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Dual 'Prices' in Plan and Market

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Abstract

(JEL classification: 020) The solution to the dual programming problem is traditionally conceived as a vector of unit values for the constraints of the primal problem, which in much economic-theoretic exposition employs dollar-denominated output (revenue; gross output) as the maximand, and in some applied literature, contribution to profit. This interpretation of the dual is unreasonable and we argue that the dual values either are the unit *premiums* over market prices, for those inputs whose acquisition costs are deducted from selling price (and only for those units which are incremental to the amounts represented by the current constraints) or, for 'common' or 'overhead' capacity constraints whose costs are not deducted to calculate the maximand, the potential profit contribution per unit activity/per unit time. The latter application requires estimation of facility life and consumption rates in the first place, which is abjured in the standard microeconomic cost model. The paper reviews other uses of duality, including that of the planned/semiplanned economy and the evolution of Kantorovich's treatment.

1. Introduction

The dual solution in mathematical programming is conventionally interpreted as a set of price imputations for the resource constraints in the maximizing problem and have been viewed as a tool for decentralized planning based on economy-wide market simulation to maximize welfare. This tradition goes back to the introduction of duality theory in the west (Kuhn–Tucker, 1950; DOSSO, 1958; Gale, 1960). But although the Kuhn–Tucker theorem underlying this valuation interpretation is unarguable as a mathematical statement, as is the equally compelling complementary slackness theorem concerning characteristics of the dual variables (Dantzig, 1963, p. 136), the underlying institutional framework of the economy in which they are to be applied has never been fully and consistently developed. Gale relates these imputations to optimization within the 'semi-planned economy' as do many Soviet writers (see Gardner, 1990; Kantorovich–Gorstko, 1972; Pita, 1984). Other writers apply the dual prices to value to inputs in market economies (Cameron, 1985; Gass, 1985; Kolman–Beck, 1980; Wu–Coppins, 1981).

Logical problems in the first interpretation relate to the consistency of the output prices, which are given, and the dual shadow prices, which are

calculated. Since some goods play a role as both outputs (for final demand as conventionally defined to include inventory accumulation or capital goods), and inputs the planner's price as output may differ from the shadow price as input. Or again, what does it mean for an input to take its value from its contribution to the production of goods whose prices are calculated on the basis of the costs of the inputs, many of which will differ from the shadow prices?

As to the second interpretation, there has been little attempt to integrate into the model a realism which recognizes the great variety of inputs so that dual valuations have a fanciful characters. Inputs to production activities in virtually any modern economy include many very different categories: (i) incrementally engaged *and* incrementally consumed factors (certain capital expenditure acquisitions); (ii) factors which are only incrementally consumed; (iii) common inputs (capital and non-capital); (iv) non-common inputs; and (v) inputs which are virtually not consumed at all.

The long history of the issue is shown by the lengthy time span of the previous citations. Their lack of realism is sometimes apologized, as in other areas of economics, by the need to walk before we can run. But, just as elsewhere, we come to treat the elementary procedure as an end in itself.

It turns out that duality theory can provide important insights into the allocation of common input costs within the context of the firm which operates outside the classroom. But since even advanced treatments of microeconomic theory (e.g. Varian, 1978) concentrate on the simplified cost model which excludes the great diversity of inputs, neither the need for, nor the contribution of, such an interpretation is recognized.

While the foregoing comments apply to the general mathematical theory of programming and duality, nothing will be lost from the analysis if this article focuses on the linear programming case.

The article moves as follows. Section 2 presents a brief summary of the conventional interpretation of the dual solution and indicates its inadequacies. Section 3 points to two important ways to modify the definition of the dual prices (after first revising the primal objective function to reflect profit contribution rather than total sales volume or revenue) for materials, semi-fabricates, and labour constraints, on the one hand, or fixed equipment capacity, on the other. (Two other non-controversial uses of the valuation interpretation for non-input constraints are also indicated.) Section 4 examines the use of dual values in the original model of Kantorovich who conceived them for a mechanical function for very limited decentralization rather than for economy-wide decentralization, although later many Soviet writers including Kantorovich came to view them as having a more fundamental economic meaning. Section 5 provides a concluding summary.

