcal{B}*$ by the weight form $\omega_{\mathcal{B}\,Fj}$.

If we can show that $\mathcal{B}$ preserves the limit of F_j, then by I being an initial functor it has to preserve the limit of F as well. Note that since Aff_K and Vct_K are small complete the existence of each limit is guaranteed.

[4]Recall that a functor $L : J \to J'$ is said to be *initial* if for each $j' \in J'$ the slice category L/j' is non-empty and connected. The notion of an initial functor is dual to that of a final functor (cf. [Mac98][section IX.3]). In particular, the dual of theorem IX.3 in [Mac98] asserts that for an initial L and functor $F : J' \to C$ such that $\varprojlim FL$ exists, $\varprojlim F$ exists and the canonical morphism $h : \varprojlim F \to \varprojlim FL$ is an isomorphism. Note that L being initial in this sense has apriori nothing to do with L being an initial element in the functor category J'^J.

(ii) Let $\mu^t : \Delta \varprojlim F_t \dot{\to} F_t$ be the limiting cone for F_t. We will show $\mathcal{B}\,\mu^t : \Delta\,\mathcal{B} \varprojlim F_t \dot{\to} \mathcal{B}\,F_t$ a limiting cone by constructing $h : V \to \mathcal{B} \varprojlim F_t$ with $\mathcal{B}\,\mu^t \circ h = \nu$ for a cone $\nu : \Delta V \dot{\to} \mathcal{B}\,F_t$. Firstly, suppose that $\omega := \nu_t : V \to K$ is not the zero map. For ν is a cone we have $\omega_{Fj} \circ \nu_j = \nu_t \neq 0$. By reverting to remark 4.2.2(a) we can consider V as the $\mathcal{B}$-image of the affine space $\omega^{-1}(1)$ with associated vector space $\ker \omega$. Due to the equality $\omega_{Fj} \circ \nu_j = \nu_t = \omega$ we can moreover consider the ν_j as $\mathcal{B}$-images of the affine maps $\nu_j|\omega^{-1}(1)$. With this the cone ν can be considered as the the $\mathcal{B}$-image of a cone $\nu' : \Delta\omega^{-1}(1) \dot{\to} F_t$ in Aff_K. The property of μ^t to be a limiting cone yields a unique affine map $h' : \omega^{-1}(1) \to \varprojlim F_t$ such that $\mu^t \circ h' = \nu'$ and so with $\mathcal{B}\,h'$ the existence of a linear map $h : V \to \mathcal{B} \varprojlim F_t$ with $\mathcal{B}\,\mu^t \circ h = \nu$. On the other hand, any such map h is compatible with the weight forms

$$\omega_{\mathcal{B} \varprojlim F_t} \circ h = \mu^t_t \circ h = \nu_t = \omega,$$

and hence is the $\mathcal{B}$-image of an affine map $h' : \omega^{-1}(1) \to \varprojlim F_t$. With $\mathcal{B}$ being faithful we can conclude the uniqueness of such h.

(iii) Now suppose that $\omega := \nu_t : V \to K$ is the zero map. For ν is a cone we have $\omega_{Fj} \circ \nu_j = \nu_t = 0$. The ν_j thus factor through $\ker \omega_{Fj}$. We will consider $V \oplus K$, extend the cone ν to $\tilde{\nu} : \Delta V \oplus K \dot{\to} \mathcal{B}\,F_t$ and reduce this case to the previous one.

Since $\varprojlim F_t$ is not the empty space by assumption, we can fix a point $O \in \varprojlim F_t$ and set $O_j := \mu^t_j(O)$ for each $j \in \mathrm{Obj}\,J$. Consider the representations $\mathcal{B} \varprojlim F_t \cong V_{\varprojlim F_t} \oplus KO$, $\mathcal{B}\,Fj \cong V_{Fj} \oplus KO_j$, whereby $V_{\varprojlim F_t} \cong \ker \mathcal{B}\,\mu^t_t$ ($\mathcal{B}\,\mu^t_t = \omega_{\varprojlim F_t}$) and $V_{Fj} \cong \ker \omega_{Fj}$. For $j \in \mathrm{Obj}\,J$ we define $\tilde{\nu}_j : V \oplus K \to \mathcal{B}\,F_t j$ as $\nu_j \oplus (1 \mapsto O_j)$, whereby $(1 \mapsto O_j)$ denotes the linear map $K \to KO_j$ with $1 \mapsto O_j$. For $t \in J_t$ we have $\mu^t_t(O) = 1$, so $\tilde{\nu}_t := \nu_t \oplus (1 \mapsto 1) = \mathrm{pr}_K : V \oplus K \to K$, which is a consequence of $\nu_t = 0$. Since ν is a cone with each ν_j factoring through V_{Fj} and since the O_j were chosen coherently, $\tilde{\nu}$ is a cone, too. In fact, note that for each $f : j \to j'$ in J if $\mathcal{B}\,Ff : \mathcal{B}\,Fj \to \mathcal{B}\,Fj'$ is considered as the respective linear map $V_{Fj} \oplus KO_j \to V_{Fj'} \oplus KO_{j'}$, then it is of the form $\widehat{Ff} \oplus (O_j \mapsto O_{j'})$, where $\widehat{Ff}$ is the associated linear map of Ff. So $\tilde{\nu}_{j'} = \mathcal{B}\,Ff \circ \tilde{\nu}_j$ follows from this and the construction of $\tilde{\nu}$. Regarding $t \in J_t$ by $\omega_{Fj}(O_j) = 1$ we have $\omega_{Fj} \circ \tilde{\nu}_j = \mathrm{pr}_K = \tilde{\nu}_t$, since each ν_j factors through $\ker \omega_{Fj}$ and $\tilde{\nu}_j(0, 1) = O_j$.

It is $\tilde{\nu}_t = \mathrm{pr}_K \neq 0$. By (ii) there is thus a unique linear map $h' : V \oplus K \to \mathcal{B} \varprojlim F_t$ such that $\mathcal{B}\,\mu^t \circ h = \tilde{\nu}$. Composing this map with the natural inclusion $i : V \hookrightarrow V \oplus K$ yields the existence of a linear map $h : V \to \mathcal{B} \varprojlim F_t$ with $\mathcal{B}\,\mu^t \circ h = \nu$, since we have $\tilde{\nu} \circ i = \nu$ by construction of $\tilde{\nu}$. As for the uniqueness of such a map h observe that $0 = \nu_t = \mu^t_t \circ h$ implies h to factor

through $\ker \mu_t^t = \ker \omega_{\varprojlim F_t} \cong V_{\varprojlim F_t}$. We can then consider the linear map

$$h \oplus (1 \mapsto O) : V \oplus K \to V_{\varprojlim F_t} \oplus KO \cong \mathcal{B} \varprojlim F_t.$$

From $\mathcal{B}\,\mu^t \circ h = \nu$ and the construction of $\tilde{\nu}$ one can see readily that $\mathcal{B}\,\mu^t \circ h \oplus (1 \mapsto O) = \tilde{\nu}$. Our preceding result yields $h \oplus (1 \mapsto O) = h'$ and thus $h = h' \circ i$.

(qed)

Remarks 4.3.2.

(a) All the assumptions of non-emptiness and connectedness of J as well as the non-emptyness of $\varprojlim F$ are necessary assumptions. Dropping the assumption of non-emptiness of J would allow to take as J the empty category $\underline{0}$. This category is connected and the limit of the unique functor $F : \underline{0} \to \mathrm{Aff_K}$ is a terminal object and hence a one-point set, which is non-empty, too. The barycentric functor does, however, not preserve terminal objects. Dropping connectedness of J would allow for binary products of non-empty affine spaces. Such a product too is non-empty, but it is not preserved by $\mathcal{B}$. Finally, if we allow for empty limits, then we can consider the equalizer of two translations $\tau_v, \tau_w : A \to A$ with $v, w \in V_A$, $v \neq w$. The respective index category J is non-empty and connected. The limit, i.e. the equalizer in $\mathrm{Aff_K}$, is the empty space. But the limit of the $\mathcal{B}$-image, i.e. the equalizer of $\mathcal{B}\,\tau_v$ and $\mathcal{B}\,\tau_w$ is $V_A \hookrightarrow \mathcal{B}\,A$, which for $\dim V_A > 0$ is not isomorphic to the zero space $\mathcal{B}\,\emptyset$.

(b) Given any index category J and a diagram $F : J \to \mathrm{Aff_K}$ that has a non-empty limit, then, although the inclusion $I : J \to J_t$ needn't to be an initial functor in general, one can see readily that F and its extension F_t to a terminal one-point space (as constructed in (i) in the proof of theorem 4.3.1) have the same limit. Now J_t is a connected, non-empty index category with a non-empty limit, so its limit is preserved by $\mathcal{B}$ by the preceding theorem.

This shows that any non-empty small limit of $F : J \to \mathrm{Aff_K}$ is preserved *as* a limit of $F_t : J_t \to \mathrm{Aff_K}$. This also explains and generalizes our initial observation that binary products of non-empty spaces are preserved as pullbacks of terminal maps. For if J is the discrete category $\{1, 2\}$, then limits of diagrams of shape J_t are pullbacks, and if we consider for each $F : J \to \mathrm{Aff_K}$ the extension F_t, then those limits become pullbacks of terminal maps.

A closer inspection of part (i) of the proof of the preceding theorem reveals that the inclusion functor $I : J \to J_t$ is an initial functor if and only if J is connected and non-empty. So, with regard to the aforementioned fact, for such J the J-limits and J_t-limits happen to coincide.

(c) $\mathcal{B}$ preserves monos. Let $f : A \to C$ be a mono in Aff_K. If A is empty then $\mathcal{B}f$ is the initial zero map to $\mathcal{B}C$, which is monic. In the other case we can use remark 3.5.4(c) and the preceding theorem. There we have seen that any mono f is regular and hence the equalizer of two affine maps, which is preserved by $\mathcal{B}$ since A is non-empty by assumption. We conclude our assertion in recalling that any equalizer is monic.

Of course, one can check $\mathcal{B}f$ being an injective map and hence a mono also directly by the explicit construction of $\mathcal{B}$ in section 4.2 and proposition 2.2.4, theorem 3.2.10.

(d) With the proof of theorem 4.3.1 we have taken a first glance at the importance of weight forms for barycentric calculi of affine spaces. What we haven't seen yet is the "abstract reason" for this importance and "why" the theorem actually holds. This "abstract reason" shall become clear in the subsequent chapter.

If we turn to S we can see a "dual" result of theorem 4.3.1 to hold for it as well[5].

Theorem 4.3.3. *Let J be a small, non-empty connected category, then S preserves J-colimits. That is, for each functor $F : J \to \mathrm{Vct}_\mathrm{K}$ and a limiting cone $\mu : F \dot{\to} \Delta \varinjlim F$, $S\mu : SF \dot{\to} \Delta S(\varinjlim F)$ is a limiting cone in Aff_K. In particular, S preserves pushouts and coequalizers.*

Proof. It is not only a "dual" result to the preceding theorem, we can also obtain a proof by "dualizing" the previous one in a certain sense. More precisely, we only need to dualize the first step in the previous proof, since in the case of S the rest follows much more easily than for $\mathcal{B}$.

We construct a category J_i by adjoining J an initial element i. This construction works again for any J. Dual to the case of J_t we have that the inclusion functor $I : J \hookrightarrow J_i$ is a final functor if and only if J is connected and non-empty.

Given an $F : J \to \mathrm{Vct}_\mathrm{K}$ we can extend it to $F_i : J \to \mathrm{Vct}_\mathrm{K}$ by mapping i to a trivial vector space 0 and each initial morphism $!_j : i \to j$, $j \in J$ to the initial zero map $0 \to Fj$. Both diagrams F and F_i have the same colimit. For I is a final functor in our case, if we can show S to preserve a colimit of F_i, then it has to preserve a colimit of F, too.

Let $\mu^i : F_i \dot{\to} \Delta \varinjlim F_i$ be a limiting cone for F_i in Vct_K and $\nu : SF_i \dot{\to} \Delta A$ a cone in Aff_K. Since $SF_i i$ is a one-point space, A is a pointed space by $\nu_i : SF_i i \to A$. As such it can be identified with SV_A, the isomorphism sending the point given by ν_i to $0 \in V_A$. Moreover, regarding this identification each ν_j, $j \in J$ is linear, i.e., it can

[5] By the term "dual" we don't mean to imply that it is dual to theorem 4.3.1 in the sense of categorical duality, but rather with respect to the adjunction $\mathcal{B} \dashv S$ in a vague sense. Strictly speaking theorem 4.3.3 is also not strictly "dual" to theorem 4.3.1, since for S there is no need to restrict to colimits that are not terminal (which would be the dual notion of non-empty limits).

be identified with $S\hat{\nu}_j$. With this ν itself can be considered as the S-image of a cone $\nu' : F_i \dot{\to} \Delta V_A$. Applying the universal property of μ^i we get a unique linear map $h' : \varinjlim F_i \to V_A$ with $h' \circ \mu^i = \nu'$. With $h = Sh'$ we have established the existence of an affine map $h : S \varinjlim F_i \to A$ with $h \circ S\mu^i = \nu$. On the other hand, due to $h \circ S\mu^i_i = \nu_i$ any such h has to respect the base point. It can thus be identified with $S\hat{h}$. But then $\hat{h} = h'$ and so $Sh' = h$, since S is faithful by proposition 3.2.6. This establishes the uniqueness and concludes the proof.

(qed)

Remarks 4.3.4.

(a) Here too the assumptions of non-emptiness and connectedness of J are necessary assumptions. Dropping the assumption of non-emptiness of J would allow to take as J the empty category $\underline{0}$. The colimit of the unique functor $F : \underline{0} \to \mathrm{Vct}_\mathrm{K}$ is an initial object and hence a trivial vector space, which is not preserved by S as an initial object. Dropping connectedness of J would allow for direct sums of vector spaces, which too are not preserved by S.

(b) Given any index category J and a diagram $F : J \to \mathrm{Vct}_\mathrm{K}$ the diagram F and its extension $F_i : J_i \to \mathrm{Vct}_\mathrm{K}$ to an initial trivial vector space (as constructed in the proof of theorem 4.3.3) have the same colimit. Now J_i is a connected, non-empty index category, so its colimit is preserved by S by the preceding theorem.

This shows that any small colimit of $F : J \to \mathrm{Vct}_\mathrm{K}$ is preserved *as* a colimit of $F_i : J_i \to \mathrm{Aff}_\mathrm{K}$. For example, binary sums of vector spaces are not preserved as direct sums, but they are preserved as pushouts of initial zero maps. For if J is the discrete category $\{1, 2\}$, then colimits of diagrams of shape J_i are pushouts, and colimits for extension $F_i : J_i \to \mathrm{Vct}_\mathrm{K}$ are pushouts of initial zero maps.

(c) In contrast to the case of $\mathcal{B}$ we already know the abstract reason for theorem 4.3.3 to hold. It is the fact that the category of pointed affine spaces is equivalent to the category Vct_K of vector spaces. An equivalence, for example, is induced by S and U_V. Like we already did before, we will use this equivalence and continue to identify vector spaces and pointed affine spaces freely.

(d) From the general theory of limits and colimits we know that the limit of a functor $F : J \to \mathrm{Aff}_\mathrm{K}$ is the product of the limits $F_k : J_k \to \mathrm{Aff}_\mathrm{K}$, where the $J_k \hookrightarrow J$ are the connected components of J (cf. [Mac98, Ex. IV.2.7(a)]). Furthermore, we know that $\mathcal{B}$ doesn't, respectively, can't preserve small products in general. The preservation of small connected J-limits can thus be interpreted as $\mathcal{B}$ *being as continuous as it possibly can be*. Dually, colimits of $F : J \to \mathrm{Aff}_\mathrm{K}$ are

coproducts of colimits $F_k : J_k \to \mathrm{Aff}_\mathrm{K}$ for the connected components J_k. S preserves the J_k-colimits but doesn't, respectively, can't preserve small coproducts in general. Thus S *is as cocontinuous as it possibly can be.*

Remark 4.3.5 (finite dimensions)**.**
The last two theorems restrict to the finite dimensional case as follows: $\mathcal{B} : \mathrm{FAff}_\mathrm{K} \to \mathrm{FVct}_\mathrm{K}$ preserves finite, non-empty J-limits for a connected, non-empty J and S preserves finite colimits for such a J. This is due to the fact that the respective inclusion functors of the subcategories create finite limits and colimits. The creation of finite limits and colimits of $\mathrm{FVct}_\mathrm{K} \hookrightarrow \mathrm{Vct}_\mathrm{K}$ can be obtained in the same manner like the creation of limits and colimits of $\mathrm{FAff}_\mathrm{K} \hookrightarrow \mathrm{Aff}_\mathrm{K}$ demonstrated in remark 3.2.5(c) and remark 3.3.9(b).

4.4 The Eilenberg-Moore Algebras of $S\mathcal{B}$ and their Geometric and Algebraic Meaning

We know that the barycentric functor has a geometric meaning. Therefore, one can expect the Eilenberg-Moore algebras of the monad $(T = S\mathcal{B}, \eta, \mu = S\varepsilon\mathcal{B})$ to hold a geometric meaning as well[6]. Recall that an Eilenberg-Moore algebra of T, or T-algebra for short, is a pair (A, h) with $A \in \mathrm{Aff}_\mathrm{K}$ and an affine map $h : TA \to A$, such that the following diagrams expressing associativity and unit law commute

$$\begin{array}{ccc} T^2A & \xrightarrow{Th} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle h} \\ TA & \xrightarrow[\mu]{} & A, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle h} \\ & & A. \end{array}$$

The diagram describing the unit law is already very restrictive on h. It yields that h is the identity on the points of A and moreover by passing to the associated vector spaces, i.e. by considering the U_V-image of the diagram, it yields $\hat{h} \circ \hat{\eta}_A = 1_{V_A}$, i.e., that $\hat{h}$ is the identity on V_A as well. Here we follow the common abuse of notation and identify V_A with its $\hat{\eta}_A$-image and A with its η_A-image in $S\mathcal{B}A$. Because $S\mathcal{B}A$ is a vector space, we have a distinguished point, namely 0 (cf. remark 4.3.4(c)). As we will see shortly, h and $\hat{h}$ are uniquely determined by $h|A = 1_A$, $\hat{h}|V_A = 1_{V_A}$ and $O_h := h(0)$. First of all, we obtain

$$\hat{h}(O_h) = \hat{h}(O_h - 0) = \hat{h}(\overrightarrow{0O_h}) = \overrightarrow{h(0)h(O_h)} = \overrightarrow{O_hO_h} = 0$$

by remark 3.1.1(f). This shows $O_h \in \ker \hat{h}$. If we consider the representation $\mathcal{B}A \cong V_A \oplus KO_h$ from remark 4.2.2(b) now, $\hat{h}|V_A = 1_{V_A}$ implies $\ker \hat{h} = KO_h$.

[6]For the notion of a monad an its category of Eilenberg-Moore algebras cf. section 2.5.3.

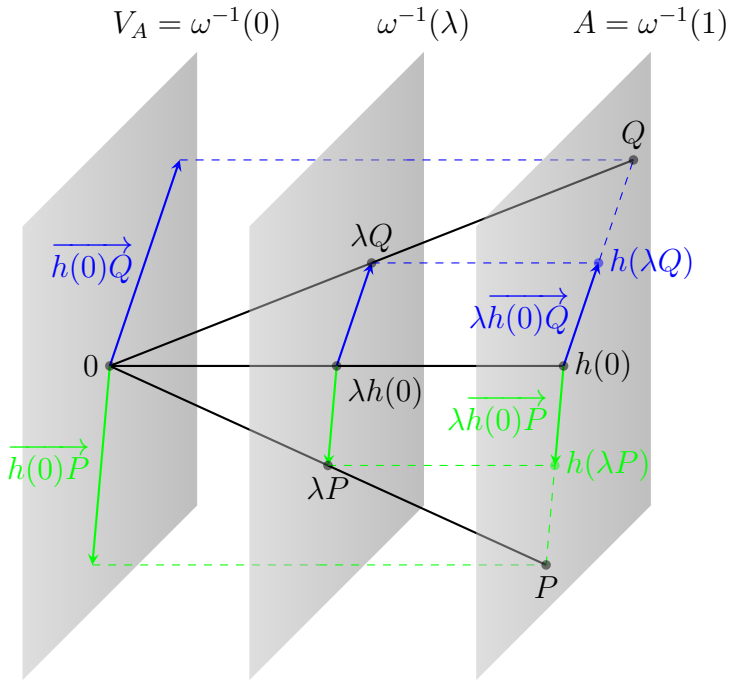


Figure 4.3: The affine multiplication map h of an Eilenberg-Moore algebra (A, h) of T is a central projection of $\mathcal{B}\,A$ to A along the line through 0 and $O_h = h(0)$. The weighted points λP and λQ are mapped to points $P' = h(\lambda P)$ and $Q' = h(\lambda Q)$ such that the ratios $\overline{O_hQ'} : \overline{O_hQ}$, $\overline{O_hP'} : \overline{O_hP}$ of the directed line segments $\overline{O_hQ'}$, $\overline{O_hQ}$, $\overline{O_hP'}$, $\overline{O_hP}$ is the weight λ.

With this and $h(0) = O_h$ we have an explicit description of h and can study the images of weighted points λQ, $\lambda \neq 0$

$$\begin{aligned} h(\lambda Q) &= h(0 + \lambda Q) = O_h + \hat{h}(\lambda Q) = O_h + \hat{h}(\lambda Q - \lambda O_h) \\ &= O_h + \lambda \hat{h}(Q - O_h) = O_h + \lambda \hat{h}(\overrightarrow{O_hQ}) \\ &= O_h + \lambda \overrightarrow{O_hQ}, \end{aligned} \tag{$*$}$$

The affine map h thus maps weighted points λQ to points Q' on the line through O_h and Q such that the ratio $\overline{O_hQ'} : \overline{O_hQ}$ of the directed line segments $\overline{O_hQ'}$, $\overline{O_hQ}$ is the weight λ. In other words h is the central projection of $\mathcal{B}\,A$ to A along the line through 0 and O_h (cf. figure 4.4).

The above discussion can be continued to obtain a complete characterization of the Eilenberg-Moore algebras of T. We have seen that each "multiplication" h of a T-algebra (A, h) is a central projection of $\mathcal{B}\,A$ to A along the line through 0 and $O_h \in A$. Conversely, we will show that given a point $P \in A$ the central projection $h_P : \mathcal{B}\,A \to A$ of $\mathcal{B}\,A$ to A along the line through 0 and P yields a multiplication of

a T-algebra (A, h). The diagram for the unit law commutes by construction of such an h and we only need to show the diagram for the associative law to commute.

Theorem 4.4.1. *Consider the monad $T = S\,\mathcal{B}$ given by the adjunction $\mathcal{B} \dashv S$. Given a $P \in A$ there is a unique affine map $h_P : S\,\mathcal{B}\,A \to A$ such that $h_P(0) = P$ and (A, h_P) is a T-algebra. The map $P \mapsto h_P$ yields a one-to-one correspondence of points in A and T-algebra structures on A.*

Proof. (i) We will show first that for a $P \in A$ there is exactly one affine map $h : TA \to A$ with $h|A = 1_A$, $\hat{h}|V_A = 1_{V_A}$ and $h(0) = P$. By remark 4.2.2(b) we have $V_{TA} = \mathcal{B}\,A \cong V_A \oplus KP$. Consider the projection $\mathrm{pr}_1 : \mathcal{B}\,A \cong V_A \oplus KP \to V_A$. By proposition 3.1.3(ii) there is a unique affine map $h : TA \to A$ with $\hat{h} = \mathrm{pr}_1$ and $h(0) = P$. We have $\hat{h}|V_A = 1_{V_A}$ by construction. Moreover, due to $TA = S\,\mathcal{B}\,A$ the point P is also a vector in V_{TA}, so $h(P) = h(0 + P) = P + \hat{h}(P) = P$ and thus

$$h(Q) = h(P) + \hat{h}(\overrightarrow{PQ}) = P + \overrightarrow{PQ} = Q, \qquad Q \in A.$$

This shows the existence of the asserted h. Conversely, given such an affine map $h : TA \to A$, under the isomorphism $\mathcal{B}\,A \cong V_A \oplus KP$ its associated linear map $\hat{h} : \mathcal{B}\,A \cong V_A \oplus KP \to V_A$ becomes the projection to the first factor. This is because we have on one hand $\hat{h}|V_A = 1_{V_A}$ and, on the other hand,

$$\hat{h}(P) = \hat{h}(P - 0) = \hat{h}(\overrightarrow{0P}) = \overrightarrow{h(0)h(P)} = \overrightarrow{PP} = 0$$

by $h|A = 1_A$, which in turn implies $\hat{h}|KP = 0$. Proposition 3.1.3(ii) yields thus the uniqueness of h.

(ii) According to the previous result (together with what we know from the unit law of a T-algebra (A, h) already) there can be at most one T-algebra (A, h_P) for each $P \in A$. To prove the theorem it suffices to show that any affine map $h_P : TA \to A$ from (i), i.e., one with $h_P|A = 1_A$, $\hat{h}_P|V_A = 1_{V_A}$ and $h_P(0) = P$, yields a T-algebra (A, h). Since the unit law is satisfied by construction, we only need to check the associativity law to hold. This amounts to show

$$h_P \circ Th_P = h_P \circ \mu_A.$$

Recall that for any affine map $f : A \to C$ the linear map $\mathcal{B}\,f$ is uniquely determined by its restriction to $A \hookrightarrow \mathcal{B}\,A$, for A spans $\mathcal{B}\,A$ as a vector space. It is thus sufficient to show the asserted identity to hold for points $x \in S\,\mathcal{B}\,A$. Any such x can be represented as a linear combination of points $Q_i \in A$: $x = \sum_i \lambda_i Q_i$. With this and $(*)$ we get

$$\begin{aligned} h_P \circ Th_P(x) &= h_P \circ Th_P(\sum_i \lambda_i Q_i) = h_P(P + \sum_i \lambda_i \overrightarrow{PQ_i}) = P + \sum_i \lambda_i \overrightarrow{PQ_i} \\ &= h_P(x). \end{aligned}$$

In remark 4.2.2 (e) we have seen that the counit ε_V restricts to an identity on $V \hookrightarrow \mathcal{B}\,SV$, and so

$$h_P \circ \mu_A(x) = h_P \circ S\varepsilon_{\mathcal{B}\,A}(x) = h_P(x).$$

This shows the associativity law, establishes (A, h_P) as a T-algebra and concludes the proof.

(qed)

Recall that any pointed affine space can be considered as a vector space by remark 4.3.4(c). In particular, $TA = S\,\mathcal{B}\,A$ is a vector space, and so becomes A for any T-algebra (A, h) with $h(0)$ as the zero vector. From this point of view we can read theorem 4.4.1 as follows: There is a bijection between T-algebra structures on A and vector space structures on A, i.e., affine isomorphisms $A \cong SV_A$. One would thus expect the categories of T-algebras $\mathrm{Aff_K}^T$ and $\mathrm{Vct_K}$ being equivalent. This is indeed the case and is a direct consequence of Beck's theorem and theorem 4.3.3.

Corollary 4.4.2. *The self-action functor S is monadic. In particular, the category of T-algebras $\mathrm{Aff_K}^T$ is equivalent to $\mathrm{Vct_K}$.*

Proof. By theorem 4.3.3 S preserves coequalizers; in particular, it preserves coequalizers of reflexive pairs. By proposition 3.2.6 S reflects isomorphisms. It is thus monadic by Beck's theorem in the version of [MM92, theorem IV.4.2].

(qed)

Remarks 4.4.3 (algebraic meaning of T-algebras)**.**

(a) We have read theorem 4.4.1 as a one-to-one correspondence between T-algebras and vector space structures on A, but only by making a detour via pointed spaces. In fact, each T-algebra (A, h_O) yields a vector space structure directly. For observe that h maps linear combinations of points $\sum_i \lambda_i P_i$ to points $O + \sum_i \lambda_i \overrightarrow{OP}$, and so assigns to $h(0) = O$ the role of the zero vector and to points the role of position vectors. Furthermore, the restriction $h_O|SV_A$ yields the ismorphism $SV_A \cong A$ determined by $0 \mapsto O$ and, conversely, h_O is determined by this isomorphism uniquely.

Of course, this is precisely what is behind the assertion that S is monadic, for the comparison functor

$$K_T : \mathrm{Vct_K} \to \mathrm{Aff_K}^T, \quad V \mapsto (SV, S\varepsilon_V), \quad f \mapsto Sf$$

is then an equivalence of categories, or, equivalently, full, faithful and isomorphism dense. Here ε is the counit of the adjunction $\mathcal{B} \dashv S$.

(b) By introducing Cartesian coordinates for an affine space A one has to fix an origin $O \in A$ first. Such a choice of origin is equivalent to giving an affine isomorphism $A \cong V_A$. In the second step one chooses a basis of V_A, or, equivalently, a maximal set of non-collinear points of A (including O). It is the first step, however, that identifies the space of points with the linear space V_A and allows for linear algebraic calculations with points of A. In this sense the T-algebras (A, h) can be regarded as precoordinatizations or linearizations of A.

(c) The category of vector spaces $\mathrm{Vct_K}$ is on one hand equivalent to the category of pointed affine spaces, and, on the other hand, to the category of T-algebras. The category $\mathrm{Aff_K}^T$ has thereby the advantage that it allows for an identification of the vector spaces as precoordinatizations of affine spaces. This is done by the forgetful functor $S_T : \mathrm{Aff_K}^T \to \mathrm{Aff_K}$, $(A, h) \mapsto A$, $f \mapsto f$ that forgets the multiplication map. For each affine space A the S_T-fiber over A, i.e., the subcategory of $\mathrm{Aff_K}^T$ given by the S_T preimage of the discrete category $\{A\}$, becomes the category of precoordinatizations of A. It is a discrete category itself.

4.5 The Strong Symmetric Monoidality of the Barycentric Functor

Comparing the dimensions of $\mathcal{B}(A \otimes C)$ with $\mathcal{B}A \otimes \mathcal{B}C$ for A, C in $\mathrm{FAff_K}$ shows that they have to be isomorphic. This suggests to ask whether $\mathcal{B}$ is strong symmetric monoidal. The answer to this is affirmative and yields our key result for the investigations to come.

Theorem 4.5.1. *The barycentric functor $\mathcal{B}$ is strong symmetric monoidal.*

Proof. We will follow our internal hom approach to tensor products to show this. The key idea is to internalize the adjunction $\mathcal{B} \dashv S$ as follows

$$S\,\underline{\mathrm{Vct}}_{\mathrm{K}}(\mathcal{B}A, V) \cong \underline{\mathrm{Aff}}_{\mathrm{K}}(A, SV), \qquad \text{natural in } A \text{ and } V. \tag{$*$}$$

For this we need to show that the natural bijection φ of the adjunction $\mathcal{B} \dashv S$ lifts to an isomorphism in $\mathrm{Aff_K}$. Indeed, each $\varphi_{A,C}$ is seen affine with itself as the associated linear map. As usual this follows from the representation of this bijection by the contravariant homming by the unit η and the pointwise structures involved (cf. (2.3) in section 2.2.2 for the representation and theorem 3.4.3 together with part (iii) of its proof for η^* to be affine with $\widehat{\eta^*} = \eta^*$):

$$\begin{aligned}\varphi_{A,V}(f+g) &= \eta_A^* S(f+g) = \eta_A^*(f+g)\\ &= \eta_A^* f + \eta_A^* g = \varphi_{A,V}(f) + \varphi_{A,V}(g), \qquad f, g \in \underline{\mathrm{Vct}}_{\mathrm{K}}(\mathcal{B}A, V).\end{aligned}$$

The adjunction $\mathcal{B} \dashv S$, the linear and affine hom-tensor adjunctions and $(*)$ yield the following chain of bijections

$$\begin{aligned}\mathrm{Vct}_{\mathrm{K}}(\mathcal{B}\,A_1 \otimes \mathcal{B}\,A_2, V) &\cong \mathrm{Vct}_{\mathrm{K}}(\mathcal{B}\,A_1, \underline{\mathrm{Vct}}_{\mathrm{K}}(\mathcal{B}\,A_2, V))\\ &\cong \mathrm{Aff}_{\mathrm{K}}(A_1, S\,\underline{\mathrm{Vct}}_{\mathrm{K}}(\mathcal{B}\,A_2, V))\\ &\cong \mathrm{Aff}_{\mathrm{K}}(A_1, \underline{\mathrm{Aff}}_{\mathrm{K}}(A_2, SV))\\ &\cong \mathrm{Aff}_{\mathrm{K}}(A_1 \otimes A_2, SV)\\ &\cong \mathrm{Vct}_{\mathrm{K}}(\mathcal{B}(A_1 \otimes A_2), V),\end{aligned}$$

natural in A_1, A_2 and V. The naturality in the latter yields by the Yoneda embedding (cf. the Yoneda argument in the proof of theorem 3.4.5)

$$\mathcal{B}_2(A_1, A_2) : \mathcal{B}\,A_1 \otimes \mathcal{B}\,A_2 \cong \mathcal{B}(A_1 \otimes A_2);$$

the naturality in the former yields once more by the Yoneda embedding that this isomorphism is natural in A_1, A_2, and hence that it is the first structure morphism of the asserted monoidality of $\mathcal{B}$. The second structure morphism is the isomorphism $\mathcal{B}_0 : K \cong \mathcal{B}\,*$ that sends 1 to the only point in $*$. Because we know the underlying maps of the structure morphisms for both tensor structures explicitly (cf. (2.4) in section 2.2.3 and corollary 3.4.13), the commutativity of the respective coherence diagrams can be verified directly by evaluation on points as well. For example, setting $V = \mathcal{B}\,A_1 \otimes \mathcal{B}\,A_2$ and following the image of 1_V under the above chain of isomorphisms yields the expected map:

$$\mathcal{B}_2(A_1, A_2)(\sigma_P \otimes \sigma_Q + \sum_i \nu_{v_i} \otimes \nu_{w_i}) = \sigma_{P \otimes Q + \sum_i v_i \otimes w_i}. \tag{4.3}$$

On the other hand, $(*)$ shows that we can use the internal homs with their structure morphisms translated by the respective hom-tensor adjunction and the Yoneda argument from the proof of theorem 3.4.5 to obtain the commutativity of the coherence diagrams, too (cf. the proof of corollary 3.4.6). As always, we refrain from chasing the diagrams explicitly here.

(qed)

5 The Slice Category $\mathrm{Vct_K}/K$ as a Proper Stage

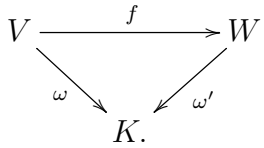
One consequence of the discussion in section 4.1 is that we can equally replace an affine space A by its barycentric calculus $\mathcal{B}\,A$, at least we can do so from the geometric point of view. The advantage of such a replacement is evident: we can do computations much more easily and with the powerful tools of linear algebra at hand, but we still keep the geometric meaning; that is in particular the distinction between points and displacement vectors.

One might argue that we could replace A equally well by its vector space of displacement vectors regarded as an affine space over itself. Here we have the tools of linear algebra at hand, too. The geometric meaning is kept also and appears as the distinction between position vectors, which are the points of SV_A, and the translation vectors, which are the displacement vectors of SV_A. The subtle difference between those two approaches lies in how they relate to the affine space A. For the barycentric calculus the space A is embedded as the subset of points of weight one. Such an embedding doesn't require extra structure; it is given naturally. In contrast, to relate A with SV_A we need to fix an origin, i.e. a particular point $O \in A$, hence we need to introduce further structure arbitrarily. This is why for an algebraic representation of A the barycentric calculus is more suitable than SV_A.

5.1 Augmented Vector Spaces

From the structural point of view the K-vector space $\mathcal{B}\,A$ can't replace A, because the geometric distinction between points and displacement vectors would be lost. According to the geometric discussion we need the additional information on the weights. The structure to replace A is thus not $\mathcal{B}\,A$ but the weight form $\omega : \mathcal{B}\,A \to K$. If one wants to put more emphasis on the domain, as we actually do in the geometric context, we speak about the *augmented K-vector space* $\omega : \mathcal{B}\,A \to K$. Recall from remark 4.2.2 (a) that the weight form can be obtained from the $\mathcal{B}$-image of the terminal map $A \to *$. It is thus this terminal map that reflects the geometric information on what points are in an affine space structurally.

The obvious notion of a morphism for augmented vector spaces is a linear map that respects the augmentations



With this the category of augmented vector spaces is seen to be the slice category $\mathrm{Vct_K}/K$. Any affine map $f : A \to C$ in $\mathrm{Aff_K}$ respects the terminal maps. Accordingly, $\mathcal{B}f$ respects the weight forms and is thus a morphism of augmented vector spaces. In summary, we can consider $\mathcal{B}$ as a functor $\mathcal{B} : \mathrm{Aff_K} \to \mathrm{Vct_K}/K$.

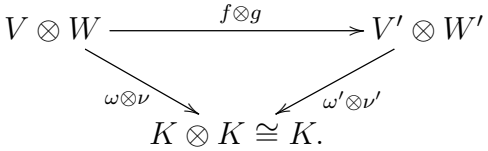
Not every augmented vector space can be considered as a barycentric calculus of an affine space though. In fact, we need the augmented vector space to be "non-degenerated". For non-empty affine spaces this means the augmentation to be a surjective map. This is because for a linear $\omega : \mathcal{B}A \to K$ it is $\omega^{-1}(1) \neq \emptyset$ if and only if $\omega : \mathcal{B}A \to K$ is surjective. Restated in categorical terms a linear map $\omega : V \to K$ in $\mathrm{Vct_K}$ can be regarded as a barycentric calculus of a non-empty affine space if and only if it is epi. Although the weight form is the zero map for an empty space $\mathcal{B}\emptyset$, and not an epi, in particular, not every zero map $0 : V \to K$ can be considered as the empty space; by remark 3.1.1 (c) and remark 4.2.2 (a) V has to be a trivial vector space. Accordingly, a *non-generated* augmented vector space is a linear form that is either epi or the initial zero-map $0 \to K$ for a trivial vector space 0. We define a full subcategory $(\mathrm{Vct_K}/K)^a < \mathrm{Vct_K}/K$ that has as objects the non-degenerated augmented vector spaces.

5.2 The Natural Monoidal Structure on $\mathrm{Vct_K}/K$

Before we will actually formulate and prove that $\mathcal{B} : \mathrm{Aff_K} \to (\mathrm{Vct_K}/K)^a$ is an equivalence of categories, we shall introduce a symmetric monoidal structure on $\mathrm{Vct_K}/K$; it is the structure induced by the linear tensor structure of $\mathrm{Vct_K}$. The tensor-product bifunctor is defined on a pair of augmented vector spaces $(\omega : V \to K, \nu : W \to K)$ as

$$\omega \otimes \nu : V \otimes W \to K \otimes K \cong K,$$

and on a pair of morphisms $(f : \omega \to \omega', g : \nu \to \nu')$ between augmented vector spaces as $f \otimes g$



Here $\omega \otimes \nu$ and $f \otimes g$ are the respective images of the pairs of linear maps (ω, ν) and (f, g) under the linear tensor bifunctor $\otimes : \mathrm{Vct_K} \times \mathrm{Vct_K} \to \mathrm{Vct_K}$. The isomorphism $K \otimes K \cong K$ is the one given by the left-unit λ_K or right-unit law ρ_K of the linear

tensor structure of $\mathrm{Vct_K}$ (recall the identity $\lambda_K = \rho_K$); it is induced by $1 \otimes 1 \mapsto 1$. The structure morphisms for this tensor-product on $\mathrm{Vct_K}/K$ are induced by the respective structure morphisms α, ρ, λ, τ of the linear tensor structure of $\mathrm{Vct_K}$ in the obvious way:

$$\begin{array}{ccc}
U \otimes (V \otimes W) & \xrightarrow{\alpha_{U,V,W}} & (U \otimes V) \otimes W \\
{\scriptstyle \omega_U \otimes (\omega_V \otimes \omega_W)}\downarrow & & \downarrow{\scriptstyle (\omega_U \otimes \omega_V) \otimes \omega_W} \\
K \otimes (K \otimes K) & \xrightarrow{\alpha_{K,K,K}} & (K \otimes K) \otimes K \\
{\scriptstyle 1_K \otimes \lambda_K}\downarrow & & \downarrow{\scriptstyle \rho_K \otimes 1_K} \\
K \otimes K & \xrightarrow{1_{K\otimes K}} & K \otimes K \\
& {\scriptstyle \lambda_K}\searrow \quad \swarrow{\scriptstyle \rho_K} & \\
& K, &
\end{array}$$

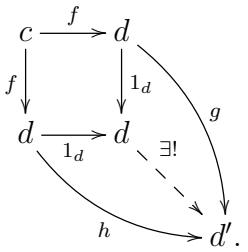
$$\begin{array}{ccc}
U \otimes V & \xrightarrow{\tau_{U,V}} & V \otimes U \\
{\scriptstyle \omega_U \otimes \omega_V}\downarrow & & \downarrow{\scriptstyle \omega_U \otimes \omega_V} \\
K \otimes K & \xrightarrow{\tau_{K,K} = 1_{K\otimes K}} & K \otimes K \\
& {\scriptstyle \lambda_K}\searrow \quad \swarrow{\scriptstyle \lambda_K} & \\
& K, &
\end{array}$$

$$\begin{array}{ccc}
K \otimes V & \xrightarrow{\lambda_V} & V \\
{\scriptstyle 1_K \otimes \omega_V}\downarrow & & \downarrow{\scriptstyle \omega_V} \\
K \otimes K & \xrightarrow[\lambda_K]{} & K,
\end{array}
\qquad
\begin{array}{ccc}
V \otimes K & \xrightarrow{\rho_V} & V \\
{\scriptstyle \omega_V \otimes 1_K}\downarrow & & \downarrow{\scriptstyle \omega_V} \\
K \otimes K & \xrightarrow[\rho_K]{} & K.
\end{array}$$

The fact that the structure morphisms do yield morphisms of augmented vector spaces is a consequence of their naturality and the coherence axioms; it can be seen as indicated in the commutative diagrams. As for the commutativity of the coherence diagrams of a symmetric monoidal category this follows from the commutativity of the respective diagrams for the linear tensor structure of $\mathrm{Vct_K}$ and the naturality of the structure morphisms. Once again we refrain from chasing the diagrams explicitly here.

As with $\mathrm{Vct_K}$ we will understand $\mathrm{Vct_K}/K$ as a symmetric monoidal category $(\mathrm{Vct_K}/K, \otimes, 1_K)$ with this monoidal structure. Note this monoidal structure to restrict to $(\mathrm{Vct_K}/K)^a$, for 1_K is epi, we have $\omega \otimes 0 \cong 0 \otimes \omega$ to be a zero map $0 \to K$ for any linear $\omega : V \to K$ and any other such initial zero map 0 with codomain K as well as $\omega \otimes \nu$ to be epi when $\omega : V \to K$ and $\nu : W \to K$ are epis. Both of the latter facts are a consequence of the hom-tensor adjunction in $\mathrm{Vct_K}$. The first fact is due to the distributivity law of the linear tensor product

(this can be seen in the same way as the distributivity of the affine tensor product, cf. part (v) of the proof of theorem 3.4.5). The second fact follows from $\omega \otimes \nu = \omega \otimes 1_K \circ 1_V \otimes \nu$ and $(-) \otimes K$, $V \otimes (-)$ to be cocontinuous and hence to preserve epis (cf. dual of [Mac98, theorem V.5.1]). For the preservation of epis note that in any category C the property of a morphism $f : c \to d$ to be epi can be expressed as the identity morphism 1_d to yield a a cokernel pair for this morphism (cf. [Mac98, exercise III.4.4]), which is a particular type of a pushout and hence a colimit



thus any cocontinuous functor is seen to preserve epis. Henceforth, we will consider $(\mathrm{Vct_K}/K)^a$ as a symmetric monoidal category with this restricted monoidal structure und thus as a strict symmetric monoidal subcategory[1] of $\mathrm{Vct_K}/K$.

5.3 The Equivalence of the Categories of Affine Spaces and Non-degenerated Augmented Vector Spaces

In this section we will formulate and give a proof of the equivalence of the categories of affine apaces and non-degenerated augmented vector spaces. This result is one of our main results and of fundamental importance from the structural point of view, for it yields a structurally very convenient description of affine spaces in terms of linear structures only. We will in particular need it to establish a similar such equivalence for affine algebras, which we recall to be the internal monoids in the monoidal $\mathrm{Aff_K}$ and thus the natural structure for a geometric algebra of an affine space.

Theorem 5.3.1. *The barycentric functor $\mathcal{B} : \mathrm{Aff_K} \to \mathrm{Vct_K}/K$ induces a symmetric monoidal adjoint equivalence $(\mathrm{Vct_K}/K)^a \simeq \mathrm{Aff_K}$ as follows. The functor $\mathcal{B}$ corestricts to a functor $\mathcal{B} : \mathrm{Aff_K} \to (\mathrm{Vct_K}/K)^a$. It has the functor $\mathcal{A} : (\mathrm{Vct_K}/K)^a \to \mathrm{Aff_K}$, which maps an augmented vector space ω to the affine space $\mathcal{A}\omega$ with point set $\omega^{-1}(1)$ and associated vector space $\ker\omega$, as a right adjoint. The unit $\eta : I_{\mathrm{Aff_K}} \dot{\to} \mathcal{A}\mathcal{B}$ and counit $\varepsilon : \mathcal{B}\mathcal{A} \dot{\to} I_{(\mathrm{Vct_K}/K)^a}$ are isomorphisms. When we use the representation of the barycentric functor from section 4.2 (cf. also theorem 4.2.1) then the components of the unit and counit can be given as follows*

$$\eta_A : A \to \mathcal{A}\mathcal{B}A, \quad P \mapsto \sigma_P, \qquad \varepsilon_\omega : \mathcal{B}\mathcal{A}\omega \to \omega, \quad \sum_i \lambda_i \sigma_{P_i} \mapsto \sum_i \lambda_i P_i$$

[1] This means the respective inclusion functor to be strict symmetric monoidal.

with $P_i \in \omega^{-1}(1)$ and $\lambda_i \in K$. Moreover, $\mathcal{B}$ and $\mathcal{A}$ are symmetric monoidal functors and η and ε are monoidal natural transformations.

Proof. To begin with recall the construction of $\mathcal{B}\,A$ as the vector subspace $\mathcal{B}\,A \leq \mathrm{Aff}_{\mathrm{K}}(A, SU_V A)$ generated by the maps $\sigma_P : Q \mapsto \overrightarrow{QP}$ for each P in A. With this and remark 4.2.2 (a) the weight form $\omega_A : \mathcal{B}\,A \to K$ is the (unique) linear map that maps each generator σ_P, $P \in A$ to $1 \in K$. Up to the monoidality of $\mathcal{A}$, η and ε the assertion then basically follows from the preceding discussion, theorem 4.2.1, theorem 4.5.1 and $\mathcal{A}$ to reflect isomorphisms.

(i) In more detail, by the above discussion we see $(\mathrm{Vct}_{\mathrm{K}}/K)^a$ to be designed in such a way that $\mathcal{B}$ can be corestricted respectively. By the construction of the monoidal structure on $\mathrm{Vct}_{\mathrm{K}}/K$, its restriction to $(\mathrm{Vct}_{\mathrm{K}}/K)^a$ and theorem 4.5.1 we see both $\mathcal{B}$, the original lift and the corestriction, to be strong symmetric monoidal.

(ii) We shall show the functoriality of $\mathcal{A}$ and its property to reflect isomorphisms. To see the functoriality of $\mathcal{A}$ observe that any morphism $f : \omega \to \nu$ in $(\mathrm{Vct}_{\mathrm{K}}/K)^a$ restricts and corestricts to a set-theoretic map $f : \omega^{-1}(1) \to \nu^{-1}(1)$ and a linear map $\hat{f} : \ker\omega \to \ker\nu$ such that $(f, \hat{f})$ is affine. Suppose now that $\mathcal{A}\,f$ is an isomorphism, then $f : \omega^{-1}(1) \to \nu^{-1}(1)$ is a bijection and $\hat{f} : \ker\omega \to \ker\nu$ is a linear isomorphism. Note a morphism $g : \omega \to \mu$ to be an isomorphism in $\mathrm{Vct}_{\mathrm{K}}/K$ if and only if the linear map $g : \mathrm{dom}\,\omega \to \mathrm{dom}\,\nu$ is an isomorphism. We thus need to show $f : \mathrm{dom}\,\omega \to \mathrm{dom}\,\nu$ to have a trivial kernel and to be surjective. The trivial kernel of f is a consequence of $\hat{f} : \ker\omega \to \ker\nu$ being injective and $0 = f(x)$ to imply $0 = \nu(f(x)) = \omega(x)$, that is $\ker f \leq \ker\omega$. Let $y \in \mathrm{dom}\,\nu$. If $\nu(y)$ equals zero, i.e. $y \in \ker\nu$, then there is an $x \in \ker\omega$ such that $f(x) = y$, for $\hat{f}$ is surjective. Otherwise there is a $Q \in \nu^{-1}(1)$ such that $y = \nu(y)Q$. Because $f : \omega^{-1}(1) \to \nu^{-1}(1)$ is surjective there is a $P \in \omega^{-1}(1)$ that is mapped to Q by f. This yields $f(\nu(y)P) = y$ by the linearity of f. With this f is seen to be bijective and an isomorphism as asserted.

(iii) The argument we gave above to show the lift of $\mathcal{B} : \mathrm{Aff}_{\mathrm{K}} \to \mathrm{Vct}_{\mathrm{K}}$ to $\mathrm{Aff}_{\mathrm{K}} \to \mathrm{Vct}_{\mathrm{K}}/K$ together with the explicit description of the mapping $\omega_A : \mathcal{B}\,A \to K$ imply that each component of the unit given in theorem 4.2.1 can be corestricted to $\omega_A^{-1}(1)$. Moreover, it shows its corestriction η_A to be still an universal morphism from A to $\mathcal{A}$. In fact, $\omega_A^{-1}(1)$ is seen to be the image of (the set-theoretic) η_A and hence η_A to be surjective. By theorem 4.2.1 it is bijective and hence an isomorphism by corollary 3.1.4. As for the component ε_ω of the counit we know it to be the left adjunct of the identity $1_{\mathcal{A}\omega}$ by (2.3) in section 2.2.2:

$$\mathcal{A}\,\varepsilon_\omega \circ \eta_{\mathcal{A}\omega} = 1_{\mathcal{A}\omega}.$$

This shows $\mathcal{A}\,\varepsilon_\omega = \eta^{-1}_{\mathcal{A}\,\omega}$ to be an ismorphism and hence ε_ω to be an ismorphism by (ii). We have thus established an adjunction $\mathcal{B} \dashv \mathcal{A}$ by [Mac98, theorem IV.I.2] and moreover an adjoint equivalence $(\mathrm{Vct_K}/K)^a \simeq \mathrm{Aff_K}$.

(iv) We shall show the assertions on monoidality next. First of all, we will use the strong symmetric monoidality of $\mathcal{B}$ together with the adjunction to obtain the strong symmetric monoidality of $\mathcal{A}$. We have the following chain of natural bijections

$$\begin{aligned}\mathrm{Aff_K}(\mathcal{A}\,\omega \otimes \mathcal{A}\,\mu, \mathcal{A}(\omega\otimes\mu)) &\cong (\mathrm{Vct_K}/K)^a(\mathcal{B}(\mathcal{A}\,\omega\otimes\mathcal{A}\,\mu), \omega\otimes\mu)\\ &\overset{(\mathcal{B}_2(\mathcal{A}\,\omega,\mathcal{A}\,\mu))^*}{\cong} (\mathrm{Vct_K}/K)^a(\mathcal{B}\,\mathcal{A}\,\omega\otimes\mathcal{B}\,\mathcal{A}\,\mu, \omega\otimes\mu)\\ &\overset{((\varepsilon_\omega\otimes\varepsilon_\mu)^{-1})^*}{\cong} (\mathrm{Vct_K}/K)^a(\omega\otimes\mu, \omega\otimes\mu),\end{aligned}$$

where the first bijection denotes the one from the adjunction $\mathcal{B} \dashv \mathcal{A}$, $\mathcal{B}_2(\mathcal{A}\,\omega, \mathcal{A}\,\mu)$ the first structure morphism of the monoidal $\mathcal{B}$ and $\varepsilon_\omega \otimes \varepsilon_\mu$ the linear tensor product of the respective components of the counit. The inverse image $\mathcal{A}_2(\omega,\mu)$ of $1_{\omega\otimes\mu}$ under this chain of bijections is thus a natural candidate for the first monoidal structure morphism for $\mathcal{A}$. Due to the naturality of each bijection it is seen to be a natural transformation $\mathcal{A}_2 : \otimes \circ \mathcal{A}\times\mathcal{A} \dot{\to} \mathcal{A}\circ\otimes$, where the first tensor-bifunctor denotes the affine tensor product and the second tensor-bifunctor the tensor-product in $(\mathrm{Vct_K}/K)^a$. Following the chain of bijections backwards it can be written explicitly as

$$\begin{aligned}\mathcal{A}_2(\omega,\mu) &= \varphi_{\mathcal{A}\,\omega\otimes\mathcal{A}\,\mu,\omega\otimes\mu}(\varepsilon_\omega\otimes\varepsilon_\mu \circ \mathcal{B}_2(\mathcal{A}\,\omega,\mathcal{A}\,\mu)^{-1})\\ &= \mathcal{A}(\varepsilon_\omega\otimes\varepsilon_\mu \circ \mathcal{B}_2(\mathcal{A}\,\omega,\mathcal{A}\,\mu)^{-1}) \circ \eta_{\mathcal{A}\,\omega\otimes\mathcal{A}\,\mu},\end{aligned}$$

where $\varphi_{\mathcal{A}\,\omega\otimes\mathcal{A}\,\mu,\omega\otimes\mu}$ denotes the respective component of the natural bijection from the adjunction and we have used its representation by the unit as given in (2.3) in section 2.2.2. By (iii) this also shows $\mathcal{A}_2(\omega,\mu)$ to be an isomorphism. The second structure morphism is the natural isomorphism $\mathcal{A}_0 : * \cong \mathcal{A}\,1_K$, which maps the unique point in $*$ to the unique point 1 in $\mathcal{A}\,1_K$.

We shall show the commutativity of the respective coherence diagrams by evaluating them on points. For that we need to know the underlying set-theoretic map of $\mathcal{A}_2(\omega,\mu)$ explicitly. By theorem 3.4.10(ii) we know the points of $\mathcal{A}\,\omega\otimes\mathcal{A}\,\mu$. Together with (iii) this yields the underlying map of $\eta_{\mathcal{A}\,\omega\otimes\mathcal{A}\,\mu}$. The underlying set-theoretic map of $\mathcal{A}(\mathcal{B}_2(\mathcal{A}\,\omega,\mathcal{A}\,\mu)^{-1})$ can be derived from (4.3) as

$$\mathcal{A}(\mathcal{B}_2(\mathcal{A}\,\omega,\mathcal{A}\,\mu)^{-1})(\sigma_{P\otimes Q+\sum_i v_i\otimes w_i}) = \sigma_P\otimes\sigma_Q + \sum_i \nu_{v_i}\otimes\nu_{w_i},$$

because $P\otimes Q + \sum_i v_i\otimes w_i$, $P\in\mathcal{A}\,\omega$, $Q\in\mathcal{A}\,\mu$, $v_i\in\ker\omega$, $w_i\in\ker\mu$ are the points of $\mathcal{A}\,\omega\otimes\mathcal{A}\,\mu$ by theorem 3.4.10(ii). Here the $\nu_{v_i}\in\mathcal{B}\,\mathcal{A}\,\omega$, $\nu_{w_i}\in\mathcal{B}\,\mathcal{A}\,\mu$

are the constant maps, which map each point to the vector indicated in the index. Recall those to represent the displacement vectors of $\mathcal{A}\,\omega$ and $\mathcal{A}\,\mu$, respectively. Moreover, they are seen to constitute the images of $\hat{\eta}_{\mathcal{A}\,\omega}$ and $\hat{\eta}_{\mathcal{A}\,\mu}$, respectively (cf. (4.1) in section 4.2 and the proof of theorem 4.2.1). It remains to see the underlying map of $\mathcal{A}(\varepsilon_\omega \otimes \varepsilon_\mu)$. The identity $\mathcal{A}\,\varepsilon_\omega = \eta_{\mathcal{A}\,\omega}^{-1}$ gives us the restriction $\varepsilon_\omega|_{\operatorname{im}\eta_{\mathcal{A}\,\omega}}$. But the set $\operatorname{im}\eta_{\mathcal{A}\,\omega}$ spans $\operatorname{dom}\mathcal{B}\,\mathcal{A}\,\omega$ as a vector space, so ε_ω is uniquely determined by this restriction. In particular, we get $\varepsilon_\omega(\nu_{\overrightarrow{P_1P_2}}) = \varepsilon_\omega(\sigma_{P_2} - \sigma_{P_1}) = P_2 - P_1 = \overrightarrow{P_1P_2}$ for $P_1, P_2 \in \mathcal{A}\,\omega$. The underlying set-theoretic map of $\mathcal{A}(\varepsilon_\omega \otimes \varepsilon_\mu)$ then follows from what we know about the underlying map of the linear tensor product of linear maps. Alltogether this yields

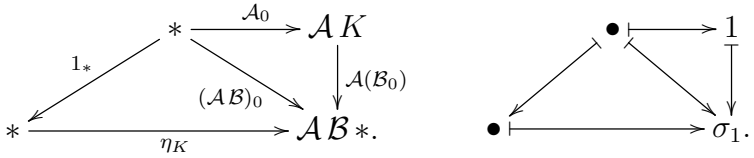
$$\begin{aligned}
&\mathcal{A}_2(\omega,\mu)(P \otimes Q + \sum_i v_i \otimes w_i)\\
=&\,\mathcal{A}(\varepsilon_\omega \otimes \varepsilon_\mu) \circ \mathcal{A}(\mathcal{B}_2(\mathcal{A}\,\omega, \mathcal{A}\,\mu)^{-1})(\sigma_{P\otimes Q + \sum_i v_i \otimes w_i})\\
=&\,\mathcal{A}(\varepsilon_\omega \otimes \varepsilon_\mu)(\sigma_P \otimes \sigma_Q + \sum_i \nu_{v_i} \otimes \nu_{w_i})\\
=&P \otimes Q + \sum_i v_i \otimes w_i, \qquad P \in \mathcal{A}\,\omega,\ Q \in \mathcal{A}\,\mu,\ v_i \in \ker\omega,\ w_i \in \ker\mu.
\end{aligned}$$

In other words, applying $\mathcal{A}_2(\omega,\mu)$ to points amounts to a substitution of the affine tensor product by the linear tensor product. By corollary 3.4.13 and (2.4) we know the underlying maps of the structure morphisms for the linear and affine tensor structure explicitly. With this the commutativity of the coherence diagrams for $\mathcal{A}_2$ and $\mathcal{A}_2$ becomes apparent.

(v) It remains to show the monoidality of η and ε. Since we know the underlying maps explicitly as well as the structure morphisms, the respective diagrams can be shown commutative by evaluation on points directly. For example, for η we have

$$\begin{array}{ccccc}
A \otimes C & & \xrightarrow{\quad 1_{A\otimes C} \quad} & & A \otimes C \\
\downarrow{\scriptstyle \eta_A \otimes \eta_C} & & & & \downarrow{\scriptstyle \eta_{A\otimes C}} \\
\mathcal{A}\,\mathcal{B}\,A \otimes \mathcal{A}\,\mathcal{B}\,C & \xrightarrow[\mathcal{A}_2(\mathcal{B}\,A, \mathcal{B}\,C)]{} & \mathcal{A}(\mathcal{B}\,A \otimes \mathcal{B}\,C) & \xrightarrow[\mathcal{A}(\mathcal{B}_2(A,C))]{} & \mathcal{A}\,\mathcal{B}(A \otimes C), \\
P \otimes Q + \sum_i v_i \otimes w_i & & \longmapsto & & P \otimes Q + \sum_i v_i \otimes w_i \\
\downarrow & & & & \downarrow \\
\sigma_P \otimes \sigma_Q + \sum_i \nu_{v_i} \otimes \nu_{w_i} & \mapsto & \sigma_P \otimes \sigma_Q + \sum_i \nu_{v_i} \otimes \nu_{w_i} & \longmapsto & \sigma_{P\otimes\sigma_Q + \sum_i v_i \otimes w_i}
\end{array}$$

and



The commutativity of the respective diagrams for the counit ε are a consequence of η and thus η^{-1} being monoidal, $\mathcal{A}$ being faithful and $\mathcal{A}\,\varepsilon_\omega = \eta^{-1}_{\mathcal{A}\,\omega}$. Both, the monoidality of η and ε can be also obtained by general abstract arguments.

(qed)

Remark 5.3.2 (finite dimensions)**.**
As usual, a respective finite dimensional result holds as well. To begin with, we observe that $\mathrm{FVct_K}/K < \mathrm{Vct_K}/K$ becomes a monoidal subcategory when we either restrict the monoidal structure of $\mathrm{Vct_K}/K$ or when we use the fact that $\mathrm{FVct_K} < \mathrm{Vct_K}$ is a strict symmetric monoidal subcategory and repeat the construction of the monoidal structure for $\mathrm{Vct_K}/K$ with $\mathrm{FVct_K}$ instead. We can define $(\mathrm{FVct_K}/K)^a$ again as the full subcategory with objects the epis $V \to K$ or initial zero maps $0 \to K$ for a trivial vector space 0. The functor $\mathcal{A}$ restricts and corestricts to a functor $\mathcal{A} : (\mathrm{FVct_K}/K)^a \to \mathrm{FAff_K}$ by definition 3.1.2(3). The functor $\mathcal{B}$ restricts and corestricts to a functor $\mathcal{B} : \mathrm{FAff_K} \to (\mathrm{FVct_K}/K)^a$ by remark 4.2.3(a). As all categories of finite dimensional structures involved are shown to be full strict symmetric monoidal subcategories, we see the adjunction $\mathcal{B} \dashv \mathcal{A}$ to restrict to finite dimensions and to yield a symmetric monoidal adjoint equivalence $\mathrm{FAff_K} \simeq (\mathrm{FVct_K}/K)^a$.

Remarks 5.3.3.

(a) In the previous theorem we have established $\mathcal{B} \dashv \mathcal{A}$ as an adjoint equivalence. This implies in particular that $\mathcal{A}$ is also the left adjoint of $\mathcal{B}$; to see the adjunction $\mathcal{A} \dashv \mathcal{B}$ just take $\eta^{-1} : \mathcal{A}\,\mathcal{B} \dot{\to} I_{\mathrm{Aff_K}}$ as its counit and $\varepsilon^{-1} : I_{(\mathrm{Vct_K}/K)^a} \dot{\to} \mathcal{B}\,\mathcal{A}$ as its unit (cf. the discussion of an adjoint equivalence in [Mac98, section IV.4]). This shows $\mathcal{A}$ and $\mathcal{B}$ to be continuous and cocontinuous. Identifying both categories via this equivalence $(\mathrm{Vct_K}/K)^a \simeq \mathrm{Aff_K}$ will hence respect the formation of small limits and colimits. Similarly, the established strong monoidality of this equivalence shows such an identification to respect the monoidal structures, too. By [Mac98, theorem IV.4.1] $\mathcal{B}$ and $\mathcal{A}$ are both full, faithfull and isomorphism-dense. This shows the equivalence to respect seperators and coseparators. In fact, the identification of $(\mathrm{Vct_K}/K)^a$ with $\mathrm{Aff_K}$ via $\mathcal{B} \dashv \mathcal{A}$ is seen to respect everything we have established for $\mathrm{Aff_K}$ up to now.

(b) With the category $(\mathrm{Vct_K}/K)^a$, the identification of $(\mathrm{Vct_K}/K)^a$ with $\mathrm{Aff_K}$ and the notion of an augmented vector space we have a pure algebraic viewpoint on the notion of an affine space. In particular, no underlying sets of points appear whatsoever and everything is cast within linear structures. It is remarkable that the very construction involved is the general construction of a slice category over a distinguished object in $\mathrm{Vct_K}$, namely an object that is a separator and coseparator; but it is exactly because of this generality we have to pay close attention to how $(\mathrm{Vct_K}/K)^a$ is identified with $\mathrm{Aff_K}$, that is to how the algebraic notion of an augmented vector space attains its geometric meaning as an affine space.

We will henceforth consider the category $\mathrm{Aff_K}$ as the *geometric presentation* and the category $(\mathrm{Vct_K}/K)^a$ as the *algebraic presentation* of the notion of an affine space.

(c) Many results of chapter 3 can be seen and proved much more easily in $(\mathrm{Vct_K}/K)^a$. We have already seen this in remark 4.2.2 (i) when we gave a representation of the coproduct of affine spaces without resorting to base points. Similarly, with the tensor-product in $\mathrm{Vct_K}/K$ the representation of the affine tensor product given in theorem 3.4.10 follows easily from what we know about the linear tensor product already. Indeed, the construction of the whole monoidal structure of $(\mathrm{Vct_K}/K)^a$ is much more easily done than the construction of the affine tensor structure. On the other hand, an explicit internal hom is easier to see in $\mathrm{Aff_K}$ than in $(\mathrm{Vct_K}/K)^a$.