2. The conventional dual interpretation

The essence of Kuhn–Tucker theory was to set up the Lagrangean form of the constraint set, maximize a function of output and these constraints, and then allow the m Lagrangean multipliers, (u_h, in their notation), which emerge in the saddlepoint solution to represent the unit values of the constraining resources. In their brief section on linear programming (p. 9), they draw explicitly the equivalence of the m-vector u and the solution values for the dual minimization problem. As this is a mathematics rather than an economics paper, the authors do not linger over the economic meaning of the dual, indicating similarities in passing with constructs of several authors, e.g. Samuelson's Lagrange multipliers (*Foundations*, ch. 8), although the multipliers in the cited chapter apply to both outputs and inputs in a one-product maximization problem, with all other output levels pre-specified. However in such a setting market prices for output would not be necessary.

To help follow the discussion, we recall the basic form of the problem. The primal problem[1] requires us to find

$$x_1, x_2, \ldots, x_J \geqslant 0; \qquad x_{J+1}, \ldots, x_n \in \boldsymbol{R}$$

which maximize

$$z = \boldsymbol{xs} = \sum_{j=1}^{n} x_j s_j$$

subject to the constraints

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n \leqslant b_1 \\
&a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n \leqslant b_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{I1}x_1 + a_{I2}x_2 \cdots + a_{In}x_n \leq b_I \\
&a_{I+1,1}x_1 + a_{I+1,2}x_2 \cdots + a_{I+1,n}x_n = b_{I+1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n = b_m .
\end{aligned}$$

To this primal, corresponds the dual: find

$$y_1, y_2, \ldots, y_J \geqslant 0; \qquad y_{J+1}, \ldots, y_m \in \boldsymbol{R}$$

which minimize

$$w = \boldsymbol{by} = \sum_{i=1}^{m} b_i y_i$$

subject to the constraints

$$
\begin{aligned}
&a_{11}y_1 + a_{12}y_2 \cdots + a_{1m}y_m \leqslant s_1 \\
&a_{12}y_1 + a_{22}y_2 \cdots + a_{2m}y_m \leqslant s_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{1J}y_1 + a_{2J}y_2 \cdots + a_{mJ}y_m \leqslant s_J \\
&a_{J+1,1}y_1 + a_{J+1,2}y_2 \cdots + a_{J+1,n}y_m = s_{J+1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{m1}y_1 + a_{m2}y_2 \cdots + a_{mn}y_m = s_m \,.
\end{aligned}
$$

Interpretation of the dual solution $y_1, y_2, \ldots, y_m$ as an equilibrium resource price vector is the equivalent of the emergence of a set of prices by the workings of a perfectly competitive market (Gale, Sec. 3.5; DOSSO, Sec. 3.6). These equilibrium prices were thought to provide the competitive market result while sidestepping the need for a market. From the beginning they held out hope for rational prices within an economy with social ownership but decentralized input decisionmaking, obviating thereby the need for a huge administrative bureaucracy. Or, as Gale puts it (p. 91):

> Perhaps the most natural situation to which our theorem applies is that of a 'semiplanned' economy in which the planners determine the prices of consumer goods but allow the competitive mechanism to determine the prices of resources. A rather spectacular consequence of our theorems is then that the planners can effectively determine the output of the economy by suitably adjusting consumer prices and still preserve the profit motion as the incentive to individual firms.

The prophetic nature of the quote is apparent in a recent Cuban example, in which daulity is said to 'permit the application of the principle for democratic centralism in management' by assigning dual shadow prices to inputs and having firms maximize profits, with profit calculations based on these dual prices (Pita, p. 108). But what this fails to address is the basis for the output values, s_j and the consistency of their structure with the dual. If they themselves reflect the planners' preferences (which in turn are conditioned by perceptions of market consumer preferences), the application is reasonable. But if the s_j are historically cost determined, i.e. based on previously existing cost relationships, we have the paradox: the input resources are now to derive their values from the contribution they make to the production of an output which is valued in the first place according to different values for those same input resources.