(d) Now that we have established a pure algebraic viewpoint on affine spaces we can look for algebraic structures involving augmented vector spaces and give them a geometric meaning. One such structure is the notion of a coalgebra or internal comonoid (V, Δ, ε) in $\mathrm{Vct_K}$ (cf. [Swe69, section 1.0]). It can be obtained by dualizing the defining diagrams of an internal monoid (cf. section 2.5). So, instead of a multiplication it has a comultiplication $\Delta : V \to V \otimes V$ and instead of a unit it has a counit $\varepsilon : V \to K$, such that the respective duals of associativity, i.e. coassociativity, and unit law, i.e. counit law, hold. Due to the counit each coalgebra is an augmented vector space. Due to the counit law, which in particular asserts $\rho_V \circ (1_V \otimes \varepsilon) \circ \Delta = 1_V$ with $\rho_V : V \otimes K \cong V$ being the right-unit law of the monoidal structure, ε can't be a zero map unless V itself is a trivial vector space. It is thus seen to be an object in $(\mathrm{Vct_K}/K)^a$ and hence a barycentric calculus of the affine space with set of points $\varepsilon^{-1}(1)$ and $\ker\varepsilon$ as vector space of displacements.

(e) (Affine structures on a vector space). We can read the equivalence of affine spaces and non-degenerate augmented vector spaces also as follows: for a given non-trivial K-vector space V a non-zero linear form $\omega : V \to K$ can be considered as an affine structure on V with $\omega^{-1}(1)$ the set of points and $\ker\omega$ the

K-vector space of displacement vectors. We have used this already in the proof of theorem 4.3.1.

This gives $V^* = \underline{\hom}(V, K)$ the interpretation as the K-vector space generated by all possible affine structures on V.

(f) (On the relation with projective spaces). We take up the final remark in section 4.1 and compare the barycentric calculus with the representation of affine spaces in projective geometry. In projective geometry an affine space A with dimension n is represented as a projective space P of projective dimension n together with a fixed projective hyperplane $H < P$, whose points are considered as the "points at infinity" for A, whereas the points in $P \setminus H$ are the "proper" points of A (cf. [KM03, Definition 8.4.6]). Indeed, it was precisely the idea to mathematically formalize the fact that we perceive any two parallel lines as to intersect at infinity when we look along these lines, which led to the notion of a projective space.

When we model projective spaces on finite dimensional vector spaces as we have done in section 3.5 then we have $P = P(V)$ for a vector space V with $\dim V = n + 1$ and $H = P(U)$ for $U < V$ with codimension 1. The geometric idea suggests to construct a simple transitive action of U on the set $A = P(V) \setminus H$. The problem is that given the one-dimensional vector spaces $P, Q \in A$ there seems to be no natural choice of $u \in U$ such that $P + u = Q$. Therefore, to define an action we will use a "coherent" choice of representants $v_P \in P$ for each $P \in A$ such that $(v_P - v_Q) \in U$ for each $P, Q \in A$ (compare e.g. with the proof of [KM03, Hilfssatz 8.4.5] or the proof of [MB99, Appendix, theorem 10.19]). Given such a coherent choice of representants an action of U on A is given as $(P, u) \mapsto K(v_P + u)$. Note this action to depend on the choice of representants. In more detail, for any other choice of representants v'_P, $P \in A$ such that $v'_P - v'_Q \in U$ for all $P, Q \in A$ there is a scalar $\lambda \in K \setminus \{0\}$ such that $v'_P = \lambda v_P$. To see this observe any such coherent choice of representants to be equivalent to a linear form $\omega : V \to K$ with $\ker \omega = U$ such that all v_P, $P \in A$ are mapped to one $\mu \in K \setminus \{0\}$. W.l.o.g. we can assume $\mu = 1$. Let ω be such a linear form for the first choice $(v_P)_{P \in A}$ and ω' for the second choice $(v'_P)_{P \in A}$. Due to $\ker \omega = U = \ker \omega'$ and since both linear forms are non-zero, there is a $\lambda \in K \setminus \{0\}$ such that $\omega' = \lambda \omega'$ by the fundamental theorem on homomorphisms for vector spaces (2.1). The one-dimensionality of each $P \in A$ implies the existence of a $\lambda_P \in K \setminus \{0\}$ such that $v'_P = \lambda_P v_P$; the previous identity for the linear forms then yields $\lambda_P = \lambda$ for each $P \in A$. This implies for the action of U on A to scale with λ, too, i.e. $P + 1/\lambda\, u = P +' u$, $P \in A$, $u \in U$ where the prime denotes the action defined in terms of the second choice $(v'_P)_{P \in A}$.

We can conclude the following two facts. First, in spite of the geometric insight,

structurally, the above construction of an affine structure on $A = P \setminus H$ for a given projective space $P = P(V)$ and projective hyperplane $P(U) = H < P$ is not natural; it involves and depends on an additional choice of a non-zero linear form[2] $\omega : V \to K$. Second, this linear form ω is an object of $(\mathrm{Vct_K}/K)^a$ and the representants $v_P \in \omega^{-1}(1)$ become the actual points of the affine space $\mathcal{A}\,\omega$. In particular, we see that the geometric passage from a projective space to an affine space involves the structure of an augmented vector space in $(\mathrm{Vct_K}/K)^a$ and hence a barycentric calculus. Conversely, given a barycentric calculus $\mathcal{B}\,A$ of an affine space A with weight form $\omega : \mathcal{B}\,A \to K$ its projectivization $P(\mathcal{B}\,A)$ yields a projective space that can be understood as the enlargement of A by the points at infinity $P(\ker \omega)$. This shows once more the barycentric calculus to keep up with its role as an algebraic representation of an affine space. Additionally, we get a geometric interpretation of the "points" of zero weight to be the points at infinity. However, this interpretation is only meaningful for an intersection theory. One should clearly distinct between the barycentric calculus as an algebraic structure and an affine space or a projective space as geometric structures.

5.4 The Category $\mathrm{Vct_K}/K$ as a Fusion of $\mathrm{Vct_K}$ and $\mathrm{Aff_K}$

We saw that affine spaces correspond only to objects of a proper full subcategory of $\mathrm{Vct_K}/K$. We shall show the other objects to correspond to vector spaces next. The objects in $\mathrm{Vct_K}/K$ that aren't epi are the zero maps $V \to K$. We define the full subcategory $(\mathrm{Vct_K}/K)^v$ that has as objects the zero maps. Accordingly, $(\mathrm{Vct_K}/K)^v$ and $(\mathrm{Vct_K}/K)^a$ have as common objects only the inital zero maps with codomain K. Furthermore, we define the following functor

$$\mathcal{V} : \mathrm{Vct_K} \to (\mathrm{Vct_K}/K)^v, \quad V \mapsto (V \xrightarrow{0} K), \quad f \mapsto f,$$

where $V \xrightarrow{0} K$ denotes the (unique) zero map from V to K. Note this to be well-defined, for any linear map respects the zero maps. This functor has an inverse. It is the restriction of the canonical projection functor

$$\mathcal{P} : \mathrm{Vct_K}/K \to \mathrm{Vct_K}, \quad \omega \mapsto \mathrm{dom}\,\omega, \quad f \mapsto f$$

to $(\mathrm{Vct_K}/K)^v$. This shows $(\mathrm{Vct_K}/K)^v \cong \mathrm{Vct_K}$. In fact, even more is true. When we recall the monoidal structure on $\mathrm{Vct_K}/K$ we can see $\mathcal{P}$ strict symmetric monoidal immediately from the construction of this monoidal structure. Regarding $\mathcal{V}$ (extended to $\mathrm{Vct_K}/K$) this is not true. However, the extension of $\mathcal{V}$ is seen to be

[2] This dependence on the choice of representants is neither mentioned in [KM03] nor in [MB99]. In both books the respective theorem is proved by even fixing a particular vector in V as additional structure and not just a non-zero linear form.

comonoidal[3]. We have the peculiar situation that despite the first structure morphism $\mathcal{V}_2 : \mathcal{V} \circ \otimes \dot{\to} \otimes \circ \mathcal{V} \times \mathcal{V}$ being the identity, the second structure morphism $\mathcal{V}_0 : \mathcal{V}(K) \to 1_K$ can't be an isomorphism, for $\mathcal{V}(K) = 0_K : K \to K$ is the zero map. Observe by the way that there is no morphism $1_K \to 0_K$ in $\mathrm{Vct_K}/K$, so there is no way to make the extension of $\mathcal{V}$ a monoidal functor.

A further structural reason for the comonoidality can be found in the induced monoidal structure of the subcategory $(\mathrm{Vct_K}/K)^v$. The distributivity law for the linear tensor product in $\mathrm{Vct_K}$ implies the tensor-bifunctor of $\mathrm{Vct_K}/K$ to restrict and corestrict to a bifuntor $\otimes : (\mathrm{Vct_K}/K)^v \times (\mathrm{Vct_K}/K)^v \to (\mathrm{Vct_K}/K)^v$, but the identity object 1_K is not an object of $(\mathrm{Vct_K}/K)^v$. When we replace 1_K by the zero map 0_K, however, then this is seen to yield an identity object for the restricted tensor-bifunctor to $(\mathrm{Vct_K}/K)^v$ and, furthermore, it is compatible with the respective associativity, symmetry and unit laws; those structure morphisms restrict to $(\mathrm{Vct_K}/K)^v$ as it is a full subcategory of $\mathrm{Vct_K}/K$. Alltogether this yields a symmetric monoidal category $((\mathrm{Vct_K}/K)^v, \otimes, 0_K)$ and henceforth we will consider $(\mathrm{Vct_K}/K)^v$ with this monoidal structure. Due to the fact that there is only one morphism $0_K \to 1_K$ but no morphism $1_K \to 0_K$, the inclusion functor $(\mathrm{Vct_K}/K)^v \hookrightarrow \mathrm{Vct_K}/K$ is comonoidal and not strict. However, with this monoidal structure the original $\mathcal{V}$ becomes strict symmetric monoidal and we can see the restriction of $\mathcal{P}$ to $(\mathrm{Vct_K}/K)^v$ to be strict symmetric monoidal, too, for it maps 0_K to K. We have thus established the following result.

Proposition 5.4.1. *The restriction of the canonical projection functor*

$$\mathcal{P} : \mathrm{Vct_K}/K \to \mathrm{Vct_K}$$

to $(\mathrm{Vct_K}/K)^v$ *yields a strict (symmetric) monoidal isomorphism*

$$(\mathrm{Vct_K}/K)^v \cong \mathrm{Vct_K}\,.$$

Its inverse is the strict (symmetric) monoidal functor $\mathcal{V} : \mathrm{Vct_K} \to (\mathrm{Vct_K}/K)^v$ *that maps each vector space* V *to the uniqe zero map* $V \xrightarrow{0} K$.

Remarks 5.4.2.

(a) With proposition 5.4.1 and theorem 5.3.1 we can consider the slice category $\mathrm{Vct_K}/K$ as the fusion of both categories $\mathrm{Vct_K}$ and $\mathrm{Aff_K}$. Up to the ambivalent initial zero maps with codomain K, which appear either as the trivial vector spaces in $(\mathrm{Vct_K}/K)^v$ or as the empty spaces in $(\mathrm{Vct_K}/K)^a$, every other object of $\mathrm{Vct_K}/K$ belongs either to $(\mathrm{Vct_K}/K)^v$ or $(\mathrm{Vct_K}/K)^a$. Given an affine space

[3]The notion of a *comonoidal* functor between two monoidal categories can be obtained by reversing all arrows in the definition of the structure morphism and in the coherence diagrams of a monoidal functor.

$\omega \in (\mathrm{Vct_K}/K)^a$ the simple transitive action of its associated vector space of displacement vectors $\ker\omega$ becomes the morphism $\mathcal{V}(\ker\omega) \hookrightarrow \omega$ of augmented vector spaces, or, equivalently, the short exact sequence

$$\ker\omega \hookrightarrow \mathrm{dom}\,\omega \xrightarrow{\omega} K.$$

These are some of the reasons why, from the structural point of view, we can regard the slice category $\mathrm{Vct_K}/K$ as the proper stage to discuss affine spaces.

(b) We list some properties of the slice category $\mathrm{Vct_K}/K$. First of all, observe that it is *not* a linear category. It is complete and cocomplete, the terminal objects are the isomorphisms $V \cong K$, where V denotes any one-dimensional vector space and the initial objects are the initial zero maps $0 \to K$, where 0 denotes any trivial vector space. As in any slice category the Cartesian product is given by the pullback and the coproduct is induced by the coproduct in the base category, i.e. the direct sum (cf. dual of [Mac98, exercise V.1.1] and remark 4.2.2(i) where we have already used it explicitly).

$\mathrm{Vct_K}/K$ has a (small) separating set[4] $\{1_K, 0_K\}$, but no separator. In contrast it has a coseparator, namely $\mathrm{pr}_2 : K \oplus K \to K$. As we will see in the subsequent proposition 5.4.3 its monoidal structure is closed.

(c) Observe for a given ω in $(\mathrm{Vct_K}/K)^a$ and any 0_V in $(\mathrm{Vct_K}/K)^v$ that, provided $\mathrm{dom}\,\omega$ is not trivial, there are no $\mathrm{Vct_K}/K$-morphisms $\omega \to 0_V$, but only morphisms $0_V \to \omega$, because any morphism $f : \nu \to 0_V$ yields that ν is an object in $(\mathrm{Vct_K}/K)^v$ by $\nu = 0_V \circ f = 0_{\mathrm{dom}\,\nu}$. We saw a particular instance of this when we discussed the comonoidality of $\mathcal{V}$.

(d) (On the inclusion functors.) We will study the preservation of limits and colimits of the full and faithful inlusion functors

$$\iota_v : (\mathrm{Vct_K}/K)^v \hookrightarrow \mathrm{Vct_K}/K \quad \text{and} \quad \iota_a : (\mathrm{Vct_K}/K)^a \hookrightarrow \mathrm{Vct_K}/K.$$

We will show that they are both cocontinuous. Regarding continuity we can see that both functors don't have this property; ι_v can't be continuous, for 0_K is terminal in $(\mathrm{Vct_K}/K)^v$ but not terminal in $\mathrm{Vct_K}/K$, and ι_a doesn't preserve equalizers in general. The latter fact can be seen, for example, in the equalizer of two different translations $\tau_v, \tau_w : \omega \to \omega$. In $(\mathrm{Vct_K}/K)^a$ their equalizer is the inclusion of the zero map $0 \to K$, but in $\mathrm{Vct_K}/K$ it is the inclusion of $\mathcal{V}(\ker\omega)$.

However, we shall see that these are the worst things to happen: the inclusion functor ι_v preserves J-limits for a small and non-empty J, i.e. it preserves every

[4] Recall that for a category C a subset $S \subset \mathrm{Obj}\,C$ is said to be a *separating set* when for any pair of C-morphisms $f, g : c \to d$, $f \neq g$ there is a $s \in S$ and a morphism $h : s \to c$, such that $fh \neq gh$. The dual notion is that of a *coseparating set*.

small limit besides the terminal object, and the inclusion functor ι_a preserves every J-limit for small J that is not an initial object, i.e., an initial zero map $0 \to K$. (Compare this and the subsequent remark with theorem 4.3.1 and theorem 4.3.3.)

The cocontinuity of ι_v follows from the existence of a right adjoint for ι_v:

$$\ker : \mathrm{Vct_K}/K \to (\mathrm{Vct_K}/K)^v.$$

This functor sends each $\omega : V \to K$ to its kernel $\mathcal{V}(\ker\omega) : \ker\omega \xrightarrow{0} K$ and each morphism $f : \omega \to \omega'$ to its restriction and corestriction $\mathcal{V}(f : \ker\omega \to \ker\omega')$. In fact, this functor is a right-adjoint-left-inverse for ι_v. To see this note the natural bijection

$$\mathrm{Vct_K}/K(\iota_v 0_V, \omega) \cong (\mathrm{Vct_K}/K)^v(0_V, \ker\omega) = \mathrm{Vct_K}/K(0_V, \ker\omega)$$

to be just a reformulation of the universal property of the kernel. The cocontinuity of ι_a follows too from the existence of a right-adjoint-left-inverse

$$r : \mathrm{Vct_K}/K \to (\mathrm{Vct_K}/K)^a.$$

It can be constructed as follows. Fix an initial zero map $0 : 0 \to K$. Define r to map each 0_V in $(\mathrm{Vct_K}/K)^v$ with non-trivial $\mathrm{dom}\, 0_V = V$ to 0, and to map each $(\mathrm{Vct_K}/K)^v$-morphism between such zero maps to the identity morphism 1_0. Furthermore, set r to be the identity on $(\mathrm{Vct_K}/K)^a$ and to map $\mathrm{Vct_K}/K$-morphism $0_V \to \omega$ for an epi ω and non-trivial V to the unique morphism $0 \to \omega$. With this and remark (a) it is seen readily to be a well-defined functor. The natural bijection

$$\mathrm{Vct_K}/K(\iota_a\omega, \nu) \cong (\mathrm{Vct_K}/K)^a(\omega, r\nu) = \mathrm{Vct_K}/K(\omega, \nu)$$

is then a consequence of the definition of r and the previous remark.

By means of the previous remark we can conclude ι_v to preserve J-limits for a non-empty small category J as follows. As there are no morphisms from a non-zero ω to a zero map, any cone $\mu : \Delta\omega \dot{\to} \iota_v F$ for any functor $F : J \to (\mathrm{Vct_K}/K)^v$ in $\mathrm{Vct_K}/K$ implies ω to be a zero map and hence μ to be a cone in $(\mathrm{Vct_K}/K)^v$.

Regarding ι_a preserving small J-limits that are non-initial this follows from essentially the same argument as in (iii) of the proof of theorem 4.3.1. Let $F : J \to (\mathrm{Vct_K}/K)^a$ be given with limiting cone $\mu : \Delta \varprojlim F \dot{\to} F$. Let $\nu : \Delta\omega \dot{\to} \iota_a F$ be a cone in $\mathrm{Vct_K}/K$. Since ι_a is full and faithful we can assume $\omega = 0_V$. Consider $\mathrm{pr}_2 : V \oplus K \to K$, then $\ker \mathrm{pr}_2 = 0_V$, i.e., $i : 0_V \hookrightarrow \mathrm{pr}_2$ and pr_2 is in $(\mathrm{Vct_K}/K)^a$. Moreover, since $\varprojlim F$ and thus any Fj for $j \in \mathrm{Obj}\, J$ is

not a zero map by assumption, we can extend the cone ν to $\nu' : \Delta \mathrm{pr}_2 \dot{\to} F$ by fixing a $P \in (\varprojlim F)^{-1}(1)$ and applying a direct sum splitting argument like in (iii) of the proof of theorem 4.3.1. There is a unique morphism $h' : \mathrm{pr}_2 \to \varprojlim F$ with $\mu \circ h' = \nu'$. With $i \circ h'$ this shows the existence of an $h : 0_V \to \varprojlim$ with $\iota_a \mu \circ h = \nu$. For the uniqueness observe that any such morphism h factors through pr_2 and so $h = i \circ h'$.

A more abstract proof of ι_a preserving small non-initial J-limits is possible, too. For $\mathcal{B} : \mathrm{Aff_K} \to \mathrm{Vct_K}$ preserves non-empty equalizers and non-empty small pullbacks by theorem 4.3.1 and $\mathcal{P}$ reflects equalizers, ι_a preserves small non-initial products and equalizers. (Recall that products in $\mathrm{Vct_K}/K$ are pullbacks of the respective morphisms in $\mathrm{Vct_K}$.) By [Mac98][theorem V.2.2] ι_a preserves small non-initial limits.

(e) When we identify $\mathrm{Vct_K}$ with $(\mathrm{Vct_K}/K)^v$ via the isomorphism $\mathcal{V}$ and $\mathrm{Aff_K}$ with $(\mathrm{Vct_K}/K)^a$ via the adjoint equivalence $\mathcal{B} \dashv \mathcal{A}$ we can also translate the natural functors $U_V : \mathrm{Aff_K} \to \mathrm{Vct_K}$, $S : \mathrm{Vct_K} \to \mathrm{Aff_K}$ and $\mathcal{B} : \mathrm{Aff_K} \to \mathrm{Vct_K}$. The functor U_V translates to the restriction of the functor ker to $(\mathrm{Vct_K}/K)^a$. The functor S translates to $(-) + 1_K : (\mathrm{Vct_K}/K)^v \to (\mathrm{Vct_K}/K)^a$, where $+$ denotes the coproduct in $\mathrm{Vct_K}/K$. This functor can be represented as follows. It maps each zero map $0_V : V \to K$ to the natural projection $\mathrm{pr}_2 : V \oplus K \to K$ and each morphism $\mathcal{V}(f : V \to V')$ to the morphism with underlying linear map $f \oplus 1_K : V \oplus K \to V' \oplus K$. In contrast, the functor $\mathcal{B}$ translates to the functor $(\mathrm{Vct_K}/K)^a \to (\mathrm{Vct_K}/K)^v$, which maps each ω to the zero map $0_{\mathrm{dom}\,\omega}$.

(f) (On the Burnside rig.) From the viewpoint of the monoidal structure the subcategory $(\mathrm{Vct_K}/K)^v$ of vector spaces appears as an "ideal", whereas the subcategory $(\mathrm{Vct_K}/K)^a$ of affine spaces appears only as a "submonoid". Both intersect in the "zero element". Indeed, due to the distributivity of the linear tensor product any tensor-product with any zero map $0_V : V \to K$ will yield a zero map again and by the same argument the initial zero maps $0 \to K$ are seen to act like a zero. Furthermore, we have already seen $(\mathrm{Vct_K}/K)^a$ to be a (strict) monoidal subcategory, so its comparison with the submonoid is meaningful as well.

When we form the Burnside rig $Burn(\mathrm{Vct_K}/K)$ of the monoidal $\mathrm{Vct_K}/K$, that is if we consider the rig-structure[5] on the set of isomorphism classes of $\mathrm{Vct_K}/K$, where the cocartesian monoidal structure induces an additive monoid structure and the monoidal structure of the tensor-product induces the multiplicative monoid structure, then, by what we have stated before, the isomorphism classes of $(\mathrm{Vct_K}/K)^v$ will indeed yield an ideal I_v and the isomorphism classes of

[5] A *rig* or *semiring* is a set together with two monoid structures, where the first is commutative and the second distributes over the first. A standard example is $(\mathbb{N}, +, 0, \cdot, 1)$. A rig is thus a ring without negatives. For the Burnside rig construction cf. also the discussion in section 2.2.3.

$(\mathrm{Vct_K}/K)^a$ will indeed yield a subrig R_a. Here the respective additive closure is a consequence of both subcategories being cocomplete and their inclusion functor being cocontinuous. Both, I_v and R_a interesect in zero only. In fact, there is a rig-homomorphism $Burn(\mathrm{Vct_K}/K) \to \mathbb{N}$ that has I_v as kernel and $R_a \setminus \{0\}$ as the the preimage of 1. To see this recall that for an epi $\omega : V \to K$ fixing a point $P \in \omega^{-1}(1)$ induces an isomorphism $V \cong \ker\omega \oplus KP \cong \ker\omega \oplus K$. Due to our choice of P this linear isomorphism lifts to an $\mathrm{Vct_K}/K$-isomorphism between ω and $\mathrm{pr}_2 : \ker\omega \oplus K \to K$. The latter was seen to be a coproduct $\mathcal{V}(\ker\omega)+1_K$ in $\mathrm{Vct_K}/K$ in the previous remark, so on the level of isomorphism classes this yields $[\omega] = [1_K] + [\mathcal{V}(\ker\omega)]$. Because $R_a \setminus \{0\}$ consists only of isomorphism classes of epis we get $R_a = [1_K] + I_v$ as asserted. We have thus shown $Burn(\mathrm{Vct_K}/K)$ to be an augmented rig in a natural way. We will see an "internal" instance of this fact in the next section.

Observe that the Cartesian structure of $\mathrm{Vct_K}/K$ yields a third monoid structure on the set $Burn(\mathrm{Vct_K}/K)$. Like in $\mathrm{Aff_K}$ the Cartesian structure can have the same identity object 1_K as the multiplicative structure but it neither distributes over the monoidal nor the cocartesian structure in general. Contrary to the situation in $\mathrm{Aff_K}$ Cartesian multiplication by the initial object doesn't yield an initial object in $\mathrm{Vct_K}/K$. In other words, the zero in $Burn(\mathrm{Vct_K}/K)$ is not a zero for the Cartesian multiplication. Indeed, in $\mathrm{Vct_K}/K$ the Cartesian product of any ω with the initial zero map $0 \to K$ yields $\mathcal{V}(\ker\omega)$, which is an initial zero map if and only if $\mathrm{dom}\,\omega$ is trivial. Nonetheless, we can see I_v to be a (monoid) ideal and $R_a \setminus \{0\}$ a submonoid of the Cartesian monoid structure on $Burn(\mathrm{Vct_K}/K)$ as follows. The completeness of $(\mathrm{Vct_K}/K)^a$ and $(\mathrm{Vct_K}/K)^v$ together with the inclusion functor of the first category to preserve binary products and the inclusion functor of the second category to preserve binary products of epis (cf. remark (d)) yield I_v and $R_a \setminus \{0\}$ to be submonoids. Recall that products in $\mathrm{Vct_K}/K$ are pullbacks in $\mathrm{Vct_K}$. Because compositions with zero maps yield zero maps, we see that pullbacks along a zero map yield zero maps and hence that a product with a zero map is a zero map in $\mathrm{Vct_K}/K$, too. This shows I_v a monoid ideal.

(g) (Finite dimensions.) Regarding limits and colimits when we replace "small" by "finitary" we get the respective translated properties of remark (b), (d) for $(\mathrm{FVct_K}/K)^a$, $(\mathrm{FVct_K}/K)^v$, their respective inclusion functors and $\mathrm{FVct_K}/K$. Here $(\mathrm{FVct_K}/K)^v$ is defined again as the full subcategory of zero maps. The other remarks carry over directly when we replace the categories and functors by the respective finite dimensional ones.

Proposition 5.4.3. *The category* $\mathrm{Vct_K}/K$ *is closed with respect to its natural monoidal structure. For* $\omega, \nu \in \mathrm{Vct_K}/K$ *the internal hom can be represented as*

$$\underline{\hom}(\omega, \nu) = \nu_* : \nu_*^{-1}(K\omega) \to K\omega \cong K$$

Proof. For $\underline{\hom}(\omega, \nu)$ to be well-defined note that

$$\nu_* : \underline{\mathrm{Vct}}_{\mathrm{K}}(\mathrm{dom}\,\omega, \mathrm{dom}\,\nu) \to \underline{\mathrm{Vct}}_{\mathrm{K}}(\mathrm{dom}\,\omega, K)$$

is a linear map between vector spaces and $K\omega$ is an one-dimensional vector subspace of $\underline{\mathrm{Vct}}_{\mathrm{K}}(\mathrm{dom}\,\omega, K)$. For we have for any morphism $f : \omega \to \nu$

$$\begin{array}{ccc} \mathrm{dom}\,\omega & \xrightarrow{\quad f \quad} & \mathrm{dom}\,\nu \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle\nu} & \\ & K, & \end{array}$$

i.e., $\nu \circ f = \omega$, $\nu_*^{-1}(K\omega) < \underline{\mathrm{Vct}}_{\mathrm{K}}(\mathrm{dom}\,\omega, \mathrm{dom}\,\nu)$ is the smallest subspace generated by the subset of $\mathrm{Vct_K}/K$-morphisms in $\underline{\mathrm{Vct}}_{\mathrm{K}}(\mathrm{dom}\,\omega, \mathrm{dom}\,\nu)$. Moreover, this set of $\mathrm{Vct_K}/K$-morphisms is precisely $\underline{\hom}(\omega, \nu)^{-1}(1)$.

By the definition of morphisms in $\mathrm{Vct_K}/K$ the bifunctoriality of $\underline{\hom}(-,-)$ is inherited from $\underline{\mathrm{Vct}}_{\mathrm{K}}(-,-)$. For example, for a $f : \nu \to \eta$ we have $f \circ \nu = \eta$ and thus $f_* \circ \nu_* = \eta_*$, and since f is a morphism in $\mathrm{Vct_K}/K$, the linear map $f_* = \underline{\mathrm{Vct}}_{\mathrm{K}}(\mathrm{dom}\,\omega, f)$ restricts and corestricts to $f_* : \nu_*^{-1}(K\omega) \to \eta_*^{-1}(K\omega)$.

The natural bijection

$$\mathrm{Vct_K}/K(\omega \otimes \nu, \eta) \cong \mathrm{Vct_K}/K(\omega, \underline{\hom}(\nu, \eta))$$

is a consequence of the construction of $\otimes$ from the linear tensor product, the construction of $\underline{\hom}(-,-)$ from $\underline{\mathrm{Vct}}_{\mathrm{K}}(-,-)$ and the hom-tensor adjunction in $\mathrm{Vct_K}$. In more detail, we can restrict the natural bijection φ of the hom-tensor adjunction of $\mathrm{Vct_K}$ to $\mathrm{Vct_K}/K(\omega \otimes \nu, \eta)$ since

$$\mathrm{Vct_K}/K(\omega \otimes \nu, \eta) \hookrightarrow \mathrm{Vct_K}(\mathrm{dom}\,\omega \otimes \mathrm{dom}\,\nu, \mathrm{dom}\,\eta).$$

We show by direct evaluation that the right adjunct $\varphi(f)$ of a morphism $f : \omega \otimes \nu \to \eta$ is a morphism $\varphi(f) : \omega \to \underline{\hom}(\nu, \eta)$, i.e.

$$\begin{array}{ccc} \mathrm{dom}\,\omega \otimes \mathrm{dom}\,\nu & \xrightarrow{\quad f \quad} & \mathrm{dom}\,\eta \\ {\scriptstyle\omega\otimes\nu}\downarrow & & \downarrow{\scriptstyle\eta} \\ K \otimes K & \xrightarrow[\lambda_K]{\cong} & K, \end{array} \qquad \begin{array}{ccc} \mathrm{dom}\,\omega & \xrightarrow{\quad \varphi(f) \quad} & \eta_*^{-1}(K\nu) \\ {\scriptstyle\omega}\downarrow & & \downarrow{\scriptstyle\eta_*} \\ K & \xrightarrow[(1\mapsto\nu)]{\cong} & K\nu, \end{array}$$

and, conversely, the left adjunct $\varphi^{-1}(g)$ of a morphism $g : \omega \to \underline{\hom}(\nu, \eta)$ is a morphism $\varphi^{-1}(g) : \omega \otimes \nu \to \eta$. This is seen easiest by passing to bilinear maps. Then for $(v, w) \in \mathrm{dom}\,\omega \times \mathrm{dom}\,\nu$ the equation $\eta \circ f = \lambda_K \circ \omega \otimes \nu$ translates to $\eta(f(v, w)) = \omega(v)\nu(w)$, and $\eta_* \circ g = (1 \mapsto \nu) \circ \omega$ translates to $(\eta \circ g(v))(w) = \omega(v)\nu(w)$. With this both claims follow immediately.

(qed)

Remark 5.4.4.

The internal hom $\underline{\hom}(-,-)$ of $\mathrm{Vct_K}/K$ restricts to $\mathcal{V}\circ\underline{\mathrm{Vct_K}}(-,-)\circ\mathcal{P}\times\mathcal{P}$ on $(\mathrm{Vct_K}/K)^v$ and its restriction is isomorphic to $\mathcal{B}\circ\underline{\mathrm{Aff_K}}(-,-)\circ\mathcal{A}\times\mathcal{A}$ on $(\mathrm{Vct_K}/K)^a$.

Basically, the second assertion (and the first one, when relaxed to an isomorphism) is a consequence of the tensor-product of $\mathrm{Vct_K}/K$ restricting to the linear and affine tensor products on $(\mathrm{Vct_K}/K)^v$ and $(\mathrm{Vct_K}/K)^a$, $\mathcal{V}$, $\mathcal{P}$ and $\mathcal{B}$, $\mathcal{A}$ forming (strong) monoidal equivalences, and both tensor structures being closed, respectively. (By the parameter theorem on adjunctions [Mac98][theorem IV.7.3] the internal hom is determined by the monoidal structure up to a natural isomorphism.) So, instead of giving isomorphisms, we look at what happens with $\underline{\hom}(-,-)$ concretely.

In $(\mathrm{Vct_K}/K)^v$ each object is a zero map. For $\nu = 0_W$ we have $\nu_* = 0$, and so $\nu_*^{-1}(K0_V) = \underline{\mathrm{Vct_K}}(V,W)$. This shows $\underline{\hom}(0_V,0_W) = 0_{\underline{\mathrm{Vct_K}}(V,W)}$, which is precisely $\mathcal{V}(\underline{\mathrm{Vct_K}}(\mathcal{P}\,0_V,\mathcal{P}\,0_W))$. In $(\mathrm{Vct_K}/K)^a$ for each ω and ν, $\underline{\hom}(\omega,\nu)$ is an epi. With what we know from the construction of $\mathcal{A}$, $\underline{\hom}(\omega,\nu)^{-1}(1)$ is an affine space with associated vector space $\ker\underline{\hom}(\omega,\nu)$, and as noted in the proof of the preceding proposition $\underline{\hom}(\omega,\nu)^{-1}(1)$ is the set of $\mathrm{Vct_K}/K$-morphisms from ω to η. We know that each such morphism f can be considered an affine map $f : \omega^{-1}(1) \to \nu^{-1}(1)$ with associated linear map $\hat{f} : \ker\omega \to \ker\nu$. Therefore, $\underline{\hom}(\omega,\nu)^{-1}(1)$ can be identified with $\underline{\mathrm{Aff_K}}(\omega^{-1}(1),\nu^{-1}(1))$. A linear map $f : \mathrm{dom}\,\omega \to \mathrm{dom}\,\nu$ is in $\ker\underline{\hom}(\omega,\nu)$ if and only if $\nu\circ f = 0$. So any such f factors through $\ker\nu$. This shows that $\ker\underline{\hom}(\omega,\nu)$ can be identified with the associated vector space of $\underline{\mathrm{Aff_K}}(\omega^{-1}(1),\nu^{-1}(1))$, which is $\mathrm{Aff_K}(\omega^{-1}(1),\ker\nu)$.

6 Affine Algebras

From the structural point of view one would expect geometric algebras of affine spaces to arise as particular internal monoids of $(\mathrm{Aff}_K, \otimes, *)$. That this is indeed the case will be shown for two particular examples, the Grassmann algebra of affine space and the Clifford algebra of affine space, in the last section. However, to be able to see this clearly we need an algebraic presentation of internal monoids of Aff_K. With our results on the algebraic presentation of affine spaces, which we have established in the previous chapter, this is obtained rather easily from what we know about monoidal functors and internal monoids in general: the category of internal monoids of Aff_K is equivalent to the category of augmented K-algebras.

On one hand, this algebraic presentation yields a large class of examples. For example, it reveals bialgebras and Hopf algebras as affine algebras and so makes them accessible to a basic geometric interpretation. On the other hand, it implies that the category of internal monoids of Aff_K is equivalent to the category of non-unitary K-algebras. Here the non-unitary K-algebra that arises from an internal monoid (A, μ, u) of Aff_K is the associated vector spaces V_A together with the associated bilinear map $\hat{\mu}$.

The general theory of internal monoids yields that we have a free internal monoid functor $\mathrm{Aff}_K \to \mathrm{Mon}(\mathrm{Aff}_K, \otimes, *)$, which we regard as an $\mathbb{N}$-graded internal monoid according to section 2.5.4. In the algebraic presentation the free monoid of A turns out to be the linear tensor algebra $T\mathcal{B}A$ with an augmentation, i.e., a K-algebra homomorphism $\omega_T : T\mathcal{B}A \to K$ that is the unique lift of the weight form $\omega : \mathcal{B}A \to K$ to a K-algebra homomorphism according to the universal property of the linear tensor algebra. But the weight form induces more than an augmentation of the tensor algebra. It also induces a unique DG structure, i.e., a (graded) differential ∂ such that $(T\mathcal{B}A, \partial)$ becomes a DG algebra.

Taking up this observation we will study the relationship of affine spaces and DG algebras in more generality. We will then see that the DG structure of any DG algebra $\mathcal{A}$ that can be described as the "quotient" of the DG algebras $(T\mathcal{B}A, \partial)$ for a particular affine space A is uniquely determined by this affine space. Interestingly, (graded) differentials can't be represented by affine maps, so there is no notion of DG structure internal to Aff_K and the category of $\mathbb{N}$-graded internal monoids. The

relation with DG structures belongs thus exclusively to the algebraic picture of affine spaces.

Our two examples, the Grassman and the Clifford algebra of affine space carry such a DG structure as well; in fact, they inherit it from the tensor algebra $T\mathcal{B}A$. It goes without saying that the DG structure is an interesting structure from a geometric and algebraic viewpoint.

We will conclude the chapter by exhibiting why those two algebras deserve the name of a geometric algebra. In particular, we will argue from a structural point of view why the Clifford algebra of affine space deserves to be called a geometric algebra rather than the Grassmann algebra and observe that the spin group of such a Clifford algebra acts on its generating affine space by translations.

Due to the internal monoids of $(\mathrm{Vct}_K, \otimes, K)$ to be the (unital) K-algebras we will call internal monoids of $(\mathrm{Aff}_K, \otimes, *)$ *affine algebras*, or *affine K-algebras* when we want to emphasize the ground field K. Accordingly, $\mathrm{Mon}(\mathrm{Aff}_K, \otimes, *)$ becomes the *category of affine K-algebras*.

6.1 Affine Algebras and Non-unital K-Algebras I

An affine algebra (A, μ, u) consists of an affine space A, a biaffine[1] multiplication map $\mu : A \otimes A \to A$ and an identity $u : * \to A$. The identity particularly shows that the affine space A is pointed and we will often refer to the point $u(\bullet) \in A$, $\bullet \in *$ as the identity (element) of the affine algebra instead. Furthermore, we shall follow the common abuse of notation and identify A with the whole structure of an affine algebra (A, μ, u) when there is no danger of confusion. Because we know the underlying (set-theoretic) maps of the structure morphisms of $(\mathrm{Aff}_K, \otimes, *)$, with the help of theorem 3.4.10(ii) we can see the diagrammatic associativity and unit law of the affine algebra to be equivalent to the usual associativity and unit law for rings and K-algebras we are used to:

$$(P \cdot Q) \cdot R = P \cdot (Q \cdot R), \qquad 1 \cdot P = P \cdot 1 = P, \qquad P, Q, R \in A.$$

Here we have used infix notation, $\cdot$ for μ and 1 for $u(\bullet)$. We have already seen two particular examples of affine algebras in the additive and multiplicative monoid of the bipointed affine line K in example 3.4.19.

Proposition 3.4.8 and remark 3.4.9(b) show the biaffine multiplication map μ of an affine algebra A to induce a bilinear map $\hat{\mu} : V_A \otimes V_A \to V_A$ on the associated vector space V_A. We have thus a notion of multiplication for V_A. In fact, with this multiplication map V_A becomes a non-unital K-algebra, i.e. an internal semigroup[2]

[1] Recall our convention from section 3.5 to call affine maps from binary tensor products biaffine too, for they are the internal representations of biaffine maps in Aff_K.

[2] The notion of an *internal semigroup* in a monoidal category arises from the notion of an internal monoid when one leaves out the structure morphism of the identity with its corresponding diagrams.

of the monoidal category $\mathrm{Vct_K}$.

Proposition 6.1.1. *Let (A, μ, u) be an affine algebra and let $\hat{\mu} : V_A \otimes V_A \to V_A$ denote the associated bilinear map of f, then $(V_A, \hat{\mu})$ is an internal semigroup for $(\mathrm{Vct_K}, \otimes, K)$. The internal semigroup V_A becomes a (unital) K-algebra if and only if A has a zero; that is a point $0 \in A$ such that $\mu(0 \otimes P) = \mu(P \otimes 0) = 0$ for all $P \in A$. In that case the identity element of V_A is given by $\overrightarrow{01}$. Conversely, if V_A has an identity element given by $u \in V_A$, then the point $1 - u$ is a zero in A.*

Proof. (i) We have to show that the associativity law holds for $\hat{\mu}$. This is, however, a consequence of the functor $U_V : \mathrm{Aff_K} \to \mathrm{Vct_K}$, which maps each affine space to its associated vector space, to be "semigroupoidal" (cf. remark 3.4.14(c)). Indeed, in the cited remark we have argued that U_V fulfills the axioms of a monoidal functor at least for the associativity law. Moreover, the associated bilinear map $\hat{\mu}$ of the biaffine map μ is the bilinear map

$$V_A \otimes V_A \hookrightarrow V_{A \otimes A} \overset{U_V \mu}{\to} V_A.$$

Like the lift of a monoidal functor to a functor of categories of internal monoids one can show U_V to lift to categories of internal semigroups (cf. the construction in section 2.5.6). In particular, it maps the internal semigroup (A, μ) of $\mathrm{Aff_K}$ to the intenral semigroup $(V_A, \hat{\mu})$.

(ii) Next we will show the second part of the assertion. For convenience, we will do the computations in the barycentric calculus $\mathcal{B}A$. That we can do to this with biaffine maps without any difficulties is a consequence of the strong monoidality of $\mathcal{B}$ (cf. theorem 4.5.1), since by the means of this strong monoidality we can consider the biaffine μ as a bilinear map $\mu : \mathcal{B}A \otimes \mathcal{A}B \to \mathcal{B}A$. To avoid confusion we will denote the zero vector by $0_v \in V_A$ in what follows.

"⇒" Suppose $0 \in A$ is a zero. First of all, it is seen to act as a zero on displacement vectors $v \in V_A$ by μ:

$$\hat{\mu}(0 \otimes v) = \mu(0 \otimes (1 + v)) - \mu(0 \otimes 1) = 0 - 0 = 0_v.$$

A similar argument shows that $1 \in A$ acts as a left-identity on $v \in V_A$ by μ:

$$\hat{\mu}(1 \otimes v) = \mu(1 \otimes (1 + v)) - \mu(1 \otimes 1) = 1 + v - 1 = v.$$

Together this shows $\overrightarrow{01}$ a left-identity

$$\hat{\mu}(\overrightarrow{01} \otimes v) = \hat{\mu}(1 \otimes v) - \hat{\mu}(0 \otimes v) = v - 0_v = v.$$

That $\overrightarrow{01}$ is a right-identity can be shown by repeating the above arguments with $v \otimes 0$, $v \otimes 1$ and $v \otimes \overrightarrow{01}$, respectively.

"⇐" Let $u \in V_A$ be an identity for the multiplication map $\hat{\mu}$ and consider the point $0 := 1 - u \in A$. In the preceding argument we have seen that 1 acts like an identity on displacement vectors $v \in V_A$. For any point $P \in A$ we can thus calculate

$$\hat{\mu}(u \otimes P) = \hat{\mu}(u \otimes (1 + \overrightarrow{1P})) = \hat{\mu}(u \otimes 1) + \hat{\mu}(u \otimes \overrightarrow{1P}) = u + \overrightarrow{1P}.$$

With this we can show 0 a left-zero.

$$\mu(0 \otimes P) = \mu(1 \otimes P) - \hat{\mu}(u \otimes P) = P - u + \overrightarrow{P1} = 1 - u = 0.$$

Repeating the arguments with $P \otimes u$ and $P \otimes 0$ shows 0 a right-zero and concludes our assertion.

(qed)

Remark 6.1.2 (additive affine algebras)**.**

Although the underlying affine space A of the affine algebra (A, μ, u) is a pointed space and hence naturally a vector space (cf. remark 4.3.4(c)), it is still more appropriate to not consider it as such in general. The reason is that the distinguished point acts as an identity for the multiplication and not as a zero, so the biaffine μ won't become a bilinear map under the respective identification. Exceptions are affine algebras like $(K, +, 0)$ in example 3.4.19, which are pointed spaces and their "multiplication" morphism indeed comes from the additive structure of their associated vector space.

In fact, all internal monoids in $\mathrm{Mon}(\mathrm{Aff}_\mathrm{K}, \times, *) \hookrightarrow \mathrm{Mon}(\mathrm{Aff}_\mathrm{K}, \otimes, *)$ are of that kind and can be characterized within $\mathrm{Mon}(\mathrm{Aff}_\mathrm{K}, \otimes, *)$ as those affine algebras whose associated bilinear maps are zero. The inclusion is thereby induced by the lift of the monoidal identity functor $I_{\mathrm{Aff}_\mathrm{K}}$ (cf. theorem 3.4.17).

To see this consider an internal monoid (A, μ, u) for the Cartesian structure. By proposition 3.2.8 U_V preserves products. It is thus strong monoidal for the Cartesian structure, and so $(V_A, \hat{\mu}, \hat{u})$ is an internal monoid in $(\mathrm{Vct}_\mathrm{K}, \oplus, 0)$. At the end of section 2.5.1 we have shown that each such internal monoid is just the underlying vector space with the "multiplication" being the vector addition. Hence $\hat{\mu}$ is the vector addition on V_A. The left and right unit laws for u and μ yield then that $A \cong SV_A$ as an internal monoid, which is just a reformulation of our claim that the μ comes from the additive structure of V_A.

Regarding the claim about the characterization by associated bilinear maps, recall the discussion in section 3.4.5 that the structure morphism $p : \otimes \dot{\to} \times$ of the monoidal $I_{\mathrm{Aff}_\mathrm{K}}$ is the projection of affine tensors to their pure point parts; in particular, for each $A, C \in \mathrm{Aff}_\mathrm{K}$ the associated bilinear map $\hat{p}_{A,C}$ is zero. Conversely, consider an internal monoid $(A, \mu, u) \in \mathrm{Mon}(A, \otimes, *)$ with associated bilinear map

$\hat{\mu}$ being the zero map. By theorem 3.4.10 and its preceding discussion we can represent each point $R \in A \times A$ as

$$R = 1 \otimes 1 + \sum_i 1 \otimes w_i + \sum_j v_j \otimes 1 + \sum_\ell v_\ell \otimes w_\ell.$$

Applying μ and using the unit law with the associated bilinear map being zero yields

$$\mu(R) = 1 + \sum_i w_i + \sum_j v_j.$$

But this shows μ to be induced by the vector addition of V_A and (A, μ, u) thus an internal monoid of the Cartesian structure.

6.2 The Equivalence of the Categories of Affine Algebras and Augmented K-Algebras

Regarding geometrical calculi we stated in the introduction of chapter 4 that we consider the barycentric calculus of points the first step. It is thus natural to ask how an affine algebra is related with the barycentric calculus of its underlying affine space. The answer lies in theorem 5.3.1: due to the there stated equivalence of categories to be monoidal it lifts to an equivalence of the respective categories of internal monoids.

Proposition 6.2.1. *The adjoint equivalence* $(\mathrm{Vct_K}\,/K)^a \simeq \mathrm{Aff_K}$ *from theorem 5.3.1 lifts to an adjoint equivalence*

$$\mathrm{Mon}((\mathrm{Vct_K}\,/K)^a, \otimes, 1_K) \simeq \mathrm{Mon}(\mathrm{Aff_K}, \otimes, *).$$

Proof. In (2.17) of section 2.5 we indicated how a monoidal functor lifts to a functor between the respective categories of internal monoids and in (2.18) how a monoidal natural transformation lifts to a natural transformation between such functors on the level of internal monoids. When we apply this to the symmetric monoidal adjoint equivalence $(\mathrm{Vct_K}\,/K)^a \simeq \mathrm{Aff_K}$ given by the adjunction $\mathcal{B} \dashv \mathcal{A}$ then it yields an adjoint equivalence

$$\mathrm{Mon}((\mathrm{Vct_K}\,/K)^a, \otimes, 1_K) \simeq \mathrm{Mon}(\mathrm{Aff_K}, \otimes, *).$$

Indeed, because of the monoidality of the unit $\eta : I_{\mathrm{Aff_K}} \dot{\cong} \mathcal{A}\,\mathcal{B}$ and counit $\varepsilon : \mathcal{B}\,\mathcal{A} \dot{\cong} I_{(\mathrm{Vct_K}\,/K)^a}$ both lift to natural isomorphisms

$$\eta : I_{\mathrm{Mon}(\mathrm{Aff_K}, \otimes, *)} \dot{\cong} \mathcal{A}\,\mathcal{B}, \qquad \varepsilon : \mathcal{B}\,\mathcal{A} \dot{\cong} I_{\mathrm{Mon}((\mathrm{Vct_K}\,/K)^a, \otimes, 1_K)},$$

whereby $\mathcal{A}$ and $\mathcal{B}$ denote the respective lifts to functors between the respective categories of internal monoids.

(qed)

Remark 6.2.2 (finite dimensions)**.**
By remark 5.3.2 the respective finite dimensional result is the lift of the respective equivalence $(\mathrm{FVct_K}\,/K)^a \simeq \mathrm{FAff_K}$ to a an equivalence $\mathrm{Mon}((\mathrm{FVct_K}\,/K)^a, \otimes, 1_K) \simeq \mathrm{Mon}(\mathrm{FAff_K}, \otimes, *)$.

To get an algebraic presentation of affine algebras it remains to figure out a more convenient representation of $\mathrm{Mon}((\mathrm{Vct_K}\,/K)^a, \otimes, 1_K)$. For this purpose we apply our explicit construction of the monoidal structure on $(\mathrm{Vct_K}\,/K)^a$ and reconsider the structure morphisms of such an internal monoid. We obtain

$$\mathrm{Mon}((\mathrm{Vct_K}\,/K)^a, \otimes, 1_K) \cong \mathrm{Mon}(\mathrm{Vct_K}, \otimes, K)/(K, \cdot, 1_K)$$

immediately. In other words, the internal monoids of $(\mathrm{Vct_K}\,/K)^a$ are precisely the augmented (unital) K-algebras.

Theorem 6.2.3. *The restriction of the canonical projection functor*

$$\mathcal{P} : \mathrm{Vct_K}\,/K \to \mathrm{Vct_K}$$

to $(\mathrm{Vct_K}\,/K)^a$ *induces a functor*

$$\mathrm{Mon}((\mathrm{Vct_K}\,/K)^a, \otimes, 1_K) \to \mathrm{Mon}(\mathrm{Vct_K}, \otimes, K)/(K, \cdot, 1_K),$$

which maps an internal monoid (ω, μ, u) *to a* K*-algebra homomorphism*

$$(\mathcal{P}(\omega), \mathcal{P}(\mu), \mathcal{P}(u)) \xrightarrow{\omega} (K, \cdot, 1_K).$$

It is an isomorphism of categories.

Proof. First of all, recall $\mathcal{P} : \mathrm{Vct_K}\,/K \to \mathrm{Vct_K}$ to be strict (symmetric) monoidal and $(\mathrm{Vct_K}\,/K)^a$ a strict monoidal subcategory, so the restriction $\mathcal{P} : (\mathrm{Vct_K}\,/K)^a \to \mathrm{Vct_K}$ is a strict (symmetric) monoidal functor. Solely from that we get the existence of a lift $\mathcal{P} : \mathrm{Mon}((\mathrm{Vct_K}\,/K)^a, \otimes, 1_K) \to \mathrm{Mon}(\mathrm{Vct_K}, \otimes, K)$. As we explained in section 2.5 this lift maps each internal monoid (ω, μ, u) to the K-algebra $(\mathcal{P}(\omega), \mathcal{P}(\mu), \mathcal{P}(u))$ and each monoid morphism $f : (\omega, \mu, u) \to (\omega', \mu', u')$ to a K-algebra homomorphism $\mathcal{P}(f) : (\mathcal{P}(\omega), \mathcal{P}(\mu), \mathcal{P}(u)) \to (\mathcal{P}(\omega'), \mathcal{P}(\mu'), \mathcal{P}(u'))$. When we write out the multiplication morphism $\mu : \omega \otimes \omega \to \omega$ and the unit law for the identity $u : 1_K \to \omega$ of the internal monoid (ω, μ, u) explicitly

$$\begin{array}{ccc} \mathrm{dom}\,\omega \otimes \mathrm{dom}\,\omega & \xrightarrow{\mu} & \mathrm{dom}\,\omega \\ {\scriptstyle \omega\otimes\omega}\downarrow & & \downarrow{\scriptstyle \omega} \\ K \otimes K & \xrightarrow[\lambda_K]{} & K, \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{u} & \mathrm{dom}\,\omega \\ {\scriptstyle 1_K}\downarrow & & \downarrow{\scriptstyle \omega} \\ K & \xrightarrow[1_K]{} & K, \end{array} \qquad (*)$$

we see these to be the defining diagrams of a monoid morphism ω between the internal monoids $(\mathcal{P}(\omega), \mathcal{P}(\mu), \mathcal{P}(u))$ and $(K, \rho_K, 1_K)$. Recall that the component ρ_K of the right-unit law equals to the respective component λ_K of the left-unit law and is just the multiplication map $\cdot$ of K (cf. the construction of the initial internal monoid out of the identity object of a monoidal structure in section 2.5). In other words, this says that the linear form $\omega : \mathcal{P}(\omega) \to K$ is a K-algebra homomorphism.

Conversely, for $(*)$ is the defining diagram of an K-algebra homomorphism and the associativity law is a natural transformation, each augmented K-algebra $\omega : (V, \mu, u) \to (K, \cdot, 1_K)$ is an internal monoid (ω, μ, u) where μ and u are considered as morphisms in $(\mathrm{Vct_K}\,/K)^a$. Futhermore, due to the same reason every morphism $f : \omega \to \omega'$ of augmented K-algebras is seen a morphism of the respective internal monoids. Because the underlying linear maps of f and $\mathcal{P}(f)$ coincide, this shows the lifted $\mathcal{P}$ full. Obviously, we have $\mathcal{P}(\omega, \mu, u) = (\omega : (V, \mu, u) \to (K, \cdot, 1_K))$, so the lifted $\mathcal{P}$ is surjective on objects. Its injectivity on objects is a direct consequence of its definition and $\mathcal{P} : \mathrm{Vct_K}\,/K \to \mathrm{Vct_K}$ being faithful. Due to $\mathcal{P} : \mathrm{Vct_K}\,/K \to \mathrm{Vct_K}$ being faithful, the lifted $\mathcal{P}$ is faithful, too. Alltogether we have shown the lifted $\mathcal{P}$ to be full, faithful and bijective on objects; it is therefore an isomorphism of categories as asserted.

(qed)

Corollary 6.2.4. *The category of affine K-algebras is equivalent to the category of augmented (unital) K-algebras*

$$\mathrm{Mon}(\mathrm{Aff_K}, \otimes, *) \simeq \mathrm{Mon}(\mathrm{Vct_K}, \otimes, K)/(K, \cdot, 1_K),$$

where the left $\otimes$ denotes the affine tensor product and the right $\otimes$ denotes the linear tensor product.

Proof. This is an immediate consequence of proposition 6.2.1 and theorem 6.2.3, for compositions of equivalences of categories yield an equivalence of categories again. This is seen easiest when one uses that a functor $F : C \to D$ is an equivalence $C \simeq D$ if and only if it is full, faithful and isomorphism-dense (cf. [Mac98, theorem IV.4.1]).

(qed)

Remark 6.2.5 (finite dimensions)**.**
In the finite dimensional case the canonical projection functor

$$\mathcal{P} : \mathrm{FVct_K}\,/K \to \mathrm{FVct_K}$$

induces an isomorphism of categories

$$\mathrm{Mon}((\mathrm{FVct_K}\,/K)^a, \otimes, 1_K) \cong \mathrm{Mon}(\mathrm{FVct_K}, \otimes, K)/(K, \cdot, 1_K),$$

which is given as in theorem 6.2.3. Together with remark 6.2.2 this shows that the category of finite dimensional affine K-algebras is equivalent to the category of finite dimensional augmented (unital) K-algebras.

To see this recall $\mathrm{FVct_K}/K$, $\mathrm{FVct_K}$ and $(\mathrm{FVct_K}/K)^a$ to be strict (symmetric) monoidal full subcategories of $\mathrm{Vct_K}/K$, $\mathrm{Vct_K}$ and $(\mathrm{Vct_K}/K)^a$, respectively. With these facts $\mathcal{P} : \mathrm{FVct_K}/K \to \mathrm{FVct_K}$ is seen to be the restriction and corestriction of $\mathcal{P} : \mathrm{Vct_K}/K \to \mathrm{Vct_K}$ and

$$\mathrm{Mon}((\mathrm{FVct_K}/K)^a, \otimes, 1_K) \hookrightarrow \mathrm{Mon}((\mathrm{Vct_K}/K)^a, \otimes, 1_K)$$

as well as

$$\mathrm{Mon}(\mathrm{FVct_K}, \otimes, K)/(K, \cdot, 1_K) \hookrightarrow \mathrm{Mon}(\mathrm{Vct_K}, \otimes, K)/(K, \cdot, 1_K),$$

are seen to be full subcategories. We just need to note the isomorphism given in theorem 6.2.3 to restrict and corestrict respectively; but this is obvious from its definition and both subcategories of internal monoids being full.

Remarks 6.2.6.

(a) As we did for affine spaces we will use the equivalence in corollary 6.2.4 and identify affine K-algebras with augmented (unital) K-algebras. More precisely, as with the notion of an affine space we will consider the category of internal monoids of $\mathrm{Aff_K}$ as the geometric presentation and the category of augmented (unital) K-algebras as the algebraic presentation of the notion of an affine K-algebra.

(b) We know the inclusion $\iota_a : (\mathrm{Vct_K}/K)^a \hookrightarrow \mathrm{Vct_K}/K$ to be strict (symmetric) monoidal and full. It thus lifts to a full inclusion

$$\iota_a : \mathrm{Mon}((\mathrm{Vct_K}/K)^a, \otimes, 1_K) \hookrightarrow \mathrm{Mon}(\mathrm{Vct_K}/K, \otimes, 1_K).$$

In fact, it is an isomorphism of categories, for there are no morphisms from 1_K to zero maps $0_V : V \to K$ (cf. remark 5.4.2 (c)). As a consequence there can't be internal monoids in $\mathrm{Mon}(\mathrm{Vct_K}/K, \otimes, 1_K)$ which underlying object is a zero map. Together with theorem 6.2.3 this shows that the internal monoids of $\mathrm{Vct_K}/K$ are the augmented (unital) K-algebras

$$\mathrm{Mon}(\mathrm{Vct_K}/K, \otimes, 1_K) \cong \mathrm{Mon}(\mathrm{Vct_K}, \otimes, K)/(K, \cdot, 1_K).$$

(c) Note theorem 6.2.3, respectively the result in the previous remark to be an "internal" instance of the natural augmentation $\alpha : \mathit{Burn}(\mathrm{Vct_K}/K) \to \mathbb{N}$ of the Burnside rig $\mathit{Burn}(\mathrm{Vct_K}/K)$ as shown in remark 5.4.2 (f). There, according to our identifications, the isomorphism classes of vector spaces appeared as the rig

ideal $\ker \alpha$ and the isomorphism classes of non-empty affine spaces appeared as $\alpha^{-1}(1)$. When we internalize the external $\otimes$-multiplication of augmented vector spaces by means of internal monoids, i.e. by affine algebras according to the previous remark, then theorem 6.2.3 says that this kind of augmentation arises internally as well. The vector space of displacement vectors becomes an ideal of the internal monoid, i.e. a K-algebra ideal, and the set of points arises as the preimage of 1 under the augmentation map. Within $\mathrm{Vct_K}/K$ and the category of affine K-algebras the augmentation as the respective terminal morphism is the only means to reflect this internally. When we pass to the category of internal semigroups of $\mathrm{Vct_K}/K$ then the ideal of displacement vectors will appear as an internal semigroup over the respective zero map. However, the augmentation will be still the only means to represent the set of points internally.

This analogy suggests to consider the sequence of functors

$$\mathrm{Vct_K} \xrightarrow{\mathcal{V}} \mathrm{Vct_K}/K \xrightarrow{\mathcal{A}or} \mathrm{Aff_K},$$

which is "exact up to equivalence" in a reasonable sense, as a "gros" notion of an affine space. Here r is the functor we defined in remark 5.4.2 (d).

Examples 6.2.7 (bi- and Hopf-algebras)**.**
With the algebraic presentation of the notion of an affine K-algebra by the category of augmented (unital) K-algebras we obtain also various classes of examples of affine K-algebras. Two such classes are given by K-bialgebras and K-Hopf algebras.

(a) A K-bialgebra in the sense of [Swe69, section 3.1] is an internal bimonoid of $(\mathrm{Vct_K}, \otimes, K)$. Another way to put it is to consider internal bimonoids as internal comonoids of the (symmetric) monoidal category of internal monoids of $\mathrm{Vct_K}$. The induced tensor-structure on internal monoids was indicated in section 2.5 in the general setting of symmetric monoidal categories. In our particular case it is given by the linear tensor product of K-algebras and the respective structure maps. For us the details of these constructions are not important here. We need only the following result (cf. [Swe69, proposition 3.1.1]): a K-bialgebra has both the structure of a (unital) K-algebra and the structure of a (counital) K-coalgebra, such that the structure maps of the coalgebra are K-algebra homomorphisms. We saw in remark 5.3.3 (d) K-coalgebras to be affine spaces; the augmentation was given by the counit. The characterization of K-bialgebras tells us now that its counit is a K-algebra homomorphism and the K-bialgebra thus an affine K-algebra. Because morphisms of K-bialgebras respect all structures involved, they are in particular K-algebra homomorphisms and respect the counit. As such they are also morphisms of affine K-algebras.

(b) A K-Hopf algebra H is a K-bialgebra $(H, \mu, u, \Delta, \varepsilon)$ with an antipode $S : H \to H$. The antipode is usually defined as the inverse of 1_H in the convolution

algebra $(\underline{\mathrm{Vct}}_{\mathrm{K}}(H,H), *, \varepsilon u)$ where the multiplication map is the convolution

$$* : \underline{\mathrm{Vct}}_{\mathrm{K}}(H,H) \otimes \underline{\mathrm{Vct}}_{\mathrm{K}}(H.H) \longrightarrow \underline{\mathrm{Vct}}_{\mathrm{K}}(H \otimes H, H \otimes H) \xrightarrow{\underline{\mathrm{Vct}}_{\mathrm{K}}(\Delta,\mu)} \underline{\mathrm{Vct}}_{\mathrm{K}}(H,H).$$

Here the first linear map is the natural one which maps the tensor product of two linear maps as vectors to the tensor product of those linear maps. By [Swe69, proposition 4.0.1] the antipode S is a contravariant endomorphism[3] of the augmented (unital) K-algebra $\varepsilon : H \to K$. Hence S is an contravariant endomorphism of the affine K-algebra H; in particular, it is an affine map.

6.3 Affine Algebras and Non-unital K-Algebras II

The slice category $\mathrm{Mon}(\mathrm{Vct}_{\mathrm{K}}, \otimes, K)/(K, \cdot, 1_K)$ arises also in a pure algebraic context, namely, as an equivalent category to the category of non-unital K-algebras; that is the category of internal semigroups $\mathrm{Sgr}(\mathrm{Vct}_{\mathrm{K}}, \otimes, K)$. Here the equivalence is given by the functor that adjoins to a non-unital K-algebra a multiplicative identity (cf. [BH96, lemma 10.6] for the case of K-algebras and [Mey80, section 3.1] for the construction of adjoining an identity for non-unital rings, i.e. non-unital $\mathbb{Z}$-algebras). This adjoin-an-identity functor

$$\mathcal{AD}_1 : \mathrm{Sgr}(\mathrm{Vct}_{\mathrm{K}}, \otimes, K) \to \mathrm{Mon}(\mathrm{Vct}_{\mathrm{K}}, \otimes, K)/(K, \cdot, 1_K)$$

has as left and right adjoint the functor

$$\mathcal{I} : \mathrm{Mon}(\mathrm{Vct}_{\mathrm{K}}, \otimes, K)/(K, \cdot, 1_K) \to \mathrm{Sgr}(\mathrm{Vct}_{\mathrm{K}}, \otimes, K)$$

that maps each augmented K-algebra $\omega : (V, \mu, u) \to (K, \cdot, 1_K)$ to the ideal $(\ker \omega, \tilde{\mu})$, where $\tilde{\mu} : \ker \omega \otimes \ker \omega \to \ker \omega$ denotes the restricted and corestricted multiplication map μ. Combining this result with corollary 6.2.5 yields that the category of affine K-algebras is equivalent to the category of non-unital K-algebras.

Corollary 6.3.1. *The category of affine K-algebras is equivalent to the category of non-unital K-algebras*

$$\mathrm{Mon}(\mathrm{Aff}_{\mathrm{K}}, \otimes, *) \simeq \mathrm{Sgr}(\mathrm{Vct}_{\mathrm{K}}, \otimes, K),$$

where the left $\otimes$ denotes the affine tensor product and the right $\otimes$ denotes the linear tensor product.

[3] In the literature on algebra a contravariant homomorphism $f : A \to B$ is also referred to as an antihomomorphism or antimorphism (cf. e.g. [Swe69]). It rather reverses than preserves the binary operation, i.e. $f(xy) = f(y)f(x)$ for all $x, y \in A$.

Remarks 6.3.2.

(a) The equivalence in the preceding corollary is in the spirit of proposition 6.1.1. Indeed, when we follow the respective functors involved we can see that the equivalence is given by a functor that maps each affine K-algebra (A, μ, u) to the non-unital K-algebra $(V_A, \hat{\mu})$.

(b) Although it is technically possible to identify affine K-algebras and non-unital K-algebras via this equivalence, it doesn't seem appropriate from the geometric point of view. The reason is as follows. Contrary to the equivalence of the category of affine K-algebras and augmented (unital) K-algebras, where both the set of points and the vector space of displacement vectors appear as part of the respective structure, a non-unital K-algebra has no inherent set of points. It is more that by the formal adjunction of an external identity the set of points can be formally reconstructed. We can only do so, for we know that an affine algebra is a pointed affine space and hence naturally isomorphic to its vector space of displacement vectors. This is a situation very similar to the one in remark 6.1.2.

Instead, if we consider the geometric presentation of an affine K-algebra by the internal monoids of Aff_K as to stress the underlying set of points then we can understand the equivalence of augmented (unital) K-algebras with the non-unital K-algebras as to stress the structure of an algebra involved. In this sense both equivalences appear as complementary pictures of augmented (unital) K-algebras.