A second objection to this treatment is the possible disagreement between the price of a good as final demand and the dual price. Inventory accumulation or machine tool production, for example, are components of final demand (investment).

3. Duality in the market economy

The mathematical theory of duality and all its attendant characteristics continue to hold in the market economy. But care must be exercised in its interpretation. In this section now assume a market framework and explore the meaning – meanings, we shall learn – of the dual prices.

Starting in the 70s many or most textbook writers used profit rather than simple market price as the money weights in the maximization problem (e.g., Cameron, Sec. 2.2; Kolman–Beck, Sec. 3.1; Gass, Sec. 6.3; Taha, Sec. 4.3; Wu-Coppins, Sec. 4.2). With the exception of the Kolman–Beck book, which uses sawmilling activity capacities to illustrate the constraints (and in which there are no intermediate inputs or labour inputs), their maximand constraints are materials and/or labour availability. Wu–Coppins introduces the important refinement of calling its primal coefficients 'contributions to profits' rather than simply 'unit profits'. This is necessary because profit is influenced by the consumption of fixed facilities, such as assembly lines and machinery, whose use the producer may be thinking of as a common cost for all products but which is not further allocated. This may be the case if it is too costly to trace and assign machinery consumption. If the machinery consumption associated with the output of product j *can* be determined, it should be deducted from the selling price s_j to derive the maximand 'price weights' as may be done for the other traceable inputs (labour, raw materials, and semifabricates, the last two to be termed 'intermediates' for simplicity in what follows). This is the background for Section 3.1, to which we now proceed. Following that we consider the use of fixed non-traceable factors as constraints (3.2) and other types of constraint such as mandatory pollution quotas.

3.1. Intermediates, labour, and traceable capital constraints

It is frequent practice to formulate the production model with intermediates, labour, and traceable machinery consumption as the resource constraints. These are purchased in the market and so prices already exist for them. The same problems about the interpretation of the dual in their context applies for all of them, but we may treat them separately since in Section 3.2 we will also want to distinguish traceable from non-traceable capital consumption.

i. Intermediates and labour inputs

It is not explained why such production needs, which are normally readily available on a day-to-day basis in the market at the going market price, should constrain production in the first place. (The Kolman–Beck formu-

lation which omits such inputs is a recognition of this fact.) However, one possibility is that an input material is available in limited quantities in the short term, possibly because of long distance shipping requirements. Furthermore – and this is the main thrust of this analysis – since market prices do already exist for these goods, it is not obvious what is the interplay between the shadow price and the market price. One thing alone is certain: comments to the contrary notwithstanding, the shadow price *does not represent* the amount that the firm would be willing to pay for the product. Moreover, it is neither the maximum amount nor the minimum amount. Rather, it constitutes a *premium over the market price* up to which a firm would be willing to pay to buy *one* (*or possibly a few*) *additional* units of this input per unit time until conditions revert to normal. It is very easy to see why this is so.

Profit is defined as selling price less cost of inputs. Let s_j denote the selling price of product j as before. Denote by c_i the unit acquisition cost of input i and by p_j the profit or contribution to profit per unit of production of product j. Then $k_j = \Sigma_m^{i=1} a_{ij}$ is the cost of the traceable inputs (intermediates, labour, and capital consumption) to produce a unit of output j. To say that p_j is to serve as the coefficient of x_j in the primal, means that the market acquisition cost, k_j, has already been subtracted from s_j.