6.4 The Affine Tensor Algebra

In section 2.5.4 we saw the forgetful functor from the category of internal monoids to the underlying monoidal category to have a left adjoint if this category has the property to be symmetric closed monoidal and cocomplete. This is the case for Aff_K and Vct_K/K (cf. theorem 3.4.5, theorem 3.3.8, remark 5.4.2(b)). If we use remark 6.2.6(b) we can restate the abstract result for these two cases as follows.

Proposition 6.4.1 (affine tensor algebra)**.**

(1) The forgetful functor

$$U : \mathrm{Mon}(\mathrm{Aff}_K, \otimes, *) \to \mathrm{Aff}_K, \qquad (A, \mu, u) \mapsto A, \qquad f \mapsto f,$$

has a left adjoint T.

(2) The forgetful functor

$$U : \mathrm{Mon}(\mathrm{Vct_K}, \otimes, *)/(K, \cdot, 1_K) \to \mathrm{Vct_K}\,/K,$$
$$(\omega : (V, \mu, u) \to (K, \cdot, 1_K)) \mapsto \omega, \qquad f \mapsto f,$$

has a left adjoint T.

In both cases we call T the affine tensor algebra functor *and TA and $T\omega$ an* affine tensor algebra*, respectively.*

Remarks 6.4.2.

(a) That we don't differentiate between both left adjoints is justified by remark 6.2.6(b) and corollary 6.2.4, for those results show the left adjoint from (1) to be a restriction of the one in (2) and the respective categories of internal monoids to be equivalent.

(b) Let $\omega : V \to K$ be an augmented vector space. From (2.12) in section 2.5.4 and remark 5.4.2(b) we can see the affine tensor algebra $T\omega$ to be the K-algebra homomorphism

$$\omega_T : T_v V \to K,$$

where

$$T_v : \mathrm{Vct_K} \to \mathrm{Mon}(\mathrm{Vct_K}, \otimes, K)$$

denotes the linear tensor algebra functor and ω_T the unique K-algebra homomorphism extending ω. Henceforth, we will not distinguish between the two tensor products and refer to both as the tensor algebra or tensor algebra functor T, when there is no danger of confusion.

(c) With the representation of an affine tensor algebra functor by a linear tensor algebra functor we can write down the set of points of $T\omega$ for an augmented vector space $\omega : V \to K$ from what we know about the underlying map of the K-algebra homomorphism $T\omega : T_v V \to K$ from linear algebra:

$$\ker T\omega + \{\lambda_0 + \sum_{k \in \mathbb{N}\setminus\{0\}} \lambda_k v_1^k \otimes \ldots \otimes v_k^k \mid \lambda_k = 0 \text{ for all, but finitely many } k \in \mathbb{N},$$
$$\forall\, k \in \mathbb{N}\ \forall\, j\ 1 \le j \le k,\ v_j^k \in \omega^{-1}(1),\ \sum_{k \in \mathbb{N}} \lambda_k = 1\}.$$

(d) Despite remark (a) it is instructive to compare a tensor algebra $T0_V$ of a vector space $0_V : V \to K$ with a tensor algebra $T\omega$ of an affine space $\omega : V \to K$. In the first case the set of points reduces to

$$\{1 + \alpha \mid \alpha \in \bigoplus_{k \in \mathbb{N}\setminus\{0\}} V^{\otimes n}\},$$

i.e. to the set of all tensors that have 1 as the 0-component, whereas in the second case the set of points is given as in the previous remark and it has this set of tensors with 0-component equal to 1 as a proper subset.

(e) (Finite dimensions.) The tensor algebra is genuinely an infinite dimensional space, so these result have no analogue in finite dimensions.

As indicated in section 2.5.4 we want to consider the tensor algebra $T\omega$ of an augmented vector space $\omega : V \to K$ as an $\mathbb{N}$-graded affine algebra. Following our discussion in section 2.5.6 we recall this to mean that we consider $T\omega$ as a monoidal functor in the functor category $(\mathrm{Vct}_K / K)^{\mathbb{N}}$. Due to the augmentation it is also suggestive to view it as a chain complex

$$\ldots \xrightarrow{0} V^{\otimes 4} \xrightarrow{0} V^{\otimes 3} \xrightarrow{0} V^{\otimes 2} \xrightarrow{0} V \xrightarrow{\omega} K.$$

A non-trivial ω may be thus regarded as a non-trivial boundary operator ∂_0. Each $V^{\otimes k}$ is a linear span of the pure tensors of degree k. We can use this and ω to define boundary operators $\partial_k : V^{\otimes(k+1)} \to V^{\otimes k}$ for each $k \in \mathbb{N}$. To do so we use the graded Leibniz rule

$$\partial_k(v_0 \otimes \ldots \otimes v_k) = \sum_{i=0}^{k} (-1)^i \omega(v_i) v_0 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes v_k, \qquad v_i \in V, \tag{6.1}$$

where the hat means the respective factor being omitted. As indiciated above the ∂_k on $V^{\otimes(k+1)}$ are then obtained by linear extension (cf. [Che56, theorem 2.3]). It is a standard exercise in algebraic topology to show $\partial_k \circ \partial_{k+1} = 0$ from such a definition (cf. [Hat02, lemma 2.1]). With this we can replace the zero maps by the respective boundary operators. This yields the chain complex

$$\ldots \xrightarrow{\partial_4} V^{\otimes 4} \xrightarrow{\partial_3} V^{\otimes 3} \xrightarrow{\partial_2} V^{\otimes 2} \xrightarrow{\partial_1} V \xrightarrow{\omega} K.$$

Even more is true. When we consider the differential $\partial : TV \to TV$, which is defined by the universal property of the direct sum for the linear maps

$$\partial_k : V^{\otimes(k+1)} \to V^{\otimes k} \hookrightarrow TV, \quad k \in \mathbb{N}, \quad \text{and} \quad \partial_{-1} : K \to \{0\} \hookrightarrow TV,$$

then TV and ∂ are seen to yield a DG algebra. This is a direct consequence of our definition of the boundary operators via the graded Leibniz rule.

Theorem 6.4.3. *Let $\omega : V \to K$ be an augmented K-vector space. The augmentation ω determines uniquely a graded derivation $\partial : TV \to TV$ of degree -1 of the K-algebra TV such that $\partial|_V = \omega$ and $\partial^2 = 0$. This shows (TV, ∂) a DG algebra.*

Proof. In the preceding discussion we have already shown the existence. It remains to show the uniqueness. This is, however, a direct consequence of the graded Leibniz rule, V to generate TV as a K-algebra and $\partial|_V = \omega$.

(qed)

Remarks 6.4.4.

(a) Let $\omega : V \to K$ be an augmented K-vector space. Consider the (linear) tensor algebra TV as an $\mathbb{N}$-graded algebra. By proposition 6.4.1 and remark 5.4.2(b) it may be either regarded as an $\mathbb{N}$-graded affine algebra $T\omega$ or, by the preceding theorem it may be regarded as a DG algebra (TV, ∂). Note that the boundary operators ∂_k induced by ∂ do not yield morphisms of augmented vector spaces, unless ω and hence all ∂_k are trivial. This is due to $\partial_k \circ \partial_{k+1} = 0$ and $\omega^{\otimes(k+1)}$ as well as $\omega^{\otimes(k+2)}$ to be non-trivial if ω is. Accordingly, there is no notion of a DG affine tensor algebra and both structures belong to different categories. We have thus once more two different presentations of one notion and both presentations are of importance for geometry.

(b) Consider the DG algebra $(T\mathcal{B}A, \partial)$ induced by the weight form of an affine space $\omega : \mathcal{B}A \to K$. To indicate a geometric relevance of the differential we will consider a basic example and suggest some possible geometric interpretations. We have

$$\partial(P \otimes Q) = Q - P = \overrightarrow{PQ} \quad \text{and} \quad \partial(P \otimes \overrightarrow{PQ}) = \overrightarrow{PQ}, \qquad P, Q \in A.$$

If we think of $P \otimes Q$ as the located directed line segment $\overline{PQ}$, then ∂ assigns to it its respective non-located line segment $\overrightarrow{PQ}$. Taking a more dynamical point of view we could also speak of $\partial(P \otimes Q)$ as the law of displacement we need to follow for one unit of time to trace out $\overline{PQ}$ starting in P.

One might be inclined to use the same interpretation for the second case. There is a difference though. From the point of view of the affine tensor algebra TA $P \otimes Q$ is a point but $P \otimes \overrightarrow{PQ}$ is a displacement vector. So, instead of a directed line segment we think of $P \otimes \overrightarrow{PQ}$ as the displacement vector $\overrightarrow{PQ}$ bound to the point P. Indeed, by bilinearity we have

$$P \otimes R + P \otimes \overrightarrow{PQ} = Q \otimes (R + \overrightarrow{PQ}) = P \otimes R',$$

which shows $P \otimes \overrightarrow{PQ}$ to displace line segments that start in P keeping P fixed.

(c) (Affine structures on V and DG structures on TV.) We can restate theorem 6.4.3 as follows: the linear map

$$\mathrm{Der}_{-1}(TV) \to V^*, \qquad \partial \mapsto \partial|_V,$$

is an isomorphism. Here $\mathrm{Der}_{-1}(TV)$ denotes the vector space of (graded) derivations of degree -1 of TV. Each of these derivations square to zero and is thus a (graded) differential. This can be seen either from our proof of theorem 6.4.3 or by considering the $\mathbb{Z}$-graded lie algebra $\mathrm{Der}(TV)$ of all graded derivations of TV. For $\mathrm{Der}(TV)$ it is $\mathrm{Der}_k(TV) \cong 0$ for all $k \leq -1$ (cf. [Che56, theorem 2.3]), and for derivations ∂ of degree -1 the graded commutator yields a derivation of degree -2 and reads $[\partial, \partial] = 2\partial^2$ (cf. [Che56, theorem 1.6]). Conversely, any graded differential is a graded derivation of degree -1 by definition.

In remark 5.3.3(e) we interpreted V^* as the vector space generated by affine structures on V. With this the above result then shows that any affine structure on V corresponds one-to-one to a differential on TV and hence to a DG structure on TV.

(d) Although the construction of the induced DG structure on TV by $\omega \in V^*$ might suggest it, the iterated tensor products $V^{\otimes k}$, $k \in \mathbb{N}$ together with the "face" maps $\delta_i^k : V^{\otimes(k+1)} \to V^{\otimes k}$ defined by the ith contraction

$$v_0 \otimes \ldots \otimes v_k \mapsto \omega(v_i) v_0 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes v_k$$

is *not* an (augmented) simplicial object, i.e., a functor $\Delta^{op} \to \mathrm{Vct_K}$ with Δ the simplicial category (cf. [Mac98][section VII.5]). What is missing are the respective degeneracy maps. Those exist if and only if we can fix a coassociative comultiplication $\Delta : V \to V \otimes V$ for which ω is a counit. Recall that this is due to the category Δ^{op} carrying (and being generated by) the universal comonoid $(1, 1 \to 1 + 1, 1 \to 0)$ (cf. the dual of proposition VII.5.1 and theorem XI.3.1 in [Mac98]).It can be still considered a functor $\Delta_{mon}^{op} \to \mathrm{Vct_K}$ though, whereby Δ_{mon} is the subcategory with the same objects as Δ but with the morphisms restricted to monos of Δ.

(e) (Comultiplications and codifferentials.) In the same way a linear form ω on a K-vector space V corresponds one-to-one to a (graded) derivation of degree -1 on the tensor algebra TV, a comultiplication Δ on V corresponds one-to-one to a (graded) derivation of degree $+1$ (cf. [Che56][theorem 2.3]). We have seen that for the $\mathbb{N}$-grading any (graded) derivation of degree -1 on TV is necessarily a differential. It is not true that every (graded) derivation of degree $+1$ yields a codifferential δ, i.e., a (graded) derivation δ of degre $+1$ with $\delta^2 = 0$. One has the following interesting relationship, however: a (graded) derivation δ of degree $+1$ is a codifferential if and only if the corresponding comultiplication $\Delta = \delta_1 : V \to V \otimes V$ is coassociative.

The abstract reason for this relationship is due to the following. Recall that the subcategory Δ_{epi} of the simplicial category Δ that has the same objects as Δ but as morphisms only epimorphisms of Δ is a strict monoidal subcategory of

$(\Delta, +, 0)$. Δ_{epi} carries (and is generated by) the universal semigroup $(1, 1+1 \to 1)$ (cf. [Mac98][exercise VII.5.3]). Any strong monoidal functor $F : \Delta_{epi} \to \mathrm{Vct_K}$ is determined by a an internal semigroup of $\mathrm{Vct_K}$ up to isomorphism. Dually, any strong monoidal functor $\Delta_{epi}^{op} \to \mathrm{Vct_K}$ is determined by an internal cosemigroup (V, Δ), i.e., by a coassociative comultiplication $\Delta : V \to V \otimes V$, up to isomorphism. The iterated tensor products $V^{\otimes k}$, $k \in \mathbb{N}$ thus lift to a strong monoidal functor $\Delta_{epi}^{op} \to \mathrm{Vct_K}$ for any coassociative comultiplication Δ on V. For any strong monoidal functor $F : \Delta_{epi}^{op} \to \mathrm{Vct_K}$ coboundaries $\delta_k : F(k) \to F(k+1)$ can be defined as the alternating sums of the k "degeneracy" maps $F((\sigma_i^k)^{op} : k \to k+1)$. So any such functor yields a cochain complex.

If (V, Δ, ε) is a coalgebra, d the differential induced by ε and δ the codifferential induced by Δ, then the counit law implies $[d, \delta] = d \circ \delta = 0$, whereby the bracket denotes the graded commutator of the $\mathbb{Z}$-graded Lie algebra $\mathrm{Der}(TV)$; but the last identity seems to be not equivalent to the counit law.

6.5 Affine Spaces and Connected DG Algebras

The affine tensor algebra turns out to carry a further structure that is uniquely determined by the affine structure: it is the (graded) differential of a DG algebra. Such a differential cannot respect the augmentation, and so the differential as well as the boundary maps can't be affine maps. This brings a new aspect into play that has to be understood. We therefore take up the observation from the preceding section and study the connections between connected DG algebras[4] and affine spaces in general. For our purposes, the category $\mathcal{DG}_c - Alg$ of connected DG algebras is defined having as morphisms the homogeneous algebra homomorphisms of degree 0 whose homogeneous components respect differentials, i.e., are chain maps. Depending on the context and whichever point of view is more convenient, we will consider a connected DG algebra $\mathcal{A}$ either as a particular functor $\mathcal{A} : (\mathbb{N}, \geq) \to \mathrm{Vct_K}$, which restricts to a monoidal functor $(\mathbb{N}, +, 0) \to (\mathrm{Vct_K}, \otimes, K)$ (cf. the respective discussion in section 2.5.6), emphasizing the gradation and boundary maps $\partial_k = \mathcal{A}(k+1 \to k)$, or the underlying K-algebra $(\mathcal{A}, \partial)$ emphasizing the total structure and the differential ∂.

According to our discussion in section 2.5.4 the affine tensor algebra functor

$$T : \mathrm{Vct_K}/K \to \mathrm{Mon}(\mathrm{Vct_K}, \otimes, *)/(K, \cdot, 1_K)$$

factors through the category $\mathrm{Mon}((\mathrm{Vct_K}/K)^{\mathbb{N}}, \otimes, \varepsilon_K)$ of $\mathbb{N}$-graded algebras, which can also be considered as the subcategory of $(\mathrm{Vct_K}/K)^{\mathbb{N}}$ of monoidal functors $(\mathbb{N}, +, 0) \to \mathrm{Vct_K}/K$ by our discussion in section 2.5.6. Theorem 6.4.3 shows us that T even factors through the category $\mathcal{DG}_c - Alg$ of connected DG algebras. If we

[4] Recall that an ($\mathbb{N}$-graded) DG algebra $(\mathcal{A}, \partial)$ is called *connected* if $\mathcal{A}_0 = K1 \cong K$.

consider the latter as a subcategory of $(\mathrm{Vct_K}/K)^{(\mathbb{N},\geq)}$ (recall $\mathrm{Vct_K} \cong (\mathrm{Vct_K}/K)^v \hookrightarrow \mathrm{Vct_K}/K$), then the factorization can be written like this

$$\begin{array}{ccc} \mathcal{DG}_c\text{-}Alg & \xrightarrow{(\mathrm{Vct_K}/K)^\iota} & \mathrm{Mon}((\mathrm{Vct_K}/K)^{\mathbb{N}}, \otimes, \varepsilon_K) \\ & \nwarrow & \uparrow T \\ & & \mathrm{Vct_K}/K, \end{array}$$

whereby $(\mathrm{Vct_K}/K)^\iota$ is the functor given by right-composition by the inclusion functor $\iota : \mathbb{N} \hookrightarrow (\mathbb{N}, \geq)$, and whose action simply amounts to forgetting the differentials. We will denote the functor $\mathrm{Vct_K}/K \to \mathcal{DG}_c\text{-}Alg$ by T again. It maps vector spaces 0_V to trivial DG algebras $(TV, 0)$ and non-zero affine spaces ω to non-trivial DG algebras $(T\,\mathrm{dom}\,\omega, \partial)$ with $\partial_0 = \omega$.

Conversely, we can consider the functor

$$E : \mathcal{DG}_c\text{-}Alg \to \mathrm{Vct_K}/K, \qquad F \mapsto F(1 \to 0) = \partial_0 : F(1) \to K, \quad \tau \mapsto \tau_1.$$

It maps DG algebras $(\mathcal{A}, \partial)$ with trivial ∂_0 to vector spaces and with non-trivial ∂_0 to non-empty affine spaces. In fact, E is a right-adjoint-left inverse of T[5].

Theorem 6.5.1. *The functor E is a right-adjoint-left inverse of T.*

Proof. The identity $E \circ T = I_{\mathrm{Vct_K}/K}$ is obvious from the definitions of T and E. To establish the natural bijection

$$\mathcal{DG}_c\text{-}Alg(T\omega, F) \cong \mathrm{Vct_K}/K(\omega, EF)$$

it is sufficient to show for each ω the morphism $1_\omega : \omega \to ET\omega$ a universal morphism from ω to E. This, however, is a consequence of the universal property of the linear tensor algebra and can be seen as follows. Let $f : \omega \to EF$ be given and let $\mathcal{A}$ be the underlying K-algebra of F. (Recall that F is an $\mathbb{N}$-gradation for $\mathcal{A}$ and $\mathcal{A} = \varinjlim(\mathrm{Vct_K}/K)^\iota F = \bigoplus_{k\in\mathbb{N}} F(k)$.) Then f is in particular a linear map

$$f : \mathrm{dom}\,\omega \to \mathrm{dom}\,EF = F(1) \hookrightarrow \mathcal{A}\,.$$

By the universality of the inclusion $\eta_{\mathrm{dom}\,\omega} : \mathrm{dom}\,\omega \hookrightarrow T\,\mathrm{dom}\,\omega$ with $T\,\mathrm{dom}\,\omega$ being the (underlying) linear tensor algebra of $T\omega$ there is a unique K-algebra homomorphism $\tilde{f} : T\,\mathrm{dom}\,\omega \to \mathcal{A}$ with $\tilde{f} \circ \eta_{\mathrm{dom}\,\omega} = f$. From the construction of $\tilde{f}$, the image of f lying in $F(1)$ and the fact that $\mathcal{A}$ is graded with gradation F, one can see readily that $\tilde{f}$ is a homogeneous algebra homomorphism of degree zero, i.e., it lifts to a natural transformation $\tilde{f} : (\mathrm{Vct_K}/K)^\iota(T\omega) \dot{\to} (\mathrm{Vct_K}/K)^\iota F$.

[5] Compare this with the evaluation-at-1 functor E_1 being a right-adjoint-left inverse to $T : \mathrm{Vct_K}/K \to \mathrm{Mon}((\mathrm{Vct_K}/K)^{\mathbb{N}}, \otimes, \varepsilon_K)$ in (2.14) in section 2.5.4.

We need to show $\tilde{f}$ a chain map, or in other words that it is a natural transformation $T\omega \dot{\to} F$. If we denote the k-th boundary maps $T\omega(k+1 \to k)$, $F(k+1 \to k)$ by ∂_k^ω and ∂_k, respectively, then the chain map condition reads $\tilde{f}_k \circ \partial_k^\omega = \partial_k \circ \tilde{f}_{k+1}$. It is sufficient to show this for pure tensors $v_0 \otimes \ldots \otimes v_k$, $v_i \in \operatorname{dom}\omega$; but for those this is an immediate consequence of $\omega = \partial_0 \circ f$, $\tilde{f}_1 = f$ and the graded Leibinz rule:

$$\begin{aligned}
\tilde{f}_k \circ \partial_k^\omega(v_0 \otimes \ldots \otimes v_k) &= \tilde{f}_k(\sum_{i=0}^{k}(-1)^i \omega(v_i) v_0 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes v_k) \\
&= \sum_{i=0}^{k}(-1)^i f(v_0) \cdot \ldots \cdot \omega(v_i) \cdot \ldots \cdot f(v_k) \\
&= \sum_{i=0}^{k}(-1)^i f(v_0) \cdot \ldots \cdot \partial_0(f(v_i)) \cdot \ldots \cdot f(v_k) \\
&= \partial_k \circ \tilde{f}_{k+1}(v_0 \otimes \ldots \otimes v_k).
\end{aligned}$$

(qed)

Remarks 6.5.2.

(a) (On the counit ε of $T \dashv E$.) For each connected DG algebra $\mathcal{A}$ the component of the counit $\varepsilon_{\mathcal{A}} : TE\,\mathcal{A} \dot{\to} \mathcal{A}$ of the adjunction $T \dashv E$ of theorem 6.5.1 is given by (6.2) as follows.

Let $\langle \mathcal{A}_1 \rangle$ be the DG subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_1$. That is,

$$\langle \mathcal{A}_1 \rangle_k = \operatorname{span}\{v_1 \cdot \ldots \cdot v_k \mid v_i \in \mathcal{A}_1\} \leq \mathcal{A}_k,$$

and each boundary operator

$$\partial_k^{\langle \mathcal{A}_1 \rangle} : \langle \mathcal{A}_1 \rangle_{k+1} \to \langle \mathcal{A}_1 \rangle_k$$

is given by the restriction and corestriction of $\partial_k : \mathcal{A}_{k+1} \to \mathcal{A}$. That this is well-defined is a consequence of the graded Leibniz rule. For each $k \in \mathbb{N}$ there is a subvector space $\mathcal{I}_k \leq \mathcal{A}_1^{\otimes k}$ such that $\mathcal{A}_1^{\otimes k} / \mathcal{I}_k \cong \langle \mathcal{A}_1 \rangle_k$. Furthermore, the $\mathcal{I}_k$, $k \in \mathbb{N}$ yield an graded ideal $\mathcal{I}$ of $TE\,\mathcal{A}$.

Consider the linear maps $p_k : (TE\,\mathcal{A})_k \to \mathcal{A}_1^{\otimes k} / \mathcal{I}_k \cong \langle \mathcal{A}_1 \rangle$. Applying the graded Leibniz rule together with $\mathcal{I}_0 = \mathcal{I}_1 = \{0\}$ as in the last step of the proof of theorem 6.5.1 one can see readily that they constitute a chain map $p_{\mathcal{A}} : TE\,\mathcal{A} \dot{\to} \langle \mathcal{A}_1 \rangle$. Let $\iota_{\mathcal{A}} : \langle \mathcal{A}_1 \rangle \dot{\to} \mathcal{A}$ denote the inclusion of $\langle \mathcal{A}_1 \rangle$ in $\mathcal{A}$, then we have[6]

$$\varepsilon_{\mathcal{A}} = \iota_{\mathcal{A}} \cdot p_{\mathcal{A}}. \tag{6.2}$$

[6] Although obvious from the contex, be aware the "$\cdot$" meaning the vertical composition of natural transformations and no multiplication here.

To see this apply the definition of the counit as the left adjunct of $1_{E\mathcal{A}}$; then one just needs to verify $1_{E\mathcal{A}} = E(\iota_{\mathcal{A}} \cdot p_{\mathcal{A}}) \circ \eta_{E\mathcal{A}}$ (cf. (2.3) in section 2.2.2), which follows from $\eta_{E\mathcal{A}} = 1_{E\mathcal{A}}$, $(\iota_{\mathcal{A}} \cdot p_{\mathcal{A}})_0 = 1_K$ ($\mathcal{I}_0 = \{0\}$) and $(\iota_{\mathcal{A}} \cdot p_{\mathcal{A}})_1 = 1_{\mathcal{A}_1}$ ($\mathcal{I}_1 = \{0\}$).

(b) From the representation (6.2) of $\varepsilon_{\mathcal{A}}$ we can read off that $\varepsilon_{\mathcal{A}}$ is epi if and only if $\langle \mathcal{A}_1 \rangle = \mathcal{A}$, which is the case if and only if $\mathcal{A}$ is generated by $\mathcal{A}_1$ as a graded algebra. Herto recall that a natural transformation is an epi in its respective functor category if and only if each component is an epi in the respective target category. (This is a consequence of the dual of [Mac98][theorem V.3.1] and the fact that f is epi if and only if $1_{\mathrm{cod}\, f}$ yields a cokernel pair of f (cf. [Mac98][exercise III.4.4]).)

(c) One can see easily that the construction $\langle \mathcal{A}_1 \rangle$ in remark (a) yields a functor

$$\langle (-)_1 \rangle : \mathcal{DG}_c - Alg \to \mathcal{DG}_c^g - Alg, \quad \mathcal{A} \mapsto \langle \mathcal{A}_1 \rangle, \quad f \mapsto f|\langle \mathcal{A}_1 \rangle,$$

whereby $\mathcal{DG}_c^g - Alg$ is the full subcategory of DG algebras $\mathcal{A}$ generated by its space $\mathcal{A}_1$ of homogeneous elements of degree 1. This is because of the fact that any homogeneous DG algebra homomorphism (of degree zero) $f : \langle \mathcal{A}_1 \rangle \dot{\to} \mathcal{B}$ is determined uniquely by $f_1 : \mathcal{A}_1 \to \mathcal{B}_1$ and the image of f being in $\langle \mathcal{B}_1 \rangle$. Moreover, from this observation we can deduce that $\langle (-)_1 \rangle$ is a right adjoint of the inclusion functor $\mathcal{DG}_c^g - Alg \hookrightarrow \mathcal{DG}_c - Alg$. Therefore, $\mathcal{DG}_c^g - Alg$ is a full coreflective subcategory[7] of $\mathcal{DG}_c - Alg$.

(d) The preceding theorem lets us consider $\mathrm{Vct_K}/K$ as a full coreflective subcategory of $\mathcal{DG}_c - Alg$. But since there are connected DG algebras $\mathcal{A}$ that aren't freely generated by $\mathcal{A}_1$ like, for example, our two constructions of the Grassmann- and Clifford algebra of a (non-empty) affine space from section 6.6, this is not an equivalence of categories. This can be also deduced from the counit ε of $T \dashv E$ being not an isomorphism in general (cf. remarks (a), (b) and [Mac98][theorem IV.3.1]).

In particular, the category of affine spaces $\mathrm{Aff_K} \simeq (\mathrm{Vct_K}/K)^a$ may be considered as a full coreflective subcategory of the category of connected DG algebras $(\mathcal{A}, \partial)$ with either $\partial_0 \neq 0$ or the DG algebra $(\varepsilon_K, 0)$.

For the next step we will consider a connected $\mathbb{N}$-graded algebra $\mathcal{A}$ that is generated by $\mathcal{A}_1$ and generalize remark 6.4.4(c) to $\mathcal{A}$; that is, we will relate the DG structures (differentials) ∂ of $\mathcal{A}$ to affine structures of $\mathcal{A}_1$. Here is the first result.

[7] Recall from [Mac98][section IV.3] that a subcategory $C \hookrightarrow D$ is called *coreflective* if the inclusion functor has a right-adjoint.

Proposition 6.5.3. *Let $\mathcal{A}$ be a connected $\mathbb{N}$-graded algebra that is generated by $\mathcal{A}_1$, then*

$$DG(\mathcal{A}) = \mathrm{Der}_{-1}(\mathcal{A}) \to \mathcal{A}_1^*, \qquad \partial \mapsto \partial|\,\mathcal{A}_1 = \partial_0$$

is injective. $DG(\mathcal{A})$ denotes the vector space of all ($\mathbb{N}$-graded) differentials of $\mathcal{A}$.

Proof. The assertion $DG(\mathcal{A}) = \mathrm{Der}_{-1}(\mathcal{A})$ follows as in remark 6.4.4(c) from the graded Lie-algebra structure on $\mathrm{Der}(\mathcal{A})$, $\mathrm{Der}_{-2}(\mathcal{A}) \cong 0$ due to the $\mathbb{N}$-grading, and $[\partial, \partial] = 2\partial^2$ for the graded commutator for derivations of degree -1. We thus have to show the restriction map $\mathrm{Der}_{-1}(\mathcal{A}) \to \mathcal{A}_1^*$ injective. But since we have $\mathcal{A} = \langle \mathcal{A}_1 \rangle$, whereby the construction of $\langle \mathcal{A}_1 \rangle$ is understood to mean the graded algebra only, the graded Leibniz rule yields the uniqueness of a DG structure ∂ that restricts to $\omega \in \mathcal{A}_1^*$ in the same way as in the proof of theorem 6.4.3 or in the proof of theorem 6.5.1.

(qed)

Proposition 6.5.4. *Let $\omega \in V^*$, $\mathcal{I}$ be a graded ideal of TV with $\mathcal{I}_0 = \mathcal{I}_1 = \{0\}$ and ∂^ω the unique DG structure induced by ω. Then the connected graded algebra $TV/\mathcal{I}$ has a DG structure $\tilde{\partial}$ induced by ω if and only if $\partial(\mathcal{I}) \subset \mathcal{I}$.*

Proof. Regarding the "if" part this is an obvious application of the fundamental theorem on homomorphisms of vector spaces, $\pi_\mathcal{I} : TV \twoheadrightarrow TV/\mathcal{I}$ being an algebra homomorphism and the definition of ∂^ω by the boundary maps ∂_k^ω (cf. also [Che56][theorem 1.5]). The "only if" part follows from the preceding proposition and remark 6.5.2(a), which shows that $\pi_\mathcal{I}$ coincides with $\varepsilon_{TV/\mathcal{I}}$ in this case.

(qed)

Remarks 6.5.5.

(a) The assumption of $\mathcal{A}$ being generated by $\mathcal{A}_1$ is necessary in proposition 6.5.3. For consider a finite dimensional non-trivial vector space V and its exterior algebra $\bigwedge V$ as a graded algebra. It is generated by V and we have $(\bigwedge V)_k = \{0\}$ for $k > n := \dim V$. Let $\mathcal{A}$ be any graded algebra (for example, $\mathcal{A} = \bigwedge V$) and consider its shift by n, i.e., the graded algebra $\mathcal{A}[n]$ with $\mathcal{A}[n]_k = \{0\}$ for $0 \leq k \leq n$ and $\mathcal{A}[n]_{k+n+1} = \mathcal{A}_k$ for $0 \leq k$. Then $F := \bigwedge V + \mathcal{A}[n]$ is a connected graded algebra, whereby "+" means the componentwise direct sum. F is so to speak the concatenation of $\bigwedge V$ and $\mathcal{A}$. It is not generated by $F_1 = V$ anymore. We can then choose differentials for $\mathcal{A}$ and $\bigwedge V$ arbitrarily and, in particular, independently of each other. Each such choice of a pair of differentials will yield a differential on F. So the DG structures are not determined by linear forms in $F_1^* = V^*$ alone.

(b) There are connected graded algebras $\mathcal{A}$ that are generated by a non-trivial $\mathcal{A}_1$ such that the restriction map of proposition 6.5.3 is (i) the zero map (ii) a true embedding and (iii) an isomorphism.

(i) Let V be a non-trivial vector space with $\dim V > 1$. The symmetric algebra SV is a connected graded algebra generated by V that allows only a trivial DG structure. To make this more obvious recall that the graded ideal $\mathcal{I} < TV$ for SV is generated by the relations $v \otimes w - w \otimes v = 0$ for all $v, w \in V$. Let ∂ be a non-trivial DG structure on TV, then

$$\partial(v \otimes w - w \otimes v) = 2(\partial_0(v)w - \partial_0(w)v).$$

Since ∂_0 is not the zero map, $\partial_0^{-1}(1)$ is not empty. Since $\dim V > 1$, this set has at least two elements. Choose v, w to be any two such elements, then $\partial(v \otimes w - w \otimes v) = 2(v - w) \neq 0$. The claim thus follows from proposition 6.5.4.

(ii) Let V be as in (i). Fix two linearly independent $v, w \in V$ and a linear form $\omega \in V^*$ with $v, w \in \ker \omega$. Consider the graded ideal $\mathcal{I}$ generated by $v \otimes w - w \otimes v$. Then $\partial^\omega(v \otimes w - w \otimes v) = 0$. Since $\mathcal{I}$ is an ideal generated by $v \otimes w - w \otimes v$ the graded Leibniz rule yields $\partial^\omega(\mathcal{I}) \subset \mathcal{I}$, and by proposition 6.5.4 there is a $\tilde{\partial}$ on $TV/\mathcal{I}$ with $\tilde{\partial}|V = \omega$. However, there are also linear forms $\nu \in V^*$ that don't map $v \otimes w - w \otimes v$ to 0, so the restriction map can't be surjective.

(iii) As we will see in the subsequent section, the Grassmann algebra has this property.

(c) (Affine spaces and connected DG algebras.) We can summarize the interrelation of affine spaces and connected DG algebras as follows. Let $\mathcal{DG}_c^f - Alg$ denote the full subcategory of connected DG algebras that are free over its space of homogeneous elements of degree 1. Each such DG algebra $\mathcal{A}$ is isomorphic to $T E \mathcal{A}$ by the counit $\varepsilon_{\mathcal{A}}$. We have in particular that T yields an equivalence $\mathrm{Vct_K}/K \simeq \mathcal{DG}_c^f - Alg$. By theorem 6.5.1 $\mathcal{DG}_c^f - Alg$ is a coreflective subcategory of $\mathcal{DG}_c^g - Alg$ and $\mathcal{DG}_c - Alg$. The coreflection is given by TE and its restriction, respectively. We have thus the following chain of coreflective subcategories

$$\mathcal{DG}_c^f - Alg \underset{\hookrightarrow}{\overset{TE}{\underset{\top}{\leftarrow}}} \mathcal{DG}_c^g - Alg \underset{\hookrightarrow}{\overset{\langle(-)_1\rangle}{\underset{\top}{\leftarrow}}} \mathcal{DG}_c - Alg\ .$$

Note that $T E \mathcal{A} = TE\langle(\mathcal{A})_1\rangle$ for any connected DG algebra $\mathcal{A}$. In analogy to $\mathrm{Vct_K}/K$ each of those subcategories is built from two full and "disjoint" subcategories: those with DG algebras $\mathcal{A}$ with trivial ∂_0 and those with non-trivial ∂_0. Each of the above functors restricts and corestricts to those two kinds

of subcategories, respectively. We will refer to the second kind of subcategories as "affine", since this kind is related to affine spaces, whereas the first kind is related to vector spaces.

We have therefore that the category of non-empty affine spaces $\mathrm{Aff_K}^{\times}$ is equivalent to the affine subcategory of $\mathcal{D}\,\mathcal{G}_c^f - Alg$. For the DG algebras $\mathcal{A}$ in the bigger affine subcategory of $\mathcal{D}\,\mathcal{G}_c^g - Alg$ we have that the underlying affine spaces ∂_0 uniquely determine the respective DG structure. Finally, $\mathrm{Aff_K}^{\times}$ is related to the affine subcategory of $\mathcal{D}\,\mathcal{G}_c - Alg$, namely as a coreflective subcategory, but for a general affine DG algebra the underlying affine space determines neither the algebra nor the DG structure.

Of course we can extend everything to include the empty spaces as well, if we extend each affine subcategory by the initial connected DG algebras O ($O_o = K$ and O_k is a trivial vector space) and the respective initial morphisms.

We conclude this section by studying when a connected $\mathbb{N}$-graded K-algebra $\mathcal{A}$ generated by $\mathcal{A}_1$ admits an augmentation induced by a given affine structure $\omega : \mathcal{A}_1 \to K$. The answer is once again given by the affine tensor algebra and the fundamental theorem on homomorphisms.

Proposition 6.5.6. *Let $\mathcal{A}$ be a connected $\mathbb{N}$-graded K-algebra that is generated by $\mathcal{A}_1$. Let $\varepsilon_{\mathcal{A}} : T E_1 \mathcal{A} \dot{\to} \mathcal{A}$ be the $\mathcal{A}$-component of the counit of the adjunction $T \dashv E_1$ from (2.14) in section 2.5.4. Then for any $\omega : \mathcal{A}_1 \to K$ there is an augmentation $\omega_{\mathcal{A}} : \mathcal{A} \to K$ with $\omega_{\mathcal{A}} | \mathcal{A}_1 = \omega$ unique with this property if and only if $\ker \varepsilon_{\mathcal{A}} \leq \ker T\omega$, whereby $T\omega : T\mathcal{A}_1 \to K$ denotes the augmentation of the tensor algebra induced by ω.*

Proof. We know that each affine tensor algebra $T\omega$ is an augmented algebra, whereby the augmentation is induced and uniquely determined by ω. Since our $\mathcal{A}$ is generated by $\mathcal{A}_1$, $\varepsilon_{\mathcal{A}}$ is epi (cf. the construction in remark 6.5.2(a) related to the graded algebra). Therefore, $\mathcal{A}$ is isomorphic to $T\mathcal{A}_1 / \ker \varepsilon_{\mathcal{A}}$ for the graded ideal $\ker \varepsilon_{\mathcal{A}}$ of $T\mathcal{A}_1$. By the fundamental theorem on homomorphisms for K-algebras and considering $\varepsilon_{\mathcal{A}}$ as the respective quotient map $T\omega$ factors through $\varepsilon_{\mathcal{A}}$ if and only if $\ker \varepsilon_{\mathcal{A}} \leq \ker T\omega$. This deals with the uniqueness and the "if" part.

For the "only if" part note that since $T\mathcal{A}_1 / \ker \varepsilon_{\mathcal{A}} \cong \mathcal{A}$ as graded algebras any augmentation $\nu : \mathcal{A} \to K$ extending ω, i.e., $\nu | \mathcal{A}_1 = \omega$, yields an augmentation $\nu \circ \varepsilon_{\mathcal{A}}$ of $T\mathcal{A}_1$ extending ω, so $\nu \circ \varepsilon_{\mathcal{A}} = T\omega$.

(qed)

Remarks 6.5.7.

(a) Since it is based on the fundamental theorem on homomorphisms on K-algebras the preceding proposition holds also for (not necessarily graded) K-algebras $\mathcal{A}$

that are generated by a subspace $\mathcal{A}_1$ as K-algebras: for any $\omega : \mathcal{A}_1 \to K$ there is an augmentation $\omega_{\mathcal{A}} : \mathcal{A} \to K$ with $\omega_{\mathcal{A}}|\,\mathcal{A}_1 = \omega$ unique with this property if and only if $\ker \varepsilon_{\mathcal{A}} \leq \ker T\omega$, whereby $T\omega : T\,\mathcal{A}_1 \to K$ denotes the augmentation of the tensor algebra induced by ω.

(b) Given an augmented vector space $\omega : V \to K$ with $\omega \neq 0$ proposition 6.5.4 and proposition 6.5.6 characterize those connected graded algebras generated by V, which admit a DG structure and an augmentation induced by ω. In the tensor algebra TV we have an example which allows both. A further such example will be the Clifford algebra of ω in the subsequent section. This algebra, however, is not $\mathbb{N}$-graded. On the other hand, the Grassmann algebra of ω, which we will consider in the subsequent section, too, is an example of a DG algebra in $\mathcal{D}\,\mathcal{G}_c^g - Alg$, where ω doesn't (cannot) induce an augmentation.

6.6 Two Examples of Affine Algebras

In this last and final section we will present two examples of affine algebras: the Grassmann algebra of affine space and the Clifford algebra of affine space. Each of those could be considered as a geometric algebra of affine space in its own right. However, we shall see the Clifford algebra to deserve this name more than Grassmann's algebra, at least it does so from the point of view of affine spaces.

6.6.1 The Grassmann Algebra of Affine Space

Given an affine space A in $\mathrm{Aff_K}$ we define *the Grassmann algebra* $\mathcal{G}\,A$ *of* A to be the exterior algebra $\bigwedge \mathcal{B}\,A$. This is Grassmann's algebra of the affine space A, which he constructs and discusses in the second part of [Gra78][8]. As a K-algebra we know it to be a quotient of the linear tensor algebra $T\,\mathcal{B}\,A$ by the (graded) ideal generated by the relations $v^2 = 0$ for all $v \in \mathcal{B}\,A$. We consider it as an $\mathbb{N}$-graded K-algebra with its natural gradation

$$\mathbb{N} \to \mathrm{Vct_K}, \qquad k \mapsto \bigwedge^k \mathcal{B}\,A.$$

The projection to the zero component

$$\mathrm{pr}_0 : \bigwedge \mathcal{B}\,A \to \bigwedge^0 \mathcal{B}\,A = K$$

is a K-algebra homomorphism, so $\mathcal{G}\,A$ is an augmented algebra and hence an affine algebra. However, with this definition we have $A \subset \ker \mathrm{pr}_0$. So A isn't an affine

[8] Grassmann actually never speaks about an affine space, but only about "Elementargrößen", which he distinguishes by their weight. He calls the Elementargröße of zero weight a line segment and of non-zero weight a wighted point only when he applies his extension theory to geometry.

subspace of $\mathcal{G}\,A$ and in consequence the points of A are not points of $\mathcal{G}\,A$ but displacement vectors. By what we saw in remark 6.4.2(d) as an affine algebra $\mathcal{G}\,A$ is rather constructed from the vector space $0_{\mathcal{B}\,A} : \mathcal{B}\,A \to K$ than from the affine space $\omega : \mathcal{B}\,A \to K$. Nonetheless, the weight form ω does play a role for $\mathcal{G}\,A$: it yields a differential ∂ uniquely determined by ω according to proposition 6.5.3 and proposition 6.5.4. This is for the ω-induced derivation ∂ of $T\,\mathcal{B}\,A$ mapping for each $v \in \mathcal{B}\,A$ the generators $v \otimes v$ of the defining ideal to zero, which is a consequence of the graded Leibniz rule. In fact, any differential on $T\,\mathcal{B}\,A$ respects this defining ideal. We can conclude:

Theorem 6.6.1. *Let V be a vector space, then the restriction map*

$$DG(\bigwedge V) = \mathrm{Der}_{-1}(\bigwedge V) \to V^*, \qquad \partial \mapsto \partial|V = \partial_0$$

is an isomorphism.

We won't differentiate between the two differentials of $\mathcal{G}\,A$ and $T\,\mathcal{B}\,A$ in the notation.

We have already seen in section 3.5.4 that for a vector space V the *extensors*, i.e., the elements of the exterior algebra $\bigwedge V$ correspond to finite dimensional subspaces of V. Recall the notion that establishes this correspondence to be the support $\mathrm{supp}\,\alpha$ of an extensor α:

$$\mathrm{supp}\,\alpha = \{v \in V \mid \alpha \wedge v = 0\}.$$

For pure extensors we have (cf. appendix A)

$$\mathrm{supp}\,v_1 \wedge \ldots \wedge v_n = \mathrm{span}\{v_1, \ldots, v_n\}.$$

When considering $V = \mathcal{B}\,A$ we have two kinds of factors: displacement vectors of V_A and points of A. Because a point $P \in A$ is linearly independent of V_A we have

$$P \wedge \alpha = 0 \quad \Longleftrightarrow \quad \alpha = 0, \qquad \alpha \in \bigwedge V_A < \mathcal{G}\,A.$$

We can define two types of supports for an extensor $\alpha \in \mathcal{G}\,A$:

$$\mathrm{supp}_p\,\alpha := \mathrm{supp}\,\alpha \cap A \quad \text{and} \quad \mathrm{supp}_v\,\alpha := \mathrm{supp}\,\alpha \cap V_A;$$

the first one is called the support of points of α, the second one is called the support of displacement vectors of α. If $\mathrm{supp}_p\,\alpha \neq \emptyset$ then $\mathrm{supp}_p\,\alpha$ is an affine subspace of A with associated vector space $\mathrm{supp}_v\,\alpha$. For, let $P, Q \in \mathrm{supp}_p\,\alpha$, we get $\overrightarrow{PQ} \in \mathrm{supp}_v\,\alpha$ from the identity

$$\alpha \wedge \overrightarrow{PQ} = \alpha \wedge (Q - P) = \alpha \wedge Q - \alpha \wedge P = 0 - 0 = 0.$$

On the other hand, $P \in \operatorname{supp}_p \alpha$ and $v \in \operatorname{supp}_v \alpha$ yield $P + v \in \operatorname{supp}_p \alpha$ because of the identity

$$\alpha \wedge (P + v) = \alpha \wedge P + \alpha \wedge v = 0 + 0 = 0.$$

This establishes a correspondence between extensor in $\mathcal{G}\, A$ and affine subspaces of A.

Any pure extensor $\beta \in \mathcal{G}\, A$, which has a point $P \in A$ as a factor, can be written as

$$\beta = P \wedge \alpha, \qquad \alpha \in \bigwedge V_A.$$

To see this suppose β has besides P the points $P_1, \ldots, P_\ell \in A$ as factors. Use the alternating law to rewrite β in the following form

$$\beta = \pm P \wedge P_1 \wedge \ldots \wedge P_\ell \wedge \alpha',$$

where $\alpha' \in \bigwedge^{k-\ell} V_A$ is a pure extensor of degree $k-\ell$. Once again by the alternating law we can subtract P from each $P_1, \ldots, P_\ell$ in the above expression without changing β

$$\beta = \pm P \wedge (P_1 - P) \wedge \ldots \wedge (P_\ell - P) \wedge \alpha' = \pm P \wedge \overrightarrow{PP_2} \wedge \ldots \wedge \overrightarrow{PP_\ell} \wedge \alpha'.$$

Since a possible minus sign vanishes if we interchange two of the vectors $\overrightarrow{PP_i}$, $\overrightarrow{PP_j}$ with $i \neq j$, this term is essentially of the asserted form $\beta = P \wedge \alpha$. In addition we have for $P, Q \in A$ and pure non-zero extensors $\alpha, \beta \in \bigwedge V_A$

$$P \wedge \alpha = Q \wedge \beta \quad \Longleftrightarrow \alpha = \beta,\ P \in \operatorname{supp} Q \wedge \beta,$$

because a point is linearly independent of V_A in $\mathcal{B}\, A$. We can thus consider a pure extensor β with a point factor as a (pure) linear extensor $\alpha \in \bigwedge V_A$ "bound to space", namely to the affine subspace $\operatorname{supp} \beta$ of $\mathcal{B}\, A$.

Grassmann calls the linear extensor α the "Ausweichung" of β and sets the ausweichung of a pure linear extensor to zero. He does so, for he thinks of the ausweichung as the natural extension of the weight of points to extensors build from them. From our preceding result we see that the ausweichung is indeed a well-defined notion. By linear extension to $\mathcal{G}\, A$ it yields a linear map that extends the weight form ω of $\mathcal{B}\, A$.

The ausweichung of

$$P \wedge Q = P \wedge \overrightarrow{PQ}, \qquad P, Q \in A$$

is $\overrightarrow{PQ}$. On the other hand we have

$$\partial(P \wedge Q) = Q - P = \overrightarrow{PQ}.$$

It is thus natural to ask how the ausweichung and the differential are related. In fact, they are one and the same thing (cf. [Law96]).

As for applications of $\mathcal{G}\,A$ to (classical) geometry and physics we refer to the literature. Regarding classical geometry many applications can be found in [Zad94], [Bro09] and [Whi98]. Grassmann himself also used his algebra to study certain types of algebraic curves.

Regarding physics the applications are mainly restricted to rigid body mechanics (cf. [Bro09]). More concretely, the application regards the theory of screws, a geometric theory of rigid body kinematics developed by Robert Stawell Ball around 1876, which can be algebraized with the help of $\mathcal{G}\,A$ in an elegant way (cf. [Bro09], [Hyd88]). One interesting aspect of this approach is that one need not to distinguish artificially between force and torque. Both arise as extensors of degree two[9]. The notion of a force is given in the original sense of Newton as a line bound vector $P \wedge f$. The vector f represents the directed quantity of the amount of the force. The displacement vector f together with the point P define the line of action of the force in space. When we want to know how the action of this force is felt at another point R of the rigid body we just need to expand $P \wedge f$ for $P = R + \overrightarrow{RP}$:

$$P \wedge f = R \wedge f + \overrightarrow{RP} \wedge f.$$

So, at the point R we feel the same amount of force f but since we deviate of the line of action of the force we also feel a torque $\overrightarrow{RP} \wedge f$. Similarly, when we add two forces $P \wedge f_1$ and $Q \wedge f_2$, we get

$$P \wedge f_1 + Q \wedge f_2 = P \wedge (f_1 + f_2) + \overrightarrow{PQ} \wedge f_2.$$

In words: the sum of two forces yields a total force and a torque that depends on the relative position of the line of actions of the forces. Like in the previous case the torque depends also on the particular point we look at.

6.6.2 The Clifford Algebra of Affine Space

Although we indicated some of the geometric significance of Grassmann's algebra $\mathcal{G}\,A$ of an affine space A, the fact that A doesn't appear as an affine subspace even though it generates the algebra may be felt as a major drawback. Our second construction of a Clifford algebra $\mathcal{C}\,A$ of an affine space A will have A as an affine space.

One way to construct such a Clifford algebra $\mathcal{C}\,A$ is to consider the quadratic form Q_ω induced by the weight form $\omega : \mathcal{B}\,A \to K$ and define $\mathcal{C}\,A$ as the Clifford algebra $Cl(\mathcal{B}\,A, Q_\omega)$, i.e. the quotient of $T\,\mathcal{B}\,A$ by the ideal generated by the relations $v \otimes v = \omega(v)^2$ for all $v \in \mathcal{B}\,A$.

We present another approach where we construct $\mathcal{C}\,A$ from $\mathcal{G}\,A$ by deformation of the exterior product. This approach is based on the construction of Clifford

[9] In the literature it is more common to call extensors multivectors. Extensors of degree two are thus usually referred to as bivectors.

algebras from respective Grassmann algebras as presented by Oziewicz in [Ozi86]. In contrast to Oziewicz we shall not restrict ourselves to the finite dimensional case here.

Consider the representation of $\mathcal{G}\,A$ on its endomorphism algebra $\mathrm{End}(\mathcal{G}\,A)$ by left multiplication operators; that is the right adjunct of the bilinear exterior multiplication under the linear hom-tensor adjunction:

$$e : \mathcal{G}\,A \to \mathrm{End}(\mathcal{G}\,A), \quad \alpha \mapsto e_\alpha, \quad \text{with} \quad e_\alpha : \beta \mapsto \alpha \wedge \beta.$$

We deform e to a K linear map $\gamma : \mathcal{G}\,A \to \mathrm{End}(\mathcal{G}\,A)$ by the differential ∂ of $\mathcal{G}\,A$ as follows. Define

$$\gamma : \mathcal{G}\,A \to \mathrm{End}(\mathcal{G}\,A)$$

as a linear map with

$$\gamma_\lambda = e_\lambda, \qquad \lambda \in K < \mathcal{G}\,A, \tag{$\mathcal{C}\,1$}$$
$$\gamma_P = e_P + \partial, \qquad P \in A \subset \mathcal{G}\,A, \tag{$\mathcal{C}\,2$}$$
$$\gamma_{\gamma_\alpha \beta} = \gamma_\alpha \circ \gamma_\beta, \qquad \alpha, \beta \in \mathcal{G}\,A. \tag{$\mathcal{C}\,3$}$$

We think of γ as the representation of the Clifford multiplication by left multiplication operators. ($\mathcal{C}\,1$) says that e and γ coincide on scalars. So for $\lambda \in K$ γ_λ is just the multiplication by the scalar λ. ($\mathcal{C}\,2$) is the actual deformation of the exterior multiplication on points. We can see that besides the extensive and constructive combinatorial aspect of the exterior product a destructive aspect of decomposition enters the Clifford multiplication by the use of the differential. ($\mathcal{C}\,3$) is the associativity law for the Clifford multiplication.

The linearity together with ($\mathcal{C}\,1 - \mathcal{C}\,3$) define and determine γ uniquely, where the idea behind the definition is a process of induction. To begin with suppose such a γ to exist. We have for any extensor $\alpha \in \mathcal{G}\,A$

$$\gamma_P \circ \gamma_\alpha \overset{(\mathcal{C}\,3)}{=} \gamma_{\gamma_P \alpha} \overset{(\mathcal{C}\,2)}{=} \gamma_{P \wedge \alpha + \partial \alpha} \overset{\text{lin.}}{=} \gamma_{P \wedge \alpha} + \gamma_{\partial \alpha},$$

and hence

$$\gamma_{P \wedge \alpha} = \gamma_P \circ \gamma_\alpha - \gamma_{\partial \alpha}. \tag{6.3}$$

Recall that $A \subset \mathcal{B}\,A$ spans $\mathcal{B}\,A$ linearly and hence generates $\mathcal{G}\,A$ as a K-algebra. So (6.3) yields the uniqueness of γ. On the other hand, since the differential ∂ decreases degrees of homogeneous extensors, we see (6.3) to yield a recursive formula to define γ. In more detail, we define γ by its components $\gamma_k : \bigwedge^k \mathcal{B}\,A \to \mathrm{End}(\mathcal{G}\,A)$. The components itselves are defined inductively by ($\mathcal{C}\,1$) and ($\mathcal{C}\,2$) both read as definitions together with (6.3) to

$$\gamma^{k+1}_{\sum_i P_i \wedge \alpha_i} := \sum_i \gamma^1_{P_i} \circ \gamma^k_{\alpha_i} - \gamma^{k-1}_{\partial \alpha_i}, \qquad P_i \in A,\ \alpha_i \in \bigwedge^k \mathcal{B}\,A.$$

The so defined γ is well-defined an can be seen to satisfy ($\mathcal{C}\,3$) in its full generality. Its left adjunct under the hom-tensor adjunction yields then an associative multiplication by ($\mathcal{C}\,3$) with the scalar $1 \in \mathcal{G}\,A$ as an identity element by ($\mathcal{C}\,1$). We call the K-algebra with underlying K-vector space $\bigwedge \mathcal{B}\,A$ together with this so called Clifford multiplication the *Clifford algebra* $\mathcal{C}\,A$ *of the affine space* A. One can show the so defined $\mathcal{C}\,A$ to be isomorphic to $Cl(\mathcal{B}\,A, Q_\omega)$ (as a K-algebra) by exhibiting $\mathcal{C}\,A$ to have the same universal property as $Cl(\mathcal{B}\,A, Q_\omega)$.

Although the Clifford algebra $\mathcal{C}\,A$ inherits the structure of an $\mathbb{N}$-graded vector space, it is not an $\mathbb{N}$-graded K-algebra anymore. That this is so can be concluded from ($\mathcal{C}\,2$) directly. Instead, like any Clifford algebra $Cl(V, Q)$ it is naturally $\mathbb{Z}_2$-graded by the eigenspaces of the K-algebra automorphism induced by $-\operatorname{id}_{\mathcal{B}\,A}$. Nevertheless, the $\mathbb{N}$-gradation of the underlying vector space $\bigwedge \mathcal{B}\,A$ is of importance for geometry and it thus shouldn't be discarded only because it is not compatible with the multiplicative structure. The underlying $\mathbb{N}$-graded vector space being the one of $\mathcal{G}\,A$ was also the main reason why we introduced $\mathcal{C}\,A$ following Oziewicz.

Theorem 6.6.2. *The K-algebra $\mathcal{C}\,A$ is an affine algebra and A is an affine subspace of $\mathcal{C}\,A$.*

Proof. This is seen the easiest when one considers the $Cl(\mathcal{B}\,A, Q_\omega)$-construction. Following this approach we can see the ideal of $T\,\mathcal{B}\,A$ generated by $v \otimes v - \omega(v)^2$ for all $v \in \mathcal{B}\,A$ to lie in $\ker T\omega$, where $\omega : \mathcal{B}\,A \to K$ denotes the weight form as usual. By the fundamental theorem on homomorphisms for K-algebras $T\omega$ descends to the quotient $Cl(\mathcal{B}\,A, Q_\omega)$ and hence to $\mathcal{C}\,A$. In fact, we can obtain the augmentation from the universal property of $Cl(\mathcal{B}\,A, Q_\omega)$ as well. Hereto, consider K as an initial K-algebra. The weight form $\omega : \mathcal{B}\,A \to K$ satisfies $\omega(v)^2 = Q_\omega(v)$ by construction and hence extends to a K-algebra homomorphism $\omega_{\mathcal{C}} : \mathcal{C}\,A \to K$. This is just a reformulation of the above construction of course, but it makes clear the appearance of the augmentation conceptually. Moreover, from this construction of $\omega_{\mathcal{C}}$ we can see the inclusion $\mathcal{B}\,A \hookrightarrow \mathcal{C}\,A$ to be an inclusion of augmented vector spaces immediately and hence A to be an affine subspace of $\mathcal{C}\,A$.

(qed)

The affine algebra $\mathcal{C}\,A$ is thus generated by the affine space A as an affine algebra. In the algebraic persentation we can denote it like the affine tensor algebra by

$$\mathcal{C}\,\omega : \mathcal{C}\,A \to K$$

to indicate the functoriality on augmented vector spaces. This is in contrast to the affine algebra $\mathcal{G}\,A$, where the point set A was merely a subset of displacement vectors and the algebraic presentation was $\mathcal{G}\,0_{\mathcal{B}\,A}$ accordingly.

Also the differential ∂ descends to $\mathcal{C}\,A$; but it does so only as a $\mathbb{Z}_2$-graded derivation of odd degree. Nonetheless, it is still the same map. This can be seen once

again from the $Cl(\mathcal{B}\,A, Q_\omega)$-construction and Chevalley's theorem 1.5 in [Che56][10], for we have $\partial(v\otimes v-\omega(v)^2)=0$. In consequence, we have for $\mathcal{C}\,A$ the same situation as for the tensor algebra TA: on one hand it is an affine algebra, on the other hand it is a DG algebra and both structures are incompatible in the sense that the DG structure can't be made internal to the affine structure. We have in particular:

Theorem 6.6.3. *Let $\omega : V \to K$ be a non-degenerated augmented vector space and $V \neq \{0\}$. Let $\mathcal{C}\,V$ denote $Cl(V, Q_\omega)$ considered as a $\mathbb{Z}_2$-graded algebra and $\mathcal{I}_Q < TV$ the defining ideal of $\mathcal{C}\,V$. Then the quotient map $TV \to TV/\mathcal{I}_Q \cong \mathcal{C}\,V$ induces an embedding*

$$V^* \cong Der_{-1}(TV) \hookrightarrow DG(\mathcal{C}\,V),$$

whereby $DG(\mathcal{C}\,V)$ is the vector space of DG structures of $\mathcal{C}\,V$ of odd degree, i.e., $\mathbb{Z}_2$-graded derivations of odd degree that square to zero.

Proof. Like in the preceding discussion this is a consequence of [Che56][theorem 1.5], for any differential ∂ of TV maps the generators $v\otimes v-\omega(v)^2$, $v\in V$ of $\mathcal{I}_Q$ to zero, and hence respects $\mathcal{I}_Q$.

(qed)

By our construction of $\mathcal{C}\,A$ and by what we have learned about the representation of extensors with non-empty support of points we can see[11]

$$\ker \mathcal{C}\,\omega > V_A \oplus \bigoplus_{k>1} \bigwedge^k \mathcal{B}\,A.$$

Moreover, the Clifford multiplication of extensors of $\bigwedge V_A$ as well as the multiplication of those extensors with points in A is the exterior multiplication. So up to the points and scalars the Grassmann algebra of A reappears within the Clifford algebra of A with the restriction that multiplication of extensors with point factors is not the exterior multiplication anymore.

In what we have seen so far $\mathcal{C}\,A$ may be considered as the proper modification of Grassmann's algebra of an affine space from the point of view of a geometric algebra of an affine space: it has the same underlying K-vector space of extensors, the Clifford multiplication coincides in many cases with the exterior multiplication, we have the same map as differential but we have additionally that the weight form ω lifts to an augmentation, so A can become an affine subspace of the affine algebra.

[10] To be able to apply Chevalley's theorem we have to consider $T\,\mathcal{B}\,A$ as a $\mathbb{Z}_2$-graded algebra, too. This gradation is given once again by the eigenspaces of the automorphism induced by $-\mathrm{id}_{\mathcal{B}\,A}$. It is related to the natural $\mathbb{N}$-gradation by word length as follows: odd elements are the sums of pure tensors of odd degree, even elements are sums of pure tensors of even degree.

[11] More precisely, we have $\ker \mathcal{C}\,\omega \cong \langle 1-A\rangle \oplus \bigoplus_{k>1} \bigwedge^k \mathcal{B}\,A$, where $\langle 1-A\rangle$ denotes the vector subspace generated by the elements $1-P$, $P\in A$.

The Clifford algebra $\mathcal{C}\,A$ of an affine space A has a further interesting feature for affine geometry. We can define a pin and a spin group and consider their action by conjugation on the affine space A. For the spin group this action turns out to be an action by translations.

In $\mathcal{C}\,A$ observe that we have $P^2 = 1$ for points $P \in A$, so points are invertible and each point is its own inverse. Also, points are the only invertible elements in $\mathcal{B}\,A$. We define the *pin group* Pin_A to be the subgroup of units of $\mathcal{C}\,A$ generated by points. Its underlying set is given by

$$\mathrm{Pin}_A = \{1\} \cup \{P_1 \cdots P_n \mid n \in \mathbb{N},\ P_i \in A,\ 1 \leq i \leq n\}.$$

We define the *spin group* Spin_A as its subgroup generated by even products of points. Its underlying set is thus

$$\mathrm{Spin}_A = \{1\} \cup \{P_1 \cdots P_{2n} \mid n \in \mathbb{N},\ P_i \in A,\ 1 \leq i \leq 2n\}.$$

Since $\mathcal{C}\,\omega : \mathcal{C}\,A \to K$ is a K-algebra homomorphism the action of any unit $\alpha \in \mathcal{C}\,A^\times$ by conjugation on $\mathcal{C}\,A$ yields an affine automorphism of $\mathcal{C}\,A$:

$$\mathcal{C}\,\omega(\alpha\beta\alpha^{-1}) = \mathcal{C}\,\omega(\alpha)\,\mathcal{C}\,\omega(\beta)\,\mathcal{C}\,\omega(\alpha)^{-1} = \mathcal{C}\,\omega(\beta), \quad \beta \in \mathcal{C}\,A.$$

Consider the particular action of a point $P \in A$ by conjugation on a point $Q \in A$ next:

$$PQP^{-1} = PQP = P(Q{\wedge}P{+}1) = P{\wedge}Q{\wedge}P{+}P{+}\partial(Q{\wedge}P){+}\partial 1 = P{+}P{-}Q = 2P{-}Q.$$

Due to $2P - Q = P - \overrightarrow{PQ}$ this shows the conjugation by P to act as a point reflection with P as its center. Take two point reflections, say the first with Q as its center and the second with P as its center. Applying the first and then the second reflection to a point R is the same as a translation of R by $2\overrightarrow{QP}$. In terms of our action this means

$$PQR(PQ)^{-1} = (PQ)R(QP) = R + 2\overrightarrow{QP}, \qquad R \in A.$$

One can obtain the same result by an explicit calculation as well

$$(PQ)R(QP) = P(2Q - R)P = 2PQP - PRP = 4P - 2Q - 2P + R = R + 2\overrightarrow{QP}.$$

Since any element of the spin group can be written as a product of binary products of points this shows the spin group to act on A by translations as asserted.

Regarding the application of the Clifford algebra of affine space to geometry and physics we can't give any right now. In fact, the author neither found an explicit construction or use nor a suggestion of this natural and universal construction of an affine algebra of an affine space in the literature so far. This comes of no suprise,

for there seems to be no obvious way to arrive at $\mathcal{C}\,A$ by a process of abstraction from geometric arguments as it can be for example done for the Grassmann algebra (cf. appendix A). In fact, two key observations seem to be necessary. The first fact is the universality of the barycentric calculus and that it is a K-vector space. This was more or less already clear to Grassmann in [Gra78] and lead him to his notion of "Elementargröße" to which he could apply his construction of linear extension to obtain the Grassmann algebra of affine space. The second fact is the equivalence of augmented K-algebras and affine algebras. It was this fact that lead us to notice the inconvenience of the Grassmann algebra of an affine space A to not have A as an affine subspace. If one wants to correct this then one needs the weight form ω to become a K-algebra homomorphism. The failure of ω to be a K-algebra homomorphism for $\mathcal{G}\,A$ lies in $\omega(P) = 1$ but P to be nilpotent ($P \wedge P = 0$). An obvious ansatz to correct this is to define a K-algebra by the relations $P^2 = 1$, $P \in A$. But these are the necessary relations to define $\mathcal{C}\,A$. Knowing the Clifford algebra construction it is not difficult to recognize $\mathcal{C}\,A$ as such from this presentation. In fact, given an augmented vector space $\omega : \mathcal{B}\,A \to K$ and $\mathcal{G}\,A$, besides the tensor algebra the Clifford algebra $\mathcal{C}\,A$ is the next best guess when asking about a universal K-algebra to be constructed from this data.

The justification to call $\mathcal{C}\,A$ a geometric algebra of affine space is threefold. Firstly, it is its tight relation with the Grassmann algebra of affine space and the structural and geometrical relevance ot the latter. Secondly, it is its property to be an affine algebra generated by A. Finally, it is the fact that Clifford algebras of Euclidean and Minkowski space have various applications as geometric algebras to physics and hence the hope that $\mathcal{C}\,A$ and other affine algebras derived from it have itneresting applications as well. A first support for this hope is the geometric meaning of the action of the spin group Spin_A by translations.