By the fundamental duality theorem, the maximand value equals the minimand value at optimality, i.e.

$$\boldsymbol{xp} = \boldsymbol{yb} .$$

This equality is the result of a mathematical theorem and is true no matter what the units of measurement of the elements p_j of the vector $\boldsymbol{p}$, provided only that the dual and primal are formulated consistently. Thus, we must shift from a maximand with $\boldsymbol{s}$ to one using $\boldsymbol{p}$, that is

$$\text{maximize } z' = \boldsymbol{x}(\boldsymbol{s} - \boldsymbol{k}) = \boldsymbol{xp}$$

and the dual variable vector will automatically be cast in terms related to $\boldsymbol{p}$ rather than to $\boldsymbol{s}$. The fundamental duality theorem then has as its equation at optimality

$$\boldsymbol{xp} = \boldsymbol{x}(\boldsymbol{s} - \boldsymbol{k}) = \sum_{j=i}^{n} x_j \left(s_j - \sum_{m}^{i=1} a_{ij} c_i \right) = \boldsymbol{yb} = \sum_{i=1}^{m} y_i b_i$$

In words, at the optimal output configuration, the total *net profit* rather than the total revenue is distributed over all the inputs, with y_i being the shadow 'price' per unit of the ith input.

Thus, the dual valuation does represent the increment to profit that could be generated by one more unit of the intermediate input. This

increment in profit is net of the normal market acquisition cost of this and all other inputs. Therefore, it does not serve as the market acquisition cost but, rather, as the *premium over* the market acquisition cost which the firm would be willing to pay for one more unit of this input. For example, if the unit market acquisition cost is \$6, and the dual shadow value for this input is \$2.50, one more unit of the input could increase profit by \$2.50 and the rational entrepreneur would be willing to pay up to \$8.50 for *the next unit*. 'The next unit' must be stressed: if the firm began to pay \$8.50 for all the previous units, the unit profit for the jth product would be $\$2.50 * a_{kj}$ less than what was used and a radically different production plan might emerge.

In view of the foregoing, the dual solution is of far more modest potential than is often assumed to be the case. There can be no dispensing with market prices. But the valuations do have a value. We now examine the circumstances under which such a dual solution might be relevant?

Imagine a sudden extra supply coming on the market, possibly because of bankruptcy of product change of another user firm. The entrepreneur knows how high an offer he can make – and, after applying sensitivity analysis to the problem to determine the limits of stability, how much he can bid for it. This is still a useful – even if much more modest – application than that envisaged in the macro planning literature or the micro literature in which profits temper the maximand.

ii. Traceable capital input constraints

One example which uses capital facilities – machinery – as constraints within the western profit maximization rather than gross output context is the Kolman–Beck sawmill example (pp. 154–157) which posits sawing and planing capacity; however, the cost of these facilities is said (p. 156) to have been

> . . . hidden in whatever computations were done to figure out that the profits were \$120 per 1000 board feet of finish grade and \$100 per . . . [1000] . . . board feet of construction-grade lumber.

This presents the same logical inconsistency that we discussed in Sec. 3.1 and the way out is similar. Here, 'whatever calculation' has been made reflects the putatively traceable capital consumption (sawing or planing equipment) and the dual valuation must represent the *premium* over the market price that the rational firm would willingly pay for *one more* unit of capacity. It is an imputation for the capital which can be used for many activities but, since it may be traced, it is not a common cost. Of course, since capital equipment must be purchased in size much larger than what is required to produce one additional unit of output, the dual price premium can be only an indicator of questions to explore.

In the Kolman–Beck example the dual prices are \$35 for an hour of sawing and \$10 for an hour of planning, and it is recognized that these are 'fictitious prices', the true acquisition costs having been taken into account in calculating the unit profits. These are said, then, to represent the addition to profits of one more unit of capacity, but this is incorrect in just the same way that it was incorrect to imput to one more unit of an intermediate good a potential profit contributation equal to the shadow price since if one did have to go out into the market to expand capacity of sawing or planing, the cost of acquiring the new unit would have to be subtracted from the potential profit contribution.