7 Directions for Future Research

We have seen with proposition 6.2.1 and theorem 6.2.3 in corollary 6.2.4 that the category of internal monoids of $(\mathrm{Aff}_K, \otimes, *)$ is equivalent to the category of augmented K-algebras as proposed in the introduction. Since the Grassmann algebra of affine space and the Clifford algebra of affine space were demonstrated to be augmented algebras, our ansatz to follow the most natural way towards the desired category of algebras, which is constructed from Aff_K and has actual geometric algebras as objects, has proven successful.

Furthermore, we observed that algebraizing affine geometry with the help of geometric algebras constructed from the barycentric calculus is structurally and conceptually different from the usual approach using rings (K-algebras) of "coordinate functions". In terms of structure the former yields manifestly covariant functors, the latter manifestly contravariant ones. Moreover, in contrast to rings of coordinate functions, the K-algebras that serve as geometric algebras tend to be non-commutative.

With the category of augmented algebras we are too given large classes of examples of affine algebras that are already known in mathematics (but not in this way).

7.1 Hopf Algebras as Affine Algebras

The most important such class of examples of affine algebras seems to be the category of Hopf algebras in the sense of [Swe69]. With our characterization we are able to give the counit of a Hopf algebra a geometric meaning as an affine structure and the antipode is seen to be an affine map. Although it is obvious from our results that the coalgebra structure of a Hopf algebra corresponds to an internal comonoid of $(\mathrm{Aff}_K, \otimes, *)$, and that for an affine space to become an internal comonoid only a coassociative comultiplication is needed that has the terminal map as a counit, it is not clear what geometric meaning such a comultiplication could have (or whether there is one at all).

While pursuing this question one would look first for bi- or Hopf-algebraic structures for geometric algebras we know already. In fact, it is well known that a

Grassmann algebra is a Hopf algebra in a natural way. However, we have also noted that a Grassmann algebra of an affine space has its generating affine space not as an affine subspace but only as a vector subspace. So, regarding the affine geometric interpretation of the coproduct, the coalgebra structure of the Grassmann algebra doesn't seem helpful. On the other hand there doesn't seem to be a natural comultiplication for an affine tensor algebra or a Clifford algebra of affine space that would make them into bialgebras.

The situation seems to change, if one considers a pointed affine space. For the pointed affine line $P : * \to K$ one can define a comultiplication Δ and antipode S such that $\mathcal{C}\,K$ becomes a Hopf algebra as follows ($v \in V_K \cong K$)

$$\begin{aligned} &\Delta P = P \otimes P, \quad \Delta v = v \otimes 1 + P \otimes v, \quad \Delta(Pv) = 1 \otimes 1 + Pv \otimes P + 1 \otimes Pv \\ &SP = P, \qquad\quad Sv = -Pv, \qquad\quad S(Pv) = v. \end{aligned}$$

With these structure maps $\mathcal{C}\,K$ is isomorphic to the smallest non-commutative, non-cocommutative Hopf algebra. Moreover, the action of the spin group can be described by the right-adjoint action. It looks like this construction can be extended to arbitrary Clifford algebras of pointed affine spaces. In the general case, however, the right-adjoint action wouldn't yield the action of the spin group anymore.

Besides the comultiplication it seems also instructive to study the affine geometric meaning of other Hopf algebraic notions like the integral, for instance.

7.2 Dynamical Algebras

We know that the Grassmann algebra and the Clifford algebra are universal objects. However, the universal property of each of those algebras doesn't really shed any light on the question why and in which way they are universal as geometric algebras. There is an interesting ansatz to tackle these questions that is due to Lawvere. For, as Lawvere in effect observed in [Law96], if one takes Grassmann's geometric construction of the exterior algebra serioulsy, one can see it arising from the "internal dynamics" of the category $\mathrm{Aff_K}$.

In more detail, Lawvere's ansatz is to make precise Grassmann's dynamical construction of the exterior product (cf. appendix A). This dynamical construction of Grassmann is based on his philosophical foundation of a "einfache Ausdehnungsform" (simple form of extension) as a form that is obtained from a generating element following a fixed, i.e. non-varying, law of "continuous change" (cf. [Gra78][Einleitung, Nr. 9-10] and [Gra78][Anhang III, I.5]). In [Law96] Lawvere calls the latter "the simple law of change". He assumes that in an affine space A a particular "motion" or particular "law of change" is described by an affine endomorphism $S : A \to A$. Then, for an affine space A, a *simple change*, which is a kind of motion, is considered as a distinguished subset $\mathcal{S}_A \subset \mathrm{Aff_K}(A, A)$ of affine endomorphisms. A simple form of extension with respect to the simple law of change

$\mathcal{S}_A$ that is generated by simple motions from P to $Q \neq P$ can then be identified with the equivalence class of (P, Q) with respect to the equivalence relation $\sim$ on $A \times A \setminus \Delta$ defined as follows[1]:

$$(P, Q) \sim (O, R) \quad :\Longleftrightarrow \quad \forall S \in \mathcal{S}_A : \; SP = Q \Leftrightarrow SO = R. \tag{7.1}$$

Building on Grassmann's construction in [Gra78][§13-14] we may regard the formal product $P \wedge Q$ as an invariant for the dynamical rule how to move from P to Q and, more generally, from a point X to $X + \overrightarrow{PQ}$ on the affine line spanned by P and Q. To have

$$(P, Q) \sim (O, R) \quad \Longleftrightarrow \quad P \wedge Q = O \wedge R, \qquad P \neq Q,$$

$\mathcal{S}_A$ has to be the set of all affine endomorphisms S with the property $S^2 = 2S - 1_A$. Besides translations one can see shear transformations to have this property, too. A further interesting fact, which was discovered by Lawvere, is the origin of those automorphisms. They are in one-to-one correspondence to internal (biaffine) actions of the internal monoid $(K, +, 0)$ on A (the "internal dynamics" of A). One can see this by applying the hom-tensor adjunction and writing these actions as internal monoid morphisms

$$\alpha : (K, +, 0) \to (\underline{\mathrm{Aff}}_{\mathrm{K}}(A, A), \circ, 1_A).$$

The correspondence is then given by $\alpha \mapsto \alpha(1)$, for example.

This dynamical approach, however, doesn't determine the exterior product and hence the Grassmann algebra uniquely. Rather than that it determines a whole class of *dynamical algebras of* A, i.e. K-algebras $\mathcal{A}$ with $\mathcal{B}\,A \hookrightarrow \mathcal{A}$ that are generated by $\mathcal{B}\,A$ as K-algebras, and that satisfy

$$(P, Q) \sim (O, R) \quad \Longleftrightarrow \quad PQ = OR, \qquad (P, Q), (O, R) \in A \times A \setminus \Delta. \tag{7.2}$$

That this class consists not only of the Grassmann algebra is due to the fact that our equivalence relation $\sim$ does not say anything about the pairs (P, P), $P \in A$. A product defined by $\sim$ yields $P^2 = Q^2$ for all $P, Q \in A$, but not necessarily $P^2 = 0$. This constitutes a further relation that has to be imposed. More precisely, besides $P^2 = Q^2$ for $P, Q \in A$ a multiplication satisfying (7.2) has the symmetric part of the product PQ to satisfy $PQ + QP = 2P^2$ and the antisymmetric part to behave like the exterior product. With this the Clifford algebra of affine space is seen to be a dynamical algebra, too.

The affine tensor algebra of A does not belong to the class of dynamical algebras of A. Yet there is an initial dynamical algebra $\mathcal{D}\,A$ for an affine space A. It is an affine algebra that has A as an affine subspace. $\mathcal{D}\,A$ is different from the Grassmann and Clifford algebra of the affine space A and its underlying vector space is also

[1] Here $\Delta \subset A \times A$ denotes the diagonal $\{(P, P) \mid P \in A\}$.

not that of the Grassmann algebra. A further dynamical algebra $\bar{\mathcal{C}}A$, which is not isomorphic to the others, can be obtained by twisting the sign in the defining relations of the Clifford algebra of affine space $\mathcal{C}A$, i.e. $x^2 = -\omega(x)^2$, $x \in \mathcal{B}A$. For the case of the one-point space $*$ the initial algebra $\mathcal{D}*$ is isomorphic to the polynomial ring $K[X]$, $\mathcal{G}*$ is the algebra of dual numbers over K, $\mathcal{C}*$ is the algebra of split-complex numbers over K and $\bar{\mathcal{C}}*$ is the algebra of complex numbers over K.

The very interesting fact about dynamical algebras is that for any dynamical algebra $\mathcal{A}$ of A the product PQ of two points $P, Q \in A$ in $\mathcal{A}$ represents the same geometric/dynamical invariant: namely, a vector $\overrightarrow{PQ}$ bound to the affine line through P and Q. (The invariant itself is defined by the equivalence class of $\sim$ and hence by our choice of $\mathcal{S}_A$.) Stated differently, we have one notion of invariant and with non-isomorphic dynamical algebras of A different possibilities to represent this invariant algebraically in terms of binary products of points. It is thus interesting to get an overview of all possible dynamical algebras of A up to isomorphy. We can use $\mathcal{D}A$ for that, since it is the initial dynamical algebra. By $\mathcal{D}A$ being initial and (7.1), there is a correspondence between dynamical algebras and ideals I of $\mathcal{D}A$ with the properties $\mathcal{B}A \cap I = \{0\}$ and

$$PQ - OR \in I \quad \Longleftrightarrow \quad PQ = OR, \qquad (P,Q),(O,R) \in A \times A \setminus \Delta.$$

Since $\mathcal{D}A$ has a natural $\mathbb{N}$-gradation, only the vector subspace $I \cap (K \oplus \mathcal{B}A \oplus \mathcal{D}_2 A)$ turns out to be "critical". Even for a principal ideal all the types of "polynomials of degree two" that are admissible for I are still not fully determined yet.

7.3 Geometric Algebra of Affine Metric Spaces

Given a finite dimensional $\mathbb{R}$-vector space V with a metric g (in the sense of a non-degenerate symmetric bilinear form) one can define the Clifford algebra $Cl(V,g)$ and (after fixing an orientation) introduce further operations on the Grassmann algebra $\bigwedge V$ like the Hodge star operator and an inner product[2]. In the case of finite dimensional affine metric spaces over $\mathbb{R}$, i.e. affine spaces whose associated $\mathbb{R}$-vector spaces carry a metric, those constructions can't be carried out in a straightforward manner. For a finite dimensional affine metric space[3] A with metric g on V_A one has to deal with the problem that g is only defined on the subspace $V_A < \mathcal{B}A$.

Since V_A is a subspace of codimension 1 an obvious approach would be to extend g to the whole space $\mathcal{B}A$. There are in essence two different ways to do so. They result either in a non-degenerate or degenerate metric $\tilde{g}$ on $\mathcal{B}A$. (In the degenerate case there are the two options to extend by $+1$ or -1 in addition.) However, both means to extend g single out a distinguished point of A. In the non-degenerate case

[2] The Hodge star operator corresponds to what Grassmann calls "Ergänzung" in [Gra62].

[3] From now on we will mean by an affine metric space a finite dimensional affine metric space over $\mathbb{R}$.

it is the intersection of the $\tilde{g}$-orthogonal complement of V_A in $\mathcal{B}\,A$ with A, and in the degenerate case it is the intersection of the radical subspace of $\tilde{g}$ with A. This kind of approach forces us basically to consider pointed affine spaces.

For both types of extensions we can embed $Cl(V_A, g)$ as a K-algebra in $Cl(\mathcal{B}\,A, \tilde{g})$. It is well-known that the natural action of the spin group[4] of $Cl(V_A, g)$ on $Cl(V_A, g)$ by conjugation leaves V_A invariant and the action corresponds to g-orthogonal transformations on V_A. After embedding in $Cl(\mathcal{B}\,A, \tilde{g})$ this action of the spin group of $Cl(V_A, g)$ on $Cl(BA, \tilde{g})$ is seen to leave A invariant and to correspond to affine g-isometries with the respective $\tilde{g}$-distinguished point $O \in A$ as "the center", i.e., O is a fixed point for each such transformation.

Of course, there are differences depending on the type of extension chosen. In the degenerate case, for example, one can not formulate Hodge duality for the Grassmann algebra of affine space. On the other hand, in the degenerate case, which we will label by $\tilde{g}_0$ from now on, it is possible to represent both g-orthogonal transformations and translations and hence the whole group of affine g-isometries within the Clifford approach. This representation, however, isn't given by the action of the spin group of $Cl(\mathcal{B}\,A, \tilde{g}_0)$ on A (or the subgroup that leaves A invariant). Instead, one has to represent points $P \in A$ by $1 + OP = 1 + O \wedge P$, which is a generalization of the application of dual quaternions to the representation of the affine isometries of the three-dimensional Euclidean space in a certain sense. Since O is a fixed point for the action of the spin group of $Cl(V_A, g)$ on A, and since this action is given by conjugation, one can see from the fact that the spin group of $Cl(V_A, g)$ acts on A by g-isometries that it does so on our representation, too. Due to $O^2 = \tilde{g}(O, O) = 0$ multiplication of $1 + O \wedge P$ on the left or right by $1 + Ov = 1 + O \wedge v$, $v \in V_A$ yields a translation by v, i.e. $1 + O(P + v)$. In fact, $1 + O \wedge P$ as well as $1 + O \wedge v = 1 + O \wedge (O + v)$ belong to the spin group of $Cl(\mathcal{B}\,A, \tilde{g}_0)$. So we actually represent A within the spin group of $Cl(\mathcal{B}\,A, \tilde{g}_0)$. One might suspect the spin group of $Cl(\mathcal{B}\,A, \tilde{g})$ to be isomorphic to the semidirect product of the additive group of V_A and the spin group of $Cl(V_A, g)$. This is indeed the case (cf. [Bro86][theorem 1]).

The action of $(1 + OP)$ on a point $Q \in A$ results in $Q + 2\tilde{g}_0(P, Q)O$. This shows that the action of $(1 + OP)$ does not leave A invariant. It "boosts" a point Q in the direction of O, instead. This observation leads to the interpretation of the spin group of $Cl(\mathcal{B}\,A, \hat{g}_0)$ as a so called "kinematical group" in certain special cases (cf. [Bro86], [Bro78]). Consider, for instance, $\mathcal{B}\,A$ with $\tilde{g}_0$ for A a three-dimensional Euclidean space as a Galilei spacetime[5] of an inertial observer with four-velocity

[4] We stick here to the usual definition of the spin group of $Cl(V, g)$ as the intersection of the subgroup of the Lipschitz group that consists of elements with spinor norm 1 with the even subalgebra. The Lipschitz group is the subgroup of the group of units of $Cl(V, g)$ whose action on $Cl(V, g)$ by twisted-conjugation leave V invariant.

[5] Following Arnold [Arn89][section 1.2 B] a Galilei space or Galilei spacetime is a four-dimensional affine space A over $\mathbb{R}$ whose associated vector space V_A is equipped with a linear form $\tau : V_A \to$

the distinguished point O. The spin group of the Clifford algebra $Cl(\mathcal{B}\,A, \tilde{g}_0)$ can then be identified with the homogeneous Galilei group, i.e. the subgroup of the Galilei group of all transformations except the spacetime translations. Within this identification, however, the Galilei boosts with velocity v don't correspond to actions of $1 + O \wedge v$ on $\mathcal{B}\,A$; one has to resort to a more complicated transformation law (cf. [Bro78][section 3]). As it is presented in [Bro78] the identification of the homogeneous Galilei group and the spin group of $Cl(\mathcal{B}\,A, \tilde{g}_0)$ is very likely not a natural one. It is still an interesting observation though. The question is thus whether there exists an appropriate geometric setting such that this identification itself or a related construction becomes natural.

The representation of points $P \in A$ by $1 + OP$ is not just a formal trick, but has a geometric interpretation, too. Recall $1 + O \wedge P$ to represent the vector $\overrightarrow{OP}$ bound to the line through O and P. Given an origin O and hence a pointed affine space, the map

$$A \to \{1 + \alpha \mid \alpha \in \bigwedge^2 \mathcal{B}\,A \text{ pure, with } O \in \operatorname{supp}_P \alpha\}, \quad P \mapsto 1 + O \wedge P$$

is a bijection with $1 + \alpha \mapsto O + \partial(1 + \alpha)$ as its inverse. Moreover, if we equip the codomain set with the obvious action of V_A by $(1 + \alpha, v) \mapsto 1 + \alpha + O \wedge v$ then the bijection becomes an isomorphism of pointed affine spaces. Because we have the origin O fixed, we can read this isomorphism as to assign to a point the directed line segment from O to P. In this sense we represent points by "rods" to the origin, and this is exactly one of the geometrical ideas behind the introduction of a Cartesian frame of reference. (The other idea is that one can choose a certain set of preferred "measuring rods" that can be used to describe all possible rods.)

Regarding the non-degenerate case it is not clear whether there is a possibility to represent translations at all. Although $1+O\wedge P$ was the algebraic object responsible for translations in the case of $\mathcal{C}\,A$ and $Cl(\mathcal{B}\,A, \tilde{g}_0)$ (yet with a different representation of points), in the non-degenerate case $1 + O \wedge v$ has a spinor norm of 1 only for g-isotropic $v \in V_A$. This can be seen with the following general formula for the spinor norm of $(1 + O \wedge v)$

$$(1 + O \wedge v)(1 + v \wedge O) = 1 + (O \wedge v)(v \wedge O) = 1 + \hat{g}(O, O)g(v, v),$$

Here O is the $\tilde{g}$-distinguished point of A. Therefore, if g is anisotropic like the Euclidean metric, for example, $(1+O\wedge v)$ doesn't belong to the spin group whatsoever. In [Cli71] Clifford, who also pursued a geometric algebra of (affine) Euclidean motions in this paper, considered this non-degenerate case (for A the three-dimensional Euclidean space) as working with elliptic geometry and not with parabolic geometry. (The parabolic geometry corresponds to the degenerate case.) All of this

$\mathbb{R}$ and $\ker \tau$ is a Euclidean vector space with metric g.

suggests that it is not possible to represent translations like in the degenerate case. We would like to have a particular impossibility result though.

For a g-anisotropic v the action of $(1 + O \wedge v)$ on a point P is given by

$$(1 + O \wedge v)P(1 + v \wedge O) = P + 2g(\overrightarrow{OS}, v)O + 2\tilde{g}(O, O)(1 + g(\overrightarrow{OS}, v))v.$$

It is thus a combination of a translation and a "boost". Even though one might be tempted to see this as an indication for kinematic groups once more, note that we have used the isotropy of the metric g here already.

There is one last observation related to the Clifford algebra $Cl(\mathcal{B}\,A, \tilde{g})$ for either type of extension. Until know we have used the main involution of the Clifford algebra $Cl(\mathcal{B}\,A, \tilde{g})$ implicitly. Regarded from an affine point of view, this is not the most natural choice, since it is derived from $-1_{\mathcal{B}\,A}$, which is not an affine map. Because there is a $\tilde{g}$-distinguished point O of A, a more natural choice would be the involution induced by the point reflection on O. Restricted to $Cl(V_A, g) \hookrightarrow Cl(\mathcal{B}\,A, \tilde{g})$ this involution coincides with the main involution. So nothing changes with regard to the action of the spin group of $Cl(V_A, g)$. What does change is the action of the spin group[6] of $Cl(\mathcal{B}\,A, \tilde{g})$, if one keeps the action as the action by twisted-conjugation. This twisted-conjugation involves the main involution. For the spin group and the usual main involution this didn't matter, for the elements of the spin group lie in the even subalgebra. This is not the case for the main involution that is induced by the point reflection anymore. For this main involution elements like $O \wedge v$ become odd and elements like $O \wedge v \wedge w$ become even. The geometric consequences of this are still to be explored.

What one might feel unsatisfactory about the approach of extending g to the whole space $\mathcal{B}\,A$ is that it needs pointed affine spaces. It is rightful to ask whether there are other types of geometric algebras of an affine metric space A that don't rely on a preferred point $O \in A$, and that in some sense extend $\mathcal{C}\,A$. Our answer to this question is as follows: we don't know yet, but it looks like that there is none so far.

In more detail, we have studied two different ways to obtain an algebra that could be reasonably called a geometric algebra of an affine metric space A. The first approach is guided by the idea of the action of the spin groups of $Cl(V_A, g) \hookrightarrow Cl(\mathcal{B}\,A, \tilde{g})$ and $\mathrm{Spin}_A \subset \mathcal{C}\,A$ on A. Both result from the fact that linear g-isometries and translations can be decomposed into an even number of reflections, namely orthogonal reflections on hyperplanes in the case of g-isometries and point reflections in the case of translations. The former fact is the content of the Cartan-Dieudonné theorem, the latter fact is a consequence of our discussion in section 6.6.2. A first observation is that $\mathcal{C}\,A$ can be presented by the following relations[7]:

$$P^2 = 1, \quad PQP = 2P - Q, \qquad P, Q \in A.$$

[6] The spin group is still defined in the old sense.

[7] In fact, only the relations $P^2 = 1$, $P \in A$ are needed to get a presentation of $\mathcal{C}\,A$.

For each $P \in A$ the first relation sets each point to be invertible such that its action by conjugation becomes an isomorphism of order two. The second relation states that the conjugation-action by P on A is a point reflection. The action of a g-unit vector $v \in V_A$ on A by twisted conjugation is a reflection on the affine hyperplane that contains the $\tilde{g}$-distinguished point $O \in A$ and has the g-orthogonal complement of v as associated vector space. To avoid a choice of origin, the idea is to represent an orthogonal reflection by Qv, where Q is a point in the hyperplane and $v \in V_A$ a normal g-unit vector. In complete analogy to the preceding case of $\mathcal{C}\,A$ this leads to the following relations

$$(Qv)^2 = 1, \quad (Qv)P(Qv) = P - 2g(P-Q,v)v, \qquad P,Q \in A,\ v \in S_{V_A},$$

where $S_{V_A} \subset V_A$ denotes the set of g-unit vectors. Even though this seems to be a promising ansatz, the resulting algebra turns out to be trivial. Basically, this is due to the fact that Q and v transform under conjugation by Qv differently. We have $QvQQv = Q$, but $QvvQv = -v$. This results in

$$Qv = QvQvQv = (QvQQv)(QvvQv) = Q(-v) = -Qv,$$

and hence in $Qv = 0$, which yields $0 = 1$ finally.

The problem of the different transformation behaviour can be circumvented by describing the affine hyperplane for the orthogonal reflection intrinsically. One way to do this goes as follows: for any point Q and g-orthonormal system $h_1, \ldots, h_n \in V_A$ ($n = \dim A - 1$) we represent the affine hyperplane that contains Q and whose associated vector space is spanned by $h_1, \ldots, h_n$ by $Qh_1 \cdots h_n$. The relations for $Qh_1 \cdots h_n$ to represent an orthogonal reflection on this hyperplane are

$$(Qh_1 \cdots h_n)^2 = 1, \quad (Qh_1 \cdots h_n)P(Qh_1 \cdots h_n) = 2Q - P + 2\sum_{i=1}^{n} g(P-Q,h_i)h_i, \quad P \in A.$$

Even the algebra presented by such relations for each $Q \in A$ and g-orthonormal system $h_1 \ldots, h_n \in V_A$ turns out to be trivial. In fact, there is a general impossibility result that renders any ansatz of this kind to yield a trivial algebra:

> Let A be an affine space over a field K, and $\operatorname{char} K \neq 2$. Let $\mathcal{A}$ be a K-algebra that is generated by $\mathcal{B}\,A$ (as a K-algebra). If there is a $Q \in A$, $x \in \mathcal{A}$ and $v \in V_A \setminus \{0\}$ such that $(Qx)^2 = 1$, $((Q+v)x)^2 = 1$ and $((Q+2v)x)^2 = 1$ then $\mathcal{A}$ is a trivial algebra[8].

This result yields the impossibility to represent the hyperplane of an orthogonal reflection by products. A further ansatz is to represent the hyperplane with the

[8] The restriction on the characteristic of K is not essential here in so far as there exist three distinct points on an affine line whose product with x square to 1. Furthermore, the same result holds, if the replace 1 by -1.

help of the additive structure, e.g., by $Q + h_1 \cdots h_n$ and $(Q + h_1 \cdots h_n)^2 = 1$. With a Euclidean metric in mind one could also try to represent simple g-"rotations" directly. If one represents the affine plane of rotation by $Q + uv$, $u, v \in S_{V_A}$ and considers uv as a directed arc-segment and Q as the center of rotation then the respective relations could be stated as follows

$$(Q + uv)(Q - uv) = 1, \quad (Q + uv)P(Q - uv) = \rho_Q(uv)P, \qquad P \in A,$$

whereby $\rho_Q(uv)$ denotes the action of the product of u and v in $Cl(V_A, g) \hookrightarrow Cl(\mathcal{B}\,A, \tilde{g}_{0,Q})$ on A, and Q is the $\tilde{g}_{0,Q}$-distinguished point in A. Whether this or the preceding additive ansatz yield a non-trivial algebra or not has still to be determined.

The second way to tackle the question on the existence of a geometric algebra of an affine metric space A builds on the idea of the simple law of change. This involves two tasks. The first task is to determine a reasonable subset $\mathcal{S}_{A,g} \subset \underline{\mathrm{Aff}}_{\mathrm{K}}(A, A)$ that can serve as a "simple change" for this case. The second task is to study whether the resulting algebra can indeed serve as a geometric algebra.

Since we have no apriori candidate for $\mathcal{S}_{A,g}$, we can only assume that $\mathcal{S}_{A,g}$ is a subset of g-isometries. The first observation is that the only g-isometries in $\mathcal{S}_A$ are the translations; $\mathcal{S}_{A,g} \cap \mathcal{S}_A$ can thus only consists of translations. The second observation is that if $\mathcal{S}_{A,g}$ is the full group of affine g-isometries then the equivalence classes with respect to the $\mathcal{S}_{A,g}$-induced equivalence relation $\sim$ become one-point sets. In other words, we have $(P, Q) \sim (O, R)$ if and only if $P = O$ and $Q = R$. The equivalence relation $\sim$ is thereby defined as in (7.1) with $\mathcal{S}_{A,g}$ in place of $\mathcal{S}_A$.

Because the obvious choices for $\mathcal{S}_{A,g}$ do not work, we have to use other means to arrive at a reasonable $\mathcal{S}_{A,g}$. One approach is to think of the geometric/dynamical invariant first and determine $\mathcal{S}_{A,g}$ with this. With the Euclidean metric in mind, one can consider the invariant to be related with to a directed arc segment. If one doesn't want to have a fixed center of rotation[9], however, then one needs a triple of points instead of a pair of points to represent it. This leads to the following modified ansatz

$$\begin{aligned}(P, Z, Q) \sim (P', R, Q') \;:\Longleftrightarrow\; & \\ \forall S \in \mathcal{S}_{A,g} :\; & SZ = Z \text{ and } SP = Q \Leftrightarrow SR = R \text{ and } SP' = Q', \\ & (P, Z, Q), (P', R, Q') \in A \times A \times A \setminus \Delta,\end{aligned} \tag{7.3}$$

whereby Δ denotes the diagonal of $A \times A \times A$. We think of (P, Z, Q) with $\|\overrightarrow{ZP}\|_g = \|\overrightarrow{ZQ}\|_g$ as the arc segment with center Z directed from P to Q. We refer to the other triples of point that don't satisfy this condition as degenerate arc segments. Like $P \wedge Q$ is a line-bound displacement vector $\overrightarrow{PQ}$, i.e. a notion of displacement

[9] We will develop the follwoing thoughts for g a Euclidean metric, for this case is the most pictorial one.

vector invariant under one-dimensional affine dynamics, the invariant related to arc segments should be related to rotational dynamics. It seems natural to relate it to simple rotations, for those are determined by a plane of rotation, a center of rotation in this plane and an angle. More precisely, simple rotations are uniquely determined by an invariant $(\dim A - 2)$-dimensional affine subspace and an angle of rotation. A simple rotation restricts to the identity map on its invariant affine subspace, and any transversal affine plane orthogonal to this subspace serves as a plane of rotation. In consequence, any two arc segments (P, Z, Q), (P', R, Q') that lie in parallel planes for which the affine line through Z and R is orthogonal and that describe the same angle should be identified. One might think that to get such an invariant the corresponding $\mathcal{S}_{A,g}$ would be the set of simple rotations. This is, however, not the case, because for this particular choice of $\mathcal{S}_{A,g}$ the arc segments (P, Z, SP) and (SP, Z, S^2P) are not equivalent for a simple rotation S with $SZ = Z$. But (P, Z, SP) and (SP, Z, S^2P) are equivalent in the above sense.

The set $\mathcal{S}_{A,g}$ that yields the desired notion of invariant has still to be determined. It is also not known, what kind of invariant we get for $\mathcal{S}_{A,g}$ being the set of simple rotations. What would also be very helpful are algebraic relations for the elements of $\mathcal{S}_{A,g}$ so that we can derive algebraic relations for the triple products of points. Otherwise, one can stick to the geometric notion of the invariant only.

A third possible way, which we have not considered yet, is the search for adjoints of the self action functor S from the category of finite dimensional $\mathbb{R}$-vector spaces with a metric to the category of metric affine spaces.

Regarding the morphisms, the natural choice would be to use isometries. But then the category of finite dimensional vector spaces is neither finitary complete nor finitary cocomplete. In particular, it is not complete (and even the category of metric $\mathbb{R}$-vector spaces of arbitrary dimension is not), so we can't use the adjoint functor theorem for an abstract existence result.

Like in the non-metric case, the self action functor can't have a right adjoint. This is due to a trivial vector space (with the one and only possible metric) being (strict) initial in the category of finite dimensional metric $\mathbb{R}$-vector spaces but the corresponding one-point space lacking this property in the category of metric affine spaces.

All in all we can't expect to get an abstract existence result. On the contrary, it looks like that one can derive that there is no left adjoint for S by considering the formal criterion [Mac98][theorem X.2] for the existence of a left adjoint.

To allow for more flexiblity in the structure one can either use the order on $\mathbb{R}$ and take as morphisms $f : (V, g) \to (W, h)$ linear maps $f : V \to W$ with $h(fu, fv) \leq g(u, v)$ for all $u, v \in V$ (and the respective affine maps for affine metric spaces), or allow for degenerate metrics, or both. Even though there might be no left adjoint for the respective S, studying those categories and their monoidal structures might help to find a suitable notion of simple change $\mathcal{S}_{A,g}$ for the metric case.

7.4 Geometric Algebra on a C^∞-Manifold

Grassmann's abstract foundation of extension on the simple form of extension that is obtained from a continuous simple law of change suggests one way to transfer his extension theory to smooth manifolds. Consider a finite dimensional connected smooth manifold M with a complete affine connection ∇. The geodesic equation becomes the simple law of change and oriented geodesic segments become the simple forms of extension. To make obvious how to proceed, we reconsider the affine case in this framework first.

If we look at a finite dimensional affine space A over $\mathbb{R}$ as a smooth manifold then the tangent bundle is $\mathrm{pr}_A : A \times V_A \to A$. The directional derivative induces the canonical (flat) affine connection ∇_A, which is complete. In this case the action map $\alpha : A \times V_A \to A$ is the exponential map of ∇_A, and the map in (3.2)

$$\theta : \ A \times V_A \xrightarrow{(\mathrm{pr}_A, \alpha)} A \times A,$$

is a diffeomorphism. In the general case of (M, ∇) and tangent bundle $\pi : TM \to M$ the respective map

$$\theta : \ TM \xrightarrow{(\pi, \exp_\nabla)} M \times M,$$

is only a local diffeomorphism, however. In particular, it doesn't need to be neither onto nor injective. This means that there may be two points which can't be connected by a ∇-geodesic, or starting from one point P in different directions one may arrive at the same target point $Q \in M$.

In the framework of smooth manifolds with affine connections we have to replace the vector space V_A of displacement vectors by the tangent bundle $\mathrm{pr}_A : A \times V_A \to A$ for an affine space A. The displacement vectors are then regarded as infinitesimal displacements and each displacement vector is bound to a particular point in A. As a consequence we can only add displacement vectors that are bound to the same point. This addition is not to be confused with the one induced by the geometric addition of directed line segments. Even though they happen to coincide in the affine case, they are still different conceptually.

In more detail, the geometric addition of directed line segments has to be restricted to those line segments that start at the same point P. It goes as follows. Consider two such directed line segments $\overline{PQ}$, $\overline{PR}$ together with their infinitesimal generators $(P, \overrightarrow{PQ})$ and $(P, \overrightarrow{PR})$, respectively. To obtain the geometric sum $\overline{PQ} + \overline{PR}$ parallel transport $(P, \overrightarrow{PQ})$ along $\overline{PR}$ first; this yields $(R, \overrightarrow{PQ})$. Consider the endpoint P' of its exponential next, i.e. $P' = R + \overrightarrow{PQ}$. Since the map θ is a bijection, there is a directed line segment $\overline{PP'}$. This is the geometric sum of $\overline{PQ}$ and $\overline{PR}$.