3.2. Non-traceable capital as constraints

We have so far explored the meaning of the dual solution within the context of inputs which are traceable – labour, intermediates, and some types of capital equipment. But nontraceably consumed capital may also constrain production. Gale's previously cited example recognizes the existence of fixed capital resource constraints, putting them into a separate constaint vector $\boldsymbol{b}$ and calculates shadow values for them within the gross output framework. The original Kantorovich formulation which pursues an entirely different approach to the primal objective function and will be discussed in Section 4, implicitly assumes the capital to be non-consumable and not to have had its acquisition cost subtracted from the selling price to calculate profit.

There may be several reasons for not attempting to deduct capacity cost from the net profit contribution of individual products. Some capital components degrade only slowly with respect to individual outputs but do degrade with respect to general periodic use – plant lighting for example or storage capacity. These *are* common cost elements which must be imputed. In this case, the shadow values do represent the gross potential contribution to the profit function which one more unit of capital to the profit function, per unit time since the profit function has not been calculated to reflect this capital input. However, it is necessary to consider the dimensions of the contribution in order to determine a maximum price that the firm can pay for this capital. In particular, the life of the machine or other production facility must be determined. Then the potential contribution per unit time must be multiplied by the number of units of time in the life – weeks, months, years. Whether this is a difficult task or not will depend on the type of equipment in use and its pattern of consumption: does it wear with respect to time, output per unit time, or not at all? It should be emphasized that if we know these things in the first place, we should be able to do a much better job of determining true unit incremental costs in our microeconomic cost theoretic structure in the first place, in particular, noting that much of the capital expenditure

should be thought of as a variable or incremental cost, as the present author has often reasoned (1983; 1985, Chs 2–3; 1987). But the fact that it is difficult should not make us reject this interpretation of duality, but, rather, encourage us to get on with the job, with the hope that doing so within the programming framework will spill over into our general intermediate microeconomic analysis.

3.3. Constraints which are not inputs

A variety of other types of constraint may be imagined and these have clear interpretation. These may impose either maximum or minimum levels. For example, maximum pollutant levels may be introduced, with pollution-generating coefficients used to represent the amount of some pollution effect contributed by a unit of output of product j. In this case, the shadow value for the constraint would indicate how much additional net profit would result from increasing the restriction by one unit (e.g. kilograms of PCBs or mercury). This value might be used as a basis for trading pollution rights.

A minimum production constraint may be illustrated by the contracted sale of an output h about the quantity of which the producing firm has come to have second thoughts. For example, if the firm wishes to renegotiate a long-term contract to provide electric power or transportation to some user, it might start from the shadow value per kilowatt-hour. This indicates the profit sacrifice that the firm makes per unit of production of this product rather than using this input to produce other products. The net impact would be then equal to this increased profit, less the revenue s_h brought in by the sale of one unit of product h (any traceable inputs must of course be subtracted from s_h).

4. The Kantorovich problem

Much of the interest in planning and operational applications of duality originates in socialist planning. The work of Kantorovich just before World War II and in his later years played a major role in this current. This work has led to and/or reinforced the great expectations for duality among writers in both the eastern and western intellectual worlds, although Kantorovich himself had rather more modest hopes, to start with at any rate.

The first (1939) Kantorovich formulation appeared in various versions, the last in a 1959 collection of essays on the use of mathematics in economics edited by V.S. Nemchinov. This paper contained numerous examples which had certain structural elements in common and reflected the overall planning and operating context of the Soviet economy, most

notably the type of objective function and constraints. Essentially, the target was to maximize an *output vector* with individual outputs required in fixed proportions. In the case of equal quantities, this may be phrased

$$\text{find } h_{11}, h_{12}, \ldots, h_{21}, \ldots, h_{nm} \geqslant 0$$

which maximize

$$z = z_1 = z_2 = \cdots = z_m$$

subject to

$$\begin{aligned}
&\alpha_{11}h_{11} + \alpha_{21}h_{21} \cdots + \alpha_{n1}h_{n1} = z_1 \\
&\alpha_{12}h_{12} + \alpha_{22}h_{22} \cdots + \alpha_{n2}h_{n2} = z_2 \\
&\alpha_{13}h_{13} + \alpha_{23}h_{23} \cdots + \alpha_{n3}h_{n3} = z_3 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\alpha_{1m}h_{1m} + \alpha_{2m}h_{2m} \cdots + \alpha_{nm}h_{nm} = z_m
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
&h_{11} + h_{12} \cdots + h_{1m} = 1.0 \\
&h_{21} + h_{22} \cdots + h_{2m} = 1.0 \\
&h_{31} + h_{32} \cdots + h_{3m} = 1.0 \\
&\cdots\cdots\cdots\cdots = 1.0 \\
&h_{n1} + h_{n2} \cdots + h_{nm} = 1.0\,.
\end{aligned} \tag{2}$$

The α_{ij} represent the amount of product j producible by machine i per unit time (week, say) and the h_{ij} the proportions of time of machine i to be devoted to production of product j.[2]

Denote by h^*_{ik}, α^*_{ik}, and z^*_i the solution *values* for the *variables* h_{ik}, α_{ik}, and z_i. A trial-and-error procedure would clearly take a very long time. But Kantorovich's method described and illustrated in the first two appendixes to the 1939 article, speeded up the process immensely. Using his method the solution of a five-product, eight-machine problem was reached through his 'method of resolving multipliers' in a series of just six tableaus with calculations based on coefficients λ^s_k, called resolving multipliers, which dictate the re-assignment of machine time at step s according to the absolute productivity advantage of machine i in producing some product k. By adjusting these coefficients at each step (through multiplication by 'correction factors' ε^s) the manager can raise or lower the measured productivities. Internal controls within the program prevent total assigned machine times from exceeding the constraints. Finally, the point is reached where all outputs are equal and the resolving multipliers have values λ^*_k.[3] (Kantorovich takes the equality of outputs itself to constitute proof that production has been maximized has been optimized

(1939, p. 305), although equality could occur even with sub-optimal programs.)

Owing, perhaps, to Kantorovich's omission of the step superscript, the nature of the resolving multiplier λ_k as an *indicator* variable has not always been distinguished from the nature of the *final values* λ_k^* which are taken on when the optimal production assignment has been reached (Two important exceptions which do stress the multipliers' role as an ancillary function are Campbell, 1961 and Charnes and Cooper, 1962.) This auxiliary role might be compared to the part played by the ϑ-ratios in the simplex method.[4]

Greater significance came to attach to the multipliers by the fifties. This was reflected in their new name – 'objectively determined valuations' (*ob'ektivnye obosluvlennye otsenki*, usually written *O.O.O.*), about which a great ideological dispute subsequently arose because of their apparent conflict with Marxian cost of production (Campbell, 1961). These *O.O.O.* could be regarded as equilibrium opportunity costs in the following sense: if instead of the physical output targets, one takes a value measure computed as the inner product of the vector of these multipliers (for the outputs) and the vector of physical outputs, an increase of one unit of output k will contribute the value λ_k to the maximand and consume resources whose use in the production of the other products is at present contributing a like amount. That is, one can only improve in one area by worsening in another.

The qualification in the phrase 'multipliers (for the outputs)' was necessitated by a new tack in the 1959 book. This approach was to assign multipliers to the *equipment being assigned* as well as the outputs. Incidentally, with a little effort the Kantorovich production problem presented above may easily be seen to fit into the transportation problem, with the output *O.O.O.* corresponding to the dual value for the destinations and the input *O.O.O.*, to those for the supplies at the origins. Their various uses are illustrated by Kantorovich. For example, he asks how the equipment assignment for various earthworks should change when a new type of excavation is assigned (1959, Chap. 2), and he shows how to determine this by reference to the multipliers for the outputs and the inputs. In this, Kantorovich is always careful to stress a major background assumption to any use of the *O.O.O.*, one which is often neglected by western writers: the dual variables will be stable only under small departures from the optimal solution.