In this way the notion of geometric sum can be generalized to oriented geodesic segments of (M, ∇). Since θ is only a local diffeomorphism the geometric sum is

defined only locally in general[10]. Whenever it is defined for two geodesic segments $\overline{PQ}$, $\overline{PR}$ with infinitessimal generators $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, the generator of $\overline{PP'}$ yields a "sum" $\overrightarrow{PQ} \oplus \overrightarrow{PR}$ of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. In the affine case this sum $\oplus$ is defined for all displacement vectors in each fiber and coincides with the sum of infinitesimal generators. In the general case, however, the only locally defined

$$\oplus : U \subset T_P M \times T_P M \to V \subset T_P M, \qquad (u, v) \mapsto u \oplus v$$

is different from the vector addition in $T_P M$. In fact, it is neither commutative, nor associative; but it has the zero vector as its neutral element and for any $v \in T_P M$ the additive inverse $-v$ is also an inverse for $\oplus$. With geodesic segments all of this can be made obvious pictorially.

Besides the geometric sum one can introduce the geometric difference $\ominus$ following the same procedure. In the affine case this difference coincides with the difference of displacement vectors (in each fiber) and can thus be reduced to the vector addition and inverse operation. In the general case one has only the following relation[11]

$$w = v \oplus u \quad \Longleftrightarrow \quad v = w \ominus u.$$

Because of the lack of associativity, $\ominus$ can't be reduced to the inverse operation and $\oplus$ in the general case. This can be seen pictorially, too. Regarding the algebraic structure we are thus faced with something like a partially defined non-associative "group" structure.

It may come of as a suprise that such a geometric addition of geodesic segments and the accompanying algebraic structure has been considered in the literature already. A discussion of the general setting of (M, ∇) one can find in A. Tarantola's book [Tar09] in the first chapter on what he calls geotensors. But Tarantola is primarily interested on the case of M being a Lie group and the exponential map being the Lie group exponential. The reason for him to consider geodesic segments he gives in the introduction to his first chapter:

> *"Even when the physical space (or space-time) is assumed to be flat, some of the "tensors" appearing in physics are not elements of a linear space, but of a space that may have curvature and torsion. For instance, the ordinary sum of two "rotation vectors", or the ordinary sum of two "strain tensors", has no interesting meaning, while if these objects are considered as oriented geodesic segments of a nonflat space, then, the (generally noncommutative) sum of geodesics exactly corresponds to the 'composition' of rotations or to the 'composition' of deformations."*

[10] Since the property of θ of being a local diffeomorphism doesn't depend on the completeness of ∇, we may drop this assertion.

[11] Compare this with Grassmann's "analytische Verknüpfung" (analytical operation) in [Gra78][§5].

Tarantola also observes that the commutator $[v, w] = v \oplus w - w \oplus v$ and the associator[12] $[u, v, w] = (u \oplus v) \oplus w - u \oplus (v \oplus w)$ for $u, v, w \in T_P M$ are related to the torsion and curvature of ∇.

Following Grassmann, it is also clear what binary products of geodesic segments should mean geometrically: oriented plane segments traced out by the parallel transport of one geodesic segment along another. What is less clear is the respective invariant. One way to determine it is to impose algebraic properties for the product as it was done for the exterior product. It has still to be determined which properties would yield a geometrically reasonable invariant for this kind of geodesic plane segment.

So far, we have resorted to the framework of classical smooth manifolds. However, with regard to Grassmann's simple law of change it seems much more compelling to use the framework of synthetic differential geometry (SDG). This is mainly for the fact that the notion of an infinitesimal law of motion, which is modelled by tangent vectors in the classical setting, is given directly by infinitesimal line (or better path) segments in SDG.

[12] Note that our definition of commutator and associator differ from Tarantola's. He defines them with respect to $\ominus$ and denotes them by curly brackets. Moreover, he uses our bracket notation for the first non-vanishing terms in the Taylor expansion of his commutator and associator, respectively.

Appendix A

The Exterior Algebra as a Geometric Algebra of Linear Extension

It seems to be appropriate to explain how the exterior algebra may be regarded as a geometric calculus. We will explain this in rather detail by developing the exterior algebra from the point of view of a geometric calculus. This will actually contain the basic geometric ideas of Grassmann behind his construction of the exterior algebra. The employed interplay between geometric and algebraic reasoning can be found in a similar but less strict form in Grassmann's [Gra78] as well[1].

Consider an affine space A with an associated $\mathbb{R}$-vector space V_A as a V_A torsor, i.e., V_A acts on the set A in a simply transitive way. For the sake of geometric arguments we will assume a euclidean structure on A when speaking about "length", "area" or "volume". However, the geometric construction given here doesn't depend on such a structure, it does depend on $\mathbb{R}$ being an ordered ring though. We choose the directed line segments for the class of geometric objects to begin with. They can be represented by maps

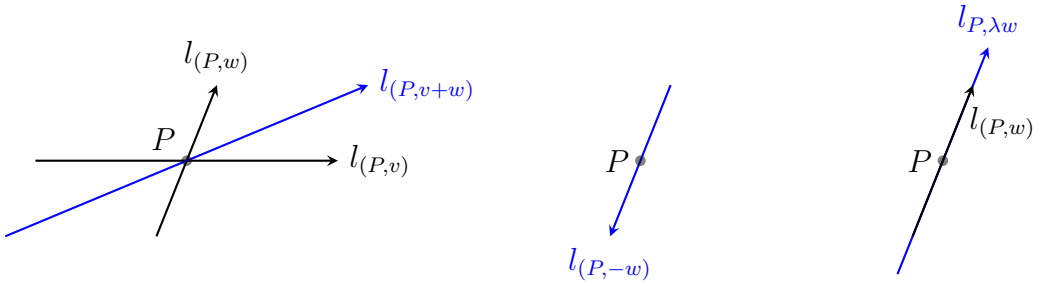
$$l_{(P,v)} : [-1/2, 1/2] \to A, \qquad t \mapsto P + tv,$$

where $P \in A$, $v \in V_A$ and $(P, v) \mapsto P + v$ is the simply transitive action of V_A on A. Because of the simply transitive action of V_A on A the map that assigns to each directed line segment $l_{(P,v)}$ the vector $v \in V_A$ is well-defined. If we allow for degenerate line segments $l_{(P,0)}$, i.e., those maps which represent points P, it will be surjective. We chose the vectors of V_A as the proper invariants for oriented line segments in A, which we will call "segments" from now on. For two such segments l and l' the equality of their vectors holds if and only if l and l' are parallel, have the same length and the same orientation. Stated differently, l and l' have equal invariants if and only if l and l' differ only by a translation, i.e., there is a translation $\tau : A \to A$ such that $l' = \tau \circ l$. The zero vector is the invariant representing the points of A.

One can say that the (non zero) vector captures the essential idea of a directed line segment. It is this something we have in mind when speaking about such an object without refering to a special spatial location of this object. Now we can

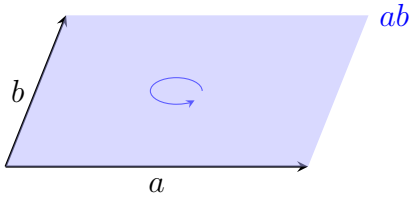
[1] Of course we present this idea from a nowadays point of view and use terminology and structures not known to Grassmann in this explicit form.

convince ourselves of the well known fact that the algebraic operation of vector addition corresponds to the geometric addition of two segments, scalar multiplication with positive scalars to scaling the segments, multiplication by -1 to reversing the orientation[2]:



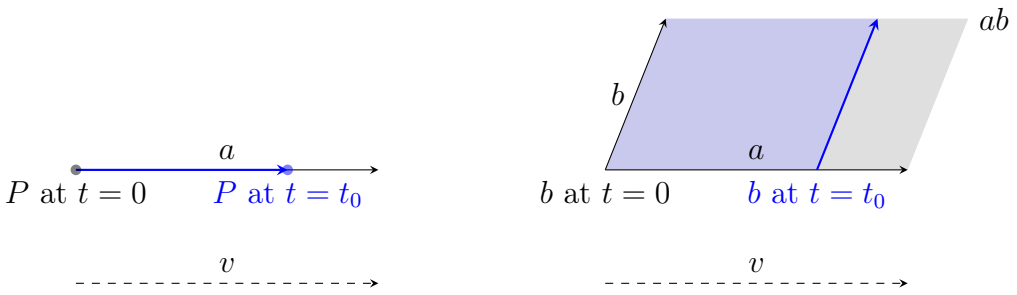
Here we actually went the opposite way of Grassmann. He considered (located) segments first, reflected on the notion of "directed line segment" and arrived by that, what Hamilton called a vector together with its algebraic properties of scalar multiplication and addition.

In the next step we ask which geometric construction involving segments a, b could be regarded as a product "$\wedge$" on the invariants. The ansatz of Grassmann was the formation of an oriented plane segment ab, i.e. an oriented parallelogram with the segments a, b as its sides as depicted below.



The idea behind this ansatz is the following one: consider the vector assigned to a segment as a rule on how to move a point along the segment. Moving the point according to this rule from the starting point of the segment in time 1 will sweep out the segment. Now having the two segments a and b with the rule $v \in V_A$ for a, we don't move a point but the segment b along a. This will sweap out the parallelogram ab:

[2]Note that if we would represent segments by maps $l_{(P,v)} : [0,1] \to A$, $t \mapsto P + tv$ then multiplication by -1 would correspond to reflection of the segment on its starting point P.



If we assume that a doesn't lie in the plane of bc then we can form the oriented parallelepiped abc in the same manner by moving the oriented parallelogramm bc along a. Note that moving the oriented parallelogram ab along c will result in the same oriented parallelepiped abc.

In principle one can generalize this type of construction to any dimension and we will see that this yields a way to make the notion of linear extension precise. Moreover, this generalizability of the construction is a necessary property for it to yield a product in the first place.

If we want to introduce a product in this way, the first thing we have to do is to extend our class of geometric objects by higherdimensional oriented parallelepipeds. Like the segments, we can represent a k-dimensional oriented parallelepiped by a map

$$e_{(P,v_1,\ldots,v_k)} : [-1/2, 1/2]^k \to A,$$
$$(t_1, \ldots, t_k) \mapsto P + \sum_{i=1}^{k} t_i v_i, \qquad v_i \in V_A,\ P \in A,\ k \in \mathbb{N},\ k > 0,$$

with the set $\{v_i \neq 0\} \subset \{v_1, \ldots, v_k\}$ of non zero vectors being linearly independent. If the $v_1, \ldots, v_k$ are linearly dependent we call $e_{(P,v_1,\ldots,v_k)}$ a degenerate oriented parallelepiped otherwise we call it nondegenerate. Note that the dimension of an oriented parallelpiped reflects the number of its arguments and not necessarily the number of the linear independent vectors spanning it.

In geometric terms a degenerate oriented parallelepiped is obtainend from a higherdimensional nondegenerate oriented parallelepiped by collapsing some of its sides. When there is only one zero vector a degenerate oriented parallelepiped can be regarded as a face of a nondegenerate oriented parallelepiped from which it induces its orientation. Note that segments and points can be regarded accordingly as 1-dimensional oriented parallelepipeds or higherdimensional oriented parallelepipeds with a sufficient degree od degeneracy. We abbreviate "oriented parallelepiped" by "parallelepiped" from now on.

Next we need to make the notion of linear extension precise. Let $l := l_{(Q,v)}$ be a (possibly degenerate) segment and $e := e_{(P,v_1,\ldots,v_k)}$ a (possibly degenerate) parallelepiped such that both have the same starting point, i.e. $l(-1/2) = e(-1/2, \ldots, -1/2)$,

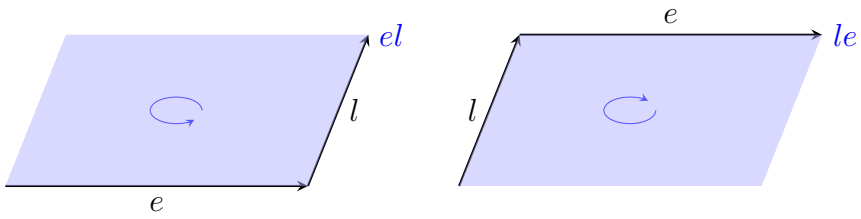
and $v \notin \operatorname{span}\{v_1, \ldots, v_k\}$ holds in the case of $v \neq 0$. The linear extension of e along l from the left is the parallelepiped

$$(le)_{(P+1/2v,v,v_1,\ldots,v_k)} : [-1/2, 1/2]^{k+1} \to A,$$
$$(s, t_1, \ldots, t_k) \mapsto (P + 1/2v) + sv + \sum_{i=1}^{k} t_i v_i.$$

In complete analogy one defines the linear extension of e along l from the right

$$(el)_{(P+1/2v,v_1,\ldots,v_k,v)} : [-1/2, 1/2]^{k+1} \to A,$$
$$(t_1, \ldots, t_k, s) \mapsto (P + 1/2v) + \sum_{i=1}^{k} t_i v_i + sv.$$

This reflects the idea of moving a parallelepiped along a segment given above with regard to the orientation. Indeed, in geometric terms the orientation is the only possible difference between el and le. We can observe this phenomenon in dimension two, i.e. for two segments forming an oriented parallelogram:



From these definitions we get that the type of associativity mentioned above for the three-dimensional parallelepiped holds more generally, i.e. $(ae)b = a(eb)$ holds for a parallelepiped e of arbitrary dimension and segments a, b, such that the respective independency conditions are satisfied. Furthermore, we obtain that each parallelepiped can be represented as a consecutive linear extension along its sides.

Contrary to the case of segments we are now truly in the position of Grassmann to find a notion of an invariant for a k-dimensional parallelepiped, which will allow for a structure of an $\mathbb{R}$-vector space such that addition and scalar multiplication reflect geometric operations to some extent. As suggested by the construction we want to see a k-dimensional parallelepiped e as a product of its (possibly degenerated) sides a_1 to a_k. Note that by the simply transitive action of V_A on A the sides a_1 to a_k are uniquely determined by e. Accordingly, we will assign to such an e the (formal) product $v_1 \wedge \ldots \wedge v_k$ as an invariant, where $v_k \in V_A$ is the respective vector assigned to the segment a_k. Again this yields a well-defined map from the k-dimensional parallelepipeds to the set of formal products of k vectors. Because of that we will exploit the consequences of the aspired idea of multiplication first. This will indeed give a reasonable addition and scalar multiplication to some degree, but as one can

guess from the knowledge of pure and mixed tensors already, the additon of the invariants related to such parallelepipeds of same dimension will be just a formal one in the generic case.

We can now turn back to the "$\wedge$"-product on the invariants. The type of associativity for linear extension from above is compatible with the associativity of the product. Associativity is a very useful algebraic property of course, so we gladly impose this relation on our product. On the level of invariants it implies that we can see a higherdimensional parallelepiped as a product of two lowerdimensional parallelepipeds that both need not to be segments. However, we can not depict this, because one has to consider at least a four-dimensional affine space to employ associativity in this way.

Until know we have mainly dealt with the nondegenerate case, i.e., with the product of two non parallel segments or more generaly with linear independent vectors. We still have to consider the linear dependent case. By linear extension and associativity everything is build up from segments; it is thus reasonable to study the case of two parallel segments first and deal with the general case later. However, parallel segments don't satisfy the condition for our definition of linear extension. Yet we are in the depictable case of two dimensions so we can try to return to more vague geometric reasoning.

An obvious first approach in this line of thought is to apply the geometric idea of the construction behind the linear extension to parallel a, b as depicted above. When a and b have the same orientation[3] then this will yield as a product a degenerate parallelogram with one side c which is the geometric sum of a and b.

If v and w are the corresponding vectors of a and b this corresponds to $v \wedge w = v + w$. We would like our product "$\wedge$" to distribute over addition and scalar multiplication of course; in other words: it should be bilinear. But then this equality can't hold even for linear dependent vectors v, w with $v = \lambda w$, $\lambda > 0$, because if one vector is not zero we have $\lambda\mu(v + w) \neq \lambda v + \mu w$ for $\lambda, \mu \geq 0$ in general.

On a second thought, the fact that we obtained a degenerate parallelogram as a product of a and b by geometric reasoning together with our insistance on the distributivity laws forces us by our above apriori assignment of invariants to parallelepipeds to set $v \wedge w = 0$. Indeed, by the map from above we assign to a degenerate parallelepiped a formal product of vectors in which one vector is the zero vector at least. By distributivity over addition, the product has to be zero. Consequently, like we did for degenerate segments, degenerate parallelepipeds have to be assigned to the zero vector as well. This imposes in particular the relation $v \wedge v = 0$ on our product and yields by bilinearity its skewsymmetry. By associativity, this extends on higher products to the alternating law.

[3]The same orientation of a and b is necessary for c to be a segment in a natural way, i.e., to be able to view the construction from a and b as a degenerate parallelogram in a natural way. We need to have a naturaly determined orientation by a and b from this construction in particular. Otherwise we would have to make an arbitrary choice.

With this and the imposed requirement of bilinearity we can deal with the case of linear dependent vectors in a universal way, i.e. independently of the actual $\mathbb{R}$-vector space structure chosen later on; but only as long as it is compatible with these imposed laws. As is well-known, for linear dependent vectors $v_1, \dots v_k$ we obtain $v_1 \wedge \ldots \wedge v_k = 0$ by expressing for example v_1 as a linear combination of $v_2, \ldots, v_k$ and employ the bilinearity together with the alternating law.

This is a rather formal algebraic argument than a geometric one, of course, and as one can see, we made a compromise between reasonable algebraic properties and the geometric interpretation of our construction of parallel segments a, b [4]. Yet we can make some sense of it from the geometric point of view. Our idea behind the product is the construction of linear extension. In the parallel case we couldn't extend anything. In other words, our construction didn't result in the proper new type of geometric object we wished for. We thus assign the algebraic zero to this result. This zero can be regarded as a kind of garbage collector for results of our synthetic geometric constructions that don't fit in the class of geometric objects we designed the construction process for. This also justifies the implicit algebraic assumption made above, namely that the zero vector of k-dimensional invariants is for all k the same zero vector and the zero vector which is assigned to degenerate segments, i.e. points, in particular. Note, however, that assuming any $\mathbb{R}$-vector space structure on the invariants it doesn't seem to make sence to interpret the identity law for k-dimensional invariants in a geometric way for $k > 1$, i.e., to relate points with k-dimensional parallelepipeds.

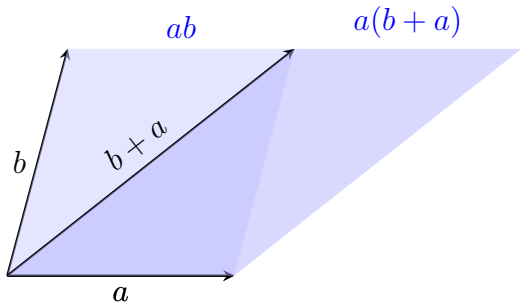
We are now in the position to study the geometric consequences of the wished for distributivity laws. For this we first construct the proper algebraic stage employing the gathered algebraic information until now. This means to construct a universal $\mathbb{R}$-vector space structure on our desired k-dimensional parallelepiped invariants $v_1 \wedge \ldots \wedge v_k$ in pure algebraic terms as $\bigwedge^k V_A$ compatible with the information we already have on our product "$\wedge$", namely the desired bilinearity and that it is alternating. From this we obtain the universal unital, associative $\mathbb{R}$-algebra for such type of product to be $\bigwedge V_A = \bigoplus_{k\in\mathbb{N}} \bigwedge^k V_A$ with $\bigwedge^0 V_A := \mathbb{R}$ together with the multiplication defined by products $\wedge_{kl} : \bigwedge^k V_A \otimes \bigwedge^l V_A \to \bigwedge^{k+l} V_A$ constructed from our particular formal "$\wedge$"-product on vectors and its imposed associativity law, the universal property of the direct sum as well as the distributivity of the tensor-product of $\mathbb{R}$-vector spaces over the direct sum. $\bigwedge V_A$ contains in particular all of our invari-

[4] Note that this is the only case where we can deal with linear dependent vectors in a geometric way using our construction and k-dimensional parallelepipeds as geometric objects. Consider for example the affine plane and three segments a, b, c with a common starting point that span the plane and that aren't pairwise parallel to each other. We could form the parallelogram ab by linear extension. However, applying the construction behind the linear extension to ab and the segment d won't yield a parallelogram. It will yield a polygone which can be regarded as the image of a projection of a (nondegenerate) three-dimensional parallelepiped standing on its face ab to the plane of ab.

ants and implies that the sum of invariants of different dimensions will be a purely formal one. This is acceptable, because there seems to be no reasonable way to give such a sum a direct geometric meaning except of a list that keeps track of invariants involved from different dimensions. Recall what we have already concluded about the interpretation of the zero vector $0 \in \bigwedge V_A$ above. It can be regarded as a kind of garbage collector for our synthetic constructions and has thus no direct geometric object corresponding to it.

Our task now is to study the geometric consequences of this pure algebraic setting. Hereto, we want to see whether it is reasonably compatible with our geometric setting or not and in consequence, whether geometric reasoning forces further relations on our product or whether it forces us to give up our alternating or distributivity laws. Although we already know that this won't be the case here, this consideration would be crucial if we wanted to incorporate more geometric structure like a metric.

As we did it in the case of segments, the first thing we will consider is what it means for two nondegenerate parallelograms ab, cd to have the same invariant. From $v \wedge v = 0$ we can conclude that ab and $a(b + a)$ have the same invariant. Pictorially, this resembles Cavalieri's principle and suggests that our parallelogram invariants can be actually regarded as area elements. This point of view will be supported by the general case as well.



Let $v_a, v_b, v_c, v_d \in V_A$ be the vectors assigned to the respective segments a, b, c, d. We observe that $v_a \wedge v_b = v_c \wedge v_d$ implies $v_c, v_d \in \operatorname{span}\{v_a, v_b\}$. This is a special case of the well known general fact that $v_1 \wedge \ldots \wedge v_k = 0$ if and only if $v_1, \ldots, v_k$ are linearly dependent. We have seen the argument for one direction of this statement already. The other direction can be seen better when passing to its contraposition: if $v_1, \ldots v_k$ are linearly independent, then $v_1 \wedge \ldots \wedge v_k \neq 0$. This is reflected in our geometric construction and we can thus construct a nondegenerate k-dimensional parallelepiped that has $v_1 \wedge \ldots \wedge v_k$ as its invariant. In algebraic terms this follows either from the explicit construction of $\bigwedge^k V_A$ from the k-fold tensor-product of V_A by an explicit examination of the subspace to be quotiened out, or from the universal property of $\bigwedge^k V_A$ as a representing object for alternating k-linear maps by the construction of such a map that doesn't vanish on $(v_1, \ldots, v_k)$ [5]. Returning

[5]Such a map can be constructed with the help of the determinant map $\det : \mathbb{R}^k \times \ldots \times \mathbb{R}^k \to \mathbb{R}$

back to our original discussion, $v_c, v_d \in \operatorname{span}\{v_a, v_b\}$ means that there are (unique) coefficients $\lambda_{ca}, \lambda_{cb}, \lambda_{da}, \lambda_{db} \in \mathbb{R}$, such that

$$v_c = \lambda_{ca} v_a + \lambda_{cb} v_b, \qquad v_d = \lambda_{da} v_a + \lambda_{db} v_b.$$

Using the bilinearity and alternating law of our product this yields (after substitution and ordering of terms)

$$v_a \wedge v_b = (\lambda_{ca}\lambda_{db} - \lambda_{cb}\lambda_{da}) v_a \wedge v_b.$$

Finally, by applying the cancellation law this results in $(\lambda_{ca}\lambda_{db} - \lambda_{cb}\lambda_{da}) = 1$. For $V := \operatorname{span}\{v_a, v_b\}$ the expression $\lambda_{ca}\lambda_{db} - \lambda_{cb}\lambda_{da}$ can be understood as the determinant of the linear change of basis isomorphism $A : V \to V$, defined by $Av_a = v_c, Av_b = v_d$. With this we can reformulate our result as

$$v_a \wedge v_b = (Av_a) \wedge (Av_b) \quad \Longleftrightarrow \quad \det A = 1 \quad \Longleftrightarrow \quad A \in SL(V).$$

If we interpret the determinant form in geometric terms as a volume or area form, then from the upper result we obtain that two nondegenerate parallelograms have equal invariants if and only if they are parallel and have the same oriented area. This is why it does indeed make sense to see the invariants as area elements.

In terms of affine maps $f : A \to A$ we can state this characterization in the following way: two nondegenerate parallelograms ab, cd have equal invariants if and only if there is an affine map $f : A \to A$ such that f restricts to $f : \operatorname{supp} ab \to \operatorname{supp} cd$ and is unimodular. Here for a k-dimensional nondegenerate parallelepiped $e = a_1 \dots a_k$ we mean by $\operatorname{supp} e \subset A$ the affine subspace equipped with a determinant form $\det_e$ such that $(v_1, \dots, v_k)$ is a unimodular basis, i.e. $\det_e(v_1, \dots, v_k) = 1$ and the $v_i \in V_A$ are the respective vectors assigned to the sides a_i of e. By the unimodularity of f we mean the associated linear map $\hat{f} : V_{\operatorname{supp} ab} \to V_{\operatorname{supp} cd}$ to respect these determinant forms, i.e. $\hat{f}^* \det_{cd} = \det_{ab}$. Note how this extends the geometric characterization of segments with equal vectors.

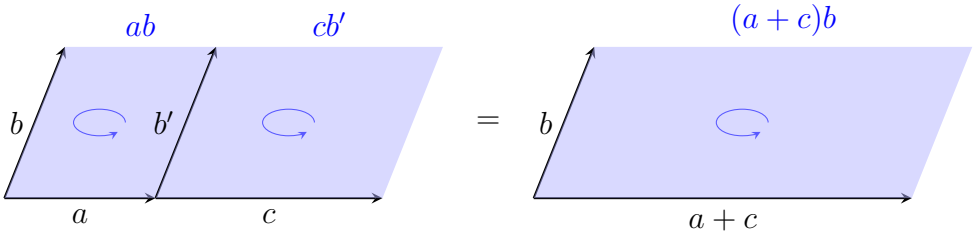
It is straightforward to generalize this result to the case of higherdimensional parallelepipeds. Namely, two nondegenerate k-dimensional parallelepipeds $e = a_1 \dots a_k$, $e' = a'_1 \dots a'_k$ have equal invariants if and only if there is an affine map $f : A \to A$ such that f restricts to $f : \operatorname{supp} e \to \operatorname{supp} e'$ and is unimodular. Hence we can see the invariants in geometric terms as k-dimensional volume elements in general. Note that such volume elements always denote the non-zero vectors in $\bigwedge^k V_A$ that can be represented as products $v_1 \wedge \ldots \wedge v_k$ for $v_i \in V_A$.

This geometric characterization also suggests a geometric way to interpret a multiplication of a parallelepiped by a positive scalar that is compatible with the respective distribution law for our product: scale the fictious volume of the parallelepiped.

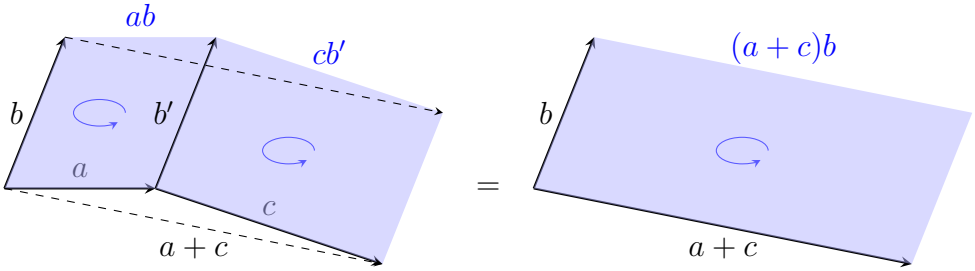
on $\mathbb{R}^k$ given by Leibniz' formula, the linear isomorphism $\Phi : \operatorname{span}\{v_1, \dots, v_k\} \to \mathbb{R}^k$ induced by the ordered basis $(v_1, \dots, v_k)$ and a projection $\operatorname{pr} : V_A \to \operatorname{span}\{v_1, \dots, v_k\}$ as $(\Phi \circ \operatorname{pr})^* \det$.

This can be obtained by scaling one of each sides and this actually expresses the distributive law of "$\wedge$" over scalar multiplication.

At last we have to study the geometric content of the addition. We begin again with the most simplest case of parallelograms ab, cb':



An addition of the respective invariants that is compatible with distributivity over vector addition corresponds in geometric terms to a fusion of the two parallelograms together to one $(a + c)b$. This is a reasonable geometric manipulation with such parallelograms. It fits in the interpretation of the invariants as area elements as well. In the case of a, c not being parallel such an addition of ab and cb' is an extension of the geometric sum of segments to parallelograms and still reasonable from a geometric point of view. It can be seen as an addition of area elements which respects the possible different relative positions of the area elements to each other in a bigger surrounding space.



However, in a space of dimension greater than two it doesn't need to correspond to the addition of areas directly like above. This type of interpetation still holds in the case of k-dimensional nondegenerate parallelepipeds e, e', when e and e' differ only in the jth side for a $1 \leq j \leq k$ and are properly located.

Due to our algebraic and geometric construction we have only the distributivity laws which can yield a geometric interpretation of the addition of two invariants assigned to nondegenerate parallelograms ab, cd. The reason is simple and holds in more generality. Only when we are able to apply the distributivity law a sum of two invariants of k-dimensional parallelepipeds yields an invariant of a k-dimensional parallelepiped again. Consequently, w.l.o.g. we need that ab has the same invariant as a parallelogramm cb'. Let $v_a, v_b, v_c, v_d \in V_A$ be the vectors assigned to the respective segments a, b, c, d. From our characterization of parallelograms with equal

invariants we infer easily that $v_c \in \operatorname{span}\{v_a, v_b\}$ is a necessary and sufficient condition for the existence of a $v_{b'} \in V_A$ such that $v_a \wedge v_b = v_c \wedge v_{b'}$ [6]. This result can be extended to nondegenerate k-dimensional parallelepipeds e, e' in a straightforward manner once again. Let $v_1 \wedge \ldots \wedge v_k$, $w_1 \wedge \ldots \wedge w_k$ be the invariants assigned to e and e'. Their sum will be an invariant $u_1 \wedge \ldots \wedge u_k$ again if and only if $\operatorname{supp} v_1 \wedge \ldots \wedge v_k$ and $\operatorname{supp} w_1 \wedge \ldots \wedge w_k$ have a common subvector space of dimension $k-1$. Here we mean by $\operatorname{supp} v_1 \wedge \ldots \wedge v_\ell$ the (oriented) vector space spanned by $v_1, \ldots, v_\ell$.

In terms of our k-dimensional volume element interpretation this amounts to the following result. The sum of two k-dimensional volume elements will be again such a volume element if and only if the volume elements differ only in one degree of freedom of their relative position to each other in the vector space V_A. Otherwise the sum will be just a formal one. Whether there are formal sums of k-dimensional volume elements or not depends on k and the dimension of V_A. For example, in the three-dimensional case the sum of two area elements will always yield an area element, because two sets of respective linearly independent vectors have at least a common one dimensional subvector space by the dimension formula for subvector spaces $\dim U \cap V = \dim U + \dim V - \dim(U+V) \geq 4 - 3 = 1$. On the other hand, formal sums of area elements do exist in four dimensions, e.g. $e_1 \wedge e_2 + e_3 \wedge e_4$ for a basis $e_1, \ldots, e_4$ of V_A. The result for area elements in the three-dimensional case can be extended to finite dimensional affine spaces A of dimension n: there are no formal sums of $(n-1)$-dimensional volume elements.

All in all the geometric interpretation of invariants as volume elements of different dimensions fits with the formal algebraic structure, which we have deduced from the geometric idea of linear extension along segments, to some acceptable extent. It is therefore reasonable to call Grassmann's exterior algebra a geometric calculus. Of course our construction doesn't show how strong this geometrical calculus is. Something like that can be only shown by its application in the discussion and solution of geometric problems.

However, in our current state of development we are still missing some important integral components. The first thing are the regressive products, which are in some sense dual to the exterior product and describe intersections (cf. [Zad94], [Bro09], [Gra78]). Equipped with one particular regressive product the Grassmann algebra becomes powerful enough to formulate and solve problems in projective geometry (cf. [Zad94]). The second thing we are missing is important for its application to affine geometry and especially physics. It is a geometric calculus of points. Grassmann deals with this issue in the second part of [Gra78] with his concept of "Elementargröße". The third thing that is not incorporated in its current state is the structure of a metric. Grassmann deals with this in [Gra62] by introducing his "Ergänzung", which we would call Hodge duality today. From the point of view

[6] Let $V := \operatorname{span}\{v_a, v_b\}$. Define a change of basis map $A : V \to V$ by $Av_a = v_c$, $Av_b = v_b$. From $v_c \wedge v_b = (Av_a) \wedge (Av_b) = \det A \; v_a \wedge v_b$ we see that $v_{b'} := 1/\det A \; v_b$ is such a vector.

of computational aspects and applications, however, Clifford products seem to be more appropriate here.

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