The background context for the use of the dual solution in Kantorovich is typical of other writers as well. For example, Birman (1959, p. 229) hypothesizes a situation in which the destination delivered prices are set and rents for each supplier under the optimal shipping plan are established and must be paid by the suppliers to the state treasury and reflected in their balance sheet. The important thing to remember is that they are using the dual solution merely as a mechanical device to bring

about the optimal way for discharging plans handed down from above, and, indeed, it is the fact that they are supposed to promote the achievement of such plans and to be conditioned by the aims of those plans, to which Kantorovich and others have made recourse to justify the use of such methods.

4. Conclusion

While the mathematical relationships between the dual and primal problems and those of the equilibrium theorem hold in a strict mathematical sense, the economic interpretations of the dual solution are often formulated in a context which is not consistent with the realities of the market and economic consistency. These violations arise in the failure to recognize that profit rather than selling price is the weighting value factor in the primal.

In particular, if the constraining resources are direct inputs, such as labour, intermediates, or traceable capital consumption, the dual variables are *premiums* over market prices and are *not* market prices, for reasons explained in Section 3.1. However, if the constraints are capacities of fixed facilities *for which, for some reason, no costs are* deducted from the selling prices (leaving them to serve as 'overheads'), the dual valuation does represent the incremental contribution to profit per unit of additional capacity *per unit time*. However, before conclusions may be drawn about expanding capacity, it is vital to determine two things: (i) the range over which expansion of this capacity can contribute linearly to the profit function; and (ii) how many units of time will the capacity expansion last.

How is it possible that such major gaps in the theoretical presentation of the dual solution have persisted so long? A number of influences may be at work.

First, economists tend to think mainly in oversimplified terms in the hope of reaching results which are essentially true, although they might deviate in small particulars. But the truth is that the nature of the problem is often changed and the truth is distorted instead. The purer the theorist, the less likely is it that he/she will engage in consulting work relying on the theoretic results and the less likely will he/she be to question the conventional wisdom. Meanwhile, those concerned with actual applications do not explicitly employ the dual variables but, rather, perform sensitivity analysis to determine the range over which the solution to the primal holds up rather than relying on the dual, and even when they write textbooks, it may be easier to accept the inherited interpretation without worrying too much about it, especially since they may find the sensitivity analysis more meaningful.

Notes

* I would like to thank two anonymous referees for helpful comments on an earlier version. Any remaining errors are mine.

[1] This formulation of the problem, which may be further generalized to include the case of 'at least' as well as 'at most' constraints among the inequalities, is called by many writers 'the general problem' (e.g. Gale; Kolman and Beck), and by others (e.g., Wu and Coppins after modifying it to eliminate the equalities), 'the canonical' problem. These expressions are reversed for the case in which all the constraints are strict equalities. On the other hand, Gale's 'canonical problem' is termed the 'standard problem' by Taha, which is an expression used by Gale for a constraint system containing no equality constraints. Other writers, other variations.

[2] It is more natural to interpret this situation of equal production as components for some overall product, which is Kantorovich's formulation, but the interpretation here helps to point up the generality of the problem.

[3] The step superscript and the final-value asterisk notation for the λ are not employed by Kantorovich in this or subsequent writings.

[4] Another indication of Kantorovich's intentions regarding the multipliers is implicit in the name for his method: '*metod razreshaiushchikh mnozhitelei*' ('method of resolving multipliers'), which he continued to employ in Kantorovich–Gorstko, 1971 (p. 15). Elsewhere the method is referred to as 'method of correcting valuations' ('*metod korrektirovki otsenok*') (Kantorovich–Gorstko, p. 24). Another name for his method was 'method of sequential correction' ('*metod possleddovatel'nyi ispravleniia*'), used by Rubinshtein in his contribution the Nemchinov volume.

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